Higher order corrections to Higgs production at hadron colliders and their phenomenological consequences

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Higher order corrections to Higgs production at hadron colliders and their phenomenological consequences

A dissertation submitted to the

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presented by

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Abstract

Next-to-leading order corrections in perturbative QCD are computed with full mass dependence for Higgs production through gluon fusion, the main production mechanism at hadron colliders. Analytic results for the virtual amplitudes are obtained for fermionic and scalar particles and provide the first independent check of known results. Together with real radiation corrections the virtual corrections are combined into a suitable expression for numerical evaluation using the FKS subtraction method.

The results are incorporated in a new fully differential Monte-Carlo program. Phenomenological important decay channels, $h \rightarrow \gamma\gamma$, $h \rightarrow WW \rightarrow \ell\nu\ell\nu$ and $h \rightarrow ZZ \rightarrow \ell\ell\ell\ell'$ are implemented and arbitrary selection cuts can be applied. It is the first code for Higgs production at next-to-leading order taking into account the full mass dependence of the matrix elements and thus, it will be an useful tool for phenomenological studies at hadron colliders. The mass effects on distributions are studied in detail for the production of a Standard Model Higgs. In agreement with similar studies, we find that mass effects are mild for most distributions but sizable for high transverse momentum tails.

Our code is then used to correct an existing Monte-Carlo program at next-to-next-to-leading order employing the approximation of an infinitely heavy quark mass. Recently computed electroweak virtual and real corrections are integrated as well and most up to date predictions for Higgs production obtained.

The most dominant decay channels are studied and the analysis of experimentally relevant distributions confirms the excellent approximation of the calculation in the effective theory.

First studies at next-to-leading order of light and heavy CP even Higgs production in the Minimal Supersymmetric Standard Model (MSSM) retaining the full mass dependence are performed, using numerical results for more complicated virtual corrections involving many mass scales. We find that in certain regions of the MSSM parameter space these mass effects alter the production rate significantly.
Zusammenfassung

Korrekturen in zweiter Ordnung Störungstheorie in QCD für die Higgs Produktion durch Gluon Fusion an Hadronenbeschleunigern werden berechnet mit voller Massenabhängigkeit. Die analytischen Ergebnisse für fermionische und skalare Teilchen erlauben die erste unabhängige Überprüfung von bekannten Resultaten. Die virtuellen Korrekturen werden dann zusammen mit Bremsstrahlungskorrekturen mittels der FKS Methode kombiniert und in eine für die numerische Auswertung geeignete Form gebracht.


Das Programm wird dazu verwendet, ein existierendes Monte-Carlo Programm zu korrigieren, das Korrekturen dritter Ordnung in der Approximation einer unendlich schweren Quarkmasse enthält. Auch kürzlich bestimmte Korrekturen durch elektroschwache Prozesse werden in das Programm integriert und damit die aktuellsten Vorhersagen zur Higgs Produktion berechnet.

Die wichtigsten Zerfälle werden dann untersucht und die Analyse bestätigt die hervorragende Näherung, die im Rahmen der effektiven Theory berechnet wird.

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Motivation

Since its very beginning mankind has been seeking for the origins of its existence and explanations for nature’s phenomenas. This striving for answers gave rise for the development of religions, culture and science in all its richest forms and eventually empowered human beings, though weak in constitution, to conquer the world and to relax their exposure to nature. Yet, we are still facing fundamental puzzles. What is the origin of the universe that allows our existence? What is the world made of and what keeps it together? Both questions turn out to be closely related and are subject to cosmology and elementary particle physics.

Ancient Greeks believed, that everything was made up from five elements, fire, water, air, earth and ether. Soon, the idea of fundamental particles, atoms, arose. It was not before Rutherford’s experiment in 1909 that the sub-structure of atoms, electrons surrounding a nucleus, was established. Later, in 1932, it was found, that the nucleus of the atom is composed out of protons and neutrons. Experiments at ever larger energies, mainly in the second half of 20th century, unveiled the constituents of these nucleons, the quarks, and many more elementary particles.

All these discoveries would not have been possible without a strong interplay of experiment and theory. Only the advances in both, experimental technology and mathematical formalism, lead to the knowledge we share now and allowed to verify the theoretical models to a marvelous precision.

But also the consistency of different theoretical concepts has proven to be equally fruitful. Maxwell’s unified description of electric and magnetic force, the Maxwell equations, and their property of Lorentz invariance suggested the existence of an universal value for the speed of light, independent of coordinate systems. Einstein promoted this insight into a more general concept, special relativity. When quantum mechanics developed in the 1920’s, it became immediately clear that it does not comply with special relativity, thus triggering Dirac to introduce the Dirac equation in 1928. Though very successful in explaining much of the fine structure of atomic spectra and correctly predicting the anti-particle of the electron, the positron, it could not account for particle creation and annihilation allowed by
special relativity. Only the correct reinterpretation of the Dirac equation lead to relativistic quantum field theory, the theoretical framework of modern elementary particle physics.

The freedom in field theories of assigning any value to the field in each point in space is promoted to a symmetry, local gauge invariance. The mathematical formulation of this invariance introduces naturally massless gauge fields. In Quantum Electrodynamics (QED), the quantized field theoretical formulation of Maxwell’s theory of electro-magnetism, this gauge field is identified with the electro-magnetic field with photons as quanta. QED is today the most precise and best verified physical theory, e.g. it predicts the magnetic moment of the electron with up to eight significant digits in agreement with experiment.

Electro-magnetism is not the only force we observe in nature. First of all, there is gravity which, yet, evades the formulation as a consistent quantum field theory. The other two forces evade every-day’s experience. These are the weak force, responsible for radioactive decay, and the strong force, binding together protons and neutrons in atomic nuclei. The discovery of dozens, if not hundreds, of strongly interacting particles with similar properties, mesons and baryons, suggested that they are composite. Eventually, the postulation of the quarks as elementary particles which carry a new quantum number, color, and whose interactions obey a $SU(3)$ symmetry, could explain mesons and baryons as bound state of quarks. The analogon of the photons in QED are the gluons, the carrier of the strong force, and the interactions of quarks and gluons is described by Quantum Chromodynamics (QCD). The strong interactions among nucleons is, equivalently to van-der-Vaals forces in molecule interactions, not a fundamental force but follows from polarization effects of compound objects.

In spirit of Maxwell’s successful attempt in unifying electric and magnetic force, Glashow, Weinberg and Salam developed a common gauge theory of electro-magnetic and weak interactions, the electroweak or GSW model. Its symmetry group is $SU(2) \times U(1)$. It was found, that the carrier of the weak force, W and Z bosons, are not massless but quite heavy and the direct observation of the Z boson at CERN in 1983 was a milestone in particle physics. Gauge theories forbid massive gauge bosons, however. This is where the Higgs-Brout-Englert mechanism comes to rescue: an additional scalar field with a symmetry breaking ground state is introduced into the theory. The so called spontaneous symmetry breaking of $SU(2) \times U(1)$ symmetry then generates mass terms for W and Z bosons. Simultaneously, this mechanism allows the introduction of mass terms for fermions (i.e. quarks and leptons) which are otherwise forbidden due to the chiral nature of the weak force.

QCD together with electroweak theory is what is known as the Standard Model (SM) of elementary particle physics. It is an extremely powerful and successful theory and its predictions are very well met by experiments. The only missing ingredient is the Higgs
boson, which has not been discovered, yet. Its discovery is the paramount purpose of large collider experiments at Tevatron and, hopefully soon, LHC.

Since the Higgs coupling to light fermions is very weak, heavy particles must first be produced in hadronic collisions which then couple to the Higgs. The main production channels are depicted as Feynman diagrams in Fig. 1. The gluon fusion process is peculiar in that the leading-order process already involves a loop. Due to the large coupling of the top quark to the Higgs boson, it is still the dominant production mechanism at hadron colliders, see Fig. 2. This peculiarity not only renders computations more difficult but implies also that the cross-section for Higgs production is sensitive to any kind of new particles, too heavy to be directly produced, that could virtually be exchanged between the gluons and the Higgs.

It is widely believed that there exists indeed new physics beyond the Standard Model, for several reasons. Though renormalizable and thus formally valid up to arbitrary mass scales, the Standard Model must be an effective theory since it does not include gravitation whose interaction with Standard Model particles gets important near the Planck scale at $10^{19}$ GeV. Already at $10^{16}$ GeV renormalization group arguments suggest the emerging of new phenomena when the strong, weak and electro-magnetic coupling become comparable in size. The sheer existence of such an upper bound for the validity of the Standard Model

![Feynman diagrams for Higgs production processes](image-url)
causes another hint for new physics at energy scales close to what might soon be tested. The Higgs mass is the only relevant operator in the Standard Model, i.e. it is quadratically dependent on the cut-off scale. Experimental data and theoretical arguments require a rather light Higgs which is in conflict with very high cut-off scales. This is the fine-tuning problem.

Many proposals for Beyond the Standard Model (BSM) physics exist addressing one or several of the above mentioned issues or trying to solve still other shortcomings. Some extend the gauge symmetry to larger symmetry groups, others introduce new space-time dimensions. Still others enlarge the group of Lorentz symmetry. This is the idea of supersymmetry. The Minimal Supersymmetric Standard Model (MSSM), although leaving many open questions, at least provides a solution for the fine-tuning problem of the Higgs mass.

In this thesis we will compute higher order corrections in perturbative QCD to Higgs production via gluon fusion taking into account the full mass dependence of particles in the loop without resorting to a simplifying approximation. These corrections are computed both for the SM as well as the MSSM and phenomenological consequences of the mass effects are studied.

The organization is as follows: First, we will cover some of the theoretical aspects in part I. In chapter 1, we review basic concepts of gauge theories and higher order computations in QCD. Also, the SM Lagrangian is presented. The MSSM is then subject of chapter 2.

Calculational details are collected in part II. In order to emphasize the difficulties in higher order corrections, we discuss the computation of the leading order result in chapter 3 before presenting the virtual corrections to Higgs production in chapter 4. Contributions, where additional particles are radiated off, are computed in chapter 5. Both, fermionic and scalar particle contributions are considered, the latter being part of the full MSSM corrections. The implementation into a flexible computer code applying Monte-Carlo algorithms is then presented in chapter 6.

Part III is devoted to examining phenomenological consequences of our calculations. Mass effects in SM Higgs production are studied in detail in chapter 7 at NLO accuracy. Most precise predictions at NNLO accuracy including corrections not considered before in the literature are presented in chapter 8. We conclude our phenomenological studies in chapter 9 with a short study on Higgs production in the MSSM retaining the complete mass effects. Finally we draw our conclusions in chapter 10.

We collect some useful technicalities in the Appendix. Feynman rules used in this thesis are found in Appendix A. The analytic continuation of a special class of functions,
harmonic polylogarithms, is discussed in Appendix B. Appendices C and D contain the analytic results for virtual corrections while results for real contributions are collected in Appendix E. Matrix elements for some of the decay channels are presented in Appendix F.
Part I

Introduction
Chapter 1

Theory overview

The Standard Model is formulated in the language of relativistic quantum field theory, a generalization of quantum mechanics. While in quantum mechanics particles, interacting with fields, are regarded as the fundamental entities, fields are the building blocks of quantum field theory and their excitations create particles. Since a field can assume distinct values in every point in space-time, quantum field theory is the generalization of quantum mechanics to a system with an infinite numbers of degrees of freedom.

This chapter, far from being complete, shall introduce the main concepts of modern elementary particle physics. We start in section 1.1 with the formulation of field theory in terms of path integrals and introduce the Lagrangian, which is the specifying object of every quantum field theory. We then proceed in section 1.2 with discussing a special class of symmetries, i.e. local gauge symmetries, and investigate some implications on possible interactions among fields. In this thesis we are concerned with determining corrections in perturbation theory involving the strong force. The latter and its interactions with fundamental particles is described in Quantum Chromodynamics, introduced in section ??, while some issues of perturbation theory are covered in section 1.5. The protagonist of this thesis is however the Higgs boson, the quantum of a scalar field breaking the gauge symmetry of electroweak theory. Electroweak theory is the unification of the electromagnetic and the weak force and subject of section 1.4. We conclude this chapter in section 1.6 by summarizing the complete Standard Model of elementary particle physics and by pointing out some unsolved puzzles, despite the astonishing success of the theory.

The content of this chapter can be found in numerous standard text books on phenomenological aspects of modern particle physics [2], quantum field theory in general [3–6] and phenomenology of QCD in particular [7, 8].
CHAPTER 1. THEORY OVERVIEW

1.1 Field theoretical formulation

Classical mechanics in its Lagrangian formulation is most elegantly expressed in the principle of stationary action. In quantum field theory, this principle is generalized to fields by replacing the Lagrange function by a Lagrange density. Considering only one scalar field \( \phi \), the principle of stationary action requires

\[
\delta S = 0 \tag{1.1}
\]

with action

\[
S[\phi] = \int d^4x \, L(\phi, \partial \mu \phi) \tag{1.2}
\]

The theory is fully determined by specifying the Lagrangian density (short: Lagrangian), \( L \), as functional of the field \( \phi \).

Quantization of the field theory is most elegantly formulated in terms of path integrals which leads us to introduce the generating functional

\[
Z[J] = \int D\phi \exp i \left( S[\phi(x)] + \int d^4x \, J(x)\phi(x) \right). \tag{1.3}
\]

The second term in the exponential is called the source term and excites quanta of the field \( \phi \). Physical observables are computed from \( n \)-point correlation functions describing the interaction of \( n \) fields. These are obtained by taking functional derivatives of \( Z[J] \). The generating functional can be cast into the form

\[
Z[J] = \mathcal{N} \exp \left( \int d^4x \, \mathcal{L}_{\text{int}} \left( \frac{\delta}{\delta J} \right) \right) \int D\phi \exp i \int d^4x \left[ L_{\text{free}} + J(x)\phi(x) \right] \tag{1.4}
\]

where we have split up the Lagrangian into a free and an interacting part, \( L = L_{\text{free}} + L_{\text{int}} \). If the interaction Lagrangian contains terms proportional to a coupling constant, we can easily derive a perturbative expansion in the coupling constant from expression (1.4) by collecting terms with the same power of the coupling constant.

The exponential containing the free Lagrangian and source terms in (1.4) can be rewritten in terms of a propagator, \( \Delta \),

\[
\int D\phi \exp i \int d^4x \left[ L_{\text{free}} + J(x)\phi(x) \right] = \exp i \int d^4x J^\dagger \Delta J. \tag{1.5}
\]

This propagator is simply the inverse of quadratic operators in \( L_{\text{free}} \). Interaction terms are obtained from \( L_{\text{int}} \) and together they form the set of Feynman rules which allow us to compute correlation functions order by order in the perturbative expansion.

But what is the Lagrangian of our theory? The Standard Model Lagrangian is established from experiments and its structure is dictated by the fundamental particles and their symmetries observed in nature. In the upcoming sections we will see how symmetries constrain our theory before presenting the Standard Model Lagrangian in full.
1.2 Gauge Theories

Symmetries are transformations of fields and coordinates which leave the action invariant. In the following we focus on a particular set of possible transformations and discuss implications of such symmetries.

1.2.1 Abelian gauge invariance

We start with a toy example, scalar Quantum Electrodynamics (QED). The Lagrangian for a complex scalar field is

\[ L = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi. \]  

(1.6)

The symmetry group of QED is \( U(1) \) with the electric charge \( Q \) as generator fulfilling \( Q \phi = Q \phi \phi \), where \( Q \phi \) is the charge quantum number of the field \( \phi \). The field \( \phi \) transforms as follows under \( U(1) \),

\[ \phi'(x) = U \phi(x) = \exp[-i\alpha(x)Q] \phi(x). \]  

(1.7)

In the future we will not write the space-time dependence explicitly anymore and leave out the argument \( x \). Since we require local invariance, the group parameter \( \alpha \) is a function of \( x \). Thus,

\[ \partial_\mu \phi' = \partial_\mu \exp[-i\alpha(x)Q] \phi = \exp[-i\alpha(x)Q] \left\{ \partial_\mu - iQ \partial_\mu \alpha(x) \right\} \phi. \]  

(1.8)

In order to achieve gauge invariance of (1.6), we replace the normal derivative by the covariant derivative,

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu - ieQA_\mu \]  

(1.9)

by introducing an additional field \( A_\mu \) which has the following transformation property

\[ A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha. \]  

(1.10)

We add a kinetic term \(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\) for \( A_\mu \) expressed through the field strength tensor

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]  

(1.11)

and obtain the Lagrangian

\[ L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D^\mu \phi^\dagger D_\mu \phi - m^2 \phi^\dagger \phi. \]  

(1.12)

The only other gauge and Lorentz invariant term involving only the new gauge field \( A \), which is renormalizable, i.e. has mass dimension \( \leq 4 \), is \( \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \). But this term
can be rewritten as a total derivative and thus vanishes when computing the action. In the presence of instanton\(^1\) field configurations, the surface terms are not negligible and the above argument is not valid. Since we consider only perturbative effects, we ignore such terms. In bypassing we note that a mass term \(-m^2 A_\mu A^\mu\) would break gauge invariance.

The construction of the QED Lagrangian for spin-1/2 fields (fermions) proceeds along the same lines starting from the Lagrangian for the free fermion field,

\[
\mathcal{L} = i\bar{\psi}\partial_\mu \gamma^\mu \psi - m\bar{\psi}\psi.
\]

(1.13)

Here, we have introduced the Dirac \(\gamma\) matrices and applied the notation \(\bar{\psi} = \psi^*\gamma^0\). We will often use the shorthand notation

\[
\partial = \partial_\mu \gamma^\mu.
\]

(1.14)

1.2.2 Non-Abelian generalization

In this section we generalize the Lagrangian from section 1.2.1 to the case of \(N\) complex scalar fields,

\[
\mathcal{L} = \partial^\mu \phi_i \partial_\mu \phi_i - (M^2)_{ij} \phi_i^\dagger \phi_j.
\]

(1.15)

We assume invariance of this Lagrangian under a \(SU(N)\) transformation,

\[
\phi_i' = U_{ij} \phi_j
\]

(1.16)

and consider only infinitesimal transformations

\[
U_{ij} = \exp \left[ -i\theta_a T^a \right]_{ij} = \delta_{ij} - i\theta_a T^a_{ij} + \mathcal{O}(\theta^2).
\]

(1.17)

The generators \(T^a\) of the group are Hermitian and fulfill a Lie algebra,

\[
\left[ T^a, T^b \right] = i f^{abc} T^c
\]

(1.18)

with structure constants \(f^{abc}\). The trace of the product of two generators is

\[
\text{Tr} \left( T^a T^b \right) = T_R \delta^{ab}.
\]

(1.19)

We choose \(T_R = \frac{1}{2}\).

The covariant derivative in the non-Abelian case is in analogy to the Abelian case given by

\[
D_\mu = \partial_\mu - ig A_\mu.
\]

(1.20)

\(^1\)Instantons are solutions of the equations of motions which are localized in space and time.
where the gauge field can be expanded in terms of generators, $A_\mu = A_\mu^a T^a$. The covariant derivative transforms as

$$D_\mu \rightarrow D'_\mu = U D_\mu U^\dagger,$$

(1.21)

while the mass matrix $M^2$ transforms as $M'^2 = U M^2 U^\dagger$. Then the invariance of the Lagrangian is apparent. Writing out the covariant derivative in the above expression for the covariant derivative, we find the transformation property of the gauge field to be

$$A'_\mu = U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger = A_\mu - \frac{1}{g} T^a \partial_\mu \theta^a + O(\theta^2) \ .$$

(1.22)

Since we have introduced a new field, we better specify its kinetic term. Analogously to the Abelian case we define the field strength as

$$G_{\mu \nu} = \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu^a T^a - \partial_\nu A_\mu^a T^a - i g A_\mu^a A_\nu^b \left[ T^a, T^b \right] .$$

(1.23)

Unlike before, the commutator no longer vanishes and leads to self interaction among the gauge fields. The commutator also spoils gauge invariance of the field strength tensor. Even the combination $G_{\mu \nu} G^{\mu \nu}$ is not invariant anymore. Luckily, we can restore gauge invariance by taking the trace,

$$\text{Tr} \left( G'_{\mu \nu} G'^{\mu \nu} \right) = \text{Tr} \left( U G_{\mu \nu} U^\dagger U G^{\mu \nu} U^\dagger \right) = \text{Tr} \left( G_{\mu \nu} G^{\mu \nu} \right) .$$

(1.24)

The Lagrangian for complex scalar fields now reads

$$\mathcal{L} = - \frac{1}{4} G_{\mu \nu}^a G^{a, \mu \nu} + \left[ (\delta_{ik} \partial^\mu + i g T^b T^b_{ik} A^{k, \mu}) \phi_i^\dagger \right] \left[ (\delta_{ij} \partial^\mu - i g T^c T^c_{ij} A^c_\mu) \phi_j \right] - (M^2)_{ij} \phi_i^\dagger \phi_j$$

(1.25)

and analogously we can derive the Lagrangian for fermion fields,

$$\mathcal{L} = - \frac{1}{4} G_{\mu \nu}^a G^{a, \mu \nu} + \bar{\psi}_i \left( i \delta_{ij} \partial - i g T^a T^a_{ij} A^a_\mu \right) \psi_j - \bar{\psi}_i (M)_{ij} \psi_j .$$

(1.26)

### 1.2.3 Gauge fixing and ghosts

So far our discussion was based on the classical Lagrangian. When we quantize our theory, more subtleties arise, some of which we will cover in this section. Quantization of gauge theories, particularly of non-Abelian type, is carried out preferably using the method of path integrals.

For simplicity we start with the generating functional for gauge fields only. It is

$$\int D A e^{i S [A]} .$$

(1.27)
Evaluating this expression, one finds divergences which are due to over-counting equivalent field configurations. These equivalent configurations are related through gauge transformations. By imposing a gauge condition $G(A)$, we separate the divergences into an irrelevant, overall normalization factor,

$$\int DA e^{iS[A]} = \left( \int D\alpha \right) \int DA e^{iS[A]} \delta(G(A)) \det \left( \frac{\delta G(A\alpha)}{\delta \alpha} \right).$$  \hspace{1cm} (1.28)

Equally valid is to express the gauge fixing condition as $G(A) - w(x)$ and to integrate over $w(x)$ with a Gaussian weight $\exp\left(i \int d^4x \frac{w^2}{2\xi}\right)$, yielding

$$Z = C' \int DA \exp \left( i \int d^4x (\mathcal{L}[A] - \frac{1}{2}G(A)^2) \right) \det \left( \frac{\delta G}{\delta \alpha} \right).$$ \hspace{1cm} (1.29)

The gauge parameter $\xi$ can be freely chosen. For $G(A) = \partial^\mu A_\mu(x)$ we find the propagator being

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} \left( g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \delta^{ab} e^{-ik(x-y)}. \hspace{1cm} (1.30)$$

Special choices for $\xi$ are for example $\xi = 0$ (Landau gauge) or $\xi = 1$ (Feynman–'t Hooft gauge), our preferred choice.

What is the determinant in (1.29)? After an infinitesimal (non-Abelian) gauge transformation, the gauge field $A^a_\mu$ is given by

$$A^a_\mu + \frac{1}{g} D_\mu \alpha^a = A^a_\mu + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c.$$ \hspace{1cm} (1.31)

In the case of Abelian gauge transformations the last term would be absent and the determinant independent of $A$. But in the case of non-Abelian gauge this is no longer true and we have to find a way how to compute the determinant. The determinant can be rewritten by introducing two anti-commuting fields in the adjoint representation, the Fadeev-Popov ghosts. Then,

$$\det \left( \frac{1}{g} \partial^\mu D_\mu \right) = \int Dc D\bar{c} \exp \left( i \int d^4x \bar{c}(\partial^\mu D_\mu)c \right).$$ \hspace{1cm} (1.32)

The ghosts are no physical particles that appear in final states and it is even possible to avoid them by a proper gauge choice (unitary gauge). However, if we choose to work with a general gauge, ghosts are essential in loop diagrams in order to preserve unitarity.

Following the above discussion, we are lead to include two additional terms into the Lagrangian,

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 \quad \text{and} \quad \mathcal{L}_{\text{ghosts}} = c^a (-\partial^\mu D^a_\mu) c^b,$$ \hspace{1cm} (1.33)

if we are working in a covariant gauge.
1.3.4 BRST symmetry

It turns out that the gauge fixed Lagrangian in a covariant gauge possesses an additional symmetry, first described by Becci, Rouet, Stora and Tyutin. It is called the BRST symmetry and involves an anti-commuting number (Grassmann variable) $\theta$ as gauge parameter. The transformations of the various fields are

\[
\delta_{\text{BRST}} \psi = -g iT^a \theta e^a \psi , \\
\delta_{\text{BRST}} \bar{\psi} = ig \bar{\psi} T^a \theta e^a , \\
\delta_{\text{BRST}} A^a_\mu = \theta (\partial_\mu e^a + gf^{abc} e^b A^c_\mu) , \\
\delta_{\text{BRST}} c^a = \frac{g}{2} f^{abc} e^b c^c \theta , \\
\delta_{\text{BRST}} c^* = -\frac{1}{\xi} \theta \partial_\mu A^\mu ,
\]

This symmetry is helpful in deriving Slavnov-Taylor identities for matrix elements as generalizations of Ward identities in QED.

1.3 Quantum Chromodynamics

1.3.1 Quark model

In order to interpret the rich spectrum of mesons and baryons, Gell-Mann and Zweig proposed in 1963 strongly interacting particles composing these states. They were called quarks. Mesons were explained as $q\bar{q}$ bound states in this model. Baryons on the other hand, should be built out of three quarks. They must have a totally symmetric wave function under interchange of quark spins and flavor quantum numbers in order to explain baryon properties. This contradicts the fermionic nature of quarks. The recovery of the Fermi-Dirac statistics is achieved by introducing a new quantum number, the color, which obey a $SU(3)$ symmetry. Promoting the global to a local symmetry leads to the nowadays well confirmed theory of strong interactions, Quantum Chromodynamics (QCD). The associated gauge bosons are called gluons - glueing together strong interacting particles. All the results discussed in the preceding sections can immediately applied here to find a quantum field theoretical description.

1.3.2 Group structure

The generators in the fundamental representation of the $SU(N_c = 3)$ gauge group, which describes the symmetry under color transformation, obey

\[
[T^a, T^b] = if^{abc} T^c
\]
and are normalized such that
\[ \text{Tr} \left( T^a T^b \right) = T_R \delta^{ab} \]  
(1.40)
with \( T_R = \frac{1}{2} \). The fundamental representation of \( SU(N_c = 3) \) has dimensionality \( d_r = N_c = 3 \). The structure constants are related to the generators in the adjoint representation,
\[ \left( T^A_a \right)_{bc} = i f^{abc} . \]  
(1.41)

The dimension of the adjoint representation is \( d_A = N_c^2 - 1 = 8 \). The quadratic Casimir operator of the fundamental representation, \( C_F \), defined through
\[ T^a T^a = C_F \cdot 1_{d_r \times d_r} , \]  
(1.42)
is given by
\[ C_F = \frac{d_A T^{-1}}{d_r T_R} = \frac{N_c^2 - 1}{2N_c} . \]  
(1.43)
The quadratic Casimir operator of the adjoint representation is
\[ C_A = N_c , \]  
(1.44)
where
\[ T^a_A T^a_A = C_A \cdot 1_{d_A \times d_A} . \]  
(1.45)

### 1.4 Electroweak theory

So far we have discussed some generic properties of gauge theories. We have also seen how strong interactions are formulated as a \( SU(3) \) gauge theory. After discussing generic examples of gauge invariant Lagrangians, we want to specialize now to the case of electroweak theory of the Standard Model. The most interesting property of this theory is that it maximally breaks chiral symmetry. With chiral symmetry breaking we mean that left handed states transform differently from right handed states. Electroweak interactions are based on \( SU(2)_L \times U(1)_Y \) gauge transformations. The subscript \( L \) of \( SU(2)_L \) already denotes that its gauge bosons selectively couples only to left handed states. In particular we have the following transformation properties for left and right handed states, respectively,
\[ \psi_L(x) \rightarrow \psi'_L(x) = \exp \left[ i \alpha(x) \sigma^j \frac{Y}{2} + i \beta(x) \frac{Y}{2} \right] \psi_L(x) , \]  
(1.46)
\[ \psi_R(x) \rightarrow \psi'_R(x) = \exp \left[ i \beta(x) \frac{Y}{2} \right] \psi_R(x) . \]  
(1.47)
Here, we have decomposed states $\psi$ into their left handed ($\psi_L$) and right handed components ($\psi_R$). The $SU(2)_L$ generators are the Pauli matrices $\sigma^j$ and $Y$ denotes the hyper-charge (generator of $U(1)_Y$). The transformation properties of gauge bosons $W^j$, $j = 1, 2, 3$, and $B$ follow directly from the discussions in previous sections.

Each term in the Lagrangian must transform as a singlet under a gauge transformation, if the gauge transformation reflects a symmetry of the Lagrangian. Therefore, mass terms for fermions, which are charged under $SU(2)_L \times U(1)_Y$, are not allowed in electroweak theory, since they mix left and right handed states, $\bar{\psi}_L m \psi_R$. Similarly gauge boson mass terms are not allowed. We will see how fermion and gauge boson masses are generated through the Higgs mechanism in the following sections.

### 1.4.1 Higgs mechanism

We start with a toy example for discussing spontaneous symmetry breaking and the mechanism of generating massive gauge bosons. For simplicity we start with an Abelian example. We consider a complex scalar field coupled to an electromagnetic field and exhibiting also self coupling. The Lagrangian is given by

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi D^\mu \phi^* - V(\phi),$$

with covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ and potential

$$V(\phi) = -\mu^2 \phi \phi^* + \lambda (\phi \phi^*)^2.$$

If $\mu^2 > 0$, the minimum of the potential happens to lie at

$$|\langle \phi \rangle| = \phi_0 = \sqrt{\frac{\mu^2}{2\lambda}}.$$

Choosing a specific direction for the vacuum state, say $\langle \phi \rangle = \phi_0$, the global $U(1)$ symmetry is spontaneously broken. We expand the scalar field around this state and introduce two real valued fields, $\phi_1$ and $\phi_2$:

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2).$$

After replacing the vacuum expectation value, the Lagrangian reads now

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 \mu^2}{2\lambda} (A_\mu)^2 + \frac{1}{2} (\partial_\mu \varphi_1)^2 - \mu^2 \varphi_1^2 + \frac{1}{2} (\partial_\mu \varphi_2)^2$$

$$+ \frac{e\mu}{\sqrt{\lambda}} A_\mu \partial_\mu \varphi_2 + O((\text{fields})^3).$$

While both, the gauge boson and the real scalar field $\varphi_1$, acquired a mass, $m_A^2 = e^2 \mu^2 / \lambda$ and $m_1^2 = 2\mu^2$, respectively, the second real scalar field $\varphi_2$ remains massless. It is the
Goldstone boson of the spontaneously broken symmetry. The last term in (1.52) describes a direct coupling of the gauge boson to the Goldstone boson. Treating the mass term for the gauge boson as an interaction vertex as well, we can compute the vacuum polarization amplitude in leading order and find the required structure to generate a pole in the gauge boson propagator at \( m_A^2 \).

A careful treatment of the gauge fixed Lagrangian using a general covariant gauge shows that the Goldstone boson’s rôle is to cancel the unphysical polarizations of the gauge boson. By choosing a special gauge (unitarity gauge), we can keep \( \phi(x) \) real in every space-time point and eliminate the Goldstone boson \( \varphi_2 \). In turn, the gauge boson receives an additional degree of freedom (the longitudinal polarization).

The generalization of the above discussed mechanism to non-Abelian gauge theories is straightforward and we will explicitly consider such a case when discussing the Glashow-Salam-Weinberg model in 1.4.2. A new phenomenon will arise through breaking the symmetry not in all directions in symmetry space. We find broken and unbroken symmetry generators and gauge bosons related to unbroken generators remain massless.

### 1.4.2 The Glashow-Salam-Weinberg model

The electroweak Lagrangian is invariant under \( SU(2) \times U(1) \) and as we have observed in sections 1.2.1 and 1.2.2 the gauge bosons cannot have mass terms without breaking gauge invariance. In nature, however, the weak gauge bosons, \( W^\pm \) and \( Z^0 \) have rather heavy masses (80.41 GeV and 91.2 GeV, respectively), while the photon remains massless. Accordingly, the weak force is short-ranged while the electromagnetic force mediated by photons is long-ranged.

In the theory of Glashow, Salam and Weinberg, the mass is generated through a Higgs mechanism as discussed in (1.4.1). We introduce a scalar field \( \Phi \), the Higgs field, whose dynamics is governed by the Lagrangian

\[
\mathcal{L}_{\text{Higgs}} = D^\mu \Phi^\dagger D_\mu \Phi - V(\Phi),
\]

\[
V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda \left( \Phi^\dagger \Phi \right)^2,
\]

\[
D_\mu = \partial_\mu + ig \sigma^a_2 W^a_\mu + ig' Y B_\mu.
\]

The covariant derivative restores invariance under \( SU(2)_L \) weak isospin transformations with generators \( \sigma^a \), \( a = 1, 2, 3 \), and corresponding gauge bosons \( W^a \) as well as invariance under \( U(1)_Y \) hyper-charge transformations with hyper-charge \( Y \) and gauge boson \( B \). The coupling constants are given by \( g \) and \( g' \), respectively. The scalar potential \( V \) describes
the famous Mexican hat potential with a minimum for $|\Phi_0|^2 = \frac{v^2}{2} = \frac{\mu^2}{2\lambda} \neq 0$, if $\mu^2 > 0$. The non-zero vacuum expectation value (vev) of $\Phi$ breaks invariance under $SU(2) \times U(1)$.

What is the ground state of $\Phi$? Since we want the photon (zero electric charge) to remain massless even after symmetry breaking, we require, by Goldstone’s theorem, the vacuum $\Phi_0$ to be invariant under the diagonal subgroup $U(1)_\text{em}$ of $SU(2)_L \times U(1)_Y$. The unbroken generator, identified as the electric charge, is given by

$$Q = T_3 + \frac{Y}{2}. \quad (1.54)$$

If $\Phi$ transforms as a doublet under $SU(2)_L$, we can choose the $T_3 = -\frac{1}{2}$ component of the $T = \frac{1}{2}$ isospin doublet with weak hyper-charge $Y = 1$ as ground state, such that $Q\Phi_0 = 0$:

$$\Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (1.55)$$

Then we introduce real fields $\chi_i, i = 1, 2, 3$ and $h$ and expand around the vacuum expectation value $v$:

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_1(x) + i\chi_2(x) \\ v + h(x) + i\chi_3(x) \end{pmatrix}. \quad (1.56)$$

Again as we have seen before in 1.4.1, we can gauge away the $\chi_i$ by a proper redefinition of $\Phi$. This is the unitarity gauge. The new field $h$, however, will remain in the spectrum and is the physical Higgs field.

### 1.4.2.1 Gauge boson masses

We substitute $\Phi$ from (1.56) back into the Lagrangian (1.53) and get terms proportional to $v$ as well as terms proportional to $h$. The former ones have the form of mass terms while the latter describe couplings of the Higgs to gauge bosons. Setting $\Phi = \Phi_0$ and writing out explicitly the Pauli matrices $\sigma^a$, we find the gauge boson mass term,

$$\frac{1}{8} \left| \begin{pmatrix} gW^3_\mu + g'B_\mu \\ g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) - gW^3_\mu + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \quad (1.57)$$

$$= \frac{1}{8} v^2 \left( g(W^1_\mu)^2 + (W^2_\mu)^2 \right) + \frac{1}{8} \left( g'B_\mu - gW^3_\mu \right)^2 \left( g'B_\mu - gW^3_\mu \right) \quad (1.58)$$

$$= \left( \frac{1}{2}vg \right)^2 W^+\mu W^-\mu + \frac{1}{8} (W^3_\mu, B_\mu) \begin{pmatrix} g^2 & -gg' \\ -gg' & g^2 \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}. \quad (1.59)$$

The first term has the form of a mass term for $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$ with mass $m_W = \frac{1}{2}vg$. After diagonalization of the second term, we obtain the physical fields for
the \( Z \) boson and the photon \( A \) as combinations of the third \( SU(2)_L \) gauge boson, \( W^3 \), and the \( U(1)_Y \) gauge boson \( B \):

\[
Z_\mu = \frac{gW^3_\mu - gB_\mu}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad A_\mu = \frac{gW^3_\mu + gB_\mu}{\sqrt{g^2 + g'^2}}. \tag{1.60}
\]

with masses \( m_Z = \frac{v}{\sqrt{g^2 + g'^2}} \) and \( m_A = 0 \). The ratio of the couplings define the weak mixing angle \( \theta_W \) via

\[
\tan \theta_W = \frac{g'}{g} \quad \text{and} \quad \cos^2 \theta_W = \frac{m_W^2}{m_Z^2}. \tag{1.61}
\]

What does happen to the \( \chi \) fields? If we choose the unitarity gauge, they are not in the spectrum anymore. The number of dynamical degrees of freedom (dof) before and after symmetry breaking is conserved, however. Before symmetry breaking, we had three massless \( SU(2) \) gauge bosons (two dofs each), one massless \( U(1) \) gauge boson (two dof) and four degrees of freedom for the Higgs field, twelve degrees of freedom in total. After symmetry breaking we find three massive gauge bosons with three degrees of freedom each, one massless boson (two dof) and one degree of freedom for the Higgs field, again twelve in total, as it has to be. In other gauges we have to take into account the Goldstone bosons in computations. As mentioned before, their effect is to cancel unphysical degrees of freedom of the gauge bosons.

### 1.4.2.2 Fermion masses

By introducing the Higgs field we were able to generate masses for the gauge bosons. Fermions, however, stayed massless so far, since mass terms of the form \( m_D \bar{\psi}_L \psi_R \) are not allowed due to the chirality breaking nature of electroweak interactions. Left handed fermions are charged under \( SU(2)_L \) and \( U(1)_Y \) while right handed fermions are only charged under \( U(1)_Y \). Therefore a Dirac mass term is not allowed. We will now see how to generate fermion masses by coupling fermions to the Higgs field.

We are allowed to add any gauge invariant term of mass dimension less or equal to four to the Lagrangian without spoiling gauge invariance and renormalizability. Therefore we can couple quarks and leptons to the Higgs field via Yukawa terms

\[
\mathcal{L}_{\text{Yukawa}} = -\lambda_e \bar{E}_L \Phi e_R - \lambda_d \bar{Q}_L \Phi d_R - \lambda_u \epsilon^{ab} \bar{Q}_L a^b u_R + \text{h.c.} . \tag{1.62}
\]

The \( SU(2) \) doublets are

\[
E_L = \begin{pmatrix} \nu_e & e_L \end{pmatrix} \quad \text{and} \quad Q_L = \begin{pmatrix} u_L & d_L \end{pmatrix}. \tag{1.63}
\]
When $\Phi$ acquires a vacuum expectation value, (1.62) reads

$$L_{\text{Yukawa}} = -\frac{\lambda_e v}{\sqrt{2}} \bar{e}_L e_R - \frac{\lambda_d v}{\sqrt{2}} \bar{d}_L d_R - \frac{\lambda_u v}{\sqrt{2}} \bar{u}_L u_R$$

$$- \frac{\lambda_e v}{\sqrt{2}} \bar{d}_L h e_R - \frac{\lambda_d v}{\sqrt{2}} \bar{d}_L h d_R - \frac{\lambda_u v}{\sqrt{2}} \bar{u}_L h u_R + \text{h.c.} .$$ (1.64)

The first line in (1.64) are fermion mass terms with masses

$$m_e = \frac{\lambda_e v}{\sqrt{2}}, \quad m_d = \frac{\lambda_d v}{\sqrt{2}} \quad \text{and} \quad m_u = \frac{\lambda_u v}{\sqrt{2}}.$$(1.65)

The second line describes the coupling of fermions to the physical Higgs field $h$ with coupling strength

$$\lambda_i = \sqrt{2 \frac{m_i}{v}}, \quad i = e, u, d.$$ (1.66)

Therefore we often say that the Higgs couples to the mass of fermions.

**CKM matrix**

In the Standard Model we introduce three generations for each fermion family. The Yukawa couplings might mix them. We define mass eigenstates of definite mass by diagonalizing the Yukawa sector,

$$u_L = \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} \quad \text{and} \quad d_L = \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}.$$(1.67)

While the electroweak interaction with the charged gauge bosons are diagonal in the original basis, $u'_L$ and $d'_L$, this is in general not the case anymore for the new basis. The two bases are related through unitary transformations

$$u'_L = U_{ij} u_L^i, \quad d'_L = U_{ij} d_L^i.$$ (1.68)

The coupling to $W^+$ is then given by

$$-i \frac{g}{\sqrt{2}} \frac{1}{2} (\tau^1 + i\tau^2) W^+_\mu u'_L \bar{u}'_L \gamma^\mu d'_L = -i \frac{g}{\sqrt{2}} \frac{1}{2} (\tau^1 + i\tau^2) W^+_\mu u'_L \bar{u}'_L \gamma^\mu \left(U_{ij}^d U_d\right)_{ij} d'_L.$$ (1.69)

and we define the Cabibbo-Kobayashi-Maskawa (CKM) matrix

$$V_{ij} = \left(U_{ij}^d U_d\right)_{ij}.$$ (1.70)

The charged current interactions (1.69) thus allows transitions between quark generations.

Are similar transitions between lepton generations possible? For a long time it was assumed, that neutrinos are massless and therefore right-handed neutrinos were assumed to...
be singlets. Nowadays we know that neutrinos are massive and thus, the mechanism of mixing is completely analogous to the one in the quark sector. There is an additional complication though. Since right-handed neutrinos are singlets, they are their own anti-particle, i.e. neutrinos could be described as Majorana fermions. The according mass term would cause a much more complicated mixing.

1.5 Perturbation theory

In most realistic models we are not able to solve (1.4) exactly. Rather, if the coupling constants appearing in $\mathcal{L}_{\text{int}}$ turn out to be small, we find rescue in doing a perturbative expansion. How this comes about, can be explained by considering a toy example [6]. Consider the ordinary integral

$$Z(J) = \int_{-\infty}^{+\infty} dq \exp \left[ -\frac{1}{2} m^2 q^2 - \frac{\lambda}{4!} q^4 + J q \right].$$

(1.71)

We can write it as

$$Z(J) = \exp \left[ -\frac{\lambda}{4!} \left( \frac{d}{dJ} \right)^4 \right] \int_{-\infty}^{+\infty} dq \exp \left[ -\frac{1}{2} m^2 q^2 + J q \right],$$

(1.72)

which has the structure of (1.4). For this toy example we could solve the integral over $q$ but we won’t. We could expand (1.71) also first in $J$ and then in $\lambda$,

$$Z(J) = \sum_{n=0}^{\infty} \frac{J^n}{n!} \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{\lambda}{4!} \right)^k \int_{-\infty}^{+\infty} dq q^{4k+n} e^{-\frac{1}{2} m^2 q^2} \right\}. \quad (1.73)$$

In quantum field theory, $J$ corresponds to sources for field excitations, i.e. particles. Therefore, (1.73) can be interpreted as an expansion in numbers of external particles (= powers of $J$) with coefficients (= Green’s functions) given as an expansion in the coupling constant $\lambda$. The $q$ integral has to be replaced by a path integral in theories, we are considering. In $\phi^4$ theory, the direct analogon to (1.73) is

$$Z[J] = \mathcal{N} \sum_{n=0}^{\infty} \frac{1}{n!} J(x_1) \cdots J(x_n) \sum_{k=0}^{\infty} \frac{1}{k!} \int D\phi \phi(x_1) \cdots \phi(x_n)$$

$$\times \left( -\frac{\lambda}{4!} \right)^k \left( \prod_{l=0}^{k} \int d^4 z_l \phi(z_l)^4 \right) e^{i \int d^4 x \left[ \frac{1}{2} ((\partial \phi)^2 - m^2 \phi^2) \right]} \quad (1.74)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} J(x_1) \cdots J(x_n) G_n(x_1, \ldots, x_n).$$

$\mathcal{N}$ is an irrelevant normalization and $G_n$ are the $n$-point Green’s functions which have an expansion in $\lambda$ that can be read off from (1.74). When computing cross sections we
1.5. PERTURBATION THEORY

actually compute Green’s functions. It turns out that it is more advantageous to work in momentum space rather than in position space, so we consider the Fourier transform of (1.74). The integrals that appear in $G_n$ can be visualized by Feynman diagrams.

Since the SM Lagrangian is much more complicated than the $\phi^4$ Lagrangian, many more features arise but we will not dwell on them any further. Feynman rules for computing higher terms in the perturbative expansion are listed in A, if they are relevant for this work.

In the upcoming subsections we will very quickly cover some issues important for perturbative computations.

1.5.1 Regularization

When working in momentum space, we encounter typically divergent integrals of the type

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2}$$

(1.75)

when computing higher order corrections to a particular $n$-point function. This spoils of course the predictive power of the theory. We could still try to regularize these integrals and hope that physical observables do not depend on this regulator. As a first attempt we introduce a UV cut-off. This is justified, if we assume that our theory is only valid up to a certain scale $\Lambda$. Then our integral is given by

$$\int_0^\Lambda \frac{\kappa^3d\kappa d\Omega_4}{(2\pi)^4} \frac{1}{\kappa^2 - m^2} = \frac{1}{6\pi^2} \left( \frac{\Lambda^2}{2} + \frac{m^2}{2} \ln \frac{\Lambda^2 - m^2}{-m^2} \right).$$

(1.76)

This is a valid approach and in certain situations this is the method of choice. However, the drawback of this method is, that the integral is not shift invariant anymore and therefore also spoils gauge invariance.

In QCD we mostly apply another regularization, dimensional regularization. Examining our toy example, we realize that it is convergent, if we evaluate it in a smaller number of dimensions, say in one dimension. Then we find

$$\int \frac{dk}{(2\pi)^4} \frac{1}{k^2 - m^2} = \frac{i}{2m}.$$

(1.77)

The idea is simple: Evaluate the divergent integrals in $d$ dimensions, where it is convergent, regard the result as an analytic function in $d$ and then analytically continue the result to the neighborhood of 4. The result is then given as a Laurent series in a small parameter
\[\epsilon = (4 - d)/2.\] In our example we find
\[
\mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} = -\mu^{2\epsilon} \frac{i}{(4\pi)^{2-\epsilon}} \Gamma (-1 + \epsilon) \left( \frac{1}{m^2} \right)^{-1+\epsilon}
\]
\[= \frac{im^2}{(4\pi)^2} \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi + 1 - \ln \frac{m^2}{\mu^2} + O (\epsilon^2) \right).\] (1.78)

In order to maintain the correct mass dimensions we have introduced the mass scale \(\mu\) to the appropriate power. The Euler-Mascheroni constant \(\gamma_E\) results from the expansion of the Gamma function in \(\epsilon\). Note that for \(d = 1\) we recover our one dimensional result (dropping the \(\mu\) scale) up to a minus sign. This minus sign is a consequence of performing a Wick rotation in computing our \(d\) dimensional integral in order to transform Minkowskian space into Euclidean space. We have ignored this subtlety in computing the one dimensional result.

1.5.2 Renormalization

Now, that we have regularized the divergences of our theory, we want to extract finite physical results. This is done by introducing renormalized fields and parameters, where bare and renormalized quantities are related through renormalization constants \(Z_i\),

\[
\psi_0 = Z_{\psi}^{1/2} \psi, \quad m_0 = Z_{m\bar{\psi}\psi}^{-1} m = Z_m m, \quad (1.79)
\]
\[
A_\mu^0 = Z_A^{1/2} A^\mu, \quad g_{s0} = Z_g \mu^\epsilon g_s, \quad (1.80)
\]
\[
c_0 = Z_c^{1/2} c, \quad \xi_0 = Z_A \xi. \quad (1.81)
\]

The subscript 0 denotes bare quantities, while quantities without subscript are renormalized. We write \(Z_i = 1 + \delta Z_i\), where \(\delta Z_i = \delta Z_i^{(1)} + \delta Z_i^{(2)} + \cdots\) is expanded in the strong coupling \(\alpha_s\). We then split up the Lagrangian in a part having the same form as the original Lagrangian but in terms of renormalized fields and another part containing counter-terms (terms modified by \(\delta Z_i\),

\[\mathcal{L}_{\text{renorm}} = \mathcal{L} + \mathcal{L}_{\text{counter}}.\] (1.82)

It can be shown that in renormalizable theories all divergences occurring in loop diagrams are cancelled by a finite number of counter-terms of the form of interaction terms of mass dimension less or equal four. In non-renormalizable theories, we have to introduce new counter-terms with higher mass dimensions at each order in perturbation theory.

The renormalization constants are determined by requiring certain quantities (quark mass, strong coupling, \ldots) at a specific scale to have determined values. These are called
1.5. PERTURBATION THEORY

renormalization conditions. These conditions are scheme dependent. In the modified minimal subtraction scheme (\(\overline{\text{MS}}\)) for example, we require for the fermion self energy

\[
\delta Z_{m}^{\overline{\text{MS}}} = \text{Re} \, \Sigma^{\text{div}} (p^2 = m^2),
\]

(1.83)

where \(\Sigma^{\text{div}}\) is the part proportional to \(\Delta_e = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi\). As a consequence, the mass in the \(\overline{\text{MS}}\) scheme is running, \(m_{\text{renorm}}^{\overline{\text{MS}}} = m_{\text{renorm}}^{\text{OS}}(Q^2)\). In contrast, when applying the on-shell scheme (OS), we choose

\[
\delta Z_{m}^{\text{OS}} = \text{Re} \, \Sigma (p^2 = m^2),
\]

(1.84)

such that \(m_{\text{renorm}} = m_{\text{pole}}\). Here, \(m_{\text{pole}}\) is the pole of the fermion propagator. We will discuss renormalization in more details for Higgs production in 4.4.

1.5.3 Running Coupling, Running Masses and Asymptotic Freedom

Because we are forced to renormalize our perturbation theory, we introduce a renormalization scale \(\mu_R\) at which our renormalization conditions are enforced. Now consider a dimensionless physical observable \(R\) which depends on a single energy scale \(Q\). Hence, \(R\) depends after renormalization on the ratio \(Q^2/\mu_R^2\) and the renormalized \(\alpha_s\). Since \(\mu_R\) is arbitrary, \(R\) fulfills the following differential equation,

\[
\mu_R^2 \frac{d}{d\mu_R^2} R \left( \frac{Q^2}{\mu_R^2}, \alpha_s \right) = \left( \mu_R^2 \frac{\partial}{\partial \mu_R^2} + \mu_R^2 \frac{\partial \alpha_s}{\partial \mu_R^2} \frac{\partial}{\partial \alpha_s} \right) R \left( \frac{Q^2}{\mu_R^2}, \alpha_s \right) = 0.
\]

(1.85)

The so called Renormalization Group Equation can be solved by introducing the running coupling \(\alpha_s(Q^2)\) via

\[
\ln \left( \frac{Q^2}{\mu_R^2} \right) = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)},
\]

(1.86)

where \(\alpha_s(\mu) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}\) is the beta function.

The \(\beta\) function is obtained from higher-order corrections to the vertices of QCD. At leading order \(\beta(\alpha_s)\) is given by

\[
\beta(\alpha_s) = -\frac{11C_A - 2n_f}{12\pi}.
\]

(1.87)

In QCD we have \(C_A = 3\). \(n_f\) is the number of active light flavor, i.e. with masses much smaller than \(Q\). We immediately conclude that for \(n_f < 16\) the \(\beta\) function is negative and that accordingly \(\alpha_s(Q^2)\) decreases with increasing energy scale \(Q^2\). This is called asymptotic freedom.
Similarly we introduce the running mass by considering the dependence of $R$ on the mass of a single quark. Applying a scheme in which the $\beta$ function and the mass anomalous dimension, $\gamma_m = -\frac{1}{2\mu_R^2}\frac{\partial m}{\partial \mu_R^2}$, are only dependent on $\alpha_s$, we find the following renormalization group equation

$$
\left(\mu_R^2 \frac{\partial}{\partial \mu_R^2} + \mu_R^2 \frac{\partial \alpha_s}{\partial \mu_R^2} \frac{\partial}{\partial \alpha_s} - \gamma_m(\alpha_s)m \frac{\partial}{\partial m}\right) R \left(\frac{Q^2}{\mu_R^2}, \alpha_s, m \frac{Q}{\mu_R^2}\right) = 0.
$$

(1.88)

Another differential equation is obtained by observing that $R$ must be independent of a rescaling of all mass scales:

$$
\left(\mu_R^2 \frac{\partial}{\partial \mu_R^2} + Q^2 \frac{\partial}{\partial Q^2} + m^2 \frac{\partial}{\partial m^2}\right) R \left(\frac{Q^2}{\mu_R^2}, \alpha_s, m \frac{Q}{\mu_R^2}\right) = 0.
$$

(1.89)

The combined equation is solved through the running mass $m(Q^2)$ such that

$$
Q^2 \frac{\partial m(Q^2)}{\partial Q^2} = -\gamma_m(\alpha_s)m(Q^2).
$$

(1.90)

1.5.4 From matrix elements to cross sections and decay widths

We are interested in the probability that a certain initial state interacts and produces a certain final state. Initial and final states are approximated by asymptotic, non-interacting particles (wave function). In a typical collider experiment, we have two colliding particles producing $n$ particles in the final state. As we go to momentum space, we actually have to compute the following object

$$
\langle p_1 p_2 \cdots | i T | q_A q_B \rangle_{\text{out}} = \langle p_1 p_2 \cdots | S | q_A q_B \rangle_{\text{in}}
$$

(1.91)

where we have introduced the S-matrix $S$. Since we are not interested in non-interacting processes, we write $S = 1 + iT$ and focus on $T$. We introduce the invariant matrix element $\mathcal{M}$ by factoring out a global delta function assuring total momentum conservation,

$$
\langle p_1 p_2 \cdots | iT | q_A q_B \rangle = (2\pi)^4\delta^{(4)}(q_A + q_B - \sum p_f) \cdot i \mathcal{M}(q_A, q_B \rightarrow p_f).
$$

(1.92)

In reality particles do not have a definite momentum since they are localized in space. Rather, they are described by wave packets and we have to integrate over momenta. This leads then to the definition of the cross section. We discuss now a further issue particular to QCD before we define the cross section in terms of matrix elements.

The elementary particles in QCD, quarks and gluons, cannot be observed isolated in nature due to color confinement. The potential energy between color charges increases linearly with distance and thus, it takes an infinite amount of energy to separate them. Only color neutral states can propagate freely and be observed. This fact is also expressed
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through the behavior of the strong coupling constant $\alpha_s$ which has a singularity at a finite energy, $\Lambda_{\text{QCD}}$. While QCD is dominated by non-perturbative effects at low energy, because of asymptotic freedom, we can compute interactions at higher energies in perturbative QCD (pQCD).

In hadron colliders, protons $p$ and/or anti-protons $\bar{p}$ get accelerated and eventually brought to collision in order to study what is produced in this interaction. These hadrons are strongly bound objects described by non-perturbative QCD. As we probe them at higher energies, more details of their inner structure gets revealed and at a certain hard scale, the interaction is well described by considering individual partons instead of the whole hadron (factorization theorem). We express the probability to find a certain parton $i$ with momentum fraction $x$ in a hadron $h$ at a scale $\mu_F$ through a parton density function (PDF), $f_{ij/h}(x; \mu_F)$. The factorization scale $\mu_F$ is to be viewed as the scale separating the long distance or soft physics from the short distance or hard physics. The cross section for two hadrons $h_1$ and $h_2$ with momenta $P_1$ and $P_2$, respectively, producing a final state $X$ is given by

$$d\sigma(P_1, P_2) = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 f_{i/h_1}(x_1, \mu_F) f_{j/h_2}(x_2, \mu_F) d\hat{\sigma}_{ij} \left( x_1 P_1, x_2 P_2; \alpha_s(\mu_R), \frac{\hat{s}}{\mu_R^2}, \frac{\hat{s}}{\mu_F^2} \right).$$

(1.93)

d$\hat{\sigma}_{ij}$ is called the partonic cross section. The center of mass energy squared of the hadrons is

$$S = (P_1 + P_2)^2$$

(1.94)

while the partons $i = 1, 2$ carry only a fraction of the hadron momenta, $p_i = x_i P_i$, and collide with the partonic center of mass energy squared

$$\hat{s} = (p_1 + p_2)^2 = x_1 x_2 S.$$  

(1.95)

The partonic cross section is computable in pQCD and we will see in Part II in detail how it is computed for Higgs production. For the process $i + j \rightarrow X$ it is decomposed as

$$d\sigma = d\Pi_X \times T(i + j \rightarrow X)$$

(1.96)

with phase space,

$$d\Pi_X = \left( \prod_f \frac{d^3 p_f}{(2\pi)^3 2 E_f} \right) (2\pi)^4 \delta^{(4)} \left( p_i + p_j - \sum_f p_f \right),$$

(1.97)
and the color and spin summed and averaged matrix element squared multiplied with the flux factor

\[ \Upsilon(i + j \rightarrow X) = \Phi_{ij} \times \frac{1}{\omega_i \omega_j} \sum_{\text{spin}} \sum_{\text{color}} |M(i + j \rightarrow X)|^2 . \]  

(1.98)

The averaging factors for the incoming partons in conventional dimensional regularization, \( \omega_i \), are given by

\[ \omega_q = \omega_{\bar{q}} = 2N_c \quad \text{and} \quad \omega_g = 2(1 - \epsilon)(N_c^2 - 1). \]  

(1.99)

We postpone the discussion of different regularization schemes to Part II. The flux factor is given by

\[ \Phi_{ij} = \frac{1}{E_i E_j |v_i - v_j|} . \]  

(1.100)

The quantities appearing in (1.100) are the energy, \( E_{i/j} \), and velocity, \( v_{i/j} \), of parton \( i/j \). The product over momenta in (1.97) is taken over all particles \( f \) in final state \( X \).

If a particle \( A \) decays into a final state \( X \), the partial decay width is obtained from

\[ d\Gamma = \frac{1}{E_A} \times d\Pi_X \times \sum_{\text{spin}} \sum_{\text{color}} |M(A \rightarrow X)|^2 \]  

(1.101)

where the phase space is basically given by (1.97) but the sum \( p_i + p_j \) replaced by the momentum of particle \( A \), \( p_A \) with energy \( E_A \).

We want to make a last remark before turning towards higher order corrections. The sums over spins and colors in (1.96) and (1.101) are only taken if spin and color is not observed. Similarly, averaging over initial state spins is performed only if incoming partons are not polarized.

1.6 The Standard Model

In the first few sections we have discussed separately the different sectors of the full theory known as the Standard Model. For completeness, we write down the full Standard Model Lagrangian in terms of physical fields in section 1.6.1. The particle content of the Standard Model is collected in Table 1.1.
We present now the Standard Model Lagrangian after electroweak symmetry and in unitarity gauge, where no unphysical fields appear. We define

\[ W_{\mu}^{\pm} = \partial_{\mu}W_{\nu}^{\pm} - \partial_{\nu}W_{\mu}^{\pm}, \]
\[ Z_{\mu\nu} = \partial_{\mu}Z_{\nu} - \partial_{\nu}Z_{\mu}, \]
\[ F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \]  

The Standard Model Lagrangian can be written as follows:

\[ \mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{electroweak}} + \mathcal{L}_{\text{strong}}, \]
\[ \mathcal{L}_{\text{Dirac}} = i\bar{e}_L^i \gamma^\alpha \partial_\alpha e_L^i + i\bar{e}_R^i \gamma^\alpha \partial_\alpha e_R^i + i\bar{\nu}_L^i \gamma^\alpha \partial_\alpha \nu_L^i + i\bar{\nu}_R^i \gamma^\alpha \partial_\alpha \nu_R^i + i\bar{d}_L^i \gamma^\alpha \partial_\alpha d_L^i + i\bar{d}_R^i \gamma^\alpha \partial_\alpha d_R^i, \]
\[ \mathcal{L}_{\text{Gauge}} = -\frac{1}{4} (\partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a)^2 - \frac{1}{2} W_{\mu\nu}^+ W_{\nu\mu}^- - \frac{1}{4} Z_{\mu\nu}^a Z_{\mu\nu}^a - \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a, \]
\[ \mathcal{L}_{\text{Higgs}} = \frac{1}{2} (\partial_{\mu}h)^2 - \frac{1}{2} m_h^2 h^2 - \sqrt{\frac{\lambda}{2}} m_h^2 h^4 - \frac{1}{4} \lambda h^4 + \left[ m_{\tilde{W}}^2 W_{\mu\nu}^+ W_{\nu\mu}^- + \frac{1}{2} m_{Z}^2 Z_{\mu}^\nu Z_{\mu}^\nu \right] \left( 1 + \frac{\tilde{h}}{v} \right)^2, \]
\[ L_{\text{Yukawa}} = -m_u^i \bar{u}^i_L c_R \left( 1 + \frac{h}{v} \right) - m_d^i \bar{d}_L u_R^i \left( 1 + \frac{h}{v} \right) - m_t^j \bar{t}_L d_R^j \left( 1 + \frac{h}{v} \right) + \text{h.c.}, \]  
(1.107)

\[ L_{\text{EW}} = ig \cos \theta_W \left[ (W_{\mu}^- W_{\nu}^+ - W_{\mu}^+ W_{\nu}^-) \partial^\mu Z^\nu + W_{\mu}^+ W_{\nu}^- Z^\nu - W_{\mu}^- W_{\nu}^+ Z^\nu \right] 
+ ie \left[ (W_{\mu}^- W_{\nu}^+ - W_{\mu}^+ W_{\nu}^-) \partial^\mu A^\nu + W_{\mu}^+ W_{\nu}^- A^\nu - W_{\mu}^- W_{\nu}^+ A^\nu \right] 
+ g^2 \cos \theta_W \left( W_{\mu}^+ W_{\nu}^- Z^{\mu} Z^\nu - W_{\mu}^+ W_{\nu}^- Z^\mu Z^\nu \right) 
+ g^2 \left( W_{\mu}^+ W_{\nu}^- A^\mu A^\nu - W_{\mu}^+ W_{\nu}^- A_{\nu} A^\mu \right) 
+ ge \cos \theta_W \left[ W_{\mu}^+ W_{\nu}^- (Z^\mu A^\nu + A^\mu Z^\nu) - 2W_{\mu}^+ W_{\nu}^- Z^\nu A^\nu \right] 
+ \frac{1}{2} g^2 \left( W_{\mu}^+ W_{\nu}^- \left( W_{\nu}^+ W_{\nu}^- - W_{\nu}^+ W_{\nu}^- \right) \right) 
- g \left( W_{\mu}^+ \gamma^\nu W_{\nu}^- J_{\mu W^+} + W_{\mu}^- J_{\mu W^-} + Z_{\mu} J_{Z}^\mu \right) - e A_{\mu} J_{\mu}^A, \]  
(1.108)

\[ L_{\text{strong}} = -\frac{g_s}{2} f^{abc} \partial_\mu A_{\mu}^a A_{\mu}^b A_{\mu}^c + g_s^2 f^{abc} f^{ade} A_{\mu}^a A_{\nu}^b A_{\mu}^c A_{\nu}^d - g_s A_{\mu} J_{A}^\mu. \]  
(1.109)

In order to abbreviate the expressions, we have introduced the following currents,

\[ J_{\mu}^{a \mu} = \bar{u}^i \gamma^\mu T^a u^i + \bar{d}^i \gamma^\mu T^a d^i, \]  
(1.110)

\[ J_{\mu W^+} = \frac{1}{\sqrt{2}} \left( \bar{\nu}^i_L \gamma^\mu \nu^i_L + V^{ij} \bar{u}^j_L \gamma^\mu d^j_L \right), \]  
(1.111)

\[ J_{\mu W^-} = (J_{\mu W^+})^*, \]  
(1.112)

\[ J_{\mu}^{A} = \left( -1 \right) \bar{\nu}^i \gamma^\mu \nu^i + \frac{2}{3} \bar{u}^i \gamma^\mu u^i + \frac{1}{3} \bar{d}^i \gamma^\mu d^i. \]  
(1.114)

From these expressions we can read off all possible particle interactions in the Standard Model and derive Feynman rules. It is impressive how simple symmetry principles can generate such a rich structure of particle interactions, which lead to an astounding accurate description of nature on the quantitative level.

