Proof of convergence conditions for the paper "Predictive power dispatch through negotiated locational pricing"

Report

Author(s):
Warrington, Joseph C.

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Proof of convergence conditions for the paper "Predictive power dispatch through negotiated locational pricing"

J. C. Warrington

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1 Introduction

This document gives further analysis and proofs of the results presented in the paper "Predictive power dispatch through negotiated locational pricing" presented at the IEEE PES Innovative Smart Grid Technologies Europe conference in October 2010 [4]. In section IV of that paper, results are given concerning convergence of a negotiation mechanism for determining short-term electricity prices. The notation used in this document is the same as that from the paper, except where stated otherwise.

The assumptions required for the proof are first clarified, and then a formal mathematical description of the algorithm is given. The case where allocations are shared equally is described, leading to the results given in the paper. Then comment is made on the more general case, where the allocation algorithm generates differing allocations ϕ.

2 Assumptions

A1. For notational clarity, it is assumed that one price-elastic unit and one price-inelastic unit is present on each of the n nodes of the network, although it can easily be shown that the analysis applies equally well when m_{e} elastic and m_{i} inelastic units are present, as in the paper.

A2. Multiple elastic generators, each with a cost function of the form in equation (2a) of the paper, with no time coupling ($\alpha_i = 0$), satisfy the load.

A3. Each generator has a strictly convex quadratic cost function $J_i(p_i) = a_i + b_i p_i + \frac{1}{2} c_i p_i^2$. $a_i, b_i, c_i > 0 \ \forall i$.

A4. There is no congestion on the network, so no price feedback on line flows is used.

A5. No generator violates its power output constraints, so that the optimal output of generator i for a given price at negotiation stage j, λ_{ij}, is $p_i^\star(\lambda_{ij}) = \frac{\lambda_{ij} - b_i}{c_i}$.

A6. It is assumed that power outputs stay positive ($\lambda_{ij} \geq b_i$), and that the sum of loads is positive.

A7. Storage is not considered here, and is assumed not to be present on the network.

Under assumption A2, each time step over the horizon can be considered independently, which means that proving convergence to a power match for one time step, as shown below, is equivalent to proving convergence for all steps over the horizon. Therefore the k-subscripts from the paper, which indicate the time step, are not used here.

3 Price update algorithm

In the paper, a general price update rule of the following form was defined:

$$\lambda_{j+1} = \lambda_j - \kappa_{\mathrm{L}}(p^\star(\lambda_j), \ell, \Sigma) - \kappa_{\mathrm{E}}(p^\star(\lambda_j), \ell, \Sigma, \overline{P}) \tag{1}$$

$\lambda_j := [\lambda_{1j} \cdots \lambda_{nj}]^{\mathrm{T}} \in \mathbb{R}^n$ is the planned vector of *nodal* prices at negotiation round j;
$p^\star(\lambda_j) := [p_1^\star(\lambda_{1j}) \cdots p_n^\star(\lambda_{nj})]^{\mathrm{T}} \in \mathbb{R}_+^n$ is the vector of power outputs given by the rule in A5;
$\ell \in \mathbb{R}_+^n$ is the vector of fixed loads for the time slot considered, $\ell := [\hat{p}_{1t} \cdots \hat{p}_{nt}]^{\mathrm{T}}$ using the notation from the paper;
$\Sigma \in \mathbb{R}_+^{n \times n}$ is the susceptance matrix that implicitly defines the network topology;
$\overline{P} \in \mathbb{R}_+^{n \times n}$ is the matrix of line limits;
$\kappa_{\mathrm{L}} : \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^{n \times n} \to \mathbb{R}^n$ is the function mapping power mismatches at loads to a change in price;
$\kappa_{\mathrm{E}} : \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^{n \times n} \times \mathbb{R}_+^{n \times n} \to \mathbb{R}^n$ is the function mapping line congestions to a change in price.

