VIRTUAL FLAT CHAINS AND HOMOLOGIES IN METRIC SPACES

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Abstract

In the first part of this thesis we analyze virtual rectifiable and virtual flat chains with coefficients in a normed group $G$ in a metric space as defined by Thierry De Pauw and Robert Hardt in [DPH09]. We show various results: rectifiability of finite mass chains and a rectifiable slices theorem (see [DPH09]), compactness results, equivalence of flat and ’inner flat’ convergence, an isoperimetric inequality for small cycles and the existence of a mass-minimizer in a fixed homology class. This is analogous to results known for Euclidean space and partially for Banach spaces, or for (integer) rectifiable currents. These outcomes depend on $G$ and sometimes on certain properties of the metric space.

The second part is a joint work with Daniel Schäppi. In a metric space, we compare the homology groups of the chain complex of integral currents with compact support with the singular Lipschitz homology and with ordinary singular homology. If the metric space satisfies certain cone inequalities all these homology theories coincide. On the other hand, for the Hawaiian Earring the homology of integral currents differs from the singular Lipschitz homology and it differs also from the classical singular homology.
Zusammenfassung


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my friends, my family, my colleagues
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Part I

Virtual flat $G$-chains in metric spaces
Chapter 1

Introduction

We will show that the theory of virtual flat chains in a metric space (with coefficients in a normed group) as defined in [DPH09] has various properties analogous to such of (rectifiable) metric currents.

1.1 History

In 1960, H. Federer and W. H. Fleming ([FF60]) give the definition of generalized manifolds with integer coefficients (called integral currents) in Euclidean space. They show e.g. a deformation theorem, a closure theorem, an isoperimetric inequality and the existence of mass-minimizing currents in homology classes. Fleming generalizes this in [Fle66], defining flat chains with coefficients in a normed (abelian) group in Euclidean space (the flat distance he uses was defined by H. Whitney [Whi57] for real coefficients). He shows that for a finite group every flat chain of finite mass is rectifiable; this result was proved by W. P. Ziemer [Zie62] for the group $\mathbb{Z}_2$ in special dimensions. Federer’s book ([Fed69]) contains many results for integral currents such as closure and compactness theorems. In 1999, B. White ([Whi99a]) generalizes Fleming’s result to the following: In Euclidean space $\mathbb{R}^m$, every $n$ dimensional finite mass $G$-chain is rectifiable for $m > n$ if and only if there exists no non-constant continuous path of finite length in $G$; here $G$ is a complete normed group. At the same time, L. Ambrosio and B. Kirchheim publish their theory of metric currents ([AK00]) which contains good generalizations of parameterized sets in complete metric spaces with integral (or real) coefficients; these are called integral (resp. rectifiable) currents. They show closure and compactness theorems.

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as in Euclidean space. S. Wenger considers such integral currents in [Wen07]. Assuming the metric space to admit local cone type inequalities he shows that the weak (i.e. pointwise) convergence of such integral currents is equivalent to convergence with respect to the flat metric (in Euclidean space, this is also in [FF60]). As shown therein, the existence of a mass-minimizing current in a fixed homology class then follows. T. Adams defines in [Ada08] flat $G$-chains in Banach spaces and proves a compactness theorem provided that the normed group $G$ is proper. The flat $G$-chains obtained by Adams are the completion with respect to the flat norm of polyhedral $G$-chains. T. De Pauw and R. Hardt in [DPH09] generalize the definition of flat $G$-chains to arbitrary metric spaces and prove in this setting that if there exists no non-constant continuous path of finite length in $G$ then every finite mass chain is rectifiable. Thus, they generalize (one direction of) White’s result to complete metric spaces. Finally, De Pauw and Hardt define virtual flat chains and note that their result holds for such chains too.

Other recent work in this area are: [Ste10] on the rectifiability of metric flat chains (here, $\mathbb{R}$-coefficients are considered); [AW09] and [AK09], both working with currents whose coefficients are in $G = \mathbb{Z}_p$ (they give a generalization of the corresponding definition of Federer [Fed69] in the Euclidean case): among other results they prove an isoperimetric inequality and bounds on the filling radius. There is an announced paper by Ambrosio, Katz and Wenger on this topic.

By [Lan08] there is a theory of local currents in locally compact metric spaces (analogous to the theory of metric currents). Our construction would apply there too; compare also [Whi09] for the Euclidean case.

1.2 Virtual flat chains

Let $(X, d)$ be a complete metric space and let $(G, \|\|)$ be a normed group, e.g. $G = \mathbb{R}$ (or $\mathbb{Z}$) with the (induced) Euclidean norm. Flat chains in Banach spaces are defined as the completion of ‘polyhedral’ chains with respect to a flat norm (actually, we do not consider polyhedral chains, but Lipschitz push-forwards of such). To define virtual flat chains in a complete metric space $X$ we embed $X$ isometrically into $l^\infty(X)$. Now, virtual $n$-dimensional flat chains in $X$ are defined as the flat chains in $l^\infty(X)$ that have support in the image of the embedding; we denote this space by $\mathcal{F}_n(X, G)$. Analogously we define virtual rectifiable chains as the rectifiable chains in $l^\infty(X)$ whose support is in the image.
1.3. OVERVIEW AND RESULTS

Virtual flat chains are convenient in various ways: For example, the definitions of mass and flat norm are invariant for isometric embeddings. Or, for a closed subset \( A \subset X \) is

\[
F_n(A,G) = F_n(X,G) \cap \{ F \mid \text{spt } F \subset A \}.
\]

Also, the definition of virtual chains allows us to work in Banach spaces, this eases many proofs.

As it turns out, virtual chains share many properties with metric currents. For real coefficients we even have: The set \( R_n(X,\mathbb{R}) \) of virtual rectifiable \( n \)-chains with coefficients in \( \mathbb{R} \) (with the Euclidean norm) we can identify isometrically with \( R_n(X) \), the set of rectifiable currents as defined in [AK00]. This holds analogously for \( R_n(X,\mathbb{Z}) \) and the integer rectifiable currents.

Remark that the definitions that we use are not exactly the ones of [DPH09]: We consider a different (but bi-Lipschitzly equivalent) mass and also a slightly different replacement of 'polyhedral' chains (compare for this Appendix A).

1.3 Overview and results

Part I is organized as follows:

Chapter 2 introduces our notation and collects facts we need later: integral currents and properties of these, normed groups, injective metric spaces, weak convergence of measures, Lévi-Prokhorov metric on finite Borel measures.

Chapter 3 gives the definition of the group \( P_n(X,G) \) for a normed group \( G \) when \( X \) is a complete metric space. This will be used in Chapter 4 to define virtual flat chains. The mass and other basic constructions and results are explained: mass, push-forward, boundary, slicing, restriction, measure, support, flat norm, lower semicontinuity of mass. This is covered by [DPH09]; our presentation is different and we use another (but bi-Lipschitz equivalent) mass.

Chapter 4 introduces virtual rectifiable and virtual flat chains with coefficients in \( G \) as defined by De Pauw and Hardt in complete metric spaces. This is done first for injective metric spaces and then for complete metric spaces. Definitions and constructions as in Chapter 3 are transferred to virtual chains, e.g., mass and boundary. We show representation theorems as in (and based on) [AK00] for virtual rectifiable chains. Again, this section is as in [DPH09].

In Chapter 5 we give different compactness results (similar to the ones for Euclidean spaces in [Fed69]; compare also [Ada08] for Banach spaces, [AK00] for metric currents). E.g., if a sequence of flat \( G \)-chains has bounded mass,
bounded boundary mass and all the supports lie in a common compact set, then there is a converging subsequence provided $\bar{G}$ (the completion of the normed group $G$) is proper.

In Chapter 6 we show that the measure of any $n$-dimensional finite mass chain is absolutely continuous with respect to the $n$-dimensional Hausdorff measure $H^n$, that rectifiability is equivalent to having a measure that is concentrated on a countably $H^n$-rectifiable set, and that a finite mass chain is rectifiable if and only if all its slices are rectifiable a.e. Also, we recall the generalization of White’s result: If there is no non-constant continuous path of finite length in $\bar{G}$ then every finite mass $G$-chain is rectifiable. These results are in [DPH09]; sometimes we give different proofs.

Chapter 7 uses a construction due to White (see e.g. [Whi99a]) to define semi-norms on $\mathcal{F}_n(X, G)$, for example the flat size. A rectifiability result is obtained, comparable to results in [Whi99a] for Euclidean space and [AK00] for rectifiable currents, e.g.: Every finite mass chain of finite flat size is rectifiable.

In Chapter 8 we consider only discrete normed groups $G$. Similar to [Wen07] for integral currents, we obtain e.g. the following for a metric space $X$ that satisfies local cone type inequalities: A sequence of flat chains with bounded masses and without boundaries is converging to zero if and only the filling volume in $X$ of the sequence goes to zero. (In Euclidean space, such a result is in [FF60] for integral currents.) Also, an isoperimetric inequality holds for chains of small mass and without boundary, exactly as in [Wen07].

Chapter 9 shows that (similar to the Euclidean case of integer rectifiable currents in [FF60]), if $X$ is compact and satisfies local cone type inequalities, and if $A \subset X$ is a Lipschitz neighborhood retract, then in each homology class $h \in H^F_n(X, A; G)$ there is a mass-minimizing chain provided $G$ is a discrete and proper normed group. This is similar to the corresponding result for integer rectifiable currents of [Wen07] and fixes a gap of the proof therein.

Chapter 10 deals with subgroups $H \subset G$, modelling the case $\mathbb{Z} \subset \mathbb{R}$ more closely. We indicate that the results obtained so far are all applicable to this case.
Chapter 2

Notation and basics

In this chapter we introduce our notation and recall definitions and properties that we need later on (e.g. rectifiable metric currents, normed groups).

2.1 Notation

We write $(X, d)$ or $X$ for a metric space with fixed metric $d$ (or $d_X$). We will always assume that the cardinality of $X$ is an Ulam number (as is done in [AK00]), see Remark 2.1.1.

We say that a sequence $(x_i)_{i \in \mathbb{N}}$ in a metric space $(X, d)$ converges rapidly if and only if $\sum_{i \in \mathbb{N}} d(x_i, x_{i+1}) < \infty$.

In Euclidean space $\mathbb{R}^n$ we denote by $||$ the usual norm that comes from the standard inner product; $\mathcal{L}^n$ is the $n$-dimensional Lebesgue-measure.

The open (resp. closed) ball of radius $r > 0$ around a non-empty subset $A \subset X$ is denoted by $U_r(A)$ or $U(A, r)$ (resp. $B_r(A)$ or $B(A, r)$). For $A \subset Z \subset X$, the closed $r$-ball in $Z$ is $B^Z_r(A) := B_r(A) \cap Z$, similarly for the open ball.

The function measuring the distance from $A$ we denote by $d_A(x) := d(A, x) := \inf_{z \in A} d(z, x)$.

Let $A \subset X$, then $\bar{A}$ denotes the closure of $A$, $\overset{\circ}{A}$ the interior of $A$. The diameter of $A$, $\text{diam } A$, is $\max\{0, \sup_{x,y \in A} d(x, y)\}$.

Let $f : X \to \mathbb{R}$ be a function; set $\{ f \leq r \} := f^{-1}((-\infty, r])$ and define similarly sets like $\{ r \leq f < r' \}$ etc.

For metric spaces $X$ and $Y$ the space of Lipschitz maps from $X$ to $Y$ is $\text{Lip}(X, Y)$, i.e. $\text{Lip}(X, Y)$ consists of maps $f : X \to Y$ such that there exists
CHAPTER 2. NOTATION AND BASICS

Let \( \lambda \in \mathbb{R} \) satisfying

\[
d_Y(f(x), f(x')) \leq \lambda d_X(x, x') \quad \forall x, x' \in X.
\] (2.1.1)

By \( \text{Lip}(X) \) we denote the set of Lipschitz functions from the metric space \( X \) to \( \mathbb{R} \) and by \( \text{Lip}^b(X) \) the subset of bounded Lipschitz functions. For \( f \in \text{Lip}(X, Y) \) the Lipschitz constant of \( f \) (i.e. the infimal \( \lambda \) satisfying (2.1.1)) is denoted by \( \text{Lip}(f) \). For \( \lambda \in \mathbb{R} \), \( \text{Lip}_\lambda(X, Y) \) is the set of \( f \in \text{Lip}(X, Y) \) with \( \text{Lip}(f) \leq \lambda \).

The image of a map \( f : A \to B \) is \( \text{im}(f) := \{ b \in B \mid \exists a \in A : f(a) = b \} \).

\( f \in \text{Lip}(X, Y) \) is bi-Lipschitz onto its image if there exists \( \lambda \in \mathbb{R} \) such that

\[
\lambda^{-1} d_X(x, x') \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x').
\] (2.1.2)

If it is clear from the context that the map is not necessarily surjective, 'bi-Lipschitz' means 'bi-Lipschitz onto its image'. An isometric embedding is a map that satisfies (2.1.2) for \( \lambda = 1 \).

The characteristic function of a subset \( A \subset B \) is denoted by \( \chi_A \), so

\[
\chi_A : B \to \{0, 1\}, \quad \chi_A(b) = 1 \quad \text{if and only if} \quad b \in A.
\]

For \( B_i \subset X \), we write \( \bigcup_{i \in \mathbb{N}} B_i \) for the union \( \bigcup_{i \in \mathbb{N}} B_i \) when we want to emphasize that the sets \( B_i \) are pairwise disjoint.

Let \( \mathcal{B}_X \) denote the \( \sigma \)-algebra of Borel sets in \( X \).

By a measure on \( X \) we mean what is sometimes called an outer measure: a monotone and countably subadditive function \( \mu : 2^X \to [0, \infty] \) with \( \mu(\emptyset) = 0 \).

A Borel measure \( \mu \) is a measure on \( X \) such that all Borel sets are measurable, i.e. for any \( B \in \mathcal{B}_X \) and any \( Y \subset X \) holds that \( \mu(Y) = \mu(Y \setminus B) + \mu(Y \cap B) \). A measure \( \mu \) on \( X \) is Borel-regular if it is a Borel measure and for every \( Y \subset X \) there is \( B \in \mathcal{B}_X \) with \( Y \subset B \) and \( \mu(B) = \mu(Y) \).

If \( \Omega \) is a \( \sigma \)-algebra in \( X \), by a measure on \( (X, \Omega) \) we mean a monotone and countably additive function \( \mu : \Omega \to [0, \infty] \) with \( \mu(\emptyset) = 0 \).

A measure \( \mu \) on \( X \) (respectively on \( (X, \Omega) \)) is concentrated on \( Y \subset X \) (resp. \( Y \in \Omega \)), if \( \mu(X \setminus Y) = 0 \). The support of \( \mu \) is the set \( \text{spt} \mu := \{ x \in X \mid \forall r > 0 : \mu(B_r(x)) > 0 \} \); note that \( \text{spt} \mu \) is closed. A measure \( \mu \) (on \( X \) or \( (X, \Omega) \)) is finite if \( \mu(X) < \infty \).

Remark 2.1.1. Let \( X \) be a complete metric space. If \( X \) is separable or if the cardinality of \( X \) is an Ulam number, then any finite Borel measure is concentrated on a \( \sigma \)-compact set (see [AK00, Lemma 2.9]); we always assume that the cardinality of \( X \) is an Ulam number.
2.2. \textbf{RECTIFIABLE CURRENTS IN METRIC SPACES}

The \textit{n-dimensional Hausdorff measure} of $Y \subset X$ is defined to be

$$H^n(Y) := \liminf_{\delta \to 0} \left\{ \sum_{k \in \mathbb{N}} \omega_n 2^{-n} (\text{diam}(B_k))^n \mid Y \subset \bigcup_{k \in \mathbb{N}} B_k, \text{diam}(B_k) < \delta \right\},$$

where $\omega_n$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. $H^n$ is a Borel-regular measure (see [Fed69]).

An $H^n$-measurable set $Y \subset X$ is called \textit{countably $H^n$-rectifiable} if there exist countably many Lipschitz maps $f_i : B_i \to X$, $B_i \subset \mathbb{R}^n$, such that

$$H^n\left(Y \setminus \bigcup_{i=1}^{\infty} f_i(B_i)\right) = 0.$$

If $X$ is complete (what we will usually assume), then we can require the sets $B_i$ to be closed. Now we find $K_i \subset \mathbb{R}^n$ compact and Lipschitz maps $f'_i : K_i \to Y$ with pairwise disjoint images such that $H^n(Y \setminus \bigcup_{i=1}^{\infty} f'_i(K_i)) = 0$.

The \textit{n-dimensional lower density} $\Theta_{*n}(\mu, x)$ of a finite Borel measure $\mu$ at a point $x \in X$ is

$$\Theta_{*n}(\mu, x) := \liminf_{0 < r \to 0} \frac{\mu(B_r(x))}{\omega_n r^n}.$$

\section{2.2 Rectifiable currents in metric spaces}

Here we repeat in few words the definitions and properties of metric currents that we use later. For the proofs see [AK00], our presentation follows partially [Wen07].

Let $X$ be a complete metric space. An \textit{n-dimensional metric current} $T$ of \textit{finite mass} on $X$ is a multi-linear functional on

$$D^n(X) := \{(f, \pi) := (f, \pi_1, \ldots, \pi_n) \mid f \in \text{Lip}^b(X), \pi_i \in \text{Lip}(X)\}$$

such that the following properties are satisfied:

(i) If $\pi_i^j \to \pi_i$ pointwise and such that $\sup_{i,j} \text{Lip}(\pi_i^j) < \infty$ then

$$T(f, \pi_1^j, \ldots, \pi_n^j) \to T(f, \pi_1, \ldots, \pi_n).$$

(ii) If $\pi_i$ is constant outside a Borel set $B_i$ and $f|_{\bigcup_{i=1}^{n} B_i} = 0$ then

$$T(f, \pi_1, \ldots, \pi_n) = 0.$$
(iii) There exists a finite Borel measure $\mu$ on $X$ such that
\[
|T(f, \pi_1, \ldots, \pi_n)| \leq \prod_{i=1}^{n} \text{Lip}(\pi_i) \int_X |f|d\mu \tag{2.2.1}
\]
for all $(f, \pi_1, \ldots, \pi_n) \in D^n(X)$.

The space of such currents is denoted by $M_n(X)$, the minimal Borel measure satisfying (2.2.1) is written as $\|T\|$. The mass of $T$, $M_{AK}(T)$, is by definition $\|T\|$. The mass of $T$, $M_{AK}(T)$, is by definition $\|T\|$. The support of $T$ is defined as the support of $\|T\|$. The restriction of $T$ to a Borel set $A \subset X$ is
\[
(T[A])(f, \pi_1, \ldots, \pi_n) := T(f\chi_A, \pi_1, \ldots, \pi_n);
\]
this is indeed well-defined (compare [AK00]). Then we have $T = T[spt(T)]$, $(T + T')[A] = T[A] + T'[A]$, $M_{AK}(T[A]) = \|T\|(A)$ and $(T[A])[B] = T[A \cap B]$. Also
\[
M_{AK}(T) = \sup \sum_{i \in \mathbb{N}} T(\chi_{B_i}, \pi_{i1}, \ldots, \pi_{in}) \tag{2.2.2}
\]
where $B_i$ are pairwise disjoint Borel sets and $\pi_{ij} \in \text{Lip}_1(X)$.

The push-forward of $T$ by $\phi \in \text{Lip}(X, Y)$ is
\[
(\phi \# T)(f, \pi_1, \ldots, \pi_n) := T(f \circ \phi, \pi_1 \circ \phi, \ldots, \pi_n \circ \phi) \in M_n(Y)
\]
where $(f, \pi_1, \ldots, \pi_n) \in D^n(Y)$. Note that this is already well-defined for a Lipschitz map $\phi$: $\text{spt} T \to Y$. Moreover, it holds that
\[
(\phi \# T)[B] = \phi \# (T[\phi^{-1}(B)]) \quad \text{and} \quad M_{AK}(\phi \# T) \leq \text{Lip}(\phi)^nM_{AK}(T). \tag{2.2.3}
\]

If $\phi: X \to Y$ is an isometric embedding then $M_{AK}(\phi \# T) = M_{AK}(T)$.

The restriction of $T \in M_n(X)$ to $(f, \pi) = (f, \pi_1, \ldots, \pi_k) \in D^k(X)$, $k \leq n$, is the current in $M_{n-k}(X)$ defined by
\[
T[(f, \pi)](f', \pi_1', \ldots, \pi_{n-k}') := T(ff', \pi_1', \pi_k, \pi_{11}', \ldots, \pi_{n-k}').
\]

Note that
\[
M_{AK}(T[(1, \pi)]) \leq M_{AK}(T) \prod_{i=1}^{k} \text{Lip}(\pi_i).
\]
2.2. **RECTIFIABLE CURRENTS IN METRIC SPACES**

The boundary of $T$ is $\partial T(f, \pi_1, \ldots, \pi_{n-1}) := T(1, f, \pi_1, \ldots, \pi_{n-1})$ for $n > 0$, else we set $\partial T = 0$. We call $T$ normal if $T \in \mathcal{M}_n(X)$ and if $n > 0$ also $\partial T \in \mathcal{M}_{n-1}(X)$. Note that $\partial \partial T = 0$ and $\phi \# (\partial T) = \partial (\phi \# T)$.

**Definition 2.2.1.** $T \in \mathcal{M}_n(X)$ is called rectifiable if

(i) $\|T\|$ is concentrated on a countably $\mathcal{H}^n$-rectifiable set,

(ii) $\|T\|$ vanishes on $\mathcal{H}^n$-negligible sets.

Furthermore, $T$ is called integer rectifiable if $T$ is rectifiable and for any $\phi \in \text{Lip}(X, \mathbb{R}^n)$ and any open set $U \subset X$ there exists $\theta \in L^1(\mathbb{R}^n, \mathbb{Z})$ such that for all $(f, \pi) := (f, \pi_1, \ldots, \pi_n) \in D^n(\mathbb{R}^n)$

$$
\phi \# (T|_U)(f, \pi) = \int_{\mathbb{R}^n} \theta f \det \left( \frac{\partial \pi_i}{\partial x_j} \right) d\mathcal{L}^n =: \|\theta\|(f, \pi) .
$$

(2.2.4)

The space of rectifiable $n$-currents is denoted by $\mathcal{R}_n(X)$, the group of integer rectifiable $n$-currents by $\mathcal{I}_n(X)$. Integer rectifiable normal currents are called integral currents, this group is denoted by $\mathcal{I}^c_n(X)$; by the boundary-rectifiability theorem [AK00, Theorem 8.6] is the boundary of an integral $n$-current an integral $(n-1)$-current. The group that we basically will work with are the integral currents with compact support, $\mathcal{I}^c_n(X)$, consisting of those integral $n$-currents whose supports are compact (note that $\partial T \in \mathcal{I}^c_{n-1}(X)$ for $T \in \mathcal{I}^c_n(X)$). Also we use $\mathcal{I}^c_n(X)$, the group of integer rectifiable $n$-currents with compact support.

Remark that the push-forward of an integer rectifiable current with compact support (resp. integral current with compact support) is again integer rectifiable with compact support (resp. integral with compact support). Also, the set of integer rectifiable currents with compact support is closed under restriction to Borel sets.

The characteristic set $S_T$ of $T \in \mathcal{R}_n(X)$ is defined by

$$
S_T := \{ x \in X \mid \Theta_{*n}(\|T\|, x) > 0 \} .
$$

(2.2.5)

$S_T$ is countably $\mathcal{H}^n$-rectifiable and $T$ is concentrated on $S_T$ (i.e. $\|T\|$ is concentrated on $S_T$), so $T|_{S_T} = T$. The size of $T \in \mathcal{R}_n(X)$ is defined as

$$
S_{AK}(T) := \mathcal{H}^n(S_T) .
$$

Let $f \in \text{Lip}(X, \mathbb{R}^k)$ and $T \in \mathcal{I}_n(X)$. The slice of $T$ by $f$ at $r \in \mathbb{R}^k$, denoted by $(T, f, r)$, is defined for $\mathcal{L}^k$-almost every $r \in \mathbb{R}^k$ through

$$
\int_{\mathbb{R}^k} \langle T, f, r \rangle \psi(r) dr = T|_{(\psi \circ f, f)} \text{ for all } \psi \in C_c(\mathbb{R}^k) .
$$

(2.2.6)
CHAPTER 2. NOTATION AND BASICS

(This equality is understood as an equality of currents.) If $T \in \mathbf{I}_n^c(X)$ (resp. $\mathcal{I}_n^c(X)$) and $B \in \mathcal{B}_X$, then for $\mathcal{L}^k$-a.e. $r \in \mathbb{R}^k$ is $\langle T, f, r \rangle \in \mathbf{I}_{n-k}^c(X)$ (resp. $\mathcal{I}_{n-k}^c(X)$), $\langle T, f, r \rangle$ is concentrated on $S_T \cap f^{-1}(r)$ and

$$
\langle T|_B, f, r \rangle = \langle T, f, r \rangle|_B.
$$

(2.2.7)

Also we have that

$$
\int_{\mathbb{R}^k} \|\langle T, f, r \rangle\|dr = \|T\|_{(1, f)}.
$$

(2.2.8)

(this has to be understood as an equality of Borel measures) and writing $f(x) = (f_1(x), \ldots, f_k(x)) \in \mathbb{R}^k$ it holds

$$
\int_{\mathbb{R}^k} S_{AK}(\langle T, f, r \rangle)dr \leq c(k, n) \prod_{i=1}^k \text{Lip}(f_i) S_{AK}(T),
$$

(2.2.9)

where $c(k, n) \in \mathbb{R}$ is constant. In the above equality is contained that $r \mapsto M_{AK}(\langle T, f, r \rangle) = \|\langle T, f, r \rangle\|_{(X)}$ is $\mathcal{L}^k$-measurable.

If $\phi \in \text{Lip}(X, Y)$ and $f \in \text{Lip}(Y, \mathbb{R}^k)$, then by (2.2.6) for a.e. $r$

$$
\langle \phi \# T, f, r \rangle = \phi \# \langle T, f \circ \phi, r \rangle.
$$

(2.2.10)

If $f \in \text{Lip}(X)$ and $T' \in \mathbf{I}_n(X)$ then for $\mathcal{L}^1$-a.e. $r$ is

$$
\langle T', f, r \rangle = \partial(T'|_{\{f<r\}}) - (\partial T')|_{\{f<r\}} = (\partial T')|_{\{r \leq f\}} - \partial(T'|_{\{r \leq f\}})
$$

$$
= (\partial T'|_{\{f \leq r\}}) - (\partial T')|_{\{f \leq r\}} = (\partial T')|_{\{r < f\}} - \partial(T'|_{\{r < f\}}).
$$

(2.2.11)

Let $f_i \in \text{Lip}(X, \mathbb{R}^k)$ and $f(x) := (f_1(x), f_2(x))$. For $\mathcal{L}^{k_1+k_2}$-a.e. $r = (r_1, r_2)$ is

$$
\langle T, f, r \rangle = \langle (T, f_1, r_1), f_2, r_2 \rangle = (-1)^{k_1k_2} \langle (T, f_2, r_2), f_1, r_1 \rangle
$$

(2.2.12)

Let $T \in \mathcal{I}_n(X)$, then there exist a sequence of compact sets $K_i \subset \mathbb{R}^n$, functions $\theta_i \in L^1(\mathbb{R}^n, \mathbb{Z})$ with spt $\theta_i \subset K_i$, and bi-Lipschitz maps $f_i: K_i \to X$ with pairwise disjoint images such that (compare (2.2.4))

$$
T = \sum_{i \in \mathbb{N}} f_i \# [\theta_i] \quad \text{and} \quad \sum_{i \in \mathbb{N}} M_{AK}(f_i \# [\theta_i]) = M_{AK}(T).
$$

(2.2.13)

This implies that $\mathcal{I}_n(X)$ is the $M_{AK}$-closure of $\mathcal{I}_n^c(X)$. Also, we see from this representation that instead of (2.2.2) it holds

$$
M_{AK}(T) = \sup \sum_{i \in \mathbb{N}} T(\chi_{U_i}, \pi^i_1, \ldots, \pi^i_n)
$$

(2.2.14)
where $U_i$ are pairwise disjoint open sets and $\pi_j^i \in \text{Lip}_1(X)$.

For $n = 0$ we can write $T \in \mathcal{I}_0(X)$ as $T = \sum_{i \in \mathbb{N}} n_i[x_i]$ with $x_i \in X, n_i \in \mathbb{Z}$. Note that we can assume that $x_i \neq x_j$ for $i \neq j$; hence it is a finite sum.

For the integral $n$-current $J$ on $\mathbb{R}^n$ where $\theta = \chi_{[a_1, b_1] \times \ldots \times [a_n, b_n]}, a_i < b_i \in \mathbb{R}$, we write $[a_1, b_1] \times \ldots \times [a_n, b_n]$.

Let $T \in \mathcal{I}_n(X)$; the area factor on the countably $\mathcal{H}^n$-rectifiable set $S_T$ is a function $\lambda_{S_T}: S_T \to (0, \infty)$, see [AK00, p. 55 and p. 58] for the definition. It satisfies

$$n^{-n/2} \leq \lambda_{S_T} \leq 2^n / \omega_n,$$

and $\lambda_{S_T} = 1$ if $X$ can be isometrically embedded into an inner product space.

For $T \in \mathcal{I}_n(X)$ there exists an $\mathcal{H}^n$-integrable function $\Theta: S_T \to \mathbb{N}$ such that

$$\|T\|(A) = \int_{A \cap S_T} \lambda_{S_T} \Theta d\mathcal{H}^n \quad \text{for all } A \subset X \text{ Borel.}$$

From [Wen07] we have: Let $T \in \mathcal{I}^c_n(X)$, give $\mathbb{R} \times X$ the Euclidean product metric. Set

$$[t] \times T(f, \pi_1, \ldots, \pi_n) := T(f(t, \cdot), \pi_1(t, \cdot), \ldots, \pi_n(t, \cdot)),$$

then $[t] \times T \in \mathcal{I}^c_n(\mathbb{R} \times X)$, $\partial([t] \times T) = [t] \times \partial T$ and $M_{AK}([t] \times T) = M_{AK}(T)$. For $s, r \in \mathbb{R}$ with $s < r$ set

$$[s, r] \times T(f, \pi_1, \ldots, \pi_{n+1}) := \sum_{i=1}^{n+1} \int_s^r \left( f(t, \cdot) \frac{\partial \pi_i(t, \cdot)}{\partial t}, \pi_1(t, \cdot), \ldots, \pi_{i-1}(t, \cdot), \pi_{i+1}(t, \cdot), \ldots, \pi_{n+1}(t, \cdot) \right) dt;$$

this is in $\mathcal{I}^c_{n+1}(\mathbb{R} \times X)$ and $M_{AK}([s, r] \times T) \leq (r - s)(n + 1)M_{AK}(T)$. Then we have

$$\partial([s, r] \times T) = [r] \times T - [s] \times T - [s, r] \times \partial T.$$

Clearly, the above construction works also if we take the supremum metric $d((s, x), (t, y)) := \max\{|s - t|, d(x, y)|$ on $\mathbb{R} \times X$. Note that again we get (directly from the definition)

$$M_{AK}([s, r] \times T) \leq (r - s)(n + 1)M_{AK}(T).$$
2.3 Normed groups

Let $G$ be an abelian group. The operation of $\mathbb{Z}$ on $G$ we write as

$$ng := g + \cdots + g$$

for $n \in \mathbb{Z}$, $g \in G$.

**Definition 2.3.1.** A norm on $G$ is a function $\|\| : G \to [0, \infty)$ such that

(i) $\|-g\| = \|g\|$, $\forall g \in G$,

(ii) $\|g + h\| \leq \|g\| + \|h\|$, $\forall g, h \in G$,

(iii) $\|g\| \geq 0$ with equality if and only if $g = 0$.

**Example 2.3.2.** Let $G$ be an abelian group; we can define a norm an $G$ by setting $\|g\| := 1$ if $g \neq 0$ and 0 else. We will later refer to this norm as size.

**Example 2.3.3.** Let $G \in \{\mathbb{Z}, \mathbb{R}\}$ with the Euclidean norm.

Note that for a group which has also a vector space structure, $(G, \|\|)$ is possibly not a normed vectorspace:

**Example 2.3.4.** Let $G \in \{\mathbb{Z}, \mathbb{R}\}$ with $\|g\| := |g|^\alpha$ for $\alpha \in (0, 1)$.

More examples can be found e.g. in [Whi96, Whi99a].

A normed group we denote by $(G, \|\|)$ or just $G$. A norm induces a metric on $G$ by $d_G(g, h) := \|g - h\|$. We will not assume that $G$ is complete with respect to this metric (in the literature, completeness is often assumed); the main results remain the same when we replace $G$ by its completion right at the beginning. The completion of $G$ with respect to $d_G$ is denoted by $\bar{G}$. This is also a normed group and we denote the extended norm also by $\|\|$.

A discrete normed group is a normed group $G$ such that there exists $\Theta > 0$ with $\|g\| \geq \Theta$ for all $g \in G \setminus 0$. Note that such groups are complete.

2.4 Injective metric spaces

**Definition 2.4.1.** A metric space $X$ is said to be injective if, whenever $X$ is isometric to a subspace $Z$ of a metric space $Y$, then $Z$ is a 1-Lipschitz retract of $Y$, that is, there exists $\phi \in \text{Lip}_1(Y,Z)$ such that $\phi|_Z = \text{id}_Z$.

Note that every injective metric space is complete.
Example 2.4.2. Let $B$ be a set; then
\[ l^\infty(B) := \{ f : B \to \mathbb{R} \mid \|f\|_\infty < \infty \} \]
with the supremum norm is an injective Banach space. First note that for any $f \in \text{Lip}(A, l^\infty(B))$ where $A \subset W$ we can define for $w \in W, b \in B$
\[
(\bar{f}(w))(b) := \inf \{ (f(a))(b) + \text{Lip}(f)d_W(a, w) \mid a \in A \}. \tag{2.4.1}
\]
One easily checks that $\bar{f} \in \text{Lip}(W, l^\infty(B))$ and $\text{Lip}(\bar{f}) = \text{Lip}(f)$ (compare e.g. [Lan05, DPH09]). Now, let $Z, Y$ be as in the definition, let $\alpha : l^\infty(B) \to Z \subset Y$ be an isometry. Then there is an 1-Lipschitz extension $\psi : Y \to l^\infty(B)$ of $\alpha^{-1}$. Now, $\alpha \circ \psi$ is as desired.

Every metric space $X$ admits an isometric embedding into $l^\infty(X)$ by $x \mapsto d_x - d_{x_0}$ where $x_0 \in X$ is a fixed point.

Injective metric spaces are important to us as we can extend Lipschitz maps into such spaces preserving the Lipschitz constant, precisely as for $l^\infty(B)$: Let $X$ be an injective metric space; then for every $f \in \text{Lip}(A, X), A \subset W$, there exists an extension $\hat{f} \in \text{Lip}(W, X)$ of $f$ with $\text{Lip}(\hat{f}) = \text{Lip}(f)$. We can achieve this by embedding $X$ in $l^\infty(X)$, extending the map to a map into $l^\infty(X)$, then retracting and mapping back to $X$. On the other hand, spaces with this extension property are clearly injective, this can be seen as in the example above.

In particular we get: Let $X, Y$ be injective metric spaces; then $X \times Y$ with the supremum metric is again injective.

Moreover, remark that the closed $r$-ball $B_r(0) \subset l^\infty(B)$ is injective as well. We see this by bounding the functions in (2.4.1):
\[
(\hat{f}(w))(b) := \min \{r, \max \{-r, (\bar{f}(w))(b)\}\}. \tag{2.4.2}
\]
So for a $\lambda$-Lipschitz map $A \to B_r(0) \subset l^\infty(B)$ we find a $\lambda$-Lipschitz extension $W \to B_r(0)$. This implies: Let $X$ be an injective metric space, then for all $x \in X, r > 0$ is $B_r(x)$ an injective metric space.

Note also that $\mathbb{R}^n$ is bi-Lipschitz equivalent to $l^\infty(\{1, \ldots, n\})$.

### 2.5 Lévi-Prokhorov metric

The following definitions and results follow directly from [Bil99] (one only has to replace probability measures by finite ones; so for nontrivial measures we normalize, then apply [Bil99], then rescale).
Definition 2.5.1. The sequence \((\mu_i)_{i \in \mathbb{N}}\) of measures on \((X, \mathcal{B}_X)\) converges weakly to the finite measure \(\mu\) on \((X, \mathcal{B}_X)\), \(\mu_i \rightrightarrows \mu\), if and only if for every bounded continuous function \(f : X \rightarrow \mathbb{R}\) holds
\[
\int_X f \, d\mu_i \to \int_X f \, d\mu, \quad (i \to \infty).
\]

Remark 2.5.2. The notation \(\mu_i \rightrightarrows \mu\) is used in [Bil99]; another common notation is \(\mu_i \rightharpoonup \mu\).

Definition 2.5.3. Let \(X\) be a complete metric space. The Lévi-Prokhorov (or Lévi-Prohorov) metric \(d_{LP}\) on the set of finite measures on \((X, \mathcal{B}_X)\) is defined as follows: Let \(\mu, \nu\) be finite measures on \((X, \mathcal{B}_X)\), then
\[
d_{LP}(\mu, \nu) := \inf \{\varepsilon > 0 \mid \text{for all } A \in \mathcal{B}(X): \mu(A) \leq \nu(U_{\varepsilon}(A)) + \varepsilon \\
\quad \quad \quad \quad \text{and} \quad \nu(A) \leq \mu(U_{\varepsilon}(A)) + \varepsilon\} .
\]

Proposition 2.5.4. Let \(X\) be a complete, separable metric space, then the set of finite measures on \((X, \mathcal{B}_X)\) is complete with respect to the metric \(d_{LP}\). Let \(\mu, \mu_i\) be finite measures on \((X, \mathcal{B}_X)\) for \(i \in \mathbb{N}\). Equivalent are:

(i) \(\mu_i \rightrightarrows \mu\) for \(i \to \infty\).

(ii) \(\mu_i(X) \to \mu(X)\) and \(\liminf_{i \to \infty} \mu_i(U) \geq \mu(U)\) for all open \(U \subset X\).

(iii) \(\mu_i(X) \to \mu(X)\) and \(\limsup_{i \to \infty} \mu_i(A) \leq \mu(A)\) for all closed \(A \subset X\).

(iv) \(d_{LP}(\mu_i, \mu) \to 0\) for \(i \to \infty\).

This is [Bil99, Theorem 2.3, Theorem 6.8 and page 73].
Chapter 3

Chains with $G$-coefficients

In this chapter the cornerstones for the definition of virtual flat chains are given. We define the group $\mathbf{P}_n(X, G)$, our replacement of the polyhedral chains in Euclidean space. The set $\mathbf{P}_n(X, G)$ is by definition dense in the set of virtual flat chains (if $X$ is an injective space). This allows us to transfer the constructions for $\mathbf{P}_n(X, G)$ that we explain here to virtual flat chains in Chapter 4.

This chapter is contained in [DPH09]; our presentation is somewhat different. A discrepancy appears in the definition of mass (see Remark 3.2.4), however both definitions are comparable. With our definition the mass is directly lower semicontinuous with respect to flat convergence (and later on also other functionals will be lower semicontinuous).