There are 19 parameters in the Standard Model. 9 of those are the fermion masses, thus determining the Yukawa couplings, 3 real angles and 1 CP-violating phase in the CKM matrix, 3 coupling constants at a certain scale (usually \( m_Z \)), the quadratic coupling of the Higgs and its self-coupling and the QCD vacuum angle, \( \theta_{\text{QCD}} \). In our form of the SM Lagrangian we have neglected the pseudo-scalar \( SU(3) \) term related to \( \theta_{\text{QCD}} \) since experimentally it must be extremely small. Apart from the fermion masses and mixing angles and phases, we have 5 parameters describing the Standard Model. We can choose
them as
\[ G_F, \ m_Z, \ m_W, \ m_h, \ \text{and} \ \alpha_s(m_Z). \] (1.115)

where \( G_F \) is Fermi’s constant. We list here some tree level relations among the parameters:
\[
\cos \theta_W = \frac{m_W}{m_Z}, \tag{1.116}
\]
\[
e = g \sin \theta_W, \tag{1.117}
\]
\[
g^2 = \frac{8m_W^2}{\sqrt{2}} G_F = \frac{4m_W^2}{v^2}, \tag{1.118}
\]
\[
v = \sqrt{\frac{\mu^2}{\lambda}}, \tag{1.119}
\]
\[
m_h = \sqrt{2 \mu}. \tag{1.120}
\]

1.6.2 Shortcomings of the Standard Model

Despite its marvelous success in explaining observations, the Standard Model still has its deficiencies. First of all, we know four fundamental forces, the electromagnetic, weak and the strong force as well as gravity, but the Standard Model describes only three of them. This is legitimate in the energy regime of modern particle physics experiments, as the gravitational force only gets comparable at energy scales close to the Planck mass, \( M_P = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19} \text{GeV} \). At the weak scale, \( M_{\text{weak}} \approx 100 \text{GeV} \), the gravitational force is about 25 magnitudes weaker than the weak force and 38 magnitudes weaker than the strong force. So far no consistent quantum field theory for gravitation exists, that would allow us to treat the gravitational interactions on the same footing as the other three forces. Does a theory of everything exists (TOE)?

It has been a great success to describe electromagnetism and weak force in a unified theory, which is the GSW theory. We are tempted to expect the same emerging with the strong force. Indeed, extrapolating the interaction strengths of the three forces to high energies, they almost get equal at the same scale of about \( \approx 10^{16} \text{GeV}/c^2 \) but not exactly. Is there a great unified theory (GUT) describing electroweak and strong force as part of one and the same theory?

For a long time we believed that neutrinos are massless. Recent experiments have established mixing of neutrino flavors. Accordingly lepton number must not be conserved any more. Yet, so far no mixing of charged leptons has been observed. Mixing can only occur, if mass and flavor eigenstates are not the same, which implies in turn a non-vanishing neutrino mass for at least one neutrino flavor. But the SM Lagrangian does not foresee neutrino masses and needs to be extended. What is the nature of neutrinos?
We have been forced to introduce a scalar field into the theory in order to account for masses while not spoiling local gauge invariance. The quanta of this field, the Higgs bosons has not been found yet. Does the Higgs boson exist?

From theoretical considerations as well as from experimental no-observations, one can set bounds on the Higgs mass. It is assumed, that the Higgs mass must lie in the range \( 114.4 \text{ GeV} < m_h \lesssim 200 \text{ GeV} \). This leads to the infamous hierarchy problem: Why is the Higgs mass so much smaller than the Planck scale?

We can also consider higher order corrections to the Higgs mass. We find that these corrections depend quadratically on the scale of new physics and must be compensated by appropriate renormalization. If we set the Planck scale as the scale of new physics, we might ask the question: How can we justify the fine tuning up to 16 digits of the Higgs mass?

Besides of the above mentioned problems and questions unsolved in the Standard Model, we also observe deviations of SM predictions and experimentally measured quantities. The most prominent deviation occurs in the measurement of the muon anomalous magnetic moment, \( g_\mu - 2 \), which is one of the most precisely measured quantities in particle physics. Its measured value deviates from its SM predictions by about 3 standard deviations. See [9] and references therein. Is \( g_\mu - 2 \) pointing to new physics just around the corner?

Astrophysical observations strongly suggest the existence of a so far unknown kind of matter, called dark matter, responsible for about 23% of the energy density in the universe [10]. Ordinary matter contributes with 4.6% only with a small fraction. What is dark matter?

In this thesis we will not pursue these questions any further. However, they hint towards the incompleteness of the Standard Model and justify the quest for models beyond the Standard Model. Even if we ignore the open questions above, it seems very unlikely that no new physics shall appear in 16 orders of magnitude spanning the range between the electroweak and the Planck scale. As one example of such a beyond the Standard Model (BSM) model, we will present the Minimal Supersymmetric Standard Model (MSSM) [11] in the next chapter, which is the most studied extension of the SM. But there exist many more BSM models, extending the gauge group (e.g. [12]) or introducing additional space-time dimensions (e.g. [13]).
Chapter 2

The Minimal Supersymmetric Standard Model

At the end of section 1.6.2 we already mentioned that the Higgs mass potentially receives large corrections in perturbation theory. This is the starting point for a motivation for supersymmetry. The discussion in this chapter follows closely [11].

Imagine that the Higgs field couples to fermion $\psi$ through a term $-\lambda_\psi h \bar{\psi} \psi$ in the Lagrangian. At one-loop the Higgs mass receives a correction through a fermion loop of the form

$$\Delta m^2_h = -\frac{|\lambda_\psi|^2}{8\pi^2} \Lambda^2_{UV} + \ldots$$

when regularized with a UV-Cutoff. $\Lambda_{UV}$ should be interpreted as the scale of new physics and if this is $M_P$, the correction must be compared to the Higgs mass $m_h$, which must be of the order of 100 GeV. Therefore, the physical Higgs mass is obtained from cancelling 32 digits in two quantities (the bare Higgs mass and the one-loop correction)! This is not very elegant and is again the hierarchy problem.

If we couple the Higgs to a complex scalar field $\phi$ through $-\lambda_\phi |h|^2 |\phi|^2$, $\phi$ would contribute to the Higgs mass correction with

$$\Delta m^2_h = \frac{\lambda_\phi}{16\pi^2} \Lambda^2_{UV} + \ldots$$

Comparing (2.2) and (2.1), we notice that the corrections come with opposite signs. If the fermion and the scalar coupling were related by $\lambda_\phi = 2|\lambda_\psi|$, the quadratic dependence on $\Lambda_{UV}$ would cancel. This suggests an additional symmetry between fermions and bosons, what we call supersymmetry.
CHAPTER 2. THE MINIMAL SUPERSYMMETRIC STANDARD MODEL

We introduce the supersymmetry operator $Q$, which relates fermions and bosons to each other,

$$Q|\text{Boson}\rangle = |\text{Fermion}\rangle, \quad Q|\text{Fermion}\rangle = |\text{Boson}\rangle.$$  \hfill (2.3)

Since $Q$ produces a spinor from a bosonic field, it carries fermionic quantum numbers, in particular spin $1/2$. It is shown that in chiral theories, $Q$ and its hermitian conjugate $Q^\dagger$ must fulfill the following commutation and anti-commutation algebra:

$$\{Q_\alpha, Q^\dagger_\dot{\alpha}\} = 2\sigma_{\alpha\dot{\alpha}} P_\mu,$$
$$\{Q_\alpha, Q_\beta\} = 0, \quad \{Q^\dagger_\dot{\alpha}, Q^\dagger_\dot{\beta}\} = 0,$$
$$[Q_\alpha, P^\mu] = 0, \quad [Q^\dagger_\dot{\alpha}, P^\mu] = 0.$$  \hfill (2.4-2.6)

Here we have introduced dotted and undotted indices, whose meaning we will explain in section 2.1. (2.6) shows that the supersymmetry generators commute with the space-time momentum operator $P^\mu = (H, P)$ and therefore supersymmetry is a symmetry of our theory.

This chapter is organized as follows: In section 2.1 we will introduce the notion of supermultiplets, which contain an equal number of fermionic and bosonic degrees of freedom of the same mass and describe how they are described. Possible terms in a supersymmetric Lagrangian are discussed in section 2.2 before we specialize in section 2.3 to a realistic realization of supersymmetry in terms of a minimal supersymmetric extension of the Standard Model, the Minimal Supersymmetric Standard Model (MSSM). Finally, we identify a region of minimal fine tuning in section 2.4.

2.1 Superfields and superpotentials

When discussing supersymmetric Lagrangians, it turns out to be advantageous to work with 2 component Weyl spinors instead of 4 component Dirac spinors. We choose a specific representation of gamma matrices,

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$  \hfill (2.7)

with Pauli matrices

$$\bar{\sigma}_0 = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\sigma}_1 = -\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\bar{\sigma}_2 = -\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \bar{\sigma}_3 = -\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hfill (2.8)$$
Then we write a Dirac spinor $\Psi_D$ in terms of Weyl spinors, $\xi$ and $\chi$:

$$\Psi_D = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger} \dot{\alpha} \end{pmatrix}$$  \hspace{1cm} (2.9)

where $\xi$ is called a left-handed Weyl spinor and $\chi^{\dagger}$ a right-handed Weyl spinor. We apply the following convention for the contraction of spinor indices:

$$\xi\chi = \xi^\alpha\chi_\alpha = \xi^\alpha\epsilon_{\alpha\beta}\chi^\beta = -\chi^\beta\epsilon_{\alpha\beta}\xi^\alpha = \chi^\beta\xi_\alpha. \hspace{1cm} (2.10)$$

Now, the Dirac Lagrangian

$$L_{\text{Dirac}} = -i\bar{\Psi}_D \gamma^\mu \partial_\mu \Psi_D - m_D \bar{\Psi}_D \Psi_D \hspace{1cm} (2.11)$$

can be expressed in the Weyl notation as

$$L_{\text{Dirac}} = -i\xi^{\dagger}\bar{\sigma}^\mu \partial_\mu \xi - i\chi^{\dagger} \bar{\sigma}^\mu \partial_\mu \chi - m_D \left( \xi\chi + \xi^{\dagger}\chi^{\dagger} \right). \hspace{1cm} (2.12)$$

Similarly, we can find a representation for a Majorana fermion by imposing $\chi = \xi$. Then the Majorana Lagrangian

$$L_{\text{Majorana}} = -\frac{1}{2}i\bar{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \frac{1}{2}m_M \bar{\Psi}_M \Psi_M \hspace{1cm} (2.13)$$

leads to

$$L_{\text{Majorana}} = -i\xi^{\dagger}\bar{\sigma}^\mu \partial_\mu \xi - m_M \left( \xi\xi + \xi^{\dagger}\xi^{\dagger} \right) \hspace{1cm} (2.14)$$

in the Weyl notation.

What supermultiplets are possible? The simplest combination is a supermultiplet with a single Weyl fermion (= 2 fermionic degrees of freedom) and one complex scalar field (= 2 bosonic degrees of freedom) or equivalently two real scalar fields (sfermions). This is called a chiral supermultiplet, or matter supermultiplet. Note that in this way, we naturally can implement different transformation properties for left- and right-handed fermions, since they belong to separate supermultiplets.

Spin-1 vector bosons are put into a gauge supermultiplets, consisting of the massless vector boson (= 2 bosonic degrees of freedom) and a Weyl fermion (gaugino) as superpartner (= 2 fermionic degrees of freedom). The Weyl fermion must transform in the same way as the vector boson, that is in the adjoint representation. In particular left- and right-handed gauginos must transform the same and are therefore of Majorana type.

It turns out, that we can understand supersymmetry by defining a superspace built up from ordinary four dimensional space-time supplemented by four anti-commuting dimensions. A point in superspace is then given by $(x^\mu, \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2)$, where $\theta_i$ and $\bar{\theta}_i$ are Grassmann variables obeying

$$\{\theta_i, \bar{\theta}_j\} = 0. \hspace{1cm} (2.15)$$
A superfield is a function of these superspace coordinates: \( \Phi(x, \theta, \bar{\theta}) \). Working out the transformation properties and using anti-commutation of the \( \theta \) coordinates, it turns out, that we can write a chiral superfield as

\[
\Phi(x, \theta) = \phi(x) + \sqrt{2} \theta^a \psi_a(x) + \theta^a \theta_a F(x) \tag{2.16}
\]

with component fields \((\phi(x), \psi(x), F(x))\). \( \phi(x) \) is identified with a complex scalar field, \( \psi(x) \) a Weyl fermion and \( F(x) \) will turn out to be an auxiliary vector field. The decomposition of a Non-Abelian gauge superfield is more involved and we only quote here the component fields out of which it can be reconstructed, \((D^a(x), \lambda^a(x), A^a(x))\) with \( D^a(x) \) as scalar auxiliary field, \( \lambda^a(x) \) a Weyl fermion and \( A^a(x) \) the vector boson.

### 2.2 Constructing a supersymmetric Lagrangian

#### 2.2.1 Chiral supermultiplets

We start with considering only chiral supermultiplets, \( \Psi_i = (\phi_i, \psi_i, F_i) \). Here, \( \phi_i \) is a scalar field, named sfermion, \( \psi_i \) a spin \( \frac{1}{2} \) fermion field, while \( F_i \) is an auxiliary complex scalar field, which ensures the supersymmetry algebra to close also off-shell. The dynamic of the superfield is governed by the following Lagrangian, which we split up in a part describing free component fields and a part describing interactions among the component fields:

\[
\mathcal{L}_{\text{chiral}} = \mathcal{L}_{\text{chiral, free}} + \mathcal{L}_{\text{chiral, int}} \tag{2.17}
\]

\[
\mathcal{L}_{\text{chiral, free}} = -\partial^\mu \phi^*_i \partial_\mu \phi_i - i \psi^i \bar{\sigma}^\mu \partial_\mu \psi_i + F^*_i F_i \tag{2.18}
\]

\[
\mathcal{L}_{\text{chiral, int}} = \left( -\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i \right) + \text{c.c.} \tag{2.19}
\]

At this point, \( W^i \) and \( W^{ij} \) are unrelated coefficients. The form of (2.19) for the interaction terms is determined by allowing only renormalizable terms (mass dimensions \( \leq 4 \)) and exploiting the supersymmetric invariance of (2.19), since (2.18) is invariant on its own. The examination of the interaction terms under supersymmetric transformations leads to the conclusion, that \( W^i \) and \( W^{ij} \) are derivatives with respect to the fields \( \phi_i \) and \( \phi_j \) of an object \( W \), which is a holomorphic polynomial in \( \phi_k \) of at most degree 3. The completely general form of \( W \), called the superpotential, is thus given by

\[
W = L^i \phi_i + \frac{1}{2} M^{ij} \phi_i \phi_j + \frac{1}{6} y^{ijk} \phi_i \phi_j \phi_k \tag{2.20}
\]

with \( M^{ij} \) being a symmetric mass matrix and \( y^{ijk} \) totally symmetric Yukawa couplings. The derivatives showing up in the Lagrangian are \( W^i = \frac{\partial W}{\partial \phi_i} \) and \( W^{ij} = \frac{\partial W}{\partial \phi_i \partial \phi_j} \).
Solving the classical equations of motion, \( F_i = -W^*_i \) and \( F^{*i} = -W^i \), the auxiliary field can be eliminated and the Lagrangian reads

\[
\mathcal{L}_{\text{chiral}} = -\partial^\mu \phi^i \partial_\mu \phi_i - V(\phi, \phi^*) - i\bar{\psi}j^{ij} \partial^\mu \psi_i - \frac{1}{2} M^{ij} \psi_i \psi_j - \frac{1}{2} M_i^* \psi_i^* \psi^*_j ,
\]
\[
V(\phi, \phi^*) = L_k^* M^{kj} \phi_j + \frac{1}{2} L_k^* y^{ijk} \phi_i \phi_j + \frac{1}{2} L_k^* y^{ijk} \phi_i \phi_j + \frac{1}{2} M_i^* y^{ijn} \phi_i \phi_j \phi_n + \frac{1}{2} M_i^* y^{ijn} \phi_i \phi_j \phi_n + \frac{1}{4} y^{ijk} y^{ijn} \phi_i \phi_j \phi_n \phi_n .
\tag{2.21}
\]

The scalar potential (2.22) is also expressible in terms of derivatives of the superpotential or through auxiliary fields \( F \), \( V(\phi, \phi^*) = W^k W^*_k = F^{*k} F_k \).

Which interactions are actually allowed in a given theory? Although we have not introduced gauge interactions, yet, we anticipate and investigate the chiral Lagrangian under gauge transformations. Considering (2.21) and (2.22) we realize that \( M^{ij} \) is non-zero only if the corresponding supermultiplets \( \Psi_i \) and \( \Psi_j \) transform under gauge groups in representations that are conjugates to each other. Otherwise the term \( \frac{1}{2} M^{ij} \psi_i \psi_j \) and its conjugate would not stay invariant. Similarly, \( L_k^* M^{kj} \phi_j \) and its conjugate only remain invariant, if \( \phi_j \) is a singlet under gauge transformations. However, since \( M^{kj} \neq 0 \) require the scalar fields \( \phi_k \) and \( \phi_j \) to transform in conjugated representations, such a term is only allowed, if \( k = j \) and therefore if \( \Psi_i \) is a singlet under the gauge groups. At last, we note that \( y^{ijk} \) is non-zero only if \( \Psi_i, \Psi_j \) and \( \Psi_k \) can combine to form a singlet.

### 2.2.2 Gauge supermultiplets

A gauge supermultiplet \( \Phi^a = (D^a, \lambda^a, A^a_\mu) \) has the following transformation properties under gauge transformations:

\[
\delta_{\text{gauge}} A^a_\mu = \partial_\mu \Lambda^a + gf^{abc} A^b_\mu \Lambda^c ,
\tag{2.23}
\]
\[
\delta_{\text{gauge}} \lambda^a = gf^{abc} \lambda^b \Lambda^c ,
\tag{2.24}
\]
\[
\delta_{\text{gauge}} D^a = gf^{abc} D^b \Lambda^c .
\tag{2.25}
\]

Here the index \( a \) runs over the number of generators of the gauge group belonging to the adjoint representation. We introduce the usual field strength,

\[
F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu ,
\tag{2.26}
\]

and define the covariant derivative in the adjoint representation,

\[
D_\mu \lambda^a = \partial_\mu \lambda^a + gf^{abc} A^b_\mu \lambda^c .
\tag{2.27}
\]
The Lagrangian for a gauge supermultiplet is then written as

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - i \lambda^a \bar{\sigma}^\mu D_{\mu} \lambda^a + \frac{1}{2} D^a D^a. \]  

(2.28)

### 2.2.3 Supersymmetric gauge interactions

As in normal field theory, supersymmetric gauge interactions are obtained by replacing normal derivatives by covariant derivatives:

\[
D_{\mu} \phi_i = \partial_{\mu} \phi_i - igA_{\mu}^a (T^a \phi)_i, \\
D_{\mu} \phi^{i*} = \partial_{\mu} \phi^{i*} + igA_{\mu}^a (\phi^* T^a)^i \quad \text{and} \\
D_{\mu} \psi_i = \partial_{\mu} \psi_i - igA_{\mu}^a (T^a \psi)_i.
\]

(2.29) \hspace{1em} (2.30) \hspace{1em} (2.31)

But now we can construct more gauge invariant, renormalizable terms involving the fields \( \lambda^a \) and \( D^a \). Following the spirit, that we must write down all allowed terms, the Lagrangian describing gauge and chiral superfields and their interactions is

\[ \mathcal{L} = \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{gauge}} - \sqrt{2} g (\phi^* T^a \psi) \lambda^a - \sqrt{2} g \lambda^a (\psi^* T^a \phi) + g (\phi^* T^a \phi) D^a \]

(2.32)

with \( \mathcal{L}_{\text{chiral}} \) given by (2.21) (but with normal derivatives replaced by covariant derivatives) and \( \mathcal{L}_{\text{gauge}} \) given in (2.28).

The \( D \) fields contribute to the scalar potential (2.22) and after eliminating the auxiliary fields, the scalar potential is now given by

\[ V(\phi, \phi^*) = F^{\star i} F_i + \frac{1}{2} \sum_a D^a D^a = W^{\star i} W_i + \frac{1}{2} \sum_a g_2^2 (\phi^* T^a \phi)^2. \]

(2.33)

### 2.2.4 Soft supersymmetry breaking interactions

Supersymmetry is clearly broken in nature at low energies, but it is natural to assume that it is restored at high energies. This is exactly what happened in electroweak spontaneous symmetry breaking. However, it is not clear how spontaneous symmetry breaking is triggered in supersymmetric theories. Without knowing the exact mechanism and being interested in the low energy phenomenology only, we add explicitly supersymmetry breaking terms to the Lagrangian.

The form of these symmetry breaking terms are restricted, if we require that quadratic divergences in mass corrections cancel even in the presence of broken supersymmetry. One condition for this to hold, is that dimensionless couplings of fermions and sfermions retain
a certain relationship given by exact supersymmetry. Then we split up our Lagrangian in a supersymmetry conserving and a supersymmetry breaking part,

\[ \mathcal{L} = \mathcal{L} \text{SUSY} + \mathcal{L} \text{soft}, \]  

(2.34)

where \( \mathcal{L} \text{soft} \) contains only mass terms and couplings with positive mass dimensions.

In a general theory, the possible terms of \( \mathcal{L} \text{soft} \) are

\[ \mathcal{L} \text{soft} = - \left( \frac{1}{2} M_a \lambda^a \lambda^a + \frac{1}{6} a^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + t^i \phi_i \right) + \text{c.c.} - (m^2)^i_j \phi^* \phi_i \]  

(2.35)

and

\[ \mathcal{L} \text{maybe soft} = - \frac{1}{2} c^{jk} \phi^* \phi_j \phi_k + \text{c.c.} \]  

(2.36)

We recognize mass terms for gauginos, scalar mass terms, (scalar)\(^3\) couplings and tadpole couplings. Soft mass terms for chiral supermultiplet fermions are not included since they can get absorbed by redefining the superpotential, \( (m^2)^i_j \) and \( c^{jk} \). This follows from properties of supersymmetry transformations of scalar fields, which are proportional to fermionic fields. \( \mathcal{L} \text{maybe soft} \) can lead to quadratic divergences if one of the scalar fields is a singlet under all gauge symmetries of the theory. Therefore it is called “maybe soft”.

Comparing (2.35) and (2.22), we find couplings of exactly the same form, \( t^i \) corresponds to \( L^i \), \( a^{ijk} \) to \( y^{ijk} \) and \( b^{ij} \) to \( M^{ij} \). Thus, the same restrictions as discussed in the case of the superpotential apply to the soft breaking terms.

The form of \( \mathcal{L} \text{soft} \) provides also an explanation, why all SM particles seem to be light compared to their superpartners. All SM particles have the property, that they would be massless, if there were no electroweak symmetry breaking. On the other hand, their superpartners receive a mass through \( \mathcal{L} \text{soft} \).

### 2.3 Minimal Supersymmetric Standard Model

After having discussed the possible interactions of supermultiplets, we now construct the Lagrangian of the Minimal Supersymmetric Standard Model. We have to ascertain that we reproduce the known particle spectrum already described in the SM while explaining why no superpartners have been found, yet. Thus, we have to assign all SM particles to a supermultiplet, write down the superpotential and define the soft breaking terms.

Every Standard Model field is part of a supermultiplet, Fermion fields belong to chiral supermultiplets and gauge bosons to gauge supermultiplets. We employ the same notation for superfields as we would do for their SM content. Without further discussing it, we
### CHAPTER 2. THE MINIMAL SUPERSYMMETRIC STANDARD MODEL

<table>
<thead>
<tr>
<th>Name</th>
<th>Spin 0</th>
<th>Spin 1/2</th>
<th>Spin 1</th>
</tr>
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<tr>
<td>Squarks, Quarks (3 generations)</td>
<td>$Q$</td>
<td>$(\bar{u}_L, d_L)$</td>
<td>$(u_L, d_L)$</td>
</tr>
<tr>
<td></td>
<td>$\bar{u}$</td>
<td>$\bar{u}_R$</td>
<td>$u_R$</td>
</tr>
<tr>
<td></td>
<td>$d$</td>
<td>$d_R$</td>
<td>$d_R$</td>
</tr>
<tr>
<td>Sleptons, Leptons (3 generations)</td>
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<td>$(\nu_L, e_L)$</td>
<td>$(\nu_L, e_L)$</td>
</tr>
<tr>
<td></td>
<td>$\bar{e}$</td>
<td>$\bar{e}_R$</td>
<td>$e_R$</td>
</tr>
<tr>
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<td>$(H^+_u, H^0_u)$</td>
<td>$(H^+_u, H^0_u)$</td>
</tr>
<tr>
<td></td>
<td>$H_d$</td>
<td>$(H^+_d, H^0_d)$</td>
<td>$(H^+_d, H^0_d)$</td>
</tr>
<tr>
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<td>$\tilde{W}^\pm$</td>
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</tr>
<tr>
<td>Winos, W Boson</td>
<td>$g$</td>
<td>$W^\pm$</td>
<td>$W^0$</td>
</tr>
<tr>
<td>Bino, B Boson</td>
<td>$B^0$</td>
<td>$B^0$</td>
<td>$\tilde{B}^0$</td>
</tr>
</tbody>
</table>

Table 2.1: Particle content of the MSSM. The particles correspond to eigenstates of the gauge groups and quantum numbers are the same as in the SM. Since in the MSSM we represent fermions in terms of left-handed Weyl fermions, we choose the anti-fermion state to represent the right-handed fermion. Physical states are obtained only after mixing.

mention here that we are forced to introduce two Higgs doublets instead of only one as in the Standard Model in order to eliminate anomalies.

In principle we could introduce Baryon- and Lepton-number violating terms in the superpotential, which contradict experimental limits on such processes. In order to prevent this, we define $R$-parity,

$$P_R = (-1)^{3(B-L)+2s},$$

(2.37)

for each particle. Then, all SM particles including the Higgs bosons have even $R$-parity ($P_R = +1$) while all supersymmetric partners have odd $R$-parity ($P_R = -1$). We will allow only terms in the Lagrangian, that are even under $R$-parity. This solves the problem of Baryon or Lepton number changing processes and helps to distinguish between ordinary particles and sparticles (supersymmetric particles). The lightest particle with $P_R = -1$ is stable because it cannot decay into lighter states without violating $R$-parity. Therefore it is a viable dark matter candidate. It is known as the LSP - lightest supersymmetric particle.

### 2.3.1 Superpotential

Which terms are allowed in the superpotential given the particle content shown in Table 2.1? Following the discussion in section 2.2.1, we write down all combinations of chiral supermultiplets (or their corresponding scalar fields) respecting restrictions from gauge
2.3. MINIMAL SUPERSYMMETRIC STANDARD MODEL

invariance, \( R \)-parity and analicity of the superpotential and find

\[
W_{\text{MSSM}} = \bar{u} y_u Q H_u - \bar{d} y_d Q H_d - \bar{e} y_e L H_d + \mu H_u H_d .
\]  

(2.38)

Family, color and isospin indices have all been omitted here. If we were to restore them, the \( y_u \) term would read for example \( \bar{u}_i^a (y_u)^j_{\alpha a} Q^j_{\alpha a} H_u^\alpha \) with family index \( i = 1, 2, 3 \), color index \( a = 1, 2, 3 \) and weak isospin index \( \alpha = 1, 2 \). The minus signs are chosen such that, after electroweak symmetry breaking, the mass terms generated through couplings to the Higgs fields have the correct signs. We note that we cannot create a coupling \( \bar{d} y_d Q H^u \) because the superpotential has to be analytic while a term of the form \( \bar{d} y_d Q H^u \) is forbidden due to gauge invariance. Therefore, we have to introduce two Higgs doublets, not only because of anomaly cancellations.

Neglecting the masses of 1\(^{\text{st}}\) and 2\(^{\text{nd}}\) generation fermions, we introduce the 3\(^{\text{rd}}\) generation approximation and set all entries of the Yukawa coupling \( y_{\phi} \) to zero except for \( (y_{\phi})_3^3 \). This yields

\[
W_{\text{MSSM}} \approx y_t (\bar{t} t H_u^0 - \bar{b} b H^u_d) - y_b (\bar{b} b H_u^0 - \bar{t} t H^u_d) - y_\tau (\bar{\tau} \nu H_d^0 - \bar{\tau} \nu H^u_d) + \mu (H_u^+ H_u^0 - H_u^0 H_u^0) .
\]  

(2.39)

### 2.3.2 Soft breaking terms

We obtain the soft breaking Lagrangian by writing down all allowed soft breaking terms involving scalar components of MSSM superfields. Hereby we introduce gaugino mass terms for the gluino, the Wino and the Bino, squark and Higgs mass terms and (scalar)\(^3\) couplings among scalar fields. The result is given by

\[
\mathcal{L}_{\text{soft}}^{\text{MSSM}} = -\frac{1}{2} \left( M_3 \tilde{g} \tilde{g} + M_2 \tilde{W} \tilde{W} + M_1 \tilde{B} \tilde{B} + \text{c.c.} \right) - (\tilde{a} a Q H_u - \tilde{d} a_d Q H_d - \tilde{e} e L H_d + \text{c.c.}) - \tilde{Q}^\dagger m_Q^2 \tilde{Q} - \tilde{L}^\dagger m_L^2 \tilde{L} - \tilde{u} m_u^2 \tilde{u}^\dagger - \tilde{d} m_d^2 \tilde{d}^\dagger - \tilde{e} m_e^2 \tilde{e}^\dagger - m_{H_u}^2 H_u^* H_u - m_{H_d}^2 H_d^* H_d - (b H_u H_d + \text{c.c.}) .
\]  

(2.40)

Note that this general form potentially introduces large flavor mixing as well as CP violation in conflict with observations. The assumption of flavor-blind squared-mass matrices, i.e. proportional to the identity matrix in family space, evades the problem of large flavor changing neutral currents. The remaining source for flavor changing processes is due to (scalar)\(^3\) couplings. By requiring them to be proportional to the Yukawa coupling matrices at a certain high scale,

\[
a_{\phi} = A_{\phi 0} y_{\phi} ,
\]  

(2.41)
large (scalar)\(^3\) couplings exist only for the third generation. In the 3\(^{rd}\) generation approximation, the renormalization group evolution respects relation (2.41) and therefore it remains valid at any scale. Large CP violating effects can be avoided by prohibiting the introduction of new complex phases.

Unlike the supersymmetry conserving part the general soft breaking Lagrangian \(\mathcal{L}_{\text{soft}}^{\text{MSSM}}\) introduces many new parameters. In total we find 105 non-CP violating irreducible parameters in addition to 19 SM parameters. When imposing the above mentioned restrictions, there remain 14 additional parameters in the MSSM.

### 2.3.3 Higgs bosons

Now we will explore the Higgs sector in more detail. We consider the scalar potential ((2.33) and (2.22)) for the MSSM superpotential (2.38) and keep only terms involving Higgs scalar fields. The Higgs \(SU(2)\) doublets are \(H_u = (H_u^+ H_u^0)\) and \(H_d = (H_d^0 H_d^-)\) and the Higgs potential in terms of scalar fields is found to be

\[
V_{\text{Higgs}} = (|\mu|^2 + m_{H_u}^2)(|H_u^0|^2 + |H_u^+|^2) + (|\mu|^2 + m_{H_d}^2)(|H_d^0|^2 + |H_d^-|^2)
+ (b(H_u^+ H_d^- - H_u^- H_d^0) + \text{c.c.}) + \frac{1}{2} g^2 |H_u^+ H_d^0| + H_u^0 H_d^-|^2
+ \frac{1}{8} (g^2 + g'^2)(|H_u^0|^2 + |H_u^+|^2 - |H_d^0|^2 - |H_d^-|^2)^2.
\]

(2.42)

The \(g^2\) and \(g'^2\) terms come from \(D\) terms in (2.33) and \(|\mu|^2\) terms result from \(F\) terms (see (2.22)). The remaining terms have their origin in the soft breaking Lagrangian. At the minimum of the potential, we are allowed to choose \(H_u^+ = 0\) due to \(SU(2)\) invariance and find for this choice also \(H_d^- = 0\). Further, by redefining phases of the Higgs fields we can choose \(b > 0\).

Electroweak symmetry breaking occurs when \(V > 0\) for some direction in the \((H_u^0, H_d^0)\) plane, where \(H_u^0\) and \(H_d^0\) are chosen both real without loss of generality. Examining the conditions for a true minimum in \(V\), we find the conditions

\[
2b < 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2 \quad \text{and} \quad b^2 > (|\mu|^2 + m_{H_u}^2)(|\mu|^2 + m_{H_d}^2).
\]

(2.43)

We introduce the following notation for the vacuum expectation values for the Higgs fields:

\[
v_u = \langle H_u^0 \rangle, \quad v_d = \langle H_d^0 \rangle, \quad \tan \beta = \frac{v_u}{v_d}, \quad 0 < \beta < \frac{\pi}{2}.
\]

(2.44)

The vacuum expectation values for the up-type and the down-type neutral Higgs boson is related to the electroweak scale by

\[
v_u^2 + v_d^2 = v^2 = \frac{2 m_Z^2}{g^2 + g'^2} \approx (174 \text{ GeV})^2.
\]

(2.45)
We write \( v_u = v \sin \beta \), \( v_d = v \cos \beta \). By minimizing the potential with respect to \( H_u^0 \) and \( H_d^0 \) we derive the relations

\[
\sin 2\beta = \frac{2b}{2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2} \quad \text{and} \quad m_Z^2 = \frac{|m_{H_u}^2 - m_{H_d}^2|}{\sqrt{1 - \sin^2 2\beta}} - 2|\mu|^2 - m_{H_u}^2 - m_{H_d}^2. \tag{2.46}
\]

For phenomenology we want to express the Higgs fields in terms of physical fields, as we did in discussing the SM Higgs. We start with discussing the degrees of freedom. Before symmetry breaking the two doublets have eight real scalar degrees of freedom. Electroweak symmetry generates three Goldstone bosons \( G^{0,\pm} \), which build the longitudinal modes of the \( W \) and \( Z \) bosons. The remaining five degrees of freedom mix to give two neutral, CP even mass eigenstates \((h^0 \text{ and } H^0)\), one neutral CP odd state \( (A^0) \) and two charged eigenstates \((H^\pm)\). We expand the gauge eigenstates in terms of the physical fields:

\[
\begin{pmatrix}
H_u^0 \\
H_d^0
\end{pmatrix}
= \begin{pmatrix}
v_u \\
v_d
\end{pmatrix}
+ \frac{1}{\sqrt{2}} R_\alpha \begin{pmatrix} h^0 \\ H^0 \end{pmatrix}
+ \frac{i}{\sqrt{2}} R_{\gamma_0} \begin{pmatrix} G^0 \\ A^0 \end{pmatrix}
\]

\[
\begin{pmatrix}
H_u^+ \\
H_d^-
\end{pmatrix}
= R_{\beta_\pm} \begin{pmatrix} G^+ \\ H^+ \end{pmatrix}
\tag{2.48}
\]

with rotation matrices

\[
R_\gamma = \begin{pmatrix}
\cos \gamma & \sin \gamma \\
-\sin \gamma & \cos \gamma
\end{pmatrix}
\]

We find \( \beta_0 = \beta_\pm = \beta \) and \( m_{G^0}^2 = m_{G^\pm}^2 = 0 \) at tree level, requiring that \( v_u \) and \( v_d \) minimize the potential. The other masses are found to be

\[
m_{A^0}^2 = \frac{2b}{\sin 2\beta} = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2,
\]

\[
m_{h^0, H^0}^2 = \frac{1}{2} \left( m_{A^0}^2 + m_Z^2 \mp \sqrt{(m_{A^0}^2 - m_Z^2)^2 + 4m_Z^2 m_{A^0}^2 \sin^2 2\beta} \right),
\]

\[
m_{H^\pm}^2 = m_{A^0}^2 + m_W^2.
\]

Also at tree level we find (by definition \( m_{H^0}^2 > m_{h^0}^2 \))

\[
\frac{\sin 2\alpha}{\sin 2\beta} = -\left( \frac{m_{h^0}^2 + m_{H^0}^2}{m_{h^0}^2 - m_{H}^2} \right) \quad \text{and} \quad \frac{\tan 2\alpha}{\tan 2\beta} = -\left( \frac{m_{h^0}^2 + m_{Z}^2}{m_{h^0}^2 - m_{Z}^2} \right). \tag{2.53}
\]

Keep in mind, that all these angles and masses squared receive corrections at higher orders which modify the relations. We will not go into details but only remark, that at tree level the lightest Higgs would have a mass \( < m_Z \), which is excluded by direct observation for most models. Only due to loop corrections, which turn out to be rather large, the Higgs has a mass exceeding experimental lower bounds.
2.3.4 Squark masses

Various pieces in the MSSM Lagrangian are quadratic in the squark fields and potentially contribute to the squark masses. We start with collecting all the relevant terms to build the squark mass Lagrangian,

\[
\mathcal{L}_{\text{squark masses}} = -\tilde{Q}^{\dagger} m^2_{\tilde{Q}} \tilde{Q} - \tilde{u} m^2_{\tilde{u}} \tilde{u}^{\dagger} - \tilde{d} m^2_{\tilde{d}} \tilde{d}^{\dagger} - (\tilde{u}_{a_d} \tilde{Q} v_u - \tilde{d}_{a_d} \tilde{Q} v_d + \text{c.c.)} \\
\quad + \left( \mu^* (\tilde{u}_{y_u} \tilde{u} v_u^* + \tilde{d}_{y_d} \tilde{d} v^*_d) + \text{c.c.)} \right) \quad \subset \mathcal{L}_{\text{susy (scalar)}^3} (2.54)
\]

The \( D \) term contribution is obtained from expanding the respective contribution to the scalar potential and keeping only terms involving squarks and neutral Higgs fields (which acquire a vacuum expectation value),

\[
\frac{1}{2} \sum_{a=1,2,3} g^2_a (\phi^* T^a \phi)^2 \geq \frac{1}{2} \left( g_1^2 \left| Y_{H_u}^0 \right|^2 + Y_{H_d}^0 \right| H^0_d \right| (\tilde{u}_L^* Y_{\tilde{u}_L}^0 \tilde{u}_L + \tilde{d}_L^* Y_{\tilde{d}_L}^0 \tilde{d}_L + \tilde{u}_R^* Y_{\tilde{u}_R}^0 \tilde{u}_R + \tilde{d}_R^* Y_{\tilde{d}_R}^0 \tilde{d}_R \right)
\]

\[
+ g^2_2 (T_{3,H_u}^0 | H^0_u \right|^2 + T_{3,H_d}^0 | H^0_d \right|^2) (\tilde{u}_L^* T_{3,\tilde{u}_L} \tilde{u}_L + \tilde{d}_L^* T_{3,\tilde{d}_L} \tilde{d}_L \right) \\
= \frac{1}{2} \left( |H_u^0|^2 - |H_d^0|^2 \right) \left\{ \tilde{u}_L^* \left( g_1^2 Y_{\tilde{u}_L}^0 \tilde{u}_L - g_2^2 T_{3,\tilde{u}_L} \tilde{u}_L \right) \tilde{u}_L + \tilde{d}_L^* \left( g_1^2 Y_{\tilde{d}_L}^0 \tilde{d}_L - g_2^2 T_{3,\tilde{d}_L} \tilde{d}_L \right) \tilde{d}_L \right.
\]

\[
\quad \left. + \tilde{u}_R^* g_1^2 Y_{\tilde{u}_R}^0 \tilde{u}_R + \tilde{d}_R^* g_1^2 Y_{\tilde{d}_R}^0 \tilde{d}_R \right\} \\
\text{(2.55)}
\]

since \( \frac{1}{2} Y_{H_u}^0 = T_{3,H_u}^0 = \frac{1}{2}, \frac{1}{2} Y_{H_d}^0 = T_{3,H_d}^0 = -\frac{1}{2} \). The couplings are \( g_1 = g', g_2 = g \) and \( g_3 = g_\beta \). Plugging in the correct quantum numbers of the squark fields and after some rewriting, we obtain

\[
\Delta m^2_{\tilde{u}_L} = \left( \frac{1}{2} - \frac{2}{3} \sin^2 \theta_W \right) \cos 2\beta m^2_Z \quad \Delta m^2_{\tilde{d}_L} = \left( -\frac{1}{2} + \frac{2}{3} \sin^2 \theta_W \right) \cos 2\beta m^2_Z \quad \Delta m^2_{\tilde{u}_R} = \frac{2}{3} \sin^2 \theta_W \cos 2\beta m^2_Z \quad \Delta m^2_{\tilde{d}_R} = -\frac{1}{3} \sin^2 \theta_W \cos 2\beta m^2_Z \quad \text{(2.56)}
\]

Neglecting mixing among \( 1^{\text{st}}/2^{\text{nd}} \) and the \( 3^{\text{rd}} \) generation, we introduce squark mass matrices for stop and sbottom,

\[
\mathcal{L}_{\text{squark masses}} = - (\tilde{t}_L^* \tilde{t}_R) M^2_t \left( \begin{array}{c} \tilde{t}_L \\ \tilde{t}_R \end{array} \right) - (\tilde{b}_L^* \tilde{b}_R) M^2_b \left( \begin{array}{c} \tilde{b}_L \\ \tilde{b}_R \end{array} \right) \\
\text{(2.57)}
\]
The eigenvalues of the mass matrices, we find

\[ M^2_t = \begin{pmatrix} m^2_{Q_3} + m^2_t + \Delta m^2_{Q_L} & v(a_t^* \sin \beta - \mu y_t \cos \beta) \\ v(a_t \sin \beta - \mu^* y_t \cos \beta) & m^2_{t_R} + m^2_t + \Delta m^2_{t_R} \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \theta_t & \sin \theta_t \\ -\sin \theta_t & \cos \theta_t \end{pmatrix} \begin{pmatrix} m^2_{t_1} & 0 \\ 0 & m^2_{t_2} \end{pmatrix} \begin{pmatrix} \cos \theta_t & -\sin \theta_t \\ \sin \theta_t & \cos \theta_t \end{pmatrix} \tag{2.58} \]

and

\[ M^2_b = \begin{pmatrix} m^2_{Q_3} + m^2_b + \Delta m^2_{Q_L} & v(a_b^* \cos \beta - \mu y_b \sin \beta) \\ v(a_b \cos \beta - \mu^* y_b \sin \beta) & m^2_{b_R} + m^2_b + \Delta m^2_{b_R} \end{pmatrix} \tag{2.59} \]

The eigenvalues of \( M^2_t \) and \( M^2_b \) are \( m^2_{t_1}, m^2_{t_2} \) and \( m^2_{b_1}, m^2_{b_2} \), respectively. We have introduced the mixing angles \( \theta_t \) and \( \theta_b \) parameterizing the squark mixing matrix

\[ \mathcal{R}^q = \begin{pmatrix} \cos \theta_q & \sin \theta_q \\ -\sin \theta_q & \cos \theta_q \end{pmatrix}. \tag{2.60} \]

From now on we assume \( a_q \) and \( \mu \) to be real. Solving the eigenvalue equation for the mass matrices, we find

\[ m_{\tilde{q}_{1,2}} = \frac{1}{2} \left( 2m^2_q + m^2_{Q_3} + \Delta m^2_{Q_L} + m^2_{q_R} + \Delta m^2_{q_R} \right) \pm \sqrt{(m^2_{Q_3} + \Delta m^2_{Q_L} - m^2_{q_R} - \Delta m^2_{q_R})^2 + (2m_q X_q)^2}. \tag{2.61} \]

where we have rewritten the off-diagonal terms of \( M^2_t \) as

\[ v(a_t \sin \beta - \mu y_t \cos \beta) = v \sin \beta y_t \left( \frac{a_t}{y_t} - \mu \cot \beta \right) = m_t (A_t - \mu \cot \beta) = m_t X_t. \tag{2.62} \]

and similarly the off-diagonal terms of \( M^2_b \):

\[ v(a_b \sin \beta - \mu y_b \cos \beta) = m_b (A_b - \mu \tan \beta) = m_b X_b. \tag{2.63} \]

Here, we have exploited the fact that within the 3rd generation approximation relation (2.41) holds at any scale.

These terms quantify the mixing of left- and right-handed states in the squark sector. We observe that the mixing is proportional to the corresponding quark mass. In particular

\[ \sin 2\theta_t = \frac{2m_t (A_t - \mu \cot \beta)}{m^2_{t_2} - m^2_{t_1}} \quad \text{and} \quad \sin 2\theta_b = \frac{2m_b (A_b - \mu \tan \beta)}{m^2_{b_2} - m^2_{b_1}}. \tag{2.64} \]

By definition \( m^2_{t_1} < m^2_{t_2} \) and \( m^2_{b_1} < m^2_{b_2} \).
Since we imposed SU(2) invariance on the soft Lagrangian, the mass parameter $m_{Q_3}^2$ is the same for the stop and the sbottom mass matrix. Therefore the mass eigenvalues are not independent. We find the following mass relation at tree level \[14\]

$$
m_{t_1}^2 \cos^2 \theta_t + m_{t_2}^2 \sin^2 \theta_t = m_{t_1}^2 \cos^2 \theta_t + m_{t_2}^2 \sin^2 \theta_t + m_b^2 - m_{\tilde{t}}^2 - m_{\tilde{b}}^2 \cos 2\beta. \quad (2.65)$$

At one loop level the mass relation gets modified \[15\]. We write the bare mass parameter for left handed squarks as

$$m_{Q_3}^2 = m_{t_1}^2 + \delta m_{t_1}^2 = m_{b_1}^2 + \delta m_{b_1}^2. \quad (2.66)$$

We can express the left-handed squark soft breaking mass either through the eigenvalues of the stop mass matrix, the stop mixing angle, the top mass and its left-handed $D$-term contribution or through the respective quantities in the sbottom mass matrix. The relation is

$$m_{Q_3}^2(\tilde{q}_L) = m_{t_1}^2 \cos^2 \theta_t + m_{t_2}^2 \sin^2 \theta_t - m_q^2 = \Delta m_{Q_3}^2 \quad \text{for} \quad q = t, b. \quad (2.67)$$

The correction at one-loop to $m_{Q_3}^2(\tilde{q}_L)$ is accordingly given by

$$\delta m_{Q_3}^2(\tilde{q}_L) = \delta m_{t_1}^2 \cos^2 \theta_t + \delta m_{t_2}^2 \sin^2 \theta_t + (m_{t_2}^2 - m_{t_1}^2) \sin 2\theta_t \delta \theta_t - 2m_q \delta m_q. \quad (2.68)$$

Note that $\Delta m_{Q_3}^2(\tilde{q}_L)$ does not get renormalized at $\mathcal{O}(\alpha_s)$. The mass relation (2.66) is obtained by substituting (2.67) and (2.68) for $q = t$ and $q = b$, respectively.

### 2.3.5 Higgs couplings to quarks and squarks

Compared to the Standard Model interactions the MSSM Higgs potential lead to a much richer structure of couplings between Higgs bosons, quarks and squarks. We are only interested in interactions involving the CP even neutral Higgs bosons and (s)quarks in the top/bottom sector. The corresponding Lagrangian reads

$$
\mathcal{L}_{Htb} = - \sum_{q=t,b} \left\{ \frac{m_q}{v} h_f(q) h^0 \bar{q}q + \frac{m_q}{v} H_f(q) H^0 \bar{q}q \right\} - \sum_{q=t,b} \sum_{i,j=1,\bar{1}} \left\{ \frac{m_{\tilde{q}_i}^2}{v} h_s(q, i,j) h^0 \tilde{q}_i \tilde{q}_j + \frac{m_{\tilde{q}_i}^2}{v} H_s(q, i,j) H^0 \tilde{q}_i \tilde{q}_j \right\}. \quad (2.69)
$$

In coming sections we will often apply the following generic shorthand notation:

$$
\Lambda_q = \frac{h_f(q)}{v}, \quad \Lambda_{\tilde{q}_i} = \frac{m_{\tilde{q}_i}^2 h_s(q, i,j)}{m_{\tilde{q}_i}^2 v}.
$$

(2.70)

The dimensionless couplings $h_f$ and $h_s$ depend on the angles $\beta$ and $\alpha$ introduced in section 2.3.3. The quark couplings are rather trivial, given by

$$
\begin{align*}
h_f(b) &= -\frac{\sin \alpha}{\cos \beta}, \\
h_f(t) &= \frac{\cos \alpha}{\sin \beta}.
\end{align*}
$$

(2.71)
2.4. THE GOLDEN REGION

The squark couplings receive additional contributions from soft breaking terms and the mixing of electroweak eigenstates has to be accounted for,

$$m^2_{h_s(q,i,j)} = \left[ R^q \left( h^q_{LL} \, h^q_{LR} \, h^q_{RR} \right) R^q \right]_{ij}$$

(2.72)

where

$$h^q_{LL} = 2 \frac{\cos \alpha}{\sin \beta} m^2_t + m^2_Z \frac{4 \sin^2 \theta_W - 3}{3} \sin(\alpha + \beta),$$

(2.73)

$$h^t_{LR} = h^t_{RL} = m_t A_t \frac{\cos \alpha + \mu \sin \alpha}{\sin \beta},$$

(2.74)

$$h^t_{RR} = 2 \frac{\cos \alpha}{\sin \beta} m^2_t - m^2_Z \frac{4 \sin^2 \theta_W - 3}{3} \sin(\alpha + \beta),$$

(2.75)

$$h^b_{LL} = -2 \frac{\sin \alpha}{\cos \beta} m^2_b - m^2_Z \frac{2 \sin^2 \theta_W - 3}{3} \sin(\alpha + \beta),$$

(2.76)

$$h^b_{LR} = h^b_{RL} = -m_b A_b \frac{\sin \alpha + \mu \cos \alpha}{\cos \beta},$$

(2.77)

$$h^b_{RR} = -2 \frac{\sin \alpha}{\cos \beta} m^2_b + m^2_Z \frac{2 \sin^2 \theta_W - 3}{3} \sin(\alpha + \beta).$$

(2.78)

The couplings $H_{f,s}$ to the heavier CP even neutral Higgs, $H^0$, are obtained by replacing $\alpha \rightarrow \alpha - \pi/2$ in all the above expressions for $h_f$ and $h_s$.

## 2.4 The golden region

The MSSM is a rather successful extension of the SM. It is promising to solve several problems of the SM: (i) It cancels the quadratic divergence of the Higgs mass, (ii) it might successfully unify the electromagnetic, the weak and the strong coupling near the Planck scale, (iii) the lightest stable supersymmetric particle provides a good candidate for dark matter. However, if one tries to make the MSSM consistent with precision measurements, a rather large amount of fine tuning in a large part of the parameter space is required.

Recently, attempts have been made to identify regions in the parameter space where fine tuning is minimal [16–19] and which are consistent with phenomenology. In particular in [16] a region in parameter space has been identified in terms of low energy parameter values without referring to a specific SUSY breaking scenario. The authors called that region “The Golden Region of the MSSM”. These studies have been refined in [18] and compared to bounds from dark matter constraints. In this section we quickly review these studies.

In [16,18] the full MSSM parameter space has been reduced through the following simplifying assumptions:
(1) All soft parameters are flavor-diagonal.

(2) A common soft mass for the first and second generation squarks, \( m_{\tilde{q}} = m_{Q_{1,2}} = m_{u_{1,2}} = m_{d_{1,2}} \) and for all sleptons, \( m_{\tilde{\ell}} = m_{L_{1,2,3}} = m_{\nu_{1,2,3}} \).

(3) All tridiagonal terms \( A \) are set to zero except for \( A_t \).

(4) The third generation right handed down-type soft mass fulfills \( m_{d_3} = m_{\tilde{q}} \), while \( m_{Q_3} \) and \( m_{u_3} \) are independent and are varied in the studies.

The remaining eleven parameters are

\[
\mu, ~ m_A, ~ \tan \beta, ~ M_1, ~ M_2, ~ M_3, ~ m_{\tilde{q}}, ~ m_{\tilde{l}}, ~ m_{Q_3}, ~ m_{u_3}, \text{ and } A_t.
\]

Instead of \( m_{Q_3}, m_{u_3}, A_t \) the physical mass of the lightest stop, \( m_{\tilde{t}_1} \), the mass difference between the two stops, \( \delta m = m_{\tilde{t}_2} - m_{\tilde{t}_1} \), and the stop mixing angle \( \theta_t \) have been chosen as parameters. Next, a scan over reasonable ranges of these parameters has been performed and checked if a certain set of constraints are fulfilled for each scanned point.

In the following sections we summarize first the constraints and then we summarize the properties of the golden region. The aim of this thesis, however, is not to investigate the golden region in detail but rather to use the generic properties of the golden region for motivating the particular choice of parameter values in our studies, presented in section 9. In case that observations hint towards the existence of supersymmetry in nature, more thorough studies along the lines presented here are necessary.

### 2.4.1 Constraints on the golden region

First, let us define measures of fine tuning and how we constrain them in order to identify the golden region according to [16,18].

- **Fine tuning of the \( Z \) mass at tree level.** The scale at which electroweak symmetry is broken is represented by the mass of the \( Z \) boson, which is fixed by experimental measurements. In the MSSM, \( m_Z \) depends on parameters of the theory, in particular on \( m_{H_u}^2, m_{H_d}^2, b, \mu \) at tree level. These parameters must be related to each other such that the physical \( Z \) mass is obtained after taking higher order corrections into account. This introduces fine tuning. We define

\[
A(\xi) = \left| \frac{\partial \ln m_Z^2}{\partial \ln \xi} \right| = \left| \frac{\xi}{m_Z^2} \frac{\partial m_Z^2}{\partial \xi} \right| \quad (2.80)
\]

where \( \xi = m_{H_u}^2, m_{H_d}^2, b, \mu \) and \( m_Z \) is the tree level \( Z \) mass. \( A(\xi) \) relates the percentage change of \( m_Z^2 \) to the percentage change in \( \xi \). The overall fine tuning of \( m_Z \) is
then defined as $\Delta = \sum_\xi A(\xi)^2$. Points in parameter space with values $\Delta > 100$ are excluded as being too much fine tuned.

- **Fine tuning of the $Z$ mass at one loop level.** The parameters defining the tree level $Z$ mass receive radiative corrections at one-loop. Retaining only the most dominant contribution we define

$$\Delta_t = \left| \frac{\delta_t m_Z^2}{m_Z^2} \right|$$

(2.81)

where $\delta_t m_Z^2$ are corrections due to top and stop loops to $m_{H_u}^2$. Again, $\Delta_t \leq 100$ is required for valid points in the parameter space.

- **Higgs mass bounds.** The LEP2 lower bound on the Higgs mass is

$$m_{h^0} > 114 \text{ GeV}.$$  

(2.82)

Higher order corrections must be taken into account in order to determine the mass of the lighter neutral Higgs boson. In [16] only an approximate formula for the higher order corrections to the Higgs mass has been applied, while in [18] the numerical package *FeynHiggs* is used for determining the appropriate corrections.

- **$\rho$ parameter.** The $\rho$ parameter is a measure for the deviation of the weak boson masses, $m_W$ and $m_Z$, from the relation

$$\sin^2 \theta_W = 1 - \frac{m_W^2}{m_Z^2}$$

(2.83)

corrected by dominant loop corrections. The expected value in the SM is $\rho = 1$. Experimental bounds from [20], used in [18], are

$$(2 - 8) \times 10^{-4} \leq \rho - 1 \leq (2 + 8) \times 10^{-4}.$$  

(2.84)

These bounds restrict the parameter range further, since $\rho$ receives corrections due to stop and sbottom loops.

- **Constraints from $b \to s\gamma$.** While [16] do not include these constraints in most of their studies, [18] do include these throughout their paper.

### 2.4.2 The golden region.

The authors of [18] perform a scan over the following parameter ranges:

- $80 \text{ GeV} < \mu < 500 \text{ GeV}$,
- $100 \text{ GeV} < m_A < 2000 \text{ GeV}$,
- $100 \text{ GeV} < M_1 < 400 \text{ GeV}$,
- $100 \text{ GeV} < M_2 < 2000 \text{ GeV}$,
- $100 \text{ GeV} < M_3 < 2000 \text{ GeV}$,
- $100 \text{ GeV} < m_{\tilde{q}} < 2000 \text{ GeV}$,
- $100 \text{ GeV} < m_{\tilde{\ell}} < 2000 \text{ GeV}$,
- $100 \text{ GeV} < m_{\tilde{t}_1} < 1000 \text{ GeV}$,
- $100 \text{ GeV} < \delta m < 600 \text{ GeV}$,
- $\tan \beta = 10$,
- $\theta_t = \frac{\pi}{4}$.
Here, the parameter are defined at the electroweak scale and no relations between parameters at high energies are assumed.

In summary, we can list the following properties of the golden region:

- Stop eigenstates have masses much below 1 TeV.
- Stops have a significant mass splitting of at least $\delta m > 150$ GeV.
- There is a large stop mixing (fixed to $\theta_t = \frac{\pi}{4}$ in these studies).

These properties will enhance the importance of contributions from virtual corrections containing gluinos computed in [21, 22] and we will illustrate these effects in the case of the total cross-section in section 9.
Part II

Calculational Details
Chapter 3

Leading order

For illustration purposes and for setting up some notations, we will discuss the computation of the Born contribution in a rather detailed way before discussing the more involved calculation of virtual corrections in chapter 4 and real corrections in chapter 5. Typical diagrams contributing at leading order are shown in Fig. 3.1.

3.1 Fermionic contribution

We consider two gluons with momenta $p_{1,2}$, color $a$ and $b$ and polarization $\lambda_{1,2}$ forming a Higgs with momentum $p_h$ via a fermion loop as depicted in the upper row of Fig. 3.1. The incoming momenta are on-shell, $p_1^2 = p_2^2 = 0$. All Feynman rules needed for the calculation are found in Appendix A. The two diagrams containing a fermion loop differ only by the charge flow and give the same contribution. We write for the matrix element

$$iM = \epsilon^{(\lambda_1)}_\mu(p_1)\epsilon^{(\lambda_2)}_\nu(p_2) C^{ab} T^{\mu\nu}$$

(3.1)

where $\epsilon^{(\lambda)}_\alpha(p_i)$ are the gluon polarization vectors. We have separated the color factor $C^{ab}$ and the Lorentz structure $T^{\mu\nu}$. The color structure is particularly easy and given by

$$C^{ab} = \text{Tr} \left[ T^a T^b \right] = \frac{1}{2} \delta^{ab}.$$ 

(3.2)

Since the Lorentz structure contains a loop which is potentially divergent, we choose the framework of dimensional regularization and work in $d$ dimensions. We will see that although the loop turns out to be finite, it is crucial to work in $d$ dimensions in order to pick up all contributions. The object $T^{\mu\nu}$ is the following loop integral,

$$T^{\mu\nu} = -g_s^2 \Lambda_q \delta^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr}(\gamma^\mu(k + p_2 + m_q)\gamma^\nu(k + m)(k + p_1 + p_2 + m_q))}{(k^2 - m_q^2 + i\varepsilon)((k + p_2)^2 - m_q^2 + i\varepsilon)((k + p_1 + p_2)^2 - m_q^2 + i\varepsilon)}.$$
First, let us examine the tensor structure of $T^{\mu\nu}$. The most general Lorentz invariant tensor that could be built from $g^{\mu\nu}$ and the momenta $p_1^\mu$ and $p_2^\nu$ is

$$T^{\mu\nu} = A g^{\mu\nu} + B (p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) + C (p_1^\mu p_2^\nu - p_1^\nu p_2^\mu) + D p_1^\mu p_2^\nu + E p_2^\mu p_1^\nu. \quad (3.4)$$

Invariance under the exchange of the two gluons requires $D = E$ and $C = 0$. The Ward identity

$$p_1^\mu \epsilon^{(\lambda_2)}_\nu (p_2) T^{\mu\nu} = \epsilon^{(\lambda_1)}_\mu (p_1) p_2^\nu T^{\mu\nu} = 0 \quad (3.5)$$

relates the coefficients $A$ and $B$ by

$$A = -B (p_1 \cdot p_2) . \quad (3.6)$$

The coefficients $D$ and $E$ are irrelevant, since they do not contribute to the matrix element, because $p_1 \cdot \epsilon_1 = p_2 \cdot \epsilon_2 = 0$. Therefore it is convenient to define a projector $P_{\mu\nu}$ in the conventional dimensional regularization scheme

$$P_{\mu\nu} = \frac{1}{2 - d} \left( -g_{\mu\nu} + \frac{p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu}}{p_1 \cdot p_2} \right) \quad (3.7)$$

such that

$$P_{\mu\nu} T^{\mu\nu} = A . \quad (3.8)$$

In particular we have

$$\epsilon^{(\lambda_1)}_\mu (p_1) \epsilon^{(\lambda_2)}_\nu (p_2) T^{\mu\nu} = ((2 - d) A) \epsilon^{(\lambda_1)}_\mu (p_1) \epsilon^{(\lambda_2)}_\nu (p_2) P_{\mu\nu} \quad (3.9)$$
and using
\[
\sum_{\lambda_i} \epsilon^{(\lambda_i)}_{\mu} \epsilon^{*(\lambda_i)}_{\mu'} = -g_{\mu\mu'} + \frac{p_{\mu n_{\mu'}} + p_{\mu' n_{\mu}}}{p_1 \cdot n}
\] (3.10)

with a light-like auxiliary vector \( n \), \( n^2 = 0 \), for the polarization sum of gluons, the contraction of polarizations with projector \( P^{\mu\nu} \) squared is given by
\[
\sum_{\lambda_1, \lambda_2} \epsilon^{(\lambda_1)}_{\mu} \epsilon^{*(\lambda_1)}_{\mu'} \epsilon^{(\lambda_2)}_{\nu} \epsilon^{*(\lambda_2)}_{\nu'} P^{\mu\nu} P^{\mu'\nu'} = \frac{1}{d - 2}
\] (3.11)
The matrix element squared is thus easily obtained and reads
\[
|M|^2 = C^{ab} C_{ab} (d - 2) |A|^2 = \frac{N^2 - 1}{4} (d - 2) |A|^2.
\] (3.12)

Next, let us determine \( A \) for the quark contribution. We start with working out the trace of \( P_{\mu\nu} T^{\mu\nu} \) and find
\[
P_{\mu\nu} T^{\mu\nu} = -g_s^2 \Lambda_q m_q \left[ (4d - 24) g_{\mu\nu} + 32 \frac{P_{\mu\nu} P_{2\nu}}{p_1 \cdot p_2} C^{\mu\nu} \right]
\] (3.13)
\[
+ (8d - 16) p_{2\mu} C^{\mu} + (4d - 8) (p_1 \cdot p_2 - m^2) C_0
\]

where we have introduced the following notation for the three point integrals:
\[
C_0 = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m_q^2 + i\epsilon)((k + p_1)^2 - m_q^2 + m^2 + i\epsilon)((k + p_1 + p_2)^2 - m_q^2 + i\epsilon)}
\] (3.14)
\[
C^{\mu} = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2 - m_q^2 + i\epsilon)((k + p_1)p_2 - m_q^2 + i\epsilon)((k + p_1 + p_2)^2 - m_q^2 + i\epsilon))}
\] (3.15)
\[
C^{\mu\nu} = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m_q^2 + i\epsilon)((k + p_1)p_2 - m_q^2 + i\epsilon)((k + p_1 + p_2)^2 - m_q^2 + i\epsilon))}
\] (3.16)

Now consider \( g_{\mu\nu} C^{\mu\nu}, p_{1\mu} p_{2\mu} C^{\mu\nu} \) and \( p_{2\mu} C^{\mu} \) and replace the scalar products in the numerator by linear combinations of the denominator. The coefficient \( A \) is then written in terms of scalar one-loop integrals, \( C_0 \) and \( B_0 \),
\[
A_q = -g_s^2 \Lambda_q m_q \left[ 4 s_{12} \left( \frac{d - 2}{2} - \tau_q \right) C_0(0, 0; s_{12}, m_q, m_q) + (4d - 16) B_0(s_{12}; m_q, m_q) \right].
\] (3.17)

\( C_0 \) is defined in (3.14), while \( B_0 \) is given by
\[
B_0(s_{12}; m_q, m_q) = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m_q^2 + i\epsilon)((k + p_1 + p_2)^2 - m_q^2 + i\epsilon)}
\] (3.18)
and the kinematic variables \( s_{12} \) and \( \tau_q \) are defined by

\[
s_{12} = (p_1 + p_2)^2 \quad \text{and} \quad \tau_q = \frac{4m_q^2}{s_{12}},
\]

respectively. The interesting point here is that \( B_0 \) is multiplied with \( 4d - 16 = -8\epsilon \) which vanishes if \( \epsilon \to 0 \). However, \( B_0 \) has a \( 1/\epsilon \) term so there remains a constant piece even if \( \epsilon = 0 \). This is the reason why it was important to work in dimensional regularization.