Under A4, congestion feedback is not considered here, so we set $\kappa_{\mathrm{E}}(\cdot, \cdot, \cdot, \cdot) := 0$ and consider only κ_{L}. Under A5, the optimal power output vector for a given price vector λ_j can be written $p^\star(\lambda_j) = C^{-1}\lambda_j - b$, where $C := \mathrm{diag}(c_1, \ldots, c_n)$ and $b := [\frac{b_1}{c_1} \cdots \frac{b_n}{c_n}]^{\mathrm{T}}$.

We choose a form for $\kappa_{\mathrm{L}}(\cdot, \cdot, \cdot)$ which gives a feedback proportional to the global power mismatch:

$$\kappa_{\mathrm{L}}(p^\star(\lambda_j), \ell, \Sigma) := k\mathbf{1}^{\mathrm{T}}(p^\star(\lambda_j) - \ell)\phi(p^\star(\lambda_j), \ell, \Sigma). \tag{2}$$

$\phi : \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^{n \times n} \to [0,1]^n$, with $\mathbf{1}^{\mathrm{T}}\phi = 1$, is a 'blame' function which controls how price changes are weighted across the nodes. Here for clarity we simply call the feedback constant k rather than the K_{L} in the paper. Define $\hat{p} := \mathbf{1}^{\mathrm{T}}\ell$, the total to be satisfied by the inelastic generation combined. Substituting (2) into (1), we get a shared feedback scaled by the size of the power mismatch $(\mathbf{1}^{\mathrm{T}}(C^{-1}\lambda_j - b) - \hat{p})$ and a gain k:

$$\lambda_{j+1} = \lambda_j - k(\mathbf{1}^{\mathrm{T}}(C^{-1}\lambda_j - b) - \hat{p})\phi(p^\star(\lambda_j), \ell, \Sigma) \tag{3}$$

To test for convergence to optimal and feasible prices, we should find the final price vector. When no congestion is present, examination of the KKT conditions shows that the price should be the same for all nodes at optimality. Then the only value of $\lambda = \lambda^\star \mathbf{1}$ which gives a power match is given by

$$\begin{aligned} \mathbf{1}^{\mathrm{T}}(C^{-1}\lambda^\star \mathbf{1} - b) &= \hat{p} \\ \mathbf{1}^{\mathrm{T}} C^{-1} \lambda^\star \mathbf{1} &= \hat{p} + \mathbf{1}^{\mathrm{T}} b \\ \Rightarrow \lambda^\star &= \frac{\hat{p} + \mathbf{1}^{\mathrm{T}} b}{\mathbf{1}^{\mathrm{T}} C^{-1} \mathbf{1}} \in \mathbb{R} \end{aligned}$$

Since there is only one feasible solution for the price vector, it is also optimal. We define a new price vector $\tilde{\lambda}_j := \lambda_j - \lambda^\star \mathbf{1}$, and substitute into the price update rule (3). the offset b disappears and we end up with the following update relationship:

$$\tilde{\lambda}_{j+1} = (I - k\phi \mathbf{1}^{\mathrm{T}} C^{-1})\tilde{\lambda}_j \tag{4}$$

3.1 Constant-ϕ price updates

Consider the feedback rule $\phi = \frac{1}{n}\mathbf{1}$. Then equation (4) becomes:

$$\tilde{\lambda}_{j+1} = (I - \tfrac{k}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}} C^{-1})\tilde{\lambda}_j$$

This is a discrete-time state space representation, and the desired convergence $\tilde{\lambda}_j \to 0$ as $j \to \infty$ depends on the eigenvalues of the matrix $A := I - \frac{k}{n}\mathbf{1}^{\mathrm{T}} C$. We wish to find necessary and sufficient conditions on the feedback scaling factor k for such convergence. A can be written out as follows:

$$A = I - \frac{k}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}} C^{-1} = \begin{bmatrix} 1 - \frac{k}{nc_1} & -\frac{k}{nc_2} & \cdots & -\frac{k}{nc_n} \\ -\frac{k}{nc_1} & 1 - \frac{k}{nc_2} & \cdots & -\frac{k}{nc_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{k}{nc_1} & -\frac{k}{nc_2} & \cdots & 1 - \frac{k}{nc_n} \end{bmatrix}. \tag{5}$$