Another difference is that our starting set (the replacement of the polyhedral chains) is made up using integral currents with compact support whereas De Pauw and Hardt consider singular Lipschitz chains. Both options lead to the same completion (with respect to the mass or the flat norm) first in injective metric spaces and through the definition of virtual chains also for all complete metric spaces (see Appendix A). The reason for our choice is that we can consider slicing without the need of looking at a completion first.

Let $(X, d)$ be a complete metric space and let $(G, \|\|)$ be a normed group. Let $\tilde{\mathbf{P}}_n(X)$ be the free abelian group on $\mathbf{I}_n^c(X)$, define

$$\tilde{\mathbf{P}}_n(X, G) := G \otimes \mathbf{Z} \tilde{\mathbf{P}}_n(X).$$

Elements of $\tilde{\mathbf{P}}_n(X, G)$ we indicate as finite formal sums: $P = \sum_{i=1}^m g_i T_i$ where $g_i \in G$ and $T_i \in \mathbf{I}_n^c(X)$. So it holds $\sum_{i=1}^m (n_i g_i) T_i = \sum_{i=1}^m g_i (n_i T_i)$ for $n_i \in \mathbf{Z}$. 

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Additionally, let
\[ \tilde{\mathcal{P}}_n(X, G) := G \otimes \mathbb{Z} \tilde{\mathcal{P}}_n(X) \]
where \( \tilde{\mathcal{P}}_n(X) \) is the free abelian group on \( \mathcal{I}_c^n(X) \). Again we write elements of \( \mathcal{P}_n(X) \) as finite formal sums as above, now with \( T_i \in \mathcal{I}_c^n(X) \).

So, \( \tilde{\mathcal{P}}_n(X, G) \) and \( \tilde{\mathcal{P}}_n(X, G) \) are abelian groups for the addition, and
\[ -P = \sum_{i=1}^{m} (-g_i)T_i = \sum_{i=1}^{m} g_i(-T_i). \]

Equivalence classes of elements in these groups will give \( \mathcal{P}_n(X, G) \), our 'polyhedral' chains (see Section 3.3).

Note that \( \tilde{\mathcal{P}}_0(X, G) = \tilde{\mathcal{P}}_0(X, G) \); for \( n > 0 \) is \( \tilde{\mathcal{P}}_n(X, G) \subset \tilde{\mathcal{P}}_n(X, G) \).

**Example 3.0.1.** For \( (G, \| \|) = (\mathbb{R}, \| \|) \), the map \( h: \tilde{\mathcal{P}}_n(X, \mathbb{R}) \to \mathcal{R}_n(X) \) given by
\[ \sum_{i=1}^{m} r_i T_i \mapsto \sum_{i=1}^{m} (r_i T_i) \]
is a homomorphism.

The restriction of \( P \in \tilde{\mathcal{P}}_n(X, G) \) to a Borel subset \( Y \subset X \) is
\[ P|_Y := \sum_{i=1}^{m} g_i(T_i|_Y) \in \tilde{\mathcal{P}}_n(X, G). \]

This is clearly well-defined. Then, \( P|_Y + P'|_Y = (P + P')|_Y \) and \( (P|_Y)|_{Y'} = P|_{Y \cap Y'} \), for \( Y' \in \mathcal{B}_X \). Note that for \( f \in \text{Lip}(X) \), \( P \in \tilde{\mathcal{P}}_n(X, G) \) and for almost all \( r \) is \( P|_{\{r < f\}} \in \tilde{\mathcal{P}}_n(X, G) \).

### 3.1 Representation

Our goal here is to find for a fixed \( P = \sum_{i=1}^{m} g_i T_i \in \tilde{\mathcal{P}}_n(X, G) \) pairwise disjoint sets \( C_j \subset X \) such that the currents \( T_i \) agree on \( C_j \) up to a constant factor. More precisely, for each \( i, i' \) we find \( n_{ii'}^{ji}, m_{ii'}^{ji} \in \mathbb{Z} \) such that \( n_{ii'}^{ji} T_i|_{C_j} = m_{ii'}^{ji} T_i'|_{C_j} \).

Also, these sets will cover \( \|T_i\|\)-almost everything of \( X \) for all \( 1 \leq i \leq m \).

We claim that for \( k \in \mathbb{N} \) there exists \( n_{ik} \in \mathbb{N} \) and \( f_k: K_k \to X \) bi-Lipschitz where \( K_k \subset \mathbb{R}^n \) is compact such that \( T_i = \sum_{k \in \mathbb{N}} n_{ik} f_k\# \langle x K_k \rangle \) and hence
\[ P = \sum_{i=1}^{m} g_i \left( \sum_{k \in \mathbb{N}} n_{ik} f_k\# \langle x K_k \rangle \right). \]

Once we achieved this, we define for \( k \in \mathbb{N} \)
\[ h_k := \sum_{i=1}^{m} n_{ik} g_i. \]
Then a representation of $P$ is given by the countable formal sum

$$R := \sum_{k \in \mathbb{N}} h_k f_k \# [\chi_{K_k}] = \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{m} n_{ik} g_i \right) f_k \# [\chi_{K_k}] .$$

(For the moment, one should consider a representation as a sequence instead of a sum. However, our notation will later have its justification as the above sum converges with respect to the mass, see Remark 3.2.2.)

Proof of the claim. From (2.2.13) applied to $T_i, i = 1, \ldots, m$, follows that there exist countably many bi-Lipschitz maps $f_{ij} : K_{ij} \to X$ with $K_{ij} \subset \mathbb{R}^n$ compact such that

$$T_i = \sum_{j \in \mathbb{N}} m_{ij} f_{ij} \# [\chi_{K_{ij}}] \quad \text{for } m_{ij} \in \mathbb{Z}, \tag{3.1.2}$$

and such that $f_{ij}$ and $f_{ij'}$ have disjoint images for $j \neq j'$.

For $I \leq m$, any subset $\Lambda = \{n_1, \ldots, n_I\} \subset \{1, \ldots, m\}$ with $n_i < n_{i+1}$, and $j_i \in \mathbb{N}$ for $1 \leq i \leq I$, consider the sets

$$E_{j_1, \ldots, j_I}^\Lambda := \left( \bigcap_{1 \leq i \leq I} \text{im}(f_{n_{ij_i}}) \right) \setminus \left( \bigcup_{1 \leq k \leq m, k \notin \Lambda, l \in \mathbb{N}} \text{im}(f_{kl}) \right).$$

There are countably many such sets; they are pairwise disjoint and cover the set $\bigcup_{i=1}^{m} \bigcup_{j \in \mathbb{N}} \text{im}(f_{ij})$. Taking the orientation of the currents $T_i |_{E_{j_1, \ldots, j_I}^\Lambda}$ into account, another decomposition of $E_{j_1, \ldots, j_I}^\Lambda$ yields a countable family $(C_k)_{k \in \mathbb{N}}$ of pairwise disjoint compact subsets of $X$ satisfying the following properties:

(i) for every $k$ there is $K_k \subset \mathbb{R}^n$ of positive $\mathcal{L}^n$-measure and a bi-Lipschitz map $f_k : K_k \to C_k$,

(ii) $\mathcal{H}^n \left( (\bigcup_{i=1}^{m} \bigcup_{j \in \mathbb{N}} \text{im}(f_{ij})) \setminus \bigcup_{k \in \mathbb{N}} C_k \right) = 0$,

(iii) $C_k \subset f_{ij}(K_{ij})$ for some $i, j$ (note that then $C_k \cap f_{ij'}(K_{ij'}) = \emptyset$ for $j \neq j'$),

(iv) if $C_k \cap f_{ij}(K_{ij}) \neq \emptyset$ then

$$f_k \# [\chi_{K_k}] = \pm (f_{ij} \# [\chi_{K_{ij}}]) \mid_{C_k} .$$
We define \((i, j, k) \mapsto l(i, j, k)\) to be \(m_{ij}\) (see (3.1.2)) if the sign in (iv) is positive, 
\(-m_{ij}\) if it is negative and 0 if \(C_k \cap f_{ij}(K_{ij}) = \emptyset\). Note that for \(i, k\) fixed, 
\(l(i, j, k) \neq 0\) implies that \(l(i, j', k) = 0\) for all \(j' \neq j\). So, \(n_{ik} := \sum_{j \in \mathbb{N}} l(i, j, k)\) is defined and then
\[
n_{ik} f_k \# [\chi_{K_k}] = T_i |c_k \quad \text{and} \quad \sum_{k \in \mathbb{N}} n_{ik} f_k \# [\chi_{K_k}] = T_i. \tag{3.1.3}
\]

\[\square\]

**Remark 3.1.1.** Having representations \(R\) of \(P\) and \(R'\) of \(P'\) we can find ‘refined representations’ of \(P\) and \(P'\): Let
\[
R = \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{m} n_{ik} g_i \right) f_k \# [\chi_{K_k}] \quad \text{and} \quad R' = \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{m'} n'_{ik} g_i' \right) f_k' \# [\chi_{K'_k}].
\]
Replacing in the above proof the metric currents \(T_i\) by the two sequences 
\((f_k \# [\chi_{K_k}])_{k \in \mathbb{N}}\) and \((f_k' \# [\chi_{K'_k}])_{k \in \mathbb{N}}\), we can apply the same procedure as in
the proof (therein we worked with the sequences \((f_i)_{j \in \mathbb{N}}\) for \(i = 1, \ldots, m\)). So we get bi-Lipschitz maps \(\tilde{f}_k : \tilde{K}_k \rightarrow \tilde{C}_k\) which satisfy (adjusted versions of) (i)-(iv). With appropriately chosen \(\tilde{n}_{ik}, \tilde{n}'_{ik} \in \mathbb{Z}\) we then get representations \(\tilde{R} = \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{m} \tilde{n}_{ik} g_i \right) \tilde{f}_k \# [\chi_{\tilde{K}_k}]\) of \(P\) as well as \(\tilde{R}' = \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{m'} \tilde{n}'_{ik} g_i' \right) \tilde{f}_k' \# [\chi_{\tilde{K}'_k}]\) of \(P'\). Note that the same maps \(\tilde{f}_k\) are used for both representations and that
(3.1.3) still holds. Moreover, remark that this argument works also for finitely
many representations instead of only two.

## 3.2 Mass

We define the mass through representations:

**Definition 3.2.1.** The mass \(M(P)\) of \(P \in \mathcal{P}_n(X, G)\) is defined by
\[
M(P) := M(R) := \sum_{k \in \mathbb{N}} \left\| \sum_{i=1}^{m} n_{ik} g_i \right\| M_{AK}(f_k \# [\chi_{K_k}]). \tag{3.2.1}
\]
where \(R = \sum_{k \in \mathbb{N}} \left( \sum_{i=1}^{m} n_{ik} g_i \right) f_k \# [\chi_{K_k}]\) is any representation of \(P\).

Note that \(M\) is finite: As \(\| \sum_{i=1}^{m} n_{ik} g_i \| \leq \sum_{i=1}^{m} |n_{ik}| \| g_i \|\) we have by (3.1.3)
\[
M(P) \leq \sum_{i=1}^{m} \| g_i \| \left( \sum_{k \in \mathbb{N}} |n_{ik}| M_{AK}(f_k \# [\chi_{K_k}]) \right) = \sum_{i=1}^{m} \| g_i \| M_{AK}(T_i) < \infty.
\]
3.2. MASS

We have shown that a representation exists. Given two representations \( R, R' \) of \( P \) we find a refined representation \( \tilde{R} \) by Remark 3.1.1. Via (3.1.3) we get equality for the mass of \( R \) and \( \tilde{R} \) as well as for \( R' \) and \( \tilde{R} \). Hence the mass is well-defined.

**Remark 3.2.2.** Let \( P \in \tilde{\mathcal{P}}_n(X, G) \) and let \( R = \sum_{i \in \mathbb{N}} h_i f_i \# [\chi_{K_i}] \) be a representation of \( P \). Then \( R_m := \sum_{i=1}^m h_i f_i \# [\chi_{K_i}] \) is in \( \mathcal{P}_n(X, G) \) and converges to \( P \) with respect to the mass, i.e., \( \mathbf{M}(P - R_m) \to 0 \) for \( m \to \infty \).

**Lemma 3.2.3.** The mass is the unique function mapping \( \tilde{\mathcal{P}}_n(X, G) \) to \( \mathbb{R}_{\geq 0} \) that satisfies

(i) For \( Y_i \in \mathcal{B}_X \) pairwise disjoint is \( \sum_{i \in \mathbb{N}} \mathbf{M}(P|Y_i) = \mathbf{M}(P|\bigcup_{i \in \mathbb{N}} Y_i) \).

(ii) If \( P = \sum_{j=1}^m g_j (n_j f \# [\chi_K]) \) where \( n_j \in \mathbb{Z}, K \subset \mathbb{R}^n \) compact and \( f : K \to \text{im}(f) \subset X \) bi-Lipschitz then

\[
\mathbf{M}(P) = \left\| \sum_{j=1}^m n_j g_j \right\| \mathbf{M}_{AK}(f \# [\chi_K]).
\]

Moreover, the mass satisfies for \( P, P' \in \tilde{\mathcal{P}}_n(X, G), g \in G \) and \( T \in \mathcal{T}^c_n(X) \)

\[
\mathbf{M}(P + P') \leq \mathbf{M}(P) + \mathbf{M}(P') \quad \text{and} \quad \mathbf{M}(gT) \leq \left\| g \right\| \mathbf{M}_{AK}(T). \quad (3.2.2)
\]

If \( G = \mathbb{R} \) with the Euclidean norm, then for \( h \) as in Example 3.0.1 holds

\[
\mathbf{M}(P) = \mathbf{M}_{AK}(h(P)). \quad (3.2.3)
\]

**Proof.** From the definition we see that (i) and (ii) are satisfied.

Let \( \tilde{R}, R, R' \) be representations of \( P + P', P, P' \), respectively. Applying Remark 3.1.1 to these three representations we get refined representations

\[
\tilde{r} = \sum_{k \in \mathbb{N}} \tilde{l}_k f_k \# [\chi_{K_k}] \quad \text{of} \quad P + P',
\]

as well as \( r = \sum_{k \in \mathbb{N}} l_k f_k \# [\chi_{K_k}] \) of \( P \) and \( r' = \sum_{k \in \mathbb{N}} l'_k f_k \# [\chi_{K_k}] \) of \( P' \). By (3.1.3) we have \( \tilde{l}_k = l_k + l'_k \). By the subadditivity of the norm on \( G \) follows

\[
\mathbf{M}(P + P') = \mathbf{M}\left( \sum_{k \in \mathbb{N}} \tilde{l}_k f_k \# [\chi_{K_k}] \right) = \sum_{k \in \mathbb{N}} \left\| \tilde{l}_k \right\| \mathbf{M}_{AK}(f_k \# [\chi_{K_k}]) \leq \sum_{k \in \mathbb{N}} \left( \left\| l_k \right\| + \left\| l'_k \right\| \right) \mathbf{M}_{AK}(f_k \# [\chi_{K_k}]) \leq \mathbf{M}(P) + \mathbf{M}(P').
\]
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This shows the first part of (3.2.2). As $\|ng\| \leq |n| \|g\|$ for $n \in \mathbb{Z}$, $g \in G$, the second part is a direct consequence of the definition of $M$.

Uniqueness: By the construction of the sets $C_k$ we have $T_i|_{X\setminus \bigcup_{k \in \mathbb{N}} C_k} = 0$ and hence $P|_{\bigcup_{k \in \mathbb{N}} C_k} = P$. For any mass $\tilde{M}$ now follows by (i), (ii) and (3.1.3)

$\tilde{M}(P) = M(P|_{\bigcup_{k \in \mathbb{N}} C_k}) = \sum_{k \in \mathbb{N}} M(P|_{C_k}) = \sum_{k \in \mathbb{N}} M(P|_{C_k}) = M(P)$.

Hence $M$ is unique.

If $\langle G, \|\cdot\| \rangle = (\mathbb{R}, \|\cdot\|)$, $M_{AK} \circ h$ has properties (i) and (ii) that define $M$. □

Remark 3.2.4. One can define the mass used in [DPH09] replacing (3.2.1) by

$M_{DH}(R) := \sum_{k \in \mathbb{N}} \left\| \sum_{i=1}^m n_{ik} g_i \right\| \mathcal{H}^n(f_k(K_k))$.

From (2.2.15) we see that $n^{-n/2}M_{DH}(R) \leq M(R) \leq 2^n/\omega_n M_{DH}(R)$. The reason for our choice is that our mass is directly lower semicontinuous with respect to flat convergence (see Theorem 3.9.2).

3.3 The group $P_n(X, G)$

Define on $\hat{\mathcal{P}}_n(X, G)$

$P \sim P' \quad$ if and only if $\quad M(P - P') = 0$.

This is clearly an equivalence relation. Set

$\mathcal{P}_n(X, G) := \hat{\mathcal{P}}_n(X, G)/\sim$.

Definition 3.3.1. An $n$-chain with $G$-coefficients is an element of

$P_n(X, G) := \{ [P] \in \mathcal{P}_n(X, G) \mid P \in \hat{P}_n(X, G) \}$.

Set $M([P]) := M(P)$; this is well-defined by (3.2.2). Note that for $T \in T_\infty^c(X)$, $g \in G$ and $n \in \mathbb{Z}$ is $[g(nT)] = [(ng)T]$.

For shorter notation we write $P$ for either $P \in \hat{P}_n(X, G)$ or $[P] \in \mathcal{P}_n(X, G)$ if it is clear what we mean. Similarly we do for $\hat{P}_n(X, G)$ and $P_n(X, G)$.

Now, $\mathcal{P}_n(X, G)$ is an abelian group and $M$ is a norm on it. Moreover, $P_n(X, G)$ is a (normed) subgroup of $\mathcal{P}_n(X, G)$.

If $G = \mathbb{R}$ with the Euclidean norm then the last lemma implies that $h$ is well-defined on $\mathcal{P}_n(X, \mathbb{R})$; it is then an isometric embedding and a homomorphism.

For Euclidean space $X = \mathbb{R}^k$ we have the following:
Lemma 3.3.2. Every $P \in \tilde{P}_n(\mathbb{R}^k, G)$ induces a rectifiable $n$-chain in the sense of Fleming, denoted by $Fl(P)$. Then $M(P) = M_{Fl}(Fl(P))$.

Proof. Let $R = \sum_{j \in \mathbb{N}} h_j f_j \# [\chi_{K_j}]$ be a representation of $P$. Let $Q_{ij}$ be a polyhedral approximation of $K_j$ such that $\mathcal{L}^n(K_j \setminus Q_{ij} \cup Q_{ij} \setminus K_j) \leq 2^{-i-j}$. Extend the Lipschitz maps $f_j$ to Lipschitz maps $F_j: \mathbb{R}^n \to \mathbb{R}^k$. Then $F_j \# Q_{ij}$ are integer rectifiable chains in the sense of Fleming and the Fleming-mass $M_{Fl}$ satisfies $M_{Fl}(F_j \# Q_{ij}) \to H^n(f_j(K_j)) = M_{AK}(f_j \# [\chi_{K_j}])$ (this last equality holds by (2.2.16) as the area factor in Euclidean space and the multiplicity function are both equal to 1). Similarly is $h_j F_j \# Q_{ij}$ a rectifiable $n$-chain with $G$-coefficients in the sense of Fleming and $M_{Fl}(h_j F_j \# Q_{ij}) \to \|h_j\| M_{AK}(f_j \# [\chi_{K_j}])$. From this we see that $R$ induces a rectifiable $n$-chain with $G$ coefficients in the sense of Fleming. So we set $Fl(P) := R$ where $R$ is a representation of $P$. This is well-defined since $M_{Fl}(R - \bar{R}) = 0$ for two representation $R, \bar{R}$ of $P$ (by the same argument as we used in order to show the well-definedness of $M$). Then clearly $M(P) = M_{Fl}(Fl(P))$. \qed

Remark 3.3.3. This lemma implies that there is an isometric embedding of $P_n(\mathbb{R}^k, G)$ into the rectifiable $n$-chains of Fleming in $\mathbb{R}^k$.

3.4 Measure

Let $P = \sum_{i=1}^m g_i T_i \in P_n(X, G)$. Then the measure of $P$ on $X$, $\mu_P$, is

$$\mu_P(Y) := \sum_{j \in \mathbb{N}} \|h_j\| \|R_j\|(Y),$$

where $\sum_{j \in \mathbb{N}} h_j R_j$ is any representation of $P$ and $\|R_j\|$ is the measure of the integer rectifiable current $R_j$. This is well-defined since for another representation we can find a common refinement (by Remark 3.1.1) covering $H^n$-almost all of $\bigcup_{j \in \mathbb{N}} \text{spt } R_j$; then we see that both measures coincide. Clearly, $\mu_P$ is a finite Borel-regular measure on $X$. Moreover, $\mu_P$ is concentrated on the countably $H^n$-rectifiable set $\bigcup_{j \in \mathbb{N}} \text{spt } R_j$. As $\|R_j\| \ll H^n$ for every $j$, we have $\mu_P \ll H^n$.

From the definition we see that

$$\mu_P(X) = M(P) \text{ and } \mu_P(Y) = M(P|_Y) \text{ for } Y \in \mathcal{B}_X.$$

If $G = \mathbb{R}$ with the Euclidean norm then we get from (3.2.3) that $\mu_P$ is the usual measure of the induced metric current, i.e. $\mu_P = \|h(P)\|$. 


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3.5 Support and diameter

The support of \( P \in \mathcal{P}_n(X,G) \), denoted by \( \text{spt} P \), is defined as the support of the measure \( \mu_P \). Note that \( P|_{\text{spt} P} = P \). The diameter of \( P \), \( \text{diam} P \), is \( \text{diam}(\text{spt} P) \).

Note that \( \text{spt} P \subset \bigcup_{i=1}^m \text{spt} T_i \) if \( P = \sum_{i=1}^m g_i T_i \); thus \( \text{spt} P \) is compact.

3.6 Push-forward

Let \( c \in \text{Lip}(X,Y) \), \( Y \) a complete metric space, and let \( P = \sum_{i=1}^m g_i T_i \in \tilde{\mathcal{P}}_n(X,G) \). Set

\[
c#P := \sum_{i=1}^m g_i c#T_i \in \tilde{\mathcal{P}}_n(Y,G).
\]

Then \( c#(P + \tilde{P}) = c#P + c#\tilde{P} \) and we have by (2.2.3) that

\[
(c#P)|_B = c#(P|_{c^{-1}(B)}) \quad \text{for } B \in \mathcal{B}(Y).
\]  

(3.6.1)

Lemma 3.6.1. The map \( c# : \mathcal{P}_n(X,G) \to \mathcal{P}_n(Y,G) \) is well-defined and

\[
\mathcal{M}(c#P) \leq \text{Lip}(c)^n \mathcal{M}(P).
\]

Moreover, \( c#|_{\mathcal{P}_n(X,G)} : \mathcal{P}_n(X,G) \to \mathcal{P}_n(Y,G) \).

If \( Z \) is a complete metric space and \( \bar{c} : Y \to Z \), then \( \bar{c}c#P = (\bar{c} \circ c)#P \).

Proof. The well-definedness is implied by the inequality for the mass. As \( c#T \in \Gamma^c_n(Y) \) if \( T \in \Gamma^c_n(X) \), then also holds \( c#|_{\mathcal{P}_n(X,G)} : \mathcal{P}_n(X,G) \to \mathcal{P}_n(Y,G) \).

Let \( \epsilon > 0 \), let \( R = \sum_{j \in \mathbb{N}} h_j f_j \#[\chi_{K_j}] \) be a representation of \( P = \sum_{i=1}^m g_i T_i \), let \( n_{ij} \) be such that \( T_i[f_j(K_j)] = n_{ij} f_j \#[\chi_{K_j}] \) and let \( N \in \mathbb{N} \) be big enough such that \( \sum_{i=1}^m g_i \|\mathcal{M}_{AK}(T_i|_{X \setminus \bigcup_{j=1}^N f_j(K_j)}) \| < \epsilon \). Set \( A := \bigcup_{j=1}^N f_j(K_j) \). Then, a representation of \( P|_A \) is given by \( \sum_{j \leq N} h_j f_j \#[\chi_{K_j}] \) and for the rest holds

\[
\mathcal{M}\left(c#\left(\sum_{i=1}^m g_i T_i|_{X \setminus A}\right)\right) \leq \sum_{i=1}^m g_i \|\mathcal{M}_{AK}(c#(T_i|_{X \setminus A})) \| \leq \text{Lip}(c)^n \epsilon.
\]

Now, let \( \sum_{k \in \mathbb{N}} \tilde{h}_k \tilde{f}_{k#}[\chi_{\tilde{K}_k}] \) be a representation of \( c#(P|_A) \). We can assume this representation to be refined with respect to fixed representations of
3.7. BOUNDARY

the (finitely many) \( c_\#(f_j\#[\chi_{K_j}]) \), \( j \leq N \), compare Remark 3.1.1. Thus, we have for every \( 1 \leq j \leq N \) and \( k \in \mathbb{N} \) an integer \( \bar{n}_{jk} \) with \( \sum_{j=1}^{N} \bar{n}_{jk} h_j = \tilde{h}_k \) and

\[
(c_\#(f_j\#[\chi_{K_j}]))|_{\tilde{f}_k(\tilde{K}_k)} = \bar{n}_{jk} \tilde{f}_k\#[\chi_{\tilde{K}_k}], \quad \sum_{k \in \mathbb{N}} \bar{n}_{jk} \tilde{f}_k\#[\chi_{\tilde{K}_k}] = c_\#(f_j\#[\chi_{K_j}]).
\]

Then

\[
\sum_{k \in \mathbb{N}} \| \tilde{h}_k \| M(\tilde{f}_k\#[\chi_{\tilde{K}_k}]) \leq \sum_{1 \leq j \leq N} \| h_j \| \sum_{k \in \mathbb{N}} |\bar{n}_{jk}| M(\tilde{f}_k\#[\chi_{\tilde{K}_k}])
\]

and so

\[ M(c_\#P) - \text{Lip}(c)^n \epsilon \leq \sum_{1 \leq j \leq N} \| h_j \| M(\tilde{f}_j\#[\chi_{\tilde{K}_j}]) \leq \text{Lip}(c)^n M(P). \]

The last assertion is clear from the definition.

\[
\square
\]

3.7 Boundary

For \( P \in P_0(X,G) \) we set \( \partial P = 0 \).

Let \( n > 0 \) and let \( \tilde{\partial} : \tilde{P}_n(X,G) \to \tilde{P}_{n-1}(X,G) \) be given by

\[
\tilde{\partial} \left( \sum_{i=1}^{m} g_i T_i \right) := \sum_{i=1}^{m} g_i \partial T_i.
\]

Clearly is \( \tilde{\partial}(P + P') = \tilde{\partial}P + \tilde{\partial}P' \). Let \( c \in \text{Lip}(X,Y) \), then

\[
\tilde{\partial} c_\#P = c_\# \tilde{\partial}P. \tag{3.7.1}
\]

**Lemma 3.7.1.** The boundary \( \partial : P_n(X,G) \to P_{n-1}(X,G) \), \( \partial[P] := [\tilde{\partial}P] \), is well-defined for \( n > 0 \).

For \( P \in \tilde{P}_n(\mathbb{R}^k,G) \) is \( \partial_F l\tilde{F}(P) = l\tilde{F}(\tilde{\partial}P) \).

For \( G = \mathbb{R}, \mathbb{Z} \) with the Euclidean norm is \( \partial \circ h = h \circ \partial \).

Here, \( \partial_F l\tilde{F} \) denotes the boundary of chains as defined by Fleming.

**Proof.** Indeed is \( \partial_F l\tilde{F}(P) = \sum_{i=1}^{m} g_i \partial T_i = l\tilde{F}(\tilde{\partial}P) \).

We show now that \( M(P) = 0 \) implies that \( M(\tilde{\partial}P) = 0 \). Assume in contrary that \( M(P) = 0 \) with \( M(\tilde{\partial}P) > 0 \) and let \( \sum_{j \in \mathbb{N}} h_j f_j\#[\chi_{K_j}] \) be a representation
of $\tilde{\partial}P$. Without loss of generality is $M_{AK}(f_1\#\zeta_{\chi_{K_1}}) > 0$ and $h_1 \neq 0$. Let $F: X \to \mathbb{R}^{n-1}$ be a Lipschitz extension of the bi-Lipschitz map $f_1^{-1}: f_1(K_1) \to K_1$ and let $\tilde{F}: X \to \mathbb{R}^{n-1} \times \mathbb{R}$ be given by

$$\tilde{F}(x) := (F(x), d_{f_1(K_1)}(x)).$$

Then $\tilde{F}$ maps $\tilde{F}^{-1}(K_1 \times \{0\}) = f_1(K_1)$ bi-Lipschitz onto $K_1 \times \{0\}$; furthermore

$$\tilde{F}(X \setminus f_1(K_1)) \cap (K_1 \times \{0\}) = \emptyset.$$

Hence

$$[(\tilde{F}_\# \tilde{\partial}P)|_{K_1 \times \{0\}}] = [\tilde{F}_\#(h_1 f_1\#\zeta_{\chi_{K_1}})] = [h_1\zeta_{\chi_{K_1}}]$$

and

$$M(\tilde{F}_\#(\tilde{\partial}P)) \geq M\left((\tilde{F}_\#(\tilde{\partial}P))|_{K_1 \times \{0\}}\right) = \|h_1\|M_{AK}(\zeta_{\chi_{K_1}}) > 0.$$

But $M(\tilde{F}_\#(P)) \leq \text{Lip}(W)^nM(P) = 0$. As $\tilde{F}_\#(\tilde{\partial}P) = \tilde{\partial}(\tilde{F}_\#P)$, Lemma 3.3.2 and $\partial_f Fl(\tilde{F}_\#P) = Fl(\partial \tilde{F}_\# P)$ imply that the trivial chain in $\mathbb{R}^n$ in the sense of Fleming has non-trivial boundary; a contradiction.

The last assertion follows from the definitions. \hfill \Box

### 3.8 Slicing

Slicing will be an essential tool in many proofs later on; it allows us to argue by induction.

Let $P = \sum_{i=1}^m g_i T_i \in \tilde{P}_n(X, G)$ and $f \in \text{Lip}(X, \mathbb{R}^k)$, $1 \leq k \leq n$; set

$$\langle P, f, r \rangle := \sum_{i=1}^m g_i \langle T_i, f, r \rangle.$$

This is in $\tilde{P}_{n-k}(X, G)$ for $\mathcal{L}^k$-almost all $r \in \mathbb{R}^k$ (see p. 11) and also $\text{spt} \mu_{\langle P, f, r \rangle} \subset f^{-1}(r) \cap \bigcup_{i=1}^m \text{spt} T_i$. Clearly, $\langle P + P', f, r \rangle = \langle P, f, r \rangle + \langle P', f, r \rangle$ for a.e. $r \in \mathbb{R}^k$. If $P \in \tilde{P}_n(X, G)$ then for a.e. $r$ is $\langle P, f, r \rangle \in \tilde{P}_{n-k}(X, G)$.

**Lemma 3.8.1.** Let $f \in \text{Lip}(X, \mathbb{R}^k)$, let $P, Q \in \tilde{P}_n(X, G)$ with $[P] = [Q]$ and let $R = \sum_{j \in \mathbb{N}} h_j R_j$ be a representation of $P$.

(i) For $\mathcal{L}^k$-a.e. $r \in \mathbb{R}^k$ is $M(\langle P, f, r \rangle - \langle Q, f, r \rangle) = 0$ and

$$[[P, f, r]] = \lim_{l \to \infty} \left[ \sum_{j=1}^l h_j \langle R_j, f, r \rangle \right]. \quad (3.8.1)$$
(ii) If \( k = 1 \) and \( P \in \tilde{P}_n(X,G) \), then for a.e. \( r \) holds
\[
\langle P, f, r \rangle = \partial (P|_{\{f < r\}}) - (\partial P)|_{\{f < r\}} = (\partial P)|_{\{r \leq f\}} - \partial (P|_{\{r \leq f\}}).
\] (3.8.2)

(iii) For \( B \subset X \) Borel and a.e. \( r \in \mathbb{R}^k \) is \( \langle P, f, r \rangle|_B = \langle P|_B, f, r \rangle \).

(iv) Let \( f_i \in \text{Lip}(X, \mathbb{R}^{k_i}) \). For \( \mathcal{L}^{k_1 + k_2} \)-a.e. \( (r_1, r_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \) holds
\[
\langle P, (f_1, f_2), (r_1, r_2) \rangle = \langle \langle P, f_1, r_1 \rangle, f_2, r_2 \rangle = (-1)^{k_1 k_2} \langle \langle P, f_2, r_2 \rangle, f_1, r_1 \rangle.
\]

(v) Let \( \pi \in \text{Lip}(Y, X) \) where \( Y \) is a complete metric space and let \( \bar{P} \in \tilde{P}_n(Y,G) \). Then \( \pi#\langle \bar{P}, f \circ \pi, r \rangle = \langle \pi#P, f, r \rangle \) for a.e. \( r \).

(vi) Let \( \bar{f}_#\llbracket \chi_K \rrbracket \in \mathcal{T}^c_n(X) \) where \( \bar{f}: K \to \bar{f}(K) \subset X \) bi-Lipschitz for \( K \subset \mathbb{R}^n \) compact and let \( g \in G \). Then for a.e. \( r \) is
\[
M(\langle \bar{f}_#\llbracket \chi_K \rrbracket, f, r \rangle) = \|g\| M_{AK}(\langle \bar{f}_#\llbracket \chi_K \rrbracket, f, r \rangle), \quad (3.8.3)
\]
\[
\mu(\langle \bar{f}_#\llbracket \chi_K \rrbracket, f, r \rangle) = \|g\| \|\langle \bar{f}_#\llbracket \chi_K \rrbracket, f, r \rangle\|. \quad (3.8.4)
\]

(vii) \( \mu_{\langle P, f, r \rangle} = \sum_{j \in \mathbb{N}} \|h_j\| \|\langle R_j, f, r \rangle\| \) for a.e. \( r \), and for \( B \subset X \) Borel is
\[
\sum_{j \in \mathbb{N}} \|h_j\| \|\langle R_j, 1, f \rangle\|(B) = \int_{\mathbb{R}^k} \mu_{\langle P, f, r \rangle}(B) dr.
\]

The map \( r \mapsto M(\langle P, f, r \rangle) \) is \( \mathcal{L}^k \)-measurable.

(viii) Let \( f_i \in \text{Lip}(X) \) and \( f := (f_1, \ldots, f_k) \), then
\[
\int_{\mathbb{R}^k} M(\langle P, f, r \rangle) dr \leq \prod_{i=1}^k \text{Lip}(f_i) M(P). \quad (3.8.5)
\]

(ix) Let \( k = 1 \), then for \( a < b \in \mathbb{R} \cup \{\pm \infty\} \) is
\[
\int_a^b M(\langle P, f, r \rangle) dr \leq \text{Lip}(f) M(P|_{\{a < f < b\}}). \quad (3.8.6)
\]

(x) Let \( P \in \tilde{P}_n(X,G) \); for a.e. \( r \in \mathbb{R}^k \) is \( \tilde{\partial}(P, f, r) = (-1)^k \langle \partial P, f, r \rangle \).
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Proof. (i) Let $P = \sum_{i=1}^{m} g_i T_i$ and let $R_j = f_j \# [\chi_{K_j}]$, $K_j \subset \mathbb{R}^n$ compact, $f_j$ bi-Lipschitz. Looking at the metric currents we have by (3.1.3) that $T_i = \sum_{j \in \mathbb{N}} n_{ij} R_j \in \mathcal{I}_n^c(X)$, and this sum is absolutely convergent for $M_{A_K}$. With (2.2.8) it follows that $(\sum_{j=1}^{l} \langle n_{ij} R_j, f, r \rangle)_{l \in \mathbb{N}}$ is a mass-convergent sequence for a.e. $r \in \mathbb{R}^k$ with limit $\langle T_i, f, r \rangle \in \mathcal{I}_n^c(X)$. This implies that

$$\left[ \sum_{i=1}^{m} \sum_{j=1}^{l} g_i \langle n_{ij} R_j, f, r \rangle \right] \rightarrow \left[ \sum_{i=1}^{m} g_i \langle T_i, f, r \rangle \right] (l \to \infty)$$

in mass, hence on $\mathcal{P}_{n-k}(X, G)$ for a.e. $r$. On the other hand is

$$\left[ \sum_{i=1}^{m} \sum_{j=1}^{l} g_i \langle n_{ij} R_j, f, r \rangle \right] = \left[ \sum_{j=1}^{l} h_j \langle R_j, f, r \rangle \right]$$

as $h_j = \sum_{i=1}^{m} n_{ij} g_i$ by (3.1.1). Thus, we have shown (3.8.1).

As the representation $R$ of $P$ was arbitrary, we can assume that $R$ is also a representation of $Q$, showing (i).

(ii) holds for normal currents a.e. and then also for $P \in \tilde{\mathcal{P}}_n(X, G)$ a.e.

(iii) By (2.2.7), we have for a.e. $r$ that $\langle T_i, f, r, \rangle|_B = \langle T_i|_B, f, r \rangle$, hence $\langle P|_B, f, r \rangle = \sum_{i=1}^{m} g_i \langle T_i|_B, f, r \rangle = \langle P, f, r \rangle|_B$.

(iv) follows from the definition and (2.2.12).

(v) By (2.2.10) such an equality holds for $T \in \mathcal{I}_n^c(X)$; then by the definition on $\tilde{\mathcal{P}}_n(X, G)$ too.

(vi) By (v) we have a.e.

$$\langle g \bar{f} \# [\chi_K], f, r \rangle = g \bar{f} \# (\langle [\chi_K], f \circ \bar{f}, r \rangle).$$

For $\langle [\chi_K], f \circ \bar{f}, r \rangle \in \mathcal{I}_{n-k}(\mathbb{R}^n)$ we can assume by [AK00, Theorem 9.7] that the absolute value of the multiplicity function takes values in $\{0, 1\}$. So let $\sum_{k \in \mathbb{N}} \bar{f}_k \# [\chi_{K_k}] = \langle [\chi_K], f \circ \bar{f}, r \rangle$ with $K_k \subset \mathbb{R}^{n-k}$ compact, $\bar{f}_k$ bi-Lipschitz with disjoint images. Then a representation of $\langle g \bar{f} \# [\chi_K], f, r \rangle$ is given by

$$\sum_{k \in \mathbb{N}} g(\bar{f} \# \bar{f}_k \# [\chi_{K_k}]).$$

Now we get (3.8.3) and (3.8.4).