If more than one quark contributes to the cross section, we have to sum over all of these. At order \( \epsilon^0 \) the matrix element squared averaged over incoming color and spin and multiplied with the flux factor \( 1/(2s_{12}) \) is given by

\[
\Upsilon_{gg\to h}(s_{12}) = \frac{1}{2s_{12}} \frac{1}{2(N_c^2 - 1)} \sum_{\text{color, spin}} \left| \sum_q \mathcal{M}_q \right|^2 = \frac{s_{12}^2}{2s_{12}} \frac{\alpha_s}{2(N_c^2 - 1)} \left( \frac{\alpha_s}{4\pi} \right)^2 \sum_q \left( \frac{\Lambda_q}{m_q \tau_q ((1 - \tau_q) f(\tau_q) + 1)} \right)^2 \]

\[
= \frac{\alpha_s^2 G_F s_{12}}{256 \sqrt{2} \pi^2} \sum_q \left( \tau_q ((1 - \tau_q) f(\tau_q) + 1) \right)^2.
\]

where in the last line we have plugged in the SM coupling of quarks to the Higgs, \( \Lambda_q = \frac{m_q}{v} = m_q \sqrt{2} G_F \), and set \( N_c = 3 \). The function \( f(\tau) \) is

\[
f(\tau) = \begin{cases} 
\arcsin^2 \frac{1}{\sqrt{\tau}} & , \text{ if } \tau > 1 , \\
-\frac{1}{4} \left( \ln \frac{1 + \sqrt{1 - \tau}}{1 - \sqrt{1 - \tau}} - i\pi \right)^2 & , \text{ if } 0 < \tau \leq 1 .
\end{cases}
\]

Although we have computed only a leading order contribution, the calculation was already rather involved. Of course, since nowadays most one-loop scalar integrals are known and provided as computer codes [23–25], the computation is simplified and boils down to performing Passarino-Veltman reduction, which can be carried out by hand. When we go to the next order, even in the case of the real radiation which is still a one-loop calculation, it gets much more tedious and we will make use of the power of computer algebra. At two-loop, as we will see when discussing the virtual corrections, the proper master integrals are often not known and computing them with the method of Feynman parameterization seems to be almost unfeasible.

We continue now with computing the scalar contributions and end the chapter with a short comment on effective theories.
3.2 Scalar contribution

In the MSSM not only quarks mediate the $ggh$ coupling at order $\alpha_s$ but also squarks. The relevant diagrams with scalar particles are shown in the lower row of Fig. 3.1. Note the additional diagram with a scalar bubble, which does not exist in the fermionic case, because there is no quartic coupling of fermions and gluons. The calculation proceeds along the same line as in the fermionic case. For further reference we quote here the results. The equivalent of (3.17) is

$$A_q = -g_s^2 \Lambda_q m_q^2 \left[ -4m_q C_0(0, 0, s_{12}; m_q, m_t, m_{\tilde{q}}) + (d - 4)B_0(s_{12}; m_q, m_{\tilde{q}}) \right]. \quad (3.22)$$

In a theory with quarks and scalar quarks, (3.20) is modified by the scalar quark contributions,

$$\Upsilon_{gg\to h}^{(0), MSSM}(s_{12}) = \frac{1}{2s_{12}} \frac{1}{(N_c^2 - 1)} \sum_{\text{color,spin}} \left| \sum_q M_q + \sum_{\tilde{q}} M_{\tilde{q}} \right|^2 \quad (3.23)$$

which in the limit $\epsilon \to 0$ reads

$$\Upsilon_{gg\to h}^{(0), MSSM}(s_{12}) = \frac{s_{12}}{4(N_c^2 - 1)} \frac{\alpha_s}{4\pi} \left( \sum_q \left( \frac{\Lambda_q}{m_q} \tau_q \left( (1 - \tau_q) f(\tau_q) + 1 \right) \right) \right)^2 \quad (3.24)$$

Let us investigate (3.24) in the limit of unbroken symmetry for one species of quarks and their superpartners. For concreteness, assume the top quark contribution. The sum over quarks contains only one term, the top contribution. Since the top has two superpartners, the sum over squarks contains two terms. In exact supersymmetry, all masses are equal, i.e. $m_t = m_{\tilde{t}_i} = m \forall i$, and the couplings to the Higgs are related by $\Lambda_t = 2m\Lambda_{\tilde{t}} = 2\lambda h m^2$. Then (3.24) simplifies to

$$\Upsilon_{gg\to h}^{(0), MSSM}(s_{12}) = \frac{s_{12}}{4(N_c^2 - 1)} \frac{\alpha_s}{4\pi} \left| \lambda_h \tau_t f(\tau_t) \right|^2 \quad (3.25)$$

3.3 Effective theory

Since we will eventually merge our results with the computation obtained in the limit of an infinitely heavy quark mass using an effective theory, we will give a short summary of how to obtain such a theory. We follow [26] and compare with the result we have obtained with the exact mass dependence by taking the limit $m_t \to \infty$. 

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3.3.1 Heavy quark decoupling

The top quark is much heavier than all other quarks and we can make the approximation that it does not propagate. However, we cannot simply drop the top quark but we have to assure that results in the full and in the effective theory agree at some scale. This is called matching and is performed by properly defining decoupling relations. The starting point is to define decoupling constants which relate bare quantities in the full and in the effective theory, closely related to multiplicative renormalization,

\[ g_s^0 = \zeta_0^0 g_s , \quad m_q^0 = \zeta_0^0 m_q , \]  

(3.26)

and similar relations for quark, gluon and ghost fields and the gauge parameter. The effective Lagrangian \( \mathcal{L}' \) contains only light degrees of freedom but has the same functional form as the full QCD Lagrangian neglecting heavy states. \( \mathcal{L}' \) additionally depends on the decoupling constants,

\[ \mathcal{L}' \left( g_s^0, m_q^0, \zeta_i^0; \psi_q^0, G_{\mu}^{a}, \bar{c}^{0}; \bar{c}^0 \right) = \mathcal{L}' \text{QCD} \left( g_s^0, m_q^0, \zeta_i^0; \psi_q^0, G_{\mu}^{a}, \bar{c}^{0}; \bar{c}^0 \right) . \]  

(3.27)

The decoupling constants \( \zeta_i^0 \) are determined by matching the \( n \)-point Green’s functions in the effective theory and in the full theory up to terms which are suppressed by inverse powers of the heavy mass. For example the decoupling constant of the quark field, \( \zeta_2^0 \), is obtained by matching the massless-quark propagator:

\[ - \frac{1}{p \left( 1 + \Sigma_\nu^0 (p^2) \right)} = - \frac{1}{\zeta_2^0 p \left( 1 + \Sigma_\nu^0 (p^2) \right)} \]  

(3.28)

where \( \Sigma_\nu^0 (p^2) \) is the vector part of the quark self-energy, \( \Sigma(p) = p\Sigma_\nu^0 (p^2) + m_q \Sigma_S^0 (p^2) \).

Since we work in the limit \( m_t \to \infty \) we can safely set \( p = 0 \) and find

\[ \zeta_2^0 = 1 + \left. \Sigma_\nu^0 (0) \right|_{\text{only heavy quarks}} . \]  

(3.29)

Analogously we derive relations for all other decoupling constants. Combining these relations with renormalization relations yields expressions for renormalized decoupling constants.

3.3.2 Effective theory for \( gg h \) interactions

If the Higgs mass \( m_h \) is much smaller than the top mass \( m_t \), we could try to describe the coupling of Higgs bosons and top quarks through an effective one. Since the top quark loop is responsible for the bulk contribution to \( gg \to h \) the effective description should be a very good approximation to the exact result. It was shown in [27] that the interaction of
the Higgs boson with light degrees of freedom in the effective Lagrangian can be written in terms of five dimension 4 operators $\mathcal{O}_i$ and corresponding coefficients $C_i$ as

$$\mathcal{L}_{\text{eff}} = -\frac{h^0}{v^0} \sum_{i=1}^{5} C_i^0 \mathcal{O}_i^\prime. \tag{3.30}$$

We are only interested in the effective couplings to gluons. Then the relevant renormalized, effective Lagrangian is given by

$$\mathcal{L}_{\text{eff}} = -\frac{h}{v} C_1 \left( \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} \right). \tag{3.31}$$

In the Standard Model $C_1$ is known up to $\mathcal{O}(\alpha_s^3)$ and reads in the $\overline{\text{MS}}$ scheme [28–30]:

$$C_{1}^{\text{SM}} = -\frac{\alpha_s}{3\pi} \left( 1 + \frac{\alpha_s}{\pi} \frac{11}{4} + \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{2777}{288} + \frac{19}{16} \ln \frac{\mu_R}{m_t^2} - n_t \left( \frac{67}{96} - \frac{1}{3} \ln \frac{\mu_R^2}{m_t^2} \right) \right] \right) + \mathcal{O}(\alpha_s^4) \tag{3.32}$$

where $n_t = 5$ is the number of light flavors.

The Feynman rule for the effective $gg h$ coupling is then readily derived to be

$$= -i\delta^{ab} \frac{\Lambda_t}{m_t} C_1 \left( p_1 \cdot p_2 \right) \left( -g^{\mu\nu} + \frac{p_1^\mu p_2^\nu}{p_1 \cdot p_2} \right). \tag{3.33}$$

Computing the Standard Model leading order color and spin averaged matrix element squared divided by the flux factor, we find

$$\Upsilon^{(0), m_t=\infty}_{gg\rightarrow h} (s_{12}) = \frac{\alpha_s^2 G_F s_{12}}{256 \sqrt{2\pi^2 \lambda}} \tag{3.34}$$

which is exactly the limit $\tau \rightarrow \infty$ of (3.20) neglecting all other quarks except for the top quark.
Chapter 4

Virtual corrections

In Fig. 4.1 some typical two-loop diagrams contributing to virtual corrections at NLO are depicted. Fig. 4.1(a) and Fig. 4.1(b) show diagrams with two mass scales, the Higgs mass and the quark/squark mass, respectively. In the SM calculation, only diagrams of the class in Fig. 4.1(a) have to be considered, in a theory with additional scalar particles coupling to the Higgs also diagrams in Fig. 4.1(b) contribute. In the MSSM, there are additional diagrams with up to 5 mass scales due to quartic couplings of squarks and couplings to gluinos. Examples are shown in Fig. 4.1(c). In this work however, only diagrams of the class Fig. 4.1(a) and Fig. 4.1(b) have been computed, while diagrams of Fig. 4.1(c), needed for phenomenological studies, were computed elsewhere [21, 22, 31].

The complexity of the next-to-leading order calculation renders the use of powerful computer algebra tools necessary. Results of this thesis were obtained through a combination of Maple [32] and form [33] scripts. Diagrams are generated with QGRAF [34], which are subsequently dressed with spin, Lorentz and color indices in a Maple script which also maps single diagrams to master topologies provided by the user. Dressed diagrams are then passed to the powerful tool form, tuned for dealing with large expressions. The form scripts require input for the reduction to master integrals, which is performed with the Maple package AIR [35]. In form, Dirac and color algebra is performed. After some simplifications the output can further be processed in Maple.

This chapter is devoted to explain the crucial steps in computing the virtual diagrams with focus on evaluating the master integrals. We start in section 4.1 with discussing the reduction to master integrals. Their analytical computation using the method of differential equation is subject of section 4.2. Results for the virtual amplitudes are presented in section 4.3 and a short description of the contributions which are not part of this work is given. Renormalization in the Standard Model as well as in the MSSM is discussed in section 4.4. In section 4.5 the final form of renormalized matrix elements is given. We con-
include this chapter in section 4.6 by the discussion of modified mass relations in the MSSM at NLO due to renormalization and how we restore these relations for phenomenological studies.

The results of this sections have been published in [36].

4.1 Reduction to master integrals

The virtual contributions obviously have the same Lorentz structure as the Born term. Thus it is advantageous to multiply the two loop part with the projector $P_{\mu\nu}$ defined in 3.7. The resulting integrals have no open Lorentz indices anymore but scalar products involving loop momenta and external momenta in the numerator. As we did in the one-loop case when applying the Passarino-Veltman reduction, we express these scalar products through denominator factors. But now we have seven scalar products involving loop momenta $k$ and $l$, but only up to six propagators, leaving one irreducible scalar product. By introducing a seventh auxiliary propagator we can express all scalar products through propagators. A certain set of propagators define a topology and we find three of them to which we can map all 2 loop diagrams. The topologies in Figure 4.2 correspond to the following momentum
4.1. REDUCTION TO MASTER INTEGRALS

\[ \begin{align*}
D_{11} &= k^2 \\
D_{12} &= (k + p_1)^2 \\
D_{13} &= (k + p_{12})^2 \\
D_{14} &= (l + p_{12})^2 - m^2 \\
D_{15} &= (l + p_1)^2 - m^2 \\
D_{16} &= l^2 - m^2 \\
D_{17} &= (k - l)^2 - m^2 \\
D_{11} &= k^2 \\
D_{12} &= (k + p_2)^2 - m^2 \\
D_{13} &= (l + p_{12})^2 - m^2 \\
D_{14} &= l^2 - m^2 \\
D_{15} &= (k - l)^2 \\
D_{16} &= k^2 - m^2 \\
D_{17} &= (k - l)^2 \\
D_{21} &= (k + p_2)^2 - m^2 \\
D_{22} &= (l + p_{12})^2 - m^2 \\
D_{23} &= l^2 - m^2 \\
D_{24} &= (k - l)^2 \\
D_{25} &= (k + p_2)^2 - m^2 \\
D_{26} &= (l + p_{12})^2 - m^2 \\
D_{27} &= l^2 - m^2 \\
D_{28} &= (k - l)^2 \\
D_{31} &= (k - l - p_1)^2 \\
D_{32} &= (k + p_1)^2 - m^2 \\
D_{33} &= (l + p_{12})^2 - m^2 \\
D_{34} &= (l + p_1)^2 - m^2 \\
D_{35} &= (k + p_2)^2 - m^2 \\
D_{36} &= (l + p_{12})^2 - m^2 \\
D_{37} &= (k - l)^2 \\
\end{align*} \]

where we write \( p_{12} \) as a shorthand for \( p_1 + p_2 \). An infinitesimal imaginary part \(+\text{i}\epsilon\) is implicitly understood in all denominators.

For exemplifying the use of auxiliary propagators consider the following integral in topol-
ogy TP

\[ \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{N(k, l, p_1, p_2)}{D_{11} D_{12} D_{13} D_{14} D_{16} D_{17}} \] (4.2)

With an numerator \( N \) containing any scalar products formed out of loop and/or external momenta. Six out of seven scalar products are expressible through denominators, e.g.

\[ k^2 = D_{11}, \quad l^2 = D_{16} + m^2, \]
\[ k \cdot l = -\frac{1}{2} (D_{17} - D_{16} + D_{11}), \quad k \cdot p_1 = \frac{1}{2} (D_{12} - D_{11} - p_1^2), \]
\[ k \cdot p_2 = \frac{1}{2} (D_{13} - D_{12} + p_1^2 - p_2^2 - 2p_1 \cdot p_2), \quad l \cdot p_2 = \frac{1}{2} (D_{14} - D_{16} - p_{12}^2 - 2l \cdot p_1). \] (4.3)

The remaining \( l \cdot p_1 \) is irreducible. Upon adding the auxiliary propagator \( D_{15} \), the irreducible scalar product becomes reducible,

\[ l \cdot p_1 = \frac{1}{2} (D_{15} - D_{16} - p_2^2) \] (4.4)

and after inserting it into the relation for \( l \cdot p_2 \), all scalar products are expressed through linear combinations of propagators. With our choice of the master topologies, no irreducible scalar products were left over. After replacing all scalar products by denominators (called trivial tensor reduction), integrals of the following form remain,

\[ TP_i (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{D_{11}^{\nu_1} D_{12}^{\nu_2} D_{13}^{\nu_3} D_{14}^{\nu_4} D_{16}^{\nu_5} D_{17}^{\nu_7}} \] (4.5)

where exponents \( \nu_k \) can assume positive or negative values. Propagators with negative powers stem from reducing scalar products.

Our goal is now to find a minimal set of independent scalar integrals, such that all integrals in our computation are given by linear combinations of these with rationals of mass \( m \), momentum invariant \( s_{12} = (p_1 + p_2)^2 \) and space-time dimension \( d \) as coefficients. This is subject of the following two sections.

### 4.1.1 Integration by parts identities

Feynman integrals obey remarkable identities, known as integration by parts identities [38] which relate different integrals within a topology. The starting point is a generalization of Gauss’ theorem in \( d \) dimensions, namely that the integral over a total derivative with respect to a loop momentum vanishes in dimensional regularization,

\[ \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k_\mu} f(k) = 0. \] (4.6)
Since \( f(k) \) is arbitrary, we can replace \( f(k) \rightarrow p^\mu f(k) \) for any momentum \( p \). In choosing combinations of momenta to derive with and momenta in the numerator we gain 7 different identities in each topology for arbitrary \( \nu_i \),

\[
\int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \frac{\partial}{\partial q^\mu} D^\nu_1 D^\nu_2 D^\nu_3 D^\nu_4 D^\nu_5 D^\nu_6 D^\nu_7 = 0, \tag{4.7}
\]

where \( q^\mu \in \{k^\mu, l^\mu\} \) and \( v^\mu \in \{k^\mu, l^\mu, p_{1}^\mu, p_{2}^\mu\} \). In general, a graph with \( m \) loops and \( n \) external momenta leads to \( m(n-1+m) \) different IBP identities. In addition to these integration by parts identities we can derive further identities exploiting invariance under Lorentz transformations. In our simple case of a \( 2 \rightarrow 1 \) process, this is not needed and we will not discuss Lorentz identities.

**Example.** For illustration purposes we derive one particular identity for topology TP\(_1\). We choose \( q^\mu = v^\mu = k^\mu \) and find

\[
\frac{\partial}{\partial k^\mu} D_{13} = \frac{1}{D^\nu_1 D^\nu_2 D^\nu_3 D^\nu_4 D^\nu_5 D^\nu_6 D^\nu_7} \left( \frac{\partial}{\partial k^\mu} k^\mu \right) + \sum_{i=1}^{7} \frac{-\nu_ik^\mu \frac{\partial}{\partial k^\mu} D_{1i}}{D^\nu_1 D^\nu_2 D^\nu_3 D^\nu_4 D^\nu_5 D^\nu_6 D^\nu_7}. \tag{4.8}
\]

Let us explicitly compute one of the numerator factors in the sum on the right hand side:

\[
k^\mu \frac{\partial}{\partial k^\mu} D_{13} = k^\mu \frac{\partial}{\partial k^\mu} (k + p_{12})^2 = 2(k^2 + p_{12} \cdot k) = D_{11} + D_{13} - s_{12} \tag{4.9}
\]

where we have used \( p_{12}^2 = p_{22}^2 = 0 \). The term proportional to this derivative is therefore

\[
\int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \left( \frac{\nu_3 s_{12}}{D^\nu_1 D^\nu_2 D^\nu_3 D^\nu_4 D^\nu_5 D^\nu_6 D^\nu_7} + \frac{-\nu_3}{D^\nu_1 D^\nu_2 D^\nu_3 D^\nu_4 D^\nu_5 D^\nu_6 D^\nu_7} \right)
\]

\[
= \nu_3 (s_{12} TP_1 (\nu_1, \nu_2, \nu_3 + 1, \nu_4, \nu_5, \nu_6, \nu_7) - TP_1 (\nu_1 - 1, \nu_2, \nu_3 + 1, \nu_4, \nu_5, \nu_6, \nu_7))
\]

\[
= \nu_3 (s_{12} 3^- - 1^- 3^+ - 1) TP_1 (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7).
\]

In writing the last line, we have introduced the lowering/raising operators \( i^\pm \) with

\[
i^\pm TP_k(\nu_1, \ldots, \nu_i, \ldots, \nu_7) = TP_k(\nu_1, \ldots, \nu_i \pm 1, \ldots, \nu_7). \tag{4.11}
\]

Finally we find the relation

\[
( -\nu_2 1^- 2^+ + \nu_3 (-1^- + s_{12}) 3^+ + \nu_7 (-1^- + 1) 7^+ 
+ (-\nu_7 - \nu_3 - \nu_2 - 2\nu_1 + d) ) TP_1 (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7) = 0. \tag{4.12}
\]
Proceeding analogously for other momentum combination, we obtain all 7 IBP identities for each topology. In principle these are infinitely large system of equations. In the following section we will see, how a complete reduction into master integrals is obtained from such classes of identities.

4.1.2 Laporta algorithm

In the previous section we have seen how to derive an infinite system of equations using integration by parts identities for arbitrary powers $\nu_i$. Laporta suggested in [39] to build a finite system of identities which is explicitly generated from suitable sets $\{\nu_i\}$. These sets must be carefully chosen for classes of integrals with a certain number of propagator factors with positive powers and a certain number of propagator factors with negative powers, i.e. propagator factors in the numerator. One can show that by enlarging the set of allowed values for the $\nu_i$'s the number of identities increase faster than the number of unknowns. Therefore considering a large enough set of identities but small enough to be solvable in practice we are able to reduce the integrals within this set to master integrals. The Laporta algorithm generates one identity at a time, solves it for the most complex integral (according to an appropriate definition) in this relation and substitutes the obtained expression in all others identities, generated at some later stage. In our work we have been using a publicly available implementation [35] of the Laporta algorithm, called AIR (Automatic Integral Reduction).

AIR. For illustrating purposes we work through an example how AIR would reduce a particular integral in our example. We consider topology TP$_1$. Then we let AIR know which topologies are zero in order to improve convergence of the algorithm. First, a two loop integral with less then two propagators raised to a positive power vanishes,

$$\sum_{i=1}^{7} \Theta(\nu_i) = 0. \quad (4.13)$$

Next, integrals which depend only on one loop momentum also vanish in dimensional regularization, i.e.

$$\sum_{i=1}^{3} \Theta(\nu_i) + \Theta(\nu_7) = 0, \quad (4.14)$$

$$\sum_{i=4}^{7} \Theta(\nu_i) = 0. \quad (4.15)$$
In dimensional regularization integrals with scale-less tadpoles or bubbles vanish:

\[ \Theta(\nu_2) + \Theta(\nu_3) + \Theta(\nu_7) = 0, \]
\[ \Theta(\nu_1) + \Theta(\nu_3) + \Theta(\nu_7) = 0, \]
\[ \Theta(\nu_1) + \Theta(\nu_2) + \Theta(\nu_7) = 0, \]
\[ \Theta(\nu_2) + \Theta(\nu_3) + \sum_{i=4}^{6} \Theta(\nu_i) = 0, \]
\[ \Theta(\nu_1) + \Theta(\nu_3) + \sum_{i=4}^{6} \Theta(\nu_i) = 0, \]
\[ \Theta(\nu_1) + \Theta(\nu_2) + \sum_{i=4}^{6} \Theta(\nu_i) = 0, \]
\[ \Theta(\nu_3) + \Theta(\nu_7) = 0, \]
\[ \Theta(\nu_1) + \Theta(\nu_7) = 0. \]

It is however not strictly necessary to provide the knowledge about vanishing topologies and the algorithm would eventually find that these topologies are zero. But it can improve efficiency considerably. If we know beforehand what integrals we consider as master integrals, we can provide a list of these and AIR will not try to reduce them any further. Thus, the user retains the choice of a certain basis. The most crucial input are the identities among integrals of arbitrary \( \nu_i \)'s of the kind (4.12). In the problem at hand, propagator \( D_{15} \) is an auxiliary propagator and therefore never appears raised to a positive power. The list of propagators which are allowed to be raised to a positive power is then provided to AIR together with the maximum number of propagators raised to a positive power, \( N_{\text{prop}} \), the sum of positive powers minus \( N_{\text{prop}} \) and the absolute value of the sum of negative powers, \( N_- \). These quantities are defined as

\[ N_{\text{prop}} = \sum_i \Theta(\nu_i) \]
\[ N_+ = \sum_i \Theta(\nu_i)(\nu_i - 1) \]
\[ N_- = -\sum_i \Theta(-\nu_i)\nu_i. \]

The algorithm then proceeds as follows:

1. **Generate a seed.** Choose the simplest set of \( \nu_i \)'s for which the integral is not vanishing. In our case for example \((0, 0, 0, 1, 0, 0, 0)\).

2. **Generate identities.** Starting with the seed, we generate all identities explicitly from the input identities valid for arbitrary \( \nu_i \)'s. Considering (4.12) we would have

\[ -TP_1(-1, 0, 0, 1, 0, 0, 2) + TP_1(0, 0, 0, 1, 0, 0, 2) + (d - 1)TP_1(0, 0, 0, 1, 0, 0, 1) = 0. \]
(3) **Integral priority.** The algorithm determines the complexity of the integrals according to the following criteria. First it selects the integrals with the largest number of propagators, $N_{\text{prop}}$. If there are more than one integral with the same $N_{\text{prop}}$, the integral with the largest sum $N_+$ of positive indices $\nu_i$ is chosen. In case there is still some ambiguity about the most complex integral, the one with the largest sum $N_-$ of negative indices $\nu_i$ is chosen and if this is not enough, one integral among the ones of the same complexity is chosen randomly. In identity $(4.27)$ it is $TP_1(-1, 0, 0, 1, 0, 0, 2)$ which has the highest priority.

(4) **Rearranging identities.** In every identity the most complex integral is determined and solved for. In case of $(4.27)$ this yields

$$TP_1(-1, 0, 0, 1, 0, 0, 2) = TP_1(0, 0, 0, 1, 0, 0, 2) + (d-1)TP_1(0, 0, 0, 1, 0, 0, 1). \quad (4.28)$$

(5) **Next seed.** In order to obtain more identities, another seed is chosen. Since we want the new identities to be coupled to the ones already processed, we need to cleverly define the priority how to set the next seed. The rules are mostly empirical and applied automatically by AIR. These seed priorities are basically the reverse of the integral priority.

(6) **Gauss elimination.** When adding identities to our system of equations, the new identities will eventually contain integrals which have been eliminated in previous steps. The known integrals are substituted from the new equations and these are then rearranged according to the integral priority. If we were choosing the seed $(-1, 0, 0, 1, 0, 0, 1)$, we would find the identity

$$-TP_1(-2, 0, 0, 1, 0, 0, 1) + TP_1(-1, 0, 0, 1, 0, 0, 2) + (d+1)TP_1(-1, 0, 0, 1, 0, 0, 1) = 0, \quad (4.29)$$

where $TP_1(-1, 0, 0, 1, 0, 0, 1)$ is already known from the first seed. Substituting its expression and eliminating $TP_1(-2, 0, 0, 1, 0, 0, 2)$ yields

$$TP_1(-2, 0, 0, 1, 0, 0, 1) = TP_1(0, 0, 0, 1, 0, 0, 2)$$

$$+ (d - 1)TP_1(0, 0, 0, 1, 0, 0, 1) + (d + 1)TP_1(-1, 0, 0, 1, 0, 0, 1). \quad (4.30)$$

(7) **Back substituting.** Eventually we will find integrals on the left-hand side of relations which were previously unknown and appeared on the right-hand side of eliminated integrals. These are then back substituted and thus eliminated from the list of potential master integrals.
In the case of virtual corrections to Higgs production through gluon fusion we find 17 master integrals shown in Figure 4.3. Note that this basis is not unique. We are free to perform a rotation among the basis elements which can be exploited to derive simpler differential equations as we will see in section 4.2. The integrals in the first two lines are factorizable, i.e. can be written as a product of one-loop integrals, while the remaining integrals are non-factorizable and thus genuine two-loop diagrams.
4.2 Method of differential equations

In chapter 3 we have seen how tedious it is already to compute a one-loop integral using Feynman-parameterization. At two-loop this will be even more cumbersome and we will follow another strategy, the method of differential equations [40–45]. The master integrals in Figure 4.3 depend on two mass scales, \( s_{12} \) and \( m^2 \). We differentiate each master integral \( \text{MI} \) with respect to one of these, say \( m^2 \), exchange the order of derivative and loop integral and then re-express the result through master integrals using the relations obtained in the reduction procedure. We obtain a system of differential equations of the form

\[
\frac{\partial}{\partial m^2} \text{MI}(s_{12}, m^2) = A(s_{12}, m^2) \cdot \text{MI}(s_{12}, m^2) \quad \text{with} \quad \text{MI} = (\text{MI}_1, \ldots, \text{MI}_{17}).
\]  

(4.31)

This is a matrix equation and the solution is not well defined due to ambiguities in the path ordering. Let us have a closer look at the differential equation of a particular master integral \( \text{MI}_i \) with a certain set of non-zero \( \nu_i \)'s. By differentiating with respect to \( m^2 \), no scalar products in the numerator are generated, which must be replaced by linear combinations of propagators. Therefore, the derivative of the \( \text{MI}_i \) remains in the same topology and the right hand side can be rewritten in terms of master integrals belonging to this topology or sub-topologies,

\[
\frac{\partial}{\partial m^2} \text{MI}_i(s_{12}, m^2) = a_{ii}(s_{12}, m^2)\text{MI}_i(s_{12}, m^2) + \sum_{j \neq i} a_{ij}(s_{12}, m^2)\text{MI}_j(s_{12}, m^2).
\]  

(4.32)

All \( \text{MI}_j \)'s with non vanishing coefficients \( a_{ij} \) have either the same number \( N_{\text{prop}} \) of propagators or fewer. Thus, we can regard the system of differential equations as a sequence of inhomogeneous differential equations, which can be solved iteratively starting with the simplest sub-topology. Some topologies contain more than one master integral, such that the inhomogeneous term in (4.32) cannot be determined from simpler topologies alone but we still have to consider a system of coupled differential equations. In cases like this it can be very advantageous to choose our basis of master integrals conveniently.

It turns out that in our computation the natural variable to express the master integrals is

\[
x = \frac{\sqrt{1 - \tau - 1}}{\sqrt{1 - \tau + 1}} + i\varepsilon \quad \text{where} \quad \tau = \frac{4m^2}{s},
\]  

(4.33)

where and in the following we write \( s = s_{12} \). The variable \( x \) is real valued in the space-like region \( (s < 0) \) as well as in the physical region above threshold \( (s > 4m) \). Below threshold \( x \) lies in the unit circle in the complex plane. In that region, we introduce the variable \( \theta \) such that \( x = e^{i\theta} \). Explicitly

\[
\theta = \begin{cases} 
\arctan \left( \frac{2\sqrt{1 - \tau - 1}}{\sqrt{1 - 2\tau - 1}} \right) & \text{if} \quad \infty > \tau > 2, \\
\frac{\pi}{2} & \text{if} \quad \tau = 2, \\
\pi - \arctan \left( \frac{2\sqrt{1 - \tau - 1}}{\sqrt{1 - 2\tau - 1}} \right) & \text{if} \quad 2 > \tau > 1.
\end{cases}
\]  

(4.34)
4.2. Method of Differential Equations

<table>
<thead>
<tr>
<th>Region</th>
<th>$s$</th>
<th>$\tau$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>space-like</td>
<td>$-\infty &lt; s &lt; 0$</td>
<td>$0 &gt; \tau &gt; -\infty$</td>
<td>$0 &lt; x &lt; 1$</td>
</tr>
<tr>
<td>below threshold</td>
<td>$0 &lt; s &lt; 4m^2$</td>
<td>$\infty &gt; \tau &gt; 1$</td>
<td>$x = e^{\theta}$ with $0 &lt; \theta &lt; \pi$</td>
</tr>
<tr>
<td>above threshold</td>
<td>$4m^2 &lt; s &lt; \infty$</td>
<td>$1 &gt; \tau &gt; 0$</td>
<td>$-1 &lt; x &lt; 0$</td>
</tr>
</tbody>
</table>

Table 4.1: Domain spanned by the variables in the different kinematic regions

For quick reference, we list the domain of each variable in the different kinematic regions in Table 4.1.

The two-loop master integrals are now written as

$$
\text{MI}_i(s, m^2) = C_\epsilon^2 (m^2)^{-a_i - 2\epsilon} F_i(x) = C_\epsilon^2 (m^2)^{-a_i - 2\epsilon} \sum_{k=-2}^{\infty} \epsilon^k F_i^k(x)
$$

(4.35)

where $a_i$ can be determined from dimensional reasons and

$$
C_\epsilon = i \frac{\Gamma(1 + \epsilon)}{(4\pi)^{2-\epsilon} (1 - \epsilon)}
$$

(4.36)

is a common loop factor, which we choose to factor out.

4.2.1 Euler’s variation of the constant

To begin with, we assume that we are able to avoid coupled differential equations such that after rewriting the differential equations in terms of the new variable $x$ they have the form

$$
\frac{dF(x)}{dx} = A(x)F(x) + B(x)
$$

(4.37)

where $B(x)$ is known from simpler topologies. In order to solve this equation we apply Euler’s method of the variation of the constant with ansatz

$$
F(x) = \eta(x)w(x)
$$

(4.38)

where $\eta(x)$ solves the homogeneous equation

$$
\frac{d\eta(x)}{dx} = A(x)\eta(x) \quad \Rightarrow \quad \eta(x) = C_0 \exp \left( \int_0^x dx' A(x') \right).
$$

(4.39)

Upon inserting the ansatz into (4.37) we find the remaining equation for $w(x)$,

$$
\eta(x) \frac{dw(x)}{dx} = B(x)
$$

(4.40)
which is solved by
\[ w(x) = \int \frac{B(x')}{\eta(x')} \, dx' + C_1 = \frac{1}{C_0} \int B(x') e^{-\int_0^{x'} \, dx'' A(x'')} + C_1. \] (4.41)

Combining the results for \( \eta(x) \) and \( w(x) \) the solution \( F(x) \) is
\[ F(x) = e^{\int_0^x \, dx' \, A(x')} \left( \int_0^x B(x') e^{-\int_0^{x'} \, dx'' A(x'')} + C_2 \right) \] (4.42)
with \( C_2 = C_0 C_1 \). Since we are dealing with a first order differential equation we are left with determining one integration constant \( C_2 \).

### 4.2.2 Expansion in \( \epsilon \)

In practice it turns out that often it is not possible and not needed to derive an all order result but an expansion in \( \epsilon \) up to some required order is enough. In certain cases it turns out that by expanding in \( \epsilon \) we can decouple differential equations which are otherwise coupled. Consider the following toy example
\[
\begin{align*}
\frac{dF_1(\epsilon, x)}{dx} &= A_{11}(\epsilon, x) F_1(\epsilon, x) + \epsilon A_{12}(\epsilon, x) F_2(\epsilon, x) + B_1(\epsilon, x), \\
\frac{dF_2(\epsilon, x)}{dx} &= A_{22}(\epsilon, x) F_2(\epsilon, x) + A_{21}(\epsilon, x) F_1(\epsilon, x) + B_2(\epsilon, x), \\
A_{ij}(\epsilon, x) &= \sum_{k=0}^{\infty} \epsilon^k A^{(k)}_{ij}(x), \quad (4.44) \\
B_i(\epsilon, x) &= \sum_{k=k_0}^{\infty} \epsilon^k B^{(k)}(x), \quad (4.45) \\
F_i(\epsilon, x) &= \sum_{k=k_0}^{\infty} \epsilon^k F^{(k)}_i(x). \quad (4.46)
\end{align*}
\]

Here, \( k_0 \) can be negative and is the lowest order in the Laurent expansion of the involved quantities. Order by order in \( \epsilon \) the system of equations reads now
\[
\begin{align*}
\frac{dF_1^{(k_0)}(x)}{dx} &= A_{11}^{(0)}(x) F_1^{(k_0)}(x) + B_1^{(0)}(x), \quad (4.47) \\
\frac{dF_2^{(k_0)}(x)}{dx} &= A_{22}^{(0)}(x) F_2^{(k_0)}(x) + A_{21}^{(0)}(x) F_1^{(k_0)}(x) + B_2^{(0)}(x), \quad (4.48) \\
\frac{dF_1^{(k_0+1)}(x)}{dx} &= A_{11}^{(0)}(x) F_1^{(k_0+1)}(x) + A_{11}^{(1)}(x) F_1^{(k_0)}(x) + A_{12}^{(0)}(x) F_2^{(k_0)}(x) + B_1^{(1)}(x), \quad (4.49) \\
\frac{dF_2^{(k_0+1)}(x)}{dx} &= A_{22}^{(0)}(x) F_2^{(k_0+1)}(x) + A_{22}^{(1)}(x) F_2^{(k_0)}(x) + A_{21}^{(0)}(x) F_1^{(k_0+1)}(x) \\
&\quad + A_{21}^{(1)}(x) F_1^{(k_0)}(x) + B_2^{(1)}(x), \quad (4.50)
\end{align*}
\]
Starting from the coupled differential equation (4.43) we have found a sequence of decoupled differential equations (4.47)-(4.50) which can be solved iteratively. From (4.47) we obtain \( F_1^{(k_0)}(x) \). It is then inserted in (4.48) where it contributes to the inhomogeneous part. (4.48) is therefore of the form (4.37) and we solve it for \( F_2^{(k_0)}(x) \). Next, we proceed to (4.49) and so on until we reached the required order in \( \epsilon \).

4.2.3 Determination of the integration constant

In order to fix the integration constants we have to find an appropriate boundary condition. It turns out that in most cases we have considered the constant could be fixed by taking the limit \( s \to 0^- \), i.e. \( x \to 1 \). This corresponds to vanishing external momenta and the boundary condition for all non-factorizable except for the double triangle in the third line of Fig. 4.3 is given by a vacuum sunset diagrams with generic powers of propagators:

\[
\lim_{x \to 1} \text{MI}^{(\text{NF})} = \left( \begin{array}{c}
\nu_1 \n\\
\nu_2 \n\\
\nu_3 \n
\end{array} \right),
\text{(4.51)}
\]

where \( \text{MI}^{(\text{NF})} \) is one of the non-factorizable master integrals. The exponent \( \nu_2 \) corresponds to the number of massless propagators in the integral whereas \( \nu_1 + \nu_3 \) is the number of massive propagators.

The computation of the vacuum diagram is carried out using Feynman parameterization and we find

\[
\left( \begin{array}{c}
\nu_1 \n\\
\nu_2 \n\\
\nu_3 \n
\end{array} \right) = -m^2(d-\nu_{123}) \frac{(-1)^{\nu_{123}} \Gamma(\nu_{123} - d)}{(4\pi)^d \Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}
\times \int_0^1 \int_0^1 \int_0^1 \text{d}x \text{d}y \text{d}z \frac{x^{\nu_2-1}X^{\nu_{23}-\frac{d}{2}}Y^{\nu_{12}-1}Y^{\nu_3-1}}{(y+XY)^{\nu_{123}-d}} \frac{1}{(y+XY)^{\nu_{123}-d}},
\text{(4.52)}
\]

where we have introduced the shorthand notations \( \nu_{j,k} = \nu_j + \cdots + \nu_k \) and \( X = 1-x \), \( Y = 1-y \). Now we use the representation

\[
\frac{1}{(y+XY)^{\nu_{123}-d}} = \frac{1}{\Gamma(\nu_{123} - d)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{y^w}{(XY)^{\nu_{123}-d+w}} \frac{1}{\Gamma(-w)\Gamma(\nu_{123}-d+w)}. \quad \text{(4.53)}
\]
Carrying out the integrals in $x$ and $y$, we obtain

\[ \int_0^1 dx \int_0^1 dy \frac{x^{d/2-\nu_1-1} y^{d/2-\nu_2-1} (y + XY)^{\nu_3-1}}{(y + XY)^{\nu_1-1}} = \frac{\Gamma \left( \frac{d}{2} - \nu_2 \right)}{\Gamma(\nu_1-1)\Gamma \left( \frac{d}{2} \right)} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dw \frac{\Gamma(-w)\Gamma(\nu_1-1+w)\Gamma \left( \frac{d}{2} - \nu_1 - w \right)\Gamma \left( \nu_1 - \frac{d}{2} + w \right)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma \left( \frac{d}{2} \right)\Gamma(\nu_1 + 2\nu_2 - d)}. \]

Finally, applying Barnes’ Lemma we get for the vacuum sunset diagram

\[ (-1)^{\nu_1} (4\pi)^{-d} m^{2(d-\nu_1-2)} \frac{\Gamma(\nu_1-1)\Gamma(\nu_2-1)\Gamma(\nu_3-1)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \]

We have observed that one could fix the solution of the differential equations by requiring simply that the $x \rightarrow 1$ limit is finite, since the homogeneous solutions usually diverge at $x = 1$. The explicit formula for the limit $x = 1$ in Eq. 4.55 was then an additional consistency check of our calculation.

In one master integral, the $x = 1$ limit does not commute with the expansion around $\epsilon = 0$, due to a collinear singularity as $s$ vanishes. For this master integral, we have used the massless limit $x \rightarrow 0$, which is well behaved:

\[ \lim_{x \rightarrow 0} = \frac{1}{(-s)^{-1/2}} C_\epsilon \left( -6 \zeta(3) - \epsilon \frac{\pi^4}{10} + O(\epsilon^2) \right). \]

### 4.2.4 Implementation

We are going to give some technical details concerning the computation of master integrals. First, we introduce a class of functions which are generalizations of polylogarithms, called harmonic polylogarithms (HPLs). When expressing the differential equations in terms of
variable $x$, the coefficients of master integrals on the right-hand side of the equation can be written in terms of

$$f(0; x) = \frac{1}{x}, \quad f(1; x) = \frac{1}{1-x} \quad \text{or} \quad f(-1; x) = \frac{1}{1+x} \quad (4.58)$$

if one chooses the basis properly. In the basis of Fig. 4.3 no other coefficients appear. This leads us to introduce harmonic polylogarithms (HPL) [46]. At weight one we define

$$H(1; x) = -\ln(1-x), \quad H(0; x) = \ln x, \quad H(-1; x) = \ln(1+x) \quad (4.59)$$

and define higher weight HPLs recursively using Kernels (4.58) and assuming that there exists one $k = 1, \ldots, n$ such that $a_k \neq 0$:

$$H(a_n, a_{n-1}, \ldots, a_1; x) = \int_0^x dx' f(a_n; x') H(a_{n-1}, \ldots, a_1; x') \quad (4.60)$$

In addition we define

$$H(0_n; x) = \frac{1}{n!} \ln^n x \quad (4.61)$$

where $0_n$ stands for a sequence of $n$ zeros. Note the following relation to ordinary polylogarithms,

$$H(0_{n-1}, 1; x) = \text{Li}_n(x) \quad (4.62)$$

We often write $a_n = a_1, \ldots, a_n$ as a shorthand dropping the subscript $n$ in many occasions.

While the computation described in [22] has been entirely implemented in Mathematica using the publicly available HPL package described in [47, 48], our implementation for this work has been realized in Maple and form using the scripting language Perl for interfacing. There exists a form package harmpol for HPLs which can be obtained from the form webpage. However, integration of HPLs is not provided and we have to implement it ourselves. Furthermore, we have written our own primitive HPL package in Maple.

We need the following identities involving HPLs for solving the master integrals:

1. **Product and integration by parts identities.** Without proof we quote here product identities up to weight 4:

   $$H(a; x)H(b; x) = H(a, b; x), \quad (4.63)$$

   $$H(a; x)H(b, c; x) = H(a, b, c; x) + H(b, a, c; x) + H(b, c, a; x), \quad (4.64)$$

   $$H(a; x)H(b, c, d; x) = H(a, b, c, d; x) + H(b, a, c, d; x) + H(b, c, a, d; x)$$

   $$+ H(b, c, d, a; x), \quad (4.65)$$

   $$H(a, b; x)H(c, d; x) = H(a, b, c, d; x) + H(a, c, b, d; x) + H(a, c, d, b; x)$$

   $$+ H(c, a, b, d; x) + H(c, a, d, b; x) + H(c, d, a, b; x). \quad (4.66)$$
Another class of identities is obtained from partial integration:

\[ H(a_1, \ldots, a_m; x) = H(a_1; x)H(a_2, \ldots, a_m; x) \]

\[ - H(a_2, a_1; x)H(a_3, \ldots, a_m; x) + \ldots + (-1)^{m+1}H(a_m, \ldots, a_1; x) . \]  

(4.67)

For more details we refer to [46]. These identities allow us to express products of HPLs in terms of single HPLs as needed for the actual integration step.

(2) **Partial fractioning.** In order to apply (4.60) we need first to apply partial fractioning to products of kernels and factors of \(x\):

\[ f(1; x)^nf(0; x)^m = f(1; x)^{n-1}f(0; x)^{m-1} [f(1; x) + f(0; x)] , \]

(4.68)

\[ f(1; x)^nf(-1; x)^m = f(1; x)^{n-1}f(-1; x)^{m-1} \left\{ f(1; x) + f(-1; x) \right\} , \]

(4.69)

\[ f(-1; x)^nf(0; x)^m = f(-1; x)^{n-1}f(0; x)^{m-1} [f(0; x) - f(-1; x)] , \]

(4.70)

\[ x^nf(a; x)^m = x^{n-1}f(a; x)^{m-1} [\alpha(a) + \beta(a)f(a; x)] \]

(4.71)

where

\[ \alpha(a) = \begin{cases} 
1 & \text{if } a = 0, -1, \\
-1 & \text{if } a = 1, 
\end{cases} \quad \text{and} \quad \beta(a) = a . \]  

(4.72)

Repeated application of these identities will produce eventually terms of the form \(f(a; x)^nH(b; x)\) or \(x^nH(b; x)\).

(3) **Integration identities involving HPLs.**

\[ \int_0^z dx f(a; x)H(b; x) = H(a, b; z) , \]

(4.73)

\[ \int_0^z dx f(a_1; x)^nH(a_2, b; x) = -\frac{\alpha(a)}{n-1} \left[ f(a; x)^{n-1}H(a_2, b; x) \right]_{x=0}^z - \int_0^z dx f(a_1; x)^{n-1}f(a_2; x)H(b; x) \]  

if \(n > 1\),

(4.74)

\[ \int_0^z dx x^nH(a, b; x) = \frac{1}{n+1} \left[ x^{n+1}H(a, b; x) \right]_{x=0}^z - \int_0^z dx \beta(a)^{n+1}H(a, b; x) \]

\[ - \alpha(a) \int_0^z dx \sum_{k=0}^n \beta(a)^{n-k}x^kH(b; x) , \]  

(4.75)

\[ \int_0^z dx H(a, b; x) = xH(a, b; z) \bigg|_{x=0}^z - \beta(a)H(a, b; z) - \alpha(a)H(b; z) . \]  

(4.76)

These identities have to be applied iteratively and partial fractioning must be performed if necessary. There is one complication, though. If \(a = 0\) and \(b = (0, \ldots, 0)\)
then (4.73) is ill defined. We should then write

\[ \int_0^z dx f(0; x) H(0_{n-1}; x) = \lim_{\delta \to \infty} \int_{\delta}^{1} dx f(0; x) H(0_{n-1}; x) + \int_{1}^{z} dx f(0; x) H(0_{n-1}; x) = C_n^\infty + H(0_n; x) \]

where \( C_n^\infty = \lim_{\delta \to \infty} \int_{\delta}^{1} dx f(0; x) H(0_{n-1}; x) \) is a divergent quantity.

(4) **Integration identities without HPLs.** Finally, only terms without HPLs must be integrated. This can be done trivially. We simply want to remind that

\[ \int_0^z dx f(a; x) = H(a; z). \]

As we implemented the solution of differential equations as outlined in the last 3 sections, we made the observation that the choice of the lower integration boundary in (4.42) was an unfortunate choice since integrals often diverge at \( x = 0 \) calling for a divergent integration constant for rendering the solution well defined. Reconsidering (4.42) we realize that the choice of 0 as lower boundary is arbitrary and we could choose any other value as lower boundary, corresponding to a simple redefinition of the undetermined integration constant \( C_2 \). It turned out in our case that choosing \( x = 1 \) as lower integration bound was much more practical leading often to vanishing integration constants [22]. The modifications of integration identities is straightforward.

Master integrals obtained in this way are valid a priori in the space-like region, \( x > 0 \). In the physical regions, the kinematic variable \( x \) lies either on the negative real axis or on the complex unit circle. We will present in Appendix B how to analytically continue HPLs into the physical region thus introducing another class of function, log-sine functions and generalizations. Soon after our original work, the extension of HPLs to arbitrary complex argument was introduced in the HPL package [48]. We find however, that our implementation of HPLs on the complex unit circle leads to a much more efficient evaluation tuned for our application.

### 4.3 Virtual amplitudes

In this section we will set up the notation for the virtual amplitudes. We adapt the notation of [22] rather than the sometimes more confusing in our original publication [36]. This work is mainly dealing with “QCD”-like contributions as opposed to “SUSY”-like contributions.
and we therefore separate these two contributions in our amplitude. We introduce form factors $A_i$ to express the amplitude

$$\mathcal{M} = \delta^{ab} \{ \epsilon_1(p_1) \cdot \epsilon_1(p_2) p_1 \cdot p_2 - p_1 \cdot \epsilon_2(p_2) p_2 \cdot \epsilon_1(p_1) \} \left\{ \sum_q A_q + \sum_q \tilde{A}_q + A_{\text{SUSY}} \right\}.$$

(4.79)

Here, $A_q$ and $\tilde{A}_q$ are form factors containing fermion and scalar loops, respectively. We will refer to them as “QCD”-like. We will come back to these contributions in 4.3.1. We denote the fermion form factor with a subscript $f$ and the scalar one with subscript $s$ and expand in the bare coupling $\alpha_s^0$.

$$A_i = \left( \frac{\alpha_s^0 \epsilon^\nu}{4\pi} \right) C_i^{(0)} + \left( \frac{\alpha_s^0 \epsilon^\nu}{4\pi} \right)^2 C_i^{(1)} + \mathcal{O}\left( (\alpha_s^0)^3 \right), \quad i \in \{f, s\}.$$

(4.80)

For later convenience we have factored out

$$S_\epsilon = (4\pi)^{\epsilon} e^{-\gamma_E \epsilon}$$

(4.81)

with each bare coupling $\alpha_s^0$. The remaining form factor $A_{\text{SUSY}}$ contains the more complicated contributions due to supersymmetric vertices and its expansion in $\alpha_s^0$ starts at $(\alpha_s^0)^2$.

We define dimensionless coefficient functions $c_i^{(k)}$ through

$$C_i^{(0)} = \Lambda_f m_{\gamma_i}^{-2} c_i^{(0)}, \quad C_i^{(1)} = \Lambda_f m_{\gamma_i}^{-4} c_i^{(1)}, \quad C_i^{(1)} = \Lambda_s m_{\gamma_i}^{-4} c_i^{(1)}.$$

(4.82)

The functions $c_i^{(k)}$ are presented in Appendix D. As before the couplings between fermions and the Higgs is given by $\mathcal{L}_{ffh} = -m_{\gamma_i} h f \bar{f}$ while the scalar coupling to the Higgs is given by $\mathcal{L}_{ssh} = -m_{\gamma_i} h_{ss}^* h_{ss}^*$.

Finally, we remark that the amplitudes presented in this thesis have been also computed analytically in [37].

### 4.3.1 The difficult diagrams

There are classes of diagrams at two-loop order to be considered in the MSSM calculation that are beyond the scope of this thesis. They contribute to $A_{\text{SUSY}}$ in (4.79). These diagrams are more complicated than the “QCD”-like diagrams discussed so far in that they exhibit dependences on more mass scales. We distinguish two classes of “difficult” diagrams:
(1) **SUSY-QCD diagrams with quartic squark couplings.** In these diagrams two
squarks of potentially different mass run in the loop since the $\tilde{q}\tilde{q}\tilde{q}\tilde{q}$ vertex mix squarks.
There are 5 independent diagrams which are all factorizable in two one-loop diagrams.

(2) **SUSY-QCD diagrams containing gluinos.** These are the truly complicated dia-
grams with up to 4 different masses of particles running in the loop. The complication
arises because the $\tilde{g}\tilde{q}$ and the $\tilde{q}\tilde{h}$ vertices are not diagonal in the squark indices.

The diagrams of class 1 are in principle calculable analytically. However, in our approach
they were evaluated numerically with the same methods as diagrams of class 2 based on
the method of sector decomposition and contour deformation [31]. For the purpose of
this thesis the numerical results for these contributions described in [21, 22] served as an
input for our phenomenological studies in section 9. Since we work in the narrow width
approximation for the Higgs, i.e. we do not integrate over the finite width of the Higgs,
this approach is perfectly valid since virtual corrections must be computed only once for
each set of quark and squark masses. On the other hand it is not completely satisfactory
since this approach requires to run two completely independent programs. Due to the lack
of an efficient numerical implementation for the evaluation of the difficult diagrams that
is easy to use by a non-expert user and that gives results within reasonable time for being
used in a Monte-Carlo program, we have not interfaced our Monte-Carlo implementation
(see chapter 6) with the sophisticated codes described in [21, 22].

### 4.4 Renormalization

#### 4.4.1 Standard Model renormalization

##### 4.4.1.1 Renormalizing the coupling constant

We choose to decouple the top quark such that it does not contribute to the running of
$\alpha_s$. This is achieved by introducing bare coupling constants in the full theory ($g_s^0$) and the
five flavor theory ($g^0$) and relate them by the decoupling constant $\zeta_g$, [26]

$$g_s^0 = \zeta_g g_s \mu^\varepsilon. \quad (4.84)$$

The decoupling constant is related to the top quark contribution to the gluon self energy,
at one-loop

$$\zeta_g = \sqrt{\frac{1}{1 + \Pi_{gg}^{tot}(p^2 = 0)}}. \quad (4.85)$$
Since we are working through 2 loop order, the one-loop relation above is sufficient. The effect of the decoupling is that contributions from diagrams with self-energy insertions on external gluon legs with top quark loops are exactly cancelled by the decoupling constant, the net effect being that we can safely ignore those diagrams and the decoupling constant. We only have to replace the strong coupling of the full theory by the corresponding coupling in the five flavor theory. In this thesis we are especially interested in the effects of bottom quarks. They could be treated analogously to the top quarks which leads to the definition of a four flavor theory. Since parton distribution functions are usually defined in the 5 flavor \( \overline{\text{MS}} \) scheme, we do not decouple the bottoms. Self energy insertions on gluon lines are cancelled by wave function renormalization and need not to be considered.

In the \( \overline{\text{MS}} \) scheme the renormalization constant \( Z_g \) introduced in 1.5.2 is given by

\[
Z_g = 1 - \frac{\alpha_s}{4\pi} \frac{\Delta \epsilon}{2}, \quad \text{with} \quad \beta_0 = \frac{11C_A - 4T_F N_f}{3}. \tag{4.86}
\]

Here, \( C_A = 3 \) is the number of colors in QCD, \( T_F = 1/2 \) the normalization of the group algebra and \( N_f = 5 \) the number of light flavors. \( \alpha_s \) is then related to the bare coupling through

\[
S \epsilon \alpha_s^0 = \left( 1 - \frac{\alpha_s(\mu)}{4\pi} \beta_0 \right) \alpha_s(\mu) \mu^2 \epsilon \tag{4.87}
\]

and we define

\[
\delta \alpha_s^{\overline{\text{MS}}} = - \frac{\alpha_s(\mu) \beta_0}{4\pi} \epsilon. \tag{4.88}
\]

### 4.4.1.2 Mass renormalization

Self-energy corrections \( i \Sigma_q \) modify the bare heavy quark propagator to take the form

\[
\frac{i}{p - m_q^0} \rightarrow \frac{i}{p - m_q^0 + \Sigma_q(p)}. \tag{4.89}
\]

There are two common choices for renormalizing the quark mass. The first one and the most popular one when dealing with heavy quarks is the pole or on-shell scheme. In this scheme the renormalized quark mass is defined as the real part of the pole in the propagator, thus

\[
m_q^{\text{OS}} = m_q^0 - \text{Re} \Sigma_q \left(p = m_q^{\text{OS}}\right). \tag{4.90}
\]

Another popular choice is the \( \overline{\text{MS}} \) scheme in which we only subtract the term proportional to \( \Delta \epsilon = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \) denoted by \( \Sigma_q^{\text{div}} \):

\[
m_q^{\overline{\text{MS}}} = m_q^0 - \Sigma_q^{\text{div}} \left(p\right). \tag{4.91}
\]
We note that \( m_q^{\overline{\text{MS}}} \) retains a residual dependence on \( p \) and \( \mu_R \) through finite terms in \( i\Sigma_q \).

The renormalization constant in 't Hooft-Feynman gauge at one-loop order reads

\[
Z_m^{\text{REN}} = 1 - \frac{\alpha_s}{4\pi} (3C_F \Delta_\varepsilon + \mathcal{F}_m^{\text{REN}})
\]

where \( \mathcal{F}_m^{\text{REN}} \) is the difference between renormalization scheme REN and the \( \overline{\text{MS}} \) scheme,

\[
\mathcal{F}_m^{\overline{\text{MS}}} = 0,
\]

\[
\mathcal{F}_m^{\text{OS}} = -C_F \left( 4 + 3 \ln \frac{\mu_R^2}{m_q^2} \right) + \mathcal{O}(\varepsilon).
\]

We define the \( \overline{\text{MS}} \) counter-term by

\[
\delta m^{\overline{\text{MS}}} = -\frac{\alpha_s}{4\pi} 3C_F \Delta_\varepsilon.
\]

### 4.4.1.3 Wave function renormalization

In principle we should consider diagrams with gluon self-energy insertion on the external lines. But these get cancelled by the wave-function renormalization as we show in the following. The gluon self energy is

\[
\Pi(p^2) = (p^2 g^{\mu\nu} - p^\mu p^\nu) \Pi(p^2).
\]

We fix the gluon wave function renormalization by requiring the residuum of the propagator being unity,

\[
\Pi(p^2 = 0) = 0
\]

when evaluated with renormalized quantities.

Insertion of top loops on the external gluon lines do not contribute because we have decoupled the top quarks. Light quarks do not contribute since in dimensional regularization scale-less loop integrals vanish. In this work we explicitly assume a non-vanishing bottom quark mass and accordingly it would be inconsistent to simply discard diagrams with gluon self-energy insertion containing bottom loops. Such diagrams have the form

\[
\mathcal{M} = \lim_{p^2 \to 0} \epsilon^\mu(p) \Pi^{\mu\nu}(p^2) \frac{-ig_{\mu\nu}}{p^2} \mathcal{M}^\rho
\]

where \( \mathcal{M}^\rho \) is the remaining diagram without the self energy insertion. Then we find

\[
\mathcal{M} = \lim_{p^2 \to 0} ( -i\epsilon^\mu(p) g^{\rho\nu} \Pi(p^2) \mathcal{M}^\rho ) \to 0
\]
CHAPTER 4. VIRTUAL CORRECTIONS

(a) Non vanishing quark self energy diagrams.

(b) Non vanishing squark self energy diagrams.

Figure 4.4: Self energy diagrams.

because of our wave function renormalization condition. In conclusion, we don’t have to consider diagrams with self energy diagrams on external legs due to wave function renormalization. The same argument holds true in the MSSM, since the only new contributions come from heavy particles which we choose to decouple.

4.4.2 Renormalization in the MSSM

In the Standard Model we found the renormalization to be rather straightforward. The structure of MSSM couplings and relations among parameters, like mixing angles and masses, renders the renormalization in the MSSM to be much more involved. Apart from strong coupling and squark mass renormalization, which proceeds along the same line as in the SM case, we also have to consider renormalization of the quark mixing angle and pay attention to the squark-squark-Higgs coupling. Furthermore there are more diagrams to be considered in computing the quark self-energy.

4.4.2.1 Renormalizing the strong coupling

As in the SM case we choose to integrate out the heavy particles, i.e. the top quark and all squarks [49]. All comments made in section 4.4.1.1 also apply here and we can use the \( \overline{\text{MS}} \) coupling in the five flavor scheme while ignoring self energy corrections to external gluon legs and the decoupling constant.