The eigenvalues are the values of s satisfying

$$\det(sI - (I - \frac{k}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}} C^{-1})) = 0. \tag{6}$$

Defining $\tilde{s} := s - 1$ an equivalent statement to (6) is

$$\det(\tilde{s}I + \frac{k}{n}\mathbf{1}\mathbf{1}^{\mathrm{T}} C^{-1}) = \det \begin{bmatrix} \tilde{s} + \frac{k}{nc_1} & \frac{k}{nc_2} & \cdots & \frac{k}{nc_n} \\ \frac{k}{nc_1} & \tilde{s} + \frac{k}{nc_2} & \cdots & \frac{k}{nc_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{k}{nc_1} & \frac{k}{nc_2} & \cdots & \tilde{s} + \frac{k}{nc_n} \end{bmatrix} = 0. \tag{7}$$

This is clearly satisfied when $\tilde{s} = 0$, since then we have a matrix with n identical rows. Since the nullspace of this matrix has dimension $n-1$, there are $n-1$ linearly independent eigenvectors with corresponding eigenvalue $s = \tilde{s}+1 = 1$. The last eigenvalue is given by the rule $\mathrm{Trace}(A) =$

$\sum_{i=1}^{n} \mu_i$ where μ_i is the i^{th} eigenvalue of A. By inspection of (5), $\mu_1 = \mu_2 = \ldots = \mu_{n-1} = 1$, and $\mu_n = 1 - \frac{k}{n}\sum_{i=1}^{n}\frac{1}{c_i}$.

The n^{th} eigenvector u_n satisfies $Au_n = \mu_n u_n$, or equivalently $(\mu_n I - A)u_n = 0$. Writing this out we see that this is satisfied iff all elements of u_n are equal. We can therefore define the normalized eigenvector $u_n = \frac{1}{\sqrt{n}}\mathbf{1}$, and arbitrarily define the first $n-1$ eigenvectors to be orthonormal vectors satisfying $u_i \perp u_n,\ i = 1, \ldots, n-1$.

Since the eigenvectors of A span the whole of $\mathbb{R}^n$, we can write any price vector $\tilde{\lambda}_j$ as $\tilde{\lambda}_j = \theta_{1j}u_1 + \ldots + \theta_{nj}u_n$. A can be diagonalized in the form $A = U\Lambda U^{-1}$, where $U := [u_1 \cdots u_n]$ and $\Lambda := \text{diag}(\mu_1, \ldots, \mu_n)$. Since the eigenvectors can be chosen to be an orthonormal set, $U^{-1} = U^{\text{T}}$. From these statements we obtain, after defining $\gamma := \frac{k}{n}\sum_i^n \frac{1}{c_i}$:

$$\begin{aligned}
\tilde{\lambda}_j &= U\Lambda U^{\text{T}}\tilde{\lambda}_{j-1} \\
&= U\Lambda^j U^{\text{T}}\tilde{\lambda}_0 \\
&= \left[u_1\ u_2 \cdots u_{n-1}\ \tfrac{1}{\sqrt{n}}\mathbf{1}\right]
\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & (1-\beta)^j \end{bmatrix}
\begin{bmatrix} u_1^{\text{T}} \\ u_2^{\text{T}} \\ \vdots \\ u_{n-1}^{\text{T}} \\ \frac{1}{\sqrt{n}}\mathbf{1}^{\text{T}} \end{bmatrix} \tilde{\lambda}_0 \\
\Rightarrow \tilde{\lambda}_j &= \theta_{10}u_1 + \theta_{20}u_2 + \ldots + \theta_{[n-1]0}u_{n-1} + \theta_{n0}(1-\beta)^j\mathbf{1}.
\end{aligned}$$