(vii) By (i) and (3.8.4), and as the supports of $\langle R_i, f, r \rangle$ are pairwise disjoint a.e. we have that $\mu(P, f, r) = \sum_{j \in \mathbb{N}} \|h_j\| \|\langle R_j, f, r \rangle\|$.
3.8. SLICING

This implies also that $r \mapsto \mathcal{M}(\langle P, f, r \rangle)$ is $\mathcal{L}^k$-measurable as this holds for currents. Then, by (2.2.8)

$$\sum_{j \in \mathbb{N}} \| h_j \| \| R_j \| \langle 1, f \rangle \| (B) = \sum_{j \in \mathbb{N}} \| h_j \| \int_{\mathbb{R}^k} \| \langle R_j, f, r \rangle \| (B) dr = \int_{\mathbb{R}^k} \mu_{\langle P, f, r \rangle} (B) dr .$$

(viii) The assertion follows from (vii) with $\| R_j \| \langle 1, f \rangle \| \leq \prod_{i=1}^k \text{Lip}(f_i) \| R_j \|.$

(ix) Since for a.e. $r \in (a, b)$ holds that $\langle P\lfloor_{\{a<f<b\}}, f, r \rangle = \langle P, f, r \rangle$, we get (3.8.6) from (3.8.5).

(x) holds for normal currents and then also for $P \in \tilde{P}_n(X,G)$.

Now for $[P] \in \mathcal{P}_n(X,G)$ we define the slice to be

$$\langle [P], f, r \rangle := \langle [P, f, r] \rangle \in \mathcal{P}_{n-k}(X,G)$$

whenever $\langle P, f, r \rangle \in \tilde{\mathcal{P}}_{n-k}(X,G)$. Note that this is not well-defined; however, for elements $P, Q \in [P]$ the slices agree a.e.

**Remark 3.8.2.** All the properties from the preceding lemma hold similarly for $\langle [P], f, r \rangle$; we will denote $\langle [P], f, r \rangle$ henceforth by $\langle P, f, r \rangle \in \mathcal{P}_{n-k}(X,G)$.

Slicing allows us to show the following useful characterisation of $\mu_P$ (similar to (2.2.14), compare also [DPH09]).

**Proposition 3.8.3.** Let $P \in \mathcal{P}_n(X,G)$ where $n > 0$. For $B \subset X$ Borel is

$$\mu_P(B) = \sup \left( \sum_{i \in \mathbb{N}} \int_{\mathbb{R}^k} \mu_{\langle P, f_i, r \rangle} (U_i \cap B) dr \right)$$

where the supremum is over all $U_i \subset X$ open and pairwise disjoint and all $f_i \in \text{Lip}_1(X)$.

**Proof.** Let $\sum_{j \in \mathbb{N}} h_j R_j$ be a representation of $P$. Below let all $f_i \in \text{Lip}_1(X)$, and let $U_i \subset X$ be open and pairwise disjoint. With Lemma 3.8.1 is

$$\mu_P(B) = \sum_{j \in \mathbb{N}} \| h_j \| \mathcal{M}_{AK}(R_j|_B) \geq \sum_{j \in \mathbb{N}} \| h_j \| \sum_{i \in \mathbb{N}} \mathcal{M}_{AK}(R_j|_{\{x \in \chi U_i \cap B, f_i \}}) \geq \sum_{i, j \in \mathbb{N}} \| h_j \| \| R_j \| \langle 1, f_i \rangle \| (U_i \cap B) \geq \sum_{i \in \mathbb{N}} \int_{\mathbb{R}^k} \mu_{\langle P, f_i, r \rangle} (U_i \cap B) dr .$$
On the other hand, using (2.2.14) and the disjointness of spt $R_j$ we see that for $\epsilon > 0$ there are good choices of $U_i$ and $f_i$ such that

$$
\mu_P(B) = \sum_{j \in \mathbb{N}} \|h_j\| \text{MAK}(R_j \lfloor_B) \leq (1 + \epsilon) \sum_{i,j \in \mathbb{N}} \|h_j\| \text{MAK}(R_j \lfloor_{(\chi_{U_i \cap B}, f_i)})
$$

$$
\leq (1 + \epsilon) \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \mu_{(P, f_i, r)}(U_i \cap B) dr.
$$

The proposition follows by sending $\epsilon \to 0$.

\[\square\]

### 3.9 Lower semicontinuity of mass

We will show in Theorem 3.9.2 that mass is lower semicontinuous for convergence with respect to some norm $\tilde{F}_X$ (defined below). Later, we define a flat norm $F$ and an inner flat norm $\tilde{F}_X$. These norms coincide with $\tilde{F}_X$ whenever $X$ is an injective metric space.

For $P \in P_n(X, G)$ set

$$
\tilde{F}_X(P) := \inf \{ M(P_0) + M(P_1) \mid P = P_0 + \partial P_1, P_i \in P_{n+i}(X, G) \}. \tag{3.9.1}
$$

Clearly is $M(P) \geq \tilde{F}_X(P)$. As in [Ada08] we have

**Lemma 3.9.1.** Let $X$ be a complete metric space. Let $f \in \text{Lip}(X)$ and let $P_j \in P_n(X, G)$ with $\sum_{j \in \mathbb{N}} \tilde{F}_X(P_j) < \infty$. Then for a.e. $r \in \mathbb{R}$ holds:

$$
\langle P_j, f, r \rangle \in P_{n-1}(X, G) \quad \text{and} \quad P_j \lfloor_{\{f < r\}} \in P_n(X, G) \quad \forall j \in \mathbb{N},
$$

$$
\sum_{j \in \mathbb{N}} \tilde{F}_X(\langle P_j, f, r \rangle) < \infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} \tilde{F}_X(P_j \lfloor_{\{f < r\}}) < \infty.
$$

(Here, to include $n = 0$, we then define the slices to vanish.)

**Proof.** Let $P_0^j, P_1^j$ be such that $P_j = P_0^j + \partial P_1^j$ and $\sum_{j \in \mathbb{N}} M(P_0^j) + M(P_1^j) < \infty$. By Lemma 3.8.1 is $\langle P_j, f, r \rangle = \langle P_0^j, f, r \rangle - \partial \langle P_1^j, f, r \rangle$ a.e. and

$$
\int_{\mathbb{R}} \sum_{j \in \mathbb{N}} \tilde{F}_X(\langle P_j, f, r \rangle) dr \leq \int_{\mathbb{R}} \left( \sum_{j \in \mathbb{N}} M(\langle P_0^j, f, r \rangle) + M(\langle P_1^j, f, r \rangle) \right) dr
$$

$$
\leq \text{Lip}(f) \sum_{j \in \mathbb{N}} (M(P_0^j) + M(P_1^j)) < \infty.
$$
As
\[ P_j^I_{\{f<r\}} = P_0^I_{\{f<r\}} - \langle P_1^I, f, r \rangle + \partial(P_1^I_{\{f<r\}}) \]
for a.e. \( r \) we have \( \sum_{j \in \mathbb{N}} \tilde{F}_X(P_j^I_{\{f<r\}}) < \infty \) a.e. So the last two assertions hold outside a set of \( L^1 \)-measure zero. Each of the first two assertions holds for \( j \) fixed outside a set of measure zero, thus we find a \( L^1 \)-negligible set outside of which all assertions are satisfied.

\[ \textbf{Theorem 3.9.2.} \text{ Let } X \text{ be a complete metric space and let } P, P_j \in \mathcal{P}_n(X, G) \text{ with } \tilde{F}_X(P - P_j) \to 0. \text{ Then } M(P) \leq \liminf_{j \to \infty} M(P_j). \]

Compare [DPH09, Lemma 5.7.1 and Corollary 5.7.2] and [Ada08].

\[ \textbf{Proof.} \text{ We show by induction that for every open set } U \subset X \text{ holds } \]
\[ \mu_P(U) \leq \liminf_{j \to \infty} \mu_{P_j}(U). \quad (3.9.2) \]

The proof is now similar to that in [Ada08, pp. 15].

\( n = 0 \): Note that if \( \tilde{P} = g_0[x_0] \) then for every \( P' \in \mathcal{P}_1(X, G) \) we have
\[ M(\tilde{P} + \partial P') \geq M(\tilde{P}) = \|g_0\| \]
by the triangle inequality for \( G \). Assume that there exist \( [P] = \sum_{i=1}^m g_i[x_i], (P_j)_{j \in \mathbb{N}} \text{ and } U \subset X \) violating (3.9.2) and with \( \sum_{j \in \mathbb{N}} \tilde{F}_X(P - P_j) < \infty \). Assume that \( g_i \neq 0 \) and \( x_i \neq x_j \) for \( i \neq j \).

As \( \mu_P(U) = \sum_{i=1}^m \mu_{g_i[x_i]}(U) \) we find (with Lemma 3.9.1) \( i \in \{1, 2, \ldots, m\} \) and \( r > 0 \) such that \( g_i[x_i], P_j^I_{B_r(x_i)} \) violate (3.9.2) as well for the open set \( X \).

We can assume as well that
\[ \mu_P(U) = M(P) = \|g_i\| \text{ and } \mu_{P_j}(U) = M(P_j^I). \]

Let \( S_k^j \in \mathcal{P}_i(X, G) \) with \( \sum_{j \in \mathbb{N}} M(S_k^j) < \infty \) for \( k \in \{0, 1\} \) be such that \( P - P_j = S_0^j + \partial S_1^j \). Assume that \( M(P_j^I) < M(P) - \delta \) for \( \delta > 0 \). So
\[ M(P) - \delta \geq \limsup_{j \to \infty} M(S_0^j + P_j^I) = \limsup_{j \to \infty} M(P - \partial S_1^j) \geq M(P) \]
which is a contradiction.

\( n > 0 \): Let \( \sum_{j \in \mathbb{N}} \tilde{F}_X(P_j - P) < \infty \), by Lemma 3.8.1 is for a.e. \( r \)
\[ \sum_{j \in \mathbb{N}} \tilde{F}_X(\langle P_j - P, f, r \rangle) < \infty. \]
Then also $\tilde{F}_X(\langle P^j, f, r \rangle - \langle P, f, r \rangle) \to 0$ a.e. By induction is for such $r$

$$\liminf_{j \to \infty} \mu_{\langle P^j, f, r \rangle}(\tilde{U}) \geq \mu_{\langle P, f, r \rangle}(\tilde{U})$$

for every $\tilde{U} \subset X$ open. By Proposition 3.8.3 we have for pairwise disjoint open sets $U_i \subset X$, $f_i \in \text{Lip}_1(X)$ and $U \subset X$ open that

$$\mu_{P^j}(U) \geq \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \mu_{\langle P^j, f_i, r \rangle}(U_i \cap U) dr.$$ 

By the lemma of Fatou now follows for $U_i$ and $f_i$ fixed

$$\liminf_{j \to \infty} \mu_{P^j}(U) \geq \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \mu_{\langle P, f_i, r \rangle}(U_i \cap U) dr.$$ 

Hence, taking the supremum over all such $f_i$ and $U_i$, by Proposition 3.8.3

$$\liminf_{j \to \infty} \mu_{P^j}(U) \geq \mu_{P}(U).$$
Chapter 4

Virtual rectifiable and virtual flat $G$-chains

Here we recall from [DPH09] the definitions of virtual flat chains, virtual rectifiable chains, the mass and the flat norm, and show basic results and helpful lemmas as therein. There are minor differences due to the varied definition of mass. However, the resulting spaces are bi-Lipschitz equivalent. We construct a measure for a finite mass chain using the Lévi-Prokhorov metric on measures; our measure would be the same as the one constructed in [DPH09] if we used the same definition of mass.

For virtual chains, one defines all the above first in injective metric spaces. For a complete metric space $X$, we embed it isometrically into an injective metric space $Y$ (e.g. $l^\infty(X)$) and define everything through such embeddings: Virtual flat chains in $X$ will be the chains that have support in the image of $X$, their flat norm is the flat norm in $Y$.

4.1 Injective metric spaces

Let $X$ be an injective metric space. We define the $n$-dimensional (virtual) rectifiable chains with $G$-coefficients as

$$\mathcal{R}_n(X, G) := \overline{P_n(X, G)}^M.$$
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Then \((\mathcal{R}_n(X,G), \mathcal{M})\) is a normed group. We easily see that \(\mathcal{P}_n(X,G) \subset \mathcal{R}_n(X,G)\), and hence

\[
\mathcal{R}_n(X,G) = \mathcal{R}_n(X,\tilde{G}). \tag{4.1.1}
\]

Moreover, we have:

**Proposition 4.1.1.** In an injective metric space \(X\) is \(P_n(X,G) \subset \mathcal{R}_n(X,G)\).

**Proof.** With Remark 3.2.2 it is enough to show that \(g f_# [\chi_K] \in \mathcal{R}_n(X,G)\) for \(g \in G\), \(f: K \to X\) bi-Lipschitz on the image, \(K \subset \mathbb{R}^n\) compact. We can extend \(f\) to a Lipschitz map \(\tilde{f}: \mathbb{R}^n \to X\) by the injectivity of \(X\) (see Section 2.4). Covering \(K \subset \mathbb{R}^n\) by smaller and smaller cubes we see that there are \(T_i \in \mathcal{I}_n(\mathbb{R}^n)\) such that \(M(\mathcal{A}K(T_i - [\chi_K])) \to 0\). Since \(f_# [\chi_K] = \tilde{f}_# [\chi_K]\) we conclude that \(M(g(\tilde{f}_# T_i - f_# [\chi_K])) \to 0\). \hfill \Box

We set as in (3.9.1) for \(P \in \mathcal{P}_n(X,G)\)

\[
\mathcal{F}(P) := \inf \{ M(P_0) + M(P_1) \mid P = P_0 + \partial P_1, P_i \in \mathcal{P}_{n+i}(X,G) \},
\]

i.e. \(\mathcal{F}(P) = \tilde{\mathcal{F}}_X(P)\). The \(n\)-dimensional (virtual) flat chains with \(G\)-coefficients in the injective metric space \(X\) are

\[
\mathcal{F}_n(X,G) := \overline{\mathcal{P}_n(X,G)}^F.
\]

We denote \(\mathcal{F}(F - F_j) \to 0\) for \(j \to \infty\) by \(F_j \to F\) or \(F = \lim_{j \to \infty} F_j\).

We define the mass of a virtual flat chain \(F\) by

\[
M(F) := \inf \left\{ \liminf_{j \to \infty} M(P_j) \mid P^j \to F, P^j \in \mathcal{P}_n(X,G) \right\}
\]

and the set of (virtual) flat chains of finite mass

\[
\mathcal{M}_n(X,G) := \{ F \in \mathcal{F}_n(X,G) \mid M(F) < \infty \} ;
\]

\(\mathcal{M}\) is well-defined by Theorem 3.9.2. Convergence with respect to the mass will be denoted as \(F = \mathcal{M}-\lim_{j \to \infty} F_j\).

Now, \(\mathcal{F}(F + F') \leq \mathcal{F}(F) + \mathcal{F}(F')\), and

\[
\mathcal{P}_n(X,G) \subset \mathcal{P}_n(X,G) \subset \mathcal{R}_n(X,G) \subset \mathcal{M}_n(X,G) \subset \mathcal{F}_n(X,G)
\]

are normed groups.
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Note that \( \mathcal{F}(\partial P) \leq \mathcal{F}(P) \). Let \( F \in \mathcal{F}_n(X, G) \). For \( n = 0 \) we set \( \partial F = 0 \); for \( n > 0 \) we define the boundary \( \partial F \in \mathcal{F}_{n-1}(X, G) \) by setting

\[
\partial F := \lim_{j \to \infty} \partial P^j, \quad \text{where } \mathcal{F}(F - P^j) \to 0.
\]

Clearly, \( \mathcal{F}(\partial P) \leq \mathcal{F}(F) \).

Finally, we set \( N(F) := M(F) + M(\partial F) \). Flat chains \( F \) with \( N(F) < \infty \) are called normal.

We have as in [Fle66, (3.1)]:

**Lemma 4.1.2.** \( \mathcal{F}_n(X, G) = \{ R_0 + \partial R_1 \mid R_i \in \mathcal{R}_{n+i}(X, G) \} \) in an injective metric space \( X \) and

\[
\mathcal{F}(F) = \inf \left\{ M(F_0) + M(F_1) \mid F_i \in \mathcal{F}_{n+i}(X, G) \text{ with } F = F_0 + \partial F_1 \right\} = \inf \left\{ M(R_0) + M(R_1) \mid R_i \in \mathcal{R}_{n+i}(X, G) \text{ with } F = R_0 + \partial R_1 \right\}.
\]

**Proof.** This proof is as in [Fle66, p. 165]: Let \( P^j \to F \) such that \( P^j - P^{j-1} = S^j_0 + \partial S^j_1 \) with \( \sum_{j \in \mathbb{N}} M(S^j_i) < \infty \) for \( i \in \{0, 1\} \). Then

\[
R_0 := P^1 + \sum_{j > 1} S^j_0 \in \mathcal{R}_n(X, G) \quad \text{and} \quad R_1 := \sum_{j > 1} S^j_1 \in \mathcal{R}_{n+1}(X, G)
\]

have \( \mathcal{F}(F - R_0 - \partial R_1) = 0 \), hence \( F = R_0 + \partial R_1 \).

Clearly is \( R_0 + \partial R_1 \in \mathcal{F}_n(X, G) \) whenever \( R_i \in \mathcal{R}_{n+i}(X, G) \), and so

\[
\mathcal{F}_n(X, G) = \{ R_0 + \partial R_1 \mid R_i \in \mathcal{R}_{n+i}(X, G) \}.
\]

Now let \( F, P^j \) and \( S^j_i \) be as above and let \( P^j = d^j_0 + \partial d^j_1, d^j_i \in \mathcal{P}_{n+i}(X, G) \) with \( \mathcal{F}(P^j) + \epsilon_j > M(d^j_0) + M(d^j_1) \) where \( \epsilon_j \to 0 \) for \( j \to \infty \). Set

\[
A^j := d^j_0 + \sum_{k > j} S^k_0 \in \mathcal{R}_n(X, G), \quad B^j := d^j_1 + \sum_{k > j} S^k_1 \in \mathcal{R}_{n+1}(X, G).
\]

Then is \( F = A^j + \partial B^j \) for each \( j \) and as \( \mathcal{F}(P^j) \to \mathcal{F}(F) \),

\[
M(A^j) + M(B^j) \leq M(d^j_0) + M(d^j_1) + \epsilon_j' \leq \mathcal{F}(F) + \epsilon_j'' \leq \mathcal{F}(F) + \epsilon_j''',
\]

where \( \epsilon_j''' \to 0 \) for \( j \to \infty \).

On the other hand, let \( F = F_0 + \partial F_1, F_i \in \mathcal{F}_{n+i}(X, G) \) and let \( P^j_i \in \mathcal{P}_{n+i}(X, G) \) converging to \( F_i \) such that \( M(P^j_i) \to M(F_i) \). Then \( W_j := P^j_0 + \partial P^j_1 \) converges to \( F \) and \( \mathcal{F}(W_j) \leq M(P^j_0) + M(P^j_1) \). Hence,

\[
\mathcal{F}(F) = \lim_{j \to \infty} \mathcal{F}(W_j) \leq M(F_0) + M(F_1)
\]

for every \( F_0, F_1 \) with \( F = F_0 + \partial F_1 \). \( \square \)
4.1.1 Push-forward

Let $X, Y$ be injective metric spaces and let $f : X \to Y$ be a Lipschitz map. If $P_n(X, G) \ni P_j \to F$, then $f\# P_j \in P_n(Y, G)$ is converging too (compare Section 3.6) and we define the limit to be $f\# F \in \mathcal{F}_n(X, G)$.

Note that $f\# : (\mathcal{F}_n(X, G), \mathcal{F}) \to (\mathcal{F}_n(Y, G), \mathcal{F})$ is $\lambda$-Lipschitz for $\lambda := \max\{\text{Lip}(f)^n, \text{Lip}(f)^{n+1}\}$.

Moreover, the map $f\# |_{\mathcal{M}_n(X, G)} : (\mathcal{M}_n(X, G), \mathcal{M}) \to (\mathcal{M}_n(Y, G), \mathcal{M})$ is $\text{Lip}(f)^n$-Lipschitz and $f\# |_{\mathcal{R}_n(X, G)} : \mathcal{R}_n(X, G) \to \mathcal{R}_n(Y, G)$.

Then is $f\# (F_1 + F_2) = f\# F_1 + f\# F_2$ and $\partial f\# F = f\# \partial F$ (from (3.7.1)). Furthermore, for another Lipschitz map $c : Y \to Z$ where $Z$ is an injective metric space, we have $c\# f\# F = (c \circ f)\# F$ (from Lemma 3.6.1).

**Lemma 4.1.3.** Let $X, Y$ be injective metric spaces, let $\pi \in \text{Lip}(X, Y)$ and let $P \in P_n(X, G)$, $F \in \mathcal{F}_n(X, G)$.

If $\pi|_{\text{spt} P}$ is an isometric embedding, then $\mathcal{F}(P) = \mathcal{F}(\pi\# P)$.

If $\pi$ is an isometric embedding then $\mathcal{F}(F) = \mathcal{F}(\pi\# F)$.

**Proof.** We can extend $\pi|_{\text{spt} P}$ to $\Pi \in \text{Lip}_1(X, Y)$, then $\pi\# P = \Pi\# P \in P_n(Y, G)$ and so $\mathcal{F}(\Pi\# P) \leq \mathcal{F}(P)$. Arguing similarly for the other direction, we get the first assertion. The second follows analogously. \qed

4.1.2 Slicing of flat chains and restriction to sublevel sets

Let $X$ be an injective metric space. Let the sequence $(P_j)_{j \in \mathbb{N}} \subset P_n(X, G)$ converge to $F$ rapidly (i.e. $\sum_{j \in \mathbb{N}} \mathcal{F}(P_{j+1} - P_j) < \infty$) and let $\tilde{f} \in \text{Lip}(X, \mathbb{R}^k)$ and $f \in \text{Lip}(X)$. Using Lemma 3.9.1, we see that

$$\langle F, \tilde{f}, r \rangle := \lim_{j \to \infty} \langle P_j, \tilde{f}, r \rangle \in \mathcal{F}_{n-k}(X, G) \quad (4.1.2)$$

is defined for $\mathcal{L}^k$-a.e. $r$ and

$$F[\{f < s\}] := \lim_{j \to \infty} P_j |_{\{f < s\}} \in \mathcal{F}_n(X, G),$$

$$F[\{s \leq f\}] := \lim_{j \to \infty} P_j |_{\{s \leq f\}} \in \mathcal{F}_n(X, G) \quad (4.1.3)$$

are defined for $\mathcal{L}^1$-a.e. $s$. 
Lemma 4.1.4. Let \( P^i \to F \) and \( p^i \to F \) rapidly. Then

\[
\lim_{i \to \infty} \langle P^i, \bar{f}, r \rangle = \lim_{i \to \infty} \langle p^i, \bar{f}, r \rangle \quad \text{for } \mathcal{L}^k \text{-a.e. } r \in \mathbb{R}^k, \tag{4.1.4}
\]

\[
\lim_{i \to \infty} (P^i |_{\{f < s\}}) = \lim_{i \to \infty} (p^i |_{\{f < s\}}) = \lim_{i \to \infty} (P^i |_{\{f \leq s\}}) \tag{4.1.5}
\]

for \( \mathcal{L}^1 \)-a.e. \( s \in \mathbb{R} \) where \( f, \bar{f} \) are as above. Furthermore, a.e. holds

\[
\langle F, f, s \rangle = \partial((F |_{\{f < s\}}) - (\partial F) |_{\{f < s\}}) = (\partial F) |_{\{s < f\}} - \partial(F |_{\{s < f\}}), \tag{4.1.6}
\]

\[
F = F |_{\{f < s\}} + F |_{\{s \leq f\}} \quad \text{and} \quad \partial(F, \bar{f}, r) = (-1)^k \partial(F, \bar{f}, r). \tag{4.1.7}
\]

For \( \bar{f} =: (\bar{f}^1, \bar{f}^2) \) where \( \bar{f}^1 \in \text{Lip}(X, \mathbb{R}^k) \), and \( \mathcal{L}^{l_1 + l_2} \)-a.e. \( (r^1, r^2) \) is

\[
\langle F, \bar{f}, (r^1, r^2) \rangle = \langle \langle F, \bar{f}^1, r^1 \rangle, \bar{f}^2, r^2 \rangle = (-1)^{l_1 l_2} \langle \langle F, \bar{f}^2, r^2 \rangle, \bar{f}^1, r^1 \rangle. \tag{4.1.8}
\]

Let \( \bar{f} =: (f_1, f_2, \ldots, f_k) \) for \( f_i \in \text{Lip}(X) \), then

\[
\int_{\mathbb{R}^k}^* \mathcal{F}(\langle F, \bar{f}, r \rangle) dr \leq \prod_{i=1}^k \text{Lip}(f_i) \mathcal{F}(F). \tag{4.1.9}
\]

If \( F \in \mathcal{M}_n(X, G) \) then \( \langle F, \bar{f}, r \rangle \in \mathcal{M}_{n-k}(X, G) \) a.e. and

\[
\int_{\mathbb{R}^k}^* \mathcal{M}(\langle F, \bar{f}, r \rangle) dr \leq \prod_{i=1}^k \text{Lip}(f_i) \mathcal{M}(F). \tag{4.1.10}
\]

If \( F \in \mathcal{R}_n(X, G) \) then \( \langle F, \bar{f}, r \rangle \in \mathcal{R}_{n-k}(X, G) \), \( F |_{\{f < r\}} \in \mathcal{R}_n(X, G) \) a.e.

If \( F \in \mathcal{M}_n(X, G) \) and \( P^j \to F \) with \( \mathcal{M}(P^j) \to \mathcal{M}(F) \) then for a.e. \( s \) holds \( F |_{\{f < s\}} \in \mathcal{M}_n(X, G) \) and \( \mathcal{M}(P^j |_{\{f < s\}}) \to \mathcal{M}(F |_{\{f < s\}}) \).

Let \( Y \) be an injective metric space, let \( \pi \in \text{Lip}(X, Y) \). Then

\[
\langle \pi^\# F, \bar{f}, r \rangle = \pi^\# \langle F, \bar{f} \circ \pi, r \rangle, \quad \text{for } \mathcal{L}^k \text{-a.e. } r \in \mathbb{R}^k. \tag{4.1.11}
\]

Proof. We can assume that \( \sum_{j \in \mathbb{N}} \mathcal{F}(P^j - p^j) < \infty \); from Lemma 3.9.1 we see that \( \sum_{j \in \mathbb{N}} \mathcal{F}(\langle P^j - p^j, \bar{f}, r \rangle) < \infty \) for a.e. \( r \) (recall that \( \bar{f}_X = \mathcal{F} \) for \( X \) injective). So, for almost every \( r \) both limits exist and agree.

Lemma 3.9.1 implies \( \lim_{j \to \infty} (P^j |_{\{f < s\}}) = \lim_{j \to \infty} (p^j |_{\{f < s\}}) \) a.e. as well. Note that \( P^j |_{\{f < s\}} = P^j |_{\{f \leq s\}} \) a.e. (as this holds for the currents), so the last equality in (4.1.5) holds too.

By (3.8.2) then follows (4.1.6).
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Clearly is \( P^j_{\{f<r\}} + P^j_{\{r\leq f\}} = P^j \) and hence this holds for the limit a.e. From (x) of Lemma 3.8.1 follows also the last part of (4.1.7).

Also, (4.1.8) follows from (iv) of Lemma 3.8.1.

By (3.8.5) holds (4.1.10) for \( P \in P_m(X,G), m \in \mathbb{N} \). Let \( P^j \to F \) rapidly and let \( P^j = P^j_0 + \partial P^j_1 \) with \( \mathcal{F}(P^j) \geq M(P^j_0) + M(P^j_1) - \epsilon_j \) where \( \epsilon_j \to 0 \) for \( j \to \infty \). Hence by the second part of (4.1.7) a.e.

\[
\langle F, \bar{f}, r \rangle = \lim \langle P^j, \bar{f}, r \rangle = \lim \left( \langle P^j_0, \bar{f}, r \rangle + (-1)^k \partial \langle P^j_1, \bar{f}, r \rangle \right)
\]

and hence this holds a.e. From (x) of Lemma 3.8.1 follows also the last part of (4.1.7).

Also, (4.1.8) follows from (iv) of Lemma 3.8.1.

By (3.8.5) holds (4.1.10) for \( P \in P_m(X,G), m \in \mathbb{N} \). Let \( P^j \to F \) rapidly and let \( P^j = P^j_0 + \partial P^j_1 \) with \( \mathcal{F}(P^j) \geq M(P^j_0) + M(P^j_1) - \epsilon_j \) where \( \epsilon_j \to 0 \) for \( j \to \infty \). Hence by the second part of (4.1.7) a.e.

\[
\langle F, \bar{f}, r \rangle = \lim \langle P^j, \bar{f}, r \rangle = \lim \left( \langle P^j_0, \bar{f}, r \rangle + (-1)^k \partial \langle P^j_1, \bar{f}, r \rangle \right)
\]

and hence this holds a.e. From (x) of Lemma 3.8.1 follows also the last part of (4.1.7).

Finally, (4.1.11) follows from (v) of Lemma 3.8.1.

\[
\int_{\mathbb{R}^k}^* \mathcal{F}(\langle F, \bar{f}, r \rangle) dr \leq \lim \inf_{j \to \infty} \int_{\mathbb{R}^k} M(\langle P^j, \bar{f}, r \rangle) dr \leq \prod_{i=1}^k \text{Lip}(f_i) \mathcal{F}(F).
\]

As a.e. slice of \( P^j \in P_n(X,G) \) is in \( P_{n-k}(X,G) \), for a rectifiable chain \( F \) we see by (3.8.5) applied to a rapidly mass-convergent sequence that a.e. slice is again rectifiable and analogously for the restriction.

As \( M(P^j_{\{f<r\}}) + M(P^j_{\{r\leq f\}}) = M(P^j) \), and by (4.1.7) we see that for \( F \in M_n(X,G) \) and \( M(P^j) \to M(F) \) we have \( M(P^j_{\{f<r\}}) \to M(F_{\{f<r\}}) \) for a.e. \( r \).

Finally, (4.1.11) follows from (v) of Lemma 3.8.1.
4.1. INJECTIVE METRIC SPACES

We set $F\lfloor_{\{r'\leq f<r\}} := \lim_{j\to\infty} P_j \lfloor_{\{r'\leq f<r\}}$ for $r' < r$, whenever this is defined (at least a.e.); then we get

$$F\lfloor_{\{r'\leq f<r\}} = (F\lfloor_{\{f<r\}})\lfloor_{\{r'\leq f\}} = F\lfloor_{\{f<r\}} - F\lfloor_{\{f<r'\}}$$

(assuming all chains above to be defined).

Let $F, \tilde{F} \in \mathcal{F}_n(X, G)$, then for a.e. $r$ is

$$(F + \tilde{F}, f, r) = \langle F, f, r \rangle + \langle \tilde{F}, f, r \rangle,$$

(4.1.13)

and similar for other sets instead of $\{f < r\}$. As a consequence we have

Lemma 4.1.5. Let $X, Y$ be injective metric spaces, $f \in \text{Lip}(X, Y)$ and let $F \in \mathcal{F}_n(X, G)$. Then, for every $c \in \text{Lip}(Y)$ and a.e. $r \in \mathbb{R}$ holds

$$f\#(F\lfloor_{\{c \circ f<r\}}) = (f\#F)\lfloor_{\{c<r\}}.$$  

Proof. Let $P^j \to F$. Then, as $f\#P^j \to f\#F$, for a.e. $r$ we have by Lemma 4.1.4 that $(f\#F)\lfloor_{\{c<r\}} = \lim_{j \to \infty} (f\#P^j)\lfloor_{\{c<r\}}$. With (3.6.1) is

$$(f\#P^j)\lfloor_{\{c<r\}} = f\#(P^j\lfloor_{f^{-1}(\{c<r\})}) = f\#(P^j\lfloor_{\{c \circ f<r\}});$$

hence we get the assertion for a.e. $r$. \qed

4.1.3 Support and diameter

We define as in [DPH09] (following [Ada08]):

Definition 4.1.6. The support of $F \in \mathcal{F}_n(X, G)$ is

$$\text{spt } F := \{x \in X \mid \text{for a.e. } r \in (0, \infty) \text{ is } F\lfloor_{\{d_x<r\}} \neq 0\}.$$  

The diameter of $F$ is defined as the diameter of the support of $F$:

$$\text{diam } F := \text{diam}(\text{spt } F).$$

One sees easily that spt $F$ is contained in the closure of $\bigcup_{j \in \mathbb{N}} \text{spt } P^j$ when $P^j \to F$ for $j \to \infty$. Hence, for $F\lfloor_{\{f<r\}} \in \mathcal{F}_n(X, G)$ where $f \in \text{Lip}(X)$ we get

$$\text{spt}(F\lfloor_{\{f<r\}}) \subset \{f \leq r\}.$$  

(4.1.15)
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This so since for $y \notin \text{spt} F$ we find $E \subset \mathbb{R}$ with $\mathcal{L}^1(E) > 0$ such that $F|_{\{d_y < r\}} = 0$ for all $r' \in E$. Then for a.e. $r' \in E$ is $F|_{\{r' \leq d_y\}} = F$, thus for a.e. $r' \in E$ holds for $\mathbf{P}_n(X, G) \ni P^j|_{\{r' \leq d_y\}} = p_j$ that $p_j|_{\{f < r\}} \to F|_{\{f < r\}}$. As $y$ is not in the closure of $\bigcup_{j \in \mathbb{N}} \text{spt}(p_j|_{\{f < r\}})$, we get $y \notin \text{spt} F|_{\{f < r\}}$, hence (4.1.15).

**Remark 4.1.7.** Let $P^j \to F$ for $j \to \infty$. Let $t \in (0, \infty)$ be such that $P^j|_{\{d < t\}} \to F|_{\{d < t\}}$. By (4.1.12), $F|_{\{d < r\}} \neq 0$ for a.e. $r \in (0, t)$ if and only if $x \in \text{spt} F$. This means that we have to check only an arbitrary small interval (instead of $(0, \infty)$) to decide whether $x$ is in the support of $F$. This argument generalizes: Let $P^j|_{\{f < t\}} \to F|_{\{f < t\}}$ and $f(x) < t$. Then for $r$ small enough is 

$$F|_{\{d < r\}} = F|_{\{d < r\}} \text{ a.e.}$$

Thus

$$x \in \text{spt} F \cap \{f < t\} \text{ if and only if } x \in \text{spt} F|_{\{f < t\}} \text{ and } f(x) < t.$$ 

For the slice we see similarly as above that for a.e. $r$ holds

$$\text{spt}(\langle F, f, r \rangle) \subset \text{spt} F \cap f^{-1}(r). \quad (4.1.16)$$

The above remark together with (4.1.7) yields that $\text{spt}(F - F|_{\{d_{\text{spt} F} < r\}}) = \emptyset$ for a.e. $r > 0$. This implies by the lemma below that

$$F = F|_{\{d_{\text{spt} F} < r\}} \text{ for a.e. } r > 0. \quad (4.1.17)$$

**Lemma 4.1.8 ([Ada08]).** $\text{spt} F = \emptyset$ if and only if $F = 0$.

For the proof, see Lemma 6.5, 6.6 and 6.7 in [Ada08] (up to constants). Remark that with (4.1.17) follows that for $r > 0$ there is a sequence $P^j \to F$ with $\text{spt} P^j \subset B_r(\text{spt} F)$, in fact we can assume that for $P^j = \sum_{i=1}^m g_{ij}^T_j$ holds that $\text{spt} P^j_i \subset B_r(\text{spt} F)$. So, for smaller and smaller $r$ we find a sequence $P^j$ such that

$$P^j_i \to F \text{ and } \text{spt} P^j \subset B_{1/j}(\text{spt} F). \quad (4.1.18)$$

This shows also that

$$\text{spt}(\partial F) \subset \text{spt} F.$$ 

If $F \in \mathcal{M}_n(X, G)$ we see as $\mathbf{M}(P^j|_{\{d_{\text{spt} F} < r\}}) \leq \mathbf{M}(P^j)$ that we can achieve (4.1.18) such that in addition $\mathbf{M}(P^j_i) \to \mathbf{M}(F)$.

For $f \in \text{Lip}(X, Y)$, (4.1.18) implies that

$$\text{spt}(f \# F) \subset \overline{f(\text{spt} F)}.$$ 

**Lemma 4.1.9.** Let $X, Y$ be injective metric spaces and suppose $F \in \mathcal{F}_n(X, G)$. 


4.1. INJECTIVE METRIC SPACES

(i) If \( f, g : X \to Y \) are Lipschitz and \( f|_{\text{spt } F} = g|_{\text{spt } F} \), then \( f#F = g#F \).

(ii) If \( p, q : X \to \mathbb{R}^k \) are Lipschitz and \( p|_{\text{spt } F} = q|_{\text{spt } F} \), then \( \langle F, p, r \rangle = \langle F, q, r \rangle \) for a.e. \( r \).

(iii) If \( p, q \) are as in (ii) for \( k = 1 \), then \( F|_{\{p < r\}} = F|_{\{q < r\}} \) a.e.

This is essentially [DPH09, Lemma 5.4.2]; see therein for the proof of (i) and (ii). The last part follows analogously.

4.1.4 Interval times a flat current

Give \( \mathbb{R} \times X \) the supremum metric; as \( X \) is injective, \( \mathbb{R} \times X \) is an injective metric space as well. Let \( P = \sum_{i=1}^{m} g_i T_i \in \tilde{P}_n(X, G) \). Define for \( t \in \mathbb{R} \)

\[
[t] \times P := \sum_{i=1}^{m} g_i ([t] \times T_i) \in \tilde{P}_n(\mathbb{R} \times X, G)
\]

and for \( r, s \in \mathbb{R} \) with \( r < s \)

\[
[r, s] \times P := \sum_{i=1}^{m} g_i ([r, s] \times T_i) \in \tilde{P}_{n+1}(\mathbb{R} \times X, G).
\]

It holds that \( [r, s] \times (P + P') = [r, s] \times P + [r, s] \times P' \), etc. Observe that \( \tilde{\partial}([t] \times P) = [t] \times \partial P \) and

\[
\tilde{\partial}([r, s] \times P) = \sum_{i=1}^{m} g_i ([s] \times T_i - [r] \times T_i - [r, s] \times \partial T_i)
= [s] \times P - [r] \times P - [r, s] \times \tilde{\partial} P.
\]

Note that

\[
M([t] \times P) = M(P) \quad \text{and} \quad M([r, s] \times P) \leq (s - r)(n + 1)M(P)
\]

(one can extend (2.2.18) to elements of \( \overline{P}_n(X) \)). Then, for a representation \( R \) of \( P \) is \( [r, s] \times R \) a representation of \( [r, s] \times P \). So these definitions are well-defined for \( P \in P_n(X, G) \) and give elements of \( P_n(\mathbb{R} \times X, G) \), respectively of \( P_{n+1}(\mathbb{R} \times X, G) \).