4.4.2.2 Renormalizing (s)quark masses and mixing angles

In addition to the single diagram with a gluon radiated off the quark in the SM case, a diagram involving a gluino contributes to the quark self-energy. The self-energy diagrams
are shown in Fig. 4.4(a). There are in fact two variants of the second diagram to be considered, one for each superpartner of the quark. For reasons becoming obvious later we separate the contributions in QCD-like and truly supersymmetric ones. The former contains a gluon loop only and is denoted by $\Sigma^{\text{QCD}}$. The latter involves squarks and a gluino and is denoted $\Sigma^{\text{SUSY}}$. As we will discuss in section 4.4.2.3, supersymmetry must be restored, since we are working in dimensional regularization (DREG). This is achieved through a second computation in a supersymmetry preserving regularization scheme, dimensional reduction (DRED). Hence, we present the result for the self-energies both in DREG and DRED,

\[ i\Sigma^{\text{QCD}}(\not{p} = m_q) = \frac{g_s^2 C_F}{(4\pi)^2} m_q \left[ -(2 - d - \delta^{\text{REG}}) B_1(m_q^2; m_q, 0) 
+ (d + \delta^{\text{REG}}) B_0(m_q^2; m_q, 0) \right], \]

\[ i\Sigma^{\text{SUSY}}(\not{p} = m_q) = \frac{g_s^2 C_F}{(4\pi)^2} m_q \left[ B_1(m_q^2; m_{\tilde{g}}, m_{\tilde{q}_i}) + B_1(m_q^2; m_{\tilde{g}}, m_{\tilde{q}_2}) \right) 
+ m_{\tilde{g}} \sin 2\theta_q \left( B_0(m_q^2; m_{\tilde{g}}, m_{\tilde{q}_1}) - B_0(m_q^2; m_{\tilde{g}}, m_{\tilde{q}_2}) \right) \] (4.100)

where $\delta^{\text{DREG}} = 0$ and $\delta^{\text{DRED}} = 4 - d$. The difference in DREG and DRED comes from the internal gluon which is treated $d$-dimensional in DREG but 4 dimensional in DRED. The scalar integral $B_0$ has been defined in (3.18) for equal masses in the propagator. $B_1$ is related to the rank one integral,

\[ \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2 - m_1^2 + i\epsilon)(k + Q)^2 - m_2^2 + i\epsilon)} = B_1(Q^2; m_1, m_2) Q^\mu. \] (4.102)

When evaluating scalar integrals $B_0$ and $B_1$ we recover of course expression (4.92) in DREG.

Next we compute the self-energy diagrams in Fig. 4.4(b). Diagrams that are diagonal in the squark indices contribute to squark mass renormalization. The non-diagonal diagrams lead to the renormalization of the squark mixing angle. The results are

\[ i\Sigma_{\tilde{q}_i\tilde{q}_i}(\not{p} = m_{\tilde{q}_i}) = \frac{g_s^2 C_F}{(4\pi)^2} \left[ -(1 + \cos^2 2\theta_q) A_0(m_{\tilde{q}_i}) - \sin^2 2\theta_q A_0(m_{\tilde{q}_i}) - d A_0(m_q^2) 
+ \{ - d(m_q^2 + (\mp 1)m_{\tilde{g}} \sin 2\theta_q) - 4m_q^2 \} B_0(m_{\tilde{q}_i}^2; m_{\tilde{g}}, m_{\tilde{q}_i}) 
- (d - 4)m_q^2 B_1(m_{\tilde{q}_i}^2; 0, m_{\tilde{q}_i}) \right], \] (4.103)

\[ i\Sigma_{\tilde{q}_i\tilde{q}_j}(\not{p} = M) = \frac{g_s^2 C_F}{(4\pi)^2} \left[ \sin 4\theta_q \frac{A_0(m_{\tilde{g}}) - A_0(m_{\tilde{q}_j})}{2} + dm_{\tilde{g}} m_q \cos 2\theta_q B_0(M^2; m_{\tilde{g}}, m_q) \right], \] (4.104)
where \( j \neq i \). The first diagram in Fig. 4.4(b) has the same structure as the QCD correction in the fermion case. We therefore define it as “QCD”-like and get

\[
i^q_{\tilde{q}, \tilde{q}}(\vec{q} = m_{\tilde{q}_i}) = -\frac{g^2 C_F}{(4\pi)^2} \left[ 4m^2_{\tilde{q}_i} B_0(m^2_{\tilde{q}_i}; 0, m_{\tilde{q}_i}) - A_0(m_{\tilde{q}_i}) - (d - 4)m^2_{\tilde{q}_i} B_1(m^2_{\tilde{q}_i}; 0, m_{\tilde{q}_i}) \right]
\]

and accordingly the pure SUSY contribution as the remainder,

\[
i^\Sigma_{\tilde{q}, \tilde{q}}(\vec{q} = m_{\tilde{q}_i}) = i^\Sigma_{\tilde{q}, \tilde{q}}(\vec{q} = m_{\tilde{q}_i}) - i^\Sigma_{\tilde{q}, \tilde{q}}(\vec{q} = m_{\tilde{q}_i}).
\]

(4.106)

Eventually we define the following mass counter-terms

\[
(\delta m^q)^{\text{QCD}} = \text{Re} \Sigma^Q_{\tilde{q}}(\vec{q} = m_q)
\]

(4.107)

\[
(\delta m^q)^{\text{SUSY}} = \text{Re} \Sigma^S_{\tilde{q}}(\vec{q} = m_q)
\]

(4.108)

\[
(\delta m^2_{\tilde{q}_1})^{\text{QCD}} = \text{Re} \Sigma^Q_{\tilde{q}_1}(\vec{q} = m_{\tilde{q}_1})
\]

(4.109)

\[
(\delta m^2_{\tilde{q}_1})^{\text{SUSY}} = \text{Re} \Sigma^S_{\tilde{q}_1}(\vec{q} = m_{\tilde{q}_1})
\]

(4.110)

\[
(\delta m^2_{\tilde{q}_2})^{\text{QCD}} = \text{Re} \Sigma^Q_{\tilde{q}_2}(\vec{q} = m_{\tilde{q}_2})
\]

(4.111)

\[
(\delta m^2_{\tilde{q}_2})^{\text{SUSY}} = \text{Re} \Sigma^S_{\tilde{q}_2}(\vec{q} = m_{\tilde{q}_2})
\]

(4.112)

In renormalizing the squark mixing angle there is an ambiguity remaining in choosing the scale \( M \) at which to renormalize. In [14] the following choice is proposed

\[
\delta \theta_q = -\frac{\text{Re} \Sigma_{\tilde{q}_1, \tilde{q}_2}(\vec{q} = m_{\tilde{q}_1}) + \text{Re} \Sigma_{\tilde{q}_1, \tilde{q}_2}(\vec{q} = m_{\tilde{q}_2})}{2(m^2_{\tilde{q}_2} - m^2_{\tilde{q}_1})}. 
\]

(4.113)

while the authors of [49, 50] choose

\[
\delta \theta_q = -\frac{\text{Re} \Sigma_{\tilde{q}_1, \tilde{q}_2}(\vec{q} = Q)}{m^2_{\tilde{q}_2} - m^2_{\tilde{q}_1}}
\]

(4.114)

with an arbitrary scale \( Q \). In this work we will apply the latter scheme.

### 4.4.2.3 Renormalizing the Higgs coupling

The \( qqh \) vertex renormalizes as in the Standard Model case, i.e. the bare quark mass in the coupling \( m_q h_f(q) \) gets expressed by its renormalized quantity. On the other hand, the renormalization squark-squark-Higgs coupling needs a little more attention. Its renormalization is obtained by Taylor expanding \( m^2_q h_s(q, i, j) \) up to first order in the parameters, which get renormalized at NLO in QCD. We only have to consider the diagonal couplings
because the off-diagonal couplings appear at two-loop for the first time. The \( \tilde{q}qh \) counter terms are found to be

\[
\delta(m_\tilde{q}^2h_s(q, 1, 1)) = \frac{h_f(q)}{4} \left\{ [16m_q^2 + \Delta m_\tilde{q}^2(1 - \cos 4\theta_q)] \frac{\delta m_\tilde{q}^2}{m_q} - 2 \sin^2 2\theta_q \Delta \delta m_\tilde{q}^2 \\
- 2\Delta m_\tilde{q}^2 \sin 4\theta_q \delta \theta_q \right\} \frac{\delta m_q^2}{m_q} \frac{m_q^2}{2} \left\{ [h_s(q, 1, 1) - h_s(q, 2, 2)] \sin^2 2\theta_q \right\}
\]

\[
+ h_s(q, 1, 2) \sin 4\theta_q \right\} + 2m_q^2h_s(q, 1, 2)\delta \theta_q .
\]

(4.115)

with \( \Delta m_\tilde{q}^2 = m_\tilde{q}^2 - m_{\tilde{q}}^2 \) and \( \Delta \delta m_\tilde{q}^2 = \delta m_\tilde{q}^2 - \delta m_{\tilde{q}}^2 \). As discussed in all details in Chapter 4.3.4 of [22], supersymmetry must be restored in DREG by introducing a shift in the renormalizing constants equivalent to expressing all counter-terms on the right hand side of (4.115) and (4.116) through their DRED values.

If we were to compute the contribution of arbitrary scalars not related through symmetry to the fermions, we would have a coupling of the form \(-m_\tilde{q}^2\Lambda\tilde{q}\tilde{q}\) and the renormalization would only involve the renormalization of the scalar mass. Thus, we define

\[
\left[ \delta(m_\tilde{q}^2h_s(q, i, i)) \right]^{QCD} = \left[ m_\tilde{q}^2h_s(q, i, i) \right] \frac{\left( \delta m_\tilde{q}^2 \right)^{QCD}}{m_q^2},
\]

(4.117)

\[
\left[ \delta(m_\tilde{q}^2h_s(q, i, i)) \right]^{SUSY} = \left[ \delta(m_\tilde{q}^2h_s(q, i, i)) \right]^{DRED} - \left[ \delta(m_\tilde{q}^2h_s(q, i, i)) \right]^{QCD}
\]

(4.118)

and remember

\[
\left( \delta m_\tilde{q}^2 \right)^{QCD} = \delta m_\tilde{q}^2 \big|_{DRED}.
\]

(4.119)

The expressions for the heavy Higgs vertices are obtained by replacing \( h_{f,s} \) with \( H_{f,s} \) everywhere.

### 4.5 Renormalized amplitudes

The renormalized amplitude is obtained from the unrenormalized one by expressing all bare quantities in terms of renormalized quantities. These quantities are the strong coupling, quark and squark masses and couplings to the Higgs. Schematically we write therefore

\[
\mathcal{A}(\alpha_s^0, m^0, \lambda^0) = \mathcal{A}(\alpha_s + \delta\alpha_s, m + \delta m, \lambda + \delta\lambda)
\]

\[
= \mathcal{A}(\alpha_s, m, \lambda) + \frac{\partial\mathcal{A}}{\partial\alpha_s}(\alpha_s, m, \lambda)\delta\alpha_s + \frac{\partial\mathcal{A}}{\partial m}(\alpha_s, m, \lambda)\delta m + \frac{\partial\mathcal{A}}{\partial\lambda}(\alpha_s, m, \lambda)\delta\lambda
\]

(4.120)
where we take the derivative with respect to $m$ for fixed coupling $\lambda$. We shall not confuse $\lambda$ with $\Lambda_f$, the latter being the coupling with factored out mass dependence, i.e. $\mathcal{L}_{ffh} = -\lambda_f f f h = -m \Lambda_f f f h$ and $\mathcal{L}_{ssh} = -\lambda_s s h h = -m_s^2 \Lambda_s s s h$. Now we carry out this replacement for all form factors $A$ in (4.79) and distinguish between QCD-like and SUSY-like contributions.

In terms of the dimensionless coefficient functions $c_t^{(0)}$ introduced in (4.82), the fermionic contribution reads

$$\frac{\partial A_f}{\partial \alpha_s}(\alpha_s, m, \lambda_f) \delta \alpha_s = \left(\frac{\alpha_s}{4\pi}\right)^2 \Lambda_f c_t^{(0)} \delta \alpha_s, \quad (4.121)$$

$$\frac{\partial A_f}{\partial m}(\alpha_s, m, \lambda_f) \bigg|_{\lambda_f} \delta m = \left(\frac{\alpha_s}{4\pi}\right)^2 \Lambda_f \left[-1 + 2\epsilon c_t^{(0)} + \frac{2x(1-x) \partial c_t^{(0)}}{1+x} \partial x \right] \frac{\delta m}{m}, \quad (4.122)$$

$$\frac{\partial A_f}{\partial \lambda_f}(\alpha_s, m, \lambda_f) \delta \lambda_f = \left(\frac{\alpha_s}{4\pi}\right)^2 \Lambda_f c_t^{(0)} \delta \lambda_f. \quad (4.123)$$

In deriving (4.122) we exploited the chain rule and rewrote the derivative acting on $c_t^{(0)}$ in terms of the dimensionless variable $x$ using

$$\frac{\partial}{\partial m} = \frac{\partial m}{\partial x} \frac{\partial}{\partial x} = \frac{2x(1-x)}{1+x} \frac{\partial}{\partial x}. \quad (4.124)$$

We then define the QCD-like and SUSY-like counter terms by

$$(\delta A_f)^{\text{QCD}} = \delta \alpha_s^{\overline{\text{MS}}} A_t^{(0)} + \left(\frac{\delta m}{m}\right)^{\overline{\text{MS}}} A_t^{(0),r}, \quad (4.125)$$

$$(\delta A_f)^{\text{SUSY}} = \left(\frac{\delta m}{m}\right)^{\text{SUSY}} A_t^{(0),r} \quad (4.126)$$

where

$$A_t^{(0)} = \frac{\alpha_s}{4\pi} \left(\frac{\mu}{m}\right)^2 \Lambda_f c_t^{(0)}, \quad (4.127)$$

$$A_t^{(0),r} = \frac{\alpha_s}{4\pi} \left(\frac{\mu}{m}\right)^2 \Lambda_f \left[-2\epsilon c_t^{(0)} + \frac{2x(1-x) \partial c_t^{(0)}}{1+x} \partial x \right]. \quad (4.128)$$

The counter-terms $\delta \alpha_s^{\overline{\text{MS}}}$, $(\delta m)^{\overline{\text{MS}}}$ and $(\delta m)^{\text{SUSY}}$ are defined in (4.88), (4.95) and (4.108), respectively. We remind the reader that in the Standard Model calculation, $\Lambda_f = 1/v$, while in the MSSM calculation, $\Lambda_f = h_f/v$. For later convenience we choose to exhibit the soft-collinear singular pieces, which remain even after renormalization, in the following form

$$A_t^{(1)} = -\frac{\alpha_s}{4\pi} \frac{S_c}{s} \left(\frac{\mu}{s}\right)^2 \left[2N_c \left(\frac{1}{c^2} - \frac{7}{12}\right) + \frac{2\beta_0}{\epsilon}\right] \Lambda_f c_t^{(0)} \quad (4.129)$$
and write for the complete, renormalized QCD amplitude

\[ A_f^r = A_f^{(0)} + A_f^{(1)} + A_f^{(1)} + 2i\pi\beta_0 A_f^{(0)} + O(\alpha_s^3) \]  

(4.130)

The finite contribution is related to the full unrenormalized amplitude \( A_f \) through

\[ A_f = A_f^r + (\delta A_f)^{QCD} \]  

(4.131)

and can be written in terms of the coefficient function \( c_{\text{fin}}^{(1)}(\delta, 0) \), given in (D.7) below threshold, as

\[ A_{f,\text{fin}}^{(1)} = \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{\mu}{m} \right)^{2\epsilon} A_f c_{\text{fin}}^{(1)} . \]  

(4.132)

Similarly, we can derive the renormalized scalar amplitude, although we have to carefully trace factors of quark and squark masses due to the more complicated coupling of squarks to the Higgs.

\[ \frac{\partial A_s}{\partial \alpha_s}(\alpha_s, m, \lambda_s) \delta \alpha_s = \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\mu}{m} \right)^{2\epsilon} A_s c_s^{(0)} \delta \alpha_s , \]  

(4.133)

\[ \frac{\partial A_s}{\partial m}(\alpha_s, m, \lambda_s) \delta m \bigg|_{\lambda_s} = \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\mu}{m} \right)^{2\epsilon} \left[ (-1 - \epsilon) c_s^{(0)} + \frac{x(1-x)}{1+x} \frac{\partial c_s^{(0)}}{\partial x} \right] \frac{\delta m^2}{m^2} , \]  

(4.134)

\[ \frac{\partial A_s}{\partial \lambda_s}(\alpha_s, m, \lambda_s) \delta \lambda_s = \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\mu}{m} \right)^{2\epsilon} c_s^{(0)} \frac{\delta \lambda_s}{m^2} . \]  

(4.135)

For separating QCD and SUSY contributions, we introduce

\[ (\delta A_s)^{QCD} = \delta\alpha_s^{\text{MS}} A_s^{(0)} + \left( \frac{\delta m^2}{m^2} \right)^{\text{QCD}} A_s^{(0)} , \]  

(4.136)

and

\[ (\delta A_s)^{\text{SUSY}} = \left( \frac{\delta \lambda_s}{\lambda_s} \right)^{\text{SUSY}} A_s^{(0)} + \left( \frac{\delta m^2}{m^2} \right)^{\text{SUSY}} A_s^{(0)} , \]  

(4.137)

with

\[ A_s^{(0)} = \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\mu}{m} \right)^{2\epsilon} A_s c_s^{(0)} , \]  

(4.138)

\[ A_s^{(0),\prime} = \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{\mu}{m} \right)^{2\epsilon} \Lambda_s \left[ -c_s^{(0)} + \frac{x(1-x)}{1+x} \frac{\partial c_s^{(0)}}{\partial x} \right] . \]  

(4.139)

Identifying \( \lambda_s = m_q^2 h_s(q, i, i)/v \) and \( \Lambda_s = \lambda_s/m_q^2 \), we obtain the counter-terms with (4.117), (4.118) and (4.109)-(4.112). The infrared contribution takes on the same form as in the fermionic case,

\[ A_{s,\text{ir}}^{(1)} = -\left( \frac{\alpha_s}{4\pi} \right) S_s \left( \frac{\mu}{s} \right)^{2\epsilon} \left[ 2N_c \left( \frac{1}{\epsilon^2} - \frac{\pi^2}{12} \right) + \frac{2\beta_0}{\epsilon} \right] \Lambda_s c_s^{(0)} . \]  

(4.140)
Eventually, we write down the renormalized scalar amplitude
\[
A^r_s = A_s^{(0)} + A_s^{(1)} + A_{s,\text{ir}}^{(1)} + 2i\pi\beta_0 A_s^{(0)} + \mathcal{O}\left(\alpha_s^3\right),
\]
with
\[
A_{s,\text{fin}}^{(1)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\mu}{m}\right)^{2\epsilon} \Lambda \, c_{s,\text{fin}}^{(1)}.
\]
\[c_{s,\text{fin}}^{(1)}\] is found in (D.8) in a representation valid below threshold.

Since \(A_{\text{SUSY}}\) has no leading-order contribution, no additional counter-terms are generated at order \(\alpha_s^2\) by replacing bare quantities through renormalized quantities. We define
\[
\mathcal{A}_{\text{SUSY}} = A_{\text{SUSY}} + \sum_q (\delta A_q)_{\text{SUSY}} + \sum_{\tilde{q}} (\delta A_{\tilde{q}})_{\text{SUSY}}
\]
whose evaluation is described in [21, 22] and serves as an input in our studies but is not part of this thesis. Note that \(\mathcal{A}_{\text{SUSY}}\) is free from divergences.

### 4.5.1 The matrix element squared

We separate contributions, that get cancelled eventually by soft-collinear counter-terms in the subtraction from finite virtual corrections, which are further divided into a QCD-like and a SUSY-like part. The color and spin averaged squared matrix elements are given by
\[
\Upsilon_{\text{ir}}^v = \frac{1}{2s_{12} \omega_g^2} 2\mathcal{N}_{\text{proj}} \Re \left\{ \left( \sum_q A_{q,\text{ir}}^{(1)} + \sum_{\tilde{q}} A_{\tilde{q},\text{ir}}^{(1)} \right) \left( \sum_{q'} A_{q',\text{ir}}^{(0),\dagger} + \sum_{\tilde{q'}} A_{\tilde{q}',\text{ir}}^{(0),\dagger} \right) \right\},
\]
\[
\Upsilon_{\text{fin}}^v = \frac{1}{2s_{12} \omega_g^2} 2\mathcal{N}_{\text{proj}} \Re \left\{ \left( \sum_q A_{q,\text{fin}}^{(1)} + \sum_{\tilde{q}} A_{\tilde{q},\text{fin}}^{(1)} \right) \left( \sum_{q'} A_{q',\text{fin}}^{(0),\dagger} + \sum_{\tilde{q'}} A_{\tilde{q}',\text{fin}}^{(0),\dagger} \right) \right\},
\]
\[
\Upsilon_{\text{SUSY}}^v = \frac{1}{2s_{12} \omega_g^2} 2\mathcal{N}_{\text{proj}} \Re \left\{ \mathcal{A}_{\text{SUSY}}^{(1)} \left( \sum_{q'} A_{q'}^{(0),\dagger} + \sum_{\tilde{q'}} A_{\tilde{q}'}^{(0),\dagger} \right) \right\}.
\]
The common pre-factor
\[
\mathcal{N}_{\text{proj}} = (N_c^2 - 1)s_{12}^2 (d - 2)/4
\]
results from squaring the projector and the color factor. The various form factors \(A\) are all defined in (4.5).
4.6 Preserving the quark/squark mass relations

So far we have ignored the fact that supersymmetry imposes relations among various parameters. This situation is now going to be remedied in view of phenomenological studies discussed in chapter 9. At tree level the following relations hold

\[ m_q A_q = \frac{m_{\tilde{q}_1}^2 - m_{\tilde{q}_2}^2}{2} \sin 2\theta_q + m_q \mu \{ \cot \beta, \tan \beta \}, \tag{4.148} \]
\[ m_{\tilde{Q}_3}^2(q) = m_{\tilde{t}_1}^2 \cos^2 \theta_q + m_{\tilde{Q}_2}^2 \sin^2 \theta_q - m_Z^2 \cos(2\beta)(T_{3L,q} - Q_q \sin^2 \theta_W) - m_{\tilde{q}_1}^2, \tag{4.149} \]

where in (4.148) the correct quantity in curly brackets for up-type (s)quarks is \( \cot \beta \) and \( \tan \beta \) for down-type (s)quarks. At tree level, \( m_{\tilde{Q}_3}^2(t) = m_{\tilde{Q}_3}^2(b) \). Applying the \( \overline{\text{MS}} \) or \( \text{DR} \) scheme for renormalizing quark and squark masses, these relations remain valid even at higher orders, since the renormalization constants are not mass dependent. In the pole scheme, however, this relationship gets modified,

\[ m_{\tilde{Q}_3}^2(t) + \delta m_{\tilde{Q}_3}^2(t) = m_{\tilde{Q}_3}^2(b) + \delta m_{\tilde{Q}_3}^2(b). \tag{4.150} \]

In [14] several mixed renormalization schemes are discussed and compared in the case of determining the Higgs mass. Their approach in the “\( m_b \) OS” scheme is to treat the \( m_{\tilde{t}_1} \) mass as dependent and derive its counter term from the mass relation

\[ m_{\tilde{b}_1}^2 \cos^2 \theta_b + m_{\tilde{b}_2}^2 \sin^2 \theta_b = m_{\tilde{t}_1}^2 \cos^2 \theta_t + m_{\tilde{t}_2}^2 \sin^2 \theta_t + m_b^2 - m_t^2 - m_W^2 \cos 2\beta \tag{4.151} \]

which yields

\[ \delta m_{\tilde{b}_1}^2 = \frac{1}{\cos^2 \theta_b} \left( \cos^2 \theta_b \delta m_{\tilde{t}_1}^2 + \sin^2 \theta_t \delta m_{\tilde{t}_2}^2 - \sin^2 \theta_b \delta m_{\tilde{t}_2}^2 - \sin 2\theta_t (m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2) \right) \delta \theta_t \]
\[ + \sin 2\theta_b (m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2) \delta \theta_b - 2m_t \delta m_t + 2m_b \delta m_b. \tag{4.152} \]

All other masses are renormalized on-shell while mixing angles are renormalized according to (4.113). This definition ensures the validity of (4.150) even at NLO but treats the mass of one of the squarks \( m_{\tilde{b}_1} \) different from the other masses, which seems to be unphysical.

In this work we rather adopt a different approach, proposed in [51,52]. Instead of ensuring (4.150) on the level of renormalization, we renormalize all masses on-shell but solve the implicit equation (4.150) for the input values \( m_{\tilde{q}}^{\text{OS}}, q = t, b \) and \( i = 1, 2 \). Note that we differ from [51,52] in the treatment of the mass in the Higgs coupling. While in [51,52] the \( \text{DR} \) mass is used, we keep the on-shell mass in the coupling. The difference is of higher order.

Solving the implicit mass relation. We describe now, how to obtain mass and mixing angle parameters for our phenomenological studies. We start with the following input
parameters: $m_t$, $m_b$, $m_{t_1}$, $m_{t_2}$, $\theta_t$, $m_{D3}$, $A_b$, $\tan \beta$, $\mu$, $M_3 = m_{\tilde{b}}$, $m_{\tilde{b}_0}$, $\mu_R$, $m_Z$, $\sin^2 \theta_W$. Then we determine $m_{t_1}$, $m_{t_2}$ and $\theta_t$ in an iterative procedure until a chosen accuracy $\delta_{\text{acc}}$ is reached:

(1) Initialize.

(i) Set $m_{D3}^{2(0)} = m_{D3}^2$

(ii) Compute left-handed squark mass parameter in the top/stop sector:

$$m_{Q_3}^{2(0)}(\tilde{t}) = m_{t_1}^2 \cos^2 \theta_t + m_{t_2}^2 \sin^2 \theta_t - \left( \frac{1}{2} - \frac{2}{3} \sin^2 \theta_W \right) m_Z^2 \cos 2\beta. \quad (4.153)$$

(iii) Compute entries of sbottom mass matrix:

$$m_{LL}^{2(0)} = m_{Q_3}^{2(0)}(\tilde{t}) + m_{t_1}^2 + \left( \frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right) m_Z^2 \cos 2\beta, \quad (4.154)$$

$$m_{RR}^{2(0)} = m_{D3}^2 + m_{t_2}^2 + \left( \frac{1}{3} \sin^2 \theta_W \right) m_Z^2 \cos 2\beta, \quad (4.155)$$

$$m_{RL}^{2(0)} = m_b (A_b - \mu \tan \beta). \quad (4.156)$$

(iv) Compute sbottom mass eigenvalues:

$$m_{b_1}^{2(0)} = \frac{m_{LL}^{2(0)} + m_{RR}^{2(0)}}{2} - \frac{1}{2} \sqrt{\left( m_{LL}^{2(0)} - m_{RR}^{2(0)} \right)^2 + 4 m_{RL}^{2(0)}}, \quad (4.157)$$

$$m_{b_2}^{2(0)} = \frac{m_{LL}^{2(0)} + m_{RR}^{2(0)}}{2} + \frac{1}{2} \sqrt{\left( m_{LL}^{2(0)} - m_{RR}^{2(0)} \right)^2 + 4 m_{RL}^{2(0)}}. \quad (4.158)$$

(v) Compute sbottom mixing angle:

$$\theta_b^{(0)} = \frac{1}{2} \arcsin \left( \frac{2 m_b (A_b - \mu \tan \beta)}{m_{b_1}^{2(0)} - m_{b_2}^{2(0)}} \right). \quad (4.159)$$

(vi) Compute finite corrections in top/stop-sector:

$$\delta m_t = \text{Re} \Sigma_{t_1}^{\text{finite}}(p = m_t), \quad (4.160)$$

$$\delta m_{t_1}^2 = \text{Re} \Sigma_{t_1 t_1}^{\text{finite}}(p = m_{t_1}), \quad (4.161)$$

$$\delta m_{t_2}^2 = \text{Re} \Sigma_{t_2 t_2}^{\text{finite}}(p = m_{t_2}), \quad (4.162)$$

$$\delta \theta_t = - \frac{\text{Re} \Sigma_{t_1 t_2}^{\text{finite}}(p = m_{t_1}) + \text{Re} \Sigma_{t_1 t_2}^{\text{finite}}(p = m_{t_2})}{2 \left( m_{t_2}^2 - m_{t_1}^2 \right)} \cdot (4.163)$$

(vii) Compute finite correction to $m_{Q_3}^{2(0)}(\tilde{t})$:

$$\delta m_{Q_3}^{2(0)}(\tilde{t}) = \delta m_{t_1}^2 \cos^2 \theta_t + \delta m_{t_2}^2 \sin^2 \theta_t - \left( m_{t_1}^2 - m_{t_2}^2 \right) \sin 2\theta_t \delta \theta_t - 2 m_t \delta m_t, \quad (4.164)$$

$$m_{Q_3}^{2(1)}(\tilde{t}) = m_{Q_3}^{2(0)}(\tilde{t}) + \delta m_{Q_3}^{2(0)}(\tilde{t}). \quad (4.165)$$
(viii) Compute finite corrections in bottom/sbottom-sector:

\[
\delta m_b^{(0)} = \text{Re} \left( \Sigma_b^{\text{finite}}(p = m_b^{(0)}) \right), \\
\delta m_{b_1}^{(0)} = \text{Re} \left( \Sigma_{b_1 b_1}^{\text{finite}}(p = m_{b_1}^{(0)}) \right), \\
\delta m_{b_2}^{(0)} = \text{Re} \left( \Sigma_{b_2 b_2}^{\text{finite}}(p = m_{b_2}^{(0)}) \right), \\
\delta \theta_b^{(0)} = -\frac{\text{Re} \left( \Sigma_{b_2 b_2}^{\text{finite}}(p = m_{b_1}^{(0)}) \right) + \text{Re} \left( \Sigma_{b_1 b_1}^{\text{finite}}(p = m_{b_2}^{(0)}) \right)}{2 \left( m_{b_2}^{(0)} - m_{b_1}^{(0)} \right)}. 
\]

(ix) Compute right handed down type mass:

\[
m_{D_3}^{(0)} = m_{b_1}^{(0)} \sin \theta_b^{(0)} + m_{b_2}^{(0)} \sin \theta_b^{(0)} - m_b^{(0)} - \left( -\frac{1}{3} \sin^2 \theta_W \right) m_Z^2 \cos 2\beta. 
\]

(2) Iterative procedure. Start with \( n = 0 \).

(i) Compute finite correction to \( m_{Q_3}^{(b)} \):

\[
\delta m_{Q_3}^{(n+1)}(\tilde{b}) = \delta m_{Q_3}^{(n)}(\tilde{b}) + \delta m_{b_1}^{(n)} \cos^2 \theta_b^{(n)} - \delta m_{b_2}^{(n)} \sin^2 \theta_b^{(n)} - \left( m_{b_1}^{(n)} - m_{b_2}^{(n)} \right) \sin 2 \theta_b^{(n)} \delta \theta_b^{(n)} - 2 m_b \delta m_b^{(n)},
\]

\[
m_{Q_3}^{(n+1)}(\tilde{b}) = m_{Q_3}^{(n)}(\tilde{b}) - \delta m_{Q_3}^{(n)}(\tilde{b}).
\]

(ii) Compute entries of sbottom mass matrix:

\[
m_{LL}^{(n+1)} = m_{Q_3}^{(n+1)}(\tilde{b}) + m_b^2 + \left( -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right) m_Z^2 \cos 2\beta,
\]

\[
m_{RR}^{(n+1)} = m_{D_3}^{(n+1)} + m_b^2 + \left( -\frac{1}{3} \sin^2 \theta_W \right) m_Z^2 \cos 2\beta,
\]

\[
m_{RL}^{(n+1)} = m_b (A_b - \mu \tan \beta).
\]

(iii) Compute sbottom mass eigenvalues and mixing angle:

\[
m_{b_1}^{(n+1)} = \frac{m_{LL}^{(n+1)} + m_{RR}^{(n+1)}}{2} - \frac{1}{2} \sqrt{\left( m_{LL}^{(n+1)} - m_{RR}^{(n+1)} \right)^2 + 4 m_{RL}^{(n+1)^2}},
\]

\[
m_{b_2}^{(n+1)} = \frac{m_{LL}^{(n+1)} + m_{RR}^{(n+1)}}{2} + \frac{1}{2} \sqrt{\left( m_{LL}^{(n+1)} - m_{RR}^{(n+1)} \right)^2 + 4 m_{RL}^{(n+1)^2}},
\]

\[
\theta_b^{(n+1)} = \frac{1}{2} \arcsin \left( \frac{-2 m_b (A_b - \mu \tan \beta)}{m_{b_1}^{(n+1)} - m_{b_2}^{(n+1)}} \right).
\]
(iv) Compute finite corrections in bottom/sbottom-sector:

\[ \delta m_b^{(n+1)} = \text{Re} \Sigma_b^{\text{finite}}(p = m_b), \]
\[ \delta m_{b_1}^{2,(n+1)} = \text{Re} \Sigma_{b_1 b_1}^{\text{finite}}(p = m_{b_1}^{(n+1)}), \]
\[ \delta m_{b_2}^{2,(n+1)} = \text{Re} \Sigma_{b_2 b_2}^{\text{finite}}(p = m_{b_2}^{(n+1)}), \]
\[ \delta \theta_b^{(n+1)} = \frac{\text{Re} \Sigma_{b_1 b_2}^{\text{finite}}(p = m_{b_1}^{(n+1)}) + \text{Re} \Sigma_{b_1 b_1}^{\text{finite}}(p = m_{b_2}^{(n+1)})}{2 \left( m_{b_2}^{2,(n+1)} - m_{b_1}^{2,(n+1)} \right)}. \]

(v) Compute right handed down type mass:

\[ m_{D_3}^{2,(n+1)} = m_{b_1}^{2,(n+1)} \sin \theta_b^{(n+1)} + m_{b_2}^{2,(n+1)} \sin \theta_b^{(n+1)} - m_b^2 - \left( \frac{1}{3} \sin^2 \theta_W \right) m_Z^2 \cos 2\beta. \]

(vi) Compute

\[ \Delta_1 = 1 - \frac{m_{D_3}^{(n+1)}}{m_{D_3}^{(n)}}, \quad \Delta_2 = 1 - \frac{m_{b_1}^{(n+1)}}{m_{b_1}^{(n)}}, \quad \Delta_3 = 1 - \frac{\theta_b^{(n+1)}}{\theta_b^{(n)}}, \quad \Delta_4 = 1 - \frac{m_{D_3}^{(n+1)}}{m_{D_3}^{(0)}}. \]

If \( \exists i \in \{1,2,3\} : \Delta_i > \delta_{\text{acc}} \implies \text{go to (i)}. \)

If \( \Delta_4 > \delta_{\text{acc}} \implies \text{Failure.} \)

Else go to END.

The check on \( \Delta_4 \) is performed for consistency reasons. Since \( m_{D_3} \) is an input value, it must not change its value during the iterative solution of the implicit mass relation, although it is related to the sbottom squark masses and the sbottom mixing angle.

In chapter 9 we will carry out a scan in the golden region of the MSSM (see section 2.4) parameterized by \( m_{\tilde{t}_2} \). The remaining parameters in squark sector have been obtained with the procedure described in this section and a table of the resulting masses and mixing angles will be presented.
Chapter 5

Real radiation

At next-to-leading order in the strong coupling constant not only virtual corrections with an additional loop contribute, but also real radiation process with an additional parton in the final state. In preceding chapters we have already mentioned that only the combination of real and virtual contributions is finite. In order to exhibit this fact explicitly we will also discuss singularity structures of the real radiation amplitudes in this chapter. The discussion of how to combine real and virtual corrections for numerical evaluation is postponed to chapter 6.

There are three different Bremsstrahlung configurations in the case of Higgs production through gluon fusion. These are $gg \rightarrow gh$, $gq \rightarrow qh$ and $\bar{q}q \rightarrow gh$. The former one is the one with the most diagrams contributing. The latter two are related by crossing to each other and consist basically from the leading order diagram where one of the external gluons split into quarks. We start with the most complex one, the gluon-gluon initial state process, and discuss it in detail. Many techniques and simplifications are directly applicable in the other cases and we will only quickly cover them.

5.1 The process $gg \rightarrow gh$

In Fig. 5.1 we show typical diagrams contributing to the gluon-initiated Bremsstrahlung process, containing fermion loops and scalar loops, respectively. We find twelve fermionic and 21 scalar diagrams. The additional diagrams in the scalar case are due to the squark-squark-gluon-gluon vertex.

The incoming gluons carry momenta $p_1$ and $p_2$, the outgoing gluon momentum $p_3$ and
the Higgs boson $p_4$. The kinematics is given by

\begin{align}
    p_1^2 &= p_2^2 = p_3^2 = 0, \quad (5.1) \\
    p_4^2 &= m_h^2, \quad (5.2) \\
    p_1 + p_2 &= p_3 + p_4. \quad (5.3)
\end{align}

The final result will depend only on the Lorentz invariant Mandelstam variables

\begin{align}
    s_{12} &= (p_1 + p_2)^2 = 2 p_1 \cdot p_2, \quad s_{13} = (p_1 - p_3)^2 = -2 p_1 \cdot p_3 \quad \text{and} \\
    s_{23} &= (p_2 - p_3)^2 = -2 p_2 \cdot p_3. \quad (5.4)
\end{align}

We compute the matrix elements for real radiation by first generating the diagrams with \texttt{QGRAF}, which are then processed by a combination of \texttt{Maple} and \texttt{form} routines. The reduction into scalar master integrals is done applying Passarino-Veltman reduction [54]. Eventually, we express the result in terms of helicity amplitudes and various checks are performed on the amplitude.

We will first discuss the color structure of our amplitude which together with Furry’s theorem provides an useful check. Next, we will introduce the spinor formalism which enables us to efficiently work with helicity amplitudes. By introducing helicity amplitudes we choose to work in the ’t Hooft-Veltman (HV) scheme, where internal gluons are treated $d$ dimensional while external ones are kept in 4 dimensions. This has to be distinguished from the conventional regularization scheme (CDR) which has been applied in the computation of the virtual corrections. In CDR also external gluons are treated $d$ dimensional. When combining virtual and real contributions we thus have to translate between the two schemes.
5.1. THE PROCESS \( GG \rightarrow GH \)

5.1.1 Color decomposition

We start in remembering the commutation and anti-commutation properties of the color generators,

\[
[T^a, T^b] = i f^{abc} T^c, \quad (5.5)
\]
\[
\{T^a, T^b\} = \frac{1}{N_c} \delta^{ab} 1 + d^{abc} T^c, \quad (5.6)
\]

where \( f^{abc} \) is totally antisymmetric while \( d^{abc} \) is totally symmetric under interchange of any two indices \( a, b \) and \( c \). Diagrams containing a triangle loop have a color factor

\[
f^{abd} \text{Tr} \left[ T^d T^c \right] = \frac{1}{2} f^{abc}. \quad (5.7)
\]

while Box diagrams are proportional to

\[
\text{Tr} \left[ T^a T^b T^c \right] = \frac{1}{2} \text{Tr} \left[ \left( [T^a, T^b] + \{T^a, T^b\} \right) T^c \right] = i \frac{1}{4} f^{abc} + \frac{1}{4} d^{abc}. \quad (5.8)
\]

**Furry’s theorem.** Furry’s theorem tells us, that the amplitude of any process with an odd number of outer photon lines vanishes due to the invariance of the Lagrangian under charge conjugation. First, we need the transformation property of the electromagnetic field under charge conjugation \( C \)

\[
CA^\mu C^{-1} = -A^\mu
\]

and note that the vacuum is an eigenstate of charge conjugation: \( C |0\rangle = |0\rangle \). The vacuum expectation value for the product of three photons is then

\[
\langle 0 | T(A^\mu A^\nu A^\rho) | 0 \rangle = \langle 0 | C^{-1} CT(A^\mu A^\nu A^\rho) C^{-1} C | 0 \rangle
\]
\[
= \langle 0 | C^{-1} T(CA^\mu C^{-1} C A^\nu C^{-1} C A^\rho C^{-1}) C | 0 \rangle
\]
\[
= (-1)^3 \langle 0 | T(A^\mu A^\nu A^\rho) | 0 \rangle = 0,
\]

where \( T \) is the Dyson operator assuring the product of \( A^\mu \) to be time ordered. Applied to Feynman diagrams it can be shown that diagrams with opposite charge flow in a closed fermion loop for a fixed sequence of an odd number of external photons cancel each other pairwise.

Based on Furry’s theorem we conclude that the diagrams proportional to \( d^{abc} \) must add up to zero, since they can be considered as QED diagrams with external photons. This provides a first check of our calculation.
5.1.2 Spinor formalism

We assume here and in the following always mass-less Dirac spinors. Every spinor $u$ (of positive energy) and anti-spinor $v$ (of negative energy) comes in two polarizations,

$$u_{\pm}(p) = \frac{1}{2} \left( 1 \pm \gamma^5 \right) u(p) \quad \text{and} \quad v_{\mp}(p) = \frac{1}{2} \left( 1 \mp \gamma^5 \right) v(p).$$

(5.9)

Since in the mass-less limit we can normalize $u$ and $v$ equally, the following shorthand notation turns out to be useful

$$v_{\pm}(p_i) = u_{\pm}(p_i) = |p_i^\pm\rangle = |i^\pm\rangle, \quad \overline{v_{\pm}(p_i)} = u_{\pm}(p_i) = \langle p_i^\pm| = \langle i^\pm|. \quad (5.10)$$

Products of spinors are written as

$$u_{-}(p_i)u_{+}(p_j) = \langle i^- | j^+ \rangle = \langle i j \rangle \quad \text{and} \quad u_{+}(p_i)u_{-}(p_j) = \langle i^+ | j^- \rangle = [i j] \quad (5.11)$$

One can derive various useful identities involving these spinor products.

- The Dirac equation implies

$$\langle i j \rangle [j i] = \langle i^- | j^+ \rangle \langle j^+ | i^- \rangle = \text{Tr} \left( \frac{1}{2} (1 - \gamma^5) p_i p_j \right) = s_{ij} \quad \text{and in particular} \quad \langle i^\pm | j^\pm \rangle = 0 \quad (5.12)$$

- Representation of the projection operator:

$$|i^\pm\rangle \langle i^\pm| = \frac{1}{2} (1 \pm \gamma^5) p_i. \quad (5.14)$$

- Antisymmetry:

$$\langle i j \rangle = - \langle j i \rangle, \quad [i j] = - [j i], \quad \langle i i \rangle = [i i] = 0 \quad (5.15)$$

- Fierz rearrangement:

$$\langle i^+ | \gamma^\mu | j^+ \rangle \langle k^+ | \gamma_\mu | l^+ \rangle = 2 [i k] \langle j l \rangle \quad (5.16)$$

- Charge conjugation of current implies:

$$\langle i^+ | \gamma^\mu | j^+ \rangle = \langle j^- | \gamma^\mu | i^- \rangle \quad (5.17)$$

- Schouten identity:

$$\langle i j \rangle \langle k l \rangle = \langle i k \rangle \langle j l \rangle + \langle i l \rangle \langle k j \rangle \quad (5.18)$$
Often we will make use of another notational simplification:

\[ \langle i^+ | \gamma^\mu | j^+ \rangle = [i] \gamma^\mu [j] \quad \text{and} \quad \langle i^- | \gamma^\mu | j^- \rangle = \langle i | \gamma^\mu | j \rangle . \]  

(5.19)

By choosing a particular representation of \( \gamma \) matrices,

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]  

(5.20)

we are able to derive explicit expressions for spinor products in the case of \( p_i^0 > 0 \) and \( p_j^0 > 0 \)

\[ \langle i j \rangle = \sqrt{|s_{ij}|} e^{i\phi_{ij}} \quad \text{and} \quad [i j] = \sqrt{|s_{ij}|} e^{-i(\phi_{ij} + \pi)} \]  

(5.21)

where

\[ \cos \phi_{ij} = \frac{p_1^i p_3^j - p_3^i p_1^j}{\sqrt{|s_{ij}| p_1^i p_3^j}} \quad \text{and} \quad \sin \phi_{ij} = \frac{p_2^i p_3^j - p_3^i p_2^j}{\sqrt{|s_{ij}| p_1^i p_3^j}}. \]  

(5.22)

In this representation the spinors read

\[ u_+ (p) = v_- (p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\phi_p} \end{pmatrix}, \quad u_- (p) = v_+ (p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p^-} e^{-i\phi_p} \\ -\sqrt{p^+} \\ -\sqrt{p^-} e^{-i\phi_p} \end{pmatrix} \]  

(5.23)

with

\[ e^{\pm i\phi_p} = \frac{p_1^1 \pm ip_2^2}{\sqrt{p^+ p^-}}, \quad p^\pm = p_0^0 \pm p_3^3. \]  

(5.24)

Identities in the case where one or both of the energies is negative are obtained by analytic continuation.

Next, we want to find a representation of polarization vectors in terms of spinors. Polarization vectors have the following properties. The polarization vector \( \varepsilon^\pm (p) \) represents a helicity of \( \pm 1 \) and is transversal to the momentum \( p \),

\[ p \cdot \varepsilon^\pm (p) = 0. \]  

(5.25)

The reversed helicity is obtained through complex conjugation

\[ (\varepsilon^\mu_+)^\ast = \varepsilon^-_{\mu} \]  

(5.26)

and the normalization is chosen such that

\[ \varepsilon^+ \cdot (\varepsilon^+)^\ast = \varepsilon^+ \cdot \varepsilon^- = -1 \quad \text{and} \quad \varepsilon^+ \cdot (\varepsilon^-)^\ast = \varepsilon^+ \cdot \varepsilon^+ = 0. \]  

(5.27)
It is easy to show that the representation
\[ \varepsilon^{+\mu}(p, q) = \pm \frac{(q^+ | \gamma_{\mu} | p^+)}{\sqrt{2}(q^+ | p^+)} \]
fulfills all the above identities. Here we have introduced an auxiliary momentum \( q \), called the reference momentum, with \( q^2 = 0 \). The choice of \( q \) corresponds to a specific gauge choice, which is obvious by considering
\[ \varepsilon^+(p, q) - \varepsilon^+(p, k) = -\frac{\langle q^- | \gamma_{\mu} q | k^+ \rangle + \langle q^- | p \gamma_{\mu} | k^+ \rangle}{\sqrt{2} (q p) \langle k p \rangle} = -p_\mu \frac{\sqrt{2} \langle q k \rangle}{\langle q p \rangle \langle k p \rangle} \]
and remembering that a gauge transformation on \( \varepsilon_\mu(p) \) is proportional to \( p_\mu \). The reference vector \( q \) can be chosen independently for every external gluon, but must be kept fixed during the calculation. When dealing with gauge invariant objects, the invariance under different choices of reference momenta is a powerful check of the calculation.

The following identities involving the reference momentum are helpful in deriving a compact expression for our final result:
\[ \varepsilon^+(p, q) \cdot \varepsilon^-(k, p) = 0, \quad q \cdot \varepsilon^+(p, q) = 0, \]
\[ \varepsilon^+(p, q) \cdot \varepsilon^-(q, k) = 0, \quad k \cdot \varepsilon^+(p, q) = \frac{\langle q k \rangle \langle k p \rangle}{\sqrt{2} (q p)}, \]
\[ \varepsilon^+(p_1, q) \cdot \varepsilon^+(p_2, k) = \frac{|p_1 p_2\rangle \langle k q |}{(q p_1) \langle k p_2 |}, \quad k \cdot \varepsilon^-(p, q) = \frac{|q k \rangle \langle k p |}{\sqrt{2} (q p)}, \]
\[ \varepsilon^-(p_1, q) \cdot \varepsilon^-(p_2, k) = \frac{|p_1 p_2\rangle \langle k q |}{(q p_1) \langle k p_2 |}, \quad \varepsilon^+(p_1, q) \cdot \varepsilon^-(p_2, k) = -\frac{|p_1 k \rangle \langle p_2 q |}{(q p_1) \langle k p_2 |}. \]

5.1.3 The amplitude

In defining definite helicities for all external particles, amplitudes decompose into Lorentz invariant sub-amplitudes, so called helicity amplitudes, with a much simpler structure than the full amplitude. In our calculation we did not exploit these simplifications throughout the calculation, but rather used them in order to derive compact expressions for the final result or to perform checks on our calculation.

The computation is therefore carried out in the ’t Hooft Veltman regularization scheme, where internal gluons are treated \( d \) dimensional, while external gluons are kept in 4 dimensions. Gluons that become soft or collinear must be treated \( d \) dimensional as well in the phase space integral.

Writing explicitly the helicity and color indices, the amplitude is given by
\[ \mathcal{M}^{abc}_{\lambda_1 \lambda_2 \lambda_3}(gg \to gh) = C^{abc} M^{\lambda_1 \lambda_2 \lambda_3} = \varepsilon^{+\mu}(p_1) \varepsilon^{+\nu}(p_2) \varepsilon^{+\rho}(p_3) C^{abc} T_{\mu\nu\rho} \]

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where the $\lambda_i$ are the helicity indices, $\lambda_i = \pm$. As discussed in 5.1.1, the amplitude has a definite color structure,

$$C^{abc} = f^{abc} \quad (5.32)$$

following from Furry’s theorem. We are again interested in writing down the matrix element squared, summed over spins and averaged over incoming color and spin, divided by the flux factor. It is given by

$$\Upsilon^{(0)}_{gg \rightarrow gh} = \frac{1}{2s_{12}} \frac{1}{\omega_g \omega_g} \sum_{\lambda_1, \lambda_2, \lambda_3 = \pm} |M^{abc}_{\lambda_1, \lambda_2, \lambda_3}(gg \rightarrow gh)|^2 . \quad (5.33)$$

Note that we first square the helicity amplitudes and then sum over helicity, which is of course much simpler than first summing over helicities and then squaring, the reason being helicity conservation. Therefore no interference terms between amplitudes with different helicity configurations are possible. Naively we would have $2^3 = 8$ different helicity amplitudes. However not all of them are unrelated. To start with we observe that under $P$ parity we find (neglecting the phase),

$$M_{\lambda_1, \lambda_2, \lambda_3} = M_{-\lambda_1, -\lambda_2, -\lambda_3}. \quad (5.34)$$

Furthermore, by exchanging an outgoing particle with an incoming while simultaneously flipping helicity of these two particles, we have simply relabeled the external particles and therefore up to a phase,

$$M_{++-} = 1 \leftrightarrow 3 \rightarrow M_{++-}(s_{12}, s_{13}, s_{23}) = M_{++-}(s_{12}, s_{13}, s_{23}), \quad (5.35)$$

$$M_{+-+} = 2 \leftrightarrow 3 \rightarrow M_{+-+}(s_{12}, s_{13}, s_{23}) = M_{+-+}(s_{23}, s_{12}, s_{13}), \quad (5.36)$$

such that eventually we have to consider only two independent helicity amplitudes, $M_{+++}$ and $M_{++-}$. Please note that we label here the helicities for incoming particles with respect to incoming momenta and helicities of outgoing particles with respect to outgoing momenta. In the literature often a different convention is used, where all momenta are either considered to be incoming or outgoing. In order to translate between that convention and ours, we simply have to invert the helicity of the outgoing parton. Equation (5.33) in terms of helicity amplitudes reads now

$$\Upsilon^{(0)}_{gg \rightarrow gh} = \frac{2N_c(N_f^2 - 1)}{2s_{12} \omega_g \omega_g} \left( |M_{++-}^{(gg \rightarrow gh)}(s_{12}, s_{13}, s_{23})|^2 + |M_{++-}^{(gg \rightarrow gh)}(s_{13}, s_{12}, s_{23})|^2 \right) + |M_{+-+}^{(gg \rightarrow gh)}(s_{23}, s_{13}, s_{12})|^2 + |M_{+-+}^{(gg \rightarrow gh)}(s_{12}, s_{13}, s_{23})|^2 . \quad (5.37)$$

The expressions for the helicity amplitudes are found in Appendix E.1.
Gauge invariance provides us with a few checks of our computation. First, if we replace a polarization vector by its corresponding momentum, the amplitude must vanish, i.e.

$$p_1^\mu \varepsilon_{\lambda_2}^\nu (p_2) \varepsilon_{\lambda_3}^{\mu \sigma} (p_3) T_{\mu \nu \rho \sigma} = p_2^\mu \varepsilon_{\lambda_1}^\nu (p_1) \varepsilon_{\lambda_3}^{\mu \sigma} (p_3) T_{\mu \nu \rho \sigma} = p_3^\mu \varepsilon_{\lambda_1}^\nu (p_1) \varepsilon_{\lambda_2}^\nu (p_2) T_{\mu \nu \rho \sigma} = 0. \quad (5.38)$$

Secondly, gauge invariance also requires that the individual helicity amplitudes must be independent of the referencing. Another useful fact is obtained by looking at the expansion in terms of master integrals. Reduction to master integrals leads to the appearance of $B_0$ functions which have a pole in $\epsilon$. The result is therefore potentially infinite. However, the $gg \to gh$ one-loop process is the leading order process for $gggh$ coupling which according to the low energy theorem is described by a renormalizable term in the Lagrangian. Hence, the $B_0$ functions must arrange themselves such that poles in $\epsilon$ cancel out.

The helicity amplitudes in (5.37) are obtained as sums over fermionic and scalar contributions,

$$M_{\lambda_1 \lambda_2 \lambda_3} = \sum_Q M_{\lambda_1 \lambda_2 \lambda_3}^{(Q)} + \sum_Q M_{\lambda_1 \lambda_2 \lambda_3}^{(Q)} \quad (5.39)$$

where the sum over $Q$ runs over all heavy quarks and the sum over $\hat{Q}$ runs over all heavy scalar quarks. Explicit expressions in terms of scalar master integrals retaining the full mass dependence is found in Appendix E.1.

In the large mass approximation, the corresponding amplitudes are extremely compact and read

$$\frac{M_{++}^{SUSY} (gg \to gh)}{\Lambda_{gg}} = -\Lambda_q \frac{2}{3} s_{12}^2, \quad (5.40)$$

$$\frac{M_{++}^{SUSY} (gg \to gh)}{\Lambda_{gg}} = -\Lambda_q \frac{2}{3} (s_{12} + s_{23} + s_{13})^2, \quad (5.41)$$

$$\frac{M_{++}^{SUSY} (gg \to gh)}{\Lambda_{gg}} = -\Lambda_q \frac{1}{12} s_{12}^2, \quad (5.42)$$

$$\frac{M_{++}^{SUSY} (gg \to gh)}{\Lambda_{gg}} = -\Lambda_q \frac{1}{12} (s_{12} + s_{23} + s_{13})^2. \quad (5.43)$$

In the supersymmetric limit with $m_\tilde{q} = m_q = m$ and $\Lambda_\tilde{q} = 2\Lambda_q$, the amplitudes simplify very much upon adding the contributions of a supermultiplet

$$\frac{M_{++}^{SUSY} (gg \to gh)}{\Lambda_{gg}} = \Lambda_q m^2 s_{12} \left[ \frac{1}{2} s_{12} s_{23} D (s_{12}, s_{23}, m) + \frac{1}{2} s_{12} s_{13} D (s_{13}, s_{12}, m) - \frac{1}{2} s_{13} s_{23} D (s_{13}, s_{23}, m) + (s_{23} + s_{13}) C_1 (m^2_{\tilde{h}}, s_{12}, m) + (s_{12} - s_{23}) C_1 (m^2_{\tilde{h}}, s_{13}, m) + (s_{12} - s_{13}) C_1 (m^2_{\tilde{h}}, s_{23}, m) \right]. \quad (5.44)$$
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\[ M_{++}^{\text{SUSY}}(gg \rightarrow gh) \frac{\Delta_{gg}}{m_{h}^{2}m_{h}^{2}} \left[ \frac{1}{2}s_{12}s_{23}D(s_{12}, s_{23}, m) + \frac{1}{2}s_{13}s_{23}D(s_{13}, s_{12}, m) + \frac{1}{2}s_{23}s_{13}D(s_{13}, s_{23}, m) + (s_{23} + s_{12})C_{1}(m_{h}^{2}, s_{12}, m) + (s_{13} + s_{12})C_{1}(m_{h}^{2}, s_{23}, m) \right] \]

(5.45)

In particular we observe the absence of a rational term.

5.2 Real radiation involving light quarks

The real radiation processes with gluon-quark and quark-anti-quark initial states are related by crossing symmetry. It suffices therefore to compute \( \bar{q}q \rightarrow gh \) and to discuss crossing. The computation proceeds very analogously to the \( gg \rightarrow gh \) contribution but now we have to deal with external fermions. This is however straightforward working with helicity amplitudes and applying spinor formalism. The matrix element is written as

\[ M_{\lambda_{1}, \lambda_{2}; \lambda_{3}}^{\bar{q}q} = C_{ij}^{a}M_{\lambda_{1}, \lambda_{2}; \lambda_{3}} = (\bar{v}_{\lambda_{2}}(p_{2})\gamma^{\mu}u_{\lambda_{1}}(p_{1}))\xi_{\lambda_{3}}^{\rho}(p_{3})C_{ij}^{a}T_{\mu \rho} \]  

(5.46)

The color factor is trivially found to be

\[ C_{ij}^{a} = T_{ji}^{a}. \]  

(5.47)

After choosing an auxiliary momentum \( q \) for the spinor representation of the gluon polarization, the helicity amplitude reads in spinorial notation

\[ M_{\lambda_{1}, \lambda_{2}; \lambda_{3}}^{\bar{q}q} = (2^{-\lambda_{2}}|\gamma^{\mu}|1^{\lambda_{1}}) \left( -\lambda_{3}\left( q^{\lambda_{3}}|\gamma^{\rho}|3^{\lambda_{3}} \right) \frac{1}{\sqrt{2}}(q^{\lambda_{3}}|3^{\lambda_{3}}) \right) T_{\mu \rho} \]  

(5.48)

Again, a priori we have eight different helicity amplitudes. As in the case of the gluonic initial state, \( P \) parity relates amplitudes with flipped helicities,

\[ M_{\lambda_{1}, \lambda_{2}; \lambda_{3}} = M_{-\lambda_{1}, -\lambda_{2}; -\lambda_{3}}. \]  

(5.49)

Applying a \( C \) parity transformation on the initial state, we get back the same amplitude but with reversed quark helicities:

\[ M_{+, -; +}(s_{12}, s_{13}, s_{23}) = M_{-+, +}(s_{12}, s_{23}, s_{13}) = M_{+, -; -}(s_{12}, s_{23}, s_{13}), \]  

(5.50)

\[ M_{+, +; -}(s_{12}, s_{13}, s_{23}) = M_{-+, -; +}(s_{12}, s_{23}, s_{13}) = M_{+, +; +}(s_{12}, s_{23}, s_{13}). \]  

(5.51)

Next, we notice that (5.48) immediately implies that

\[ M_{+, +; -} = 0 \]  

(5.52)
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due to helicity conservation,

\[ \langle p_2^- | \gamma^\mu | p_1^+ \rangle = 0. \]  \hspace{1cm} (5.53)

Thus, it remains only one independent helicity amplitude, \( M_{+,-,-} \). \( \Upsilon_{q\bar{q} \rightarrow gh} \) is therefore given by

\[
\Upsilon_{q\bar{q} \rightarrow gh}^{(0)} = \frac{N_c^2 - 1}{2s_{12}\omega_{\bar{q} q}} \left( \left| M_{+,--}^{(q\bar{q} \rightarrow gh)} (s_{12}, s_{13}, s_{23}) \right|^2 + \left| M_{+,--}^{(q\bar{q} \rightarrow gh)} (s_{12}, s_{23}, s_{13}) \right|^2 \right). \]  \hspace{1cm} (5.54)

**Crossing.** The \( gq \rightarrow qh \) amplitude is related to the \( \bar{q}q \rightarrow gh \) amplitude by swapping the incoming anti-quark line to an outgoing quark line and switching the outgoing gluon leg with an incoming gluon leg. We choose to call the incoming momenta still \( p_1 \) and \( p_2 \) and the outgoing momentum \( p_3 \). The two amplitudes are related as follows

\[
M_{\lambda_1,\lambda_3;\lambda_2}^{(gg \rightarrow gh)} (s_{12}, s_{13}, s_{23}) = M_{\lambda_1,-\lambda_3;\lambda_2}^{(qq \rightarrow gh)} (s_{13}, s_{12}, s_{23}) \]  \hspace{1cm} (5.55)

where we have neglected phases. Thus we have

\[
\Upsilon_{qg \rightarrow qh}^{(0)} = \frac{N_c^2 - 1}{2s_{12}\omega_{\bar{q} q}} \left( \left| M_{+,--}^{(q\bar{q} \rightarrow gh)} (s_{13}, s_{12}, s_{23}) \right|^2 + \left| M_{+,--}^{(q\bar{q} \rightarrow gh)} (s_{13}, s_{23}, s_{12}) \right|^2 \right). \]  \hspace{1cm} (5.56)

Results for the helicity amplitudes are given in Appendix E.2 and in the large mass limit, the matrix elements read

\[
\frac{M_{+,--}^{SUSY}}{\Delta_{qg}} (q\bar{q} \rightarrow gh) = 2\Lambda_q s_{23} \]  \hspace{1cm} (5.57)

\[
\frac{M_{+,--}^{SUSY}}{\Delta_{qg}} (q\bar{q} \rightarrow gh) = -\Lambda_q \frac{1}{12} s_{23} \]  \hspace{1cm} (5.58)

The supersymmetric limit is again extremely simple:

\[
\frac{M_{+,--}^{SUSY}}{\Delta_{qg}} (q\bar{q} \rightarrow gh) = 2\Lambda_q s_{23} m^2 C_1 (m_0^2, s_{12}, m) \]  \hspace{1cm} (5.59)

### 5.3 Collinear and soft limits

The matrix elements we have computed in the last few sections exhibit singularities in certain phase space regions. These singularities are related to regions where the additional parton in the final state either becomes collinear to one of the incoming partons (i.e. \( s_{i3} \rightarrow 0 \), while \( s_{i3} \neq 0 \) for \( i \neq j \)) or if it becomes soft (i.e. \( s_{13}, s_{23} \rightarrow 0 \) simultaneously).