In other words, an initial price vector λ_0 will therefore only be regulated to the optimum when it starts parallel to u_n, i.e. all the nodal prices are the same. In this case we have:

$$\tilde{\lambda}_j = \frac{\theta_{n0}}{\sqrt{n}}(1-\beta)^j\mathbf{1} = (1-\beta)^j\tilde{\lambda}_0 \quad \Rightarrow \quad \lambda_j = \lambda^\star\mathbf{1} + (1-\beta)^j(\lambda_0 - \lambda^\star\mathbf{1}). \tag{8}$$

For convergence we therefore need to choose $|1-\gamma| < 1 \Leftrightarrow 0 < \gamma < 2$. Therefore the feedback gain k must be chosen according to

$$0 < k < 2n\left[\sum_{i=1}^{n}\frac{1}{c_i}\right]^{-1} \quad \Leftrightarrow \quad 0 < k < \frac{2n}{\mathbf{1}^{\text{T}}C^{-1}\mathbf{1}}. \tag{9}$$

Using the notation of the paper and substituting m_{e} as explained before, we have the same condition as given in the paper. Clearly if we knew the parameter c_i of each generator we would be able to set $\gamma = 1$ and we would have convergence in one step to a power match. Deviation from this ideal in practice is not desired, but at least choosing a very small k would give convergence.

Convergence rate

Rearranging equation (8), we have:

$$\lambda_j - \lambda^\star \mathbf{1} = (1-\beta)^j (\lambda_0 - \lambda^\star \mathbf{1})$$

From A5 we can write this in terms of power mismatches $\epsilon_j := \left\| C^{-1}(\lambda_j - \lambda^\star \mathbf{1}) \right\|$, where $\|\cdot\|$ is any desired norm:

$$\begin{aligned} C^{-1}(\lambda_j - \lambda^\star \mathbf{1}) &= (1-\beta)^j C^{-1}(\lambda_0 - \lambda^\star \mathbf{1}) \\ \epsilon_j &= \left|(1-\beta)^j\right| \epsilon_0 \end{aligned}$$

Rearranging this, we have:

$$\begin{aligned} \frac{\epsilon_j}{\epsilon_0} &= \left|(1-\beta)^j\right| \\ \ln\left(\epsilon_j/\epsilon_0\right) &= j \ln|1-\beta| \end{aligned}$$

Therefore for a desired accuracy ϵ, N_{conv} steps are required, as given by:

$$N_{\text{conv}} = \left\lceil \frac{\ln\left(\epsilon/\epsilon_0\right)}{\ln|1-\beta|} \right\rceil,$$

where the ceiling operator reflects the fact that N_{conv} must be an integer.

Comparison with Jokić's result

The KKT controller of [3] used separate control of two price components to obtain a feasible and optimal solution to the power flow problem. Firstly a controller for the common base price "λ_0" was used to bring about a power match globally (via an assumption on how a power mismatch would affect frequencies separately at nodes), and then congestion prices "$\Delta\lambda$" were used to add price differences that bring about flows that relieve line congestion.

Here the behaviour of the "simple" $\phi = \frac{1}{n}\mathbf{1}$ rule is very similar in effect to the part of that continuous-time controller for the base price, except that the stability criteria are different because we operate in discrete time and under slightly different assumptions ([3] gave no closed-form stability conditions on the controller gains, but gave a means of using an LMI solver to check for stability for any given set of gains).

3.2 Variable-ϕ price updates

Consider again the proportional update rule (2):

$$\kappa_{\mathrm{L}}(p^\star(\lambda_j), \ell, \Sigma) := k(\mathbf{1}^{\mathrm{T}} p^\star(\lambda_j) - \hat{p})\phi(p^\star(\lambda_j), \ell, \Sigma)$$

We now consider the case where ϕ is chosen in response to the state of the network. Bialek [2] presented an algorithm for assigning flows on electricity lines and supply to loads to the generators on the network. Conversely it is possible to trace line flows and generator outputs to the loads present using essentially the same procedure. We use such an algorithm to choose ϕ for the price updates.

Since analysis of this method is very difficult (some complicated analytic results for ϕ as a function of the generators and loads were only given this year [1], 14 years after the original paper), here is an alternative strategy based on Lyapunov functions.