Let \( F \in \mathcal{F}_n(X, G) \) and let \( P^j \to F \) where \( P^j - P^{j-1} = S^j_0 + \partial S^j_1 \) such that \( \sum_{j \in \mathbb{N}} M(S^j_i) < \infty, i \in \{0, 1\} \). Clearly, \([t] \times P^j\) converges and we define the limit to be \([t] \times F\).
Now set for the moment $\bar{P}^j := [r, s] \times P^j$ and $\bar{S}^j_i := [r, s] \times S^j_i$. Then

$$\bar{P}^j - \bar{P}^{j-1} = \bar{S}^j_0 + [s] \times \partial S^j_1 = \bar{S}^j_0 + [s] \times S^j_1 - [r] \times S^j_1 - \partial \bar{S}^j_1$$  (4.1.19)

with

$$\sum_{j \in \mathbb{N}} M(\bar{S}^j_0) + M([s] \times S^j_1) + M([r] \times S^j_1) + M(\bar{S}^j_1) < \infty.$$  

Hence $\bar{P}^j$ converges and we define this limit to be $[r, s] \times F$. Note that then

$$M([r, s] \times F) \leq (s - r)(n + 1)M(F)$$  (4.1.20)

and

$$\partial([r, s] \times F) = [s] \times F - [r] \times F - [r, s] \times \partial F.$$  

Clearly, if $R \in \mathcal{R}_n(X, G)$ then $[t] \times R \in \mathcal{R}_n(\mathbb{R} \times X, G)$ and also $[r, s] \times R \in \mathcal{R}_{n+1}(\mathbb{R} \times X, G).

Furthermore, $\mathcal{F}([t] \times F) = \mathcal{F}(F)$ and with Lemma 4.1.2 we see (as in (4.1.19))

$$\mathcal{F}([r, s] \times F) \leq (2 + (s - r)(2n + 3))\mathcal{F}(F).$$

### 4.2 Complete metric spaces

Let now $X$ be a complete metric space and let $l: X \to Y$ be an isometric embedding into an injective metric space $Y$ (e.g. $Y = l^\infty(X)$). As in [DPH09], the virtual flat chains in $X$ (with respect to $l$) are

$$\mathcal{F}_n(X, G) := \left\{ F \in \mathcal{F}_n(Y, G) \mid \text{spt } F \subset l(X) \right\}.$$  

([DPH09] denote this group by $\tilde{\mathcal{F}}_n(X, G).$) Note that $\partial F$, $F|_{\{f < r\}}$ as well as $\langle F, f, r \rangle$ are virtual flat chains in $X$ (the latter for a.e. $r$). We show below that for different isometric embeddings into different injective metric spaces the resulting spaces are isometric and therefore we will omit usually the embedding.

We define the virtual rectifiable chains

$$\mathcal{R}_n(X, G) := \left\{ R \in \mathcal{R}_n(Y, G) \mid \text{spt } F \subset l(X) \right\}$$

and the virtual finite mass chains

$$\mathcal{M}_n(X, G) := \left\{ F \in \mathcal{M}_n(Y, G) \mid \text{spt } F \subset l(X) \right\}.$$  

As flat norm (respectively mass norm) on $\mathcal{F}_n(X, G)$ (resp. $\mathcal{M}_n(X, G)$) we take the respective norm for $Y$.

The support of $F \in \mathcal{F}_n(X, G)$ is $l^{-1}(\text{spt } F) \subset X$. 

4.2. COMPLETE METRIC SPACES

**Lemma 4.2.1.** For \( i = 1, 2 \) let \( l_i: X \to Y_i \) be isometric embeddings of the complete metric space \( X \) into the injective metric spaces \( Y_i \); denote the virtual spaces as \( \mathcal{F}_{n}^{1}(X, G), \mathcal{R}_{n}^{1}(X, G), \) and \( \mathcal{M}_{n}^{1}(X, G) \).

(i) There exists an isometry \( \phi: (\mathcal{F}_{n}^{1}(X, G), \mathcal{F}) \to (\mathcal{F}_{n}^{2}(X, G), \mathcal{F}) \) such that \( \phi|_{\mathcal{M}_{n}^{1}(X, G)} \) is an isometry between \( (\mathcal{M}_{n}^{1}(X, G), \mathcal{M}) \) and \( (\mathcal{M}_{n}^{2}(X, G), \mathcal{M}) \); moreover \( \phi|_{\mathcal{R}_{n}(X, G)} \) is an isometry between \( \mathcal{R}_{n}^{1}(X, G) \) and \( \mathcal{R}_{n}^{2}(X, G) \).

(ii) It holds that \( \phi \circ \partial = \partial \circ \phi \).

(iii) Let \( f \in \operatorname{Lip}(X), g \in \operatorname{Lip}(X, \mathbb{R}^{k}) \). Let \( f_{i} \in \operatorname{Lip}(Y_{i}) \) be extensions of \( f \circ l_{i}^{-1}: l_{i}(X) \to \mathbb{R} \), and similarly for \( g_{i} \). Then the above isometries satisfy

\[
\phi(F|_{\{f_{1} < r\}}) = (\phi(F))|_{\{f_{2} < r\}} \quad \text{for a.e. } r \in \mathbb{R}, \quad \text{and}
\]

\[
\phi((F, g_{1}, s)) = (\phi(F), g_{2}, s) \quad \text{for a.e. } s \in \mathbb{R}^{k}.
\]

Once this lemma is proven, we usually think of \( X \) as subset of \( l^\infty(X) \).

**Proof.** Let \( \alpha_{ij} \in \operatorname{Lip}_{1}(Y_{i}, Y_{j}) \) be extensions of the maps \( l_{i}(x) \mapsto l_{j}(x), x \in X \). Then \( \alpha_{21} \circ \alpha_{12}|_{l_{1}(X)} = \operatorname{id}_{l_{1}(X)} \) and \( \alpha_{12} \circ \alpha_{21}|_{l_{2}(X)} = \operatorname{id}_{l_{2}(X)} \). Thus for \( F_{i} \in \mathcal{F}_{n}^{i}(X, G) \), we have by Lemma 4.1.9 that \( F_{1} = \alpha_{21}\#(\alpha_{12}\#F_{1}) \) and \( F_{2} = \alpha_{12}\#(\alpha_{21}\#F_{2}) \); hence \( \phi := \alpha_{12}\#: \mathcal{F}_{n}^{1}(X, G) \to \mathcal{F}_{n}^{2}(X, G) \) is bijective. As all maps are 1-Lipschitz, we also get \( \mathcal{F}(F) = \mathcal{F}(\alpha\#F) \), i.e. \( \phi \) is an isometry. Also we see that \( \mathcal{M}(F) = \mathcal{M}(\alpha\#F) \). Clearly, the push-forward of a rectifiable chain is rectifiable; together, this implies (i). Clearly, \( \phi \circ \partial = \partial \circ \phi \), hence (ii). (iii) follows by Lemma 4.1.5 and Lemma 4.1.9. \( \square \)

**Remark 4.2.2.** We will often omit the word ‘virtual’; in the following, all flat chains are virtual chains.

As a corollary of Lemma 4.1.9 we get

**Corollary 4.2.3.** Let \( X, Z \) be complete metric spaces, let \( F \in \mathcal{F}_{n}(X, G), \) let \( f \in \operatorname{Lip}(\operatorname{spt} F, Z) \). Then \( f\#F \in \mathcal{F}_{n}(Z, G) \) is well-defined, \( f\#(\partial F) = \partial f\#F \),

\[
\mathcal{F}(f\#F) \leq \max\{\operatorname{Lip}(f)^{n}, \operatorname{Lip}(f)^{n+1}\}\mathcal{F}(F) \quad \text{and} \quad \mathcal{M}(f\#F) \leq \operatorname{Lip}(f)^{n}\mathcal{M}(F).
\]

**Example 4.2.4** (Finite dimensional vector space). Let \( E_{m} := (\mathbb{R}^{m}, \| \cdot \|) \) be an \( m \)-dimensional normed vector space. Then \( E_{m} \) is bi-Lipschitz equivalent to \( l^\infty(\{1, \ldots, m\}) \), hence

\[
(\mathcal{F}_{n}(E_{m}, G), \mathcal{F}) \quad \text{is bi-Lipschitz equivalent to} \quad \left( \mathcal{F}_{n}(l^\infty(\{1, \ldots, m\}), G), \mathcal{F} \right).
\]
Example 4.2.5 (Euclidean space $\mathbb{R}^m$). In Euclidean space $\mathbb{R}^m$ we have a bi-Lipschitz equivalency as above. Also, $\mathcal{F}_n(\mathbb{R}^m, G)$ is bi-Lipschitz equivalent to the completion of $\mathcal{P}_n(\mathbb{R}^m, G)$ with respect to the norm $\tilde{F}_{\mathbb{R}^m}$ (see (3.9.1)). And this space is exactly the space of Fleming’s flat chains (compare [DPH09] and Appendix A).

4.3 Homotopies

Let $H : [r, s] \times X \to X$ be a Lipschitz map, set $H_t(x) := H(t, x)$ for $t \in [r, s]$. Let $F \in \mathcal{F}_n(X, G)$. Then $H_\#([r, s] \times F) \in \mathcal{F}_{n+1}(X, G)$ with

$$\partial H_\#([r, s] \times F) = H_s\#F - H_r\#F - H_\#([r, s] \times \partial F). \quad (4.3.1)$$

Note that if $F \in \mathcal{M}_n(X, G)$ and $\partial F \in \mathcal{M}_{n-1}(X, G)$ then (for $\text{Lip}(H) \geq 1$)

$$\mathcal{F}(H_s\#F - H_r\#F) \leq (s - r)(n + 1)\text{Lip}(H)^{n+1}\mathcal{N}(F). \quad (4.3.2)$$

4.4 Zero slices

Theorem 4.4.1 ([DPH09], Theorem 5.6.1). Let $0 \neq F \in \mathcal{F}_n(X, G)$ for $n > 0$, then there exists $f \in \text{Lip}(X, \mathbb{R}^n)$ and an $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(E) > 0$ such that $\langle F, f, r \rangle \neq 0$ for all $r \in E$.

Proof. We recall the proof from [DPH09]:

First note that White proved (see [Whi99a]) such a result for the space of Fleming’s flat chains in Euclidean space, so it holds for $\mathcal{F}_n(l^\infty(\{1, \ldots, m\}, G)$ by Example 4.2.5. Now the result follows for any finite dimensional normed vector space by Example 4.2.4.

By the definition of $\mathcal{F}_n(X, G)$ it is enough to consider an injective metric space $X$. Let $F \in \mathcal{F}_n(X, G)$ be such that for every $f \in \text{Lip}(X, \mathbb{R}^n)$ almost every slice of $F$ vanishes.

Let $\mathcal{P}_n(X, G) \ni P^j \to F$, Let $Z := \text{spt} F \bigcup_{j \in \mathbb{N}} \text{spt} P^j$; note that $(Z, d_X)$ is a separable metric space. Now we find an isometric embedding $l : Z \to A$ where $A := (C([0, 1]), \|\cdot\|_{\infty})$. By Corollary 4.2.3, $l_\#F \in \mathcal{F}_n(A, G)$ and $\mathcal{F}(l_\#(F - P^j)) \to 0$; moreover we see that $\mathcal{F}(l_\#F) = \mathcal{F}(F)$ as $l$ is an isometry on the support of $F$.

The space $A$ has the metric approximation property, i.e. for every $K \subset A$ compact and every $\epsilon' > 0$ there is a linear map $B : A \to A$ with $\|B\| \leq 1$, $\|a - B(a)\| < \epsilon'$ for all $a \in K$ and $\dim(\text{im} B) < \infty$ (see e.g. in [Meg98]).
4.5. REPRESENTATION OF RECTIFIABLE CHAINS

Let \( \epsilon > 0 \), let \( j \) be big enough with \( \mathcal{F}(l_\#(F - P^j)) < \epsilon \). Choose \( B \) for \( K := \text{spt } l_\# P^j \) and \( \epsilon' > 0 \) is small enough. Then \( \mathcal{F}(l_\# P^j - B_\# l_\# P^j) < \epsilon \) (by a linear homotopy filling and (4.3.2)).

We have for every \( f \in \text{Lip}(\text{im } B, \mathbb{R}^n) \) that

\[
\langle B_\# l_\# F, f, r \rangle = B_\# l_\# \langle F, \bar{f}, r \rangle = 0 \quad \text{for a.e. } r, 
\]

where \( \bar{f} : X \to \mathbb{R}^n \) is any Lipschitz extension of \( f \circ B \circ l : Z \to \mathbb{R}^n \). Thus, \( B_\# l_\# F \) has only vanishing slices and is therefore trivial by the remark at the beginning of the proof. Clearly is \( \mathcal{F}(B_\# l_\# P^j - B_\# l_\# F) < \epsilon \). Together we have \( \mathcal{F}(l_\# F) < 3\epsilon \), hence \( \mathcal{F}(F) < 3\epsilon \) and so \( F = 0 \). \( \square \)

4.5 Representation of rectifiable chains

Here we show that rectifiable chains are in fact push-forwards of integrable \( \hat{G} \)-valued maps, similarly as for currents.

A map \( g : \mathbb{R}^n \to \hat{G} \) is in \( L^1(\mathbb{R}^n, \hat{G}) \) if \( g \) is \( \mathcal{L}^n \)-measurable and \( \| g \| (x) := \| g(x) \| \in L^1(\mathbb{R}^n) \); i.e.

\[
\| g \|_{L^1} := \int_{\mathbb{R}^n} \| g \| d\mathcal{L}^n < \infty .
\]

As usual, define \( L^1(\mathbb{R}^n, \hat{G}) \) to be the group of equivalence classes of such maps, where \( g \sim g' \) if and only if \( g = g' \) \( \mathcal{L}^n \) a.e. Denote the \( \| \|_{L^1} \)-closure of the equivalence classes of simple maps in \( L^1(\mathbb{R}^n, \hat{G}) \) (i.e. elements of \( L^1(\mathbb{R}^n, \hat{G}) \) with countable image) by \( L^1_s(\mathbb{R}^n, \hat{G}) \). (If \( G \) is separable then \( L^1_s(\mathbb{R}^n, \hat{G}) = L^1(\mathbb{R}^n, \hat{G}) \).)

An element \( g \in L^1_s(\mathbb{R}^n, \hat{G}) \) corresponds to a rectifiable \( n \)-chain \( [g] \) as follows: Let \( h^j \) be simple functions converging to \( g \). For each \( j \) we can cover almost all of \( \mathbb{R}^n \) by countably many pairwise disjoint compact sets \( K_{jk} \) such that \( h^j|_{K_{jk}} =: g_{jk} \in \hat{G} \) is constant. With \( \| h^j \|_{L^1} = \sum_{k \in \mathbb{N}} \| g_{jk} \| \mathcal{L}^n(K_{jk}) < \infty \) is clearly

\[
R^j := \sum_{k \in \mathbb{N}} g_{jk} [\chi_{K_{jk}}] \in \mathcal{R}_n(\mathbb{R}^n, \hat{G}) = \mathcal{R}_n(\mathbb{R}^n, G). 
\]

(By Example 4.2.5, \( \mathcal{R}_n(\mathbb{R}^n, G) \) is the \( M \)-completion of \( \mathbf{P}_n(\mathbb{R}^n, G) \).)

We can without loss of generality assume that when \( j \) and \( j' < j \) are given, for \( k \in \mathbb{N} \) there is \( k' = k'(k) \) such that \( K_{jk} \subset K_{j'k'} \). Now,

\[
M(R^j - R^{j'}) = \sum_{k \in \mathbb{N}} \| g_{jk} - g_{j'k'(k)} \| \mathcal{L}^n(K_{jk}) = \| h^j - h^{j'} \|_{L^1} .
\]
So we see that \( R^l \) converges in mass to a rectifiable chain which we define to be \( [g] \in \mathcal{R}_n(\mathbb{R}^n, G) \). Note that

\[
\mathbf{M}([g]) = \lim_{j \to \infty} \mathbf{M} \left( \sum_{k \in \mathbb{N}} g_{jk} [\chi_{K_{jk}}] \right) = \|g\|_{L^1}.
\]

We write \( g \in L^1(K, \mathcal{G}) \) for a subset \( K \subset \mathbb{R}^n \) if \( g \in L^1(\mathbb{R}^n, \mathcal{G}) \) and \( \text{spt} g \subset K \).

**Theorem 4.5.1** (Parametric representation of rectifiable chains). Let \( X \) be a complete metric space considered as subset of \( l^\infty(X) \), let \( R \in \mathcal{F}_x(X, \mathcal{G}) \). Then \( R \in \mathcal{R}_n(X, G) \) if and only if there are \( f_i \in \text{Lip}(K_i, X) \), \( K_i \subset \mathbb{R}^n \) compact, \( f_i \) bi-Lipschitz and with pairwise disjoint images, and \( g_i \in L^1_n(K_i, G) \) such that

\[
R = \sum_{i \in \mathbb{N}} f_i\#[g_i] \quad \text{and} \quad \mathbf{M}(R) = \sum_{i \in \mathbb{N}} \mathbf{M}(f_i\#[g_i]) < \infty. \tag{4.5.1}
\]

For a rectifiable chain we can directly give the definition of its measure (for general \( F \in \mathcal{M}_n(X, G) \) we will do this later): Let \( R \in \mathcal{R}_n(X, G) \), let \( R = \mathbf{M}\lim_{j \to \infty} P_j \); for \( B \in \mathcal{B}_X \) set \( \mu_R(B) := \lim_{j \to \infty} \mu_{P_j}(B) \). This is well-defined. Note that from (4.5.1) we get (as \( f_i \) have disjoint images) that

\[
\mu_R = \sum_{i \in \mathbb{N}} \mu_{f_i\#[g_i]} . \tag{4.5.2}
\]

**Proof.** Let \( R \) satisfy (4.5.1). Clearly implies \([g_i] \in \mathcal{R}_n(\mathbb{R}^n, G)\) that \( f_i\#[g_i] \in \mathcal{R}_n(X, G) \) and then \( R \in \mathcal{R}_n(X, G) \).

Now let \( R \in \mathcal{R}_n(X, G) \), i.e. \( R \in \mathcal{R}_n(l^\infty(X), G) \) and \( \text{spt} R \subset X \). Let \( \check{P}^l \in \mathcal{P}_n(l^\infty(X), G) \) such that \( \mathbf{M}(R - \check{P}^l) \to 0 \). Then \( \mu_{\check{P}^l} \) is concentrated on a countably \( \mathcal{H}^n \)-rectifiable Borel set \( W_j \); with \( \mu_R(B) = \lim_{j \to \infty} \mu_{P_j}(B) \) for \( B \in \mathcal{B}_{l^\infty(X)} \) we see that \( \mu_R \) is concentrated on the countably \( \mathcal{H}^n \)-rectifiable set \( \bigcup_{j \in \mathbb{N}} W_j \) and that \( \mu_R \) is absolutely continuous with respect to \( \mathcal{H}^n \). So there are bi-Lipschitz maps \( f_i: K_i \to C_i \subset l^\infty(X), K_i \subset \mathbb{R}^n \) compact, with disjoint images covering \( \mu_R \)-almost all of \( l^\infty(X) \). We can assume that \( C_i \subset X \) (otherwise replacing \( C_i \) by \( C_i \cap X \)). Set \( W := \bigcup_{i \in \mathbb{N}} C_i \), then \( R = R\vert_W \) and \( \mathbf{M}(R - \check{P}^l\vert_W) \to 0 \). Set \( \check{P}^l\vert_W := P^l \in \mathcal{P}_n(l^\infty(X), G) \). Let \( V^j := \sum_{k \in \mathbb{N}} g_{jk}f_{jk}\#[\chi_{K_{jk}}] \) be a representation of \( P^j \); denote \( f_{jk}(K_{jk}) =: C_{jk} \). We can assume (after a refinement of the representations \( V^j \) and then possibly a sign and orientation change) that

\[
\text{either} \quad (f_i\#[\chi_{K_i}])\vert_{C_{jk}} = f_{jk}\#[\chi_{K_{jk}}] \quad \text{or} \quad (f_i\#[\chi_{K_i}])\vert_{C_{jk}} = 0 .
\]

Furthermore, we can assume the representation \( V^j \) to be refined with respect to the representations \( V^l \), when \( l \leq j - 1 \). I.e., for \( k, m \in \mathbb{N} \) is

\[
\text{either} \quad (f_{im}\#[\chi_{K_{im}}])\vert_{C_{jk}} = f_{jk}\#[\chi_{K_{jk}}] \quad \text{or} \quad (f_{im}\#[\chi_{K_{im}}])\vert_{C_{jk}} = 0 .
\]
4.5. REPRESENTATION OF RECTIFIABLE CHAINS

Set

\[ h_{ij}(z) := \sum_{k \in \mathbb{N}} \chi_{f_i^{-1}(C_j \cap C_i)}(z) g_{jk}. \]

Note that \( spt h_{ij} \subset K_i \) and that \( h_{ij} \) is equal to \( g_{jk} \in G \) (and hence constant) on each of the compact sets \( \tilde{K}_{ijk} := f_i^{-1}(C_j \cap C_i) \), \( k \in \mathbb{N} \), and zero else. Thus \( h_{ij} \) is a simple Borel map, in particular it is \( \mathcal{L}^n \)-measurable. Extend \( f_i^{-1} \) to a Lipschitz map on \( l^\infty(X) \) that we still denote by \( f_i^{-1} \). From the definition of \( h_{ij} \) we get that

\[ \|h_{ij}\|_{L^1} = M((f_i^{-1})_\#(V^j | C_i)) < \infty \quad \text{and} \quad (f_i^{-1})_\#(V^j | C_i) = [h_{ij}]. \]

Thus, \( h_{ij} \in L^1_s(K_i, \tilde{G}) \). Now, let \( l \leq j \) and set \( \bigcup_{k \in \mathbb{N}} \tilde{K}_{ijk} =: K^{ij} \); since \( V^j | C_i \setminus f_i(K^{ij}) = 0 \) we have

\[
\|h_{il} - h_{ij}\|_{L^1} = \sum_{k \in \mathbb{N}} \|g_{lk} - g_{jk}\|_{\mathcal{L}^n(\tilde{K}_{ijk})} + \sum_{k' \in \mathbb{N}} \|g_{lk'}\|_{\mathcal{L}^n(\tilde{K}_{ikl'} \setminus K^{ij})}
\leq \text{Lip}(f_i^{-1})^n \left( M((V^l - V^j) | f_i(K^{ij})) + M((V^l - V^j) | C_i \setminus f_i(K^{ij})) \right)
\leq \text{Lip}(f_i^{-1})^n M(V^l - V^j).
\]

As \( (V^j)_{j \in \mathbb{N}} \) is a Cauchy sequence for the mass, \( (h_{ij})_{j \in \mathbb{N}} \subset L^1_s(K_i, \tilde{G}) \) is Cauchy too and so \( g_i := \lim_{j \to \infty} h_{ij} \in L^1_s(K_i, \tilde{G}) \) exists by completeness. From the construction follows that

\[ R | C_i = \lim_{j \to \infty} f_i | (f_i^{-1}(V^j | C_i)) = f_i | [g_i]. \]

By the disjointness of \( C_i \) we then get (4.5.1). \( \square \)

Similarly, we define \( L^1_{(s)}(W, \tilde{G}) \) for a countably \( \mathcal{H}^n \)-rectifiable set \( W \subset X \).

**Theorem 4.5.2** (Measures of rectifiable chains). Let \( X \) be a complete metric space considered as subset of \( l^\infty(X) \). Then for \( R \in \mathcal{R}_n(X, G) \) there is a countably \( \mathcal{H}^n \)-rectifiable set \( S_R \subset X \) and \( g \in L^1_s(S_R, G \setminus 0) \) with

\[ \mu_R = \lambda_{S_R} \|g\| \mathcal{H}^n | S_R \]

where \( \lambda_{S_R} \) is the area factor of \( S_R \).

**Proof.** Such an equality holds for integer rectifiable currents (see p. 13 or [AK00, p. 58]), i.e. for \( f : K \to X \) bi-Lipschitz and \( B \in \mathcal{B}_X \) is

\[ \|f_\# [\chi_K] \|_B = \int_{f(K) \cap B} \lambda_{f(K)} d\mathcal{H}^n. \]
Now for a simple $h \in L^1(K, \widetilde{G})$ also holds
\[ \|f_#[h]\|(B) = \int_{f(K) \cap B} f \circ f^{-1} \|h\| d\mathcal{H}^n; \]
and by continuity also for $\tilde{g} \in L^1_s(K, \widetilde{G})$. Using Theorem 4.5.1 we have $R = \sum_{i \in \mathbb{N}} f_i#[g_i]$ with $g_i \in L^1_s(K_i, \widetilde{G})$ and we define
\[ g(x) := \sum_{i \in \mathbb{N}} \chi_{f_i(K_i)}(x) g_i(f_i^{-1}(x)) \]
(here we used as convention $\chi_{f_i(K_i)}(x') g_i(f_i^{-1}(x')) := 0 \in G$ for $x' \notin f_i(K_i)$).
Then $g \in L^1_s(\bigcup_{i \in \mathbb{N}} f_i(K_i), \widetilde{G})$. For $T \in \mathcal{I}_n(X)$ let $\tilde{S}_T$ be the characteristic set of $T$ (see p. 11). Then, set
\[ S_R := \left( \bigcup_{i \in \mathbb{N}} S_{f_i}(\chi_{K_i}) \right) \cap \{ x \in X \mid g(x) \neq 0 \}. \]
The sets $f_i(K_i)$ are pairwise disjoint, so for $\mathcal{H}^n$-a.e. $x \in f_i(K_i)$ is $\lambda_{S_R}(x) = \lambda_{S_{f_i(K_i)}}(x)$ or $g(x) = 0$. With (4.5.2) we conclude
\[ \mu_R(B) = \sum_{i \in \mathbb{N}} \int_{f_i(K_i) \cap B} \lambda_{S_{f_i(K_i)}} g_i(f_i^{-1}) d\mathcal{H}^n = \int_{S_R \cap B} \lambda_{S_R} g d\mathcal{H}^n. \]
\[ \square \]

4.6 Flat chains of finite mass

4.6.1 Injective metric spaces

Recall that the subset $A \subset Z$ of a metric space $Z$ is **totally bounded** if and only if for all $r > 0$ there exist $N \in \mathbb{N}$ and $z_1, \ldots, z_N \in Z$ such that $A \subset \bigcup_{i=1}^{N} U_r(z_i)$. If $Z$ is complete, then the closure of a totally bounded subset is compact.

**Lemma 4.6.1.** Let $X$ be an injective metric space. Let $F \in \mathcal{M}_n(X, G)$, $P_j \to F$ with $M(P_j) \to M(F)$. Then the set $\{\mu_{P_j}\}_{j \in \mathbb{N}}$ is totally bounded with respect to the Lévi-Prokhorov metric $d_{LP}$.

**Proof.** Assume that this set is not totally bounded, so there exists $\delta > 0$ and a sequence $P_j \to F$ with $F(F - P_j) < 1/(j + 1)$ such that $d_{LP}(\mu_{P_j}, \mu_{P_i}) \geq 2\delta$ for all $i \neq j$. Set $\mu_j := \mu_{P_j}$. 

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Possibly after reordering $2j \leftrightarrow 2j + 1$ there are $A_j \subset X$ Borel with
\[
\mu_{2j}(A_j) \geq \mu_{2j+1}(U_\delta(A_j)) + \delta.
\] (4.6.1)

Let $F - P^j = S_0^j + \partial S_1^j$ where $S_i^j \in \mathcal{R}_{n+i}(X, G)$ with $M(S_0^j) + M(S_1^j) < 1/j$. Then we find $r_j \in (\delta/3, 2\delta/3)$ such that the following conditions hold:
\[
F = F\{d_{A_j} < r_j\} + F\{r_j \leq d_{A_j}\},
\]
\[
M(F) = M(F\{d_{A_j} < r_j\}) + M(F\{r_j \leq d_{A_j}\}),
\]
\[
F\{d_{A_j} < r_j\} - P^{2j+1} S_0^{2j+1}\{d_{A_j} < r_j\} - \langle S_1^{2j+1}, d_{A_j}, r_j \rangle
+ \partial(S_1^{2j+1}\{d_{A_j} < r_j\}),
\]
\[
F\{r_j \leq d_{A_j}\} - P^{2j} S_0^{2j}\{r_j \leq d_{A_j}\} + \langle S_1^{2j}, d_{A_j}, r_j \rangle
+ \partial(S_1^{2j}\{r_j \leq d_{A_j}\}),
\]
(each of these conditions holds outside an $L^1$-negligible set) and such that
\[
M(\langle S_1^{2j}, d_{A_j}, r_j \rangle) + M(\langle S_1^{2j+1}, d_{A_j}, r_j \rangle) \leq \frac{3}{\delta} (M(S_1^{2j}) + M(S_1^{2j+1}))
\]
(by (4.1.10)). Define
\[
Q^j := P^{2j} S_0^{2j}\{r_j \leq d_{A_j}\} + P^{2j+1} S_0^{2j+1}\{d_{A_j} < r_j\} \in \mathcal{R}_n(X, G).
\]

Then,
\[
\mathcal{F}(F - Q^j) \leq \mathcal{F}(F\{r_j \leq d_{A_j}\} - P^{2j} S_0^{2j}\{d_{A_j} < r_j\})
+ \mathcal{F}(F\{d_{A_j} < r_j\} - P^{2j+1} S_0^{2j+1}\{d_{A_j} < r_j\})
\leq M(S_0^{2j}) + M(S_1^{2j}) + M(\langle S_1^{2j}, d_{A_j}, r_j \rangle)
+ M(S_0^{2j+1}) + M(S_1^{2j+1}) + M(\langle S_1^{2j+1}, d_{A_j}, r_j \rangle)
\leq \frac{1}{j} \left(1 + \frac{3}{\delta}\right),
\]
hence $Q^j \to F$ for $j \to \infty$. But $\{r_j \leq d_{A_j}\} = X \setminus U_{r_j}(A_j)$, so by (4.6.1)
\[
M(Q^j) \leq M(P^{2j} S_0^{2j}\{r_j \leq d_{A_j}\}) + M(P^{2j+1} S_0^{2j+1}\{d_{A_j} < r_j\})
\leq \mu_{2j}(X) - \mu_{2j}(U_{r_j}(A_j)) + \mu_{2j+1}(U_{\delta}(A_j))
\leq \mu_{2j}(X) + \mu_{2j}(A_j) + \mu_{2j}(A_j) - \delta
\leq M(P^{2j}) - \delta.
\]

Thus, we get the contradiction $M(F) \leq M(F) - \delta$. \qed
Now, let $P^j \to F$ with $M(P^j) \to M(F)$, and let $\mu_j := \mu_{P^j}$. By the lemma there is a Cauchy-subsequence (say $\mathbb{N}$) for $d_{LP}$. Set $X_0 := \bigcup_{j \in \mathbb{N}} \text{spt} \mu_j \subset X$ and consider $\mu_j$ as measure on $(X_0, \mathcal{B}_{X_0})$. $X_0$ is complete and separable. Then, the set of finite measures on $(X_0, \mathcal{B}_{X_0})$ is complete with respect to $d_{LP}$ so there is a measure on $(X_0, \mathcal{B}_{X_0})$ such that

$$\mu_F := \lim_{j \to \infty} \mu_j.$$  

Then, $\mu_F(X_0 \cap \cdot)$ is a measure on $(X, \mathcal{B}_X)$. We will show in Remark 4.6.2 below that $\mu_F$ is uniquely determined by $F$.

A Borel-regular measure on $X$ (also denoted by $\mu_F$) is defined as follows:

$$\mu_F(A) := \inf \{ \mu_F(U) \mid A \subset U, \ U \subset X \text{ open} \}.$$  

(Note that $\mu_F|_{\mathcal{B}_X}$ as measure on $(X, \mathcal{B}_X)$ equals the above $\mu_F$.)

For an open set $U \subset X$, $U \notin \{\emptyset, X\}$, and a strictly decreasing sequence of reals $(r_k)_{k \in \mathbb{N}}$ converging to zero we can write $U = \bigcup_{i \in \mathbb{N}} V_i$ where

$$V_1 := \{ r_1 \leq d_{X\setminus U} \}, \ V_i := \{ r_i \leq d_{X\setminus U} < r_{i-1} \} \quad \text{if } i > 1. \quad (4.6.2)$$

As $V_i$ are pairwise disjoint Borel sets, we then have

$$\mu_F(U) = \sum_{i \in \mathbb{N}} \mu_F(V_i). \quad (4.6.3)$$

Now let the sequence $(r_k)_{k \in \mathbb{N}}$ be non-exceptional in the following sense: For every $k \in \mathbb{N}$ is $\mu_F(\{d_{X\setminus U} = r_k\}) = 0$, $P_j|_{V_k} \to F|_{V_k}$ with $M(P_j|_{V_k}) \to M(F|_{V_k})$, and for every $m \in \mathbb{N}$ holds

$$F = F|_{\{d_{X\setminus U} < r_m\}} + \sum_{i=1}^{m} F|_{V_i}, \quad (4.6.4)$$

$$M(F) = M(F|_{\{d_{X\setminus U} < r_m\}}) + \sum_{i=1}^{m} M(F|_{V_i}) \quad (4.6.5)$$

(note that, having chosen $r_k$ for $k = 1, \ldots, N$, almost every $r \in (0, r_N)$ is non-exceptional). With Proposition 2.5.4 is

$$\limsup_{j \to \infty} \mu_j(\check{V}_k) \leq \mu_F(\check{V}_k) \leq \mu_F(\check{V}_k) \leq \liminf_{j \to \infty} \mu_j(\check{V}_k).$$

As $\mu_j(\check{V}_k) \geq \mu_j(\check{V}_k)$, we have equality above and then also

$$M(F|_{V_k}) \leftarrow M(P_j|_{V_k}) = \mu_j(V_k) \to \mu_F(V_k) \quad (j \to \infty).$$
Remark 4.6.2. Now, having another sequence $P'_j \to F$ with $M(P'_j) \to M(F)$, we find a sequence of non-exceptional reals as above which we now can assume to be such that also $\lim_{j \in \mathbb{N}} M(P'_j|_{V_k}) = \mu_F(V_k) = \lim_{j \in \mathbb{N}} M(P|_{V_k})$ (using Lemma 4.1.4). So, for $\mu'_F$ constructed from the sequence $P'_j$ we have by (4.6.3) equality on open sets (clearly is $\mu_F(\emptyset) = \mu'_F(\emptyset) = 0$ and $\mu_F(X) = \mu'_F(X) = M(F)$); this implies equality of the measures on Borel sets and hence the well-definedness of $\mu_F$.

We have seen that $
abla \mu_F(U) = \sum_{k \in \mathbb{N}} \mu_F(V_k) = \sum_{k \in \mathbb{N}} M(F|_{V_k})$ when we assume that $V_k$ (defined as above) come from a non-exceptional sequence. Taking such $V_k$, we define for $U \notin \{\emptyset, X\}$ open

$$F|_{U} := \sum_{k \in \mathbb{N}} F|_{V_k}, \quad \text{moreover} \quad F|_{X} := F \quad \text{and} \quad F|_{\emptyset} := 0. \quad (4.6.6)$$

This is well-defined as is shown in Remark 4.6.3. Clearly, $M(F|_{U}) = \mu_F(U) \leq M(F)$, so $F|_{U} \in \mathcal{M}_n(X, G)$. Note that $F|_{U}$ does not depend on the particular choice of non-exceptional reals since we can split (for a.e. $s \in (r_k, r_{k-1})$)

$$V_k = \{r_k < d_X \setminus U \leq r_{k-1}\}$$

$$= \{r_k < d_X \setminus U \leq s\} \sqcup \{s < d_X \setminus U \leq r_{k-1}\} = V_k^1 \sqcup V_k^2$$

with $F|_{V_k^1} + F|_{V_k^2} = F|_{V_k}$ and $M(F|_{V_k^1}) + M(F|_{V_k^2}) = M(F|_{V_k})$, and add together similarly. So having two sequences, we can find a third one, yielding equality in the definition of $F|_{U}$.

Now let $F, F' \in \mathcal{M}_n(X, G)$ and $P_j \to F, P'_j \to F'$; we can choose a non-exceptional sequence of reals such that also $(F + F')|_{V_k} = F|_{V_k} + F'|_{V_k}$. Then we get from the definition that

$$(F + F')|_{U} = F|_{U} + F'|_{U}.$$ 

Note that for $X \ni y \notin \text{spt } F$, there is $r > 0$, $P^j \to F$ as above such that $P^j|_{\{d_y < r}\} = 0$, thus $y \notin \text{spt } \mu_F$. If $y \in \text{spt } F$, $\lim_{j \to \infty} M(P^j|_{\{d_y < r\}}) > 0$ for a.e. $r > 0$; so we see that

$$\text{spt } \mu_F = \text{spt } F. \quad (4.6.7)$$

Remark 4.6.3. $F|_{U}$ is well-defined: If the open set is of the form $\{f < r\}$ for a Lipschitz function $f : X \to \mathbb{R}$, then we have for a.e. $r \in \mathbb{R}$ that $F|_{\{f < r\}}$ coincides with the other definition given in Section 4.1.2. To show this, we write
for the moment \( F_\{f<r\} \) for the restriction defined in this section here and as before \( F_\{f<r\} \) for the earlier one. We can assume that \( F = F_\{f<r\} + F_\{r\leq f\} \),

\[
M(F) = M(F_\{f<r\}) + M(F_\{r\leq f\}) \quad \text{and} \quad \lim_{i \to \infty} M(F_\{r-s_i \leq f<r\}) = 0,
\]
as well as \( \lim_{i \to \infty} M(F_\{r \leq f < r+s_i\}) = 0 \), where \( s_i > 0 \) are elements outside a \( L^1 \)-negligible set decreasing to 0. From these equations follows that

\[
F_\{f<r\} = F_\{r-s_i < f \leq r-s_i+1\},
\]
as well as \( F_\{r\leq f\} = F_\{r+s_i < f\} + \sum_{i \in \mathbb{N}} F_\{r+s_i < f < r+s_i+1\} \) and these sums are mass-convergent (for \( s_i \) chosen appropriately). Let \( F_\{f<r\} = \sum_{j \in \mathbb{N}} F_\{V_j\} \) as before; we can assume that the \( V_j \) (defined similar to (4.6.2)) are non-exceptional (i.e. the sequence of reals is non-exceptional) for \( F_\{r-s_i < f < r-s_i+1\} \) as well, etc. Then we see that

\[
F_\{r-s_i < f \leq r-s_i+1\} = \sum_{j \in \mathbb{N}} (F_\{r-s_i < f \leq r-s_i+1\})_{V_j} \quad \text{and} \quad \sum_{j \in \mathbb{N}} (F_\{r+s_i+1 \leq f < r+s_i\})_{V_j} = 0, \quad \text{for all} \ i \in \mathbb{N}.
\]

Together, this implies that \( F_\{f<r\} = F_\{f<r\} \).