For illustration purposes we compute a specific collinear limit for the \( gg \rightarrow gh \) matrix element in the fermionic case and compare to the well known universal form.
5.3. Collinear and Soft Limits

5.3.1 Collinear limit

We consider the matrix element of real radiation with gluon-gluon initial state. The helicity amplitudes are given in (E.6) and (E.7). A collinear singularity could arise if the outgoing gluon becomes collinear with either of the incoming gluons, let us assume gluon 1, i.e. \( p_1 \cdot p_3 \to 0 \) while \( p_2 \cdot p_3 \neq 0 \). We start with discussing limits of scalar integrals. We find it convenient to introduce a new kinematic variable \( x(v) \) in analogy to the \( x \) variable we have been using in discussing the virtual contribution. Its definition as function of the kinematic invariant \( v \) and implicit dependence on the heavy quark mass \( m \) is

\[
x(v) = \begin{cases} 
\frac{\sqrt{1 - 4m^2}}{\sqrt{1 - 4m^2} + 1} & \text{if } v < 0 \text{ (spacelike)}, \\
\exp i\theta & \text{if } 0 < v < 4m^2 \text{ (below threshold)}, \\
\frac{\sqrt{1 - 4m^2}}{\sqrt{1 - 4m^2} + 1} & \text{if } v > 4m^2 \text{ (above threshold)}. 
\end{cases}
\] (5.60)

In terms of this variable the bubble function \( B(v, m) \) and the triangle function \( C(v, m) \) are

\[
B(v, m) = -\gamma_E + \ln 4\pi + \ln \frac{m^2}{\mu^2} + 2 + \frac{1 + x(v)}{1 - x(v)} \ln x(v),
\] (5.61)

\[
C(v, m) = -\frac{x(v)}{2m^2(1 - x(v))^2} \ln^2 x(v).
\] (5.62)

The second three point function, \( C_1(u, v, m) \) is at \( O(\epsilon) \) related to \( C(u, m) \) and \( C(v, m) \) by

\[
C_1(u, v, m) = \frac{1}{u - v} \left( uC(u, m) - vC(v, m) \right).
\] (5.63)

We are interested in the limits where \( v \to 0 \):

\[
\lim_{v \to 0} B(v, m) = -\gamma_E + \ln 4\pi + \ln \frac{m^2}{\mu^2} \quad \text{and}
\]

\[
\lim_{v \to 0} C(v, m) = -\frac{1}{2m^2}.
\] (5.65)

For finding the limits of vanishing kinematics invariants in the case of box functions we have to work a little bit more. First we remind ourselves that \( D(u, v, m) \) implicitly depends also on \( m_h^2 \) and we define \( w = m_h^2 - u - v \). Then the box function reads

\[
D(u, v, m) = \frac{1}{uv} \left( J(u) + J(v) - J(m_h^2) \right) \quad \text{with}
\]

\[
J(v) = \frac{2(1 - x(\kappa))}{1 + x(\kappa)} \left[ \text{Li}_2 \left( \frac{x(\kappa)(1 - x(v))}{1 - x(\kappa)x(v)} \right) - \text{Li}_2 \left( \frac{(1 - x(v))}{1 - x(\kappa)x(v)} \right) \\
+ \text{Li}_2 \left( \frac{x(\kappa)(1 - x(v))}{x(\kappa) - x(v)} \right) - \text{Li}_2 \left( \frac{(1 - x(v))}{x(\kappa) - x(v)} \right) \\
+ \ln x(\kappa) \ln \left( 1 - \frac{x(\kappa)(1 - x(v))^2}{x(v)(1 - x(\kappa))^2} \right) \right].
\] (5.67)

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Here, we have introduced $\kappa = -w/uv$. We have to consider two different collinear limits of the box function. One is when either $u$ or $v$ goes to zero, the other limit is if $w$ approaches zero. Expanding the dilogarithms we find that the collinear limits are expressable in terms of bubble and triangle functions. This is not surprising since in this limit we do not have three independent momenta anymore but only two as in the case of three point functions. The expressions read

$$\lim_{v \to 0} D(u, v, m) = \frac{1}{m^2 m} \left( B(u, m) - B(m^2, m) \right), \quad (5.68)$$

$$\lim_{w \to 0} D(u, v, m) = \frac{2}{uv} \left( uC(s, m) + vC(t, m) - m^2 m \right). \quad (5.69)$$

Now, we parametrize the Mandelstam variables as follows,

$$s_{12} = \frac{m^2}{z}, \quad s_{13} = -\frac{m^2}{2} \frac{1 - z}{z} (1 - y), \quad s_{23} = -\frac{m^2}{2} \frac{1 - z}{z} (1 + y). \quad (5.70)$$

In order to find the collinear limits of helicity amplitudes, it is not enough to consider the above quoted limits of the scalar integrals but we have to expand in $(1 - y)$ up to a suitable order since scalar integrals happen to be multiplied e.g. by factors of $1/s_{13}$. A careful analysis lead to the following helicity amplitudes in the collinear limit:

$$\lim_{y \to 1} M^{(g g - g h)}_{++ \pm \pm} (s_{12}, s_{13}, s_{23}) = \frac{z^{\frac{3}{2}}}{(1 - z)^{\frac{1}{2}}} \frac{16m^2}{m^2} (2 + (4m^2 - m^2) m) \quad (5.71)$$

$$\lim_{y \to 1} M^{(g g - g h)}_{++ - \pm} (s_{12}, s_{13}, s_{23}) = -\frac{z^{\frac{3}{2}}}{z^2 (1 - z)^{\frac{1}{2}}} \frac{16m^2}{m^2} (2 + (4m^2 - m^2) m) \quad (5.72)$$

$$\lim_{y \to 1} M^{(g g - g h)}_{++ \pm -} (s_{13}, s_{12}, s_{23}) = 0, \quad (5.73)$$

$$\lim_{y \to 1} M^{(g g - g h)}_{++ - -} (s_{23}, s_{13}, s_{12}) = -\frac{z^{\frac{3}{2}}(1 - y)^{\frac{1}{2}}}{z^2 (1 - z)^{\frac{1}{2}}} \frac{16m^2}{m^2} (2 + (4m^2 - m^2) m) \quad (5.74)$$

Then the matrix element squared properly summed and averaged reads in the collinear limit

$$\lim_{y \to 1} \bar{Y}^{(0)}_{(g g - g h)} = \frac{96\alpha_s g^3 m^4}{\pi s_{13}} \left( \frac{1}{2 z s_{12} \omega(g)} \right) \left( \frac{z^4 + 1 + (1 - z)^3}{z(1 - z)} \right) \left| 2 + (4m^2 - m^2) m \right|^2$$

$$= \frac{8\pi \alpha_s}{s_{13}} P_{g/g}(z) \bar{Y}^{(0)}_{g g - h}(z s_{12}). \quad (5.75)$$

The function $P_{g/g}(z)$ is called the Altarelli-Parisi gluon-gluon splitting function which will be discussed in the following section. There we will also see that the structure of the above
expression is universal and that we can immediately deduce the limiting behavior of matrix elements in the soft and collinear limits.

The explicit computation of the collinear limit in this section has been performed for the fermionic case only. The computation for the scalar case proceeds completely analogously and we recover (5.75) for the complete MSSM matrix element squared. Of course this is a direct consequence of the universality.

### 5.3.2 Universal form of collinear and soft limits

In [55] (see also [56]) the limit of a squared matrix elements is obtained from considering a slightly off-shell mass-less outgoing parton $i$ that splits into two mass-less partons $k$ and $l$ with momenta

$$p_k^\mu = zp_i^\mu + k_{\perp}^\mu - \frac{k_{\perp}^2}{2(p \cdot n)} n^\mu,$$

$$p_l^\mu = (1-z)p_i^\mu - k_{\perp}^\mu - \frac{k_{\perp}^2}{1-z} n^\mu - \frac{z}{1-\frac{1}{z}} (p \cdot n).$$

The invariant mass of the incoming parton is $p_i^2 = 2(p_k \cdot p_l) = -k_{\perp}^2/(z(1-z))$ which vanishes in the collinear limit, $k_{\perp} \to 0$. We have introduced the auxiliary momenta $p$ and $n$ such that $p^2 = n^2 = 0$ and such that $p$ and $n$ are perpendicular to $k_{\perp}$. In the case where a gluon splits $i$ into two collinear gluons $k$ and $l$, the collinear limit of the matrix element squared in a regularization scheme RS is given by

$$\lim_{p_k \parallel p_l} \Gamma_{rs}^{RS}(a_1, \ldots, g_k, g_l, \ldots, a_{n+1}) = \frac{4\pi\alpha_s}{p_k \cdot p_l} \frac{\delta_{ab}}{\omega(a_1) \omega(a_2) 4p_1 \cdot p_2} \mathcal{M}_a^\mu \mathcal{P}_r^{RS,\mu\nu}\mathcal{M}_b^\nu.$$

Here, $\mathcal{M}_a^\mu$ denotes the matrix element of the parent process with stripped of polarization vector $\varepsilon^\mu(p_i)$ of gluon $i$ with color $a$. By averaging over the transverse momentum of the collinear gluons we can extract the usual Altarelli-Parisi splitting functions in scheme RS. Carrying out the computation for different splittings one finds the following splitting functions in CDR:

$$P_{g/g}^<(z, \epsilon) = 2C_A \left( \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right),$$

$$P_{q/g}^<(z, \epsilon) = T_F \left( 1 - \frac{2z(1-z)}{1-\epsilon} \right),$$

$$P_{q/g}^<(z, \epsilon) = C_F \left( \frac{1 + (1-z)^2}{z} - \epsilon z \right).$$

The splitting functions in HV are equal to the one in CDR. All remaining splitting functions are obtained through the relation $P_{S_i(a,b)/a}^<(z, \epsilon) = P_{b/a}^<(1-z, \epsilon)$. $S(c, d)$ is the flavor of the
The complete splitting distribution, also valid for $\epsilon = 0$, are given by

$$
S(g,g) = g, \quad S(g,q_i) = S(q_i,g) = q_i, \quad S(g,\bar{q}_i) = S(\bar{q}_i,g) = \bar{q}_i,
$$

$$
S(q_i,q_j) = S(\bar{q}_i,\bar{q}_j) = \{\}, \quad S(q_i,\bar{q}_i) = S(\bar{q}_i,q_i) = g, \quad S(q_i,q_j) = S(\bar{q}_i,\bar{q}_j) = \{\}, \quad (5.82)
$$

$S(q_i,\bar{q}_j) = S(\bar{q}_i,q_j) = \{\}$.

By definition $P^{<}_{S(c,d)/a}$ vanishes if $S(c,d) = \{\}$. The splitting functions obey the following crossing relation

$$
P^{<}_{b/a}(z) = (-1)^{\sigma(f)+1} \frac{\omega(b)}{\omega(a)} z P^{<}_{a/b} \left( \frac{1}{z} \right), \quad (5.83)
$$

where $\sigma(f)$ is the number of crossed fermions. The averaging factors of spins and colors are given by

$$
\omega(g) = 2(1 - \epsilon)(N_c^2 - 1) \quad \text{and} \quad \omega(q) = 2N_c \quad (5.84)
$$

in CDR scheme while $\omega(g)$ in the HV scheme differs from the one in CDR by setting $\epsilon = 0$. For later use we introduce the following notation, where $P^{<}_{a/b}(z,0)$ is the derivative of $P^{<}_{a/b}(z,\epsilon)$ with respect to $\epsilon$, evaluated at $\epsilon = 0$:

$$
P^{<}_{a/b}(z,\epsilon) = P^{<}_{a/b}(z,0) + \epsilon P^{<\prime}_{a/b}(z,0) + O(\epsilon^2). \quad (5.85)
$$

The complete splitting distribution, also valid for $z = 1$, is defined by

$$
P_{a/b}(z,\epsilon) = \frac{(1 - z)P^{<}_{a/b}(z,\epsilon)}{(1 - z)_+} + \delta_{ab}\gamma(b)\delta(1 - z) \quad (5.86)
$$

where $\gamma(b)$ is determined through momentum sum rules,

$$
\int_0^1 dz \left( P_{g/q}(z,\epsilon) + 2n_f P_{g/q}(z,\epsilon) \right) = 0, \quad (5.87)
$$

$$
\int_0^1 dz \left( P_{q/g}(z,\epsilon) + P_{q/g}(z,\epsilon) \right) = 0. \quad (5.88)
$$

The resulting $\gamma$ factors in CDR and HV are then found to be

$$
\gamma(g) = \frac{\beta_0}{2} + \epsilon \frac{3 T_F n_f}{3(1 - \epsilon)} \quad \text{and} \quad \gamma(q) = \frac{3}{2} C_F + \epsilon \frac{C_F}{2} \quad (5.89)
$$

with

$$
\beta_0 = (11N_c - 4T_F n_f)/3. \quad (5.90)
$$

In our calculations we encounter only initial state collinear singularities. Following [57], the limit where the $j$-th parton becomes collinear to the initial parton 1 can be written as

$$
\lim_{p_j \parallel p_1} \Upsilon_n \left( \{a_i\}_{i=1}^{n+2}, \{p_i\}_{i=1}^{n+2} \right) = \frac{4\pi\alpha_s \mu^{2\epsilon}}{p_1 \cdot p_j} \Delta \left( \{a_i\}_{i=1}^{n+2}, \{p_i\}_{i=1}^{n+2} \right)
$$

$$
+ \frac{4\pi\alpha_s \mu^{2\epsilon}}{p_1 \cdot p_j} P^{<}_{S(a_1,\bar{a}_j)/a_1}(z,\epsilon) \Upsilon_{n-1} \left( S(a_1,\bar{a}_j), a_2, \{a_i\}_{i=1}^{n+2,\bar{j}}, zp_1, p_2, \{p_i\}_{i=3}^{n+2,\bar{j}} \right). \quad (5.91)
$$
5.3. COLLINEAR AND SOFT LIMITS

The $\Delta$ term in (5.91), which we have discarded so far, vanishes upon integration over the azimuthal angle but is useful in constructing local counter-terms in order to improve convergence. Its explicit form in terms of helicity amplitudes is found in Appendix B of [57]. In the case of real radiation processes we are considering, it is identically zero, however. The reason is the following. $\Delta$ is proportional to a particular helicity sum of the Born amplitude,

$$\Delta \propto \sum_{\lambda} M_{\pm,\lambda}^{(0), (gg \rightarrow h)} \left( M_{-,\lambda}^{(0), (gg \rightarrow h)} \right)^* \quad (5.92)$$

but the only non vanishing helicity amplitudes of the Born matrix element are $M_{+,\pm}^{(0), (gg \rightarrow h)}$ and $M_{-,\pm}^{(0), (gg \rightarrow h)}$. Therefore the helicity sum vanishes.

Soft limits of real radiation matrix elements are related to insertions of soft gluon lines between external parton lines of the Born amplitude. If a soft gluon lines connects two hard lines $i$ and $j$ with color $c_i$ and $c_j$, respectively, the amplitude after insertion of the gluon reads

$$M_{c_1 c_2 \ldots c_{n+2}} \rightarrow \sum_{a, c'_i, c'_j} T^a_{c_i c'_i} T^a_{c_j c'_j} M_{c_1 \ldots c'_i \ldots c'_j \ldots c_{n+2}}. \quad (5.93)$$

Again, in our case this simplifies to the Born amplitude up to some normalization since we are dealing in $gg \rightarrow h$ with a very trivial color structure. The observation above applies to a virtual diagram where a soft gluon connects two external lines of the amplitude. But the same structure appears in the squared real radiation diagrams and the universal soft limit for a matrix element squared where the $j$-th parton of type $a_j$ becomes soft is

$$\lim_{p_j \rightarrow 0} \Upsilon \left( \{a_i\}_{i=1}^{n+2}; \{p_i\}_{i=1}^{n+2} \right) = \delta_{gaj} \frac{\alpha_s \mu^2}{2\pi} \sum_{mn \leq m} \frac{p_n \cdot p_m}{p_n \cdot p_j p_m \cdot p_j} \Upsilon_{mn} \left( \{a_i\}_{i=1}^{n+2}; \{p_i\}_{i=1}^{n+2} \right). \quad (5.94)$$

$\Upsilon_{mn}$ on the right hand side is usually called color linked Born amplitude squared. In our case we have only one such amplitude which is proportional to the Born amplitude,

$$\Upsilon^{(0)}_{12} (g, g, h; p_1, p_2, p_h) = 16\pi^2 N_c \Upsilon^{(0)} (g, g, h; p_1, p_2, p_h). \quad (5.95)$$
Chapter 6

Constructing a Monte Carlo program

For phenomenological studies to assess the production rate of Higgs bosons in hadron colliders, we need to combine the different pieces computed in previous chapters into a numerical computer code. Since these individual pieces are still singular, we have to define a framework to construct finite integrands which yield the physical cross-section, defined in section 6.1, upon integration over phase-space. This is the subject of section 6.2. We will conclude the part on calculational details in section 6.3 with discussing some issues regarding the actual implementation into a Monte-Carlo program. This code will also be used to correct existing programs at NNLO accuracy for finite mass effects.

6.1 Physical cross-section

The hard scattering cross-section between hadrons $H_1$ and $H_2$ is given by an integral of the partonic cross-section multiplied by parton densities:

$$d\sigma_{\text{phys}}^{(H_1H_2)}(P_1, P_2) = \sum_{a_1a_2} \int dx_1 dx_2 f^{(H_1)}_{a_1}(x_1)f^{(H_2)}_{a_2}(x_2)d\sigma_{a_1a_2}(x_1 P_1, x_2 P_2). \tag{6.1}$$

The quantities in this expression are bare quantities in the sense that they still contain divergences related to initial state collinear singularities. $d\sigma$ is what is straightforwardly obtained in pQCD by computing renormalized matrix elements squared and integrating over phase-space. We work in the conventional dimensional regularization scheme (CDR). At $\mathcal{O}(\epsilon^0)$ the CDR and the HV scheme are equivalent, such that, in our implementation, we can directly use the real radiation matrix elements computed before. For practical purposes we prefer to express the physical cross-section in terms of finite quantities and introduce the subtracted partonic cross-sections, $d\tilde{\sigma}$, and finite parton number densities,
\[ \tilde{f}, \text{ related to the bare parton densities } f \text{ through} \]

\[ \tilde{f}_a^{(H)}(x) = \sum_b \int_0^1 dy dz f_b^{(H)}(y) \Gamma_{ab}(z) \delta(x - yz). \quad (6.2) \]

The functions \( \Gamma_{ab} \) are given in the \( \overline{\text{MS}} \) scheme up to NLO by

\[ \Gamma_{ab}(x) = \delta_{ab} \delta(1 - x) - \frac{\alpha_s}{2\pi} \left( \frac{4\pi}{\epsilon} \right) \frac{1}{\epsilon} \left( \frac{\mu_F^2}{\mu_R^2} \right)^{-\epsilon} P_{a/b}(x, 0) + O(\alpha_s^2). \quad (6.3) \]

It is important to realize, that the splitting functions appearing on the right hand side are the CDR splitting functions without the \( \epsilon \) dependent piece, as denoted by writing \( P_{a/b}(x, 0) \). Keep in mind that the parton distribution functions must be evaluated at factorization scale \( \mu_F \) which separates soft and hard physics. Loosely speaking, long ranged physics at energy scales \( Q \ll \mu_F \) is described by PDFs while short distance processes at larger energy scales, \( Q \gg \mu_F \), are described by the partonic hard scattering cross-section \( d\sigma_{ab} \). This is called factorization. \( \Gamma_{ab}(x) \) describes either the case where parton \( b \) does not split and therefore parton \( a = b \) carries the whole momentum fraction, \( x = 1 \), of the parent parton or the case where parton \( b \) splits into a parton \( a \) carrying momentum fraction \( x < 1 \) with a probability proportional to the splitting kernel \( P_{a/b}(x, 0) \).

The physical cross-section is now written in terms of finite quantities as

\[ d\sigma_{\text{phys}}^{(H_1H_2)}(P_1, P_2) = \sum_{a_1a_2} \int d\sigma_{a_1a_2}^{(H_1)}(x_1) \tilde{f}_{a_2}^{(H_2)}(x_2) d\tilde{\sigma}_{a_1a_2}(x_1 P_1, x_2 P_2). \quad (6.4) \]

Here, all quantities are finite on their own and (6.4) is suited for numerical evaluation. Note that we suppress the dependence on the renormalization scale \( \mu_R \) and factorization scale \( \mu_F \) and assume all quantities being renormalized.

We expand the partonic cross-sections in \( \alpha_s \) and write

\[ d\sigma_{ab} = d\sigma_{ab}^{(0)} + d\sigma_{ab}^{(1)} \quad \text{and} \quad d\tilde{\sigma}_{ab} = d\tilde{\sigma}_{ab}^{(0)} + d\tilde{\sigma}_{ab}^{(1)}. \quad (6.5) \]

Starting from (6.4), applying (6.2) and plugging in (6.3) we find by comparing to (6.1)

\[ d\sigma_{ab}^{(0)}(p_1, p_2) = d\tilde{\sigma}_{ab}^{(0)}(p_1, p_2), \quad (6.6) \]

\[ d\sigma_{ab}^{(1)}(p_1, p_2) = d\tilde{\sigma}_{ab}^{(1)}(p_1, p_2) - \frac{\alpha_s}{2\pi} \left( \frac{4\pi}{\epsilon} \right) \frac{1}{\epsilon} \int d\delta \left[ \sum_c P_{c/a}(z, 0) d\tilde{\sigma}_{ac}^{(0)}(zp_1, p_2) + \sum_d P_{d/b}(z, 0) d\tilde{\sigma}_{ad}^{(0)}(p_1, zp_2) \right]. \quad (6.7) \]

This can be viewed as the defining relation for subtracted, \( d\tilde{\sigma}_{ab} \), and unsubtracted, \( d\sigma_{ab} \), partonic cross-sections. The second line in (6.7) defines the collinear counter terms in the \( \overline{\text{MS}} \) scheme.
The cross-section for $ab \to nX$ at next-to-leading order is the combination of real and virtual contributions,

$$
\frac{d\sigma^{(1)}}{d\Sigma}(p_1, p_2) = \Upsilon_{\text{virt}, ab} \, d\Pi_n + \Upsilon_{\text{real}, ab} \, d\Pi_{n+1}
$$

(6.8)

where both contributions are divergent on their own. $\Upsilon_{\text{virt}}$ contains explicitly poles in $\epsilon$ originating in loop integration, while $\Upsilon_{\text{real}}$ is singular in certain phase space regions leading to $\epsilon$ poles upon phase space integration in $d = 4 - 2\epsilon$ dimensions. The combination still contains poles in $\epsilon$ which are canceled by the collinear subtraction terms in (6.7). In aiming for a numerical evaluation, we would like to deal with virtual contributions, where we have decomposed the integration over the body final state, and

$$
\int d\sigma = \int \left[ d\sigma_{\text{real}} - d\sigma_{\text{sing}} \right] + \int \left[ d\sigma_{\text{LO}} + \left( d\sigma_{\text{virtual}} + \int_1^{\prime \prime} d\sigma_{\text{sing}} + \int_1^{\prime \prime} d\sigma_{\text{coll}} \right) \right]
$$

(6.9)

where we have decomposed the integration over the $n + 1$ body phase space, $\int_1^{\prime \prime} = \int_1^{\prime} + \int_1^{''}$, and added the collinear counter term $\int_1^{''} d\sigma_{\text{coll}}$. By properly choosing $d\sigma_{\text{sing}}$ such that it cancels the divergences in the real radiation but being simple enough to compute analytically $\int_1^{'} d\sigma_{\text{sing}}$, the integrands of the $n + 1$ and $n$ phase space are individually suited for numerical evaluation.

Often we are not only interested in the inclusive cross-section but also in more exclusive observables such as rapidity distributions or distributions of the transverse momentum or we want to apply cuts on the external particles as it is done in real experiments. Such observables are denoted $\mathcal{O}_{n+1}(p_a, p_b; p_1, \ldots, p_n, p_{n+1})$, if they are defined on the $(n + 1)$-body final state, and $\mathcal{O}_n(p_a, p_b; p_1, \ldots, p_n)$, if they are defined on the $n$-body final state. Here, $p_a$ and $p_b$ denote the initial state momenta. Consistent theoretical predictions can only be obtained if we consider infrared safe observables, i.e. $\mathcal{O}$ must exhibit the following limiting behavior,

$$
\lim_{p_i \parallel p_j} \mathcal{O}_{n+1}(p_a, p_b; p_1, \ldots, p_{n+1}) = \mathcal{O}_n(p_a, p_b; p_1, \ldots, p_{[i]}, \ldots, p_{[j]}, \ldots, p_{n+1}),
$$

(6.10)

$$
\lim_{p_i \parallel p_a} \mathcal{O}_{n+1}(p_a, p_b; p_1, \ldots, p_n, p_{n+1}) = \mathcal{O}_n(p_a, p_b; \tilde{p}_1, \ldots, p_{[i]}, \ldots, \tilde{p}_{n+1}),
$$

(6.11)

$$
\lim_{p_i \parallel p_b} \mathcal{O}_{n+1}(p_a, p_b; p_1, \ldots, p_n, p_{n+1}) = \mathcal{O}_n(p_a, \tilde{p}_b; \tilde{p}_1, \ldots, p_{[i]}, \ldots, \tilde{p}_{n+1}),
$$

(6.12)

$$
\lim_{p_i \rightarrow 0} \mathcal{O}_{n+1}(p_a, p_b; p_1, \ldots, p_n, p_{n+1}) = \mathcal{O}_n(p_a, p_b; p_1, \ldots, p_{[i]}, \ldots, p_{n+1}),
$$

(6.13)

where $p_{[i]}$ means that momentum $p_i$ is removed from the set of final state momenta, $p_{ij} = p_i + p_j$ and $\tilde{p}$ signals that in case of an initial state collinearity momentum conservation must be restored by proper rescaling of the remaining momenta. The expectation value of
observable $\mathcal{O}$ at NLO accuracy is given by

\[
\langle \mathcal{O} \rangle = \int dx_1 dx_2 \sum_{ab} \left[ (\Upsilon_{\text{born},ab} + \Upsilon_{\text{virt},ab}) \mathcal{L}_{ab} \mathcal{O}_n d\Pi_n + \Upsilon_{\text{coll},ab} \mathcal{L}_{ab} \mathcal{O}'_n d\Pi'_n + \Upsilon_{\text{real},ab} \mathcal{L}_{ab} \mathcal{O}_{n+1} d\Pi_{n+1} \right].
\]  

(6.14)

The primed quantities remind us that the collinear counter-term involves a folding over variable $z$ and a corresponding rescaling of momenta, see (6.7). $\mathcal{L}_{ab}$ is the luminosity function,

\[
\mathcal{L}_{ab}(x_1, x_2) = f_a^{(H_1)}(x_1) f_b^{(H_2)}(x_2).
\]  

(6.15)

The total cross-section is obtained by setting $\mathcal{O} \equiv 1$. A differential distribution in quantity $X = X(p_a, p_b; p_1, \ldots, p_n)$, e.g. rapidity or transverse momentum of the Higgs, is obtained by defining

\[
\mathcal{O}_n(p_a, p_b; p_1, \ldots, p_n) = \delta(X(p_a, p_b; p_1, \ldots, p_n) - x).
\]  

(6.16)

In practical situations, like in experiments or in numerical evaluation using Monte-Carlo methods, it is more feasible to approximate the differential distribution by introducing finite bins. If we consider bins with lower and upper boundaries $x_{\text{lower}}^i$ and $x_{\text{upper}}^i$, respectively, the observable for bin $i$ is expressed through $\Theta$ functions,

\[
\mathcal{O}'_n(p_a, p_b; p_1, \ldots, p_n) = \Theta \left(X(p_a, p_b; p_1, \ldots, p_n) - x_{\text{lower}}^i\right) \cdot \Theta \left(x_{\text{upper}}^i - X(p_a, p_b; p_1, \ldots, p_n)\right).
\]  

(6.17)

For simplicity we will assume in the following that there is only one region of phase space where a parton can become soft or collinear. Then the subtraction method is introduced by writing the $n + 1$-body phase space as $d\Pi_{n+1} = d\Pi_n d\Phi_{\text{rad}}$, where $d\Phi_{\text{rad}}$ is the phase space of the potentially singular parton, and (6.14) is rewritten as

\[
\langle \mathcal{O} \rangle = \int dx_1 dx_2 \sum_{ab} \left[ (\Upsilon_{\text{born},ab} + \Upsilon_{\text{virt},ab}) \mathcal{L}_{ab} \mathcal{O}_n d\Pi_n + \Upsilon_{\text{coll},ab} \mathcal{L}_{ab} \mathcal{O}'_n d\Pi'_n + \Upsilon_{\text{real},ab} \mathcal{L}_{ab} \mathcal{O}_{n+1} d\Pi_{n+1} \right].
\]  

(6.18)

In setting $\mathcal{O} = 1$ we recognize the structure schematically given in (6.9).

In the following we will focus on the simple case of Higgs production through gluon fusion at next-to-leading order and work out the subtraction terms $d\sigma_{\text{sing}}$, or equivalently $\Upsilon_{\text{sing}}$, applying the FKS [57] subtraction method.
6.2 FKS subtraction for Higgs production

In an arbitrary process we could have several partons in the final state which create soft and/or collinear singularities. In the FKS subtraction method the cross-section is split up in different regions such that only one type of singularity can occur in each region,

$$d\sigma = \sum_\alpha d\sigma^\alpha .$$  \hfill (6.19)

Such a decomposition is obtained by first separating different flavor configurations and then defining a measure of collinearity and softness for all partons in the process. Using this measure and introducing suitable weights (e.g. $\Theta$ functions or continuous generalizations) we can separate out the different regions. Next, a suitable parametrization of the phase space for the emitted parton, which potentially becomes soft or collinear, is chosen for each region. The integral over this radiation phase space is then regularized in terms of $\delta$- and Plus-distributions.

In this thesis, we only have to deal with rather trivial flavor configurations exhibiting no final-state singularities. Since there is only one parton in the final state in addition to the Higgs, the decomposition into regions is trivial. Let us immediately start with introducing the radiation phase space parametrization for the Bremsstrahlung process $a_1a_2 \rightarrow a_3h$ with cross-section

$$d\sigma_{\text{real},a_1a_2} = T_{\text{real},a_1a_2} d\Pi_2 .$$  \hfill (6.20)

The starting point for identifying singular contributions from soft and collinear regions in the phase space is to choose a specific reference frame. Since the phase space integral is Lorentz invariant the result will not depend on the actual choice of the parametrization. Thus, we choose a particularly convenient frame, the center-of-mass system, and write

$$p_1 = \frac{\sqrt{s}}{2} (1, 0, 0, 1) , \quad p_2 = \frac{\sqrt{s}}{2} (1, 0, 0, -1) , \quad p_3 = \frac{\sqrt{s}}{2} \xi (1, 0, \sin \theta, \cos \theta) , \quad p_h = \frac{\sqrt{s}}{2} (2 - \xi, 0, -\xi \sin \theta, -\xi \cos \theta) .$$  \hfill (6.21)

Note that we suppress the azimuthal dependence on $\phi$ since we can always choose the reference frame such that the above parametrization holds. The Mandelstam variables of the partonic process are as usual

$$s_{12} = s = (p_1 + p_2)^2 , \quad s_{13} = (p_1 - p_3)^2 , \quad s_{23} = (p_2 - p_3)^2 .$$  \hfill (6.22)

We write $y = \cos \theta$ and remark that soft limits correspond to $\xi \rightarrow 0$ while the two collinear limiting regions are defined by $(1 - y^2) \rightarrow 0$. We will often use the variable $z$ instead of $\xi$, related through $\xi = 1 - z$. Before proceeding in deriving the subtracted real radiation
cross-sections, we define the underlying Born event in the collinear limit. In the partonic center-of-mass system we have defined parametrization (6.21). In the limit $y \to 1$, $p_3$ becomes collinear to $p_1$. The underlying $2 \to 1$ configuration is then defined by replacing $p_1$ with $p_1 + p_3$, thus we define the underlying Born event for the collinear plus configuration

$$p_1^\square = \frac{z\sqrt{s}}{2}(1, 0, 0, 1), \quad p_2^\square = \frac{\sqrt{s}}{2}(1, 0, 0, -1),$$

and

$$p_h^\square = \frac{\sqrt{s}}{2}(1 + z, 0, 0, -(1 - z)).$$

Similarly we define an underlying event for $y \to -1$,

$$p_1^\square = \frac{\sqrt{s}}{2}(1, 0, 0, 1), \quad p_2^\square = \frac{z\sqrt{s}}{2}(1, 0, 0, -1),$$

$$p_h^\square = \frac{\sqrt{s}}{2}(1 + z, 0, 0, (1 - z)).$$

while the common Born configuration is trivially given by

$$\tilde{p}_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1), \quad \tilde{p}_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -1),$$

$$\tilde{p}_h = \frac{\sqrt{s}}{2}(1, 0, 0, 0).$$

The momenta in the laboratory frame are obtained by boosting into the rest frame of the colliding hadrons. We write $\{p\}$ for the boosted momenta in the real configuration (6.21), $\{\tilde{p}^\square\}$ for the collinear plus configuration (6.23) and analogously for collinear minus and Born configurations. Of course, these boosted momenta are only defined, if we specify $p_1$ and $p_2$ in the laboratory frame through variables $x_1$ and $x_2$.

Since the azimuthal dependence is trivial, the phase-space measure can be decomposed as follows

$$d\Pi_2 = d\Pi_h d\Phi(p_3)$$

(6.26)

where $d\Pi_h$ is the $2 \to 1$ phase space of the Higgs boson,

$$d\Pi_h((1 - \xi) s) = \frac{d^{d-1} p_h}{(2\pi)^{d-1} 2 p_h^\square} (2\pi)^d \delta(d) (p_1 + p_2 - p_3 - p_h) = 2\pi \delta (s(1 - \xi) - m_h^2).$$

Even if we were working in the HV scheme, we would have to treat the radiated parton $d$ dimensional radiation phase space, $d\Phi(p_3)$, is given by

$$d\Phi(p_3) = \frac{d^{d-1} p_3}{(2\pi)^{d-1} 2 p_3^\square} = \left[ \left( \frac{s}{4} \right)^{1-\epsilon} \frac{\Omega^{1-2\epsilon}}{2(2\pi)^{d-1}} \right] [\xi^{1-2\epsilon}(1 - y^2)^{-2\epsilon} d\xi dy].$$

(6.28)

We rearrange some factors and write

$$d\sigma = \tilde{\Gamma}_{\text{real, }a_1a_2} d\Pi_h dw$$

(6.29)
where
\[ \tilde{T}_{\text{real,}a_1a_2} = \left( \frac{2}{\pi} \right)^{1-\epsilon} \xi^2 (1 - y^2) T_{\text{real,}a_1a_2}, \]
\[ d\Pi_h = \frac{\Omega^{1-2\epsilon}}{2(2\pi)^{d-1}} d\Pi_h(\xi) = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{1}{8\pi^2} d\Pi_h ((1 - \xi)s), \]
\[ dw = \xi^{-1-2\epsilon} (1 - y^2)^{-1-\epsilon} d\xi dy. \]

In section 5.3.2 we have seen the structure of soft and collinear limits. Particularly, we note that collinear limits show up as $1/(1 \pm y)$ in matrix elements squared, while soft limits are characterized by $1/\xi^2$ factors. These singularities are regularized in (6.30). In turn the phase space measure (6.32) is singular. We regularize the integral in $\int \tilde{T} d\Pi dw$ applying the $+-$-prescription which we will introduce in the following paragraph.

The $+$-prescription. Consider a regular function $f(\xi)$ and the integral
\[ I = \int_0^b d\xi \frac{f(\xi)}{\xi^{1+\epsilon}}. \]
This integral is in general not well defined and exhibits a singularity for $\xi \to 0$ (integrable singularity if $\epsilon > 0$). Certainly it is not suited for numerical evaluation. We regularize the integral by adding and subtracting another term,
\[ I = \int_0^b d\xi \frac{f(0)}{\xi^{1+\epsilon}} + \int_0^b d\xi \frac{f(\xi) - f(0)}{\xi^{1+\epsilon}} \]
\[ = \frac{\epsilon}{\epsilon} f(0) + \int_0^b d\xi \frac{f(\xi) - f(0)}{\xi} \sum_{n=0}^\infty \frac{1}{n!} (\epsilon \ln \xi)^n. \]

The $+-$-prescription is defined for $b = 1$ and is given by the identity
\[ \frac{1}{\xi^{1+\epsilon}} \to \frac{\delta(\xi)}{\epsilon} + \sum_{n=0}^\infty \frac{1}{n!} (\epsilon \ln \xi)^n \left( \frac{1}{\xi} \right)_+ \]
in the sense of distributions, that is
\[ I = \int_0^1 d\xi \left[ \frac{\delta(\xi)}{\epsilon} + \sum_{n=0}^\infty \frac{1}{n!} (\epsilon \ln \xi)^n \left( \frac{1}{\xi} \right)_+ \right] f(\xi) \]
\[ = \frac{f(0)}{\epsilon} + \int_0^1 d\xi \sum_{n=0}^\infty \frac{1}{n!} (\epsilon \ln \xi)^n \left( \frac{f(\xi) - f(0)}{\xi} \right)_+ \]
(6.36)

When evaluating cross-sections we will encounter the case where $b \neq 1$ and we have to be careful in dealing with $+-$-prescriptions. We will need the following two identities,
\[ \int_0^b d\xi \left( \ln \xi \right)_+ f(\xi) = f(0) \frac{1}{2} \ln^2 b + \int_0^b d\xi \frac{f(\xi) - f(0)}{\xi}, \]
\[ \int_0^b d\xi \left( \frac{\ln \xi}{\xi} \right)_+ f(\xi) = f(0) \frac{1}{2} \ln^2 b + \int_0^b d\xi \frac{\ln \xi (f(\xi) - f(0))}{\xi}. \]
As an example we prove the first identity. We map the integration interval \([0, b]\) onto the interval \([0, 1]\) by substituting \(\xi = b\lambda\),

\[
\int_0^b d\xi \left( \frac{1}{\xi} \right) f(\xi) = \lim_{\epsilon \to 0} \int_0^b d\xi \left[ \xi^{-1+\epsilon} - \frac{1}{\epsilon} \delta(\xi) \right] f(\xi) \\
= \lim_{\epsilon \to 0} \left\{ \int_0^1 d\lambda b \left[ b^{-1+\epsilon} \lambda^{-1+\epsilon} f(\xi(\lambda)) - \frac{1}{\epsilon} f(0) \right] \right\}.
\]

The right-hand side can now be expanded in \(\epsilon\) applying the usual + prescription,

\[
\int_0^b d\xi \left( \frac{1}{\xi} \right) f(\xi) = \lim_{\epsilon \to 0} \left\{ \int_0^1 d\lambda b \left[ \frac{1}{b} \left( \frac{\delta(\lambda)}{\epsilon} + \delta(\lambda) \ln b + \left( \frac{1}{\lambda} \right)_+ \right) \right] f(\xi(\lambda)) - \frac{1}{\epsilon} f(0) \right\} \\
= \int_0^1 d\lambda b \left[ \frac{1}{b} \left( \delta(\lambda) f(\xi(\lambda)) \ln b + \left( \frac{f(\xi(\lambda)) - f(0)}{\lambda} \right)_+ \right) \right].
\]

Evaluating the first term using the \(\delta\)-function and reparametrizing the second term in \(\xi\) yields

\[
\int_0^b d\xi \left( \frac{1}{\xi} \right) f(\xi) = f(0) \ln b + \int_0^b d\xi \frac{f(\xi) - f(0)}{\xi}.
\]

This concludes the proof. The second identity is obtained analogously starting from

\[
\int_0^b d\xi \left( \frac{\ln \xi}{\xi} \right) + f(\xi) = \lim_{\epsilon \to 0} \int_0^b d\xi \frac{1}{\epsilon} \left[ \xi^{-1+\epsilon} - \frac{1}{\epsilon} \delta(\xi) - \left( \frac{1}{\xi} \right)_+ \right] f(\xi).
\]

The radiation phase-space \(dw\) in (6.32) is now expressed in terms of + distributions:

\[
dw = \left[ -\frac{1}{2\epsilon} \delta(\xi) + \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!} \left( \frac{\ln^n \xi}{\xi} \right)_+ \right] \\
\times \left[ \frac{\Gamma(-\epsilon)\sqrt{\pi}}{\Gamma(\frac{1}{2} - \epsilon)} \delta(1 - y^2) + \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!} \left( \frac{\ln^n(1 - y^2)}{1 - y^2} \right)_+ \right].
\]

The coefficient of the \(\delta\)-function in the second square bracket is the value of the integral \(\int_{-1}^1 dy(1 - y^2)^{-1-\epsilon}\). Now we disentangle the limits \(y \to +1\) and \(y \to -1\), expand in \(\epsilon\) and find

\[
dw = \left[ -\frac{4\pi^{-\epsilon}}{2\epsilon} \left\{ \delta(1 - y) + \delta(1 + y) \right\} + \frac{1}{2} \left\{ \left( \frac{1}{1 - y} \right)_+ + \left( \frac{1}{1 + y} \right)_+ \right\} + \epsilon A_{\text{fin}} \right] dy \\
\times \left[ \frac{1}{2\epsilon} \delta(\xi) + \left( \frac{1}{\xi} \right)_+ - 2\epsilon \left( \frac{\ln \xi}{\xi} \right)_+ \right] d\xi \\
+ \mathcal{O}(\epsilon).
\]
Up to $O(\epsilon)$, $A_{\text{fin}}$ only appears in conjunction with $\delta(\xi)$, corresponding to the soft limit. In that limit however, the matrix element is independent on $y$ and therefore only the integrated value of $A_{\text{fin}}$ is required. Since

$$
\int_{-1}^{1} (1 - y^2)^{-1-\epsilon} dy = \frac{\Gamma(-\epsilon)\sqrt{\pi}}{\Gamma(\frac{1}{2} - \epsilon)} = -\frac{4}{\epsilon} + \frac{1}{6} \epsilon \pi^2 + O(\epsilon^2)
$$

we find that

$$
\int_{-1}^{1} A_{\text{fin}} dy = \frac{1}{6} \pi^2.
$$

Terms, containing both, $\delta(\xi)$ and $\delta(1 \pm y)$, are called soft-collinear; terms containing only one or the other obviously are termed soft or collinear, respectively. The universal form of soft and collinear limits allows then to identify singularities which eventually cancel against $\epsilon$ poles in the virtual contribution. For future reference we recall the soft limit

$$
\delta(\xi) \tilde{\Upsilon}_{\text{real,}a_1a_2} = \delta_{a_3} \alpha_s \frac{\mu^2}{2\pi} \frac{p_1 \cdot p_2}{(p_1 \cdot p_3)(p_2 \cdot p_3)} 16\pi^2 N_c \Upsilon_{\text{born,}a_1a_2} \left(\{\bar{p}\}\right)
$$

or in the notation of (6.30),

$$
\delta(\xi) \tilde{\Upsilon}_{\text{real,}a_1a_2} = \delta_{a_3} \alpha_s \frac{\mu^2}{2\pi} \left(\frac{\mu}{s}\right) 4\epsilon 32 \pi^2 N_c \Upsilon_{\text{born,}a_1a_2} \left(\{\bar{p}\}\right).
$$

It turns out to be convenient to define

$$
p^{S}_{a_1a_2}(z, \epsilon) = (1 - z) P^{S}_{a_1a_2}(z, \epsilon)
$$

for writing down the collinear limits:

$$
\delta(1 - y) \Upsilon_{\text{real,}a_1a_2} = \frac{8\pi^2}{z(p_1 \cdot p_3)} \frac{\alpha_s}{2\pi} \mu^2 P^{S}_{S(a_1,\bar{a}_3)/a_1}(z, \epsilon) \Upsilon_{\text{born,}S(a_1,\bar{a}_3)a_2} \left(\{\bar{p}\}\right),
$$

$$
\Rightarrow \delta(1 - y) \tilde{\Upsilon}_{\text{real,}a_1a_2} = \frac{\alpha_s}{2\pi} 16\pi^2 4\epsilon \left(\frac{\mu^2}{s}\right) P^{S}_{S(a_1,\bar{a}_3)/a_1}(z, \epsilon) \Upsilon_{\text{born,}S(a_1,\bar{a}_3)a_2} \left(\{\bar{p}\}\right),
$$

$$
\delta(1 + y) \Upsilon_{\text{real,}a_1a_2} = \frac{8\pi^2}{z(p_2 \cdot p_3)} \frac{\alpha_s}{2\pi} \mu^2 P^{S}_{S(a_2,\bar{a}_3)/a_2}(z, \epsilon) \Upsilon_{\text{born,}S(a_1,\bar{a}_3)a_2} \left(\{\bar{p}\}\right),
$$

$$
\Rightarrow \delta(1 + y) \tilde{\Upsilon}_{\text{real,}a_1a_2} = \frac{\alpha_s}{2\pi} 16\pi^2 4\epsilon \left(\frac{\mu^2}{s}\right) P^{S}_{S(a_2,\bar{a}_3)/a_2}(z, \epsilon) \Upsilon_{\text{born,}S(a_1,\bar{a}_3)a_2} \left(\{\bar{p}\}\right).
$$

### 6.2.1 Partonic real radiation contribution

Our goal is to obtain a completely exclusive result and therefore we explicitly keep trace of the observable function $O_n$ and introduce the short-hand notation

$$
\left[\Upsilon_{ab} O_2\right] \left(\{p\}\right) = \Upsilon_{ab} \left(\{p\}\right) O_2 \left(\{p\}\right),
$$

$$
\left[\Upsilon_{0} O_1\right] \left(\{\bar{p}\}\right) = \Upsilon_{gg} \left(\{\bar{p}\}\right) O \left(\{\bar{p}\}\right).
$$
and similarly for collinear configurations. Working out all terms in \( O(\epsilon) \) and resorting to the limits (6.51) and (6.53) yields the following expression for the partonic cross-section

\[
\begin{align*}
\frac{d\sigma_{\text{real},gg}}{dy} &= \frac{\alpha_s}{2\pi} \left(\frac{\mu^2}{s}\right)^\epsilon \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left\{ 2N_c \left(\frac{1}{\epsilon^2} - \frac{\pi^2}{6} \right) \left[ \Sigma_0 (\{p\}) \right] \right. \\
&\quad - \left[ \frac{1}{\epsilon} \left(\frac{1}{1-z}\right) + \frac{2}{\epsilon} \left(\frac{\ln(1-z)}{1-z}\right) \right] p_{g/g}^\epsilon(z,\epsilon) \left[ \Gamma_0 O_1 \right] (\{\tilde{p}\}) \right\} dy \\
&\quad - \left[ \frac{1}{\epsilon} \left(\frac{1}{1-z}\right) + \frac{2}{\epsilon} \left(\frac{\ln(1-z)}{1-z}\right) \right] p_{g/g}^\epsilon(z,\epsilon) \left[ \Gamma_0 O_1 \right] (\{\tilde{p}\}) dy \\
&\quad + \left[ \frac{1}{\epsilon} \left(\frac{1}{1-z}\right) + \frac{2}{\epsilon} \left(\frac{\ln(1-z)}{1-z}\right) \right] p_{g/g}^\epsilon(z,\epsilon) \left[ \Gamma_0 O_2 \right] (\{p\}) dy \\
&\quad + \left[ \frac{1}{\epsilon} \left(\frac{1}{1-z}\right) + \frac{2}{\epsilon} \left(\frac{\ln(1-z)}{1-z}\right) \right] p_{g/g}^\epsilon(z,\epsilon) \left[ \Gamma_0 O_2 \right] (\{p\}) dy.
\end{align*}
\]

It is implicitly understood, that all momenta in the above expression are functions of integration variables \( z \) and \( y \) for given \( p_1 \) and \( p_2 \) and \( s \) is the partonic center of mass energy squared, \( s = (p_1 + p_2)^2 \). In order to make the cancellation of \( \epsilon \) poles explicit, it is convenient to rewrite the second two lines originating in the collinear contributions using

\[
P_{g/g}^\epsilon(z,\epsilon) = P_{g/g}^\epsilon(z,0)
\]

and expressing the splitting function for \( z < 1 \) in terms of the full splitting function,

\[
\left(\frac{1}{1-z}\right) p_{g/g}^\epsilon(z,0) = \left(\frac{1}{1-z}\right) (1-z) P_{g/g}^\epsilon(z,0) = P_{g/g}^\epsilon(z,0) - \beta_0 \delta(1-z).
\]

For the collinear-plus term this results in

\[
\begin{align*}
- \frac{\alpha_s}{2\pi} \left(\frac{\mu^2}{s}\right)^\epsilon &\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left\{ \left[ \frac{1}{\epsilon} \left(\frac{1}{1-z}\right) + \frac{2}{\epsilon} \left(\frac{\ln(1-z)}{1-z}\right) \right] \left[ \Gamma_0 O_1 \right] (\{\tilde{p}\}) \right\} \\
&\quad + \left[ \frac{1}{\epsilon} \left(\frac{1}{1-z}\right) + \frac{2}{\epsilon} \left(\frac{\ln(1-z)}{1-z}\right) \right] \left[ \Gamma_0 O_2 \right] (\{p\}) + \mathcal{O}(\epsilon)
\end{align*}
\]

and an analogous expression for the collinear-minus term. In this form the cancellation of the first two terms on the right-hand side of this equation is made apparent when combined with the virtual contribution and the \( \overline{\text{MS}} \) collinear counter-term, respectively.

Another contribution is the one with a quark and a gluon in the initial state. This process exhibits no soft singularities but a collinear singularity, if the outgoing (anti-)quark becomes collinear with the incoming (anti-)quark. We have to distinguish the two cases, where the initial state is \( qg \) and \( \bar{q}g \), and to take into account that the initial state (anti-)quark carries
either momentum $p_1$ or $p_2$. Crossing symmetry, (5.83), of the Altarelli Parisi splitting kernels implies the equality of splitting functions $P_{g/q}$ and $P_{q/g}$. Furthermore, $\Upsilon_{qg} = \Upsilon_{gq}$. Thus, there are two different cases remaining corresponding to which momentum the incoming quark is carrying. If the incoming quark carries momentum $p_1$, the partonic cross-section reads

$$d\sigma_{\text{real}, qg} = \frac{\alpha_s}{2\pi} \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \times \left\{ - \left( \frac{\mu^2}{\mu_F^2} \right)^\epsilon \frac{1}{\epsilon} P_{g/q}(z, 0) \right\} \cdot [\Upsilon_0 \mathcal{O}_1] \{ \{ \bar{p} \} \} \, d\Pi_h(z) \, dz$$

$$- \frac{1}{(1-z)^+} \left( \ln \frac{\mu^2}{s} \right) P_{g/q}(z, 0) \left[ \Upsilon_0 \mathcal{O}_1 \right] \{ \{ \bar{p} \} \} \, d\Pi_h(z) \, dz$$

$$+ 2 \left( \frac{\ln(1-z)}{1-z} \right) P_{g/q}(z, 0) \left[ \Upsilon_0 \mathcal{O}_1 \right] \{ \{ \bar{p} \} \} \, d\Pi_h(z) \, dz$$

$$+ \frac{1}{(1-z)^+} \left( \frac{1}{1-y} \right) \left( 1 - y \right) \frac{(1-z)^2(1-y)^2}{32\pi} \left[ \Upsilon_{qg} \mathcal{O}_2 \right] \{ \{ p \} \} \, d\Pi_h(z) \, dz \, dy \right\}$$

(6.60)

where we recognize in the first term on the right-hand side the form of the $\overline{\text{MS}}$ collinear counter-term for the $qg$ initial state. The partonic cross-section for the $qg$ initial state is obtained from this expression by replacing $\bar{p} \rightarrow p_1$ and $p_2 \rightarrow \bar{p}_2$ as well as $\Upsilon_{qg} \rightarrow \Upsilon_{gq}$.

The quark initiated real radiation process is finite and therefore requires no subtraction. Thus, the partonic cross-section is simply

$$d\sigma_{\text{real}, qg} = \frac{(1-z)s}{16\pi} \left[ \Upsilon_{qg} \mathcal{O}_2 \right] \{ \{ p \} \} \, d\Pi_h(z) \, dz \, dy \right\} \cdot$$

(6.61)

### 6.2.2 Hadronic cross-section

The hadronic or physical cross-section is obtained by multiplying the partonic cross-section with the luminosity function and integrating over $x_1$ and $x_2$, which specify also the boost from the partonic center-of-mass system into the laboratory frame. We parametrize the $x_1$ value through the rapidity variable $Y$ and introduce $\tau_h = m_h^2 / S$ with $S = (P_1 + P_2)^2$, the center-of-mass energy squared of the hadronic collision. We distinguish between born-like partonic momentum fractions $\bar{x}_1$ and $\bar{x}_2$ and real event-like momentum fractions $x_1$ and $x_2$. In terms of $Y$, integration variable $z$ and $\tau_h$ these are expressed through

$$\bar{x}_1 = \sqrt{\tau_h} e^Y,$$

$$x_1 = \sqrt{\tau_h / z} e^Y.$$

(6.62)

$$\bar{x}_2 = \tau_h / \bar{x}_1,$$

$$x_2 = \tau_h / x_1.$$

(6.63)

The momenta are parametrized as in (6.21), (6.23), (6.24) and (6.25).
We define the following functions describing Born-like contributions:

\[
\begin{align*}
\mathcal{B}(Y) &= \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{gg}(\bar{x}_1, \bar{x}_2) \mathcal{O}_1(\{\bar{p}\}), \\
\mathcal{V}(Y) &= \left[ \frac{\alpha_s}{2\pi} \left( \frac{2N_c}{3} \pi^2 - 2\beta_0 \ln \frac{s}{\mu_R^2} + 2\mathcal{F}^{\text{REN}}_g \right) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \\
&\hspace{1cm} + \frac{\alpha_s}{2\pi} \left( -3C_F \ln \frac{s}{\mu_R^2} + 2\mathcal{F}^{\text{REN}}_m \right) \mathcal{Y}^{(0),r}_{gg \to h}(\{\bar{p}\}) \\
&\hspace{1cm} + \mathcal{Y}^{(1),\text{fin}}_{gg \to h}(p_1, p_2, s) \mathcal{L}_{gg}(\bar{x}_1, \bar{x}_2) \mathcal{O}_1(\{\bar{p}\}) \right].
\end{align*}
\]

Terms, that result from subtraction and still involve some dependence on integration variable \( z \) but no dependence on \( y \), will get expressed through

\[
\mathcal{G}^{\oplus}_{gg}(Y, z) = \left[ \delta(1 - z) \left( \beta_0 \ln \frac{s}{\mu_F^2} + \ln(1 - \tau_h) \ln \frac{m_h^2}{\mu_F^2} + \ln^2(1 - \tau_h) \right) \\
- \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} - 2 \ln(1 - z) \right] p_{g/g}(1, 0, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{gg}(\bar{x}_1, \bar{x}_2) \mathcal{O}_1(\{\bar{p}\}) \\
+ \left[ \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} + 2 \ln(1 - z) \right] p_{g/g}(z, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{gg}(x_1, x_2) \mathcal{O}_1(\{\bar{p}\}^\oplus),
\]

\[
\mathcal{G}^{\ominus}_{gg}(Y, z) = \left[ \delta(1 - z) \left( \beta_0 \ln \frac{s}{\mu_F^2} + \ln(1 - \tau_h) \ln \frac{m_h^2}{\mu_F^2} + \ln^2(1 - \tau_h) \right) \\
- \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} - 2 \ln(1 - z) \right] p_{g/q}(1, 0, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{qq}(\bar{x}_1, \bar{x}_2) \mathcal{O}_1(\{\bar{p}\}) \\
+ \left[ \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} + 2 \ln(1 - z) \right] p_{g/q}(z, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{qq}(x_1, x_2) \mathcal{O}_1(\{\bar{p}\}^\ominus),
\]

\[
\mathcal{G}^{\equiv}_{gg}(Y, z) = \left[ \delta(1 - z) \left( \beta_0 \ln \frac{s}{\mu_F^2} + \ln(1 - \tau_h) \ln \frac{m_h^2}{\mu_F^2} + \ln^2(1 - \tau_h) \right) \\
- \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} - 2 \ln(1 - z) \right] p_{g/g}(1, 0, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{gg}(\bar{x}_1, \bar{x}_2) \mathcal{O}_1(\{\bar{p}\}) \\
+ \left[ \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} + 2 \ln(1 - z) \right] p_{g/q}(z, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{qq}(x_1, x_2) \mathcal{O}_1(\{\bar{p}\}^\equiv),
\]

\[
\mathcal{G}^{\equiv}_{qq}(Y, z) = 0,
\]

\[
\mathcal{G}^{\equiv}_{gg}(Y, z) = 0,
\]

\[
\mathcal{G}^{\equiv}_{qq}(Y, z) = \left[ \delta(1 - z) \left( \beta_0 \ln \frac{s}{\mu_F^2} + \ln(1 - \tau_h) \ln \frac{m_h^2}{\mu_F^2} + \ln^2(1 - \tau_h) \right) \\
- \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} - 2 \ln(1 - z) \right] p_{g/g}(1, 0, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{gg}(\bar{x}_1, \bar{x}_2) \mathcal{O}_1(\{\bar{p}\}) \\
+ \left[ \ln \frac{m_h^2}{\mu_F^2} \frac{1}{1 - z} + 2 \ln(1 - z) \right] p_{g/q}(z, 0) \mathcal{Y}^{(0)}_{gg \to h}(\{\bar{p}\}) \mathcal{L}_{qq}(x_1, x_2) \mathcal{O}_1(\{\bar{p}\}^\equiv).
\]
Ultimately, we introduce the following functions for writing out the remaining contributions due to real radiation:

\[
\mathcal{R}_g^\odot(Y, z, y) = \frac{1}{2} \frac{1}{21 - z} \frac{1}{1 - y} \left[ \frac{s(1 - z)^2(1 - y^2)}{(4\pi)^3} \right] Y_{gg \to gh}^\odot \{ \{ p \} \} O_2 \{ \{ p \} \} \tag{6.72}
\]

\[- \frac{\alpha_s}{2\pi} 2N_c \langle 0 \rangle T_{gg \to h}^\odot \left( \{ \bar{\tilde{p}} \}^\odot \right) O_1 \{ \{ \bar{\tilde{p}} \}^\odot \} \mathcal{L}_{gg}(x_1, x_2),
\]

\[
\mathcal{R}_g^\odot(Y, z, y) = \frac{1}{2} \frac{1}{21 - z} \frac{1}{1 + y} \left[ \frac{s(1 - z)^2(1 - y^2)}{(4\pi)^3} \right] Y_{gg \to gh}^\odot \{ \{ p \} \} O_2 \{ \{ p \} \} \tag{6.73}
\]

\[- \frac{\alpha_s}{2\pi} 2N_c \langle 0 \rangle T_{gg \to h}^\odot \left( \{ \bar{\tilde{p}} \}^\odot \right) O_1 \{ \{ \bar{\tilde{p}} \}^\odot \} \mathcal{L}_{gg}(x_1, x_2),
\]

\[
\mathcal{R}_g^\odot(Y, z, y) = \frac{1}{2} \frac{1}{21 - z} \frac{1}{1 - y} \left[ \frac{s(1 - z)^2(1 - y^2)}{(4\pi)^3} \right] Y_{gg \to qh}^\odot \{ \{ p \} \} O_2 \{ \{ p \} \} \mathcal{L}_{gg}(x_1, x_2),
\]

\[
\mathcal{R}_g^\odot(Y, z, y) = \frac{1}{2} \frac{1}{21 - z} \frac{1}{1 - y} \left[ \frac{s(1 - z)^2(1 - y^2)}{(4\pi)^3} \right] Y_{gg \to qh}^\odot \{ \{ p \} \} O_2 \{ \{ p \} \} \mathcal{L}_{gg}(x_1, x_2),
\]

\[
\mathcal{R}_g^\odot(Y, z, y) = \frac{1}{2} \frac{1}{21 - z} \frac{1}{1 + y} \left[ \frac{s(1 - z)^2(1 - y^2)}{(4\pi)^3} \right] Y_{gg \to qh}^\odot \{ \{ p \} \} O_2 \{ \{ p \} \} \mathcal{L}_{gg}(x_1, x_2),
\]

\[
\mathcal{R}_g^\odot(Y, z, y) = \frac{1}{2} \frac{1}{21 - z} \frac{1}{1 + y} \left[ \frac{s(1 - z)^2(1 - y^2)}{(4\pi)^3} \right] Y_{gg \to qh}^\odot \{ \{ p \} \} O_2 \{ \{ p \} \} \mathcal{L}_{gg}(x_1, x_2).\tag{6.78}
\]

Now, after combining \( d\sigma_{\text{real,gg}}, d\sigma_{\text{real,qq}}, d\sigma_{\text{real,qq}} \) with Born and virtual contributions, \( d\sigma_{\text{Born,gg}} \) and \( d\sigma_{\text{virt,gg}} \), the expectation value of the observable \( O \) is written as follows:

\[
\langle O \rangle = \frac{2\pi}{S} \int_{\ln \sqrt{\tau_h}}^{\ln \sqrt{\tau_h}} dY \left\{ B(p_1, p_2, s) + Y(p_1, p_2, s) \right\} \delta(s - m_h^2)
\]

\[
+ \frac{2\pi}{S} \int_{\tau_h}^{1} dz \int_{-\ln \sqrt{\tau_h}}^{\ln \sqrt{\tau_h}} dY \left\{ G_{gg}^{\odot}(Y, z) + G_{gg}^{\odot}(Y, z) + G_{qq}^{\odot}(Y, z) + G_{qq}^{\odot}(Y, z) + G_{qq}^{\odot}(Y, z) \right\}
\]

\[
+ \frac{2\pi}{S} \int_{\tau_h}^{1} dz \int_{-\ln \sqrt{\tau_h}}^{\ln \sqrt{\tau_h}} dY \int_{-1}^{+1} dy \left\{ R_{gq}(Y, z, y) + R_{gq}(Y, z, y) + R_{gq}(Y, z, y) + R_{gq}(Y, z, y) + R_{gq}(Y, z, y) \right\}. \tag{6.79}
\]
CHAPTER 6. CONSTRUCTING A MONTE CARLO PROGRAM

6.3 Implementation

In the last few chapters we have discussed all necessary ingredients in order to compute fully exclusive cross-sections at next-to-leading order for Higgs production via gluon fusion. It is time now to put them to work. We have written a fully differential Monte-Carlo code called \texttt{HPro}, whose main features are described in this chapter. \texttt{HPro} is implemented fully in FORTRAN 77.

6.3.1 Performance issues

Computing massive corrections is computationally much more costly compared to the calculation in the effective theory. Therefore we devoted some effort on performance issues. If we neglect the width of the Higgs, the $2 \rightarrow 1$ phase space restricts the partonic center of mass energy to $m_h$. Therefore we have to compute the Born and the virtual contribution only once in the first \texttt{VEGAS} iteration. Thus, these contributions are not relevant for performance reasons. The evaluation of the corresponding functions is described in B.2.3.

Much more crucial for performance reasons is the evaluation of real radiation matrix elements, which must be recomputed in every \texttt{VEGAS} call. Instead of a straightforward approach of evaluating the corresponding one-loop master integrals whenever they appear in the matrix elements, we have chosen to compute all of the scalar integrals once per iteration and store them in common blocks.

Another issue is the evaluation of dilogarithms which are the only functions, apart from common logarithms, that appear in the one-loop scalar integrals at finite order. The standard representation of the dilogarithm as a series of convergence radius 1 is

$$\text{Li}_2(x) = - \int_0^x dt \frac{\ln(1-t)}{t} = \sum_{k=0}^{\infty} \frac{x^k}{k}.$$  \hfill (6.80)

It turns out, however, that the following representation in terms for Bernoulli numbers $B_k$ converges much faster.

$$\text{Li}_2(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!k} (- \ln(1-x))^k.$$  \hfill (6.81)

These two measures amount in a gain in speed of approximately 30%.

Further, we can exploit the knowledge of possible kinematics configurations for each \texttt{VEGAS} call. We distinguish between real, collinear-plus, collinear-minus and Born events (see section 6.2). For each configuration we generate the four momenta once per \texttt{VEGAS} call and collect the weight from all channels corresponding to these configurations. Therefore
the jet function with all its cuts and decay matrix elements has to be called only once for each configuration per VEGAS call. Similarly the number of calls to the luminosity function is minimized.

For longer runs aimed for high precision it is advantageous to interpolate the parton density functions rather than calling them in each VEGAS call. That part has been taken over from FEHiP and improved in speed.

6.3.2 Histogramming

There are different approaches of how to obtain differential distributions. In early versions of FEHiP, one had to rerun the code for every bin for each distribution by computing the observable and setting cuts appropriately. This is of course not very efficient but allows to assess the statistical uncertainty for each bin very accurately.