Given a price evolution $\tilde{\lambda}_{j+1} = A(\phi_j)\tilde{\lambda}_j$, where $A(\phi_j) := I - k\phi_j \mathbf{1}^{\mathrm{T}} C$ from (4), we take a Lyapunov function $V(\tilde{\lambda}_j) := \tilde{\lambda}_j^{\mathrm{T}} \tilde{\lambda}_j$. Since such a function fulfils the other required properties of a Lyapunov function, convergence to an optimal set of prices ($\tilde{\lambda}_j = 0$) can be guaranteed by taking ϕ_j at each step and ensuring that:

$$\begin{aligned} V(\tilde{\lambda}_j) - V(\tilde{\lambda}_j) &< 0 \\ \Leftrightarrow (A(\phi_j)\tilde{\lambda}_j)^{\mathrm{T}}(A(\phi_j)\tilde{\lambda}_j) - \tilde{\lambda}_j^{\mathrm{T}}\tilde{\lambda}_j &< 0 \\ \Leftrightarrow \tilde{\lambda}_j^{\mathrm{T}}(I - A(\phi_j)^{\mathrm{T}} A(\phi_j))\tilde{\lambda}_j &> 0 \quad \forall \tilde{\lambda}_j \in \mathbb{R}^n \end{aligned} \tag{10}$$

This last condition is equivalent to requiring that $Q(\phi_j) := I - A(\phi_j)^{\mathrm{T}} A(\phi_j)$ is positive definite. Since Q is also a function of the feedback gain k, we can find bounds on k in terms of ϕ_j such that Q remains positive definite and the negotiation always converges.[1]

It turns out that element q_{ij} of Q is given simply by:

$$q_{ij} = \frac{k}{c_i}\phi_j + \frac{k}{c_j}\phi_i - \frac{k^2}{c_i c_j}\phi^{\mathrm{T}}\phi$$

Now a necessary condition for $Q(\phi)$ to be positive definite, and therefore for $V(\tilde{\lambda}_j) := \tilde{\lambda}_j^{\mathrm{T}}\tilde{\lambda}_j$ to qualify as a Lyapunov function, is that $q_{ii} > 0$ for $i = 1, \ldots, n$. This is the case if

$$0 < k < \frac{2c_i\phi_i}{\phi^{\mathrm{T}}\phi}, \quad i = 1, \ldots, n$$

so given a vector ϕ resulting from the allocation algorithm, we have a necessary condition for having found a successful Lyapunov function that looks similar to the constant-ϕ result (9). Just as before though, we don't know the coefficients c_i, and because we only found a necessary condition for a sufficient one to hold, we didn't guarantee anything.

Since Q is symmetric, another approach may be to find a full-rank matrix M such that $Q(\phi) = M(\phi)^{\mathrm{T}} M(\phi)$. If such an $M(\phi)$ exists, then $Q(\phi)$ is positive definite.

[1]Note that k could be chosen more elaborately (e.g. a weighted diagonal marix) but this would be analytically more difficult and not fair - the ϕ vector is already supposed to be weighting the price changes.

Another approach is to find the eigenvalues of Q. Since Q is symmetric, it is enough to show that all eigenvalues are strictly positive for Q to positive definite. Since from (3.2) we have

$$\mathrm{Trace}(Q) = \sum_{i=1}^{n} q_{ii} = \sum_{i=1}^{n} \left(\frac{2k}{c_i}\phi_i - \frac{k^2}{c_i^2}\phi^{\mathrm{T}}\phi \right)$$

which is equal to the sum of Q's eigenvalues, we could claim using a symmetry argument (not checked yet) that the eigenvalues of Q are simply the q_{ii}. In this case we have

$$q_{ii} > 0 \Leftrightarrow 0 < k < \frac{2c_i\phi_i}{\phi^{\mathrm{T}}\phi}$$

which is the same condition as before. It seems possible that this could be a *sufficient* condition for $V(\tilde{\lambda}_j)$ to be a Lyapunov function for the price evolution. In this case, we can simply set k in response to the ϕ vector obtained from Bialek's algorithm.

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