By the definition of \( F\{f\} \) we see that \( \mu_{F\{f\}} \) is concentrated on \( U \cap \text{spt} F \), that \( M(F\{f\}) = \mu_F(U) \) and that \( M(F - F\{f\}) = \mu_F(X \setminus U) \). Moreover, for open sets \( V, U \subset X \) with \( U \subset V \) holds (directly from the construction) that

\[
F\{f\} = (F\{V\})\{f\} \quad \text{(4.6.8)}
\]

and so \( \mu_{F\{f\}}(U) = \mu_F(U) \). Together this gives

\[
M(F\{V\} - F\{f\}) = \mu_{F\{f\}}(X \setminus U) = \mu_F(V \setminus U). \quad \text{(4.6.9)}
\]

Now, let \( Y \subset X \) be \( \mu_F \)-measurable. Then we set

\[
F\{Y\} := M - \lim_{k \to \infty} F\{U_k\} \in \mathcal{M}(X, G) \quad \text{(4.6.10)}
\]

where \( U_k \subset X \) are open with \( U_k \supset U_{k+1} \supset Y \) and such that \( \mu_F(U_k \setminus Y) \to 0 \). The limit is well-defined by (4.6.9). Such a sequence \( \{U_k\}_{k \in \mathbb{N}} \) is called a defining sequence for \( F\{Y\} \).
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Lemma 4.6.4. Let \( Y, Z, Z_1, Z_2 \subset X \) be \( \mu_F \)-measurable sets. Then
\[
\mu_F(Y) = \mathcal{M}(F|_Y), \quad F|_Y + F|_{X \setminus Y} = F \quad \text{and} \quad F|_{Y \cap Z} = (F|_Y)|_Z. \tag{4.6.11}
\]
Let \( Z_1 \) and \( Z_2 \) be disjoint, then
\[
F|_{Z_1 \cup Z_2} = F|_{Z_1} + F|_{Z_2}, \quad \mathcal{M}(F|_{Z_1 \cup Z_2}) = \mathcal{M}(F|_{Z_1}) + \mathcal{M}(F|_{Z_2}). \tag{4.6.12}
\]
Let \( F' \in \mathcal{M}_n(X, G) \) and \( B \in \mathcal{B}(X) \). Then
\[
(F + F')|_B = F|_B + F'|_B \quad \text{and} \quad \mu_{F + F'}(B) \leq \mu_F(B) + \mu_{F'}(B).
\]

Proof. The first assertion is clear from the definition of \( F|_Y \).

Let \( (r_i)_{i \in \mathbb{N}} \) be a strictly decreasing zero-sequence; let \( C \subset X \) be closed. Then \( \mu_F\{d_C < r_i\}|C) \to 0 \) and so \( F|_C = \mathcal{M}\lim_{i \to \infty} F|_{\{d_C < r_i\}} \). Let \( P_j \to F \) rapidly; we find a non-exceptional sequence \((r_i)_{i \in \mathbb{N}}\) with \( P_j|_{\{r_i \leq d_C < r_i - 1\}} \to F|_{\{r_i \leq d_C < r_i - 1\}} \) (compare Remark 4.6.3) and also

\[
\mathcal{M}\lim_{i \to \infty} F|_{\{r_i \leq d_C\}} = F|_{\{r_i \leq d_C\}} + \mathcal{M}\lim_{i \to \infty} \sum_{k=2}^{i} F|_{\{r_k \leq d_C < r_{k-1}\}}.
\]

So for the open set \( X \setminus C \) holds \( F|_{X \setminus C} = \mathcal{M}\lim_{i \to \infty} F|_{\{r_i \leq d_C\}} \). We can assume that \( F = F|_{\{d_C < r_i\}} + F|_{\{r_i \leq d_C\}} \), so we get \( F = F|_C + F|_{X \setminus C} \) for \( C \) closed.

Set \( F' := (F|_U)|_V \) for \( U, V \) open (we assume that \( U, V \notin \{\emptyset, X\} \)). Let \( U = \bigcup_{i \in \mathbb{N}} U_i \) and \( V = \bigcup_{i \in \mathbb{N}} V_i \) (similar as in the definition of \( F|_U \)); thus, \( F|_U = \sum_{i \in \mathbb{N}} F|_{U_i} \) and \( (F'|_U)|_V = \sum_{j \in \mathbb{N}} (F|_U)|_{V_j} \). Set \( A_{ij} := (F|_{U_i})|_{V_j} \). Assuming all the sets to be non-exceptional (first \( U_i \), then \( V_j \)) we then have

\[
F' = \sum_{i, j \in \mathbb{N}} A_{ij} \quad \text{with \text{spt}(A_{ij}) \subset \{d_X \setminus U \geq r_i\} \cap \{d_X \setminus V \geq \tilde{r}_j\}}
\]

for appropriate sequences \((r_i)_{i \in \mathbb{N}}\) for \( U \), respectively \((\tilde{r}_j)_{j \in \mathbb{N}}\) for \( V \).

Let \( U \cap V = \bigcup_{k \in \mathbb{N}} W_k \) (as for \( U, V; W_k \) non-exceptional for \( A_{ij}, i, j \in \mathbb{N} \)). Assume that \( W_k \supset \{d_X \setminus (U \cup V) \geq 1/k\} \). Note that if \( 1/M < \max\{r_i, \tilde{r}_j\} \) then

\[
A_{ij}|_{W_k} = A_{ij} \quad \text{and} \quad A_{ij}|_{W_k} = 0 \quad \text{for} \quad k > M.
\]

Let \( \epsilon > 0 \), choose \( m \in \mathbb{N} \) such that \( \mathcal{M}(F' - \sum_{i,j=1}^{m} A_{ij}) < \epsilon \). As
\[
\left( \sum_{i,j=1}^{m} A_{ij} \right)|_{U \cap V} = \sum_{k \in \mathbb{N}} \left( \sum_{i,j=1}^{m} A_{ij} \right)|_{W_k} = \sum_{k \in \mathbb{N}} \sum_{i,j=1}^{m} A_{ij}|_{W_k} \geq \sum_{i,j=1}^{m} A_{ij},
\]
we get \( M(F'|_{U\cap V} - \sum_{i,j\in\mathbb{N}} A_{ij}) < 2\epsilon \), hence \( F'|_{U\cap V} = F' \). With (4.6.8),

\[
F' = F'|_{U\cap V} = (F|_U)|_{U\cap V} = F|_{U\cap V} \quad \text{for } U, V \text{ open.} \quad (4.6.13)
\]

Now, let \((V_i)_{i\in\mathbb{N}}\) be a defining sequence of open sets for \( F|_Z \) and \((U_i)_{i\in\mathbb{N}}\) one for \( F|_Y \); note that \((V_i)_{i\in\mathbb{N}}\) is then also a defining sequence for \( (F|_Y)|_Z \), as \( \mu_{F|_Y} \leq \mu_F \). Then the sequence \((V'_i := V_i \cap U_i)_{i\in\mathbb{N}}\) satisfies 
\[
\mu_F(V'_i \setminus (Y \cap Z)) \leq \mu_F(U_i \setminus Y) + \mu_F(V_i \setminus Z) \to 0,
\]
hence it is a defining sequence for \( F|_{Y\cap Z} \).

With (4.6.13), we find for \( \epsilon > 0 \) an integer \( i \) great enough such that

\[
M(F|_{Y\cap Z} - (F|_Y)|_Z) \leq M(F|_{V'_i} - (F|_{U_i})|_{V_i}) + 2\epsilon = 2\epsilon.
\]

So \((F|_Y)|_Z = F|_{Y\cap Z} \).

For \( C \subset Y \) closed we then have

\[
M(F|_Y - F|_C) = M(F|_{Y\cap C} - F|_{X\setminus C}) = M((F|_{Y} - F|_{C})|_{X\setminus U_i}) = M(F|_{Y\cap (X\setminus C)}) = \mu_F(Y\setminus C).
\]

This implies that if \((U_i)_{i\in\mathbb{N}}\) is a defining sequence for \( F|_Y \), then the sequence \((F|_{X\setminus U_i})_{i\in\mathbb{N}}\) converges to \( F|_{X\setminus Y} \). The general equality \( F|_Y + F|_{X\setminus Y} = F \) follows now as \( F|_{U_i} + F|_{X\setminus U_i} = F \) (as shown above); we have (4.6.11).

Also (4.6.12) follows from the above. Finally, we know that for open sets in \( X \) holds that \( (F + F')|_U = F|_U + F'|_U \), this implies immediately the inequality for Borel sets and then the inequality for the measures follows. \( \square \)

### 4.6.2 Complete metric spaces

Let now \( X \) be a complete metric space. The measure \( \mu_F \) on \( X \) of \( F \in \mathcal{F}_n(X, G) \) is defined via the isometric embedding \( l: X \to Y \) used in the definition of \( F \in \mathcal{F}_n(X, G) \): \( \mu_F := \mu_F \circ l \). Clearly, the measure \( \mu_F \) on \( X \) is independent of the isometric embedding.

Still \( \mu_F(X) = M(F) \) and \( \mu_F \) is a a Borel-regular measure on \( X \).

**Remark 4.6.5.** Let \( F \in \mathcal{M}_n(X, G) \) and let \( i: X \to Z \) be an isometric embedding, \( Z \) a complete metric space. Then \( \mu_F(A) = \mu_{i\#F}(i(A)) \) for \( A \in B_X \). In particular, if \( F \) is concentrated on the set \( B \) (i.e. \( F|_{X\setminus B} = 0 \)) then \( i\#F \) is concentrated on \( i(B) \).

**Lemma 4.6.6 ([DPH09]).** Let \( F \in \mathcal{M}_n(X, G) \) and let \( f \in \text{Lip}(X, \mathbb{R}^k) \), then for any \( \mu_F \)-measurable \( A \subset X \) is a.e. \( \langle F, f, r \rangle|_A = (F|_A, f, r) \).
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Proof. The proof is similar to the one in [DPH09] written in our notation.

We can assume that $X$ is an injective metric space. Let $h \in \text{Lip}(X)$. We show the lemma first for open sets of the form $A = \{h < s\}$ where $\mu_F(\{h = s\}) = 0$; this holds for a.e. $s \in \mathbb{R}$. Note that then $\mu_F(\{s - \epsilon < h < s + \epsilon\}) \to 0$ for $0 < \epsilon \to 0$. We can write

$$F = F\lfloor_{\{h<s-\epsilon\}} + F\lfloor_{\{s-\epsilon\leq h<s\}} + F\lfloor_{\{s\leq h<s+\epsilon\}} + F\lfloor_{\{s+\epsilon\leq h\}};$$

then we have $\langle F\lfloor_A, f, r \rangle = \langle F\lfloor_{\{h<s-\epsilon\}}, f, r \rangle + \langle F\lfloor_{\{s-\epsilon\leq h<s\}}, f, r \rangle + \langle F\lfloor_{\{s\leq h<s+\epsilon\}}, f, r \rangle$.

On the other hand, by (4.1.16), for a.e. $r$ holds:

$$\langle F, f, r \rangle\lfloor_A = \langle F\lfloor_{\{h<s-\epsilon\}}, f, r \rangle\lfloor_A + \langle F\lfloor_{\{s-\epsilon\leq h<s\}} + F\lfloor_{\{s\leq h<s+\epsilon\}}, f, r \rangle\lfloor_A.$$

Considering a converging sequence $P_j \to F$ we see by Lemma 3.8.1 that

$$\langle F\lfloor_{\{h<s-\epsilon\}}, f, r \rangle = \langle F\lfloor_{\{h<s-\epsilon\}}, f, r \rangle\lfloor_A \text{ a.e.}$$

Then, for $C \in \mathbb{R}$ depending on $f$,

$$\int_{\mathbb{R}^*} M(\langle F\lfloor_A, f, r \rangle - \langle F, f, r \rangle\lfloor_A) dr \leq 2CM(F\lfloor_{\{s-\epsilon<h<s+\epsilon\}}) \to 0 \ (\epsilon \to 0).$$

Now let $A$ be open. Then we find open sets $V_i \subset A$ as above such that $\mu_F(A\backslash V_i) \to 0$; from (4.1.10) follows then this case.

For general $A$ we consider $A \subset V_i$, $V_i$ open such that $\mu_F(V_i \backslash A) \to 0$; then we conclude again by (4.1.10).

From the lemma we get

**Corollary 4.6.7.** Let $F \in M_n(X,G)$ and $a < b$ where $a, b \in \mathbb{R} \cup \{\pm \infty\}$, let $f \in \text{Lip}(X)$. Then

$$\int_a^b M(\langle F, f, r \rangle) dr \leq \text{Lip}(f) M(F\lfloor_{\{a<f<b\}}). \quad (4.6.14)$$

Moreover, we have

$$M(\langle F, f, r \rangle) \leq \text{Lip}(f) (M(F\lfloor_{\{f\leq r\}}))' \text{ for a.e. } r. \quad (4.6.15)$$
4.7 Flat chains of dimension 0

As in [DPH09] we have the following (compare [Whi99a] for Euclidean spaces):

**Theorem 4.7.1.** Let $X$ be a complete metric space. There is a homomorphism $\chi: \mathcal{F}_0(X, G) \to \bar{G}$ such that

(i) $\chi(\sum_{i=1}^m g_i[x_i]) = \sum_{i=1}^m g_i$.

(ii) $\chi(\partial F) = 0$ for $F \in \mathcal{F}_1(X, G)$.

(iii) $\|\chi(F)\| \leq \mathcal{F}(F)$.

(iv) For $F \in M_0(X, G)$ is

$$\mathcal{F}(F) \leq \|\chi(F)\| + M(F) \text{diam } F.$$  \hfill (4.7.1)

(v) Let $F' \in M_0(X, G)$, then $\chi(F'|_B) = 0$ for all $\mu_{F'}$-measurable sets $B$ if and only if $F' = 0$.

(vi) For $Z$ a complete metric space and $f \in \text{Lip}(X, Z)$ holds $\chi(f\# F) = \chi(F)$.

(This homomorphism corresponds to the augmentation for singular Lipschitz chains, compare (12.2.3).)

**Proof.** The proof from [Whi99a] generalizes directly:

By the definition of $\mathcal{F}_0(X, G)$, respectively $\mathcal{F}_1(X, G)$, it is clear that it suffices to check the theorem in injective metric spaces. So let $X, Z$ be injective metric spaces. Define $\chi$ for $P = \sum_{i=1}^m g_i(n_i[x_i]) \in \mathcal{P}_0(X, G)$ by

$$\chi(P) := \sum_{i=1}^m n_i g_i \in G.$$

We easily see that $\|\chi(P)\| \leq M(P)$, so $\chi: \mathcal{P}_0(X, G) \to \bar{G}$ is well-defined.

Clearly, $\chi(P_1 + P_2) = \chi(P_1) + \chi(P_2)$. Moreover, $\chi(\partial \bar{P}) = 0$ for every $\bar{P} \in \mathcal{P}_1(X, G)$. Thus, for $P = P_0 + \partial P_1$ where $P_i \in \mathcal{P}_i(X, G)$ is

$$\|\chi(P)\| = \|\chi(P_0) + \chi(\partial P_1)\| = \|\chi(P_0)\| \leq M(P_0).$$

Taking the infimum over all such $P_0, P_1$ we get $\|\chi(P)\| \leq \mathcal{F}(P)$. Now, we can extend $\chi$ uniquely to $\mathcal{F}_0(X, G)$ with values in $\bar{G}$; (ii), (iii) and (vi) follow by continuity (for this, note that (vi) holds for $P \in \mathcal{P}_0(X, G)$ and $f \in \text{Lip}(X, Z)$).
Let now $P' = \sum_{i=1}^{m} h_i [y_i] \in P_0(X, G)$. We can assume that $y_i \neq y_j$ for $i \neq j$, so $M(P') = \sum_{i=1}^{m} \|h_i\|$. Let $c_i : [0, 1] \to X$ be a path from $y_1$ to $y_i$ of length $L(c_i) = d(y_1, y_i)$ (this is possible since $X$ is injective); then $M_{AK}(c_i [0, 1]) = L(c_i) \leq \text{diam } P'$.

Let

$$S := \sum_{i=1}^{m} h_i c_i [0, 1] \in P_1(X, G).$$

Then $\partial S = P' - \chi(P')[x_1]$, so

$$\mathcal{F}(P') \leq M(\chi(P')[x_1]) + M(S) \leq \|\chi(P')\| + \sum_{i=1}^{m} \|h_i\| M_{AK}(c_i [0, 1]) \leq \|\chi(P')\| + M(P') \text{diam } P'.$$

For a flat 0-chain $F$ the claim holds by continuity since we can approximate $F$ by $P_j$ such that $\text{diam } P_j \leq \text{diam } F + \epsilon_j$, with $\epsilon_j \to 0$ and $M(P_j) \to M(F)$; this proves (4.7.1).

Now let $F' \in M_0(X, G)$ and assume that $\chi(F'|Y) = 0$ for all sets $Y$ that are $\mu_{F'}$-measurable. Let $(B_i)_{i \in \mathbb{N}}$ be a covering of $\text{spt } F'$ (which is separable) by pairwise disjoint Borel sets of diameter at most $\epsilon > 0$. Then

$$\mathcal{F}(F') \leq \sum_{i \in \mathbb{N}} \mathcal{F}(F'|B_i) \leq \sum_{i \in \mathbb{N}} \|\chi(F'|B_i)\| + \sum_{i \in \mathbb{N}} \epsilon M(F'|B_i) \leq \epsilon M(F').$$

Since $\epsilon$ was arbitrary, $\mathcal{F}(F') = 0$ hence $F' = 0$. \qed

### 4.8 Cone fillings

Assume that $Y$ is an injective Banach space, let $X \subset Y$. Let $F \in M_n(X, G)$ with $\text{spt } F \subset B_r(x_0)$ for $x_0 \in X$ and $\partial F = 0$, respectively $\chi(F) = 0$ if $n = 0$. Define

$$\phi : [0, 1] \times Y \to Y, \quad (t, x) \mapsto tx + (1-t)x_0.$$

Then $\phi_\#([0, 1] \times F) \in M_{n+1}(Y, G)$ has $\partial \phi_\#([0, 1] \times F) = F - \phi_0 F$. If $n = 0$, (4.7.1) implies that $\phi_0 F = 0$. If $n > 0$, almost every slice of $\phi_0 F$ vanishes (as almost every slice is necessarily supported on a subset of $\{x_0\} = \text{spt } \phi_0 F$,
hence almost every slice vanishes). Then we see that there exists a constant \( \tilde{C}_n \in \mathbb{R} \) (not depending on \( r \)) with

\[
\partial \phi_\#([0, 1] \times F) = F \quad \text{and} \quad \mathcal{M}(\phi_\#([0, 1] \times F)) \leq \tilde{C}_n r \mathcal{M}(F). \quad (4.8.1)
\]

Note that we find a filling with (4.8.1) in any injective metric space that contains \( X \), Banach or not. We refer to such a filling of \( F \) as \textit{cone filling}. 
Chapter 5

Compactness results

Adams proved in [Ada08] a similar result as we do below for the therein defined flat chains in Banach spaces (which are the flat-completion of polyhedral chains). For metric currents such a result is proved in [AK00] with respect to weak convergence. The prerequisites of Adams include that all the chains have support in a common compact set; here we weaken this assumption. Our proof follows the one of Adams, nevertheless.

Recall that a metric space is said to be \textit{proper} if and only if every closed ball is compact.

**Theorem 5.0.1.** Let $Q \in \mathbb{R}$, let $X$ be a complete metric space. Let $G$ be a normed group such that $\bar{G}$ is proper. Let $(F_i)_{i \in \mathbb{N}} \subset \mathcal{F}_n(X, G)$ be a sequence such that $\mathcal{N}(F_i) \leq Q$ for all $i \in \mathbb{N}$ and let $C_j \subset X$ be compact with

$$\mu_{F_i}(X \setminus C_j) + \mu_{\partial F_i}(X \setminus C_j) < 1/j \quad \text{for all } i \in \mathbb{N}.$$ 

Then there is an $\mathcal{F}$-convergent subsequence.

Compare [Ada08].

If $\bar{G}$ is not proper there is $r > 0$ such that $B_r(0) \subset \bar{G}$ is not compact. Then, we easily find a mass bounded sequence in $\mathcal{F}_0(X, G)$ with support in $x \in X$ that has no convergent subsequence.

**Proof.** The proof is similar to [Ada08]: We argue by induction; let $n = 0$. First, fix $C \subset X$ compact; we show that the set

$$Z := Z(C) := \{ F \in \mathcal{F}_0(X, G) \mid \text{spt } F \subset C \text{ and } \mathcal{M}(F) \leq Q \}$$

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is totally bounded.

Let $1 > \epsilon > 0$ and let $D > 0$ such that $12(Q + 1)D < \epsilon$. Then there exist
\[ m \in \mathbb{N}, \ x_1, \ldots, x_m \in C \text{ with } C \subset \bigcup_{i=1}^{m} U_D(x_i); \] and there exist $M \in \mathbb{N}$ and $g_1, \ldots, g_M \in G$ such that $\bigcup_{j=1}^{M} U_{\frac{\epsilon}{3m}}(g_j)$ covers $B_Q(0) \subset G$.

Now let $F \in Z$ be given, set
\[ F_i := F|_{U_D(x_i) \setminus \bigcup_{k=1}^{n} U_D(x_k)}. \]
Since for any Borel set $A \subset X$ holds $\|\chi(F|_A)\| \leq M(F|_A) \leq Q$, we can choose for $i = 1, \ldots, m$ elements $h_i \in \{g_1, \ldots, g_M\}$ such that $\|\chi(F_i) - h_i\| < \frac{\epsilon}{3m}$. Now we have by (4.7.1)
\[ F \left( F - \sum_{i=1}^{m} h_i[x_i] \right) < \frac{\epsilon}{3} + 2D \left( \sum_{i=1}^{m} M(F_i) + \sum_{i=1}^{m} M(h_i[x_i]) \right). \]
Since
\[ M(h_i[x_i]) = \|h_i\| \leq \|\chi(F_i)\| + \frac{\epsilon}{3m} \leq M(F_i) + \frac{\epsilon}{3m} \]
we have
\[ \sum_{i=1}^{m} M(F_i) + \sum_{i=1}^{m} M(h_i[x_i]) \leq 2Q + \epsilon/3. \]
By our choice of $D$, $\mathcal{F}(F - \sum_{i=1}^{m} h_i[x_i]) < \epsilon$ and so, $Z$ is totally bounded.

Now, assume that there exist sequences $(F_i)_{i \in \mathbb{N}}$ and $(C_j)_{j \in \mathbb{N}}$ as in the theorem such that $\mathcal{F}(F_i - F_k) \geq \delta > 0$ for $i \neq k$. Choose $j > 3/\delta$. Then $Z(C_j)$ is totally bounded and $F_i|_{C_j} \in Z(C_j)$, so we find a converging subsequence - say $\mathbb{N}$ - for the sequence $(F_i|_{C_j})_{i \in \mathbb{N}}$. But
\[ \mathcal{F}(F_i - F_k) \leq \mathcal{F}(F_i|_{C_j} - F_k|_{C_j}) + \mu_{F_i}(X \setminus C_j) + \mu_{F_k}(X \setminus C_j) < \tilde{\delta} \]
for $i, k$ great enough, in contradiction to our assumption.

Now let $n > 0$. Consider $X$ as subset of $l^\infty(X)$. Choosing a subsequence (say $\mathbb{N}$) of the compact sets there exists $\alpha \in \mathbb{R}$ with
\[ \sum_{j \in \mathbb{N}} (\mu_{F_i}(X \setminus C_j) + \mu_{\partial F}(X \setminus C_j)) \leq \alpha \quad \text{for all } i \in \mathbb{N}. \]
Assume that there is no converging subsequence, so there exist $1 > \epsilon > 0$ and a subsequence (say $\mathbb{N}$) with $\mathcal{F}(F_k - F_j) > \epsilon$ for $k \neq j$.

Let $0 < \tilde{\epsilon}$ be such that $16(Q + 1)(\tilde{C}_n + 1)\tilde{\epsilon} < \epsilon/3$ (here, $\tilde{C}_n$ as in (4.8.1)). Cover $\bigcup_{j \in \mathbb{N}} C_j$ by countably many open balls $U_{\tilde{\epsilon}}(x_l)$ such that for every $j \in \mathbb{N}$
there is \( m_j \in \mathbb{N} \) with \( C_j \subseteq \bigcup_{l \leq m_j} U_{\bar{r}}(x_l) \). For each \( i \in \mathbb{N} \) we find \( r_{li} \in (\bar{r}, 2\bar{r}) \) (inductively for \( l \in \mathbb{N} \)) such that for \( U_{\bar{r}_i} := U_{r_{li}}(x_l) \) holds

\[
\langle F_i, d_{x_l}, r_{li} \rangle = \partial(F_i|_{U_{\bar{r}_i}}) - (\partial F_i)|_{U_{\bar{r}_i}},
\]

\[
\langle F_i|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}, d_{x_l}, r_{li}} \rangle = \partial(F_i|_{U_{\bar{r}_i} \cap (\bigcup_{k=1}^{l-1} U_{\bar{r}_i})}) - (\partial(F_i|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}}))|_{U_{\bar{r}_i}},
\]

\[
\langle F_i|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}, d_{x_l}, r_{li}} \rangle = \langle F_i, d_{x_l}, r_{li} \rangle|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}} \text{ and such that }
\]

\[
M(\langle F_i, d_{x_l}, r_{li} \rangle) + \sum_{j \in \mathbb{N}} M(\langle F_i, d_{x_l}, r_{li} \rangle|_{X \setminus C_j}) \leq \frac{Q + \alpha}{\bar{r}}, \tag{5.0.1}
\]

This is possible by Lemma 4.6.6 and (4.14). Set

\[
A_{\bar{r}_i} := U_{r_{li}} \setminus \left( \bigcup_{k=1}^{l-1} U_{r_{ki}} \right) \quad \text{and} \quad F_i^l := F_i|_{A_{\bar{r}_i}}.
\]

Then, \( \sum_{l \in \mathbb{N}} F_i^l = F_i, \ F_i^l = F_i|_{U_{\bar{r}_i} \setminus (\bigcup_{k=1}^{l-1} U_{r_{ki}})} \) and so (with (4.6.11))

\[
\partial F_i^l = (\partial F_i)|_{U_{\bar{r}_i}} + \langle F_i, d_{x_l}, r_{li} \rangle - (\partial(F_i|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}}))|_{U_{\bar{r}_i}} - \langle F_i|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}}, d_{x_l}, r_{li} \rangle
\]

\[
= (\partial(F_i)|_{U_{\bar{r}_i}} + \langle F_i, d_{x_l}, r_{li} \rangle|_{X \setminus \bigcup_{k=1}^{l-1} U_{\bar{r}_i}} - (\partial(F_i|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}}))|_{U_{\bar{r}_i}}. \tag{5.0.2}
\]

As \( \partial(F_i|_{\bigcup_{k=1}^{l-1} U_{\bar{r}_i}}) = \sum_{k=1}^{l-1} \partial F_i^k \) we can repeat this argument; we get (using (5.0.1)) as a rough upper bound that

\[
M(\partial F_i^l) \leq l! \left( M(\partial F_i) + \sum_{k=1}^{l} M(\langle F_i, d_{x_k}, r_{ki} \rangle) \right) \leq (l+1)! \left( Q + \frac{Q + \alpha}{\bar{r}} \right) := Q_l
\]

for all \( i \in \mathbb{N} \) with \( Q_l < \infty \). Similarly by (5.0.2) and (5.0.1),

\[
\sum_{j \in \mathbb{N}} \mu_{\partial F_i^l}(X \setminus C_j) \leq l! \sum_{j \in \mathbb{N}} \left( \mu_{\partial F_i}(X \setminus C_j) + \sum_{k=1}^{l} \mu(\langle F_i, d_{x_k}, r_{ki} \rangle)(X \setminus C_j) \right)
\]

\[
\leq (l+1)! \left( \alpha + \frac{Q + \alpha}{\bar{r}} \right) \quad \text{for all } i \in \mathbb{N}.
\]

By induction we have for each \( l \in \mathbb{N} \) a converging subsequence for \( \partial F_i^l \) (say \( N \)) and we can assume that \( \sum_{i \in \mathbb{N}} F(\partial F_i^l - \partial F_{i-1}^l) < \infty \) for every \( l \). So we take from these a diagonal sequence that we call again \( \mathbb{N} \).
Now there are $R_i^l \in \mathcal{F}_{n-1}(l^\infty(X), G)$ and $\tilde{R}_i^l \in \mathcal{F}_n(l^\infty(X), G)$ such that $\partial F_i^l - \partial F_{i-1}^l = R_i^l + \partial \tilde{R}_i^l$ and $\sum_{i \in \mathbb{N}} (M(R_i^l) + M(\tilde{R}_i^l)) < \infty$ (by Lemma 4.1.2). Furthermore, $\partial \tilde{R}_i^l = 0$ and without loss of generality, the supports of $R_i^l$ and $\tilde{R}_i^l$ lie in the ball of radius $3\tilde{r}$ around $x_l \in L^\infty(X)$ (restricting to $U_r(x_l)$ where $r \in (2\tilde{r}, 3\tilde{r})$, as in Lemma 3.9.1). Then we have a cone filling $w_i^l$ of $R_i^l$ (compare (4.8.1)) with $M(w_i^l) \leq 3\tilde{C}_{n-1} \tilde{r} M(R_i^l)$ and $\text{spt } w_i^l \subset U_{4\tilde{r}}(x_l)$.

Now, $V_i^l := w_i^l + \tilde{R}_i^l$ has $\partial V_i^l = \partial F_i^l - \partial F_{i-1}^l$ and $M(V_i^l) \leq 3\tilde{C}_{n-1} \tilde{r} M(R_i^l) + M(\tilde{R}_i^l)$.

Then there is a cone filling $\tilde{V}_i^l$ of $F_i^l - F_{i-1}^l - V_i^l$ with $M(\tilde{V}_i^l) \leq 8\tilde{C}_n \tilde{r} M(F_i^l - F_{i-1}^l - V_i^l)$.

By construction is $\sum_{k \in \mathbb{N}} M(F_i^k) = M(F_i)$; hence we have

$$\sum_{k \leq m_j} M(\tilde{V}_i^k) \leq 8\tilde{C}_n \tilde{r} \left(2Q + \sum_{k \leq m_j} M(V_i^k)\right).$$

Now, for $j$ such that $\mu_{F_i}(X \setminus C_j) < \epsilon/6$ for all $i \in \mathbb{N}$, choose $i$ great enough such that $3\tilde{C}_{n-1} \tilde{r} M(R_i^l) + M(\tilde{R}_i^l) < \epsilon/(3m_j)$ for all $l = 1, \ldots, m_j$. This means

$$\sum_{k \leq m_j} M(V_i^k) < \frac{\epsilon}{3}$$

and we have as $\epsilon < 1$ by our choice of $\tilde{r}$ and since $C_j \subset \bigcup_{l \leq m_j} U_{\tilde{r}}(x_k)$

$$\mathcal{F}(F_i - F_{i-1}) \leq M((F_i - F_{i-1})|_{X \setminus C_j}) + \sum_{k \leq m_j} \mathcal{F}(F_i^k - F_{i-1}^k)$$

$$\leq \epsilon/3 + \sum_{k \leq m_j} (M(\tilde{V}_i^k) + M(V_i^k)) < \epsilon,$$

in contradiction to our assumption. Thus, there is a converging subsequence. \hfill \Box

**Corollary 5.0.2.** For $X$, $Q$ and $G$ as above, $C \subset X$ compact, the set

$$\{F \in \mathcal{F}_n(X, G) \mid \text{spt } F \subset C, \ N(F) \leq Q\}$$

is compact with respect to $\mathcal{F}$. 
To admit sequences with infinite mass boundary we recall a weaker 'flat
norm' (a possibly infinite semi-norm, actually). This is done in [Fed69, p. 470]
for currents in \( \mathbb{R}^m \); we use a different notation. For \( Z \subset X \) closed, set

\[
\mathcal{F}(F; Z) := \int_0^{\infty} \mathcal{F}(F_{\{r<d\}}) dr.
\]

Note that if \( \text{spt} \ F \) is bounded, we find \( F \in \mathcal{R}_{n+i}(l^\infty(X), G) \) (for \( X \subset l^\infty(X) \))
with bounded support such that \( F = R_0 + \partial R_1 \). Then \( \mathcal{F}(F; Z) < \infty \).

Remark 5.0.3. \( \mathcal{F}(F; Z) = 0 \) if and only if \( \text{spt} \ F \subset Z \). One direction is clear. If \( \mathcal{F}(F; Z) = 0 \), we can assume that there exists \( x \in X \setminus Z \), then \( d(x, Z) =: d > 0 \).

For a.e. \( r \in (0, d) \) is \( F_{\{r<d\}} = 0 \) and then \( F_{\{d\leq r\}} = F_{\{d<s\}} = 0 \), hence \( x \notin \text{spt} \ F \) by Remark 4.1.7.

Theorem 5.0.4. Let \( Q \in \mathbb{R} \), let \( X \) be a complete metric space. Let \( G \) be a
normed group such that \( \overline{G} \) is proper. Let \( C \subset X \) be compact. For \( Z \subset X \) closed, the set

\[
\{ F \in \mathcal{F}_n(X, G) \mid M(F) \leq Q, \text{spt} \ F \subset C, \text{spt} \partial F \subset Z \}
\]
is compact with respect to \( \mathcal{F}(\cdot; Z) \).

Proof. This follows using Theorem 5.0.1, exactly as in [Fed69, 4.4.4]. We give
a sketch, for more details see therein.

Let \( (F_i)_{i \in \mathbb{N}} \) be a sequence in the above set. For \( i, j \in \mathbb{N} \) we find an
\( r_{ij} \in (0, 1/j) \) such that \( M(\partial(F_i_{\{r_{ij}<d\}})) < jQ \) (by (4.1.10), and as \( \text{spt}(\partial F) \subset Z \)). Thus, \( (A_{ij} := F_i_{\{r_{ij}<d\}})_{i \in \mathbb{N}} \) has a converging subsequence for each \( j \).
We take a diagonal sequence. The limit of \( A_{ij} \) we denote by \( \bar{F}_j \), clearly is
\( M(\bar{F}_j) \leq Q \). Furthermore, we can assume that for \( V_j := X \setminus U_{1/j}(Z) \) holds
that \( \bar{F}_j^{j+1}|_{V_j} = \bar{F}_j^j|_{V_j} \). Then, the sum below is convergent with respect to the mass:

\[
\bar{F} := \bar{F}_1^{j} + \sum_{j \in \mathbb{N}} (\bar{F}_j^{j+1}|_{V_{j+1}} - \bar{F}_j^j|_{V_j}) \in \mathcal{M}_n(X, G)
\]

(5.0.3)

and then \( M(\bar{F}) \leq Q \). Note that \( \mathcal{F}(\bar{F} - F_i; Z) \to 0 \) for \( i \to \infty \). Since \( \text{spt} \partial \bar{F}_j \subset U_{1/j}(Z) \), we get \( \text{spt} \partial \bar{F} \subset Z \) and therefore is \( \bar{F} \) contained in the relevant set. \( \square \)

Finally, we drop the condition that \( G \) is proper. As indicated after Theorem 5.0.1 we will not find a converging subsequence for a normally bounded
sequence even if we consider rectifiable chains only. However, if we assume
the coefficients of the considered rectifiable chains to lie in a compact set (and
similarly for the boundaries), we still have a converging subsequence:

**Theorem 5.0.5.** Let \( Q_j \in \mathbb{R} \) for \( j \in \mathbb{N} \), let \( X \) be a complete metric space. Let \( G \) be a normed group. Let \( (F_i)_{i \in \mathbb{N}} \subset \mathcal{M}(X,G) \) with \( M(F_i) \leq Q_1 \) for all \( i \in \mathbb{N} \). Let \( C_j \subset X \) be compact and let \( G_j \subset \bar{G} \) be compact. Assume that
\[ \mu_{F_i}(X \setminus C_j) < \frac{1}{j} \]
and that \( F^{ij} := F_i|_{C_j} \) fulfills
\[ F^{ij} = \sum_{k \in \mathbb{N}} f^{ij}_{k} [g^{ij}_{k}] \in \mathcal{R}_{n}(X,G) \quad \text{as in (4.5.1) with } g^{ij}_{k} \in L^{1}(K_k,G_j), \]
and if \( n > 0 \) also \( M(\partial F^{ij}) \leq Q_j \) and
\[ \partial F^{ij} = \sum_{k \in \mathbb{N}} \bar{f}^{ij}_{k} [\bar{g}^{ij}_{k}] \in \mathcal{R}_{n-1}(X,G) \quad \text{as in (4.5.1) with } \bar{g}^{ij}_{k} \in L^{1}(\bar{K}_k,G_j). \]

Then there is a converging subsequence of \((F_i)_{i \in \mathbb{N}}\).

Above, the condition for \( n > 0 \) is quite restrictive: In general we can not
expect the boundary of such a restricted chain (as in the theorem) to be rect-
ifiable again. However, in locally compact (or proper) spaces such boundaries
may easily arise.

**Proof.** First fix \( j \in \mathbb{N} \). Note that \( N(F^{ij}) \leq Q_1 + Q_j =: Q^j < \infty \) and
\( \text{spt } F^{ij} \subset C_j \) for all \( i \in \mathbb{N} \). If \( n = 0 \) we have as in the proof of Theorem 5.0.1 that
\[ \left\{ \sum_{k \in \mathbb{N}} g_k[x_k] \in \mathcal{R}_0(X,G) \cap \{ M \leq Q_1 \} \mid x_k \in C_j, x_k \neq x_{k'} \text{ if } k \neq k', g_k \in G_j \right\} \]
is totally bounded as \( G_j \subset \bar{G} \) is compact. (For this, note that \( \{ \sum_{k \in \mathbb{N}} g_k \in \bar{G} \mid g_k \in G_j, \sum_{k \in \mathbb{N}} \| g_k \| \leq Q_1 \} \) is totally bounded.)

For \( n > 0 \) we proceed similarly to the other proof. Note that the slices of \( R \)
are rectifiable and for a representation of the slice the group coefficients take
values in the compact set \( G_j \cup (-G_j) \cup \{ 0 \} \) for a.e. \( r \). This holds for simple maps and then for all \( g^{ij}_{k} \in L^{1}(K_k,G_j) \), compare the proof of Lemma 3.8.1 (vi) respectively [AK00, Theorem 9.7]. Thus, in order to apply the inductive argu-
ment in the proof of Theorem 5.0.1, we can assume that all slices are rectifiable
and that their coefficients take values in the compact set \( G_j \cup (-G_j) \cup \{ 0 \} \).
Then, also the boundaries of the restricted chains (see in the proof of Theorem 5.0.1) are rectifiable and their coefficients lie in appropriate compact subsets of $G$ (defined inductively over the finite number of balls used to cover $C_j$ using (5.0.2)). By induction we find a converging subsequence for the boundaries, and then we can fill by the cone inequality as before. Thus, for each $j \in \mathbb{N}$, we find a converging subsequence of $(F^{ij})_{i \in \mathbb{N}}$. Then we can conclude as in the other theorem.
Chapter 6

Rectifiable chains and
slicing theorem

In this section we prove that a finite mass chain whose slices are rectifiable is again rectifiable (compare the corresponding result for metric currents in [AK00], as well as for $G$-chains in Euclidean spaces in [Whi99a], note that this result is contained but not stated in [DPH09]); that all finite mass $G$-chains are rectifiable if $G$ contains no non-constant continuous path of finite length (compare [Fle66] for finite and integral coefficients, [Whi99a] for general coefficients in Euclidean space, [DPH09] for general coefficients in complete metric spaces). Also we show that the measure of an $n$-dimensional finite mass chain is absolutely continuous with respect to $\mathcal{H}^n$ and that an $n$-dimensional finite mass chain is rectifiable if and only if its measure is concentrated on a countably $\mathcal{H}^n$-rectifiable set. These results are in [DPH09], our presentation is sometimes different.

**Proposition 6.0.1 ([DPH09]).** Let $X$ be a complete metric space and let $F \in \mathcal{M}_n(X,G)$. Then $\mu_F \ll \mathcal{H}^n$.