Another traditional approach is histogramming for which several histogram packages in FORTRAN exist. A distribution in an observable $X$ is obtained as follows: For each kinematic configuration a weight $d\sigma(x)$ is computed from the matrix elements and a Monte-Carlo weight $w(x)$ is passed. The total cross section is given by $\sigma = \int dx d\sigma(x)w(x)$. If the observable evaluated for the Monte-Carlo point $x$, $X(x)$, belongs to bin $i$, then the product of the weights is added to the value of that bin. Thus, the estimate for bin $i$ after $N$ calls is

$$h^{(i)} = \sum_{k=1}^{N} w(x_k)d\sigma(x_k)\chi_i(X(x_k))$$

(6.82)

where $\chi_i$ is the characteristic function for bin $i$, i.e. $\chi_i = 1$ if $X(x_k)$ belongs to this bin, otherwise $\chi_i = 0$. In order to obtain an error estimate, a second histogram is filled with the squared cross-section weights,

$$h_{2}^{(i)} = \sum_{k=1}^{N} w(x_k)[d\sigma(x_k)]^2\chi_i(X(x_k))$$

(6.83)

After $N$ Monte-Carlo calls, the error estimate is given by

$$\Delta h^{(i)} = \frac{1}{\sqrt{N(N-1)}}\sqrt{Nh_{2}^{(i)}-h^{(i)}_2}.$$  

(6.84)

This works fine as long no large cancellations take place. But exactly this is what happens in a typical NLO computation, where divergent real-radiation contributions are canceled by appropriate counter-terms, which might not have the same kinematics. Therefore, this cancellation does not happen in each bin.
There exist VEGAS implementations supporting vector-like functions, which allows us to define a separate component function for each bin for our distribution. The CUBA library [58] provides for example such an implementation. The CUBA variant of the VEGAS algorithm aims at optimizing the grid such that the requested precision is reached for every component. In many cases this approach slows the convergence considerably down or spoils it if the different component functions have a very different behavior. For that reason we introduced a switch in the CUBA VEGAS to prevent the adaptation to all components but restrict the adaptation only to the first component.

Following the second approach one cannot prevent the notorious problem of bin-bin fluctuation. This occurs when the observable for a real event happens to end up in a different bin than the observable for the corresponding counter-term. This is a problem in the divergent region where the counter-term should compensate the divergent behavior of the real event. If adaptation is performed only for a single integrand, the other integrands still suffer from these fluctuations which can only be reduced by choosing more Monte-Carlo calls per iteration. There might be other strategies like "smearing" trying to enforce the large $N$ limit by assigning real and counter-event to the same bin with a certain probability. We have not implemented this strategy because it occurs to us that it is not general enough.

### 6.3.3 Decays

We consider several decay modes of the Higgs in our work. These are the di-photon decay, $h \rightarrow \gamma\gamma$, and leptonic decays with intermediate weak boson states, $h \rightarrow W^+W^- \rightarrow \ell^+\nu\ell^-\bar{\nu}_\ell$, $h \rightarrow Z^0Z^0 \rightarrow \ell^+\ell^-\bar{\nu}_\ell\nu_\ell$, $h \rightarrow Z^0Z^0 \rightarrow \ell^+\ell^-\ell'^+\ell'^-$. In the latest case we have to distinguish the two cases where $\ell' = \ell$ and $\ell' \neq \ell$.

All decays are implemented within the $F_{\text{cut}}$ function, which takes the invariants of the particles involved in the production process as well as additional integration variables for the decay phase space as arguments.

The di-photon decay is very simple to implement. In the narrow width approximation, which is certainly valid in the region of low Higgs masses where this channel is sizable at all, the decay and the production completely factorizes,

$$
\sigma_{pp \rightarrow h \rightarrow \gamma\gamma} = \sigma_{pp \rightarrow h} \times \text{BR}_{h \rightarrow \gamma\gamma},
$$

since the decay amplitude solely depends on the Higgs momentum squared. We compute the branching ration $\text{BR}_{h \rightarrow \gamma\gamma}$ with $\text{HDECAY}$ [59, 60] and generate the photon momenta by integrating over the $1 \rightarrow 2$ phase space in the Higgs rest frame and boosting back into the laboratory frame.
The decay into leptonic final states via electroweak bosons is more involved. First, the $1 \rightarrow 4$ phase space is considerably more complicated than the trivial $1 \rightarrow 2$ phase space. It can be parametrized in an iterative manner as the product of $1 \rightarrow 2$ phase spaces and efficiently mapped to a flat integration region as described in detail in the appendix of [61]. Second, the matrix element is not constant anymore over the decay phase space. Therefore it must be explicitly evaluated for every Monte-Carlo call. We will present the results of the various decay matrix elements in the Appendix F.

### 6.3.4 FEHiP becomes FeHiPro

There exist publicly available codes which compute the NNLO QCD corrections in the $m_t = \infty$ approximation [62, 63] for fully differential quantities. No computation of the exact mass dependence is available at NNLO so far. While the code HIGLU can predict total cross-sections at NLO with full mass dependence, no code is so far available which incorporates NLO mass effects in a fully flexible Monte-Carlo and could correct existing NNLO codes as FEHiP. We remedy this situation by merging HiPro and FEHiP into FeHiPro.

Recently, electroweak corrections have been computed which lead to changes in the total cross-section of similar size as the NLO mass effects. If one wants to obtain the best predictions for Higgs production in gluon fusion, one has to consider these corrections, too. Consequently, we included these corrections in the current version of FeHiPro, which will be published in the near future.

In this section we are going to explain the principle of the merging and how the additional corrections have added to the code.

#### 6.3.4.1 Decomposition of the cross-section

FEHiP employs the method of sector decomposition and is able to compute fully differential distributions at NNLO in the heavy top mass approximation while retaining the exact top mass dependence at leading order, thus

$$
\sigma^{m_t=\infty}_{(N)NLO} = \sigma^t_{LO} \times \lim_{m_t \rightarrow \infty} \left( \frac{\sigma^{(N)NLO}}{\sigma^t_{LO}} \right).
$$

Here, superscripts $t$ denote the contribution of top loops only.

Originally, only the $h \rightarrow \gamma\gamma$ decay mode has been implemented. For a pedagogical explanation of computational techniques and details on the early code we refer to [62]. Recently also the leptonic decay mode $h \rightarrow WW \rightarrow \ell\nu\ell\nu$ has been added to the code [64–66].
At NNLO, 94 sectors have been identified which need to be run separately in order to obtain a satisfying convergence of the Monte-Carlo algorithm for each sector. Our approach to add the massive corrections is a pragmatic one: we identify contributions through NLO and add the difference to the computation in the effective theory. Since we want to be able to study top and bottom loops effects separately, we have chosen to decompose the partonic cross section as follows

\[ \sigma_{\text{NNLO}} = \sigma_{\text{LO}} + \Delta \sigma_{m=\infty} + \Delta \sigma_{\text{NNLO}} \]
\[ + \Delta \sigma_{\text{th}} + \Delta \sigma_{\text{NLO}} + \Delta \sigma_{\text{th}} \]

where terms in the first line are computed in the original FEHiP code and terms in the second line are correction terms computed by the sub-code HPro. As mentioned before, the first line has been computed using 94 sectors in the original implementation. For our purposes we single out \( \sigma_{\text{LO}} \) and \( \Delta \sigma_{m=\infty} \), defining two more sectors. Next, the terms in the second line become new "sectors" each, totaling in 99 sectors in the new code. Note that these corrections terms are not sectors in the sense of sector decomposition but from the point of view of the organization of the computation. The corrections terms in (6.87) are the following

\[ \Delta \sigma_{m=\infty} = \sigma_{m=\infty} - \sigma_{\text{LO}}, \]
\[ \Delta \sigma_{\text{NNLO}} = \sigma_{\text{NNLO}} - \sigma_{m=\infty}, \]
\[ \Delta \sigma_{\text{th}} = \sigma_{\text{th}} - \sigma_{\text{LO}}, \]
\[ \Delta \sigma_{\text{NLO}} = \sigma_{\text{NLO}} - \sigma_{m=\infty}, \]
\[ \Delta \sigma_{\text{th}} = \sigma_{\text{th}} - \sigma_{\text{NLO}}. \]

The superscript \( \text{tb} \) denotes contributions from both, top and bottom loops. We have modified the code HPro such that it is able to directly compute the differences in the above relations. Nevertheless, the decomposition (6.87) seems to be unnecessarily complicated. One could think of combining the LO and NLO terms to contain the exact mass dependence. However, keep in mind that the terms in the \( m_t = \infty \) limit are obtained through an effective theory computation. In section 6.3.4.2 we will modify the effective theory to include mixed QCD-electroweak corrections. It is not clear a priori how such modifications in the effective theory would affect the computation in the exact theory. In the decomposition (6.87) we can always identify contributions from the effective theory and the full theory. Furthermore we can distinguish between a theory with and without bottoms.

6.3.4.2 Mixed QCD-electroweak effects

At order \( \mathcal{O}(\alpha_s^2) \) there exist also contributions from two loop diagrams where weak bosons - \( W \) or \( Z \) - couple to the Higgs. These contributions can increase the pure QCD leading
order cross section by $+5 - 6\%$ for $m_h = 120 - 160\text{GeV}$. These contributions have been studied in [67–70]. The authors of [71] computed mixed QCD-electroweak corrections to these contributions in an effective theory approach in the limit of $m_h = 0$. It has been found that total cross sections at NNLO in the strong coupling including the mixed corrections and approximate inclusion of bottom quark effects were $4 - 6\%$ lower than previous best predictions.

We denote the contribution from two loop diagrams with light quarks and weak boson fusion by $\sigma_{t,ff}^{(0)}$. This is an excellent approximation to the complete result including heavy quarks for Higgs masses below $200\text{GeV}$. In the range $200\text{GeV} < m_h < 350\text{GeV}$ the agreement is still better than $1\%$ worsening around the threshold for producing two top quarks but not exceeding $4\%$ [72]. The mixed QCD-electroweak higher order corrections are given by a modified Wilson-coefficient $C_1$ describing an effective coupling of gluons and the Higgs,

$$L_{\text{eff}} = -\frac{1}{4\pi} \alpha_s C_1 \frac{h G_{\mu\nu}^a G^{a\mu\nu}}{}.$$  \hfill (6.93)

The Wilson-coefficient is found to be [71]

$$C_1 = \frac{1}{3\pi} \left\{ 1 + \lambda_{\text{EW}} \left[ 1 + \frac{\alpha_s}{\pi} C_{1w} + \left( \frac{\alpha_s}{\pi} \right)^2 C_{2w} \right] + \frac{\alpha_s}{\pi} C_{1q} + \left( \frac{\alpha_s}{\pi} \right)^2 C_{2q} \right\}$$ \hfill (6.94)

with heavy quark Wilson coefficients

$$C_{1q} = \frac{11}{4} \quad \text{and} \quad C_{2q} = \frac{2777}{288} + \frac{19}{16} \ln \frac{\mu^2}{m_t^2} + N_f \left( -\frac{67}{96} + \frac{1}{3} \ln \frac{\mu^2}{m_t^2} \right)$$ \hfill (6.95)

and the leading contribution from $\sigma_{t,ff}^{(0)}$ is

$$\lambda_{\text{EW}} = \frac{3\alpha}{16\pi \sin^2 \theta_W} \left\{ \frac{2}{\cos^2 \theta_W} \left[ \frac{5}{4} - \frac{7}{3} \sin^2 \theta_W + \frac{22}{9} \sin^2 \theta_W \right] + 4 \right\}.$$ \hfill (6.96)

Here, $\alpha = g^2/4\pi$ is the weak coupling constant. The coefficient $C_{1w}$ is given by

$$C_{1w} = \frac{7}{6}$$ \hfill (6.97)

while coefficient $C_{2w}$ is unknown. However, the cross section is rather insensitive to reasonable values of $C_{2w}$, i.e. of the same order of magnitude as $C_{2q}$. We introduce the correction factor $\delta_{\text{EW}}$ such that

$$\sigma_{t,ff}^{(0)} = \delta_{\text{EW}} \sigma_{t,LO}^{(0)}.$$ \hfill (6.98)

The cross-section 6.87 is then modified by electroweak contributions and reads

$$\sigma_{\text{NNLO}} = \left\{ 1 + \delta_{\text{EW}} \left[ 1 + \frac{\alpha_s}{4\pi} (C_{1w} - C_{1q}) + \left( \frac{\alpha_s}{4\pi} \right)^2 (C_{2w} - C_{2q} + C_{1q}(C_{1q} - C_{1w})) \right] \right\} \sigma_{t,LO}^{(0)}$$

$$+ \left\{ 1 + \delta_{\text{EW}} \left[ 1 + \frac{\alpha_s}{4\pi} (C_{1w} - C_{1q}) \right] \right\} \Delta \sigma_{NLO}^{m_{t,\mu = \infty}} + \{ 1 + \delta_{\text{EW}} \} \Delta \sigma_{NNLO}^{m_{t,\mu = \infty}}$$

$$+ \Delta \sigma_{LO}^{tb} + \Delta \sigma_{NLO} + \Delta \sigma_{NLO}^{tb}.$$ \hfill (6.99)
6.3.4.3 Electroweak real radiation processes

Considering electroweak corrections the processes $q\bar{q} \rightarrow g h$ and $gg \rightarrow q\bar{q}$ receive contributions from diagrams containing weak bosons. These corrections have very recently been computed in [73]. In contrast to our results in Appendix E, matrix elements are expressed in terms of spin structures,

$$M_{qq\rightarrow gh} = \frac{1}{16\pi^2} \sum_{x=W,Z,b,t} F^x(s_{13}, s_{23}, s_{12}, m_h, m_x) \Gamma^x_1 + F^x(s_{23}, s_{13}, s_{12}, m_h, m_x) \Gamma^x_2,$$  

(6.100)

with

$$\Gamma^x_1 = T_{ij}^x \left[ p_1 \cdot p_3 \bar{v}_2^j v_3^i (v_x + a_x \gamma_5) u_4^j - p_1 \cdot c_3^a v_3^j p_2^i (v_x + a_x \gamma_5) u_4^j \right],$$

$$\Gamma^x_2 = T_{ij}^x \left[ p_2 \cdot p_3 \bar{v}_2^j v_3^i (v_x + a_x \gamma_5) u_4^j - p_2 \cdot c_3^a v_3^j p_2^i (v_x + a_x \gamma_5) u_4^j \right].$$

(6.101)

and couplings

$$v_W = g_s g M_W \left[ (w_{\ell q})^2 + (w_{\ell a})^2 \right], \quad a_W = 2 g_s g M_W w_{\ell q}^a, \quad v_Z = g_s \frac{g}{c_W} M_Z \left[ (z_{\ell q})^2 + (z_{\ell a})^2 \right],$$

$$a_Z = 2 g_s \frac{g}{c_W} M_Z z_{\ell q}^q z_{\ell a}^q, \quad v_{l,b} = -g_s g \frac{m_{l,b}}{2 M_W}, \quad a_{l,b} = 0, \quad w_v^u = w_v^d = \frac{g}{2 \sqrt{2}},$$

$$w_a^u = -\frac{g}{2 \sqrt{2}}, \quad z_a^u = \frac{g}{4 c_W} \left( 1 - \frac{2}{3} s_W^2 \right), \quad z_a^d = -\frac{g}{4 c_W}.$$  

(6.102)

$s_W$ and $c_W$ are the sine and cosine of the Weinberg angle, respectively. It is straightforward to relate these spin structures to helicity amplitudes by using the identities of section 5.1.2. To incorporate these contributions in our code was therefore rather simple in particular since the authors of [73] provided us with their FORTRAN 77 code.

6.3.4.4 A fast option of computing inclusive cross sections

The authors of [71] have implemented their findings into a fast FORTRAN 77 code for computing the inclusive cross section through NNLO. This has been achieved by integrating analytically over parts of the phase space at NNLO. We have incorporated this option into FeHiPro. We added the option of including the mass effects through NLO in this mode. However, we did not integrate analytically over parts of the phase space for these contributions, therefore the inclusive calculation is slowed down by a factor of about 10 (only top contributions) up to 20 (including top, bottom and weak boson correction to real radiation). Therefore the inclusive mode is efficient only at NNLO. If we do not want to apply any cuts or do not consider decays, this speeds up the NNLO calculation enormously.
Part III

Higgs phenomenology
Chapter 7

Mass effects at NLO

In this and the following chapters we will study phenomenological consequences of the contributions we have computed in Part II. We start in this chapter with reviewing the cross-section at NLO. First, we study the effects from massive quark loops and their relative contributions in the case of total cross-sections in section 7.1. In section 7.2 we investigate how differential distributions are affected. These studies have been entirely carried out in the Standard Model. The results of this chapter have been published in [74].

7.1 Total cross-section at NLO

In this section, we revisit the gluon fusion cross-section at NLO. This serves as a check of our Monte-Carlo HPro against the predictions of HIGLU [59], and to emphasize the importance of finite quark mass effects in Higgs boson production. For the numerical results of this paper we use MSTW 2008 parton distribution functions [75].

We begin our study by revisiting the total cross-section in the LO and NLO approximation as a function of the Higgs boson mass (Fig. 7.2).

As it is well known [76–78], NLO QCD perturbative corrections are substantial. We note here that the perturbative corrections are slightly smaller with the latest parton densities [75], mainly due to the higher value of $\alpha_s$ used at leading order. Scale variation remains rather large at NLO, see Fig. 7.1. NNLO corrections stabilize the perturbative expansion and reduce the scale variation to the $\sim 10\%$ level as we will see in section 8. We also note that NNLO QCD corrections are still rather large [28–30]. However, NNLO computations rely on an approximate treatment of heavy quark loops.

In Fig. 7.2, we show the effects of different treatments for the heavy quarks. With HPro, we compute the exact LO and NLO cross-sections, where all loop diagrams with massive top
and bottom quarks are evaluated exactly (we denote with “top+bottom” the corresponding cross-sections in the plots of this paper). An approximation which can be made, is to consider a vanishing bottom Yukawa coupling and to evaluate exactly only the top-quark loops; in our plots, we denote this approximation as “top-only”. NNLO computations are performed in what is known as the infinitely heavy top-quark approximation. In the “$m_t = \infty$” approximation, the bottom Yukawa coupling is set to zero, and the cross-section at higher orders is estimated by the formula,

$$\sigma_{m_t=\infty}^{(N)NLO} = \sigma_{\text{top-only}} \times \lim_{m_t \to \infty} \left( \frac{\sigma_{\text{top-only}}^{(N)NLO}}{\sigma_{\text{top-only}}^{\text{LO}}} \right).$$

(7.1)

where bottom quark loops are ignored, and the leading order cross-section is reweighted with the ratio of the cross-sections at higher orders and the leading order in the limit of an infinite top-quark mass. In order to compare the different approximations we introduce

$$\delta X^i = \frac{X^i - X_{m_t=\infty}}{X_{m_t=\infty}}$$

(7.2)

where $X$ is the inclusive cross section or a normalized differential cross section and $i$ is labeling the contribution (top+bottom, top-only, bottom-only, top×bottom).

In Fig. 7.3, we show the cross-section deviations from the $m_t = \infty$ approximation of (6.86) when “top-only” (blue) and the complete “top-bottom” mass effects are taken into account. “Top-only” contributions are approximated within a couple of a percent up to the $m_h = 2m_t$ threshold. However, for a light Higgs boson, bottom quark contributions are important and can reach $\sim -8\%$. It is then important that bottom loops are taken into account for a precise evaluation of the total cross-section [71]. We also observe that the contribution from bottom-quark loops decreases at NLO in comparison to LO.
7.1. TOTAL CROSS-SECTION AT NLO

Figure 7.2: Total Cross-Section at Tevatron and LHC.

Figure 7.3: Percent differences of the exact NLO (LO) total cross-section with finite top bottom masses or the NLO (LO) total cross-section with exact top mass effects but zero bottom Yukawa coupling with respect to the usual approximation at Tevatron and the LHC.

In Fig. 7.4, we show the relative contributions to the NLO total cross-section from top-quark loops only, bottom quark-loops only, and from the interference of top and bottom loops. Top-only contributions are dominant, while bottom-only contributions are negligible over the whole Higgs mass range. Top-bottom interference terms are important at the few percent level and are negative for a light Higgs boson. It should be noted that the relative importance of the three contributions for a heavy Higgs boson or a pseudo-scalar Higgs boson in the MSSM may be drastically different than in the Standard Model [79].

The magnitude of QCD corrections depends strongly on the mass value of the heavy quark in the loops if we use the pole mass for the $q\bar{q}H$ coupling. In Fig. 7.5 we consider the cross-section for the gluon fusion cross-section at the Tevatron and the LHC, considering only one heavy quark with a mass $m_q$. We find that, in the pole scheme, the K-factor is reduced significantly for small values of the quark mass consistent with neglecting the
Figure 7.4: Ratio in percentage of top-only, bottom-only and top-bottom interference components with respect to the total cross-section at NLO, for Tevatron and LHC.

Figure 7.5: NLO cross-section and K-factor for the gluon fusion process via one only heavy quark at the Tevatron and the LHC, as a function of the quark mass.

running of the quark mass in the Higgs coupling at leading order.

For the Standard Model where both top and bottom quark loops contribute to the gluon fusion process, we study separately the magnitude of QCD corrections for the top-only, bottom-only, and top-bottom interference terms. The corresponding K-factors at NLO
Figure 7.6: NLO K-factors for the top-only, bottom-only, and top-bottom interference contributions. We restrict to a range of \( m_h \) where top-bottom interference contributions are still sizable.

are plotted in Fig. 7.6. While the top-only contributions receive a large K-factor, NLO QCD corrections to the top-bottom interference and bottom-only terms are milder. As a consequence, the importance of the bottom-quark loops is smaller at NLO than at LO.

The small NLO QCD corrections to the top-bottom interference contribution, which is also a very small fraction of the top-only contribution, suggests that a more precise evaluation at NNLO is not necessary. The top-only contribution receives however large NLO corrections and it requires an evaluation at NNLO. As shown in Fig. 7.3, this contribution can be approximated using (6.86) better than 2\% for a light Higgs boson, and better than 10\% for a Higgs boson with a mass above the top-pair threshold. It appears to us, that the combination of the NLO cross-section with full dependence on the top and bottom quark masses and the NNLO correction using the approximation of (6.86) yields a very precise estimate of the gluon fusion cross-section, where differences with a NNLO calculation with exact finite quark mass effects should be quite small.

7.2 NLO Differential cross-sections with finite quark masses

The search for a Higgs boson at hadron colliders is complicated due to the large cross-sections of background processes. Sophisticated experimental analyses are required, where it is essential to find optimized selection cuts. In addition, it is often necessary to perform a detailed probabilistic comparison of measured shapes for kinematic distributions with theoretical predictions for the signal and background processes. The role of very accurate Monte-Carlo programs which are fully differential is very important for these purposes.

The fully differential NNLO Monte-Carlo’s, \texttt{FEH} [64, 80, 81] and \texttt{HNNLO} [63, 82], are
Figure 7.7: NLO normalized rapidity distribution at the Tevatron and the LHC. Finite quark mass effects do not affect the shape of the distribution. The vertical grey line marks $\frac{1}{\sigma} \frac{d\sigma}{dy}$ = $10^{-3}$.

available for the gluon fusion Higgs boson production process. Given the complexity of NNLO computations, these programs employ the approximation of (6.86). In some cases, experimental cuts lead to a significantly smaller scale variation than in the total cross-section. This enhances the importance of other uncertainties, such as the one due to unaccounted finite quark mass effects. A characteristic example is the accepted cross-section for $pp \rightarrow H \rightarrow WW \rightarrow ll\nu\nu$ where a jet-veto and other cuts reduce the uncertainty due to scale variations by a factor of about two with respect to the total cross-section [64, 66, 82].

With our exact NLO Monte-Carlo $HPro$, we can correct the predictions of FEHiP and $HNNLO$ for finite quark mass effects through NLO. In this section, we illustrate the shapes of a few kinematic distributions and compare them with the corresponding predictions in the “$m_t = \infty$” approximation.

The rapidity of a particle is a measure of its direction. We denote the momentum in beam direction with $p_z$ and $E$ is the 0-component of the particle’s four momentum, its
7.2. NLO DIFFERENTIAL CROSS-SECTIONS WITH FINITE QUARK MASSES

Figure 7.8: Normalized $p_T$ distribution at Tevatron, $m_h = 120$ GeV. Massive corrections are important for large $p_T$. However only a very small fraction of events exists in this range.

energy. The rapidity $Y$ is defined by

$$Y = \ln \frac{E - p_z}{E + p_z}.$$  (7.3)

In Fig. 7.7, we study the normalized NLO rapidity distribution of a Higgs boson with mass $m_h = 150$ GeV at the Tevatron and the LHC. We observe that the shape of the rapidity distribution is unchanged when we include finite quark mass effects. The shape of the rapidity distribution is insensitive to such effects, even for Higgs boson mass values above the top-pair threshold illustrated by the lower plot in Fig. 7.7. The exact calculation and the infinite top quark mass approximation are in an excellent agreement.

Another important differential distribution is the transverse momentum of the Higgs boson. Finite quark mass effects for the $p_T$ distribution have also been studied in [79, 83, 84], and recently both electroweak and finite quark mass corrections were computed and combined [73].

In Fig 7.8 (left panel) we present the normalized cross-section at the Tevatron in $p_T$ bins of 2 GeV, for a Higgs boson with mass $m_h = 120$ GeV. At small values of $p_T$ the bin cross-sections cannot be computed accurately in perturbation theory, see Fig. 7.9, and an all orders resummation is required [85, 86]. A meaningful result is obtained, however, when the bins at low $p_T$ are added up together. In order to study the effect of finite quark masses it is more convenient if we demonstrate uncombined low $p_T$ bins. For this purpose, we present the $p_T$ distribution in the approximation of (6.86) (cyan), in the “top-only” approximation (blue) where the bottom loops are ignored but the top-loops are evaluated exactly, and with the complete “top-bottom” mass dependence (red). We have compared our results with the authors of [73] and found full agreement within numerical errors. In
Figure 7.9: First few bins of $p_T$ distribution at Tevatron, $m_h = 120$ GeV. The values of these bins are unphysical and require resummation.

Figure 7.10: Normalized $p_T$ distribution at Tevatron, $m_h = 180$ GeV. Compared to $m_h = 120$ GeV, mass corrections lead to smaller deviations from the “$m_t = \infty$” approximation. The spectrum of the bottom-only contribution is much softer (green, left panel).

In the right panel of Fig. 7.8 we show the percent deviations of the complete result (red) and the “top-only” approximation for the normalized $p_T$ distribution from the approximation of (6.86). At small $p_T$, there are very small differences due to finite quark-mass effects. We observe some important shape deviations due to the effect of top and bottom quark loops at intermediate $p_T$. As it has already been observed in Ref. [73] finite quark effects are very large at high $p_T$, where the quark production channel becomes dominant. Note that additional electroweak corrections affect the shape considerably [73]. At a higher Higgs boson mass value of $m_h = 180$ GeV (Fig. 7.10) we find an even milder effect at low $p_T$, while the magnitude of the deviations at a large $p_T$ is somewhat reduced but still large. For phenomenological purposes, these large deviations concern a tiny fraction of potential Higgs signal events for both $m_h = 120$ GeV and $m_h = 180$ GeV mass values. We show in Fig. 7.10 also the “bottom-only” contribution and observe that in this case the $p_T$ spectrum is much softer, as pointed out in Ref. [79].
7.2. NLO DIFFERENTIAL CROSS-SECTIONS WITH FINITE QUARK MASSES

![Graphs showing differential cross-sections with finite quark masses.]

Figure 7.11: Normalized $p_T$ distribution at LHC for $m_h = 120$ GeV and $m_h = 180$ GeV. Mass corrections are much more modest than at Tevatron where the $q\bar{q}$ channel plays a much bigger rôle.

At LHC energy (14 TeV), Fig. 7.11, we observe significant bottom-loop effects for a light Higgs boson ($m_h = 120$ GeV). These are reduced for a heavier Higgs boson with mass $m_h = 180$ GeV. Shape deviations due to finite quark mass effects can reach up to 10% at high $p_T$. It is interesting that bottom quark loops for a light Higgs boson change the shape at low $p_T$. As we explained, the fixed order $p_T$ spectrum is not physical at low $p_T$. However, these deviations may also survive after a complete resummation is performed via the matching procedure. It is interesting to examine the $p_T$ spectrum for a Higgs mass where the heavy top approximation is formally invalid. In Fig. 7.12 we plot the normalized distribution for $m_h = 400$ GeV at the LHC. Deviations of the “top-only” contributions from the infinitely heavy top-quark approximation are small for $p_T < 80$ GeV. At higher $p_T$ the difference increases.

Finally, we present normalized distributions for Higgs decay final state. In Fig. 7.13 we present the pseudo-rapidity difference and average $p_T$ distribution of the two photons in the process $pp \rightarrow H \rightarrow \gamma\gamma$. Finite quark-mass effects do not affect these distributions. At higher Higgs boson masses the process $pp \rightarrow H \rightarrow WW \rightarrow ll\nu\nu$ is dominating and we show as an example for this decay mode the $\phi_{ll}$ distribution in Fig. 7.14. $\phi_{ll}$ is the angle in
CHAPTER 7. MASS EFFECTS AT NLO

Figure 7.12: Normalized $p_T$ distribution at LHC for $m_h = 400$ GeV. Mass effects become important as $m_t = \infty$ approximation is formally invalid.

Figure 7.13: On the left: Normalized distribution of the average photon transverse momentum, $p_T^{\text{avg}} = (p_T^1 + p_T^2)/2$. On the right: Normalized distribution of photon pseudo-rapidity difference, $Y^* = |\eta^1 - \eta^2|/2$. In both plots $m_h = 120$ GeV and we assume LHC energies.

the transverse plane between the two charged leptons in the final state and we find again, that the shape is very well reproduced by the “$m_t = \infty$” approximation.

In summary, we have found that the shapes of distributions for leptons and photons from the decay of a Higgs boson are very well approximated by (6.86). In addition, accepted cross-sections after the application of cuts on jets are affected consistently with the expectations from the shapes of the Higgs $p_T$ spectra.
7.2. NLO DIFFERENTIAL CROSS-SECTIONS WITH FINITE QUARK MASSES

Figure 7.14: Normalized $\phi_{ll}$ distribution for $m_h = 170$ GeV at Tevatron and LHC. $\phi_{ll}$ is the angle in the transverse plane between the charged final state leptons, $\phi_{ll} = (p_{l1,\perp} \cdot p_{l2,\perp})/|p_{l1,\perp}| |p_{l2,\perp}|$. 

NLO top + bottom
NLO top only
Chapter 8

Best predictions at NNLO

Experiments are becoming for the first time sensitive of detecting a Higgs in its preferred range 115 – 200 GeV. Recently a SM Higgs boson in the mass range 160 – 170 GeV has been excluded in a combined preliminary study of two experiments at Tevatron, CDF and D0, at the 95% confidence level [88, 89]. In the very near future, LHC might cover the full range of most likely Higgs masses and allow us to gain more insight in the nature of electroweak symmetry breaking. The extraction of exclusion limits crucially depends on theoretical predictions for the rate a Higgs boson is produced and how it decays.

In section 8.1 we will discuss the inclusive cross-section including various sources of uncertainties. Then, in section 8.2, we consider differential distributions in variables which are relevant for experimental searches.

The results of this chapter are work in progress [87].

8.1 Inclusive cross-section

We start with examining the inclusive cross-section in this section. Due to the large corrections observed at NLO, the NNLO computation is indispensable in order to both, obtaining an accurate prediction for the total rate as well as reducing remaining theoretical uncertainties. The numbers in this chapter have been obtained with the Monte Carlo program FeHiPro, which incorporates QCD corrections in the \( m_t = \infty \) limit up to NNLO in \( \alpha_s \), heavy quark mass effects up to NLO, mixed QCD and electroweak virtual corrections up to order \( \mathcal{O}(\alpha_s^2\alpha) \) and electroweak real radiation effects of order \( \mathcal{O}(\alpha^2\alpha) \). We want to stress that the combination of all these corrections are studied for the first time in this thesis. In [71] mixed electroweak and QCD corrections have been computed and studied.
in combination of an approximate treatment of mass effects through NLO and in [73] electroweak real radiation has been studied for Higgs plus jet production.

Different sources make up the total theoretical uncertainty of determining the inclusive cross section. First, the size of unknown higher order corrections can be assessed by varying the renormalization and factorization scales, $\mu_R$ and $\mu_F$, respectively. The all orders cross-section must be independent of these scales and a small scale variation signals good convergence of the perturbative calculation. Second, parton distribution functions cannot be computed from first principles but must be fitted from experimental measured quantities. This leads to another ambiguity, the PDF uncertainty. Lastly, the strong coupling $\alpha_s$ must be obtained from experiments as well and leads to a third source of uncertainty. In particular, processes, as $gg \to h$, that are governed by QCD already at the leading order, suffer from this. However, the extraction of PDFs from experimental data and their evolution is not independent from $\alpha_s$ and some care must be taken in order to estimate the combined PDF+$\alpha_s$ error.

We devote section 8.1.1 to a short discussion of assessing the PDF and the combined PDF+$\alpha_s$ uncertainty before we present best predictions for Higgs production cross sections in section 8.1.2 together with estimates of their theoretical uncertainty.

8.1.1 PDF errors and $\alpha_s$ uncertainty

Parton distribution functions are obtained from fitting a certain number of free parameters to a set of experimental data. In our studies, we have used parametrization from the MRST/MSTW group [75, 90, 91]. This group provides for the more recent PDF sets additional sets for determining the PDF uncertainty. Roughly speaking, these error sets are obtained from diagonalizing the Hessian matrix obtained from minimizing the $\chi^2$ value with respect to the fitting parameters and imposing a maximal tolerance. Considering $n$ parameters, pairs of eigenvector PDF sets $S_k^\pm$ for $k=1,\ldots,n$ are introduced at fixed value of $\alpha_s$. The uncertainty of a quantity $F$ depending on the PDFs can then be calculated as a symmetric error with

$$\Delta F = \frac{1}{2} \sqrt{\sum_{k=1}^{n} [F(S_k^+) - F(S_k^-)]^2} \quad (8.1)$$

or as asymmetric errors with

$$(\Delta F)_+ = \sqrt{\sum_{k=1}^{n} \left[\max\left(F(S_k^+), F(S_k^-) - F(S_0), 0\right)\right]^2}, \quad (8.2)$$
\[
\langle \Delta F \rangle_\pm = \sqrt{\sum_{k=1}^{n} \left[ \max \left( F(S_0) - F(S_k^+), F(S_0) - F(S_k^-), 0 \right) \right]^2} \quad (8.3)
\]

where \( S_0 \) is the central set, i.e. the best-fit set. For more details we refer to [75].

As mentioned before, \( \alpha_s \) exhibits another source of uncertainty correlated to the PDF error since the evolution of PDFs in \( x \) and \( Q^2 \) depends on \( \alpha_s \). For being consistent, we need to use a different PDF set whenever we choose another value of \( \alpha_s(m_Z) \). The MSTW group provides such sets with their update of the MSTW2008 PDF sets in [92]. They suggest the following procedure in order to assess the combined PDF+\( \alpha_s \) error:

1. Compute the quantity \( F(S) \) for five different \( \alpha_s \) values corresponding to best-fit \( \alpha_s \), \( \alpha_s \) fixed at the two \( 1 - \sigma \) (= 68% confidence level) limits and \( \alpha_s \) fixed at half of these two limits, respectively. For each fixed \( \alpha_s \) there are \( 2n + 1 \) PDF sets, \( S_0 \) and \( S_k^\pm \) for \( k = 1, \ldots, n \). We obtain \( 5 \times (2n + 1) \) different values for \( F \).

2. For each fixed \( \alpha_s \), use (8.2) and (8.3) to compute the asymmetric PDF uncertainties, \( \langle \Delta F \rangle_+^{\alpha_s} \) and \( \langle \Delta F \rangle_-^{\alpha_s} \).

3. The total PDF+\( \alpha_s \) uncertainty is given by

\[
\langle \Delta F \rangle_{\text{PDF+}\alpha_s}^+ = \max_{\alpha_s} \left( F_{\alpha_s}^0(S_0) + \langle \Delta F \rangle_+^{\alpha_s}(S_0) \right)
\]

\[
\langle \Delta F \rangle_{\text{PDF+}\alpha_s}^- = F_{\alpha_s}^0(S_0) - \min_{\alpha_s} \left( F_{\alpha_s}^0(S_0) - \langle \Delta F \rangle_-^{\alpha_s} \right). \quad (8.5)
\]

Here, \( \alpha_s^0 \) is the best-fit value of \( \alpha_s \).

In the MSTW2008 PDF set, \( n = 20 \) and we have to compute the cross-section 205 times for each Higgs mass in order to assess the PDF+\( \alpha_s \) uncertainty. In our implementation of FeHiPro however, we can compute all \( 2n + 1 \) PDF sets for each \( \alpha_s \) simultaneously, reducing the computational cost enormously.

### 8.1.2 Results for total cross section

We present now numbers for the total cross-section obtained with FeHiPro incorporating NNLO QCD corrections in the \( m_t = \infty \) limit, heavy quark mass effects up to NLO, mixed QCD and electroweak virtual corrections and electroweak real radiation effects. We have applied the \( \overline{\text{MS}} \) scheme for the renormalization of \( m_b \) with input value

\[
m_b^{\overline{\text{MS}}}(10 \text{ GeV}) = 3.609 \text{ GeV} \quad (8.6)
\]
and leading order evolution,
\[ m_b^{\overline{MS}}(\mu_R) = m_b^{\overline{MS}}(10 \text{ GeV}) \cdot \exp \left[ -2 \frac{0.179}{\pi} \ln \frac{\mu_R}{10 \text{ GeV}} \right] . \] (8.7)

For the top quark pole mass we choose the most recent value from direct observation, [93]
\[ m_t = 173.1 \text{ GeV} . \] (8.8)

It was shown in [29], that the total cross-section is dominated by the soft region where the partonic center-of-mass energy is comparable to the Higgs mass. This leads to logarithmically enhanced terms of the form
\[ \ln \left( \frac{\mu_R}{m_h(1 - z)} \right) \] (8.9)
where \( z = m_h^2/s \). Choosing \( \mu_R = m_h(1 - z) \) would minimize these terms, but at fixed order we cannot make such a choice. Rather, we must choose a fixed scale. The authors of [29] suggest to achieve the before mentioned choice in “average” and hence, we choose as default scale
\[ \mu_R = \mu_F = \frac{1}{2} m_h . \] (8.10)

The scale uncertainty is estimated by computing the cross-section for \( \mu_R = \mu_F = m_h/4 \) and \( \mu_R = \mu_F = m_h \). Finally, PDF and PDF + \( \alpha_s \) uncertainties are determined as discussed in 8.1.1 with PDF sets for 90% confidence level.

The coefficient \( C_{2w} \) in the Wilson coefficient for mixed electroweak-QCD corrections in (6.94) is so far unknown but the authors of [71] found, that the cross-section is insensitive to the variation of this coefficient within “reasonable” limits. With reasonable it is meant of the same order of magnitude as the corresponding coefficient for pure QCD corrections, \( C_{2q} \). If not stated otherwise, all results in this work have been obtained with the choice
\[ C_{2w} = -10 . \] (8.11)

It is interesting to examine the effect of the individual corrections applied to the usual approximation (6.86) for the NNLO cross-section. We list the cross section for four different approximations A, B, C and D at leading order (LO), next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) in Table 8.1. Approximation A corresponds to (6.86), while the cross-section corrected by the exact mass dependence for top and bottom contributions through NLO is labeled B. C denotes the additional inclusion of virtual mixed QCD-electroweak effects and D is the best prediction available, taking into account electroweak real radiation. We show the cross-section for three different values of the Higgs mass, \( m_h = 120 \text{ GeV} \), \( m_h = 150 \text{ GeV} \) and \( m_h = 200 \text{ GeV} \).
8.1. INCLUSIVE CROSS-SECTION

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
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<td>LO</td>
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<td>480 fb</td>
<td>503 fb</td>
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<tr>
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<td>903 fb</td>
<td>944 fb</td>
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<td>1103 fb</td>
<td>1057 fb</td>
<td>1103 fb</td>
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<td>255 fb</td>
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<tr>
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<tr>
<td></td>
<td>NNLO</td>
<td>536 fb</td>
<td>522 fb</td>
<td>552 fb</td>
</tr>
<tr>
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<td>91.3 fb</td>
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<td>167.4 fb</td>
</tr>
<tr>
<td></td>
<td>NNLO</td>
<td>196.4 fb</td>
<td>194.8 fb</td>
<td>191.6 fb</td>
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</table>

Table 8.1: Comparing approximations for the total cross-section at Tevatron. A: $m_t = \infty$ limit; B: retaining exact mass dependences on top and bottom through NLO; C: B + additional mixed QCD-electroweak virtual corrections; D: C + additional electroweak real radiation contributions.

The leading virtual electroweak contribution, $\delta_{EW}$, is maximal and positive close to $m_h = 150\,\text{GeV}$ and close to minimal and negative around $m_h = 200\,\text{GeV}$. Quark mass effects (approximation B) are most important for a light Higgs mass, but very small for $m_h = 200\,\text{GeV}$. The bulk of the mass effect is due to the bottom quark which vanishes for increasing Higgs boson masses. The top quark contribution is very well reflected by approximation A in the mass range below $2m_t$, as already seen in chapter 7.

The electroweak corrections (C and D) almost completely compensate the quark mass effects for $m_h = 120\,\text{GeV}$. For an intermediate Higgs, electroweak corrections are more important than quark mass effects and increase the total cross-section by 2.5% compared to approximation A. For a Higgs of mass $m_h = 200\,\text{GeV}$, both, quark mass effects and electroweak effects decrease the cross-section by about $-3.6\%$ in total. Here, electroweak effects are dominating quark mass effects.

In Table 8.2 we have compiled the cross-section at Tevatron for a range of Higgs masses together with its uncertainties. Note that with the newest MSTW2008 PDF sets the combined PDF+$\alpha_s$ uncertainty is even larger than the residual theoretical uncertainty reflected in the scale variation. We stress the fact, that we are not allowed to combine the different uncertainties in quadrature since neither the scale variation uncertainty nor the PDF uncertainty are Gaussian distributed. Note that the statistical error from numerical integration is $\lesssim 3\%$.

Predictions for LHC for a center-of-mass energy of $\sqrt{S} = 10\,\text{TeV}$ are presented in Table 8.3. Here, the uncertainty due to PDFs is significantly lower, since we probe lower $x$ values of the parton distribution functions, which is constrained stronger by data and therefore PDFs are more accurately determined in this region. Similar observations ap-
Table 8.2: Best predictions for total the cross-section at Tevatron ($\sqrt{S} = 1.96$ TeV). Uncertainties due to scale variation, parametrization of parton distribution functions and additional uncertainty due to $\alpha_s$ are indicated.

We conclude that at the moment the uncertainties due unknown higher order corrections,
uncertainty due to scale variation, parametrization of parton distribution functions and the strong coupling are of the same size. The total uncertainty can be estimated to be of the order of 20 – 25% at Tevatron and 12 – 15% at LHC. As

reflected in the scale variation, and the uncertainties due to missing knowledge of parton distribution functions and the strong coupling are of the same size. The total uncertainty can be estimated to be of the order of 20 – 25% at Tevatron and 12 – 15% at LHC. As

Table 8.3: Best predictions for total cross-section at LHC with $\sqrt{S} = 10$ TeV. Uncertainties due to scale variation, parametrization of parton distribution functions and additional uncertainty due to $\alpha_s$ are indicated.

<table>
<thead>
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<th>$m_h$ [GeV]</th>
<th>$\sigma$ [pb]</th>
<th>Scale</th>
<th>PDF</th>
<th>PDF+$\alpha_s$</th>
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<td>115</td>
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<tr>
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<td>+7.5%</td>
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<tr>
<td>145</td>
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<td>+2.5%</td>
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<td>160</td>
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<td>+7.5%</td>
</tr>
<tr>
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<td>11.17</td>
<td>+6.8%</td>
<td>+2.8%</td>
<td>+7.5%</td>
</tr>
</tbody>
</table>
Due to scale variation, parametrization of parton distribution functions and additional uncertainty due to $\alpha_s$ are indicated.

Soon as data taking is started at LHC it is expected that the uncertainty on the PDFs decreases significantly.

Recently another group presented updated predictions for Higgs cross-sections at hadron
colliders [94]. In that study the cross-section has been computed in the $m_t = \infty$ limit through NNLO and a resummation of large logarithms at next-to-next-to-leading order (NNLL) has been performed. Further, corrections due to the bottom quark contributions have been incorporated and virtual electroweak effects accounted for in the complete factorization approach, i.e. $C_{iq} = C_{iw}$ in (6.94). The heavy quark masses has been set to $m_t = 170.9 \text{ GeV}$ and $m_b = 4.75 \text{ GeV}$, respectively. As pointed out in [94], the scale choice $\mu_F = \mu_R = m_h/2$ reproduces the result after resummation of large logarithms and scale choice $\mu_F = \mu_R = m_h$ within a few permille at small Higgs masses and within $\approx 2.5\%$ for $m_h = 200 \text{ GeV}$ at Tevatron.

We find that our best predictions with slightly different quark masses and without resummation but with additional electroweak real radiation corrections agree within approximately $1.5\%$ at Tevatron. Our prediction turns out to be slightly larger at Higgs masses $\lesssim 165 \text{ GeV}$ and slightly smaller for $m_h \gtrsim 165 \text{ GeV}$ than the numbers quoted in [94]. At LHC, the differences are larger and always positive varying between $\approx +2.9\%$ for $m_h = 110 \text{ GeV}$ and $\approx +1.8\%$ for $m_h = 200 \text{ GeV}$. We note here that the difference increases if we were to choose a smaller top quark mass. The separate influence from resummation and the choice of the factorization scheme for mixed QCD-electroweak corrections remains to be assessed.
Figure 8.1: Branching ratios for SM Higgs decays as function of the Higgs mass, obtained with HDECAY.

8.2 Differential distributions

The Higgs boson does not live long enough to be detected in a collider experiment and we have to identify its decay products. Since Higgs couplings to other particles are proportional to their mass, it preferably decays into heavy particles, if this decay is kinematically allowed. The branching ratios for different Higgs masses are found in Fig. 8.1. While $b\bar{b}$ is dominating below the threshold to produce two on-shell $W$ bosons, the decay into vector bosons dominates above. The decay into $t\bar{t}$ becomes sizable above $2m_t$.

Although $b\bar{b}$ seems to be the most promising detection channel for a light Higgs, $m_h \lesssim 140$ GeV, a careful study shows that this channel suffers from an enormous QCD background. It turns out, that the decay into two photons provides a much cleaner signal and is therefore the main search channel for light Higgs bosons despite the small branching ratio. This is because photons are detected rather easily in a detector and in particular CMS (Compact Muon Solenoid), an experiment at LHC, has an excellent photon energy resolution.

For higher masses $m_h \gtrsim 140$ GeV, the decay into vector bosons is favored followed by their decay into leptons and/or quarks. For large Higgs masses, $m_h \gtrsim 180$ GeV, the process $h \rightarrow ZZ \rightarrow \ell\ell\ell'$ delivers a very clean signal which allows to reconstruct the Higgs mass peak above background. If the Higgs has an intermediate mass, $140$ GeV $\lesssim m_h \lesssim 180$ GeV,
\[ h \rightarrow WW \rightarrow \ell\nu\ell\nu \] is the main search channel. In this case however, sophisticated search strategies with dedicated experimental cuts on the lepton kinematics is required, since the neutrinos in the final state lead to missing energy preventing the reconstruction of the Higgs mass. These cuts must be designed such, that the signal gets enhanced with respect to the background.

In section 8.2.1 we examine the effects from exact mass dependence and electroweak corrections in the di-photon decay. We repeat these studies for the decay into weak vector bosons in 8.2.2, where we will also take into account the interference effects from \( Z \) bosons when considering the \( W \) decay channel.

### 8.2.1 Di-photon decay

The decay into two photons is kinematically very simple and proceeds through heavy quark and \( W \) loops. We are working in the narrow width approximation for the Higgs width such that the cross-section \( pp \rightarrow h + X \rightarrow \gamma\gamma + X \) is given by

\[
\sigma_{gg\rightarrow\gamma\gamma X} = \text{BR}_{h\rightarrow\gamma\gamma} \cdot \left( 8\pi \int d\sigma_{gg\rightarrow hX} \cdot d\Pi_{\gamma\gamma} \right).
\]  

(8.12)

Note that the Higgs mass is directly obtained from the invariant mass of the photons, \( m_h = m_{\gamma\gamma} \). The branching ratio is computed with the program \texttt{HDECAY} [59] which we have linked with \texttt{FeHiPro}.

Several studies have been carried out concerning this decay channel, e.g. [81, 95, 96]. In this thesis, we focus on studying the effect of contributions that have been neglected in previous studies. How the total cross section is affected was subject of section 8.1 and it is obvious, that the same findings apply for the inclusive cross-section including decays before applying cuts. Thus, we examine the shape

\[
\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{dX}
\]  

(8.13)

of relevant distributions \( X \). If these get not modified by massive and electroweak corrections, we obtain the corrected distribution by simply normalizing to the updated prediction for the total cross section. In particular, we will consider only distributions before experimental cuts have been applied.

For completeness we list typical cuts for this discovery channel:

- The photons must be detected in the central region of the detector, \( |\eta_{\gamma1,2}| < 2.5 \).
- At least one of the photons must have a transverse energy larger than 40 GeV while the second one must have a minimal transverse energy of 25 GeV.
CHAPTER 8. BEST PREDICTIONS AT NNLO

Figure 8.2: Shape of $p_T^{\text{lead}}$ distribution.

- Require isolated photons, i.e. the additional energy from partons in a cone $\Delta R = \sqrt{\Delta \eta^2 + \Delta \phi^2} < 0.4$ around each photon must not exceed 15 GeV.

These cuts can easily be applied in FeHiPro, as in its predecessor, FEHiP. The first cut above refers to the pseudo-rapidity of the photons,

$$\eta = \ln \left( \frac{\cot \theta}{2} \right)$$  \hspace{1cm} (8.14)

with $\theta$ the polar angle with respect to the beam line. For mass-less particles, the pseudo-rapidity is equal to the rapidity (7.3).

First, we study the distribution of the harder photon$^1$, $p_T^{\text{lead}}$. In Fig. 8.2 we compare the shape of approximations A ("$m_t = \infty$"), B ("top+bottom") and D ("top+bottom+ewk") at NLO and NNLO for a Higgs of mass $m_h = 120 \text{ GeV}$ produced at the LHC. We denote the approximations as in 8.1.2. Note, that our fixed order computation cannot be trusted for $p_T^{\text{lead}} \sim 60 \text{ GeV}$ which receives contributions from the region close to vanishing transverse momentum of the Higgs. The bin just above 60 GeV is even negative. For a more trustful result, we have to apply re-summation. We do not find any change in the shape of the distribution due to mass effects for the range we are considering here.

$^1$By the harder photon we mean the one with a larger transverse momentum.
very few events. The application of selection cuts does not c
predicts a slightly harder distribution. But note that this h
right-hand side with cuts applied.

Fig. 8.4. We study the pseudo-rapidity di
Figure 8.4: Shape of 
Figure 8.3: Shape of $Y^*$ distribution. On the left-hand side without any cuts applied, on the right-hand side after cuts applied.

Distributions of more complicated kinematic variables are shown in Fig. 8.3 and in Fig. 8.4. We study the pseudo-rapidity difference of the two photons,

$$Y^* = \frac{|\eta_\gamma_1 - \eta_\gamma_2|}{2} \quad (8.15)$$

and the average transverse momentum,

$$p_{T}^{\text{avg}} = \frac{p_{T\gamma_1} + p_{T\gamma_2}}{2}. \quad (8.16)$$

On the right-hand side of Fig. 8.4 we present the distribution after the application of the above mentioned cuts. We do not find any mass effects in the case of the pseudo-rapidity while for very large averaged transverse momenta, we find that the $m_t = \infty$ approximation predicts a slightly harder distribution. But note that this happens in a region with only very few events. The application of selection cuts does not change this finding.

A last remark concerns the difference between NLO and NNLO. As can be inferred from
both, the $p_T^{\text{lead}}$ and the $p_T^{\text{avg}}$ distribution, the additional possible radiation of a parton in
the final state at NNLO allows for photons with a higher transverse momentum leading to
a slightly harder spectrum.

### 8.2.2 Decay into vector bosons

Weak bosons in more than two out of three cases decay into hadrons. These are however
superseded by an extremely high background. Much better suited for detection is the
decay into leptons. The probability that a $W$ boson decays into one of the charged leptons
and its corresponding neutrino is about 11% each. All other decay modes are negligible.
20% of the decays of the $Z$ bosons are invisible, mostly due to neutrinos, while it decays
in approximately 10% of the cases into a pair of charged leptons.

The decay mode $h \rightarrow ZZ \rightarrow \ell^+\ell^-\ell'^+\ell'^-$ is particularly clean because charged leptons
are rather well measurable in the detector. We distinguish the two modes

\[ h \rightarrow ZZ \rightarrow \ell^+\ell^-\ell'^+\ell'^- \quad \text{with} \quad \ell' \neq \ell, \quad (8.17) \]

and

\[ h \rightarrow ZZ \rightarrow \ell^+\ell^-\ell'^+. \quad (8.18) \]

There is one diagram contributing to (8.17) but two in the case of (8.18). In the latter
case we have to account for the symmetry in the final state, yielding a factor 1/4 in the
phase space integration. It turns out that the interference term is rather small. Hence, the
cross-section for the same flavor final state, (8.18), is about half of (8.17) as can be read
off from the plot on the right hand side of Fig. 8.5.
As first pointed out in [97], for intermediate masses below the ZZ threshold, the $WW$ decay mode with subsequent decay into charged leptons and missing energy,

$$h \to WW \to \ell^+\ell^-\nu\bar{\nu},$$ \hfill (8.19)

provides a much better chance of detecting the Higgs despite the absence of a re-constructable mass peak. Flavor $\ell$ and $\ell'$ are not correlated but charge conservation must be respected. If $\ell = \ell'$, the $WW$ decay mode shares the same final state with

$$h \to ZZ \to \ell^+\ell^-\nu\bar{\nu}.$$ \hfill (8.20)

Since neutrinos cannot be detected, their flavor remains completely undetermined and we have to sum over all three flavors, enhancing this contribution. In Fig. 8.5 we compare the total cross-section for the $h \to VV \to \ell^+\ell^-\nu\bar{\nu}$ final state with and without the interference from the ZZ intermediate state. Bear in mind, however, that these plots are obtained without any cuts applied.

Leading order matrix elements for all the decays discussed in this section are found in Appendix F. Higher order corrections turn out to be small and therefore these leading order matrix elements are a good description of the decay.

### 8.2.2.1 The $\ell\nu\ell\nu$ final state

In order to cut down the background, the following parton level cuts have been considered in [64] based on the originally studies in [97]:

- Charged leptons must have a pseudo-rapidity $|\eta| < 2$ and transverse momentum $p_T^{\ell} > 25$ GeV.
- Leptons must be isolated from hadrons: hadronic energy within a cone of $R = 0.4$ around each lepton must be less than 10% of the corresponding lepton transverse momentum.
- The di-lepton mass must fulfill $12$ GeV < $m_{\ell\ell}$ < $40$ GeV.
- Missing energy in the event must be $E_T^{\text{miss}} > 50$ GeV.
- The two leptons should be close in direction to each other in the transverse plane, $\phi_{\ell\ell} < 45^\circ$.
- No jets in the central region, $\eta^{\text{jet}} < 2.5$ are allowed with transverse momentum $p_T^{\text{jet}} > 25$ GeV.
Figure 8.6: On the left: $p_T^{\text{hard}}$. On the right: $p_T^{\text{soft}}$. The plots in the upper row are obtained without any cuts applied, the plots in the lower row are obtained after applying all selection cuts mentioned in the text.

- The harder lepton is required to have $30 \text{ GeV} < p_T^{\ell,\text{hard}} < 55 \text{ GeV}$.

In Fig. 8.6 we compare the shapes of the softer and the harder transverse momentum. We do not find any significant modifications due to mass effects. We do observe a slightly harder spectrum at NNLO than at NLO, due to the additional parton in the final state.

Next we consider the angle between the charged leptons in the transverse plane,

$$
\phi_{\ell\ell} = \frac{p_{\ell_1} \cdot p_{\ell_2}}{|p_{\ell_1}||p_{\ell_2}|},
$$

(8.21)

and the invariant lepton mass,

$$
m_{\ell\ell} = \sqrt{|p_{\ell_1} + p_{\ell_2}|^2}.
$$

(8.22)

Again, the shape of distributions in these variables seems not to exhibit effects due to the exact mass dependence, as can be read off from Fig. 8.7 apart from small bin-bin fluctuations.
From Fig. 8.8 we determine however, that the region cuts mentioned in the text.

As we have seen in Fig. 8.5, the contribution from ZZ intermediate states can be sizable. From Fig. 8.8 we determine however, that the region $\phi_{\ell\ell} < 45^\circ$ is dominated by the WW channel. Therefore we can safely neglect the ZZ contribution to the $\ell\nu\ell\nu$ final state.
8.2.2.2 The four lepton final state

Without considering selection cuts, we present in Fig. 8.9 results for the $ZZ \rightarrow e^+e^−\mu^+\mu^−$ decay mode. We display shape distributions for the transverse momentum of the hardest, the second hardest, the third and the softest lepton for a Higgs boson of mass $m_h = 190\text{ GeV}$ at LHC. Again, we find excellent description of the shape by the $m_t = \infty$ limit. Note that the NNLO distribution is slightly harder for the hardest transverse momentum and slightly softer for the softest transverse momentum than the NLO distribution.

Figure 8.9: Lepton transverse momenta for $h \rightarrow ZZ \rightarrow ee\mu\mu$. From left to right, upper to lower: hardest, second hardest momentum etc.
Chapter 9

Higgs production in the MSSM

The non-CP violating MSSM introduces 105 parameters in addition to the SM parameters, while the CP violating MSSM gives rise to even more parameters. It would be a formidable task to carry out detailed studies in the full parameter space. Regarding the enormous complications due to many mass scales and particle mixings such studies are rendered practically impossible.

Higgs production is commonly studied with simplifying assumptions like degenerate squark masses, neglecting the mixing of squarks or in the limit of very heavy supersymmetric particles. The limit of very heavy top quarks and squarks is known to provide an approximation to the full NLO result of within $20 \text{ – } 30\%$ for $\tan \beta \lesssim 5$ [99]. Full SUSY-QCD corrections for heavy supersymmetric particles up to NNLO have been obtained in [100–103]. In [104], the framework for computing the Higgs boson cross-section in the limit of heavy gluino masses has been clarified. The full mass dependence has been computed so far only in [21].

In this chapter we will make use of the results of [21] for the first study of Higgs boson production at NLO retaining the full mass dependence and explicitly taking finite gluino masses into account. These studies have to be considered as pilot studies for illustrating the importance of mass effects. For that reason we have chosen a region in parameter space which seems to be phenomenological preferred and where these effects are enhanced. We will not discuss other regions in parameter space, although their consideration would serve as a check for the validity of simplifying assumptions and might be part of future work.

If supersymmetry is discovered in future experiments, the precision measurement will certainly require detailed studies which cannot be carried out without the methods presented in this thesis. However, in that case, observations will have reduced the possible parameter space and one can focus on a very limited set of parameters. This is more
feasible than a scan over a large part of the MSSM parameter space.

Another application of computing the exact mass dependence is a proposal made in [105]. The authors of [105] suggest to use the Higgs boson mass and the ratio of the cross-section in the MSSM to the one expected in the SM to probe the stop sector. It remains to be studied, however, to what extent non-negligible contributions from the sbottom sector, as we observe in our work, allows the extraction of squark parameters.

A detailed and comprehensive review of Higgs physics in the MSSM can be found in [106]. The presented results are part of work in progress [107]. We conclude this introductory section by remarking that similar complications could also arise in other models than the MSSM where relatively light new particles exist and that our methods could directly be applied to these cases.

9.1 Scan in the golden region

The golden region has been introduced in section 2.4. Its characteristic is one very light stop squark and a large mixing in the stop sector. Furthermore, the gluino is likely to be rather light. Both effects are enhancing the contributions we have computed in this work and therefore the golden region represents an interesting part of the parameter space to apply our methods.

For our numerical studies we choose the following input values as in [21],

\[
\begin{align*}
m_t &= 172.5 \text{ GeV}, \\
m_b &= 5 \text{ GeV}, \\
\mu &= 300 \text{ GeV}, \\
\mu_{\theta} &= 200 \text{ GeV}, \\
\alpha_{\text{eff}} &= 3^\circ.
\end{align*}
\] (9.1)

Here, \(\mu_{\theta}\) is the renormalization scale for the squark mixing angles and \(\alpha_{\text{eff}}\) is the effective mixing angle of the CP even neutral Higgs bosons including higher order corrections. We fix the mass of the first stop quark \(m_{\tilde{t}_1} = 150 \text{ GeV}\) and the stop mixing angle \(\theta_t \approx 45^\circ\). Our scan of the golden region is done by selecting typical values of the mass splitting in the stop sector, according to the results in [18]. The remaining squark masses and mixing angles, listed in Table 9.1, are then obtained as described in section 4.6.

We define dimensionless couplings by normalizing to the top quark mass, i.e. we consider

\[
\begin{align*}
\frac{m_q h_f(q)}{m_t}, & \quad \frac{m_q H_f(q)}{m_t}, \quad \frac{m^2 h_s(q, i, j)}{m_t^2} \quad \text{and} \quad \frac{m^2 H_s(q, i, j)}{m_t^2}
\end{align*}
\] (9.2)

where we refer to the notation in (2.69). These normalized couplings are presented in Fig. 9.1 as function of \(m_{\tilde{t}_2}\) representing the scan in the golden region. The green (cyan) solid line is the coupling to the top (bottom) quark, clearly independent of the varying
squark coupling is drawn with a dashed-dotted line. Interestingly, the non-diagonal Higgs-squark couplings are drawn with dashed lines, red for stop squarks, blue for sbottom squarks. The non-diagonal Higgs-squark coupling is drawn with a dashed-dotted line.

In the case of light Higgs couplings, we notice a rather strong variation of the diagonal stop couplings with varying $m_{	ilde{t}_2}$ while the non-diagonal coupling is small and constant. The couplings indicate a relative dominance of (s)top couplings. Interestingly, the non-diagonal coupling for sbottom quarks is comparable in magnitude to the diagonal couplings, if not slightly enhanced.

<table>
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<th>$m_{t_2}$ [GeV]</th>
<th>$\theta_t$</th>
<th>$m_{b_1}$ [GeV]</th>
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Table 9.1: Mass parameters and mixing angles for stop and sbottom sector used as input for golden region scans.
Chapter 9. Higgs Production in the MSSM

Figure 9.1: Normalized couplings of (s)quarks to a light CP even neutral Higgs (left-hand side) and to a heavy CP even neutral Higgs (right-hand side). Fermion couplings are drawn with solid lines, (non-)diagonal squark couplings with dashed(-dotted) lines. Couplings in the (s)top sector are colored red; couplings in the (s)bottom sector blue.