Let $A \in \mathcal{B}_X$. If for every $f \in \text{Lip}(X,\mathbb{R}^n)$ holds that $\mathcal{L}^n(f(A)) = 0$, then

\[ \mu_F(A) = 0. \]

**Proof.** Clearly this holds for $n = 0$.

For $n > 0$, let first $A$ be closed. Set $F' := F|_{A} \in \mathcal{M}_n(X,G)$. For every $f \in \text{Lip}(X,\mathbb{R}^n)$ and a.e. $r$ is spt($\langle F', f, r \rangle$) $\subset f^{-1}(r) \cap A$, so we see that

\[ \{ r \in \mathbb{R}^n \mid \langle F', f, r \rangle \neq 0 \} \subset f(A) \]
(up to an $\mathcal{L}^n$-negligible set). But $\mathcal{L}^n(f(A)) = 0$, hence almost every slice vanishes. By Theorem 4.4.1 is $F' = 0$ and so $\mu_F(A) = \mu_{F'}(A) = 0$. For general $A \in \mathcal{B}_X$ we can approximate: $\mu_F(A) = \lim_{j \to \infty} \mu_F(A_j)$ where $A_{j-1} \subset A_j \subset A$ are closed sets (clearly is then $\mathcal{L}^n(f(A_j)) = 0$ for every $f \in \text{Lip}(X, \mathbb{R}^n)$).

Now let $N \subset X$ be $\mathcal{H}^n$-negligible. As $\mathcal{H}^n$ is Borel-regular we find $N' \supset N$ Borel with $\mathcal{H}^n(N') = 0$. For every Lipschitz map is then $\mathcal{L}^n(f(N')) = \mathcal{H}^n(f(N')) \leq \text{Lip}(f)^n \mathcal{H}^n(N') = 0$, hence $\mu_F(N) \leq \mu_F(N') = 0$.

If $R \in \mathcal{R}_n(X, G)$, then $\mu_R$ is concentrated on a countably $\mathcal{H}^n$-rectifiable set: Let $\mathcal{M}(P^j - R) \to 0$. Let $W := \bigcup_{j \in \mathbb{N}} W^j$ where $W^j$ is a countably $\mathcal{H}^n$-rectifiable set on which $\mu_{P^j}$ is concentrated. Then $W$ is countably $\mathcal{H}^n$-rectifiable and $\mu_R(X \setminus W) = \lim_{j \to \infty} \mu_{P_j}(X \setminus W) = 0$. On the other hand we have the following:

**Proposition 6.0.2.** Let $X$ be a complete metric space, let $F \in \mathcal{M}_n(X, G)$. Then $F \in \mathcal{R}_n(X, G)$ if and only if $\mu_F$ is concentrated on a countably $\mathcal{H}^n$-rectifiable set.

**Proof.** One direction is shown above.

The following is as in [DPH09, p. 53, 54]. Let $W \subset X$ be a countably $\mathcal{H}^n$-rectifiable set on which $\mu_F$ is concentrated. We can assume that $X$ is an injective metric space. We can parameterize $W$ by bi-Lipschitz maps $f_i$ from compact sets $K_i \subset \mathbb{R}^n$ to $W$ with pairwise disjoint images such that $\mu_F(W \setminus \bigcup_{i \in \mathbb{N}} f_i(K_i)) = 0$ (by Proposition 6.0.1). Then $F = \sum_{i \in \mathbb{N}} F|_{f_i(K_i)}$ and $\mathcal{M}(F) = \sum_{i \in \mathbb{N}} \mathcal{M}(F|_{f_i(K_i)})$.

It remains to show that $\bar{F} := F'|_{f_i(K_i)}$ is rectifiable: Let $H: X \to \mathbb{R}^n$ be a Lipschitz extension of $f_i^{-1}: f_i(K_i) \to \mathbb{R}^n$; then

$$H_\# \bar{F} \in \mathcal{F}_n(\mathbb{R}^n, G) = \mathcal{R}_n(\mathbb{R}^n, G).$$

Let $Q: \mathbb{R}^n \to X$ be a Lipschitz extension of $f_i: K_i \to X$. Clearly

$$(Q \circ H)_\# \bar{F} = Q_\# H_\# \bar{F} \in \mathcal{R}_n(X, G).$$

By Lemma 4.1.9 is then $(G \circ H)_\# \bar{F} = \bar{F} \in \mathcal{R}_n(X, G)$. □

### 6.1 Rectifiable slices theorem

The rectifiable slices theorem is not mentioned but contained in [DPH09].
Theorem 6.1.1 (Rectifiable slices). Let $X$ be a complete metric space, let $F \in \mathcal{M}_n(X,G)$. Then $F \in \mathcal{R}_n(X,G)$ if and only if for every $f \in \text{Lip}(X,\mathbb{R}^n)$ holds: $\langle F, f, r \rangle \in \mathcal{R}_0(X,G)$ for a.e. $r \in \mathbb{R}^n$.

Note that if for a finite mass chain holds that for every $f \in \text{Lip}(X,\mathbb{R}^k)$ $\mathcal{L}^k$-a.e. slice is rectifiable (where $1 \leq k \leq n$), then for every $\bar{f} \in \text{Lip}(X,\mathbb{R}^n)$ $\mathcal{L}^n$-a.e. slice is rectifiable. Thus, the above theorem holds in these cases too.

We first have to collect facts, the actual proof is in Subsection 6.1.3. Our proof is as in [DPH09] of De Pauw and Hardt; it is based on the one for currents of Ambrosio and Kirchheim (see [AK00]). We show that slicing has bounded variation, this allows us to construct a countably $\mathcal{H}^n$-rectifiable set on which the rectifiable slices live. Then it is enough to show that the chain is concentrated on this set by Proposition 6.0.2.

6.1.1 Slicing has bounded variation

This is similar to the case of currents in [AK00] and was done by Hardt and De Pauw for chains with group coefficients ([DPH09]); note that we use the definitions from [AK00]; the proof is as in Hardt and De Pauw.

Definition 6.1.2. A metric space $(X,d)$ is said to be weakly separable if there exists a sequence $(\varphi_i)_{i \in \mathbb{N}} \subset \text{Lip}_1(X)$ such that

$$d(x,y) = \sup_{i \in \mathbb{N}} |\varphi_i(x) - \varphi_i(y)|. \quad (6.1.1)$$

Note that one can assume that $\varphi_i \in \text{Lip}^b(X)$ (i.e. $\varphi_i$ are bounded).

Any separable metric space is weakly separable (set $\varphi_i := d_{x_i}$ for a dense sequence $(x_i)_{i \in \mathbb{N}} \subset X$) and any subset of a weakly separable space is again weakly separable.

A function $u$ belonging to $L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be of locally bounded variation, $u \in BV_{\text{loc}}(\mathbb{R}^n)$, if for every open $\Omega \subset \mathbb{R}^n$ with compact closure the local variation of $u$ is finite, i.e., if

$$V(u, \Omega) := \sup \left\{ \int_{\Omega} u \text{ div } \phi \, d\mathcal{L}^n \mid \phi \in C^1_c(\Omega, \mathbb{R}^n), \|\phi\|_{L^{\infty}} \leq 1 \right\} < \infty.$$

Note that for $u \in BV_{\text{loc}}(\mathbb{R}^n)$, we can consider the above integral as a continuous linear functional on $C^0_c(\mathbb{R}^n, \mathbb{R}^n)$ (by the Hahn-Banach theorem); now it defines a vector Radon measure $Du$. The total variation measure of $Du$ is

$$|Du|(B) := \sup \left\{ \sum_{i \in \mathbb{N}} |Du(B_i)| \mid B = \bigsqcup_{i \in \mathbb{N}} B_i, \ B_i \in B_{\mathbb{R}^n} \right\}.$$
For \( n = 1 \) and \( I := (a, b) \subset \mathbb{R} \), the pointwise variation of a function \( u \) is
\[
pV(u, I) := \sup \left\{ \sum_{i=1}^{m} |u(t_{i+1}) - u(t_i)| \mid t_1 < \cdots < t_{m+1}, \ t_i \in (a, b) \right\}.
\]

Let \( \Omega \subset \mathbb{R} \) be open. Let \( \Omega = \bigsqcup_{i \in \mathbb{N}} I_i \) where \( I_i \) are pairwise disjoint open intervals. Set \( pV(u, \Omega) := \sum_{i \in \mathbb{N}} pV(u, I_i) \). The essential variation is then
\[
eV(u, \Omega) := \inf \{ pV(v, \Omega) \mid u = v \ \mathcal{L}^1\text{-a.e. in } \Omega \}.
\]

It holds:
\[
V(u, \Omega) = eV(u, \Omega). \tag{6.1.2}
\]

For \( n > 1 \) and \( \Omega \subset \mathbb{R}^n \) holds
\[
V(u, \Omega) \leq \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} eV(u \circ c_{i,z}, \Omega_{i,z}) \, dz. \tag{6.1.3}
\]

Here, \( c_{i,z}(t) := (z_1, \ldots, z_{i-1}, t, z_i, \ldots, z_{n-1}) \) and
\[
\Omega_{i,z} := \Omega \cap \left\{ (x_1, \ldots, x_n) \mid x_j = z_j, \ j < i; \ x_k = z_{k-1}, \ k > i \right\}.
\]

(See [DPH09, AFP00] and the references therein.)

**Definition 6.1.3.** Let \((X, d)\) be a weakly separable metric space and let the sequence \((\varphi_i)_{i \in \mathbb{N}} \subset \text{Lip}_1(X) \cap \text{Lip}^b(X)\) satisfy (6.1.1). A function \( u : \mathbb{R}^n \to X \) is a function of metric bounded variation if \( \varphi_i \circ u \in \text{BV}_{loc}(\mathbb{R}^n) \) for any \( i \in \mathbb{N} \) and

\[
\| Du \|(B) := \sup \left\{ \sum_{i \in \mathbb{N}} |D(\varphi_i \circ u)|(B_i) \mid B = \bigsqcup_{i \in \mathbb{N}} B_i, \ B_i \in \mathcal{B}_{\mathbb{R}^n} \right\} < \infty
\]

for all Borel \( B \subset \mathbb{R}^n \). Then we write \( u \in \text{MBV}(\mathbb{R}^n, X) \).

**Lemma 6.1.4 ([DPH09]).** Let \( X \) be a complete metric space and \( n \geq 1 \). For \( F \in \mathcal{F}_n(X, G) \) and \( \pi = (\pi_1, \pi_2, \cdots, \pi_k) \in \text{Lip}(X, \mathbb{R}^k) \), \( k \leq n \), the slicing map
\[
S : \mathbb{R}^k \to (\mathcal{F}_{n-k}(X, G), \mathcal{F}), \quad S(r) := \langle F, \pi, r \rangle
\]
is \( \mathcal{L}^k \)-measurable.
Let now $F ∈ F_n(X,G)$ with $N(F) < ∞$ and let $k = n$. Then there exists $W ⊂ F_0(X,G)$ weakly separable such that for a.e. $r ∈ \mathbb{R}^n$ is $S(r) ∈ W$ and $S ∈ MBV(\mathbb{R}^n, W)$ with

$$∥DS∥ ≤ n \prod_{j=1}^n (\text{Lip}(π_j) + 1) (∑ i∈N g_j^i R_j^i) .$$

**Proof.** The proof of the measurability is from [DPH09, Theorem 5.2.1 (4)], we give only the idea: Note that we can assume that $X$ is injective and that it is enough to consider $P ∈ P_n(X,G)$. Through a representation of $P$ it is enough to show the measurability for $g_j^{i} \chi_{K_j^i} ∈ I_c(X)$. Let $W_m$ be $\mathcal{F}$-closed set for each $m ∈ \mathbb{N}$, $W$ is (weakly) separable. Clearly almost every slice $⟨P^j, π, r⟩$ is in $W$, hence the same holds for the slices of $F$. Thus, the image of $S$ lies in a weakly separable set for a.e. $r ∈ \mathbb{R}^n$.

Let $k = n$ and $N(F) < ∞$. We show that almost all of the image of $S$ lies in a weakly separable set: Let $P^j → F$, let $R^j_i := ∑_{i∈N} g_j^i R_j^i$ be a representation of $P^j$ such that $R^j_i = f_j^i \chi_{K_j^i} ∈ I_c(X)$. Let $N ⊂ \mathbb{R}^n$ be $\mathcal{H}^n$-negligible such that for all $r ∈ \mathbb{R}^n \setminus N$ holds

$$⟨P^j, π, r⟩ → ⟨F, π, r⟩ ∈ F_0(X,G), \quad \text{spt}(R^j_i, π, r) ⊂ π^{-1}(r) \cap \text{spt} R^j_i ,$$

$$⟨P^j, π, r⟩ = ∑_{i∈N} g_j^i (R^j_i, π, r) ∈ P_0(X,G) \quad \text{and} \quad ⟨R^j_i, π, r⟩ ∈ I_c^c(X).$$

Let $(z_i)_{i∈N} ⊂ ∪_{j∈N} \text{spt} R^j_i$ be dense and set

$$W_m := \left\{ ∑_{k=1}^m h_k [x_k] \mid h_k ∈ \{0\} ∪ \{±g_j^i \mid i,j ≤ m\}, x_k ∈ \{z_1, \ldots, z_m\} \right\} .$$

We define $W$ to be the $\mathcal{F}$-closure of $∪_{m∈N} W_m ⊂ F_0(X,G)$. As $W_m$ is a finite set for each $m ∈ N$, $W$ is (weakly) separable. Clearly almost every slice $⟨P^j, π, r⟩$ is in $W$, hence the same holds for the slices of $F$. Thus, the image of $S$ lies in a weakly separable set for a.e. $r ∈ \mathbb{R}^n$.

The proof that $S ∈ MBV(\mathbb{R}^n, W)$ is from [DPH09, Theorem 8.1.1]: Let $n = 1$, then for a.e. $r < s$ is $⟨F, π, s⟩−⟨F, π, r⟩ = \partial(F |_{r ≤ π < s})−(\partial F) |_{r ≤ π < s}$, hence a.e.

$$\mathcal{F}(S(s)−S(r)) ≤ M(F |_{r < π < s}) + M((\partial F) |_{r < π < s}) .$$
Let $\varphi \in \text{Lip}_1((W,F), \mathbb{R})$. For $I = (a, b) \subset \mathbb{R}$ is with (6.1.2)

$$V(\varphi \circ S, I) \leq \sup\left\{ \sum_{i=1}^{m} \mathcal{F}(S(t_{i+1}) - S(t_i)) \mid t_1 < \cdots < t_{m+1}, t_i \in I \setminus N \right\}$$

$$\leq \mu_F(\pi^{-1}(I)) + \mu_{\partial F}(\pi^{-1}(I)) = \pi_{\#}(\mu_F + \mu_{\partial F})(I);$$

where $N$ is an appropriate $\mathcal{L}^1$-negligible set. Approximating $B \in \mathcal{B}_\mathbb{R}$ from outside by open sets, we get

$$\|D(\varphi \circ S)(B)\| \leq \pi_{\#}(\mu_F + \mu_{\partial F})(B).$$

Now let $n > 1$ and $\pi = (\pi_1, \ldots, \pi_n)$; denote $(\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n)$ by $\hat{\pi}_i$ and analogously for $r \in \mathbb{R}^n$. With (4.1.8) and (6.1.3) we get for an open cube $(a_1, b_1) \times \cdots \times (a_n, b_n) =: A_1 \times \cdots \times A_n =: Q \subset \mathbb{R}^n$

$$V(\varphi \circ S, Q) \leq \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} eV(\varphi \circ S \circ c_{i,z}, Q_{i,z}) dz$$

$$\leq \sum_{i=1}^{n} \int_{Q_{i}} \left( M(\langle F, \hat{\pi}_i, \hat{r}_i \rangle |_{\pi_i^{-1}(A_i)}) + M(\langle \partial F, \hat{\pi}_i, \hat{r}_i \rangle |_{\pi_i^{-1}(A_i)}) \right) d\hat{r}_i$$

$$\leq n \prod_{j=1}^{n} (\text{Lip}(\pi_j) + 1) \left( M(F|_{\pi^{-1}(Q)}) + M((\partial F)|_{\pi^{-1}(Q)}) \right)$$

$$= n \prod_{j=1}^{n} (\text{Lip}(\pi_j) + 1) (\pi_{\#}(\mu_F + \mu_{\partial F})(Q)).$$

This implies that $|D(\varphi \circ S)(B)| \leq n(\prod_{j=1}^{n} (\text{Lip}(\pi_j) + 1))(\pi_{\#}(\mu_F + \mu_{\partial F})(B))$; then as above: $\|DS\|(B) \leq n(\prod_{j=1}^{n} (\text{Lip}(\pi_j) + 1))(\pi_{\#}(\mu_F + \mu_{\partial F})(B)) < \infty.$

### 6.1.2 Rectifiable sets from slicing

Let $S \in MBV(\mathbb{R}^n, W)$. The maximal function of the measure $\|DS\|$ is

$$M_{\|DS\|}(r) := \sup_{\rho > 0} \frac{\|DS\|(U_\rho(r))}{\omega_n \rho^n}.$$ 

For $\mathcal{L}^n$-a.e. $r$ is $M_{\|DS\|}(r) < \infty$ (by a covering argument, see [AK00, p. 42]).
Lemma 6.1.5 ([AK00]). Let \((W, d)\) be a weakly separable metric space. Then for any \(S \in MBV(\mathbb{R}^n, W)\) there exists an \(\mathcal{L}^n\)-negligible set \(N \subset \mathbb{R}^n\) such that
\[
d(S(x), S(y)) \leq c(n)(M_{\|DS\|}(x) + M_{\|DS\|}(y))|x - y|
\]
for all \(x, y \in \mathbb{R}^n \setminus N\) with \(c(n)\) depending only on \(n\).

For the proof see [AK00].

Lemma 6.1.6. Let \(K \subset X\) compact, \(X\) an injective metric space. Let \(f \in \text{Lip}(X, \mathbb{R}^n)\) and let \(F \in \mathcal{F}_n(X, G)\) with \(N(F) < \infty\). Then there is an \(\mathcal{L}^n\)-negligible set \(N \subset \mathbb{R}^n\) such that
\[
M_K := \bigcup_{z \in \mathbb{R}^n \setminus N} \{x \in K | \mu_{\langle F, f, z \rangle}({\{x\}}) > 0\}
\]
is contained in a countably \(H^n\)-rectifiable set.

Compare [DPH09, Corollary 8.1.2] and [AK00].

Proof. First we explain how bounded Lipschitz functions on \(X\) give Lipschitz functions on \((\mathcal{F}_0(X, G), d_{\mathcal{F}})\) where \(d_{\mathcal{F}}(F, \tilde{F}) := \mathcal{F}(F - \tilde{F})\): Let \(D \in (0, \infty)\) and let \(\alpha : X \to [0, D] \subset \mathbb{R}\) be a Lipschitz function. Let \(P \in \mathcal{P}_0(X, G)\), set
\[
\Phi_\alpha(P) := \inf \left(\sum_{i=1}^m \|g_i\|r_i + M(P_1)\right) \tag{6.1.4}
\]
where we take the infimum over all \(P_0 := \sum_{i=1}^m g_i[r_i] \in \mathcal{P}_0([0, D], G)\) and \(P_1 \in \mathcal{P}_1([0, D], G)\) such that \(\alpha \# P = [P_0] + \partial[P_1] \in \mathcal{P}_0([0, D], G)\).

We claim that \(\Phi_\alpha : (\mathcal{P}_0(X, G), d_{\mathcal{F}}) \to \mathbb{R}\) is Lipschitz with Lipschitz constant at most \(C := \max\{\|\alpha\|, \text{Lip}(\alpha)\}\). To prove the claim it is enough to show that for \(P = p_0 + \partial p_1\) holds \(\Phi_\alpha(P) \leq C(M(p_0) + M(p_1))\).

Let \(P = p_0 + \partial p_1\) with \(\alpha \# p_0 = \sum_{i=1}^m g_i[r_i]\); then \(r_i \leq \|\alpha\|\). As we consider the infimum, we can assume that \(r_i \neq r_j\) for \(i \neq j\). So
\[
\Phi_\alpha(P) \leq \sum_{i=1}^m \|g_i\|r_i + M(\alpha \# p_1) \leq \|\alpha\|\infty M(\alpha \# p_0) + \text{Lip}(\alpha)M(p_1)
\]
\[
\leq \|\alpha\|\infty M(p_0) + \text{Lip}(\alpha)M(p_1) \leq C(M(p_0) + M(p_1)) ,
\]
proving the claim.
CHAPTER 6. RECTIFIABLE CHAINS AND SLICING THEOREM

We can extend the Lipschitz map \( \Phi_\alpha \) to \( \mathcal{F}_0(X, G) \) by continuity (with the same Lipschitz constant). Note that \( \Phi_\alpha \) is non-negative and subadditive, and \( \Phi_\alpha(-F) = \Phi_\alpha(F) \). Also, for a finite mass \( F \) holds \( \Phi_\alpha(F) \leq \|\alpha\|_\infty M(F) \leq DM(F) \) as this is true for all \( P \in \mathcal{P}_0(X, G) \).

Now we can proceed as in [AK00]: Set \( S(r) := \langle F, \pi, r \rangle \). There is a \( \mathcal{L}^n \)-negligible set \( N \) such that for \( r \in \mathbb{R}^n \setminus N \) we can apply Lemma 6.1.5,

\[
S(r) \in \mathcal{M}_0(X, G), \quad \text{spt}(S(r)) \subset f^{-1}(r) \quad \text{and} \quad M_{\|DS\|}(r) < \infty.
\]

We write \( \mu_r \) for \( \mu_{S(r)} \). Let \( a, b > 0 \) and let \( Z_{a,b} \) be defined as the set of points \( z \in \mathbb{R}^n \setminus N \) such that \( M_{\|DS\|}(z) < 1/a \),

\[
\mu_z(\{x\}) \geq a \quad \text{implies} \quad \mu_z(U_{2b}(x) \setminus \{x\}) \leq a/3
\]

for any \( x \in X \) (note that if \( \mu_z(x) > 0 \), then necessarily \( f(x) = z \)). Set

\[
R_{a,b} := \{ x \in K \mid f(x) \in Z_{a,b} \quad \text{and} \quad \mu_f(\{x\}) \geq a \}.
\]

Note that \( M_K = \bigcup_{a,b > 0} R_{a,b} \). Furthermore, it is enough to consider \( a, b \in \mathbb{Q} \), so in order to show that \( M_K \) is countably \( \mathcal{H}^n \)-rectifiable we will show this below for \( R_{a,b} \).

Let \( B \subset R_{a,b} \) be any subset such that \( \text{diam} B < \min\{a, b, 1\} \). Let \( x, x' \in B \) with \( x \neq x' \); note that then \( f(x) \neq f(x') \) by (6.1.6) and the diameter bound. Set \( d := d(x, x') \) and \( z := f(x), z' := f(x') \in Z_{a,b} \). Define

\[
\beta(y) := \max\{0, d - d(x, y)\},
\]

then \( \beta : X \to [0, d] \) is 1-Lipschitz and \( \beta_{\#}(g[x]) = g[d] \). Let

\[
P_0 := \sum_{i=1}^m g_i r_i \in \check{\mathbf{P}}_0([0, d], G) \quad \text{and} \quad P_1 := \sum_{j=1}^{m'} h_j [a_j, b_j] \in \check{\mathbf{P}}_1([0, d], G)
\]

be such that \( g[d] = [P_0] + \partial [P_1] \in \mathcal{P}_0([0, d], G) \). We can assume that \( g_i, h_j \neq 0, r_1 < r_2 < \cdots < r_m \leq d \) and \( a_1 < b_1 \leq a_2 < b_2 \leq \cdots < b_{m'} \leq d \). If \( m' > 0 \), necessarily \( a_1 = r_1 \) and \( g_1 = h_1 \) and we have one of the following cases:

(i) \( b_1 = d, r_2 = d \) (then \( g = g_2 + h_1 \)) or \( m = 1 \) (then \( g = h_1 \),
Lemma 6.1.5) So, we have (as max \{∥F \} ∈ M)

\[ \sum_{i=1}^{m} |g_i| r_i + \mathcal{M}(P_1) \leq \sum_{i=1}^{m} g_i r_i + \mathcal{M}(P_1) \]. Therefore we can assume that \( m' = 0 \), i.e. \( [P_0] = g[d] \) and \( P_1 = 0 \). Then

\[ \Phi_\beta(g[x]) = \|g\| = \mathcal{M}(g[x]) \].

By subadditivity and continuity of \( \Phi_\beta \) we have

\[ \Phi_\beta(S(z)) \geq \Phi_\beta(S(z)|_{x}) - \Phi_\beta(S(z)|_{x \cup \{x\}}) - \Phi_\beta(S(z)|_{x \cup \{x\}}) \]
\[ \geq d\mathcal{M}(S(z)|_{x}) - d\mathcal{M}(S(z)|_{x \cup \{x\}}) - 0 \geq 2ad/3, \]
\[ \Phi_\beta(S(z')) \leq \Phi_\beta(S(z'|_{x \cup \{x\}}) + \Phi_\beta(S(z'|_{x \cup \{x\}}) + d\mathcal{M}(S(z'|_{x \cup \{x\}})) \]
\[ \leq d\mathcal{M}(S(z'|_{x \cup \{x\}}) + 0 \leq da/3. \]

So, we have (as \( \max\{\|\beta\|, \text{Lip}(\beta)\} = \max\{d, 1\} \) and \( d = d(x, x') \leq 1 \) and by Lemma 6.1.5)

\[ d(x, x') \leq \frac{3|\Phi_\beta(S(z)) - \Phi_\beta(S(z'))|}{a} \leq \frac{3F(S(z) - S(z'))}{a} \leq \frac{6c(n) |z - z'|}{a^2} \].

Now this implies that the inverse \( (f|_B)^{-1} : f(B) \to B \) is Lipschitz. \( \square \)

6.1.3 Proof of the rectifiable slices theorem

We can assume that \( X \) is an injective metric space. As every Borel measure is concentrated on a \( \sigma \)-compact set, we find compact sets \( K_i \subset X \) with \( \mu_F(X \cup i \in \mathbb{N} K_i) = 0 \). For each \( i \), take a countably \( \mathcal{H}^a \)-rectifiable Borel set \( M_i \subset X \) maximizing \( \mu_F(K_i \cap M_i) \).

Fix \( i \). If \( F' := F|_{K_i \setminus M_i} \neq 0 \), we find \( f \in \text{Lip}(X, \mathbb{R}^n) \) and an \( \mathcal{L}^n \)-measurable set \( E \subset \mathbb{R}^n \) such that

\[ \int_E \mathcal{M}(F', f, r) d\mathcal{L}^n = \int_E \mathcal{M}(F', f, r|_{K_i}) d\mathcal{L}^n > 0. \]

Writing \( F' = R_0 + \partial R_1 \) for \( R_i \in \mathcal{R}_{n+i}(X, G) \) we can apply Lemma 6.1.6 to \( \partial R_1 \in \mathcal{M}n(X, G) \) and \( K_i \) to get a countably \( \mathcal{H}^a \)-rectifiable set \( M \) such that
\( \langle \partial R_1, f, r \rangle |_{K_i} = \langle \partial R_1, f, r \rangle |_M \). Let \( S \) be a countably \( H^n \)-rectifiable set on which \( R_0 \) is concentrated. We can assume that \( M \) and \( S \) are Borel sets, hence the same holds for \( W := M \cup S \).

Then
\[
\langle F', f, r \rangle |_{W \cap K_i} = (\langle F', f, r \rangle |_W) |_{K_i} = \langle R_0, f, r \rangle |_{K_i} + \langle \partial R_1, f, r \rangle |_{K_i} = \langle F', f, r \rangle |_{K_i}
\]
a.e. and so, for a constant \( C \) depending on \( f \),
\[
CM(F'|_W) \geq CM(F'|_{W \cap K_i}) \geq \int_{\mathbb{R}^n} M(\langle F', f, r \rangle |_{K_i}) dr > 0.
\]
Now, \( M_i \cup W \) is a countably \( H^n \)-rectifiable Borel set and
\[
\mu_F(K_i \cap (W \setminus M_i)) = M((F'|_{K_i \setminus M_i}) |_W) = M(F'|_W) > 0.
\]
So we get a contradiction to the maximality of \( M_i \). Thus, it holds \( F|_{K_i} = F|_{M_i} \) for all \( i \in \mathbb{N} \). Then \( M := \bigcup_{i \in \mathbb{N}} M_i \) is again countably \( H^n \)-rectifiable and \( F = F|_M \). By Proposition 6.0.2 is then \( F|_M \in R_n(X, G) \).

## 6.2 Rectifiability of finite mass chains

The following is proved in [Whi99a] in Euclidean space. The general case of a complete metric space is treated in [DPH09, Theorem 7.2.2 and Theorem 8.2.2], our proof is only slightly different. Compare also the corresponding result of [AW09] for flat chains modulo \( p \) in Banach spaces.

**Theorem 6.2.1** ([DPH09]). Let \( X \) be a complete metric space, let \( G \) be a normed group such that there is no non-constant continuous path of finite length in \( \bar{G} \). Then every flat \( n \)-chain of finite mass in \( X \) is rectifiable.

**Corollary 6.2.2.** Let \( X \) be a complete metric space and let \( G \) be a normed group. Assume that there exists \( f : U \rightarrow f(U) \subset X \) bi-Lipschitz where \( \emptyset \neq U \subset \mathbb{R}^{n+1} \) is open. Equivalent are:

(i) Every flat \( n \)-chain of finite mass in \( X \) is rectifiable.

(ii) There is no non-constant continuous path of finite length in \( \bar{G} \).

Compare [Whi99a].
6.2. RECTIFIABILITY OF FINITE MASS CHAINS

Proof of the corollary. Assuming the theorem, (ii) implies (i).

(i) implies (ii): If there is a non-constant continuous path of finite length in \( \tilde{G} \) there exists a non-rectifiable finite mass \( \tilde{G} \)-chain with compact support (as defined by Fleming) by [Whi99a]. This gives also an \( F \in \mathcal{M}(\mathbb{R}^{n+1}, \tilde{G}) \subset \mathcal{F}_n(\mathbb{R}^{n+1}, G) \) which is not rectifiable. We can assume that \( \text{spt} F \subset U \). Now, \( f_\# F \in \mathcal{F}_n(X, G) \) has finite mass. We can extend \( f^{-1} \) to a Lipschitz map from \( X \) to \( \mathbb{R}^{n+1} \). Then \( f^{-1}_\# F = F \notin \mathcal{R}_n(\mathbb{R}^{n+1}, G) \), hence \( f_\# F \notin \mathcal{R}_n(X, G) \).

Proof of the theorem. The proof is a generalization of the proof of White; our version is similar to the one given in [DPH09].

By the rectifiable slices theorem, we only have to verify the rectifiability of the slices. Thus, it is enough to consider \( n = 0 \).

Let \( \tilde{F} \) be a flat 0-chain of finite mass which is not rectifiable, so \( \tilde{F} \neq 0 \). We can assume that \( \tilde{F} \) is purely unrectifiable: Since \( \mathcal{M}(\tilde{F}) \) is finite there exist at most countably many atoms of \( \muF \) which we can subtract (thus replacing \( \tilde{F} \) by \( \tilde{F} - \sum_{i \in \mathbb{N}} \chi(\tilde{F}[x_i]) \)).

Furthermore, replacing \( \tilde{F} \) by \( \tilde{F}|_A : = F \) for a suitable Borel set \( A \), we can by Theorem 4.7.1 assume that \( \chi(F) \neq 0 \). Note that still \( \muF \) has no atoms.

We will now construct a non-constant Lipschitz map \( \gamma : [0, 1] \to \bar{G} \): As \( \muF \) is a Borel regular measure without atoms we find a Borel set \( B \) such that \( \muF(B) = \mathcal{M}(F)/2 = \muF(X \setminus B) \).

(That such a set \( B \) exists we can see as follows (or compare e.g. [DPH09]): As there are no atoms for \( \muF \), for \( C := \mathcal{M}(F) \) and \( s_1 > 0 \) we can find a Borel set \( B_1 \) such that \( \muF(B_1) \in [C/2, C/2 + s_1] \). Now we can repeat this argument for \( F|_{B_1} \) and \( s_2 := s_1/2 \) and so on. Then \( B := \bigcap_{i \in \mathbb{N}} B_i \).

Set \( F_0^1 := F \) and

\[
F_1^1 := F_0^1|_B \quad \text{and} \quad F_1^2 := F_0^1|_{X \setminus B}.
\]

Then also \( \muF_1^1 \) and \( \muF_1^2 \) have no atoms as they are absolutely continuous with respect to \( \muF \). Proceeding inductively, we find from \( F_m^i \) the flat chains \( F_{m+1}^{2i-1} \) and \( F_{m+1}^{2i} \) with

\[
\mathcal{M}(F_{m+1}^{2i-1}) = \mathcal{M}(F_{m+1}^{2i}) = 2^{-m-1}\mathcal{M}(F) \quad \text{and} \quad F_{m+1}^{2i-1} + F_{m+1}^{2i} = F_m^i.
\]

Define a map \( \gamma \) from the reals of the form \( \frac{i}{2^{m}} \) for \( m \in \mathbb{N}, \mathbb{N} \ni i \leq 2^m \), to \( \bar{G} \) by

\[
\gamma\left(\frac{i}{2^{m}}\right) := \chi\left(\sum_{k=1}^{i} F_m^k\right) \quad \text{and set} \quad \gamma(0) := 0 \in \bar{G}.
\]
Since \( F_{m+1}^{2i-1} + F_{m+1}^{2i} = F_{m}^{i} \), we see that \( \gamma \) is well defined and \( \gamma(1) = \chi(F) \neq 0 \). Now, \( \gamma \) is defined on a dense subset of \([0, 1]\). Furthermore, for \( i > j \) is

\[
\| \gamma\left( \frac{i}{2^m} \right) - \gamma\left( \frac{j}{2^m} \right) \| = \| \chi\left( \sum_{k=j+1}^{i} F_{m}^{k} \right) \| \leq \sum_{k=j+1}^{i} M(F_{m}^{k}) = \frac{i-j}{2^m} M(F),
\]

hence \( \gamma \) is Lipschitz on this set. As \( \bar{G} \) is complete there exists a Lipschitz extension also denoted by \( \gamma \) from \([0, 1]\) to \( \bar{G} \) with \( \gamma(0) = 0 \) and \( \gamma(1) = \chi(F) \neq 0 \); but such a \( \gamma \) can not exist. \( \Box \)
Chapter 7

Size and other weighted area functionals

Compare [Whi99a] for the Euclidean case. Let $\phi : G \to [0, \infty)$ be a function such that

$$\phi(0) = 0, \quad \phi(-g) = \phi(g), \quad \phi(g + h) \leq \phi(g) + \phi(h)$$

and such that $\phi$ is lower semicontinuous with respect to $\|\|$.

Let $X$ be an injective metric space, then define for $P \in \tilde{\mathcal{P}}_n(X,G)$

$$\phi(P) := \sum_{i \in \mathbb{N}} \phi(h_i)M_{AK}(R_i)$$

where $\sum_{i \in \mathbb{N}} h_i R_i$ is a representation of $P$. We call $\phi(P)$ the $\phi$-mass of $P$.

**Proposition 7.0.1.** Let $X$ be an injective metric space. For any function $\phi$ as above, $\phi$ is well-defined on $\mathcal{P}_n(X,G)$, finite and it is lower semicontinuous on $(\mathcal{P}_n(X,G), \mathcal{F})$.

**Proof.** Similar to the ordinary mass, $\phi$ is well-defined first on $\tilde{\mathcal{P}}_n(X,G)$ as for two representations of $P \in \tilde{\mathcal{P}}_n(X,G)$ there is a refined representation and now all the $\phi$-masses agree. Clearly is $\phi(P + P') \leq \phi(P) + \phi(P')$ and $\phi(0) = 0$.

Now, let $P, p \in \tilde{\mathcal{P}}_n(X,G)$ with $[P] = [p] \in \mathcal{P}_n(X,G)$. Then there is a representation of both simultaneously; thus,

$$\phi([P]) := \phi(P) = \phi(p)$$

is well-defined.
As $\phi(\sum_{i=1}^{m} g_i T_i) \leq \sum_{i=1}^{m} \phi(g_i) M_{AK}(T_i) < \infty$ we get the finiteness.
Clearly $\phi$ satisfies $\phi(P_{\bigcup_{i \in \mathbb{N}} B_i}) = \sum_{i \in \mathbb{N}} \phi(P_{B_i})$ for pairwise disjoint Borel sets $B_i$. Moreover, we get for a representation $\sum_{i \in \mathbb{N}} h_i R_i$ as for the mass (compare (3.8.3)) that
\[
\phi(h_i \langle R_i, f, r \rangle) = \phi(h_i) M_{AK}(\langle R_i, f, r \rangle) \text{ a.e.}
\]
Now the proof of the lower semicontinuity proceeds exactly along the lines of the proof of the lower semicontinuity of the mass: We get (similar to Proposition 3.8.3) for $B \subset X$ Borel that
\[
\phi(P_{\mid_B}) = \sup \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \phi(\langle P, f_i, r \rangle_{\mid U_i \cap B}) dr
\]
where the supremum is over all $f_i \in \text{Lip}_1(X)$ and all $U_i \subset X$ open and pairwise disjoint.

The assertion then follows by induction as in Theorem 3.9.2. \quad \square

**Definition 7.0.2.** Let $X$ be a complete metric space isometrically embedded in an injective metric space $Y$, let $F \in \mathcal{F}_n(X,G)$. The $\phi$-mass of $F$ is
\[
\phi(F) := \inf \left\{ \liminf_{k \to \infty} \phi(P^k) \mid P^k \to F, P^k \in \mathbf{P}_n(Y,G) \right\}.
\]
Again, this definition is neither depending on the injective metric space nor on the embedding.

Arguing as for mass, we see that the following slicing inequality is valid for $F \in \mathcal{F}_n(X,G)$: Let $f = (f_1, \ldots, f_k) \in \text{Lip}(X, \mathbb{R}^k)$, then
\[
\int_{\mathbb{R}^k} \phi(\langle F, f, r \rangle) dr \leq \prod_{i=1}^{k} \text{Lip}(f_i) \phi(F). \quad (7.0.1)
\]
We extend $\phi$ to $\bar{G}$ (lower semicontinuously) and define new norm on $\bar{G}$ by
\[
\|g\|_{\phi} := \|g\| + \phi(g) = \|g\| + \inf \left\{ \liminf_{i \to \infty} \phi(g_i) \mid g_i \in G, g_i \to g \right\}.
\]

**Theorem 7.0.3.** Let $X$ be a complete metric space, let $G$ be a normed group, let $\phi: G \to [0, \infty)$ be as above. If there is no non-constant continuous path in $(\bar{G}, \| \|_{\phi})$ of finite length then every $F \in \mathcal{F}_n(X,G)$ with $\phi(F) + M(F) < \infty$ is rectifiable.
See [Whi99a] for the Euclidean case.