The situation represents itself rather different in the case of heavy Higgs couplings. Here, the couplings to top quarks is negligible compared to the bottom coupling. Also, the sbottom quark couplings are much more important than in the case of the coupling to the lighter Higgs boson. In particular the non-diagonal sbottom coupling is largest in magnitude for almost all $m_{\tilde{t}}$ values.

For a complete assessment of the relative importance of the (s)quark species one has to study the matrix elements as well. But the study of the couplings alone already signals that the (s)bottom sector is clearly enhanced despite the small $m_{b}$ mass and that the approximation of diagonal couplings does certainly not hold in the heavy Higgs case nor in the case of a light Higgs.

9.2 Light Higgs

The direct Higgs bound from LEP 2 is $m_{h} > 114.4$ GeV while MSSM requires the lighter CP even neutral Higgs, $h^{0}$, to have a maximal mass of $\approx 135$ GeV. Typically, the lighter Higgs has a mass of about $115 - 120$ GeV in the allowed parameter space of the MSSM. We choose here $m_{h^{0}} = 115$ GeV for illustration purposes.

Bottom quarks account for $\sim 7\%$ of the total cross-section at most in the Standard Model and can be neglected in a first approximation. This situation changes in the MSSM where intermediate and large $\tan \beta$ values are favored, leading to a relatively stronger coupling of bottom quarks (and squarks) to the Higgs boson such that the bottom sector cannot be neglected as illustrated in the upper plot of Fig. 9.2. While “top only” (green) and
at NLO. For a meaningful result we must perform re-summation general K factors are rather sensitive on the approximation Interestingly, the “SUSY” diagrams render the K factor even factors for the various approximations in comparison to the beyond the scope of this thesis. The plot on the right-hand si e the discovery potential, also decay channels must be studie section is given as a function of larger values, the ratio quickly approaches the ratio of the total c In Fig. 9.3 the e modify the small pT veto effects taken into account. These can alter branching ratios qe remain. In the lower row of Fig. 9.3, we present the acce “top+bottom” (red) contributions in the SM differ only by a few percent, the approximation without a bottom sector (yellow) deviates from the full result (blue) by up to 15%. Furthermore we find that “SUSY” contributions (as defined in section 4.3) are substantial in the region of the MSSM parameter space, that we are considering.

In Fig. 9.2 the complete cross-section is drawn with (blue and yellow) solid lines, while the “QCD”-like approximation is drawn with dashed lines. The difference is expected to be more pronounced for larger tan β values. Note also that the MSSM cross-section is more than a magnitude lower than in the Standard Model for large m_{t_2}. For assessing the discovery potential, also decay channels must be studied carefully and supersymmetric effects taken into account. These can alter branching ratios quite a bit. This is however beyond the scope of this thesis. The plot on the right-hand side of Fig. 9.2 shows the K factors for the various approximations in comparison to the Standard Model K factors. Interestingly, the “SUSY” diagrams render the K factor even smaller than one and in general K factors are rather sensitive on the approximation made.

In Fig. 9.3 the effect of vetoing on jets with transverse momentum larger than p_T^{veto} is investigated for m_{t_2} = 500 GeV. In the plot on the left in the upper row the vetoed cross section is given as a function of p_T^{veto}. The plot in the upper row on the right-hand side illustrates the effect of “SUSY” virtual corrections, which modify the small p_T region. For larger p_T^{veto} values, the ratio quickly approaches the ratio of the total cross sections. Note that for p_T^{veto} = 0 the cross section is negative which is due to our fixed order calculation at NLO. For a meaningful result we must perform re-summation. However, the feature for p_T^{veto} ≲ 10 GeV might remain. In the lower row of Fig. 9.3, we present the acceptance, that
Figure 9.3: $p_T^{\text{veto}}$ plots at LHC for $m_{\tilde{t}_2} = 500$ GeV. Top row, on the left: Cross section with a veto applied on the transverse momentum of the Higgs. Top row on the right: Ratio of vetoed cross sections. Bottom row: Acceptance.

is the ratio of the vetoed cross section and the total cross section. Standard Model and "QCD"-like cross sections do not exhibit large deviations for any value of $p_T^{\text{veto}}$. For very small $p_T^{\text{veto}}$ we observe a discrepancy of almost 20% between SM acceptance and MSSM acceptance.

In the case of a lighter second stop squark with mass $m_{\tilde{t}_2} = 420$ GeV, Fig. 9.4, the SM and MSSM cross sections are much more similar and also the difference in the acceptance is decreased to $\approx 10\%$ for $p_T^{\text{veto}} = 10$ GeV.

9.3 Heavy Higgs

Apart from the production of a light Higgs, we can also study the production of the heavier CP even neutral Higgs boson, $H^0$. While the mass of the lighter Higgs is rather well constrained within the MSSM, the mass of the heavier neutral Higgs can be varied in much larger range without violating constraints. In Fig. 9.5 we present the cross section for heavy Higgs production as a function of its mass for a point in the golden region,
9.3. HEAVY HIGGS

Figure 9.4: $p_T^{\text{veto}}$ plots at LHC for $m_{\tilde{t}_2} = 420$ GeV. Top row, on the left: Cross section at LHC with a veto applied on the transverse momentum of the Higgs. Top row on the right: Ratio of vetoed cross sections. Bottom row: Acceptance.

Figure 9.5: Total cross-section and K factors at LHC for producing a heavier CP even neutral Higgs. The threshold region close to $m_{H^0} = 300$ GeV should be discarded.

corresponding to $m_{\tilde{t}_2} = 400$ GeV in Table 9.1. Around $m_{H^0} = 300$ GeV our numerical evaluation cannot be trusted, since we hit a singularity there, due to a $0^{++}$ resonance of two light stop quarks of mass $m_{\tilde{t}_1} = 150$ GeV. For larger Higgs masses we see the effect of the next threshold emerging, corresponding to the production of two on-shell $b_1$. The plot
on the right-hand side in Fig. 9.5 displays the K factors.

Note that the contribution of the most difficult diagrams with up to three or four different, very distinct \( m_b/m_{\tilde{q}} \sim 10^{-2} \) (s)quark masses running in the loops is strongly enhanced in the case of heavy Higgs production due to the coupling. If we were to neglect these contributions, which are very hard to compute, we would underestimate the total cross-section by up to \( \approx 15\% \) in the range, that we are considering here.
Chapter 10

Conclusions

In this thesis I have computed the exact mass dependence at next-to-leading order in perturbative QCD for both, real radiation and virtual amplitudes, in the case of a fermionic or a scalar particle running in the loop in the Standard Model and Minimal Supersymmetric Standard Model. Real radiation corrections were computed using standard techniques while virtual matrix elements have been computed applying the method of differential equations. I could derive analytic results for the virtual amplitudes which serves as a first independent check of results by Spira, Djouadi, Graudenz and Zerwas from 1993. The analytic results are given in the Euclidean region in terms of harmonic polylogarithms. I carried out the necessary analytic continuation of these formulae onto all relevant kinematic regions. This leads to the introduction of generalized log-sine functions and variants, whose numerical evaluation I have implemented.

Real radiation and virtual corrections have been combined using the FKS subtraction method. I derived expressions, which are suitable for numerical evaluation and used them to construct the first fully differential Monte-Carlo program for Higgs production taking into account the exact mass dependence at NLO. In addition, I have incorporated the most important decay channels $h \rightarrow \gamma\gamma$, $h \rightarrow WW/ZZ \rightarrow \ell\nu\ell\nu$ and $h \rightarrow ZZ \rightarrow \ell\ell\ell\ell'$ rendering the code potentially useful for experimental studies.

With this Monte-Carlo I have studied mass effects in detail for differential distributions at NLO. It has been observed that the rapidity of the Higgs boson is perfectly described by the infinite mass approximation, while the shape of the transverse momentum distribution is modified by up to 50% for large transverse momenta. The low $p_T$ range on the contrary is accurately described by the effective calculation. Larger deviations occur for Higgs masses above the top quark pair threshold, where the effective theory is formally not valid anymore. Distributions of experimental observables of the decay products are affected much less by mass effects.
To correct NNLO results calculated by other groups before, I have merged the NLO Monte Carlo program including exact mass dependence into the existing NNLO program FEHiP. Additionally, I have incorporated recently calculated contributions involving light quarks and electroweak gauge bosons and their mixed QCD and electroweak corrections into the new NNLO code.

With this new code I have obtained most up-to-date predictions for inclusive and exclusive cross-sections. The examination of the inclusive results stresses the importance of including both, mass effects and electroweak corrections, the latter being more important for intermediate and large Higgs masses. I find good agreement with results presented by another group, which do include soft gluon resummation but no electroweak real radiation. Further, I have assessed the theoretical uncertainty arising from the fixed order calculation of the matrix elements as well as the uncertainty due to the determination of $\alpha_s$ and the parametrization of parton distribution functions. I have found them to be of comparable size, leading to a residual combined uncertainty of the order of 20% at Tevatron and 15% at LHC.

The Higgs boson cannot be observed directly in collider experiments but rather its decay products. These decay products are particles that are also abundantly produced through other processes. Therefore it is essential to study exclusive observables which might distinguish signal from background. I have investigated the effect of massive and electroweak corrections at NNLO on certain observables in $h \rightarrow \gamma\gamma$, $h \rightarrow WW/ZZ \rightarrow \ell\nu\ell\nu$ and $h \rightarrow ZZ \rightarrow \ell\ell\ell\ell'$ decays. These investigations reveal no sensitivity on mass and electroweak effects. This suggests the use of the $m_t = \infty$ approximation for distributions improved by a normalization to the inclusive cross-section including all corrections.

Finally, in order to demonstrate the importance of exact mass dependence, I have computed the cross-section for the production of a light as well as of a heavy CP even neutral Higgs boson in the MSSM in a region of parameter space with large stop mixing and light gluinos. For that purpose I have amended the calculation of scalar contributions with numerical results for virtual contributions involving more mass scales subject of another thesis. I find that the latter contributions are not negligible and alter the cross-section significantly. This is the first complete study in the MSSM taking the full mass dependence into account. It can be considered as a pilot study which lays down the solutions of all the theoretical and computational problems of a more complete study needed in the case of the discovery of supersymmetry. The method is also directly applicable to other beyond the Standard Model models.

Since the virtual contributions with many mass scales only affect the $p_T = 0$ kinematics I have studied the effect of a veto on the Higgs transverse momentum for the production of a light CP even neutral Higgs boson. I have found differences in the acceptance compared
to the Standard Model only for small values of the transverse momentum. In that region however, a proper treatment requires resummation which is beyond the scope of this thesis. The importance of including both, the top/stop and the bottom/sbottom sector is further illustrated in the case of heavy CP even neutral Higgs production.

I conclude that the sensitivity of the Higgs cross-section on additional heavy states renders the exact computation even more important and the complete phenomenology at NLO has become feasible with the methods I have presented in this thesis.
Part IV

Appendix
Appendix A

Feynman rules

Feynman rules relevant for this thesis in Feynman gauge ($\xi = 1$).

A.1 Propagators

\[ i p_{j} = \begin{cases} \delta_{ij} & \text{if } p^2 - m^2 + i\epsilon \end{cases} \]  
(A.1)

\[ i p_{j} = \begin{cases} \delta_{ij} & \text{if } p - m + i\epsilon \end{cases} \]  
(A.2)

\[ \mu p_{\mu} = -i g^{\mu\nu} \frac{1}{p^2 - m^2 + i\epsilon} \]  
(A.3)

\[ \mu, a p_{\mu, b} = -i g^{\mu\nu} \delta_{ab} \frac{1}{p^2 + i\epsilon} \]  
(A.4)
A.2 SM-like vertices

A.2.1 QCD vertices

\[ j^{a,\mu} i = -ig_s \gamma^\mu (T^a)_{ji} \]  
(A.5)

\[ b^{c,\lambda} p_2^{c,\lambda} = -g_s f^{abc} \left[ g^{\mu\nu} (p_1 - p_2)^\lambda + g^{\nu\lambda} (p_2 - p_3)^\mu + g^{\mu\lambda} (p_3 - p_1)^\nu \right] \]  
(A.6)

\[ b^{c,\lambda} p_2^{c,\lambda} = -g_s f^{abc} \left[ f^{ace} f^{bde} \left( g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\lambda\nu} \right) \right] \]  
(A.7)

A.2.2 Electroweak vertices

\[ f^{W} = -i g \frac{2}{2\sqrt{2}} \gamma^\mu (1 - \gamma^5) \]  
(A.8)
A.3 SUSY-like vertices

The squark mixing matrix is given by

\[ R^q = \begin{pmatrix} \cos \theta_q & \sin \theta_q \\ -\sin \theta_q & \cos \theta_q \end{pmatrix} \]

and we define

\[ S^q = \begin{pmatrix} \cos 2\theta_q & -\sin 2\theta_q \\ -\sin 2\theta_q & -\cos 2\theta_q \end{pmatrix} \].

Further, we introduce coefficients \( A(q,s) \) and \( B(q,s) \):

\[ A(q,s) = (R^q)_{s2} - (R^q)_{s1}, \]
\[ B(q,s) = -(R^q)_{s2} - (R^q)_{s1} \]
\[ j = -i g_s (p_1 + p_2)^\mu (T^a)_{ji} \quad (A.16) \]

\[ j = i g_s^2 \left[ T^a T^b + T^b T^a \right]_{ji} g^{\mu \nu} \quad (A.17) \]

\[ q_1, r, i \quad q_4, u, l \]

\[ = -i g_s \left\{ \delta_{q_1 q_2} \delta_{q_3 q_4} \left[ T_{ij} T_{kl} \delta_{q_1} S_{q_3} + \delta_{q_1 q_3} T_{kl} \delta_{q_2} S_{q_4} \right] + \delta_{q_1 q_4} \delta_{q_2 q_3} (1 - \delta_{q_1 q_2}) \left[ T_{il} T_{kj} \delta_{q_2} S_{q_4} + \delta_{q_1 q_3} T_{ij} \delta_{q_2} S_{q_3} \right] \right\} \quad (A.18) \]

\[ q_1, s_1, i \]

\[ a = i g_s \frac{\sqrt{n}}{2} \delta_{q_1 q_2} T_{ij}^a \left( A(q_1, s_1) - B(q_1, s_1) \gamma_5 \right) \quad (A.19) \]

\[ q_1, s_1, i \]

\[ a = i g_s \frac{\sqrt{n}}{2} \delta_{q_1 q_2} T_{ji}^a \left( A(q_1, s_1) + B(q_1, s_1) \gamma_5 \right) \quad (A.20) \]

\[ q_1, s_1 \]

\[ h = -\frac{1}{v} m^2 h_s (q_1, s_1, s_2) \delta_{q_1 q_2} \quad (A.21) \]
Appendix B

Harmonic polylogarithms

Harmonic polylogarithms are explicitly real valued in the range $0 < x < 1$. In the physical region, however, we have $-1 < x < 0$ above threshold and $x = \exp i\theta$ with $0 < \theta < \pi$.

We start with discussing the analytic continuation for real valued arguments in B.1. Analytic continuation for $x$ on the complex unit circle is worked out in B.2. In B.2.1 we collect properties and special values of functions introduced in B.2 and integration identities necessary for computing the analytic continuation of HPLs up to weight four are listed in B.2.2. Finally, we discuss the numerical evaluation in B.2.3.

B.1 Analytic continuation of HPLs for $-1 < x < 0$

For the region above threshold, $-1 < x < 0$, we exploit transformation properties [46,108] of HPLs under $x \to -x$. We start with observing that for weight 1, HPLs with real argument $x$ in the range $(-\infty, \infty)$ and and infinitesimal imaginary part $\sigma \varepsilon$, $\sigma = \pm 1$, are related to HPLs with real argument $-x$ through

$$H(1; x + i\sigma\varepsilon) = -H^*(-1; -x + i\sigma\varepsilon), \quad (B.1)$$
$$H(0; x + i\sigma\varepsilon) = [H(0; -x + i\sigma\varepsilon) - i\sigma\pi]^*, \quad (B.2)$$
$$H(-1; x + i\sigma\varepsilon) = -H^*(1; -x + i\sigma\varepsilon), \quad (B.3)$$

Note that for $-1 < x < 1$ both, $H(1; x)$ and $H(-1; x)$ are real valued. We can choose a minimal basis of HPLs, in which all HPLs of weight 2 and higher have no rightmost index 0 by exploiting product and IBP identities. Simultaneously we can choose the basis such that the leftmost index is not 1. This choice guarantees that the HPLs are regular at $x = 0$ and $x = 1$ except for $H(0; x)$ and $H(1; x)$. For instance integration by parts allows us to
write:

\[ H(1, 0; x) = H(0; x)H(1; x) - H(0, 1; x). \]  \hspace{1cm} (B.4)

Thus, the non-regular behavior for \( x \geq 0 \) is fully captured by \( H(1; x) \) and \( H(0; x) \), whose analytic continuation is given by (B.1) and (B.2), respectively. By induction follows for \( a_1 \neq 0 \)

\[ H(a_n, \ldots, a_1; x + i\sigma \varepsilon) = (-1)^{a_1 + \cdots + a_n} H^*(-a_n, \ldots, -a_1; -x + i\sigma \varepsilon). \]  \hspace{1cm} (B.5)

These identities are implemented in the \texttt{hplog FORTRAN 77} routines \cite{108} for evaluating harmonic polylogarithms up to weight 4. We use these routines for the evaluation of HPLs with real argument in our implementation.

### B.2 Analytic continuation of HPLs for \( x = \exp i\theta \)

For arguments on the complex unit circle the analytic continuation proceeds along the lines of the procedure proposed in \cite{109}. We have carried out this procedure for all HPLs up to weight four and give a few details in this section.

We first express our results in terms of the variable \( \theta \), defined by \( x = \exp i\theta \), and introduce the following notation for the harmonic polylogarithms as functions of \( \theta \)

\[ H_c(a_n, \ldots, a_1; \theta) \overset{\text{def}}{=} H(a_n, \ldots, a_1; e^{i\theta}) \]  \hspace{1cm} (B.6)

with

\[ H_c(1; \theta) = -\ln 2 \left| \sin \frac{\theta}{2} \right| + i \left( \text{sign}(\theta) \frac{\pi}{2} - \frac{\theta}{2} \right), \]  \hspace{1cm} (B.7)

\[ H_c(0; \theta) = i\theta, \]  \hspace{1cm} (B.8)

\[ H_c(-1; \theta) = \ln 2 \left| \cos \frac{\theta}{2} \right| + i \left( \text{sign}(\theta - \pi) \frac{\pi}{2} - \frac{\theta - \pi}{2} \right). \]  \hspace{1cm} (B.9)

We choose a basis of \( H_c \) without left most index being one and define for \( a_n \neq 1 \) recursively

\[ H_c(a_n, a_{n-1}, \ldots, a_1; \theta) = H(a_n, a_{n-1}, \ldots, a_1; 1) + i \int_0^\theta d\theta' g(a_n; \theta') H_c(a_{n-1}, \ldots, a_1; \theta'). \]  \hspace{1cm} (B.10)

The kernel functions \( g(a; \theta) \) are given by

\[ g(1; \theta) = \frac{e^{i\theta}}{1 - e^{i\theta}} = -\frac{1}{2} + \frac{i}{2} \cot \frac{\theta}{2}, \]  \hspace{1cm} (B.11)

\[ g(0; \theta) = \frac{e^{i\theta}}{e^{i\theta}} = 1, \]

\[ g(-1; \theta) = \frac{e^{i\theta}}{1 + e^{i\theta}} = \frac{1}{2} + \frac{i}{2} \tan \frac{\theta}{2}. \]
The integrals in (B.10) are expressed through the following functions known in the literature [109–112]

\[ Cl_1(\theta) = -\ln 2 \left| \sin \frac{\theta}{2} \right|, \quad \text{(B.12)} \]

\[ Cl_n(\theta) = \begin{cases} S_n(\theta) = \frac{1}{2i} \left[ Li_n(e^{i\theta}) - Li_n(e^{-i\theta}) \right] & \text{if } n \text{ even}, \\ C_n(\theta) = \frac{1}{2i} \left[ Li_n(e^{-i\theta}) + Li_n(e^{i\theta}) \right] & \text{if } n \text{ odd}, \end{cases} \quad \text{(B.13)} \]

\[ Ls_j^{(k)}(\theta) = -\int_0^{\theta} d\theta' \theta'^{k-1} \ln^{j-1} 2 \left| \sin \frac{\theta}{2} \right|, \quad \text{(B.14)} \]

\[ Lsc_{i,j}(\theta) = -\int_0^{\theta} d\theta' \ln^{i-1} 2 \left| \sin \frac{\theta}{2} \right| \ln^{j-1} 2 \left| \cos \frac{\theta}{2} \right|, \quad \text{(B.15)} \]

\[ LsLsc_{n,i,j}^{(k)} = \int_0^{\theta} d\theta' \theta'^{k} Ls_{n+1}^{(0)} \ln^{i-1} 2 \left| \sin \frac{\theta}{2} \right| \ln^{j-1} 2 \left| \cos \frac{\theta}{2} \right|, \quad \text{(B.16)} \]

and additional auxiliary functions,

\[ TLsc_{n,i,j}^{(k)} = \int_0^{\theta} d\theta' \theta'^{k} \tan^n \frac{\theta'}{2} \ln^{i-1} 2 \left| \sin \frac{\theta'}{2} \right| \ln^{j-1} 2 \left| \cos \frac{\theta'}{2} \right|, \quad \text{(B.17)} \]

\[ Lic_{n}(\theta) = Li_n \left( \cos^2 \frac{\theta}{2} \right), \quad \text{(B.18)} \]

\[ LLic_n^{(k)}(\theta) = \int_0^{\theta} d\theta' \theta'^{k} Li_n \left( \cos^2 \frac{\theta'}{2} \right), \quad \text{(B.19)} \]

\[ U_{n,i,j}^{(k)}(\theta) = \int_0^{\theta} d\theta' \theta'^{k} Lic_{n}(\theta') Cl_1(\theta') \frac{1}{i} Cl_1(\theta' - \pi)^j. \quad \text{(B.20)} \]

\( TLsc_{n,i,j}^{(k)} \) functions are not well defined. They have a singularity at \( \theta = \pi \) for \( k = 0 \) and \( n > 0 \) and one at \( \theta = 0 \) for \( k = 0 \) and \( n < 0 \). These singularities have the form of products of \( Cl_1(\theta) \) and \( Cl_1(\theta - \pi) \) functions and may cancel in the full expression.

The analytic continuation of the minimal basis up to weight 4 is expressed through the following functions introduced above:

- **Weight 1**: \( Cl_1 \).
- **Weight 2**: \( Cl_1, Ls_j^{(k)}, Lic_2 \).
- **Weight 3**: \( Cl_1, Ls_j^{(k)}, Lsc_{2,2}, LLic_2^{(0)}, Lic_2, Lic_3 \).
- **Weight 4**: \( Cl_1, Ls_j^{(k)}, Lsc_{2,2}, Lsc_{2,3}, Lsc_{3,2}, LLic_2^{(0)}, LLic_2^{(1)}, Lic_3, LLic_3^{(0)}, Lic_4, LsLsc_{0,3,2}^{(0)}, LsLsc_{1,1,2}^{(0)}, LsLsc_{1,2,1}^{(0)}, TLsc_{1,3,2}^{(0)}, TLsc_{1,4,1}^{(0)}, U_{2,0,1}^{(0)}, U_{2,1,0}^{(0)}, U_{2,1,0}^{(1)} \).

The explicit expressions can be obtained from the author as text or Maple files. Interestingly, these functions are a much bigger set than the functions that appear in the analytic
continuation of the matrix elements, which are
\[
\text{Cl}_1(\theta), \quad \text{L}^{(k)}_j(\theta), \quad \text{Lsc}_{i,j}(\theta), \quad \text{Lsc}_{1,1,2}(\theta).
\]

B.2.1 Properties

B.2.1.1 Generalized log-sine function

\[
\text{Ls}^{(0)}_2 = \text{Cl}_2(\theta),
\]
\[
\frac{d}{d\theta} \text{Ls}^{(k)}_{k+2}(\theta) = \theta^k \frac{d}{d\theta} \text{Cl}_2(\theta),
\]
\[
\text{Ls}^{(2j)}_{2j+2}(\theta) = \sum_{k=0}^{2j} \frac{(2j)!}{(2j - k)!} \theta^{2j-k} \text{Cl}_{2+k}(\theta)(-1)^{k(k-1)/2},
\]
\[
\text{Ls}^{(2j+1)}_{2j+3}(\theta) = (-1)^j (2j+1)! \left( \text{Cl}_{2j+3}(\theta) - \zeta_{2j+3} \right)
+ \sum_{k=0}^{2j} \frac{(2j+1)!}{(2j+1-k)!} \theta^{2j+1-k} \text{Cl}_{2+k}(\theta)(-1)^{k(k-1)/2},
\]
\[
\text{Ls}^{(j-1)}_j(-\theta) = -\frac{1}{\theta^j},
\]
\[
\text{Ls}^{(k)}_j(-\theta) = (-1)^{k+1} \text{Ls}^{(k)}_j(\theta) \quad \text{if } \theta > 0.
\]

B.2.1.1.1 Special values:

\[
\text{Ls}^{(0)}_{n+1}(\pi) = -\pi \left( \frac{d}{dx} \right)^n \frac{\Gamma(1+x)}{\Gamma(1+x/2)^2} \bigg|_{x=0},
\]
\[
\text{Ls}^{(0)}_2(\pi) = 0,
\]
\[
\text{Ls}^{(0)}_3(\pi) = -\frac{1}{2} \pi \zeta_2,
\]
\[
\text{Ls}^{(0)}_4(\pi) = \frac{3}{2} \pi \zeta_3,
\]
\[
\text{Ls}^{(1)}_3(\pi) = -\frac{7}{4} \zeta_3,
\]
\[
\text{Ls}^{(1)}_4(\pi) = -\frac{1}{6} \ln^4 2 + \frac{\pi^2}{6} \ln^2 2 - \frac{7}{2} \zeta_3 \ln 2 + \frac{19}{720} \pi^4 - 4 \text{Li}_4 \left( \frac{1}{2} \right),
\]
\[
\text{Ls}^{(2)}_4(\pi) = -\frac{3}{2} \zeta_3.
\]
B.2.1.2 The function $\text{LsLsc}_{n,i,j}^{(k)}$

\[
\begin{align*}
\text{LsLsc}_{0,0,i,j}^{(-1)}(\theta) &= \text{Lsc}_{i,j}(\theta), \\
\text{LsLsc}_{0,j,0}^{(0)}(\theta) &= \text{Ls}_{j+1}^{(1)}(\theta), \\
\text{LsLsc}_{0,1,j}^{(0)}(\theta) &= \text{Ls}_{j+1}^{(1)}(\theta - \pi) - \text{Ls}_{j+1}^{(1)}(\pi) - \pi \left( -\text{Ls}_{j}^{(0)}(\theta - \pi) - \text{Ls}_{j}^{(0)}(\pi) \right), \\
\text{LsLsc}_{n,1,0}^{(0)}(\theta) &= \theta \text{Ls}_{n+1}^{(0)}(\theta) - \text{Ls}_{n+2}^{(1)}(\theta), \\
\text{LsLsc}_{1,1,n}^{(0)}(\theta) &= \frac{1}{k + 1} \theta^{k+1} \text{Ls}_{n+1}^{(0)}(\theta) - \frac{1}{k + 1} \text{Ls}_{n+k+2}^{(k+1)}(\theta), \\
\text{LsLsc}_{0,2,0}^{(1)}(\theta) &= \text{Ls}_{2}^{(2)}(\theta), \\
\text{LsLsc}_{0,1,2}^{(1)}(\theta) &= \text{Ls}_{2}^{(0)}(\theta - \pi) \pi^2 + 2 \text{Ls}_{3}^{(1)}(\theta - \pi) \pi + \text{Ls}_{4}^{(2)}(\theta - \pi) + 2 \zeta_3 \pi, \\
\text{LsLsc}_{1,2,0}^{(1)}(\theta) &= -\frac{1}{2} \left( \text{Ls}_{2}^{(0)}(\theta) \right)^2, \\
\text{LsLsc}_{0,2,2}^{(0)}(\theta) &= \frac{1}{8} \text{Ls}_{4}^{(1)}(2\theta) - \frac{1}{2} \text{Lsc}_{1}^{(1)}(\theta) - \frac{1}{2} \left( \text{Lsc}_{1}^{(1)}(\theta - \pi) - \text{Lsc}_{1}^{(1)}(\theta) \right) \\
&\quad + \frac{\pi}{2} \left( -\text{Lsc}_{3}^{(0)}(\theta - \pi) - \text{Lsc}_{3}^{(0)}(\pi) \right).
\end{align*}
\]

B.2.1.3 The function $\text{TLsc}_{n,i,j}^{(k)}$

\[
\begin{align*}
\text{TLsc}_{0,0,i,j}^{(k)}(\theta) &= -\text{LsLsc}_{0,0,i,j}^{(-1)}(\theta), \\
\text{TLsc}_{1,0,1}^{(k)}(\theta) &= 2(-1)^{n-1} \frac{\theta^k}{n} \text{C} \text{l}_1(\theta - \pi) \text{C} \text{l}_1(\theta)^{n-1} + (n - 1) \text{TLsc}_{-1,0,1}^{(k)}(\theta) \\
&\quad - 2k \text{LsLsc}_{0,2,0}^{(k-2)}(\theta) \quad \text{for } k > 1, \\
\text{TLsc}_{1,1,n}^{(k)}(\theta) &= (-1)^{n-1} \frac{\theta^k}{n} \text{C} \text{l}_1(\theta - \pi)^n - 2 \frac{k}{n} \sum_{l=0}^{k-1} \binom{k-1}{l} \pi^{k-1-l} \text{Ls}_{n+l+1}^{(0)}(\theta - \pi) \\
&\quad - \text{Ls}_{n+l+1}^{(l)}(\theta - \pi) \quad \text{if } k > 0, \\
\text{TLsc}_{1,1,1}^{(0)}(\theta) &= 2 \text{C} \text{l}_1(\theta - \pi) + 2 \ln 2, \\
\text{TLsc}_{1,1,2}^{(0)}(\theta) &= -\text{C} \text{l}_1(\theta - \pi)^2 + \ln^2 2, \\
\text{TLsc}_{1,2,1}^{(0)}(\theta) &= \frac{1}{2} \text{Li}_2(\theta) + 2 \ln 2 \text{C} \text{l}_1(\theta - \pi) - \frac{\pi^2}{12} + 2 \ln^2 2, \\
\text{TLsc}_{1,2,1}^{(1)}(\theta) &= \frac{\theta}{2} \text{Li}_2(\theta) + 2 \theta \ln 2 \text{C} \text{l}_1(\theta - \pi) - 2 \ln 2 \text{Ls}_{2}^{(0)}(\theta - \pi) - \frac{1}{2} \text{L} \text{Li}_2^{(0)}(\theta), \\
\text{TLsc}_{1,2,2}^{(0)}(\theta) &= -\frac{1}{2} \text{C} \text{l}_1(\theta - \pi) \text{Li}_2(\theta) - \frac{\pi^2}{12} \ln 2 + \frac{1}{4} \left( \text{Li}_3(0) - \text{Li}_3(\theta) \right) \\
&\quad - \ln 2 \left( \text{C} \text{l}_1(\theta - \pi)^2 - \text{C} \text{l}_1(\pi)^2 \right),
\end{align*}
\]
APPENDIX B. HARMONIC POLYLOGARITHMS

B.2.2 Integration identities

The following identities are sufficient to analytically continue HPLs up to weight 4 on the complex unit circle, if applied iteratively and in conjunction with identities in B.2.1.

B.2.2.1 Identity involving $\text{Cl}_1$

$$\int_0^\theta d\theta' \theta'^j \tan^2 \theta' \frac{\theta'}{2} \text{Cl}_1(\theta')^j \text{Cl}_1(\theta' - \pi)^k = (-1)^{j+k} \text{TLsc}^{(i)}_{i, j+1, k+1} (\theta).$$

(B.52)

The $\text{TLsc}$ function has in many cases representations in terms of Clausen functions or generalized log-sine functions, see B.2.1.

B.2.2.2 Identities involving $L_s$

$$\int_0^\theta d\theta' \theta'^k \text{Ls}_j^{(p)}(\theta') = \frac{1}{k+1} \left\{ \theta^{k+1} \text{Ls}_j^{(p)}(\theta) - \text{Ls}_{j+1}^{(p+1)}(\theta) \right\},$$

(B.53)

$$\int_0^\theta d\theta' (\theta - \pi)^k \text{Ls}_j^{(p)}(\theta - \pi) = \frac{1}{k+1} \left\{ (\theta - \pi)^{k+1} \text{Ls}_j^{(p)}(\theta - \pi) - \text{Ls}_{j+1}^{(p+1)}(\theta - \pi) \right\}$$

(B.54)

$$\int_0^\theta d\theta' \theta'^k \text{Ls}_j^{(p)}(\theta' - \pi) = \frac{\theta^{k+1} - \pi^{k+1}}{k+1} \text{Ls}_j^{(p)}(\theta - \pi) + \frac{\pi^{k+1}}{k+1} \text{Ls}_j^{(p)}(\pi)$$

$$+ \sum_{l=0}^k \binom{k}{l} \frac{\pi^{k-l}}{l+1} \left( \text{Ls}_{j+1}^{(p+l+1)}(-\pi) - \text{Ls}_{j+1}^{(p+l+1)}(\theta - \pi) \right),$$

(B.55)

$$\int_0^\theta d\theta' \tan \frac{\theta'}{2} \text{Ls}_j^{(k)}(\theta' - \pi) = 2 \left\{ \text{Cl}_1(\theta - \pi) \text{Ls}_j^{(k)}(\theta - \pi) - \text{Cl}_1(-\pi) \text{Ls}_j^{(k)}(-\pi) - \text{Ls}_{j+1}^{(k)}(-\pi) + \text{Ls}_{j+1}^{(k)}(\theta - \pi) \right\},$$

(B.56)

$$\int_0^\theta d\theta' \tan \frac{\theta'}{2} \text{Ls}_j^{(k)}(\theta' - \pi) = 2 \left\{ \text{Cl}_1(\theta - \pi) \text{Ls}_j^{(k)}(\theta) - \text{Cl}_1(-\pi) \text{Ls}_j^{(k)}(0) + \text{LsLsc}_{0,j-k,2}^{(k-1)}(\theta) \right\},$$

(B.57)
B.2. ANALYTIC CONTINUATION OF HPLS FOR $X = \exp i\theta$

\[ \int_0^\theta d\theta' \theta'^k \tan \frac{\theta'}{2} L_s^{(0)}(\theta' - \pi) = 2 \theta^k \text{Cl}_1(\theta - \pi) L_s^{(0)}(\theta - \pi) \]
\[ + 2 \sum_{l=0}^{k} \binom{k}{l} \pi^{k-l} L_s L_s c_{l+1,1}^{(l-1)}(\theta) + 2k \sum_{l=0}^{k-1} \binom{k-1}{l} \pi^{k-l-1} L_s L_s c_{l+2,1}^{(l)}(\theta), \]

(B.58)

\[ \int_0^\theta d\theta' \theta'^k \tan \frac{\theta'}{2} L_s^{(0)}(\theta' - \pi) = 2 \theta^k \text{Cl}_1(\theta - \pi) L_s^{(0)}(\theta) + 2k L_s L_s c_{l-1,2}^{(k-1)}(\theta) \]
\[ + 2 L_s L_s c_{l+1,2}^{(k-1)}(\theta), \]

(B.59)

\[ \int_0^\theta d\theta' \theta'^k \text{Cl}_1(\theta' - \pi)^i \text{Cl}_1(\theta' - \pi)^j L_s^{(0)}(\theta' - \pi) = \]
\[ (-1)^{i+j} \sum_{l=0}^{k} \binom{k}{l} \pi^{k-l} \left(L_s L_s c_{l+1,i+1}^{(l)}(\theta - \pi) - L_s L_s c_{l+1,i+1}^{(l)}(-\pi) \right), \]

(B.60)

\[ \int_0^\theta d\theta' \theta'^k \text{Cl}_1(\theta')^i \text{Cl}_1(\theta' - \pi)^j L_s^{(0)}(\theta') = (-1)^{i+j} L_s L_s c_{l-1,i+1}^{(k)}(\theta), \]

(B.61)

\[ \int_0^\theta d\theta' \text{Cl}_1(\theta')^i \tan \frac{\theta'}{2} L_s^{(k)}(\theta' - \pi) = \left\{(-1)^i L_s^{(k)}(\theta' - \pi) TL_s^{(0)}_{l+1,0}(\theta') \right\}^{\theta}_{\theta'=0} \]
\[ - \int_0^\theta d\theta' \left\{(-1)^i (\theta' - \pi)^k \text{Cl}_1(\theta' - \pi)^j \pi^{j-k-1} TL_s^{(0)}_{l+1,0}(\theta') \right\} \quad \text{for } i > 0, \]

(B.62)

\[ \int_0^\theta d\theta' \text{Cl}_1(\theta')^i \tan \frac{\theta'}{2} L_s^{(k)}(\theta') = \left\{(-1)^i L_s^{(k)}(\theta') TL_s^{(0)}_{l+1,0}(\theta') \right\}^{\theta}_{\theta'=0} \]
\[ - \int_0^\theta d\theta' \left\{(-1)^i \theta^k \text{Cl}_1(\theta')^j \pi^{j-k-1} TL_s^{(0)}_{l+1,0}(\theta') \right\} \quad \text{for } i > 0, \]

(B.63)

\[ \int_0^\theta d\theta' \text{Cl}_1(\theta - \pi)^i \tan \frac{\theta'}{2} L_s^{(k)}(\theta' - \pi) = \frac{2}{i+1} \left\{\text{Cl}_1(\theta - \pi)^i+1 L_s^{(k)}(\theta - \pi) \right. \]
\[ - \left. \text{Cl}_1(-\pi)^i+1 L_s^{(k)}(-\pi) - \left(Ls s c_{l-1,k+1}^{(k-1)}(\pi) - Ls s c_{l-1,k+1}^{(k-1)}(\pi - \theta) \right) \right\}. \]

(B.64)

\[ \int_0^\theta d\theta' \text{Cl}_1(\theta' - \pi)^i \tan \frac{\theta'}{2} L_s^{(k)}(\theta') = \]
\[ \frac{2}{i+1} \left\{\text{Cl}_1(\theta - \pi)^i+1 L_s^{(k)}(\theta) + (-1)^i Ls s c_{l-1,k+1}^{(k-1)}(\theta) \right\}. \]

(B.65)

B.2.2.3 Identities involving $L_s$

\[ \int_0^\theta d\theta' L_s^{i,j}(\theta') = \theta L_s^{i,j}(\theta) - Ls s c_{l-1,i+1}^{(k)}(\theta), \]

(B.66)
\[
\int_0^\theta d\theta' \theta^k Lsc_{i,j}(\theta') = \frac{1}{k + 1} \left\{ \theta^{k+1} Lsc_{i,j}(\theta) - LsLsc_{0,i,j}^{(k)}(\theta) \right\}.
\] (B.67)

### B.2.2.4 Identities involving LsLsc

\[
\int_0^\theta d\theta' \theta^k LsLsc_{n,i,j}^{(p)}(\theta') = \frac{1}{k + 1} \left\{ \theta^{k+1} LsLsc_{n,i,j}^{(p)}(\theta) - LsLsc_{n,i,j}^{(p+k+1)}(\theta) \right\},
\] (B.68)

\[
\int_0^\theta d\theta' \tan \frac{\theta'}{2} LsLsc_{n,i,j}^{(k)}(\theta) = 2 Cl_1(\theta - \pi) LsLsc_{n,i,j}^{(k)}(\theta) + 2 LsLsc_{n,i,j+1}^{(k)}(\theta).
\] (B.69)

### B.2.2.5 Identities involving TLsc

\[
\int_0^\theta d\theta' TLsc_{n,i,j}^{(k)}(\theta') = \theta TLsc_{n,i,j}^{(k)}(\theta) - TLsc_{n,i,j}^{(k+1)}(\theta),
\] (B.70)

\[
\int_0^\theta d\theta' Cl_1(\theta' - \pi) \tan \frac{\theta'}{2} TLsc_{n,i,j}^{(k)}(\theta) = Cl_1(\theta - \pi)^2 TLsc_{n,i,j}^{(k)}(\theta) - TLsc_{n,i,j+2}^{(k)}(\theta).
\] (B.71)

### B.2.2.6 Identities involving Lic and LLic

\[
\int_0^\theta d\theta' \theta^k Lic_{n}(\theta' - \pi) = \sum_{l=0}^{k} \binom{k}{l} \pi^{k-l} \left\{ LLic_{n}^{(l)}(\theta - \pi) - LLic_{n}^{(l)}(-\pi) \right\},
\] (B.72)

\[
\int_0^\theta d\theta' \theta^k Lic_{n}(\theta') = LLic_{n}^{(k)}(\theta),
\] (B.73)

\[
\int_0^\theta d\theta' \theta^k \tan \frac{\theta'}{2} Lic_{n}(\theta') = \theta^k \left\{ Lic_{n+1}(0) - Lic_{n+1}(\theta) \right\}
+ k LLic_{n+1}^{(k)}(\theta) + Lic_{n+1}(0) \theta^k \quad \text{for } k \geq 0,
\] (B.74)

\[
\int_0^\theta d\theta' LLic_{n}^{(l)}(\theta') = \theta LLic_{n}^{(l)}(0) - LLic_{n}^{(l)}(\theta),
\] (B.75)

\[
\int_0^\theta d\theta' \tan \frac{\theta'}{2} LLic_{n}^{(0)}(\theta') = 2 Cl_1(\theta - \pi) LLic_{n}^{(0)}(\theta) - 2 U_{n,0,1}^{(k)}(\theta),
\] (B.76)

\[
\int_0^\theta d\theta' \tan \frac{\theta'}{2} Cl_1(\theta') Lic_{2}(\theta') = \ln 2 Lic_3(\theta) - \frac{1}{4} Lic_2(\theta)^2 - \zeta_3 \ln 2 + \frac{\pi^4}{144},
\] (B.77)

\[
\int_0^\theta d\theta' \tan \frac{\theta'}{2} Cl_1(\theta' - \pi) Lic_{2}(\theta') = Cl_1(\theta - \pi)^2 Lic_2(\theta) - Cl_1(\pi)^2 Lic_2(1) - \int_0^\theta d\theta' Cl_1(\theta' - \pi)^2 \tan \frac{\theta'}{2} (-2 Cl_1(\theta') - 2 \ln 2),
\] (B.78)
\[\int_0^\theta d\theta' \tan \frac{\theta'}{2} \text{Lic}_2(\theta' - \pi) = (\text{Lic}_3(\theta) - \text{Lic}_3(0))\]

\[+ \int_0^\theta d\theta' \tan \frac{\theta'}{2} \left\{ \frac{\pi^2}{6} + (2 \text{Cl}_1(\theta' - \pi) + 2 \ln 2) (2 \text{Cl}_1(\theta') + 2 \ln 2) \right\},\]

\[\int_0^\theta d\theta' \tan \frac{\theta'}{2} \text{Lic}_3(\theta' - \pi) = 2 \text{Cl}_1(\theta - \pi) \text{Lic}_3(\theta - \pi) + 2 \ln 2 \text{Lic}_3(\theta - \pi) - \frac{1}{2} \text{Lic}_2(\theta - \pi)^2.\]  

**(B.79)**

**(B.80)**

**B.2.3 Numerical evaluation of \(L_s, L_{sc}\) and \(L_sL_{sc}\) functions**

There exists a code in C++ for evaluating \(L_s\) functions described in [111]. We have implemented the algorithm in FORTRAN 77 and extended it for evaluating \(L_{sc}\) and \(L_sL_{sc}\) functions. In the coming sections we derive a suitable series expansion valid in a certain region to which all other arguments can be mapped.

We have found full agreement with [113], an implementation of HPLs in C++ using the strategy of [48] for extending HPLs to complex arguments.

**B.2.3.1 \(L_s\) function**

First, we map the range \(-2\pi < \theta < 2\pi\) to the range \(0 \leq \theta \leq \pi\) using

\[L_s^{(k)}(-\theta) = (-1)^{k+1} L_s^{(k)}(\theta),\]

\[L_s^{(r)}(2\pi - \theta) = L_s^{(r)}(2\pi) + (-1)^{(r-1)} L_s^{(r)}(\theta) - \sum_{p=1}^r (-1)^{r-p}(2\pi)^p \binom{r}{p} L_s^{(r-p)}(\theta).\]

**(B.81)**

**(B.82)**

Then we follow [111] to find a series expansion around \(\theta = 0\). In doing so, we need the series expansion of \(\text{arcsin}^n y\), which is given by

\[
\frac{\text{arcsin}^n y}{n!} = \begin{cases} 
\sum_{m=1}^\infty \frac{((m-1)!)^2}{(2m)!} 4^{m-1} y^{2m} B\left(m, \frac{n+1}{2}\right) & \text{if } n \text{ is even}, \\
\sum_{m=0}^\infty \binom{2m}{m} \frac{y^{2m+1}}{4^m(2m+1)} C\left(m, \frac{n+1}{2}\right) & \text{if } n \text{ is odd}
\end{cases}
\]

**(B.83)**

\[
\frac{\text{arcsin}^{2k} y}{(2k)!} = \sum_{m=1}^\infty \frac{((m-1)!)^2}{(2m)!} 4^{m-1} y^{2m} B(m, k),
\]

**(B.84)**

\[
\frac{\text{arcsin}^{2k+1} y}{(2k+1)!} = \sum_{m=0}^\infty \binom{2m}{m} \frac{y^{2m+1}}{4^m(2m+1)} C(m, k+1)
\]

**(B.85)**
with coefficients $B(m, k)$ and $C(m, k)$ given in [111] in terms of harmonic sums. We will also apply

$$\int dx x^a \ln^b x = (-1)^b b! x^{a+1} \sum_{p=0}^{b} \frac{(-\ln x)^{b-p}}{(b-p)!(a+1)^{p+1}}. \quad (B.86)$$

Defining $y = \sin \frac{\theta}{2}$ we can write

$$L_{s}^{(k)}(\theta) = -2^{k+1} \int_{0}^{\sin \theta/2} dy \arcsin^{k} y \sqrt{1-y^2} \ln^{j-k-1}(2y). \quad (B.87)$$

After integration by parts, inserting the series expansion for $\arcsin$ and applying the integration identity above, we find

$$L_{s}^{2k}(\theta) = (-1)^j \left(2 \sin \frac{\theta}{2}\right)^{2k} (2k)!(j-2k-1)! \sum_{p=0}^{j-2k-1} \frac{(-\ln (2 \sin \frac{\theta}{2}))^p}{p!} \times \sum_{m=0}^{\infty} \frac{(2m)^m C(m, k+1)}{16^m (2m+1)^{2k-p}}. \quad (B.88)$$

$$L_{s}^{2k+1}(\theta) = (-1)^{j-1} 2^{2k+2} (2k+1)!(j-2k-2)! \sum_{p=0}^{j-2k-2} \frac{(-\ln (2 \sin \frac{\theta}{2}))^p}{p!} \times \sum_{m=1}^{\infty} \frac{1}{(2m)^m} \frac{(2 \sin \frac{\theta}{2})^{2m} B(m, k+1)}{(2m)^{j-2k-p}}. \quad (B.89)$$

This series expansion suffers from a low convergence for $\theta \approx \pi$. Therefore we choose to use another approach for such $\theta$ close to $\pi$. We write

$$L_{s}^{(k)}(\theta) = -\theta^{k+1} \int_{0}^{1} dx x^k \ln^{j-k-1} 2 \left| \sin \frac{x\theta}{2} \right| \quad (B.90)$$

and expand in $\theta$ around $\theta = \pi$. We then evaluate the coefficients numerically for that region.

### B.2.3.2 $L_{sc}$ function

Again, we first map the range $-2\pi < \theta < 2\pi$ to a smaller range. In this case the smaller range is given by $0 \leq \theta \leq \pi$ and the required symmetry properties are:

$$L_{sc_{i,j}}(-\theta) = L_{sc_{i,j}}(\theta), \quad (B.91)$$

$$L_{sc_{i,j}}(2\pi - \theta) = L_{sc_{i,j}}(2\pi) - L_{sc_{i,j}}(\theta), \quad (B.92)$$

$$L_{sc_{i,j}}(\pi - \theta) = L_{sc_{i,j}}(\pi) - L_{sc_{i,j}}(\theta). \quad (B.93)$$
For $0 < \theta < \frac{\pi}{2}$ we can write

$$L_{sc_{i,j}}(\theta) = -2 \int_0^{\sin \theta/2} \frac{1}{\sqrt{1-y^2}} \ln^{i-1}(2y) \ln^{j-1}(2\sqrt{1-y^2})$$

(B.94)

and after integration by parts, we insert the following expansions

$$\arcsin y = \sum_{m=0}^{\infty} \binom{2m}{m} \frac{y^{2m+1}}{4^m (2m+1)},$$

(B.95)

$$\frac{1}{1-y^2} = \sum_{m=0}^{\infty} y^{2m},$$

(B.96)

$$\ln^k 4(1-y^2) = \sum_{m=0}^{\infty} A_{k,m} y^{2m}$$

(B.97)

with

$$A(m,0) = \delta_{0,m},$$

(B.98)

$$A(m,1) = \begin{cases} \ln 4 & \text{if } m = 0, \\ -\frac{1}{m} & \text{if } m > 0, \end{cases}$$

(B.99)

$$A(m,2) = \begin{cases} \ln^2 4 & \text{if } m = 0, \\ -2\ln4 + \frac{2}{m} \sum_{l=1}^{m-1} \frac{1}{l} & \text{if } m > 0. \end{cases}$$

(B.100)

We write the central binomial values as follows

$$\binom{2m}{m} = \frac{2^m (2m-1)!!}{m!}$$

(B.101)

and find eventually the following series expansion,

$$L_{sc_{i,j}}(\theta) = -2^{1-j} \theta \ln^{i-1} 2 \sin \frac{\theta}{2} \ln^{j-1} 4 \cos^2 \frac{\theta}{2} + 2^{1-j} (-1)^{i-2}(i-1)!$$

$$\times \sum_{p=0}^{i-1} \frac{(-\ln 2 \sin \frac{\theta}{2})^{i-1-p}}{(i-1-p)!} \sum_{n=0}^{\infty} \left( \sin \frac{\theta}{2} \right)^{2n+1} \left( a_n^{(j,p)} + (j-1) b_n^{(j,p)} \right)$$

where

$$a_n^{(j,p)} = \sum_{m=0}^{n} \frac{(2m-1)!! (n-m, j-1)}{m!(2n+1)^p (2m+1) 2^{m-1}},$$

$$b_n^{(j,p)} = \sum_{m=0}^{n} \frac{(2m-1)!!}{m!(2n+1)^{p+1} (2m+1) 2^{m-2} \sum_{l=0}^{n-m} A(l, j-2)}.$$  

(B.102)

This is now suited for numerical implementation.
B.2.3.3 $LsLsc$ function

We only consider the $LsLsc_{1,2} = LsLsc_{1,2}(0)$ function here, since it is the only one appearing in the master integrals. Also in this case we can map $-2\pi < \theta < 2\pi$ to $0 \leq \theta \leq \frac{\pi}{2}$ using

\[
LsLsc_{1,2}(-\theta) = LsLsc_{1,2}(\theta),
\]

\[
LsLsc_{1,1,2}(2\pi - \theta) = LsLsc_{1,1,2}(\theta),
\]

\[
LsLsc_{1,1,2}(\pi - \theta) = Ls(0)\theta Ls(0)\theta + LsLsc_{1,1,2}(\pi) - LsLsc_{1,1,2}(\theta).
\]

To derive a series expansion around $\theta = 0$, we first integrate by parts and then substitute $y = \sin \theta'/2$ and find

\[
LsLsc_{1,1,2}(\theta) = \theta Ls(0)\theta \ln 2 \left| \cos \frac{\theta'}{2} \right| + \frac{1}{4} \theta^2 \ln^2 2 \sin \frac{\theta}{2} \ln 4 \cos \frac{\theta}{2}
\]

\[
- \int_0^{\sin \theta/2} dy \frac{\arcsin^2 y}{y} \ln 4(1 - y^2) + 2 \int_0^{\sin \theta/2} dy \frac{y \arcsin^2 y}{1 - y^2} \ln(2y)
\]

\[
+ 2 \int_0^{\sin \theta/2} dy \frac{y \arcsin y}{1 - y^2} Ls(0)\theta (\arcsin y)
\]

We then proceed by inserting series expansions for $\arcsin$ and $Ls(0)\theta$. After some rearrangements we find

\[
LsLsc_{1,1,2}(\theta) = \frac{\theta}{2} Ls(0)\theta \ln 4 \cos^2 \frac{\theta}{2} + \frac{\theta^2}{4} \ln^2 2 \sin \frac{\theta}{2} \ln 4 \cos^2 \frac{\theta}{2}
\]

\[
+ \sum_{n=1}^{\infty} \left( 2 \sin \frac{\theta}{2} \right)^{2n} \left( \alpha_1^{(n)} + \alpha_2^{(n)} + \alpha_3^{(n)} \right),
\]

\[
+ \sum_{n=2}^{\infty} \left( 2 \sin \frac{\theta}{2} \right)^{2n} \left( \beta_1^{(n)} + \beta_2^{(n)} \right),
\]

where for $n \geq 0$

\[
\alpha_1^{(n)} = \begin{cases} 0, & \text{if } n < 2, \\ \sum_{m=0}^{n-2} \sum_{k=0}^{n-m-2} \frac{(2m-1)!!(2k-1)!!(2n+2m+1)}{m!k!2^{2n+2m+1}(2m+1)(2k+1)n} & \text{if } n \geq 2, \end{cases}
\]

\[
\alpha_2^{(n)} = \begin{cases} 0, & \text{if } n < 2, \\ -\sum_{m=1}^{n-1} \frac{((2m-1)!!)^2}{(2m)!2^{2m-2m+2}} & \text{if } n \geq 2, \end{cases}
\]

\[
\alpha_3^{(n)} = \begin{cases} 0, & \text{if } n < 2, \\ -\sum_{m=1}^{n-1} \frac{((2m-1)!!)^2}{(2m)!2^{2m-2m+2}} & \text{if } n \geq 2, \end{cases}
\]

\[
\beta_1^{(n)} = -\sum_{m=0}^{n-2} \sum_{k=0}^{n-m-2} \frac{(2m-1)!!(2k-1)!!}{m!k!2^{2n+2m+k-1}(2m+1)(2k+1)n},
\]

\[
\beta_2^{(n)} = \sum_{m=1}^{n-1} \frac{((2m-1)!!)^2}{(2m)!2^{2n-2m+1}n},
\]

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Appendix C

Scalar two loop integrals

In this Appendix we collect the results for all master integrals in terms of HPLs. For reference we repeat here the definitions of the propagator factors, equation (4.1)

\[
\begin{align*}
D_{11} &= k^2 \\
D_{12} &= (k + p_1)^2 \\
D_{13} &= (k + p_{12})^2 \\
D_{14} &= (l + p_{12})^2 - m^2 \\
D_{15} &= (l + p_1)^2 - m^2 \\
D_{16} &= l^2 - m^2 \\
D_{17} &= (k - l)^2 - m^2 \\
D_{21} &= k^2 - m^2 \\
D_{22} &= (k + p_2)^2 - m^2 \\
D_{23} &= (k + p_{12})^2 - m^2 \\
D_{24} &= (l + p_{12})^2 - m^2 \\
D_{25} &= (l + p_2)^2 - m^2 \\
D_{26} &= l^2 - m^2 \\
D_{27} &= (k - l)^2 \\
D_{31} &= k^2 - m^2 \\
D_{32} &= (k - l - p_1)^2 \\
D_{33} &= (k + p_{12})^2 - m^2 \\
D_{34} &= (l + p_{12})^2 - m^2 \\
D_{35} &= (l + p_1)^2 - m^2 \\
D_{36} &= l^2 - m^2 \\
D_{37} &= (k - l)^2
\end{align*}
\]

(C.1)

C.1 One loop integrals

\[
\begin{align*}
\int \frac{d^d k}{i \pi^{d/2}} & \frac{1}{k^2 - m^2 + i \epsilon} = \frac{\Gamma(1 + \epsilon)}{1 - \epsilon} \left( m_T^2 \right)^{-\epsilon + 1} \frac{1}{\epsilon} \\
\int \frac{d^d k}{i \pi^{d/2}} \frac{1}{D_{11} D_{13}} & = \frac{\Gamma(1 + \epsilon)}{1 - \epsilon} \left( m_T^2 \right)^{-\epsilon} \cdot \frac{\Gamma(2 - \epsilon)}{\epsilon} \frac{\Gamma(1 - \epsilon)}{\Gamma(2 - 2\epsilon)} \left( \frac{(1 - x)^2}{x} - i \epsilon \right)^{-\epsilon}
\end{align*}
\]
\[ \int \frac{d^4k}{1+\epsilon} \frac{1}{2} = \int \frac{d^4k}{1+\epsilon} \frac{1}{D_{21}D_{22}D_{23}} = \frac{\Gamma(1+\epsilon)}{1-\epsilon} \frac{(m_t^2)^{-\epsilon} - \epsilon \sum_{i=-1}^{3} \epsilon^i F_{mub}^i(x) + O(\epsilon^4)} {m_t^2} \] (C.4)

\[ F_{mub}^{-1}(x) = 1 \] (C.5)
\[ F_{mub}^{0}(x) = \frac{1}{1-x} \left\{ -x + (x+1)H(0;x) + 1 \right\} \] (C.6)
\[ F_{mub}^{1}(x) = \frac{1}{1-x} \left\{ \frac{1}{6} (\pi^2 x - 12x - \pi^2 + 12) + (x+1)H(0;x) 
- 2(x+1)H(-1,0;x) + (x+1)H(0,0;x) \right\} \] (C.7)
\[ F_{mub}^{2}(x) = \frac{1}{1-x} \left\{ -\frac{1}{6} \pi^2(x+1) - 2((2 + \zeta(3)x + \zeta(3) - 2) 
+ \frac{1}{3} \pi^2(x+1)H(-1;x) - \frac{1}{6} (-12 + \pi^2)(x+1)H(0;x) 
- 2(x+1)H(0,-1,0;x) - 2(x+1)H(-1,0;x) + (x+1)H(0,0;x) 
+ 4(x+1)H(-1,-1,0;x) - 2(x+1)H(-1,0,0;x) + (x+1)H(0,0,0;x) \right\} \] (C.8)
\[ F_{mub}^{3}(x) = \frac{1}{1-x} \left\{ -\frac{1}{40} \pi^4(x+1) - \frac{1}{3} \pi^2(x+1) - 2((4 + \zeta(3)x + \zeta(3) - 4) 
+ \frac{1}{3} \pi^2(x+1)H(0,-1;x) + \frac{1}{3} (x+1)H(-1;x) (\pi^2 + 12\zeta(3)) 
- \frac{1}{6} (x+1) (\pi^2 + 12(-2 + \zeta(3)))H(0;x) - 2(x+1)H(0,0,-1,0;x) 
- 2(x+1)H(0,-1,0;x) - \frac{2}{3} \pi^2(x+1)H(-1,-1;x) 
+ \frac{1}{3} (-12 + \pi^2) (x+1)H(-1,0;x) - \frac{1}{6} (-12 + \pi^2)(x+1)H(0,0;x) 
+ 4(x+1)H(0,-1,-1,0;x) - 2(x+1)H(0,-1,0,0;x) 
+ 4(x+1)H(-1,-1,0,0;x) + 4(x+1)H(-1,-1,0;x) 
- 2(x+1)H(-1,0,0;x) + (x+1)H(0,0,0;x) 
+ 4(x+1)H(-1,-1,0,0;x) - 8(x+1)H(-1,-1,0,0;x) + (x+1)H(0,0,0,0;x) \right\} \] (C.9)
C.2. FACTORIZABLE INTEGRALS

\[ F_{\text{mtri}_0}(x) = -\frac{xH(0, 0; x)}{(x - 1)^2} \]  
\[ F_{\text{mtri}_1}(x) = \frac{x}{(1 - x)^2} \left\{ \frac{1}{6}\pi^2 H(0; x) + 2H(0, -1, 0; x) + H(0, 0; x) + 3\zeta(3) \right. \]
\[ - H(0, 0, 0; x) \right\} \]  
\[ F_{\text{mtri}_2}(x) = \frac{x}{(1 - x)^2} \left\{ -\frac{1}{3}\pi^2 H(0, -1; x) + H(0; x) \left( -\frac{\pi^2}{6} + 2\zeta(3) \right) - 3\zeta(3) + \frac{\pi^4}{72} \right. \]
\[ + 2H(0, 0, -1, 0; x) - 2H(0, -1, 0; x) + \frac{1}{6}\pi^2 H(0, 0; x) \]
\[ - 4H(0, -1, -1, 0; x) + 2H(0, -1, 0, 0; x) + H(0, 0, 0; x) \]
\[ - H(0, 0, 0; x) \right\} \]  

C.2 Factorizable integrals

\[ \oint = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_{15} D_{17}} = \oint \times \oint \]  

\[ = \int \frac{d^d k}{i\pi^{d/2}} \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{D_{11} D_{13} D_{14} D_{16}} \]
\[ = \oint \times \oint \]  

\[ = \int \frac{d^d k}{i\pi^{d/2}} \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{D_{11} D_{13} D_{16}} \]
\[ = \oint \times \oint \]  

\[ = \int \frac{d^d k}{i\pi^{d/2}} \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{D_{14} D_{16} D_{17}} \]
\[ = \oint \times \oint \]  

\[ = \int \frac{d^d k}{i\pi^{d/2}} \int \frac{d^d l}{i\pi^{d/2}} \frac{1}{D_{14} D_{15} D_{16} D_{17}} \]
\[ = \oint \times \oint \]
\[ \begin{align*}
\text{APPENDIX C. SCALAR TWO LOOP INTEGRALS} \\
\end{align*} \]

\[ \begin{align*}
\mathcal{C}_1 & = \int \frac{d^d k}{i \pi^{d/2}} \int \frac{d^d l}{i \pi^{d/2}} \frac{1}{D_{21} D_{23} D_{24} D_{26}} \\
& = \mathcal{C}_1 \\
& = \mathcal{C}_1 \quad \text{(C.19)}
\end{align*} \]

\[ \begin{align*}
\mathcal{C}_2 & = \int \frac{d^d k}{i \pi^{d/2}} \int \frac{d^d l}{i \pi^{d/2}} \frac{1}{D_{21} D_{23} D_{24} D_{25} D_{26}} \\
& = \mathcal{C}_2 \\
& = \mathcal{C}_2 \quad \text{(C.20)}
\end{align*} \]