**Proof.** The proof is a modification of the proof of Theorem 6.2.1. We can assume that $X$ is an injective metric space. By Theorem 6.1.1, one only has to verify the rectifiability of the 0-dimensional slices of $F$ (by (7.0.1), almost every slice has simultaneously finite mass and finite $\phi$-mass). So the higher dimensional case reduces to 0-dimensional case.

Define a measure $\mu_F^\phi$ for $F \in F_0(X, G)$ with $\phi(F) < \infty$ using $\phi$ instead of $M$ (i.e. $\mu_F^\phi(B) = \phi(P_B)$ for $P \in P_0(X, G)$ and then the construction as for the mass). If $F$ is in addition in $M_0(X, G)$, we set

$$\bar{\mu}_F(Y) := \mu_F(Y) + \mu_F^\phi(Y).$$

Using subadditivity and lower semicontinuity we see as in Section 4.7 that $\phi(\chi(F)) \leq N\phi(F)$, in particular is $\|\chi(F)\|_\phi \leq N(F) + N(F)$. Now we apply the proof of Theorem 6.2.1 using $\bar{\mu}$ instead of $\mu$, $N + \phi$ instead of $N$ and $\|\|_\phi$ instead of $\|\|$ (note that $N(F_{\{x\}}) > 0$ implies that $N(F_{\{x\}}) > 0$).

In the special case where $\phi(g) = 1$ for all $g \neq 0$, we call $\phi(P)$ the flat size of $P$ (compare [Whi99a]), denote the function $\phi$ on $G$ by $\phi_s$ and write

$$S(P) := \phi_s(P).$$

For the flat size there are no non-constant continuous paths of finite length in $(\bar{G}, \|\|_{\phi_s})$, so we have (compare [Whi99a]):

**Corollary 7.0.4.** Every $F \in F_n(X, G)$ of finite flat size and finite mass is rectifiable.

Moreover, as for currents (see [AK00, Theorem 8.5]):

**Corollary 7.0.5.** Let $F_i \to F \in F_n(X, G)$ with $\sup_{i \in \mathbb{N}}(N(F_i) + S(F_i)) < \infty$. Then $F \in R_n(X, G)$.

Note that for currents, only weak convergence is required but the currents need to have bounded boundary mass.

Now we look at (integer) rectifiable currents. We know that $h : R_n(X, \mathbb{R}) \to R_n(X)$ is an isometry for the mass. By the definition of the size of currents and (2.2.15) we see that

$$\frac{1}{\sqrt{n}} S_{AK}(h(R)) \leq S(R) \leq \frac{2^n}{\omega_n} S_{AK}(h(R)) \quad \text{for } R \in R_n(X, \mathbb{R}). \quad (7.0.2)$$
Thus, for a sequence of rectifiable currents $R_i$ of bounded mass and size we find rectifiable chains $S_i \in h^{-1}(R_i) \in \mathcal{R}_n(X, \mathbb{R})$ of bounded mass and size. Assuming that $S_i \rightarrow S$, then $S \in \mathcal{R}_n(X, \mathbb{R})$ of finite size. Moreover, $R_i \rightarrow h(S)$ (current-flat which implies weakly) and $S_{AK}(h(S)) < \infty$.

**Example 7.0.6 (Size minimization).** Solutions to certain problems often tend to minimize size rather than mass, compare e.g. [Mor89, DPH03]. Difficulties arise since one has in general no compactness theorems for the size. Also, size-minimizing chains do not have to be rectifiable (as the mass can be infinite) what is often desired. An example of a boundary that bounds no size-minimizing integer rectifiable current is given by [Mor89, Theorem 2.3].

As a partial solution to this problem we take the size as norm on $G$ (or on $\bar{G}$) right from the beginning. Thus, one considers

$$\mathcal{F}_n(X, (G, \phi_s))$$ instead of $$\mathcal{F}_n(X, (G, \|\|))$$

if this makes sense for the specific problem. Now, every finite mass (which is then equal to size) chain is rectifiable. For a bounded sequence of chains in $\mathcal{R}_n(X, (G, \phi_s))$ one can sometimes apply Theorem 5.0.5. Note that - although we dropped the norm $\|\|$ - using Theorem 4.5.1 we still know the $G$-coefficients (respectively $\bar{G}$-coefficients) of rectifiable chains. So we still have a quite accurate picture of the rectifiable chains.

Applied to the example presented in [Mor89], Theorem 5.0.5 shows that the size-minimizing sequence considered in [Mor89] has a convergent subsequence, hence a limit - say $R$ - exists and has finite mass (that is: size). It is then rectifiable which means that $R \in \mathcal{R}_1(\mathbb{R}^2, (\mathbb{Z}, \phi_s))$ and we see that we can represent $R$ as

$$R = \sum_{i \in \mathbb{N}} i f_i\#[0, 1] \quad \text{with} \quad S(R) = \sum_{i \in \mathbb{N}} M_{AK}(f_i\#[0, 1]) < \infty.$$
Chapter 8

Inner flat norm

Here we show that if $G$ is discrete, then flat convergence is equivalent to 'inner flat convergence' for normal chains, assuming the complete metric space $X$ to fulfill some local conditions; see Theorem 8.4.1 and Theorem 8.4.2. Moreover, cycles of small mass satisfy an isoperimetric inequality (Theorem 8.3.1). This is the analogue of [Wen07] for integral currents (there, weak convergence is equivalent to inner flat convergence). Recall that a discrete normed group is complete and it holds by Theorem 6.2.1 that $\mathcal{R}_n(X, G) = \mathcal{M}_n(X, G)$.

The first two sections contain the necessary definitions.

8.1 Inner flat norm and filling volume

For $R \in \mathcal{R}_n(X, G)$, define the inner flat norm by

$$F_X(R) := \inf \left\{ \mathcal{M}(R_0) + \mathcal{M}(R_1) \mid R_i \in \mathcal{R}_{n+i}(X, G), \ R = R_0 + \partial R_1 \right\}.$$ 

By Lemma 4.1.2 is $\mathcal{F}(R) \leq F_X(R) \leq \mathcal{M}(R)$.

Rectifiable chains $R \in \mathcal{R}_n(X, G)$ are called cycles if $\partial R = 0$ for $n > 0$, and if $\chi(R) = 0$ for $n = 0$.

If $R$ is a cycle, the filling volume in $X$ is

$$\text{Fillvol}_X(R) := \inf \left\{ \mathcal{M}(V) \mid V \in \mathcal{R}_{n+1}(X, G), \ \partial V = R \right\}.$$ 

For $X$ isometrically embedded in an injective metric space $Y$, we consider $X$ as subset of $Y$ and define the absolute filling volume of $R \in \mathcal{R}_n(X, G)$ by

$$\text{Fillvol}(R) := \inf \left\{ \mathcal{M}(V) \mid V \in \mathcal{R}_{n+1}(Y, G), \ \partial V = R \right\}.$$
Note that this is not dependent of the isometrical embedding or of the injective space $Y$.

For cycles holds

$$\mathcal{F}_X(R) \leq \min\{\text{Fillvol}_X(R), M(R)\} \quad \text{and} \quad \text{Fillvol}(R) \leq \text{Fillvol}_X(R).$$

### 8.2 Quasiconvexity, cone type inequalities

**Definition 8.2.1.** A metric space $(X,d)$ is $\delta$-quasiconvex if there exists $C < \infty$ such that every two points $x, y \in X$ with $d(x,y) \leq \delta$ can be joined by a Lipschitz curve $\gamma: [0, 1] \to X$ of length $L(\gamma)$ at most $Cd(x,y)$.

A quasiconvex metric space is a $\delta$-quasiconvex space for some $\delta > 0$. For example Banach spaces are quasiconvex.

Similar to (4.7.1) for $F$ we have for $\mathcal{F}_X$:

**Lemma 8.2.2.** Suppose that the complete metric space $X$ is $\delta$-quasiconvex with constant $C$. Then for $R \in \mathcal{R}_0(X,G)$ with $\text{diam} R < \delta$ holds

$$\mathcal{F}_X(R) \leq \|\chi(R)\| + C \text{ diam } R \ M(R). \quad (8.2.1)$$

**Proof.** This follows as (4.7.1), now taking paths in $X$. \qed

Following [Wen07] we set:

**Definition 8.2.3.** Let $n \in \mathbb{N}$, $\delta > 0$ and $C > 0$. $X$ is said to admit a $\delta$-cone type inequality for $\mathcal{R}_n(X,G)$ with constant $C$ if for every $R \in \mathcal{R}_n(X,G)$ with $\partial R = 0$ and $\text{diam } R \leq \delta$ there exists an $S \in \mathcal{R}_{n+1}(X,G)$ satisfying $\partial S = R$ and

$$M(S) \leq C \text{ diam}(R) \ M(R).$$

We say that a space $X$ admits a local cone type inequality for $\mathcal{R}_n(X,G)$ if it admits a $\delta$-cone type inequality for $\mathcal{R}_n(X,G)$ for some $\delta > 0$. If $X$ admits a $\delta$-cone type inequality with constant $C$ for all $\delta > 0$ then we say that it admits a global cone type inequality.

Wenger ([Wen07]) calls a bounded set $B \subset X$ $(\beta, \gamma)$-Lipschitz contractible in $X$ if there exists $\phi \in \text{Lip}([0,1] \times B, X)$ satisfying $\phi(1,\cdot) = \text{id}_B$ and $\phi(0,x) = x_0$ for all $x \in B$ for some $x_0 \in X$ and moreover

$$d\left(\phi(t,x), \phi(t',x')\right) \leq \beta |t - t'| + \gamma d(x,x') \quad \forall x, x' \in B, \forall t, t' \in [0,1]. \quad (8.2.2)$$

A bounded set $B \subset X$ which is $(\gamma \text{ diam } B, \gamma)$-Lipschitz contractible is called $\gamma$-Lipschitz contractible and the corresponding map $\phi$ a $\gamma$-contraction of $B$. 
Proposition 8.2.4. Let $\delta, \gamma > 0$, $n \geq 1$, let $X$ be a complete metric space such that all its subsets of diameter less than $2\delta$ are $\gamma$-Lipschitz contractible. Then $X$ admits a $\delta$-cone type inequality for $R_n(X,G)$ with constant $(n+1)\gamma^{n+1}$.

Compare [Wen07, Prop. 3.4]. Examples of such metric spaces are Banach spaces and CAT($\kappa$)-spaces (see also therein).

Proof. Let $0 \neq R = \sum_{i \in \mathbb{N}} f_i \# [g_i] \in R_n(X,G)$ be as in Theorem 4.5.1, then $0 < \text{diam } R$. Assume that $\partial R = 0$ and $\text{diam } R \leq \delta$.

Now, $V := [0, \text{diam } R] \times R \in R_{n+1}([0, \delta] \times X,G)$ has

$$M(V) \leq (n+1) \text{diam } R M(R)$$

(see (4.1.20)).

Let $\phi$ be a $\gamma$-contraction of $\text{spt } R$ and let $\bar{\phi}: [0, \text{diam } R] \times \text{spt } R \to X$ be defined by $\bar{\phi}(t,x) := \phi(t \text{diam } R, x)$. Then, $\bar{\phi}$ is $\gamma$-Lipschitz, so for $S := \bar{\phi} \# V \in R_{n+1}(X,G)$ is

$$M(S) \leq (n+1)\gamma^{n+1} M(R).$$

Clearly is $\bar{\phi}(0,\cdot) \# R = 0$ and so $\partial S = R$. \hfill \Box

8.3 Isoperimetric inequality

An essential tool later on is the following result which is also of independent interest. Again, compare [Wen07] for integral currents; see also [AK09, Section 8] for an axiomatic approach for currents modulo $p$ to global isoperimetric inequalities (this also builds on [Wen07]).

Theorem 8.3.1 (Isoperimetric inequality). Let $C > 0$ and $n \in \mathbb{N}$, let $G$ be a discrete normed group. Let $X$ be a complete metric space that admits local $\delta$-cone type inequalities for $R_k(X,G)$, $k = 1, \ldots, n$. Then there exist constants $C', D > 0$ such that the following holds: If $R \in R_n(X,G)$ satisfies $M(R) < C'$ and $\partial R = 0$ then there exists a filling $S \in R_{n+1}(X,G)$ with

$$M(S) \leq DM(R)^{\frac{n+1}{n}} \quad \text{and} \quad \text{spt } S \subset B(\text{spt } R, 3D(n+1)M(R)^{\frac{1}{n}}).$$

The constant $D$ depends only on $n$ and on the constants of the $\delta$-cone inequalities, but not on $\delta$, whereas $C'$ also depends on $\delta$.

As in [Wen07], we get $C' = \infty$ if $X$ admits global cone type inequalities. Note furthermore, that we can settle the case $n = 0$ easily as follows: Let $\Theta(G) := \inf\{\|g\| \mid g \in G \setminus \{0\}\}$, then $\Theta(G) > 0$ and $M(R) < \Theta(G)$ implies directly that $R = 0$.

For convenience we include a sketch of the main steps of the proof; this is exactly as in the case of integral currents considered in [Wen07].
CHAPTER 8. INNER FLAT NORM

8.3.1 Sketch of the proof of Theorem 8.3.1

**Definition 8.3.2.** Let $X$ be a complete metric space and $k \in \mathbb{N}$, let $G$ be a discrete normed group. Then $X$ is said to admit an isoperimetric inequality of Euclidean type with constant $C$ for cycles in $R_k(X,G)$ with mass no larger than $C'$ if for every cycle $R \in R_k(X,G)$ with $M(R) \leq C'$ there exists an $S \in R_{k+1}(X,G)$ with $\partial S = R$ and

$$M(S) \leq CM(R)^{\frac{k+1}{k}}.$$ 

The next lemma shows that for $X$ as in the definition above there also exists a filling of $R$ that stays close to $R$ (this is [Wen07, Lemma 5.3], now for chains):

**Lemma 8.3.3.** Let $X$ be a complete metric space and $k \geq 2$, let $G$ be a discrete normed group. Let $X$ admit an isoperimetric inequality of Euclidean type with constant $C$ for cycles in $R_k(X,G)$ of mass no larger than $C'$. Then for every $R \in R_k(X,G)$ with $\partial R = 0$ and $M(R) \leq C'$ there exists an $S \in R_k(X,G)$ satisfying $\partial S = R$ as well as

$$M(S) \leq CM(R)^{\frac{k}{k-1}} \quad \text{and} \quad \text{spt } S \subset B(\text{spt } R, 3CkM(R)^{\frac{1}{k-1}}).$$

**Sketch of the proof of Lemma 8.3.3.** We follow [Wen07], compare therein for more details.

Let $\mathcal{M}$ be the space consisting of all $S \in R_k(X,G)$ with $\partial S = R$. The metric $d_\mathcal{M}(S, S') := M(S - S')$ makes $\mathcal{M}$ a non-empty, complete metric space. Then there exists $S \in \mathcal{M}$ with $M(S) \leq CM(R)^{\frac{k}{k-1}}$ such that the function

$$\mathcal{M} \ni S' \mapsto M(S') + \frac{1}{2}M(S' - S)$$

is minimal at $S' = S$ (by the Ekeland variation principle). Let $x \in \text{spt } S \setminus \text{spt } R$ and set $\beta(r) := \mu_S(B_r(x))$. As in Wenger’s proof, for almost every $0 < r < d(x, \text{spt } R)$ the slice $\langle S, \beta, r \rangle$ exists, has zero boundary, belongs to $R_{k-1}(X,G)$ and satisfies $M(\langle S, \beta, r \rangle) \leq \beta'(r)$ (see (4.6.15)). As therein follows that

$$\beta(r) \geq \frac{r^k}{(3C)^{k-1}k^k} \quad \text{for all } 0 \leq r \leq d(x, \text{spt } R).$$

Now we see that $S$ satisfies also the second estimate in the lemma. □
Lemma 8.3.4. Let $X$ be a complete metric space and $k \in \mathbb{N}$, let $G$ be a discrete normed group such that $\Theta(G) := \inf\{\|g\| \mid g \neq 0\} \geq 1$. If $k \geq 2$ then suppose furthermore that $X$ admits an isoperimetric inequality of Euclidean type with constant $C > 0$ for cycles in $\mathcal{R}_{k-1}(X, G)$ of mass no larger than $C'$. Let $\lambda \in (0, 1/6)$ and $R \in \mathcal{R}_k(X, G)$ with $\partial R = 0$ and assume that $r_0, r_1 : X \to [0, \infty)$ are functions such that for $\mu_R$-almost every $y \in X$ we have

(a) $0 < r_0(y) < r_1(y) \leq \frac{4}{3} r_0(y)$

(b) $\mu_R \left( B(y, r_0(y)) \right) = Q r_0(y)^k$

(c) $\mu_R \left( B(y, r_1(y)) \right) < Q r_1(y)^k$

(d) $\mu_R \left( B(y, 5r_0(y)) \right) < \min\{K, 5^k Q r_0(y)^k\}$,

where $Q := 1$ and $K := \infty$ if $k = 1$ and $Q := \lambda^{k-1} C^{1-k} k^{-k}$ and $K := C(C')^{\frac{k}{k-1}} \lambda^{-1}$ if $k \geq 2$. Then there exist finitely many points $y_1, \ldots, y_N \in X$ and a decomposition $R = R_1 + \cdots + R_N + R'$ such that $R_i, R' \in \mathcal{R}_k(X, G)$ are cycles with the following properties:

(i) $\text{spt } R_i \subset B(y_i, 2r_0(y_i))$ and $\text{diam } R_i \leq \frac{4}{(Q(1-\lambda))^\frac{1}{k}} M(R_i)^\frac{1}{k}$

(ii) $\sum_{i=1}^N M(R_i) \leq (1 + \lambda) M(R)$

(iii) $M(R') \leq (1 - \frac{1}{5^k} (1 - \lambda)) M(R)$

(iv) $\mu_R \left( B(y, \epsilon) \right) \leq \mu_R \left( B(y, \epsilon) \right) + \lambda M(R)$ for all $y \in X$ and all $\epsilon > 0$.

This is Wenger’s Lemma 5.5 in [Wen07] now for chains.

Proof. Replacing currents by chains we can apply exactly the proof of [Wen07, Lemma 5.5]: We find $y_1, \ldots, y_N \in X$ such that (a) - (d) hold and such that the balls $B(y_i, 2r_0(y_i))$ are pairwise disjoint and

$$\sum_{i=1}^N \mu_R \left( B(y_i, r_0(y_i)) \right) \geq \frac{1}{5^k} M(R)$$

(by a Vitali type argument). Fix $i$ for the moment. Then the function $\beta(r) := \mu_R(B(y_i), r)$ is non-decreasing, $\beta(r_0(y_i)) = Q r_0(y_i)^k$ and $\beta(r_1(y_i)) < Q r_1(y_i)^k$. 

For $\rho(x) := d_{y_i}(x)$ then $\langle R, \rho, r \rangle = \partial(r|_{\{\rho<r\}})$ exists, is rectifiable and a cycle a.e. It satisfies (by (4.6.15))

$$\mathcal{M}(\langle R, \rho, r \rangle) \leq \beta'(r) \quad \text{for a.e. } r > 0.$$  

In the case $k = 1$, we get by (b), (c) that $\langle R, \rho, r \rangle = 0$ for some $r_i \in (r_0(y_i), r_1(y_i))$ where all the above holds (recall that $\Theta(G) \geq 1$). Then

$$R_i := R|_{B(y_i, r_i)}$$  

has diam $R_i \leq 4r_0(y_i) \leq 4\mathcal{M}(R_i)$ by (a) and (b); so (i) is satisfied.

For $k \geq 2$, Lemma 5.1 of [Wen07] implies that there is a subset $\Omega \subset [r_0(y_i), r_1(y_i)]$ with $\mathcal{L}^1(\Omega) > 0$ such that the above holds for all $r \in \Omega$ and with $C(\beta'(r))^{\frac{1}{k-1}} < \lambda \beta(r)$ for all $r \in \Omega$. By (d) we get

$$\mathcal{M}(\langle R, \rho, r \rangle) \leq \beta'(r) \leq \left(\frac{\lambda \beta(r)}{C}\right)^{\frac{k-1}{k}} < C'$$

for almost every $r \in \Omega$. Fixing such an $r =: r_i \in \Omega$, by assumption and by Lemma 8.3.3, there exists an $S_i \in \mathcal{R}_k(X, G)$ with $\partial S_i = \langle R, \rho, r_i \rangle$ which satisfies

$$\mathcal{M}(S_i) \leq C \mathcal{M}(\langle R, \rho, r_i \rangle)^{\frac{k}{k-1}}$$

and $\text{spt } S_i \subset B(\text{spt } \langle R, \rho, r_i \rangle, 3Ck \mathcal{M}(\langle R, \rho, r_i \rangle)^{\frac{1}{k-1}})$. Now set

$$R_i := (R|_{B(y_i, r_i)}) - S_i, \quad R' := R - \sum_{i=1}^{N} R_i$$

and check that this decomposition fulfills all the necessary properties (exactly as in Wenger’s proof). \hfill \square

**Sketch of the proof of Theorem 8.3.1.** The proof works exactly as for integral currents in [Wen07] (see also the references therein), so we follow Wenger. For this, we can easily assume that $\Theta(G) \geq 1$. Note that we also have a density bound by Theorem 4.5.2.

Defining the constants as Wenger does, the argument is inductive on the dimension. Successively applying Lemma 8.3.4 we find for $m \in \mathbb{N}$ a partition $R = \sum_{i=1}^{N_m} R_i + R'_m$ into cycles with

(i) diam $R_i \leq \delta$ and diam $R_i \leq (Q(1 - \lambda))^{-\frac{1}{k}} \mathcal{M}(R_i)^{\frac{1}{k}}$,

(ii) $\mathcal{M}(R'_m) \leq (1 - \nu)^m \mathcal{M}(R)$ where $\nu := \frac{1}{5r}(1 - \lambda)$,
(iii) \( \sum_{i=1}^{N_m} M(R_i) \leq \frac{1+\lambda}{\nu} M(T) \).

Now we can fill each \( R_i \) by a cone \( S_i \) such that (by (i))

\[
M(S_i) \leq \frac{4C_k}{(Q(1-\lambda))^k} M(R_i)^{k+1 \over k} \cdot (8.3.2)
\]

Set \( S^m := \sum_{i=1}^{N_m} S_i \); note that this is a Cauchy sequence for \( M \) and satisfies \( \partial S^m = R - R'_m \). Since \( R'_m \to 0 \) for \( m \to \infty \), \( S := \lim_{m \to \infty} S^m \in \mathcal{R}_{k+1}(X,G) \) has \( \partial S = R \). Moreover,

\[
M(S) \leq \sum_{i \in \mathbb{N}} M(S_i) \leq \frac{4C_k}{(Q(1-\lambda))^k} \left( \frac{1+\lambda}{\nu} \right)^{k+1 \over k} M(T)^{k+1 \over k} .
\]

By Lemma 8.3.3, we now find a filling satisfying the bound on the support.

\[ \square \]

### 8.3.2 Isoperimetric inequality for mass and size

In Euclidean space there exists an improved isoperimetric inequality comparing mass and size of the filling to that of the cycle, compare [Alm86] and [Whi99b]. Unless for dimension one, we are at the moment not able to produce a similar result. However, in dimension one we get the following nice corollary of the proofs of Theorem 8.3.1 and Lemma 8.3.4:

**Corollary 8.3.5.** Let \( X \) be a complete metric space such that all its subsets of diameter less than \( \delta \) are \( \lambda \)-Lipschitz contractible, let \( G \) be a discrete normed group. Then there exist \( C', D > 0 \) such that for any \( \phi: G \to \mathbb{R}_{\geq 0} \) as in Section 7 holds: If \( R \in \mathcal{R}_1(X,G) \) with \( \partial R = 0 \) and \( M(R) < C' \) then there exists \( S \in \mathcal{R}_2(X,G) \) with \( \partial S = R \),

\[
M(S) \leq DM(R)^2 \quad \text{and} \quad \phi(S) \leq DM(R) \phi(R) .
\]

**Proof.** Note that if subsets of diameter less than \( \delta \) are \( \lambda \)-Lipschitz contractible, then for a cycle \( R \) with \( \text{diam} \, R < \delta \) holds for the filling \( S := \phi_#([0,1] \times R) \):

\[
M(S) \leq C \text{diam} \, R \, M(R) \quad \text{and} \quad \phi(S) \leq C \text{diam} \, R \, \phi(R) .
\]

Now the proof is as above; the partition in the proof of Theorem 8.3.1 for \( k = 1 \) (see also (8.3.1)) gives actually

\[
M(R) = \sum_{i=1}^{N_m} M(R_i) + M(R') \quad \text{and then} \quad \phi(R) = \sum_{i=1}^{N_m} \phi(R_i) + \phi(R') .
\]
By (i) in the proof of Theorem 8.3.1 we finally have
\[ \phi(S) \leq \sum_{i \in \mathbb{N}} \phi(S_i) \leq D \sum_{i \in \mathbb{N}} \phi(R_i) M(R_i) \leq DM(R) \phi(R). \]

\[ \square \]

### 8.4 Inner flat norm

**Theorem 8.4.1.** Let \( X \) be a complete quasiconvex metric space and \( R \in \mathcal{R}_0(X,G) \) where \( G \) is a discrete normed group. Let \((R_i)_{i \in \mathbb{N}} \subset \mathcal{R}_0(X,G)\) be an \( M \)-bounded sequence. Then
\[ \mathcal{F}(R - R_i) \to 0 \quad \text{if and only if} \quad \text{Fillvol}_X(R - R_i) \to 0. \]

**Theorem 8.4.2.** Let \( X \) be a complete quasiconvex metric space and let \( G \) be a discrete normed group, let \( n \geq 1 \). Suppose that \( X \) admits local cone type inequalities for \( \mathcal{R}_n(X,G) \) for \( j = 1, \ldots, n \). Let \((R_i)_{i \in \mathbb{N}} \subset \mathcal{R}_n(X,G)\) be an \( N \)-bounded sequence and let \( R \in \mathcal{R}_n(X,G) \). Then
\[ \mathcal{F}(R - R_i) \to 0 \quad \text{if and only if} \quad \mathcal{F}_X(R - R_i) \to 0. \]

Moreover, if \( \partial R_i = 0 \) for all \( i \in \mathbb{N} \), then
\[ \mathcal{F}(R - R_i) \to 0 \quad \text{if and only if} \quad \text{Fillvol}_X(R - R_i) \to 0. \]

Compare [Wen07, Theorem 1.2 and Theorem 1.4] for integral currents. The proofs from there generalize directly to discrete groups.

With Theorem 5.0.1 this leads to

**Corollary 8.4.3.** Let \( X, G \) be as above, let \( G \) in addition be proper; let \( K < \infty \). If \( C \subset X \) compact, then
\[
\{ R \in \mathcal{R}_n(X,G) \mid N(R) \leq K, \ \text{spt} R \subset C \}
\]
is compact with respect to \( d_{\mathcal{F}_X}(R, R') := \mathcal{F}_X(R - R') \).

**Proof of Theorem 8.4.1.** We follow [Wen07, p. 148, 149]. One direction is clear since \( \text{Fillvol}_X(S) \geq \mathcal{F}(S) \). For the other, set \( \Theta(G) := \inf \{ \| g \| \mid g \in G \setminus 0 \} \); note that \( \Theta(G) > 0 \). We can assume that \( R = 0 \) (replacing \( R_i \) by \( R_i - R \)). Furthermore, we assume that \( R_i \to 0 \) rapidly. Now we show the existence of a
subsequence for which the assertion holds. (Thus, we find such a subsequence for every given subsequence, proving the theorem.)

Let \( \epsilon > 0 \) be small. The idea is to find a subsequence and a decomposition \( R_i = T_i + R_i' \) such that \( M(T_i) \geq \Theta(G) \), \( M(R_i) = M(T_i) + M(R_i') \),

\[
\text{Fillvol}_X(T_i) \leq 3C\epsilon M(T_i), \quad \text{and} \quad R_i' \to 0 \quad \text{as} \quad i \to \infty.
\]

Repeating this for \( R_i' \), eventually we get \( M(R_i') < \Theta(G) \), hence \( R_i' = 0 \). Theorem 8.4.1 then follows by sending \( \epsilon \) to zero.

For the decomposition, first assume that there exists \( x \in X \) such that \( R_i \lfloor_{B_{2\epsilon}(x)} \neq 0 \) for a subsequence; we consider below only elements of this subsequence. We find \( r \in (2\epsilon, 3\epsilon) \) such that \( T_i := R_i \lfloor_{B_{r}(x)} \to 0 \) rapidly (hence also \( R_i' := R_i - T_i \to 0 \) rapidly). We eventually get \( \Theta(G) > F(T_i) \geq \|\chi(T_i)\| \), i.e. \( \chi(T_i) = 0 \). Then there are fillings of \( T_i \) in \( X \) of mass \( \leq 3C\epsilon M(T_i) \) (for \( \epsilon \) small enough, by the proof of Lemma 8.2.2).

If there exists no such \( x \in X \), we find a subsequence \( \Lambda \) and \( x_j \in X \) for \( j \in \Lambda \) such that \( d(x_i,x_j) \geq 2\epsilon \) and \( R_i \lfloor_{B_{\epsilon}(x_j)} = 0 \) if \( i \neq j \), \( i,j \in \Lambda \), as well as

\[
R_i \lfloor_{B_{3\epsilon}(x_i)} \neq 0, \quad \text{whenever} \quad i \in \Lambda.
\]

Then we find \( r \in (\frac{\epsilon}{2}, \epsilon) \) such that \( T_i := R_i \lfloor_{\bigcup_{j \in \Lambda} B_{r}(x_j)} = R_i \lfloor_{B_{r}(x_i)} \to 0 \) for \( i \in \Lambda \). Again, \( \|\chi(T_i)\| \to 0 \) so we eventually find fillings. Then \( R_i' := R_i - T_i' \) is a desired decomposition.

\[\Box\]

### 8.4.1 Sketch of the proof of Theorem 8.4.2

The proof is again as in [Wen07], we sketch the main steps.

For \( R \in \mathcal{R}_n(X,G) \) and \( \epsilon > 0 \) set

\[
\rho_R(\epsilon) := \sup\{\mu_R(B_\epsilon(y)) \mid y \in X\}.
\]

**Proposition 8.4.4.** Let \( X \) be a complete metric space, let \( G \) be a discrete normed group and let \( \delta > 0 \). Assume that \( X \) admits \( \delta \)-cone type inequalities for \( R_j(X,G) \), \( j = 1, \ldots, n \). Then for every mass-bounded sequence \( (T_m) \subset \mathcal{R}_n(X,G) \) of cycles and for every \( \epsilon > 0 \) the property

\[
\rho_{R_m}(\epsilon) \to 0 \quad \text{as} \quad m \to \infty
\]

implies that \( \text{Fillvol}_X(R_m) \to 0 \) as \( m \to \infty \).

This is Proposition 5.8 in Wenger’s paper, the proof is just as there.
CHAPTER 8. INNER FLAT NORM

Proposition 8.4.5. Let \( n \geq 1 \) and let \( X \) and \( G \) be as in Theorem 8.4.2. If \( n \geq 2 \), suppose that for \( 1 \leq k \leq n-1 \) every \( M \)-bounded sequence \( (S_m) \subset \mathcal{R}_k(X,G) \) of cycles with \( \mathcal{F}(S_m) \to 0 \) holds that \( \text{Fillvol}_X(S_m) \to 0 \).

Let \( (R_m) \subset \mathcal{R}_n(X,G) \) be an \( M \)-bounded sequence of cycles with \( \mathcal{F}(R_m) \to 0 \). Then for every \( \epsilon > 0 \) small enough there exists a subsequence \( (R_{m_j}) \) and decompositions of \( R_{m_j} \) into the sum of rectifiable cycles

\[
R_{m_j} = U_1^j + \cdots + U_{M_j}^j + V_j
\]

satisfying the following properties:

(i) \( \text{Fillvol}_X(U_i^j) \leq \hat{C} \epsilon M(U_i^j) \) for every \( i \in \{1, \ldots, M_j\} \) and \( j \in \mathbb{N} \) and for a constant \( \hat{C} \) depending only on the constants of the \( \delta \)-cone type inequalities.

(ii) \( \sum_{i=1}^{M_j} M(U_i^j) + M(V_j) - M(R_{m_j}) \to 0 \) as \( j \to \infty \).

(iii) \( M(V_j) \leq M(R_{m_j}) - \frac{1}{2} \rho_{R_{m_j}}(\epsilon/2) \) for every \( j \in \mathbb{N} \).

(iv) \( \mathcal{F}(V_j) \to 0 \) as \( j \to \infty \).

This is [Wen07, Proposition 5.9] for chains. Compare therein for the proof.

Remark that we have as well

\[
\int_{\mathbb{R}} \liminf_{m \to \infty} M(\langle R_m, f, r \rangle) dr \leq \text{Lip}(f) \liminf_{m \to \infty} M(R_m) < \infty.
\]

So at a.e. \( r \) we find a subsequence such that the slices are mass-bounded cycles.

Proof of Theorem 8.4.2. We claim that if \( \mathcal{F}(R_i) \to 0 \) as in Theorem 8.4.2 with \( \partial R_i = 0 \), then \( \text{Fillvol}_X(R_i) \to 0 \).

The proof of the claim is by induction applying Proposition 8.4.5, exactly as in [Wen07].

The first statement of Theorem 8.4.2 then follows: Let \( R'_i := R - R_i \) then \( \mathcal{F}(R'_i) \to 0 \). Then also \( \mathcal{F}(\partial R'_i) \to 0 \), so we find a \( S_i \in \mathcal{R}_n(X,G) \) with \( \partial S_i = \partial R'_i \) and \( M(S_i) \to 0 \) (by the claim or by Theorem 8.4.1). Then \( \partial(R'_i - S_i) = 0 \) and \( \mathcal{F}(R'_i - S_i) \to 0 \), so we find \( V_i \in \mathcal{R}_{n+1}(X,G) \) with \( \partial V_i = R'_i - S_i \) and \( M(V_i) \to 0 \). Thus,

\[
\mathcal{F}_X(R - R_i) \leq M(S_i) + M(V_i) \to 0.
\]

\( \square \)
Chapter 9

Minimizing chains in homology

Using the compactness results as well as the equivalence of flat and inner flat convergence we find mass-minimizing chains in homology classes. Compare [Fed69] and [Wen07] for analogous results for currents. We end this chapter with some remarks about other definitions for homology groups for flat chains or integral currents (compare also Corollary 11.1.3 in Part II).

Let $X$ be a complete metric space, let $G$ be a normed group. We denote by
\[ R_c^e(X,G) := \{ R \in R_n(X,G) | \text{spt } R \text{ compact} \} \]
the rectifiable chains with compact support. For $A \subset X$ closed, let
\[ Z^F_n(X,A;G) := \{ R_0 + \partial R_1 | R_i \in R_{n+i}^c(X,G), \text{spt}(\partial R_0) \subset A \text{ or } n = 0 \} \]
be the group of $n$ dimensional flat $G$-cycles (relative $A$), and let
\[ B^F_n(X,A;G) := \{ R_0 + \partial R_1 | R_i \in R_{n+i}^c(X,G), \text{spt } R_0 \subset A \} \]
be the subgroup of $n$ dimensional flat $G$-boundaries (relative $A$). The $n$ dimensional (flat) homology group (relative $A$) with coefficients in $G$ is
\[ H^F_n(X,A;G) := Z^F_n(X,A;G)/B^F_n(X,A;G). \]
(See remarks 9.0.2, 9.0.4 and 9.0.5 as well as Corollary 11.1.2 for comparisons with other homologies.)
Let $h \in H^F_n(X, A; G)$, then set

$$M(h) := \inf \{ M(F) \mid F \in h \}.$$ 

Clearly, $M(h) < \infty$ as $[R + \partial \tilde{R}] = [R] \in H^F_n(X, A; G)$ and $R \in R_n(X, G)$.

A subset $A \subset X$ is a Lipschitz neighborhood retract if there exist a neighborhood $N$ of $A$ in $X$ and a map $\phi \in \text{Lip}(N, A)$ with $\phi|_A = \text{id}_A$. We say that $A$ is a Lipschitz $r$-neighborhood retract if it is a Lipschitz neighborhood retract for the neighborhood $N := B_r(A)$.

**Theorem 9.0.1.** Let $G$ be a discrete and proper normed group. Let $X$ be a compact, quasiconvex metric space, let the closed subset $A \subset X$ be a Lipschitz neighborhood retract. Let $n \geq 0$, assume that $X$ admits local cone type inequalities for $R_k(X, G)$ for $k = 1, \ldots, n + 1$. Then for each $h \in H^F_n(X, A; G)$ there exists

$$R \in R_n(X, G) \quad \text{such that} \quad R \in h \quad \text{and} \quad M(R) = M(h).$$

Note that our assumptions are stronger than the ones in [Wen07]; however, the proof in [Wen07] seems to cover only the special case $A = \emptyset$ (as for the other case the mass of the boundaries could diverge).

**Proof.** By Theorem 6.2.1, if there is a minimizing chain then it is rectifiable.

First, let $A = \emptyset$ or $n = 0$. Then there is a converging subsequence for a mass-minimizing sequence $R_i \in h$ by Theorem 5.0.1, say $\lim_{i \in \mathbb{N}} R_i =: R$. By lower semicontinuity is $M(R) \leq M(h)$. Then, theorems 8.4.1 and 8.4.2 imply that there is $V_i \in R_{n+1}(X, G)$ with $\partial V_i = R_i - R$ for $i$ great enough. Hence $R \in h$.

Now let $A \neq \emptyset$ and $n > 0$. Let $\phi: N \rightarrow A$ be a Lipschitz neighborhood retraction; by compactness of $A$ there exists $r > 0$ such that $B_r(A) \subset N$. Note that via the Lipschitz retraction, $A$ is quasiconvex and admits local cone type inequalities for $R_k(A, G)$, $k = 1, \ldots, n + 1$.

Let $R_i \in h$ be a mass-minimizing sequence. By Theorem 5.0.4, we find a subsequence (say $N$) and $\tilde{F}$ such that $R_i$ converges to $\tilde{F}$ with respect to $\mathcal{F}(\cdot, A)$, and such that $M(\tilde{F}) \leq M(h)$. Thus, $\tilde{F} \in R_n(X, G)$; moreover is $\text{spt} \partial \tilde{F} \subset A$.