\section*{C.3 Three propagator integrals}

\[ \begin{align*}
\mathcal{C}_3 & = \int \frac{d^d k}{i \pi^{d/2}} \int \frac{d^d l}{i \pi^{d/2}} \frac{1}{D_{11} D_{14} D_{17}^2} \\
& = \left( \frac{\Gamma(1 + \epsilon)}{1 - \epsilon} \right)^2 \frac{1}{(m_2^2)^{2\epsilon - 1}} \sum_{\epsilon_i} e^{i F_i(x)} + O(\epsilon^3) \quad \text{(C.21)}
\end{align*} \]

\[ \begin{align*}
F_1^0(x) & = -\frac{2x H(0, 0; x)}{(x - 1)^2} \quad \text{(C.22)} \\
F_1^1(x) & = \frac{x}{(1 - x)^2} \left\{ \frac{1}{3} \pi^2 H(0; x) + 12 H(0, -1, 0; x) + 4 H(0, 0; x) + 6 \zeta(3) \\
& - 4 H(0, 1, 0; x) - 6 H(0, 0, 0; x) + 4 H(1, 0, 0; x) \right\} \quad \text{(C.23)} \\
F_1^2(x) & = \frac{x}{(1 - x)^2} \left\{ -2 \pi^2 H(0, -1; x) - 12 \zeta(3) + \frac{13 \pi^4}{180} \\
& + \left( \frac{2 \pi^2}{3} + 16 \zeta(3) \right) H(0; x) + \frac{2}{3} \pi^2 H(0, 1; x) - 12 H(1; x) \zeta(3) \\
& + 36 H(0, 0, -1, 0; x) - 24 H(0, -1, 0; x) + (-2 + \pi^2) H(0, 0; x) \\
& - \frac{2}{3} \pi^2 H(1, 0; x) + 8 H(0, 1, 0; x) - 12 H(0, 0, 1, 0; x) \\
& - 72 H(0, -1, -1, 0; x) + 48 H(0, -1, 0, 0; x) + 24 H(0, -1, 1, 0; x) \\
& + 12 H(0, 0, 0, 0; x) - 24 H(1, 0, -1, 0; x) - 8 H(1, 0, 0; x) \right\}
\end{align*} \]
C.3. Three Propagator Integrals

\[ +8H(1, 0, 1, 0; x) + 24H(0, 1, -1, 0; x) - 20H(0, 1, 0, 0; x) \]
\[ -8H(0, 1, 1, 0; x) - 14H(0, 0, 0, 0; x) + 12H(1, 0, 0, 0; x) \]
\[ -8H(1, 1, 0, 0; x) \}\right) \) \quad (C.24)

\[ \int \frac{d^d l}{i \pi^{d/2}} \int \frac{d^d l}{i \pi^{d/2}} \frac{1}{D_{11}^2 D_{14}^2 D_{17}^2} \]
\[ = \left( \frac{\Gamma(1 + \epsilon)}{1 - \epsilon} \right)^2 (m_1^2)^{-2\epsilon - 1} \sum_{i=-1}^{2} \epsilon^i F_2^i(x) + O(\epsilon^3) \) \quad (C.25)

\[ F_2^{-1}(x) = \frac{xH(0; x)}{x^2 - 1} \] \quad (C.26)

\[ F_2^0(x) = \frac{x}{(x - 1)^3(x + 1)} \left\{ -2H(0; x)(x - 1)^2 - \frac{1}{6} \pi^2(x - 1)^2 \\
+ (x - 1)(5x - 3)H(0; x) - 6(x - 1)^2H(-1; 0; x) \right\} \] \quad (C.27)

\[ F_2^1(x) = \frac{x}{(x - 1)^3(x + 1)} \left\{ \frac{1}{3} (x - 1) (6\zeta(3)(4 - 7x) + \pi^2(x - 1)) \\
+ \pi^2(x - 1)^2H(-1; x) - \frac{1}{6} (x - 1) (-6x + \pi^2(5x - 3) + 6) H(0; x) \\
- \frac{1}{3} \pi^2H(1; x)(x - 1)^2 - 6(5x - 3)H(0, -1, 0; x)(x - 1) \\
+ 12(x - 1)^2H(-1, 0; 0) - 2(x - 1)(5x - 3)H(0, 0; x) \\
+ 2(x - 1)(5x - 3)H(0, 1, 0; x) - 4(x - 1)^2H(1, 0; x) \\
+ 36(x - 1)^2H(-1, -1, 0; x) - 24(x - 1)^2H(-1, 0; 0; x) \\
+ (x - 1)(13x - 7)H(0, 0, 0; x) - 12(x - 1)^2H(-1, 1, 0; x) \\
+ 2(x - 1)(3x - 5)H(1, 0, 0; x) - 12(x - 1)^2H(1, -1, 0; x) \\
+ 4(x - 1)^2H(1, 1, 0; x) \} \right\} \] \quad (C.28)

\[ F_2^2(x) = \frac{x}{(x - 1)^3(x + 1)} \left\{ \frac{x - 1}{360} (60\pi^2(x - 1) + \pi^4(61x - 35) \\
- 1440(7x - 4)\zeta(3)) + \pi^2(x - 1)(5x - 3)H(0, -1; x) \\
- 2(x - 1)^2H(-1; x) (\pi^2 - 33\zeta(3)) + \frac{1}{3} (x - 1)H(0; x)(6\zeta(3))(9 - 17x) \\
+ \pi^2(5x - 3)) + \frac{2}{3} (x - 1)H(1; x) (6\zeta(3)(7 - 4x) + \pi^2(x - 1)) \\
- \frac{1}{3} \pi^2(x - 1)(5x - 3)H(0, 1; x) - 6(x - 1)(13x - 7)H(0, 0, -1, 0; x) \]
C.4 Four propagator integrals

\[ C = \mathcal{O}(\epsilon^4) \]

\[ \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \frac{1}{D_{11}D_{14}D_{15}D_{17}} \]

\[ = \left( \frac{\Gamma(1+\epsilon)}{1-\epsilon} \right)^2 (m^2_0)^{-2\epsilon} \sum_{i=-2}^{1} \epsilon^i F^i_n(x) + \mathcal{O}(\epsilon^2) \]
\[ F_3^{-2}(x) = \frac{1}{2} \]  
\[ F_3^{-1}(x) = -\frac{1}{2} \]  
\[ F_3^0(x) = \frac{1}{(1-x)^2} \left\{ -3x^2 + (6 - 4\zeta(3))x + 2\left(x^2 - 1\right) H(0; x) - 3 \right. \\
\left. - H(0, 0; x)(x - 1)^2 + 2x H(0, 0, 0; x) + 4x H(1, 0, 0; x) \right\} \]  
\[ F_3^1(x) = \frac{1}{(1-x)^2} \left\{ 3(-4 + \zeta(3))x^2 - 2(-12 + \zeta(3))x + \frac{2\pi^4 x}{45} - \frac{1}{3}\pi^2 (x^2 - 1) \right. \\
\left. + 3(-4 + \zeta(3)) + \frac{1}{6} H(0; x) (\pi^2 (x - 1)^2 + 12 (4x^2 - 3\zeta(3)x - 4)) \right. \\
\left. - 12x H(1; x)\zeta(3) + 6H(0, -1, 0; x)(x - 1)^2 - 12x H(0, 0, -1, 0; x) \right. \\
\left. - 12 (x^2 - 1) H(-1, 0; x) + \left(11x^2 - \frac{1}{3}(6 + \pi^2)x - 5\right) H(0, 0; x) \right. \\
\left. + \left(4x^2 - \frac{2\pi^2 x}{3} - 4\right) H(1, 0; x) - 2H(0, 1, 0; x)(x - 1)^2 \right. \\
\left. + 4x H(0, 0, 1, 0; x) + (-3x^2 + 4x - 3) H(0, 0, 0; x) \right. \\
\left. - 24x H(1, 0, -1, 0; x) + 2 (x^2 - 4x + 1) H(1, 0, 0; x) \right. \\
\left. + 8x H(1, 0, 1, 0; x) - 4x H(0, 1, 0, 0; x) + 6x H(0, 0, 0, 0; x) \right. \\
\left. + 12x H(1, 0, 0, 0; x) - 8x H(1, 1, 0, 0; x) \right\} \]  
\[ = \int \frac{d^4k}{(2\pi)^d} \int \frac{d^4l}{(2\pi)^d} \frac{1}{D_{12}D_{14}D_{16}D_{17}} \]  
\[ = \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m_1^{2\epsilon})^{-2\epsilon} \sum_{i=-2}^{1} \epsilon^i F_4^i(x) + \mathcal{O}(\epsilon^2) \]  
\[ F_4^{-2}(x) = \frac{1}{2} \]  
\[ F_4^{-1}(x) = \frac{1}{(1-x)^2} \left\{ \frac{3}{2}(x - 1)^2 + (1 - x^2) H(0; x) \right\} \]  
\[ F_4^0(x) = \frac{1}{(1-x)^2} \left\{ 5x^2 + 2(-5 + 2\zeta(3))x + \left(-3x^2 + \frac{\pi^2 x}{3} + 3\right) H(0; x) + 5 \right. \\
\left. + 2 (x^2 - 1) H(-1, 0; x) - (x - 1)(x + 2) H(0, 0; x) \right. \\
\left. + (1 - x^2) H(1, 0, x) + 2x H(0, 1, 0, x) + x H(0, 0, 0, x) \right\} \]  
\[ F_4^1(x) = \frac{1}{(1-x)^2} \left\{ - (\zeta(3) - 16)x^2 + \left(-32 - \frac{11\pi^4}{90} + \zeta(3)\right)x - 4(\zeta(3) - 4) \right\} \]
\[ + H(0; x) \left( -10x^2 - \frac{1}{6} \pi^2 (x + 1)x - 3\zeta(3)x + 10 \right) - \frac{2}{3} \pi^2 x H(0, -1; x) + \frac{1}{3} \pi^2 x H(0, 1; x) + \frac{1}{6} H(1; x) \left( -48\zeta(3)x - \pi^2 (x^2 - 1) \right) - 2x H(0, 0, -1, 0; x) + 2(x - 1)(x + 2) H(0, -1, 0; x) + 6 \left( x^2 - 1 \right) H(-1, 0; x) + \left( -3x^2 + \frac{1}{6} \left( -18 + \pi^2 \right) x + 6 \right) H(0, 0; x) + \left( -3x^2 - \frac{2\pi^2 x}{3} + 3 \right) H(1, 0; x) - 2 \left( x^2 + x - 1 \right) H(0, 1; x) + 2x H(0, 0, 1, 0; x) - 2x H(0, -1, 0, 0; x) - 4x H(0, -1, 1, 0; x) + \left( 4 - 4x^2 \right) H(-1, -1, 0; x) + 3 \left( x^2 - 1 \right) H(-1, 0, 0; x) + 2 \left( x^2 - 1 \right) H(1, -1, 0; x) + \left( -x^2 - 2x + 3 \right) H(1, 0, 0; x) + \left( 2 - 2x^2 \right) H(1, 1, 0; x) - 4x H(1, 0, 1, 0; x) - 4x H(0, 1, -1, 0; x) + 4x H(0, 1, 0, 0; x) + 4x H(0, 1, 1, 0; x) + 3x H(0, 0, 0, 0; x) - 2x H(1, 0, 0, 0; x) \right) \] (C.39)

\[
\int \frac{d^d k}{i \pi^{d/2}} \int \frac{d^d l}{i \pi^{d/2}} \frac{(k + p_1) \cdot (l - k)}{D_1 D_2 D_4 D_6 D_7} = \left( \frac{\Gamma(1 + \epsilon)}{1 - \epsilon} \right)^2 (m_t^2)^{1-2\epsilon} \sum_{i=-2}^{1} \epsilon^i F_5^i(x) + \mathcal{O}(\epsilon^2) \] (C.40)

\[
F_5^{-2}(x) = \frac{(x - 1)^2}{8x} \] (C.41)

\[
F_5^{-1}(x) = \frac{1}{(1 - x)^2 x} \left\{ \frac{5}{16} (x - 1)^4 + \left( -\frac{x^4}{4} + x^3 - x + \frac{1}{4} \right) H(0; x) \right\} \] (C.42)

\[
F_5^{0}(x) = \frac{1}{(1 - x)^2 x} \left\{ \frac{1}{32} (31x^4 - 140x^3 + (218 - 64\zeta(3))x^2 - 140x + 31) \\
+ \frac{1}{24} (-15x^4 + 54x^3 - 4\pi^2 x^2 - 54x + 15) H(0; x) \\
+ \frac{1}{2} (x^4 - 4x^3 + 4x - 1) H(-1, 0; x) + \left( 1 - \frac{x^4}{4} + x^3 - x \right) H(1, 0; x) \\
+ \frac{1}{4} (2 - x^4 + 4x^3 + 3x^2 - 8x) H(0, 0; x) \\
- H(0, 1, 0; x)x^2 - \frac{1}{2} H(0, 0, 0; x)x^2 \right\} \] (C.43)
\[ F_0^1(x) = \frac{1}{(1-x)^2} \left\{ \frac{189}{64} - \frac{\zeta(3)}{4} \right\} x^4 + \left( -\frac{233}{16} + \frac{\pi^2}{24} + \zeta(3) \right) x^3 \]
\[ + \left( \frac{743}{32} + \frac{11\pi^4}{180} - \frac{3\zeta(3)}{4} \right) x^2 - \frac{1}{48} (699 + 2\pi^2 - 192\zeta(3)) x - \zeta(3) \]
\[ + \frac{189}{64} + \frac{1}{3} \pi^2 x^2 H(0, -1; x) + \frac{1}{48} H(0; x) (-(93 + 2\pi^2) x^4 \]
\[ + (342 + 8\pi^2) x^3 + 6 (\pi^2 + 12\zeta(3)) x^2 - 342 x + 93 \]
\[ + \frac{1}{24} H(1; x) (96x^2\zeta(3) - \pi^2 (x^4 - 4x^3 + 4x - 1)) - \frac{1}{6} \pi^2 x^2 H(0, 1; x) \]
\[ + x^2 H(0, 0, -1, 0; x) + \frac{1}{2} (x^4 - 4x^3 - 3x^2 + 8x - 2) H(0, -1, 0; x) \]
\[ + \frac{1}{4} (5x^4 - 18x^3 + 18x - 5) H(-1, 0; x) \]
\[ + \left( -\frac{5x^4}{8} + 2x^3 - \frac{1}{24} (-57 + 2\pi^2) x^2 - 5x + \frac{5}{4} \right) H(0, 0; x) \]
\[ + \frac{1}{24} (-15x^4 + 60x^3 + 8\pi^2 x^2 - 60x + 15) H(1, 0; x) \]
\[ + \frac{1}{2} (-x^4 + 4x^3 + 3x^2 - 4x + 1) H(0, 1, 0; x) - x^2 H(0, 0, 1, 0; x) \]
\[ + H(0, -1, 0, 0; x) x^2 + 2H(0, -1, 1, 0; x) x^2 \]
\[ + (-x^4 + 4x^3 - 4x + 1) H(-1, -1, 0; x) \]
\[ + \frac{3}{4} (x^4 - 4x^3 + 4x - 1) H(-1, 0, 0; x) \]
\[ + \frac{1}{2} (x^4 - 4x^3 + 4x - 1) H(-1, 1, 0; x) \]
\[ + \left( -\frac{x^4}{2} + 2x^3 + \frac{9x^2}{4} - 4x + 1 \right) H(0, 0, 0; x) \]
\[ + \frac{1}{2} (x^4 - 4x^3 + 4x - 1) H(1, -1, 0; x) \]
\[ + \left( -\frac{x^4}{4} + x^3 + \frac{3x^2}{2} - 3x + \frac{3}{4} \right) H(1, 0, 0; x) \]
\[ + 2H(1, 0, 1, 0; x) x^2 + \frac{1}{2} (-x^4 + 4x^3 - 4x + 1) H(1, 1, 0; x) \]
\[ + 2H(0, 1, -1, 0; x) x^2 - 2H(0, 1, 0, 0; x) x^2 - 2H(0, 1, 1, 0; x) x^2 \]
\[ + x^2 H(1, 0, 0, 0; x) - \frac{3}{2} x^2 H(0, 0, 0, 0; x) \} \]

(C.44)
APPENDIX C. SCALAR TWO LOOP INTEGRALS

C.5 Five propagator integrals

\[
F^0_6(x) = \frac{xH(0, 0; x)}{2(x-1)^2} \tag{C.46}
\]

\[
F^1_6(x) = \frac{x}{(1-x)^2} \left\{ -\frac{1}{4} \pi^2 H(0; x) - H(0, -1, 0; x) - H(0, 0; x) - \frac{9\zeta(3)}{2} \right.
\]
\[
- H(0, 1, 0; x) + \frac{1}{2} H(0, 0, 0; x) + H(1, 0, 0; x) \right\} \tag{C.47}
\]

\[
F^2_6(x) = \frac{x}{(1-x)^2} \left\{ \frac{1}{2} \pi^2 H(0, -1; x) + \frac{1}{2} \pi^2 H(0; x) + 3\zeta(3) - \frac{1}{6} \pi^2 H(0, 0, -1; x) + 2H(0, 0, 0; x) \right.
\]
\[
+ \frac{1}{12} (6 - \pi^2) H(0, 0; x) + \frac{1}{2} \pi^2 H(1, 0; x) + 2H(0, 1, 0; x)
\]
\[
+ 2H(0, -1, -1, 0; x) + 2H(0, -1, 1; 0; x) - H(0, 0, 0; x)
\]
\[
- 2H(0, 0, -1, 0; x) - 2H(1, 0, 0; x) + 4H(0, 1, 0; x) + 2H(0, 1, 0; x) \right\} \tag{C.48}
\]

\[
\int \frac{d^d k}{i \pi^{d/2}} \int \frac{d^d l}{i \pi^{d/2}} \frac{1}{D_{22}D_{23}D_{24}D_{26}D_{27}} = \left( \frac{\Gamma(1+\epsilon)}{1-\epsilon} \right)^2 (m_i^2)^{-2\epsilon} 2\epsilon F^0_7(x) + O(\epsilon^3) \tag{C.49}
\]

\[
F^0_7(x) = \frac{1}{(1-x)^2} \cdot \frac{1}{6} \pi^2 H(0, 0; x) x - \frac{1}{3} \pi^2 H(1, 0; x) x - \frac{\pi^4 x}{36}
\]
\[
- xH(0, 0, 1, 0; x) - 2xH(1, 0, 1, 0; x) - 2xH(0, 1, 0, 0; x)
\]
\[
- 3xH(1, 0, 0, 0; x) - 4xH(1, 1, 0, 0; x) \right\} \tag{C.50}
\]

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C.6 Six propagator integrals

\[
\begin{align*}
F_9^0(x) &= \frac{x}{(1-x)^2} \left\{ -2H(0,0,1;x) - 2H(0,1,0;x) + 4H(1,0,0;x) - 6\zeta(3) \right\} \\
F_8^1(x) &= \frac{x}{(1-x)^2} \left\{ -12\zeta(3)H(0;x) + \frac{1}{3}\pi^2 H(0,1;x) - 24H(1;x)\zeta(3) - \frac{\pi^4}{10} \\
&\quad - 8H(0,0,0,1;x) - 10H(0,0,-1,0;x) + 4H(0,-1,0,1;x) \\
&\quad - \frac{2}{3}\pi^2 H(1,0;x) - 4H(1,0,0,1;x) - 4H(0,1,0,1;x) - 4H(0,0,1,0;x) \\
&\quad - 4H(0,0,1,1;x) + 4H(0,-1,0,0;x) + 4H(0,-1,1,0;x) \\
&\quad - 24H(1,0,-1,0;x) + 4H(1,0,1,0;x) + 4H(0,1,-1,0;x) \\
&\quad - 6H(0,1,0,0;x) - 4H(0,1,1,0;x) + 12H(1,0,0,0;x) \right\}
\end{align*}
\]

C.6 Six propagator integrals

\[
\begin{align*}
F_9^0(x) &= \frac{x^2}{(1-x)^2(x+1)} \left\{ 8\zeta(3)H(0;x) + 16H(0,0,-1,0;x) + \frac{\pi^4}{10} \\
&\quad + \frac{2}{3}\pi^2 H(0,0;x) - 4H(0,0,1,0;x) - 8H(0,-1,0,0;x) \\
&\quad + 14H(0,1,0,0;x) + H(0,0,0,0;x) \right\}
\end{align*}
\]
\[ \begin{align*}
\text{APPENDIX C. SCALAR TWO LOOP INTEGRALS} \\
\int \frac{d^4 k}{i \pi^{d/2}} \int \frac{d^4 l}{i \pi^{d/2}} \frac{1}{D_{31} D_{32} D_{33} D_{34} D_{35} D_{37}} &= \left( \frac{\Gamma(1 + \epsilon)}{1 - \epsilon} \right)^2 (m^2_i)^{-2} \sum_{i=-1}^{0} \epsilon^i F_{10}^i(x) + \mathcal{O}(\epsilon^1) \\
F_{10}^{-1}(x) &= \frac{x^2}{(1 - x)^4} \left\{ -\frac{2}{3} \pi^2 H(0; x) - 8H(0, -1, 0; x) + 4H(0, 0, 0; x) - 12\zeta(3) \right\} \\
F_{10}^{0}(x) &= \frac{x^2}{(1 - x)^4} \left\{ \frac{8}{3} \pi^2 H(0, -1; x) + 24\zeta(3) - \frac{16\pi^4}{45} \\
&\quad + \frac{4}{3} (\pi^2 - 33\zeta(3)) H(0; x) - \frac{4}{3} \pi^2 H(0, 1; x) - 48H(1; x)\zeta(3) \\
&\quad - 56H(0, 0, -1, 0; x) + 16H(0, -1, 0; x) - \frac{10}{3} \pi^2 H(0, 0; x) \\
&\quad - \frac{8}{3} \pi^2 H(1, 0; x) + 8H(0, 0, 1, 0; x) + 64H(0, -1, -1, 0; x) \\
&\quad - 40H(0, -1, 0, 0; x) - 16H(0, -1, 1, 0; x) - 8H(0, 0, 0; x) \\
&\quad - 32H(1, 0, -1, 0; x) - 16H(0, 1, -1, 0; x) + 8H(0, 1, 0, 0; x) \\
&\quad + 12H(0, 0, 0, 0; x) + 16H(1, 0, 0, 0; x) \right\} \\
\end{align*} \]
Appendix D

Virtual amplitudes

D.1 Amplitudes in terms of master integrals

In this Appendix we express the virtual amplitudes in terms of master integrals given in Appendix C.

D.1.1 Leading order amplitudes

The fermionic coefficient function $c_{f}^{(0)}$ defined in (4.82) is given by

$$
\begin{align*}
c_{f}^{(0)} &= \frac{e^{\epsilon \gamma E} m^{2\epsilon}}{1 - \epsilon} \left\{ \frac{8x}{(1 - x)^{2}} \epsilon \right\} + 4 \left( \frac{(1 + x)^{2}}{(1 - x)^{2} - \epsilon} \right) m^{2} \\
&= \epsilon^{\gamma E} E_{m}^{2\epsilon} \left\{ \frac{8x}{(1 - x)^{2}} \epsilon \right\} + 4 \left( \frac{(1 + x)^{2}}{(1 - x)^{2} - \epsilon} \right) m^{2}.
\end{align*}
$$

The scalar coefficient function $c_{s}^{(0)}$ defined in (4.83) is given by

$$
\begin{align*}
c_{s}^{(0)} &= -\frac{e^{\epsilon \gamma E} m^{2\epsilon}}{1 - \epsilon} \frac{4x}{(1 - x)^{2}} \left\{ \epsilon + m^{2} \right\}.
\end{align*}
$$

D.1.2 Next-to-leading order amplitudes

The fermionic coefficient function $c_{f}^{(1)}$ defined in (4.82) is given by

$$
\begin{align*}
c_{f}^{(1)} &= e^{2\gamma \epsilon E} m^{4\epsilon} \left\{ \frac{N_{c}}{\epsilon s^{2} x^{2}} \right\} \left[ - 24 x (1 + x)^{2} - \epsilon (1 + 56 x + 238 x^{2} + 56 x^{3} + x^{4}) \right].
\end{align*}
$$
APPENDIX D. VIRTUAL AMPLITUDES

\[
+ \epsilon^2 (9 - 360x - 994x^2 - 360x^3 + 9x^4) + 2 \epsilon^3 (19 - 900x - 2014x^2
- 900x^3 + 19x^4) \\
+ \frac{C_F}{s^2(x+1)^2} \left[ 4(1 - 12x - 25x^2 + 8x^3 - 25x^4 - 12x^5 + x^6) \\
- 8 \epsilon (1 + 10x + 51x^2 + 36x^3 + 51x^4 + 10x^5 + x^6) \\
- 8 \epsilon^2 (5 + 37x + 203x^2 + 214x^3 + 203x^4 + 37x^5 + 5x^6) \right] \bigg] \\
+ \left\{ \frac{N_c}{s(1-x)^2} \left[ 24(1+x)^2 + 20 \epsilon^2 (17 + 46x + 17x^2) + 4 \epsilon (23 + 42x + 23x^2) \right] \\
+ \frac{C_F}{s(1-x)^2(1-x)^2} \left[ 8 (1-x)^2 (1 - 6x + x^2) + 8 \epsilon (9 - 8x + 30x^2 - 8x^3 \\
+ 9x^4) + 8 \epsilon^2 (21 + 8x + 102x^2 + 8x^3 + 21x^4) \right] \right\} \\
+ \frac{N_c}{(1-x)^2} \left[ 16x - 16 \epsilon x - 16 \epsilon^2 x \right] \\
+ \left\{ \frac{N_c}{\epsilon^2(1-x)^4} \left[ 4s x (1+x)^2 + 2 \epsilon s x (3 + x) (1 + 3x) \\
+ \epsilon^2 s (1 + 2x + 122x^2 + 2x^3 + x^4) \right] \\
+ \frac{4CFs}{\epsilon(1-x)^3} \left[ 2x (1+x)^2 - \epsilon (1 - 8x - 10x^2 - 8x^3 + x^4) \right] \right\} \\
+ \left\{ \frac{N_c (1+x)^2}{(1-x)^4} \left[ 4 \epsilon (1 - 26x + x^2) \right] - \frac{CF s(1+x)^2}{(1-x)^2} \left[ 16 \epsilon \right] \right\} \\
+ \left\{ \frac{N_c}{\epsilon (1-x)^4} \left[ - 24 (1+x)^2 (1 - 4x + x^2) - 4 \epsilon (1 - 4x + x^2) (7 + 26x + 7x^2) \\
- 4 \epsilon^2 (43 - 46x - 442x^2 - 46x^3 + 43x^4) \\
- 4 \epsilon^3 (1 - 4x + x^2) (201 + 550x + 201x^2) \right] \\
+ \frac{8CF}{(1-x)^4} \left[ - 2 (1+x)^2 (1 - 4x + x^2) - \epsilon (7 - 16x - 30x^2 - 16x^3 + 7x^4) \\
- 2 \epsilon^2 (10 - 19x - 70x^2 - 19x^3 + 10x^4) \right] \right\} \\
+ \left\{ \frac{8s^2x^2 N_c}{\epsilon^2(1-x)^6} \left[ - 4 (1+x)^2 - 2 \epsilon (5 + 14x + 5x^2) - \epsilon^2 (41 + 118x + 41x^2) \right] \right\}
\]
D.1. AMPLITUDES IN TERMS OF MASTER INTEGRALS

\[ + \frac{16 s^2 x^2 C_F}{\epsilon (1 - x)^6} \left[ - (1 + x)^2 - 2 \epsilon (3 + 4 x + 3 x^2) \right] \]

\[ + \left\{ \frac{N_c}{\epsilon (1 - x)^4} \left[ - 8 s x (1 + x)^2 + 16 s x (1 + x^2) \right] \right\} \]

\[ + \left\{ \frac{N_c}{(1 - x)^2} \left[ 32 \epsilon (1 + x)^2 + 4 (1 + 6 x + x^2) \right] \right\} \]

\[ - \frac{C_F}{(1 - x)^2} \left[ 8 (1 + 14 x + x^2) + 96 \epsilon x \right] \]

\[ + \left\{ \frac{N_c}{(1 - x)^4} \left[ - 80 \epsilon^2 (1 + x)^2 + 2 (1 - 18 x + x^2) - 16 \epsilon (1 + 4 x + x^2) \right] \right\} \]

\[ + \frac{C_F}{(1 - x)^2} \left[ - 8 (1 - 6 x + x^2) + 8 \epsilon (1 + 6 x + x^2) \right. \]

\[ + 8 \epsilon^2 (5 + 6 x + 5 x^2) \]

\[ \left. + \frac{C_F}{(1 - x)^2 (1 + x)} \left[ - 32 \epsilon^2 x^2 + 16 \epsilon (1 + x^2) \right. \right. \]

\[ - 16 \epsilon x (1 + 4 x + x^2) \]

\[ \left. + \frac{N_c}{(1 - x)^4} \left[ 8 s x (1 + x)^2 + 24 \epsilon s x (1 + x^2) \right] \right. \]

\[ + \frac{C_F}{(1 - x)^4} \left[ - 16 \epsilon s (1 - x)^2 x + 8 s x (1 + x)^2 \right] \]

\[ + \frac{C_F}{(1 - x)^6} \left[ - 8 s^2 x (1 + x)^2 (1 + x^2) \right] \]

\[ + \left\{ \frac{2 s^2 x N_c}{(1 - x)^4} \left[ - 2 (1 + x)^2 - \epsilon (3 + x)(1 + 3 x) \right] \right\} \]

\[ + \left\{ \frac{N_c}{\epsilon s (1 - x)^2} \left[ - 96 (1 + x)^2 - 32 \epsilon (5 + 16 x + 5 x^2) \right] \right. \]
\[-8 \epsilon^2 (105 + 326 x + 105 x^2) - 24 \epsilon^3 (161 + 446 x + 161 x^2) \]
\[-16 C_F \frac{s}{(1-x)^2} \left[ 4 (1+x)^2 + 8 \epsilon (2+3x+2x^2) \right] \]
\[+ 3 \epsilon^2 (17 + 38 x + 17 x^2) \right\} + \mathcal{O}(\epsilon). \quad (D.3)\]

The scalar coefficient function $c_s^{(1)}$ defined in (4.83) is given by

\[c_s^{(1)} = e^{2 \gamma_E} m^4 e \left\{ \frac{4 N_c}{\epsilon s^2 x} \left[ 6 x - \epsilon^2 (1 - 108 x + x^2) - \epsilon (1 - 24 x + x^2) + \epsilon^3 (5 + 462 x + 5 x^2) \right] \right. \]
\[+ \left. \frac{4 C_F}{s^2 x (1+x)^2} \left[ x (9 - 2x + 9x^2) + \epsilon (1 + 23 x + 32 x^2 + 23 x^3 + x^4) + 2 \epsilon^2 (1 + 48 x + 78 x^2 + 48 x^3 + x^4) \right] \right\} \]
\[+ \left\{ \frac{N_c}{s (1-x)^2} \left[ -24 x - 88 \epsilon x - 400 \epsilon^2 x \right] \right. \]
\[+ \left. \frac{C_F}{s (1-x)^2 (1+x)^2} \left[ -8 (1-x)^2 (2 - 3 x + 2 x^2) - 8 \epsilon (1 + 3 x + 3 x^3 + x^4) - 16 \epsilon^2 (1 + x)^2 (1 + 3 x + x^2) \right] \right\} \]
\[+ \frac{N_c}{(1-x)^2} \left[ -4 x + 4 \epsilon x + 4 \epsilon^2 x \right] \]
\[+ \left\{ \frac{N_c}{\epsilon^2 (1-x)^4} \left[ -4 s x^2 - 8 \epsilon s x^2 + 4 \epsilon^2 s (1 - 10 x + x^2) \right] \right. \]
\[+ \left. \frac{C_F}{\epsilon (1-x)^4} \left[ -8 s x^2 - 24 \epsilon s x^2 \right] \right\} \]
\[+ \frac{N_c}{(1-x)^4} \left[ 24 \epsilon s x (1 + x)^2 \right] \]
\[+ \left\{ \frac{8 N_c}{\epsilon (1-x)^4} \left[ 3 x (1 - 4 x + x^2) + 5 \epsilon x (1 - 4 x + x^2) + \epsilon^2 x (27 - 110 x + 27 x^2) + 119 \epsilon^3 x (1 - 4 x + x^2) \right] \right. \]
\[+ \left. \frac{C_F}{(1-x)^4} \left[ 16 x (1 - 4 x + x^2) + 8 \epsilon x (5 - 22 x + 5 x^2) + 8 \epsilon^2 x (21 - 86 x + 21 x^2) \right] \right\} \]
\[ + \left\{ \frac{N_c}{\epsilon^2 (1 - x)^6} \left[ 32 s^2 x^3 + 96 \epsilon s^2 x^3 + 400 \epsilon^2 s^2 x^3 \right] \right\} \]

\[ + \frac{C_F}{\epsilon (1 - x)^6} \left[ 16 s^2 x^3 + 80 \epsilon s^2 x^3 \right] \]

\[ + \frac{N_c}{\epsilon (1 - x)^4} \left[ 8 s x^2 - 8 \epsilon s x^2 \right] \]

\[ + \left\{ \frac{N_c}{(1 - x)^2} \left[ -8 x - 32 \epsilon x \right] + \frac{C_F}{(1 - x)^2} \left[ 20 x + 32 \epsilon x \right] \right\} \]

\[ + \frac{N_c}{(1 - x)^2} \left[ 8 x + 24 \epsilon x + 80 \epsilon^2 x \right] \]

\[ + \frac{C_F}{(1 - x)^2} \left[ -8 x - 16 \epsilon x - 32 \epsilon^2 x \right] \]

\[ + \frac{2 x C_F}{(1 - x)^2 (1 + x)^2} \left[ \epsilon (1 - x)^2 + \epsilon^2 (1 - x)^2 - 2 (1 + x^2) \right] \]

\[ + \left\{ \frac{N_c}{(1 - x)^4} \left[ -8 s x^2 - 24 \epsilon s x^2 \right] \right\} \]

\[ + \frac{C_F}{(1 - x)^4} \left[ 4 s x^2 + 4 \epsilon s x^2 \right] \]

\[ + \frac{C_F}{(1 - x)^6} \left[ 8 s^2 x^2 (1 + x^2) \right] \]

\[ + \left\{ \frac{N_c}{(1 - x)^4} \left[ s x^2 \right] + \frac{C_F}{(1 - x)^4} \left[ -4 s x^2 \right] \right\} \]

\[ + \frac{N_c}{(1 - x)^4} \left[ 4 s^2 x^2 + 8 \epsilon s^2 x^2 \right] \]

\[ + \left\{ \frac{N_c}{\epsilon s (1 - x)^2} \left[ 96 x + 208 \epsilon x + 1072 \epsilon^2 x + 4608 \epsilon^3 x \right] \right\} \]
\[
\begin{align*}
+ \frac{C_F}{s(1-x)^2} \left[ 64x + 224\epsilon x + 864\epsilon^2 x \right] + O(\epsilon). & \quad \text{(D.4)}
\end{align*}
\]

### D.2 Coefficient functions

Below threshold, \(0 < s < 4m^2\), the amplitudes are expressed in terms of \(\text{Cl}\), \(\text{Ls}\), \(\text{Lsc}\) and \(\text{LsLsc}\) functions.

#### D.2.1 Leading order amplitudes

Below threshold, the leading-order coefficient functions are given by

\[
c_l^{(0)} = \frac{1}{4} (4 - \theta^2 - (4 + \theta^2) \cos(\theta)) \csc^4 \left( \frac{\theta}{2} \right)
+ \epsilon \csc^2 \left( \frac{\theta}{2} \right) \left[ -2 (2\pi - \theta) \cot \left( \frac{\theta}{2} \right) Ls_2^{(0)} (\theta - \pi) - 4 \cot^2 \left( \frac{\theta}{2} \right) Ls_3^{(1)} (\theta - \pi) 
\right.
\]

\[
+ \frac{1}{2} \csc^2 \left( \frac{\theta}{2} \right) \left[ 6 - \theta^2 - 2\theta \sin(\theta) - 7\zeta(3) (1 + \cos(\theta)) - 6 \cos(\theta) \right]
\]

\[
+ \epsilon^2 \csc^2 \left( \frac{\theta}{2} \right) \left[ -4 \theta \cot \left( \frac{\theta}{2} \right) \text{Cl}_1 (\theta - \pi) + 2 \cot^2 \left( \frac{\theta}{2} \right) Ls_2^{(0)} (\theta - \pi)^2 
\right.
\]

\[
+ 2\theta \cot^2 \left( \frac{\theta}{2} \right) Ls_3^{(0)} (\theta - \pi) - 4 \csc^2 \left( \frac{\theta}{2} \right) Ls_3^{(1)} (\theta - \pi)
\]

\[
- 2 \csc^2 \left( \frac{\theta}{2} \right) (2\pi - \theta - \sin(\theta)) Ls_2^{(0)} (\theta - \pi)
\]

\[
+ \frac{1}{48} \csc^2 \left( \frac{\theta}{2} \right) (336 + 4\pi^2 - 4\pi^3 \theta - 24\theta^2 - \pi^2 \theta^2)
\]

\[
- (336 + 4\pi^3 \theta + \pi^2 (4 + \theta^2) \cos(\theta) - 144\theta \sin(\theta) - 336\zeta(3)) \right] + O(\epsilon^3),
\]

\[
c_s^{(0)} = -\frac{1}{8} (2 - \theta^2 - 2 \cos(\theta)) \csc^4 \left( \frac{\theta}{2} \right)
+ \frac{\epsilon}{8} \csc^4 \left( \frac{\theta}{2} \right) \left[ 4(2\pi - \theta) Ls_2^{(0)} (\theta - \pi) + 8 Ls_3^{(1)} (\theta - \pi)
\right.
\]

\[
+ (-6 + \theta^2 + 6 \cos(\theta) + 2\theta \sin(\theta) + 14\zeta(3)) \right]
\]

\[
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\]

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\[ + \frac{\epsilon^2}{96} \csc^2 \left( \frac{\theta}{2} \right) \left[ 96 \theta \cot \left( \frac{\theta}{2} \right) \text{Cl}_1(\theta - \pi) - 48 \csc^2 \left( \frac{\theta}{2} \right) \text{Ls}_{2}^{(0)}(\theta - \pi)^2 \\
- 48 \theta \csc^2 \left( \frac{\theta}{2} \right) \text{Ls}_{3}^{(0)}(\theta - \pi) + 96 \csc^2 \left( \frac{\theta}{2} \right) \text{Ls}_{3}^{(1)}(\theta - \pi) \\
- 48 \csc^2 \left( \frac{\theta}{2} \right) (-2\pi + \theta + \sin(\theta)) \text{Ls}_{2}^{(0)}(\theta - \pi) \\
- \csc^2 \left( \frac{\theta}{2} \right) (168 + 2\pi^2 - 4\pi^3 \theta - 12\theta^2 - \pi^2 \theta^2 - 2(84 + \pi^2) \cos(\theta) \\
- 72 \theta \sin(\theta) - 168 \zeta(3) \right] + \mathcal{O}(\epsilon^3). \] (D.6)

D.2.2 Next-to-leading order amplitudes

The finite coefficient function at next-to-leading order are below threshold

\[ c_{\text{f,fin}}^{(1)} = \frac{1}{N_c} \left[ \frac{27}{2} \cos \left( \frac{\theta}{2} \right) \cos(\theta) \text{Ls}_{4}^{(2)}(\theta) + 18 \cos \left( \frac{\theta}{2} \right) \cos(\theta) \text{Ls}_{4}^{(2)}(\theta - \pi) \\
+ \theta \cos \left( \frac{\theta}{2} \right) (\theta \cos(\theta) - \sin(\theta)) \text{Ls}_{2}^{(0)}(\theta) \\
+ 4 \cos \left( \frac{\theta}{2} \right) ((9\pi - 5\theta) \cos(\theta) + 2 \sin(\theta)) \text{Ls}_{3}^{(1)}(\theta - \pi) \\
+ 2 \cos \left( \frac{\theta}{2} \right) ((9\pi^2 - 10\pi \theta + 2\theta^2) \cos(\theta) \\
+ 2 (2\pi - \theta) \sin(\theta)) \text{Ls}_{2}^{(0)}(\theta - \pi) \\
- \sin \left( \frac{\theta}{2} \right) ((6 + 4\theta^2) \cos(\theta) + 3(-2 + \theta^2 - \theta \sin(\theta))) \text{Cl}_1(\theta) \\
- \frac{1}{2} \left( 11 \theta \cos \left( \frac{\theta}{2} \right) + 11 \theta \cos \left( \frac{3\theta}{2} \right) + 9 \sin \left( \frac{\theta}{2} \right) \\
- 7 \sin \left( \frac{3\theta}{2} \right) \right) \text{Ls}_{3}^{(1)}(\theta) \\
+ \frac{1}{8} \left( \cos \left( \frac{\theta}{2} \right) (-3\theta^3 + \theta (4 - 140 \zeta(3)) + 144\pi \zeta(3)) \\
- 4 \sin \left( \frac{\theta}{2} \right) (18 - 9\theta^2 - 28\zeta(3) - 2 \cos(\theta) (9 + \theta^2 + 14 \zeta(3))) \\
- \cos \left( \frac{3\theta}{2} \right) (\theta^3 - 144\pi \zeta(3) + 4 \theta (1 + 35 \zeta(3))) \right] \csc^5 \left( \frac{\theta}{2} \right) \\
- N_c \left[ - 16 \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \text{LsL}_{3,1,2}(\theta) \right] \]
\[
\begin{align*}
&+ \frac{27}{2} \cos \left( \frac{\theta}{2} \right) \cos(\theta) Ls_4^{(2)}(\theta) + 18 \cos \left( \frac{\theta}{2} \right) \cos(\theta) Ls_4^{(2)}(\theta - \pi) \\
&+ 8 \pi \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) Ls_3^{(1)}(\theta - \pi) + 8 \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) Ls_4^{(1)}(\theta) \\
&- 2 \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) Ls_4^{(1)}(2\theta) + 8 \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) Ls_4^{(1)}(\theta - \pi) \\
&- \left( 16 \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \right) Ls_2^{(0)}(\theta - \pi) \\
&- \theta \cos \left( \frac{\theta}{2} \right) \left( \theta \cos(\theta) - \sin(\theta) \right) Ls_2^{(0)}(\theta) \\
&- 4 \left( \frac{8 C_{11}(\theta)}{\theta} \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \right) \\
&- \cos \left( \frac{\theta}{2} \right) \left( (9 \pi - 5 \theta) \cos(\theta) + 2 \sin(\theta) \right) Ls_3^{(1)}(\theta - \pi) \\
&- 2 \left( \frac{8 (2 \pi - \theta) C_{11}(\theta)}{\theta} \cos^2 \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \right) \\
&- \cos \left( \frac{\theta}{2} \right) \left( (9 \pi - 10 \pi \theta + 2 \theta^2) \cos(\theta) \right) \\
&+ 2 \left( 2 \pi - \theta \sin(\theta) \right) Ls_2^{(0)}(\theta - \pi) \\
&- \frac{1}{2} \left( 11 \theta \cos \left( \frac{\theta}{2} \right) + 11 \theta \cos \left( \frac{3\theta}{2} \right) - 5 \sin \left( \frac{\theta}{2} \right) \right) \\
&- 13 \sin \left( \frac{3\theta}{2} \right) Ls_3^{(1)}(\theta) \\
&- \sin \left( \frac{\theta}{2} \right) \left( - 6 + 8 \theta^2 - 3 \theta \sin(\theta) + 28 \zeta(3) \right) \\
&+ \cos(\theta) \left( 6 + 7 \theta^2 + 28 \zeta(3) \right) C_{11}(\theta) \\
&- \frac{1}{8} \cos \left( \frac{\theta}{2} \right) \left( 9 \theta^3 - 4 \theta (1 - 35 \zeta(3)) - 144 \pi \zeta(3) \right) \\
&- \frac{1}{8} \cos \left( \frac{3\theta}{2} \right) \left( 3 \theta^3 + 4 \theta (1 + 35 \zeta(3)) - 144 \pi \zeta(3) \right) \\
&- \sin \left( \frac{\theta}{2} \right) \left( - \frac{7}{2} \theta^2 + 30 + \cos^2 \left( \frac{\theta}{2} \right) \left( \frac{79}{90} \pi^4 + \frac{4}{3} \pi^2 \log(2)^2 - 3 \theta^2 \right. \right. \\
&+ \frac{1}{4} \theta^4 - \frac{2}{3} (45 + 2 \log(2)^4 + 48 \text{Li}_4 \left( \frac{1}{3} \right) + 42 \zeta(3) \\
&+ 42 \log(2) \zeta(3) \left. \right) \right) \right] \csc^5 \left( \frac{\theta}{2} \right) + O(\epsilon), \quad (D.7)
\end{align*}
\]

\( c_{s,\text{fin}}^{(1)} = - \frac{1}{N_c} \left[ \frac{27}{4} \cos(\theta) Ls_4^{(2)}(\theta) + 9 \cos(\theta) Ls_4^{(2)}(\theta - \pi) \right] \)
\[- \frac{1}{4} \sin \left( \frac{\theta}{2} \right) \left( -3 + 8 \theta^2 \right) \cos \left( \frac{\theta}{2} \right) + 3 \left( \cos \frac{3\theta}{2} \right) \]
\[- 2 \theta \sin \left( \frac{\theta}{2} \right) \right] \left( C_1(\theta) - \frac{1}{2} \left( 11 \theta \cos(\theta) - 3 \sin(\theta) \right) L_{13}^{(1)}(\theta) \right) \]
\[+ 2 \left( (9\pi - 5 \theta) \cos(\theta) + 2 \sin(\theta) \right) L_{13}^{(1)}(\theta - \pi) \]
\[+ \left( (9\pi^2 - 10 \pi \theta + 2 \theta^2) \cos(\theta) + 2 \left( 2 \pi - \theta \right) \sin(\theta) \right) L_{2}^{(0)}(\theta - \pi) \]
\[+ \frac{1}{2} \theta \left( \theta \cos(\theta) - \sin(\theta) \right) L_{2}^{(0)}(\theta) + \frac{1}{16} \left( -11 \theta + 3 \theta - 3 \theta \cos(2\theta) \right) \]
\[- 28 \sin(\theta) + 17 \theta^2 \sin(\theta) + 14 \sin(2\theta) + 112 \sin(\theta) \zeta(3) \]
\[+ \cos(\theta) \left( \theta^3 + \theta \left( 14 - 280 \zeta(3) \right) + 288 \pi \zeta(3) \right) \right] \csc \left( \frac{\pi}{2} \right) \]
\[- N_c \left( 8 \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) L_{2}^{(0)}(\theta) \right) \]
\[- \frac{27}{4} \cos(\theta) L_{2}^{(2)}(\theta) - 9 \cos(\theta) L_{2}^{(2)}(\theta - \pi) \]
\[- 4 \pi \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) L_{3}^{(1)}(\theta - \pi) - 4 \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) L_{2}^{(1)}(\theta) \]
\[+ \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) L_{3}^{(1)}(\theta) \]
\[+ \frac{1}{2} \theta \left( \theta \cos(\theta) - \sin(\theta) \right) L_{3}^{(1)}(\theta) + \frac{11}{2} \left( \theta \cos(\theta) - \sin(\theta) \right) L_{3}^{(1)}(\theta) \]
\[+ 2 \left( 4 \cos(\theta) \sin(\theta) - ((9\pi - 5 \theta) \cos(\theta) + 2 \sin(\theta)) \right) L_{3}^{(1)}(\theta - \pi) \]
\[+ \left( 4 (2 \pi - \theta) C_1(\theta) \sin(\theta) + 4 \sin(\theta) L_{2}^{(0)}(\theta) \right) \]
\[- \left( (9\pi^2 - 10 \pi \theta + 2 \theta^2) \cos(\theta) + 2 \left( 2 \pi - \theta \right) \sin(\theta) \right) L_{2}^{(0)}(\theta - \pi) \]
\[+ \frac{1}{4} \sin \left( \frac{\theta}{2} \right) \left( 3 \cos \left( \frac{3\theta}{2} \right) - 6 \theta \sin \left( \frac{\theta}{2} \right) \right) \]
\[+ \cos \left( \frac{\theta}{2} \right) \right) \left( 3 - 24 \theta^2 - 112 \zeta(3) \right) \right] C_1(\theta) \]
\[+ \frac{1}{16} \left( 11 \theta + 3 \theta^3 + 3 \theta \cos(2\theta) + 52 \sin(\theta) + \frac{158}{45} \pi^4 \sin(\theta) \right) \]
\[- 21 \theta^2 \sin(\theta) + \theta^4 \sin(\theta) + \frac{16}{3} \pi^2 \log^2(2) \sin(\theta) - \frac{16}{3} \log^4(2) \sin(\theta) \]
\[- 128 \zeta(3) \left( \frac{1}{2} \right) \sin(\theta) - 26 \sin(2\theta) - 112 \sin(\theta) \zeta(3) \]
\[- 112 \log(2) \sin(\theta) \zeta(3) + \cos(\theta) \left( 3 \theta^3 - 14 \theta (1 - 20 \zeta(3)) \right) \]
\[- 288 \pi \zeta(3) \right) \right] \csc \left( \frac{\pi}{2} \right) + \mathcal{O}(\epsilon) . \quad (D.8) \]
Appendix E

Real matrix elements

In this appendix, we apply the following shorthand notation:

\[ B(s, m) = -i(4\pi)^2 B^\text{fin}_0(s; m), \]  
(E.1)

\[ C(s, m) = -i(4\pi)^2 C_0(0, 0, s; m, m, m), \]  
(E.2)

\[ C_1(s, t, m) = -i(4\pi)^2 C_0(0, s; t, m, m, m), \]  
(E.3)

\[ D(s, t, m) = -i(4\pi)^2 D_0(0, 0, 0, m^2, s, t; m, m, m, m). \]  
(E.4)

where \( B^\text{fin}_0 \) is the finite part of the \( B_0 \) function, i.e.

\[ B_0(s; m) = \frac{i}{(4\pi)^2} \left( \frac{1}{\epsilon} + B^\text{fin}_0(s; m) \right). \]  
(E.5)

E.1 Matrix elements for \( gg \to gh \)

In the limit \( \epsilon \to 0 \) and up to a phase, the fermionic amplitudes are given by

\[
\frac{M_{gg \to gh}^{++-}}{\Delta_{gg}} = \Lambda_q m^2_q \left[ -\frac{1}{2} \frac{s_{12}s_{23}(4m^2_q s_{13} - s_{12}s_{13})}{s_{13}} D(s_{12}, s_{23}, m_q) 
- \frac{1}{2} \frac{s_{23}}{s_{12}} D(s_{13}, s_{12}, m_q) 
+ \frac{1}{2} \frac{s_{12}}{s_{23}} D(s_{13}, s_{23}, m_q) 
- \frac{4s_{13}(2s_{12}s_{23} + s_{23})}{(s_{23} + s_{12})^2} B(s_{13}, m_q) 
- \frac{4s_{23}(2s_{12}s_{13} + s_{13})}{(s_{13} + s_{12})^2} B(s_{23}, m_q) 
- \frac{2s_{13}s_{23}}{(s_{13} + s_{12})^2} \left( -4s_{13}^2 s_{12} - 2s_{23}s_{13}^2 - 8s_{13}s_{12}s_{23} 
- 10s_{13}s_{12}^2 - 2s_{23}^2 s_{13} - 10s_{23}s_{13}^2 - 4s_{12}s_{23}^2 - 8s_{12}^3 \right) B(m^2_n, m_q) \right].
\]
APPENDIX E. REAL MATRIX ELEMENTS

The diagrams containing scalar loops are

\[\begin{align*}
&+ \frac{1}{s_{23} + s_{12}} \left(4 s_{13} s_{23}^2 - s_{12} s_{23}^2 + 2 s_{23}^2 s_{13}^2 + 2 s_{13} s_{23} s_{12} + s_{12}^2\right) \text{C}_1 \left(m_{h}^2, s_{13}, m_{q}\right) \\
&+ \frac{1}{s_{13} + s_{23}} \left(4 s_{13} s_{12}^3 - s_{12} s_{13}^3 + 2 s_{13}^3 s_{23}^2 + 2 s_{23} s_{13} s_{12} + s_{12}^2\right) \text{C}_1 \left(m_{h}^2, s_{23}, m_{q}\right) \\
&- 2 \frac{s_{13}^2 s_{23}}{s_{12}} \text{C} \left(s_{13}, m_q\right) - 2 \frac{s_{23}^2 s_{13}}{s_{12}} \text{C} \left(s_{23}, m_q\right) \\
&- \frac{4 s_{12} s_{23} - s_{13} s_{23}}{s_{23} + s_{12}} \left(s_{12} + s_{13}\right) \right], \quad \text{(E.6)}
\end{align*}\]

\[
\frac{M^{gg\rightarrow gh}_{\Delta_{gg}}}{\Delta_{gg}} = \frac{g_s^2 \sqrt{2}}{(4\pi)^2 \sqrt{|s_{12} s_{13} s_{23}|}}. \quad \text{(E.8)}
\]

where we have factored out the following prefactor

\[
\Delta_{gg} = - \frac{g_s^2 \sqrt{2}}{(4\pi)^2 \sqrt{|s_{12} s_{13} s_{23}|}}. \quad \text{(E.8)}
\]

The diagrams containing scalar loops are

\[\begin{align*}
&+ \frac{1}{2} s_{12} s_{23} m_{q}^2 \text{D} \left(s_{12}, s_{23}, m_q\right) + \frac{1}{2} s_{13} s_{12} m_{q}^2 \text{D} \left(s_{13}, s_{12}, m_q\right) \\
&+ \frac{1}{4} s_{12} \text{D} \left(s_{13}, s_{23}, m_q\right) \\
&+ m_{q}^2 \left(s_{23} + s_{13}\right) \text{C}_1 \left(m_{h}^2, s_{12}, m_q\right) + \frac{1}{2} s_{13}^2 s_{23} \text{C} \left(s_{13}, m_q\right) + \frac{1}{2} s_{23}^2 s_{13} \text{C} \left(s_{23}, m_q\right) \\
&- \frac{1}{4} \left(4 s_{13} s_{23}^2 + 2 s_{13} s_{23} s_{12} + 2 s_{23}^2 s_{13}\right) \text{C}_1 \left(m_{h}^2, s_{13}, m_q\right) \\
&+ 4 m_{q}^2 \left(s_{13} - 2 s_{23}^2 + 2 s_{23} s_{12}\right) \text{C}_1 \left(m_{h}^2, s_{13}, m_q\right)
\end{align*}\]
E.2 Matrix elements for $q\bar{q} \to gh$

We define

$$\Delta_{qq} = \frac{ig^3\sqrt{2}}{(4\pi)^2 \sqrt{|s_{12}|}}. \quad (E.11)$$

Up to a phase, the amplitudes are given in the fermionic case by

$$\frac{M_{-;\rightarrow}^{q\bar{q} \to gh}}{\Delta_{qq}} = \frac{\Lambda_q m_q^2 s_{23}}{s_{23} + s_{13}} \left[ \frac{2(s_{13} + s_{23} - 4m_q^2)}{s_{23} + s_{13}} C_1 \left( m_h^2, s_{12}, m_q \right) \right. \right.$$  

$$\left. + \frac{4s_{12}}{s_{23} + s_{13}} \left( B(s_{12}, m_q) - B \left( m_h^2, m_q \right) \right) \right] - 4. \quad (E.12)$$

The diagrams containing scalar loops are

$$\frac{M_{-;\rightarrow}^{q\bar{q} \to gh}}{\Delta_{qq}} = \frac{\Lambda_q m_q^2 s_{23}}{s_{23} + s_{13}} \left[ \frac{2m_q^2 C_1 \left( m_h^2, s_{12}, m_q \right)}{s_{23} + s_{13}} \right. \right.$$  

$$\left. - \frac{s_{12}}{s_{23} + s_{13}} \left( B(s_{12}, m_q) - B \left( m_h^2, m_q \right) \right) + 1 \right]. \quad (E.13)$$

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Appendix F

Decay matrix elements

F.1 Matrix element squared for $h \rightarrow ZZ \rightarrow \ell^+ \ell^- \ell'^+ \ell'^-$

First, we consider the decay

$$h(p_1) \rightarrow Z(p_{Z_1})Z(p_{Z_2}) \rightarrow \ell^+(p_2)\ell^-(p_3)\ell'^+(p_4)\ell'^-(p_5) \tag{F.1}$$

where $\ell \neq \ell'$. We define as a shorthand

$$t_{ij,V} = s_{ij} - m_V^2. \tag{F.2}$$

The matrix element squared is given by

$$|M_{\ell\ell'\ell'\ell'}|^2 = \frac{m_Z^4 g_w^4 \Lambda_Z^2 s_{24} s_{35} \left(c_{V}^4 + 6c_{A}^2 c_{V}^2 + c_{A}^4 \right) + s_{25} s_{34} (c_{A}^4 - c_{V}^4)(c_{A}^4 + c_{V}^4)}{8 \cos^2 \theta_W \left[t_{45,Z}^2 + (\Gamma_Z m_Z)^2\right] \left[t_{23,Z}^2 + (\Gamma_Z m_Z)^2\right]} \tag{F.3}$$

with (in the Standard Model)

$$g_w^4 = \frac{16 m_W^4 G_F^2}{\pi^3}, \tag{F.4}$$

$$\Lambda_Z = \frac{1}{v} = \sqrt{\frac{2G_F}{\sqrt{2}}} \tag{F.5}$$

The vector and axial coupling of leptons to the Z are

$$c_{V} = -1 + 4 \sin^2 \theta_W, \tag{F.6}$$

$$c_{A} = -1. \tag{F.7}$$
The same flavor final state, where $\ell = \ell'$, leads to a rather more complicated matrix element squared, since both sub-processes, $Z(p_{Z_1}) \rightarrow \ell^+(p_2)\ell^-(p_3)$ and $Z(p_{Z_1}) \rightarrow \ell^+(p_2)\ell^-(p_5)$ are possible and interfere. It reads

$$|\mathcal{M}_{\ell\ell\ell\ell}|^2 = \frac{m_Z^4 g_W^4 A_{Z}^2}{8 \cos^4 \theta_W} \left( s_{24835} \left( c_V^4 + 6 c_A^2 c_V^2 + c_A^4 \right) + s_{25834} (c_A - c_V)^2 (c_A + c_V)^2 \right)$$

$$+ 2 \text{Re} \left[ \frac{s_{24835} (c_V^4 + 6 c_A^2 c_V^2 + c_A^4)}{t_{23W}^2 + (\Gamma_{WmW})^2} \left( t_{25W}^2 + (\Gamma_{ZmZ})^2 \right) \right]$$

$$+ s_{24835} (c_V^4 + 6 c_A^2 c_V^2 + c_A^4) + s_{23845} (c_A - c_V)^2 (c_A + c_V)^2 \right) \frac{t_{34Z}^2 + (\Gamma_{ZmZ})^2}{t_{25Z}^2 + (\Gamma_{ZmZ})^2} \}$$

(F.8)

Note, that in this case also the symmetry factor for the phase-space is different. When integrating over phase space, we must multiply with $1/4$ due to the symmetry of the final state.

### F.2 Matrix element squared for $h \rightarrow VV \rightarrow \ell^+ \nu\ell^- \bar{\nu}$

The Higgs can decay leptonically into a state with neutrinos via either $W$ bosons or $Z$ bosons. Both processes can interfere. We consider the processes

$$h(p_1) \rightarrow W(p_{W_1})W(p_{W_2}) \rightarrow \ell^-(p_2)\bar{\nu}_\ell(p_3)\nu_\ell(p_4)\ell^+(p_5), \quad (F.9)$$

$$h(p_1) \rightarrow Z(p_{Z_1})Z(p_{Z_2}) \rightarrow \ell^-(p_2)\bar{\nu}_\ell(p_3)\nu_\ell(p_4)\ell^+(p_5). \quad (F.10)$$

Note that the neutrino flavor in the $Z$ mediated case is undetermined in contrast to the $W$ mediated case. Therefore the former has to be multiplied by a factor of 3.

The purely $W$ mediated process is described by the matrix element squared

$$|\mathcal{M}_{\ell\nu\ell\nu}^W|^2 = \frac{4m_W^4 g_W^4 A_W^2 s_{24835}}{t_{23W}^2 + (\Gamma_{WmW})^2} \frac{t_{25W}^2 + (\Gamma_{ZmZ})^2}{t_{34W}^2 + (\Gamma_{ZmZ})^2}$$

while the matrix element squared for the $Z$ mediated decay reads

$$|\mathcal{M}_{\ell\nu\ell\nu}^Z|^2 = \frac{3m_Z^4 g_W^4 A_Z^2}{8 \cos^4 \theta_W} \left( s_{24835} \left( (c_V^c c_A^d - c_V^d c_A^c)^2 + (c_A^c c_V^d - c_A^d c_V^c)^2 \right) \right)$$

$$+ \frac{s_{24835} \left( (c_V^c c_A^d + c_V^d c_A^c)^2 + (c_A^c c_V^d + c_A^d c_V^c)^2 \right)}{t_{25Z}^2 + (\Gamma_{ZmZ})^2} \frac{t_{34Z}^2 + (\Gamma_{ZmZ})^2}{t_{34Z}^2 + (\Gamma_{ZmZ})^2} \}$$

(F.12)
Note the factor of 3 for the three possible neutrino flavors. Besides of the vector and axial couplings of the leptons, the matrix element depends on the neutrino couplings
\[ c'_V = 1 \quad \text{and} \quad c'_A = 1. \quad (F.13) \]

Lastly, the interference term
\[ |M_{t\nu\nu}^{ZW}|^2 = 2\text{Re} \left\{ M_{t\nu\nu}^W M_{t\nu\nu}^{Z,\dagger} \right\} \quad (F.14) \]
is given by
\[ |M_{t\nu\nu}^{ZW}|^2 = -\frac{m_Z^2 m_W^2 g_w^4 \Lambda_Z \Lambda_W}{2 \cos^2 \theta_W} \]
\[ \times 2\text{Re} \left\{ \left( c'_A + c'_V \right) \left( c'_V - c'_A \right) s_{35} s_{24} \right\} \]
\[ \times \left[ t_{45,Z} + i\Gamma_W m_W \right] \left[ t_{23,W} + i\Gamma_W m_W \right] \left[ t_{34,Z} - i\Gamma_Z m_Z \right] \left[ t_{25,Z} - i\Gamma_Z m_Z \right]. \quad (F.15) \]
Bibliography


[93] [Tevatron Electroweak Working Group and CDF Collaboration and D0 Collab], arXiv:0903.2503 [hep-ex].


[113] D. Maitre, private C++ implementation for HPLs with complex argument.
# List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>BR</td>
<td>Branching Ratio</td>
</tr>
<tr>
<td>LHC</td>
<td>Large Hadron Collider</td>
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<tr>
<td>LL</td>
<td>Leading Logarithms</td>
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<tr>
<td>LO</td>
<td>Leading Order</td>
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<tr>
<td>MC</td>
<td>Monte Carlo</td>
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<tr>
<td>NLL</td>
<td>Next-To-Leading Logarithms</td>
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<td>NLO</td>
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<td>NNLO</td>
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<td>PDF</td>
<td>Parton Distribution Function</td>
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<td>Quantum Chromodynamics</td>
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<td>Quantum Electrodynamics</td>
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<td>Quantum Field Theory</td>
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<td>Renormalization Group Equation</td>
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<td>SM</td>
<td>Standard Model</td>
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<tr>
<td>MSSM</td>
<td>Minimal Supersymmetric Standard Model</td>
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