Choose $r_i \in (0, r)$ such that $(\partial(R_i|_{\{r_i<d_A\}}))_{i \in \mathbb{N}}$ is a mass-bounded sequence. Set $S_i := R_i|_{\{r_i<d_A\}}$. Thus, $\text{spt}(\partial S_i) \subset B_r(A)$ and $\partial S_i \in R_{n-1}(X, G)$. Set

$$F_i := R_i - \phi\#(R_i - S_i) \in R_n(X, G). \quad (9.0.1)$$
From Lemma 4.1.9 we get that $\partial F_i = \phi_\# \partial S_i$. Hence $\partial F_i \in \mathcal{R}_{n-1}(A, G)$ is a mass-bounded sequence, and therefore is $F_i$ an $N$-bounded sequence. Hence we find by Theorem 5.0.1 an element $R \in \mathcal{R}_n(X, G)$ and a subsequence (say $N$) such that $F_i \to R$. Then $\partial F_i \to \partial R$ and so $\text{spt}(\partial R) \subset A$. By Theorem 8.4.1 or 8.4.2 we find $\alpha_i \in \mathcal{R}_n(A, G)$ (provided $i$ to be great enough) with $\partial \alpha_i = \partial F_i - \partial R$ and $M(\alpha_i) \to 0$. By Theorem 8.4.2 we find $\beta_i \in \mathcal{R}_{n+1}(X, G)$ such that $\partial \beta_i = F_i - R - \alpha_i$ and $M(\beta_i) \to 0$. Together,

$$R = F_i - \alpha_i - \partial \beta_i = R_i - \phi_\#(R_i - S_i) - \alpha_i - \partial \beta_i .$$

As $\text{spt}(\phi_\#(R_i - S_i) + \alpha_i) \subset A$ we get $R \in h$ and

$$\mathcal{F}(\tilde{F} - R; A) \leq \mathcal{F}(\tilde{F} - R_i; A) + \mathcal{F}(\partial \beta_i; A)$$

(see p. 63 for the definition of $\mathcal{F}(\cdot; A)$). With $D := \text{diam}(X)$ we get

$$\mathcal{F}(\partial \beta_i; A) \leq \int_0^D \left( \mathcal{F}(\partial (\beta_i|_{\{d_A > s\}})) + \mathcal{F}(\langle \beta_i, d_A, s \rangle) \right) ds \leq M(\beta_i)(D + 1).$$

As $\mathcal{F}(\tilde{F} - R_i; A) \to 0$ and $M(\beta_i) \to 0$, we get therefore $\mathcal{F}(\tilde{F} - R; A) = 0$. So $\text{spt}(\tilde{F} - R) \subset A$ by Remark 5.0.3 and $\tilde{F} = R + (\tilde{F} - R) \in h$, concluding the proof. \hfill\Box

Remark 9.0.2. For $A \subset X$ and $X \subset l^\infty(X)$ both being local Lipschitz neighborhood retracts (that is, the retraction map is only locally Lipschitz), the straightforward generalizations of Federer’s definition ([Fed69, p. 464]) are as follows: Cycles (rel. $A$) are defined as all flat chains $F$ in $l^\infty(X)$ with compact support in $X$ and boundary supported in $A$, and similarly for the boundaries. Denote this homology by $H_n^F(X, A; G)$. Then it holds: For $X$ compact and $X, A$ as above, there is an isomorphism from $H_n^F(X, A; G)$ to $H_n^F(X, A; G)$ induced by the inclusion. This can be seen as follows: Writing $F = R_0 + \partial R_1$ for $R_i \in \mathcal{R}_{n+i}(l^\infty(X), G)$ we can assume that $\text{spt} R_i \subset B_r(X)$. Applying the retraction $\phi$, we can push these rectifiable chains to $X$ to get $F = \phi_\# R_0 + \phi_\# R_1$ for $\phi_\# R_i \in \mathcal{R}_{n+i}(X, G)$. Extending this argument, we see that the inclusion gives an isomorphism on the homologies.

An example for which we get different homology groups is the following:

**Example 9.0.3.** Let $X \subset \mathbb{R}^2$ be the Koch snowflake. Then $X$ is a compact metric space which contains no non-trivial Lipschitz curve. Hence we have $H_1^F(X; \mathbb{Z}) = 0$. But we easily see that

$$H_1^F(X; \mathbb{Z}) \cong H_1^F(S^1; \mathbb{Z}) \cong \mathbb{Z} .$$
Remark 9.0.4. Our definitions do not agree with the ones given in [Wen07] for currents (which are used in part II). Only integral (in particular: normal) currents are considered there, whereas we allow arbitrary mass and boundary mass. However, we have the following (in analogy to the Euclidean case, see [Fed69, 4.4.5]): If the closed set \( A \) is a Lipschitz \( r \)-neighborhood retract in the complete metric space \( X \), then there exists an isomorphism between \( H^F_n(X,A;G) \) and the group

\[
H^N_n(X,A;G) := \begin{cases} 
F \in Z^F_n(X,A;G) &| N(F) < \infty \\
F \in B^F_n(X,A;G) &| N(F) < \infty \end{cases}.
\]

This so since for \( R_0 + \partial R_1 \in Z^F_n(X,A;G) \) holds that \( R_0 \in R^c_n(X,G) \) and \( [R_0] = [R_0 + \partial R_1] \in H^F_n(X,A;G) \). Now we can argue similar as for equation (9.0.1) (slice, retract and add; note that this is possible for all normed groups \( G \), discrete or not) to get \( R \in R^c_n(X,G) \) with \( M(\partial R) < \infty \) and \( \text{spt}(R_0 - R) \subset A \) (hence \( [R] = [R_0] \in H^F_n(X,A;G) \)). Thus, there is a normal chain in every \( h \in H^F_n(X,A;G) \). It remains to show that if \( [A] = [B] \in H^F_n(X,A;G) \) for normal chains \( A \) and \( B \), then also \( [A] = [B] \in H^N_n(X,A;G) \). This follows since if \( A = B + r_0 + \partial r_1 \) for \( r_i \in R^c_{n+i}(X,G) \), then \( N(r_i) < \infty \). So we have \( [A] = [B] \in H^N_n(X,A;G) \) as well; clearly this gives then an isomorphism.

Remark 9.0.5. Similar arguments as in the remark above (again using (9.0.1), note that this is possible for all normed groups \( G \)) show that with the same assumptions as above, we have also an isomorphism from \( H^F_n(X,A;G) \) to

\[
H^R_n(X,A;G) := \frac{\{ R \in R^c_n(X,G) | \partial R \in R_{n-1}(A,G) \}}{\{ R \in B^F_n(X,A;G) \cap R^c_n(X,G) | \partial R \in R_{n-1}(A,G) \}}.
\]
Chapter 10
Subgroups

In this chapter we analyze what we can say about chains for subgroups \( H \subset G \). As a guiding line we think of \( \mathbb{Z} \subset \mathbb{R} \) and try to establish similar properties as for integer rectifiable currents as subset of the rectifiable currents.

Let \( H \subset G \) be a subgroup of the normed group \( G \). Let \( X \) be a complete metric space, consider it as subset of an injective metric space \( Y \). Then the rectifiable \( n \)-chains with values in \( H \) are

\[
\mathcal{R}_n(X, H \subset G) := \{ R \in \mathcal{R}_n(X, G) \mid R \in \mathcal{P}_n(Y, H)^M \}.
\]

Clearly, we find an equivalent description using the representation theorem (Theorem 4.5.1): \( R \in \mathcal{R}_n(X, H \subset G) \) if and only if there are \( f_i \in \text{Lip}(K_i, X) \), \( K_i \subset \mathbb{R}^n \) compact, \( f_i \) bi-Lipschitz and with pairwise disjoint images, and \( h_i \in L^1(K_i, \bar{H}) \) such that

\[
R = \sum_{i \in \mathbb{N}} f_i \# [h_i] \quad \text{and} \quad M(R) = \sum_{i \in \mathbb{N}} M(f_i \# [h_i]) < \infty.
\]

Thus, we find a map \( J : \mathcal{R}_n(X, H) \rightarrow \mathcal{R}_n(X, H \subset G) \) such that

\[
(\mathcal{R}_n(X, H), M) \quad \text{is isometrically isomorphic to} \quad (\mathcal{R}_n(X, H \subset G), M).
\]

Note that slicing respects the subgroup:

**Lemma 10.0.1.** Let \( R \in \mathcal{R}_n(X, G) \). Then \( R \in \mathcal{R}_n(X, H \subset G) \) if and only if for all \( f \in \text{Lip}(X, \mathbb{R}^n) \) holds: \( \langle R, f, r \rangle \in \mathcal{R}_0(X, H \subset G) \) for a.e. \( r \).
Proof. Clearly is a.e. slice in \( R_0(X, H \subset G) \) when \( R \in \mathcal{R}_n(X, H \subset G) \). On the other hand, let a.e. slice of \( R = \sum_{i \in \mathbb{N}} f_i \# [g_i] \) be in \( \mathcal{R}_0(X, H \subset G) \). If \( R \notin \mathcal{R}_n(X, H \subset G) \) there exists \( i \) such that \( g := g_i \notin L^1_{\#}(K_i, \bar{H}) \), i.e. there is \( \Omega \subset K_i \subset \mathbb{R}^n \) with \( \mathcal{L}^n(\Omega) > 0 \) and \( g(r) \notin \bar{H} \) for all \( r \in \Omega \). Now, \( R' := R|_{f_i(\Omega)} \in \mathcal{R}_n(X, G) \backslash \mathcal{R}_n(X, H \subset G) \), in particular \( R' \neq 0 \). Note that its slices are also in \( \mathcal{R}_0(X, H \subset G) \) a.e. as this holds for \( R \). Let \( f: X \to \mathbb{R}^n \) be a Lipschitz extension of \( f_{i^{-1}} \). Then

\[
\langle R', f, s \rangle = \langle f_i \# [g | \Omega], f, s \rangle = f_i \# ([g | \Omega], \text{id}_{\mathbb{R}^n}, s) = f_i \# [g | \Omega(s)] \quad \text{a.e.}
\]

Thus, \( \langle R', f, s \rangle \notin \mathcal{R}_0(X, H \subset G) \) for a.e. \( s \in \Omega \), in contradiction to the above. \( \square \)

Using Lemma 4.1.2 we see immediately that

\[ \mathcal{F}_n(X, H) \ni F \mapsto F \in \mathcal{F}_n(X, G) \]

is a 1-Lipschitz map. However, we do not know if this map is injective. In particular, we do not know if there is \( F \in \mathcal{F}_n(X, H) \setminus \mathcal{M}_n(X, H) \) with image in \( \mathcal{M}_n(X, G) \).

**Corollary 10.0.2** (Boundary rectifiability). Let \( H \subset G \) be a subgroup of the normed group \( G \), let \( X \) be a complete metric space. Assume that there exists no non-constant continuous path of finite length in \( \bar{H} \). Then

\[ R \in \{ R' \in \mathcal{R}_n(X, H \subset G) \mid N(R') < \infty \} \]

implies that \( \partial R \in \mathcal{R}_{n-1}(X, H \subset G) \).

Proof. Of course, we want to use the proof of Theorem 6.2.1; we have to make some adjustments to make sure that everything works in \( \bar{H} \):

First let \( n = 1 \). Note that for \( f \in \text{Lip}(X) \) and a.e. \( r \) holds that

\[ \chi((\partial R)|_{\{f < r\}}) = \chi(-\langle R, f, r \rangle) \in \bar{H} \].

By continuity follows that \( \chi((\partial R)|_U) \in \bar{H} \) for all open \( U \subset X \), and then this also holds for all subsets \( Y \subset X \). Now we can proceed as in the other proof: We assume that \( R \) does not satisfy the conclusion of the corollary. Then \( \partial R \neq 0 \). Also we can assume that \( \bar{F} := (\partial R)|_{X \setminus \bigcup_{i \in \mathbb{N}} \{x_i\}} \) is purely unrectifiable; note
that still \( \chi(\tilde{F}[Y]) \in \bar{H} \) for all \( Y \subset X \). As by our assumption \( \tilde{F} \neq 0 \), we can find a Borel set \( A \) such that
\[
F := \tilde{F}[A] \text{ has } \chi(F) \in \bar{H} \setminus 0
\]
(compare the proof of (v) in Theorem 4.7.1). Similarly as before, we find a Borel set \( B \) such that
\[
\mu_F(B) = M(F)/2 \text{ and now } \chi(F|_B) \in \bar{H}.
\]
Set \( F_1 := F|_B \) and \( F_2 := F|_{X \setminus B} \), define \( \gamma(1/2) := \chi(F_1) \in \bar{H} \) and proceed as in the other proof. Again, \( \gamma \) has a Lipschitz extension \( \gamma: [0,1] \to \bar{H} \) with \( \gamma(0) = 0 \) and \( \gamma(1) = \chi(F) \neq 0 \), in contradiction to our assumption.

The cases \( n > 1 \) reduce to the other case through Lemma 10.0.1.

\[ \square \]

**Corollary 10.0.3** (Closedness of \( \mathcal{R}_n(X,H \subset G) \)). Let \( X \) be a complete metric space, let \( Q \in \mathbb{R} \). Let \( G \) be a normed group and let \( H \subset G \) be a subgroup such that there exists no non-constant continuous path of finite length in \( \bar{H} \). Let \( R_i \in \mathcal{R}_n(X,H \subset G) \) with \( M(R_i) \leq Q \) be such that \( R_i \to F \in \mathcal{F}_n(X,G) \). Then \( F \in \mathcal{R}_n(X,H \subset G) \).

**Proof.** Clearly, \( F \in \mathcal{M}_n(X,G) \). If the corollary holds for \( n = 0 \) and any \( Q \) then a.e. slice of \( F \) is in \( \mathcal{R}_0(X,H \subset G) \); by Lemma 10.0.1 then \( F \in \mathcal{R}_n(X,H \subset G) \).

For \( n = 0 \), assume that \( F \in \mathcal{M}_0(X,G) \setminus \mathcal{R}_0(X,H \subset G) \). We have for any \( f \in \text{Lip}(X) \) that
\[
F|_{\{f<r\}} = \lim_{i \to \infty} R_i|_{\{f<r\}} \text{ a.e. } r,
\]
and hence \( \chi(F|_{\{f<r\}}) \in \bar{H} \) a.e. So again, \( \chi(F|_B) \in \bar{H} \) for any Borel set \( B \). Remarking that whenever \( x \in X \) is an atom of \( \mu_F \) (i.e. \( \mu_F(x) > 0 \)) then \( \chi(F|_{\{x\}}) \in \bar{H} \), we find an \( F' := F|_{X \setminus \bigcup_{i \in \mathbb{N}} \{x_i\}} \in \mathcal{M}_0(X,G) \setminus \mathcal{R}_0(X,H \subset G) \) without atoms and such that \( \chi(F'|_B) \in \bar{H} \) for all \( B \in \mathcal{B}_X \). Now we can proceed as in the proof of Corollary 10.0.2 to show that \( F' = 0 \), in contradiction to our assumption. \[ \square \]

As a consequence of the closedness we get:

**Corollary 10.0.4** (Compactness). Let \( X \) be a complete metric space and let \( C \subset X \) be compact; let \( Q \in \mathbb{R} \) and \( n \geq 0 \). Let \( G \) be a normed group. If \( H \subset G \) is a subgroup such that there exists no non-constant continuous path of finite length in the proper group \( \bar{H} \), then
\[
\{ R \in \mathcal{R}_n(X,H \subset G) \mid \mathcal{N}(R) \leq Q, \text{ spt } R \subset C \} \text{ is } \mathcal{F}-\text{compact}.
\]
Remark 10.0.5. We knew before that \( \{ R' \in \mathcal{R}_n(X, H) \mid N(R') \leq Q, \text{spt} R' \subset C \} \) is compact for the flat norm on \( \mathcal{F}_n(X, H) \). However, for \( R' \in \mathcal{R}_n(X, H) \), it is not clear that \( N(\mathcal{J}(R')) < \infty \) implies that \( N(R') < \infty \). Hence, the set considered in the corollary above is possibly bigger than this second set.

Similarly, we get the results of Chapter 8: Define \( \text{Fillvol}_X^H \) and \( \mathcal{F}_X^H \) analogously as before but using only elements of \( \mathcal{R}_j(X, H \subset G) \); the same for cone type inequalities. E.g. for \( R \in \mathcal{R}_n(X, H \subset G) \) is \( \mathcal{F}_X^H(R) := \inf \{ M(R_0) + M(R_1) \mid R = R_0 + \partial R_1 \in \mathcal{F}_n(X, G), R_i \in \mathcal{R}_{n+i}(X, H \subset G) \} \).

Note that by a discrete subgroup of a normed group we mean a subgroup that is itself a discrete normed group, in particular it is complete. Then we have:

**Corollary 10.0.6.** Let \( X \) be a complete quasiconvex metric space and \( R \in \mathcal{R}_0(X, H \subset G) \) where \( H \) is a discrete subgroup of the normed group \( G \). Let \( (R_i)_{i \in \mathbb{N}} \subset \mathcal{R}_0(X, H \subset G) \) be an \( M \)-bounded sequence. Then

\[
\mathcal{F}(R - R_i) \to 0 \quad \text{if and only if} \quad \text{Fillvol}_X^H(R - R_i) \to 0 .
\]

**Corollary 10.0.7.** Let \( X \) be a complete quasiconvex metric space and let \( H \) be a discrete subgroup of the normed group \( G \); let \( n \geq 1 \). Suppose that \( X \) admits local cone type inequalities for \( \mathcal{R}_j(X, H \subset G) \) for \( j = 1, \ldots, n \). Let \( (R_i)_{i \in \mathbb{N}} \subset \mathcal{R}_n(X, H \subset G) \) be an \( \mathbb{N} \)-bounded sequence and let \( R \in \mathcal{R}_n(X, H \subset G) \). Then

\[
\mathcal{F}(R - R_i) \to 0 \quad \text{if and only if} \quad \mathcal{F}_X^H(R - R_i) \to 0 .
\]

Moreover, if \( \partial R_i = 0 \) for all \( i \in \mathbb{N} \) then

\[
\mathcal{F}(R - R_i) \to 0 \quad \text{if and only if} \quad \text{Fillvol}_X^H(R - R_i) \to 0 .
\]

The proofs are as in Chapter 8.
Part II

Singular (Lipschitz) homology and homology of integral currents
Chapter 11

Introduction and results

Building on the theory of metric currents (see [AK00]), Stefan Wenger introduced in [Wen07] the homology of integral metric currents with compact support in complete metric spaces. As the axioms of Eilenberg–Steenrod for a homology theory are satisfied for the Lipschitz category, these homology groups are isomorphic to the singular (Lipschitz) homology groups on finite CW-complexes.

Here, we compare this homology of integral currents with the singular homology and the singular Lipschitz homology for integer coefficients. If local cycles can be filled locally, all three theories are identical (this is due to Christian Riedweg). In the special case of the Hawaiian Earring we show that the latter homology theories do not coincide with the first one (this was contributed by Daniel Schäppi).

Considering coefficients in a normed group $G$ we get analogously: The homology groups of flat $G$-chains (as in Chapter 9) are isomorphic to the (Lipschitz) homology groups with coefficients in $\overline{G}$ if the metric space is locally Lipschitz contractible.

A recent work in this direction is De Pauw’s [DP07]; there, various homology theories are compared.

11.1 Main results

We call a metric space $X$ locally Lipschitz contractible if for each point $x \in X$ there exists a neighborhood $U_x$ and $\gamma_x > 0$ such that every subset $S \subset U_x$
is $\gamma_x$-Lipschitz contractible (see page 84; we will also come back to this in Subsection 12.3.1).

Let $C_k(X;G)$ be the group of singular $k$-chains with coefficients in the abelian group $G$ and denote by $C^L_k(X;G)$ the subgroup of singular Lipschitz $k$-chains with coefficients in $G$. For $G = \mathbb{Z}$ we write $C^L_k(X)$ instead of $C^L_k(X;\mathbb{Z})$ and $C_k(X)$ for $C_k(X;\mathbb{Z})$.

Recall also the definition of integral currents with compact support from Section 2.2. Let $A \subset X$ be a closed subset of the complete metric space $X$. We write $H^IC_n(X,A)$ for the homology of the chain complex of integral currents with compact support of a metric space $X$ (relative $A$), $H^L_n(X,A)$ for the singular Lipschitz homology (relative $A$) and $H_n(X,A)$ for ordinary singular homology (relative $A$). Again, see Chapter 12 for the definitions. For $A = \emptyset$, we write $H^IC_n(X)$ for $H^IC_n(X,A)$ and so on.

First, we compare integral currents with compact support with Lipschitz chains. Every singular Lipschitz chain induces an integral current with compact support, this gives a homomorphism $[] : C^L_k(X) \to I^C_k(X)$ (see (12.2.1)). The result about the homology theories appears as a corollary of a more general fact (Proposition 13.1.1):

**Corollary 11.1.1.** If the complete metric space $X$ and the closed subset $A \subset X$ are both locally Lipschitz contractible, then the homology of integral Lipschitz chains (relative $A$) is isomorphic to the homology of integral currents with compact support (relative $A$):

$$H^L_k(X,A) \cong H^IC_k(X,A).$$

The isomorphism is induced by the map $[] : C^L_k(X) \to I^C_k(X)$.

Now we consider continuous and Lipschitz chains with coefficients in an abelian group $G$. Again, we will prove a more general fact (Proposition 13.2.1), from which follows:

**Corollary 11.1.2.** If the metric space $X$ and the subset $A \subset X$ are both locally Lipschitz contractible, then the homology of Lipschitz chains (relative $A$) is isomorphic to the singular homology (relative $A$):

$$H^L_k(X,A;G) \cong H_k(X,A;G).$$

The isomorphism is induced by the inclusion $C^L_k(X;G) \subset C_k(X;G)$.

As a consequence of the proofs, we get for the homology of flat $G$-chains (as defined in Chapter 9) and the singular (Lipschitz) homologies:
Corollary 11.1.3. Let the complete metric space $X$ be locally Lipschitz contractible, let the closed subset $A \subset X$ be a Lipschitz neighborhood retract. Then for any normed abelian group $G$ with completion $\bar{G}$:

$$H_k^F(X, A; G) \cong H_k^L(X, A; \bar{G}) \cong H_k(X, A; \bar{G}).$$

The Hawaiian Earring $\mathbb{H} \subset \mathbb{R}^2$ is given by the countable union of the circles

$$L_n := \{ x \in \mathbb{R}^2 \mid 1/n = |x - (1/n, 0)| \}$$

with radius $1/n$ and center $(1/n, 0)$, $n \in \mathbb{N}$. As metric on $\mathbb{H}$ we set

$$d(x, y) := \begin{cases} 
| x - y |, & \text{if } \exists n \in \mathbb{N} : x, y \in L_n, \\
|x| + |y|, & \text{otherwise}
\end{cases}$$

(we could as well take the length metric of $\mathbb{H}$). Note that any neighborhood of $(0, 0)$ in $\mathbb{H}$ contains all but finitely many of the $L_n$. Thus $\mathbb{H}$ is not locally contractible, and in particular not a CW-complex.

We show in Chapter 14 that the maximal divisible subgroup of $H_1^{IC}(\mathbb{H})$ is trivial whereas the maximal divisible subgroups of $H_1^L(\mathbb{H})$ and $H_1(\mathbb{H})$ are non-trivial. This implies that

**Theorem 11.1.4.** $H_1^{IC}(\mathbb{H})$ is isomorphic neither to $H_1^L(\mathbb{H})$ nor to $H_1(\mathbb{H})$. 
Chapter 12

Definitions

Here we define the mentioned homology theories and explain basic properties of singular (Lipschitz) chains, respectively of currents. Local cones are defined and their relation to Lipschitz contractibility is discussed.

12.1 Currents

Let \( X \) be a complete metric space. Recall definitions and properties of integral currents from Section 2.2.

Let \( A \subset X \) be a closed subset, we define (see [Wen07, p. 159])

\[
\mathcal{Z}_{k}^{IC}(X, A) := \{ T \in \mathcal{I}_{k}^{c}(X) \mid \partial T \in \mathcal{I}_{k-1}^{c}(A) \} \\
\mathcal{B}_{k}^{IC}(X, A) := \{ R + \partial S \mid R \in \mathcal{I}_{k}^{c}(A), S \in \mathcal{I}_{k+1}^{c}(X) \}.
\]

The \( k \)-th homology group of integral currents with compact support (relative \( A \)) is defined by

\[
H_{k}^{IC}(X, A) := \mathcal{Z}_{k}^{IC}(X, A)/\mathcal{B}_{k}^{IC}(X, A).
\]

If \( A = \emptyset \) we write \( H_{k}^{IC}(X) \).

12.2 Singular (Lipschitz) chains

Let

\[
\Delta_{k} := \left\{ (s_{0}, \ldots, s_{k}) \in \mathbb{R}^{k+1} \mid \sum_{j=0}^{k} s_{j} = 1 \text{ and } 0 \leq s_{j}, \forall j \right\} \subset \mathbb{R}^{k+1}
\]
be the standard $k$-simplex. Sometimes it is more convenient to consider $\Delta^k$ as a subset of $\mathbb{R}^k$; for this we choose an isometry $\phi : \mathbb{R}^k \to \{ s \in \mathbb{R}^{k+1} \mid \sum_{j=0}^k s_j = 1 \}$ and define $\tilde{\Delta}^k = \phi^{-1}(\Delta^k)$.

Let $X'$ be a metric space, let $G$ be an abelian group.

**Definition 12.2.1.** A singular $k$-simplex $c$ is a continuous map $c : \Delta^k \to X'$.

The group of singular $k$-chains is the free abelian group on the $k$-simplices; we denote this group by $C_k(X')$.

The group of singular Lipschitz $k$-chains is the free abelian group on the $k$-simplices that are Lipschitz; it is denoted by $C^L_k(X')$.

The group of singular $k$-chains over $G$ is $C_k(X'; G) := G \otimes \mathbb{Z} C_k(X')$ and the group of singular Lipschitz $k$-chains over $G$ is $C^L_k(X'; G) := G \otimes \mathbb{Z} C^L_k(X')$.

Again we denote elements of such groups by finite formal sums. We use mostly $G = \mathbb{Z}$ and speak then of integral (Lipschitz) $k$-chains.

Let $[e_0, e_1, \ldots, e_k]$ be the vertices of the standard $k$-simplex for $k \geq 1$, then the boundary of a singular (Lipschitz) $k$-simplex $c$ is the (Lipschitz) $(k - 1)$-chain

$$bc := \sum_{j=0}^k (-1)^j c_{[e_0, \ldots, \hat{e}_j, \ldots, e_k]}$$

where $\hat{e}_j$ means that $e_j$ is omitted. For a $k$-chain $c = \sum_{i=1}^m g_i c_i$ we set $bc := \sum_{i=1}^m g_i bc_i$, and we define $bc := 0$ for 0-chains. Then we get homomorphisms $b : C_k^L(X'; G) \to C_{k-1}^L(X'; G)$ such that $b(bc) = 0$.

For a subset $A \subset X'$ let

$$Z_k^L(X', A; G) := \{ c \in C_k^L(X'; G) \mid bc \in C_{k-1}^L(A; G) \}$$

$$B_k^L(X', A; G) := \{ c + b\tilde{c} \mid c \in C_k^L(A; G), \tilde{c} \in C_{k-1}^L(X'; G) \}$$

and the $k$-th singular (Lipschitz) homology group with coefficients in $G$ (relative $A$) is

$$H_k^L(X', A; G) := Z_k^L(X', A; G) / B_k^L(X', A; G) .$$

(These relative homology groups are easily seen to be isomorphic to the homology groups of the chain complex $K^L_m := C_m^L(X; G) / C_m^L(A; G)$ for $m \geq 0$, and $K_m := \{ 0 \}$ otherwise, with the induced boundary map.)

If $G = \mathbb{Z}$, we write $Z_k^L(X', A)$, $B_k^L(X', A)$ and $H_k^L(X', A)$. If $A = \emptyset$ we write $H_k^L(X'; G)$, and so for $A = \emptyset$ and $G = \mathbb{Z}$ we write $H_k^L(X')$. 
There is a natural comparison homomorphism $H_n^L(X, A; G) \to H_n(X, A; G)$ which is induced by the inclusions $C_k^L(S; G) \to C_n(S; G)$ for $S \in \{A, X\}$.

For the complete metric space $X$, let $c = \sum_{i=1}^m n_i c_i \in C_k^L(X)$ be an integral Lipschitz $k$-chain, so $n_i \in \mathbb{Z}$. Then $c$ induces an integral current $[c]$ with compact support defined by

$$[c](f, \pi_1, \ldots, \pi_k) := \sum_{i=1}^m n_i \int_{\Delta^k \subset \mathbb{R}^k} f \circ \tilde{c}_i \det\left(\frac{\partial(\pi_j \circ \tilde{c}_i)}{\partial s_l}\right) d\mathcal{L}^k(s), \quad (12.2.1)$$

where $\tilde{c}_i = c_i \circ \phi$. The maps $[\cdot]: C_k^L(X) \to \mathbf{I}_k^L(X)$ are homomorphisms. By Stokes' theorem we get $[bc] = \partial[c]$, so $[\cdot]$ is a chain map for these complexes. We also refer to the induced maps between the homology groups of the respective complexes as comparison maps.

For $c = \sum_{i=1}^m g_i c_i \in C_k(X'; G)$ with $g_i \neq 0$ and $c_i \neq c_j$ for $i \neq j$ we define $\text{im}(c) := \bigcup_{i=1}^m \text{im}(c_i)$ (this is well-defined since $C_k(X'; G)$ is free over the $k$-simplices, so the representation of $c$ as sum of different simplices is unique). The diameter of a chain is defined as the diameter of its image:

$$\text{diam}(c) := \text{diam}(\text{im}(c)).$$

The push-forward of $c \in C_k(X'; G)$ to the metric space $Y$ by the continuous map $h: X' \to Y$ is the chain $h_\# c := \sum_{i=1}^m g_i (h \circ c_i) \in C_k(Y; G)$. Again, $b(h_\# c) = h_\#(bc)$. If $c \in C_k^L(X'; G)$ and $h: X' \to Y$ is Lipschitz then $h_\# c \in C_k^L(Y; G)$; if $X$ and $Y$ are complete and $h: X \to Y$ is Lipschitz, $[h_\# c] = h_\# [c]$ for all $c \in C_k^L(X)$.

In order to get smaller simplices (i.e. simplices with smaller diameter) we use barycentric subdivision. This standard construction can be found in [Hat02] or [Mun84]. We only list the facts that we need later. Let $m$ be given and let $sd^m(c)$ denote the singular (Lipschitz) $k$-chain resulting from the $m$-th barycentric subdivision of the singular (Lipschitz) $k$-chain $c$, then:

(i) For $k \geq 0$ there is a homomorphism $D_{m, X'}: C_k^L(X'; G) \to C_{k+1}^L(X'; G)$ such that for each $k$-chain $c$

$$b(D_{m, X'}(c)) + D_{m, X'}(bc) = sd^m(c) - c. \quad (12.2.2)$$

Furthermore, $D_{m, X'}$ is natural: For $f: X' \to Y$ continuous (respectively Lipschitz) is $f_\# \circ D_{m, X'} = D_{m, Y} \circ f_\#$.

(ii) Applying iterated barycentric subdivision, we can get arbitrary small diameter of the image of the resulting simplices.
(iii) For $c \in C^L_k(X')$ holds $[sd^m(c)] = [c]$.

(iv) $b(sd^m(c)) = sd^m(b(c))$.

The augmentation is the map $\chi : C^{(L)}_0(X';G) \to G$ defined by

$$\chi \left( \sum_{i=1}^m g_i c_i \right) := \sum_{i=1}^m g_i . \tag{12.2.3}$$

Note that $\chi(bc') = 0$ for all $c' \in C^{(L)}_1(X';G)$. Moreover, $\chi(h\#c) = \chi(c)$.

**Definition 12.2.2.** Let $c = \sum_{i=1}^m g_i c_i \in C^{(L)}_n(X;G)$. By a part of $c$ we mean a chain $c' = \sum_{i=1}^m g'_i c'_i$ such that $c'_i = c_i$ and either $g'_i = g_i$ or $g'_i = 0$.

**Remark 12.2.3.** Let $c \in C^{(L)}_n(X;G)$ as above, let $n \geq 1$ and $U \subset X$. For $\epsilon > 0$ there exist a chain $\bar{c} \in C^{(L)}_n(U_\epsilon(U);G)$ that is a part of $sd^m(c)$ for some $m \in \mathbb{N}$ and such that $\text{im}(sd^m(c) - \bar{c}) \subset X \setminus U$. (In particular, for $G = \mathbb{Z}$ and $U$ open is $	ext{spt}[c - \bar{c}] \subset X \setminus U$, thus $[c]|_U = [\bar{c}]|_U$. To see this, it is enough to consider a singular $n$-simplex $c$. Let $m \in \mathbb{N}$ be such that $sd^m(c) = \sum_{i=1}^k g'_i c'_i$ with $\text{diam}(c'_i) < \epsilon$ for all $i$. Set

$$\bar{c} := \sum_{i=1}^k h_i c'_i \quad \text{where} \quad h_i := \begin{cases} g'_i & \text{if } \text{im}(c'_i) \cap U \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

This remark will be used in the proofs e.g. to construct a simplicial boundary close to a slice of a simplicial cycle with a filling of the difference: Let $G = \mathbb{Z}$. Let $bc = 0$ for $c \in C^{(L)}_n(X)$, and let $U = \{d < r\}$ for a Lipschitz function $d : X \to \mathbb{R}$ and $r \in \mathbb{R}$. Then, for $\bar{c}$ as in the remark above, with $[c]|_U = [\bar{c}] - [\bar{c}]|_{X \setminus U}$ we have $\langle [c], d, r \rangle = \partial([c]|_U) = [b\bar{c}] - \partial([\bar{c}]|_{X \setminus U})$ a.e. Thus,

$$\langle [c], d, r \rangle = [b\bar{c}] - \partial([\bar{c}]|_{X \setminus U}) . \tag{12.2.4}$$

### 12.3 Cone inequalities

Here we give the definitions of miscellaneous cone inequalities. These inequalities are used later on (see propositions 13.1.1 and 13.2.1). In Proposition 12.3.3 we show that spaces which are locally Lipschitz contractible are examples of spaces admitting such cone inequalities.
12.3. CONE INEQUALITIES

Definition 12.3.1. Let \( k \geq 1 \) and let

\[ \mathcal{E} := \{ F : \mathbb{R} \rightarrow \mathbb{R} \mid F \text{ is continuous and non-decreasing with } F(0) = 0 \} . \]

- \( X \) admits a local cone inequality for \( \mathbf{I}_k^L(X) \) if for every \( x \in X \) there exists \( r_x > 0 \) and \( F_x \in \mathcal{E} \) such that for every \( T \in \mathbf{I}_k^L(X) \) with \( \partial T = 0 \) and \( \text{spt}(T) \subset U_{r_x}(x) \) there exists a \( \bar{T} \in \mathbf{I}_{k+1}^L(X) \) satisfying
  \[ \partial \bar{T} = T \quad \text{and} \quad \text{diam}(\bar{T}) \leq F_x(\text{diam}(T)). \]

- \( X \) admits a local cone inequality for \( C^L_k(X; G) \) if for every \( x \in X \) there exists \( r_x > 0 \) and \( F_x \in \mathcal{E} \) such that for every \( c \in C^L_k(X; G) \) with \( bc = 0 \) and \( \text{im}(c) \subset U_{r_x}(x) \) there exists a \( \bar{c} \in C^L_{k+1}(X; G) \) satisfying \( b\bar{c} = c \) and \( \text{diam}(\bar{c}) \leq F_x(\text{diam}(c)) \).

- \( X \) admits a local cone inequality for \( C^L_0(X; G) \) if for every \( x \in X \) there exists \( r_x > 0 \) and \( F_x \in \mathcal{E} \) such that for every \( c \in C^L_0(X; G) \) with \( \chi(c) = 0 \) and \( \text{im}(c) \subset U_{r_x}(x) \) there exists an \( \bar{c} \in C^L_1(X; G) \) satisfying \( b\bar{c} = c \) and \( \text{diam}(\bar{c}) \leq F_x(\text{diam}(c)) \).

Note that this definition of cone type inequalities for currents is different from the one used in [Wen07]; there one has the condition that locally there exists a filling with controlled mass whereas we consider only compactly supported currents and our condition is that locally there exists a filling with controlled diameter.

The following lemma is from [Mun84, Lemma 30.6] (therein it is stated for the singular theory in topological spaces, see below).

Lemma 12.3.2. There exists, for each metric space \( X \), each abelian group \( G \) and each non-negative integer \( k \), a homomorphism

\[ K_X : C^L_k(X; G) \rightarrow C^L_{k+1}([0,1] \times X; G) \]

having the following property: If \( c \in C^L_k(X; G) \) is a singular simplex, then

\[ bK_X(c) + K_X(bc) = j_\#(c) - i_\#(c). \]  \hspace{1cm} (12.3.1)

Here the map \( i : X \rightarrow [0,1] \times X \) carries \( x \) to \((0, x)\) and the map \( j : X \rightarrow [0,1] \times X \) carries \( x \) to \((1, x)\).
We only adumbrate the proof (carried out in [Mun84] for the continuous case) to indicate that this holds for Lipschitz chains too: One wants to look at $[0, 1] \times c$ as a chain in $C_{k+1}^L([0, 1] \times X; G)$ whenever $c$ is in $C_k^L(X; G)$. To do this, one first gives a decomposition of $[0, 1] \times \Delta^k$ into an integral $(k+1)$-chain consisting of (regular) simplices in $R^{k+2}$. Then one carries this decomposition over to $[0, 1] \times X$ for every simplex in an intuitive way, producing a chain $\bar{c}$ in $C_{k+1}^L([0, 1] \times \text{im}(c); G)$. Clearly, this construction respects the Lipschitz continuity.

12.3.1 Lipschitz contractions and cone inequalities

Examples of spaces that are locally Lipschitz contractible are Banach spaces or CAT($\kappa$)-spaces for $\kappa \in R$. This is discussed on pp. 146 and 147 in [Wen07]. As well from there follows the last statement of the following proposition.

**Proposition 12.3.3.** Let $X$ be a locally Lipschitz contractible metric space. Then $X$ admits local cone inequalities for $C_k^L(X; G)$ and $C_j(X; G)$ for all $j \geq 0$.

If, in addition, $X$ is complete then it also admits local cone inequalities for $I_k^c(X)$, $k \geq 1$.

**Proof.** Denote by $y^n$ the constant $n$-simplex with image $y \in X$. Note that $by^n = 0$ if $n$ is odd or zero and $by^n = y^{n-1}$ if $n$ is even and positive.

Let $x \in X$, let $r_x, \gamma_x > 0$ be such that every subset of $B_{r_x}(x)$ is $\gamma_x$-Lipschitz contractible. Note that then for any $S \subset B_{r_x}(x)$ with $\gamma_x$-contraction $\phi$ holds

$$\text{diam}(\phi([0, 1] \times S)) \leq 2\gamma_x \text{diam}(S).$$

(12.3.2)

Now we use the homomorphism $K := K_X : C_k^L(X; G) \rightarrow C_{k+1}^L([0, 1] \times X; G)$ from Lemma 12.3.2. Let $c \in C_k^L(X; G)$ with im($c$) $\subset B_{r_x}(x)$. Let $\phi : [0, 1] \times \text{im}(c) \rightarrow X$ be a $\gamma_x$ contraction and let $\phi(0, x) = x_0$. The push-forward of $K(c)$ by $\phi$ is clearly in $C_{k+1}^L(X; G)$. Let $bc = 0$ (respectively $\chi(c) = 0$ for $n = 0$); then we have $K(bc) = 0$, and for $c = \sum_{i=1}^m g_i c_i$ we have by (12.3.1)

$$b(\phi#K(c)) = c - \sum_{i=1}^m g_i x_0^k.$$  

Since $bc = 0$ we have $b(\sum_{i=1}^m g_i x_0^k) = 0$ (resp. $\chi(c) = \sum_{i=1}^m g_i = 0$ for $n = 0$). Thus, for $k$ even holds $\sum_{i=1}^m g_i = 0$; so $\bar{c} := \phi#K(c)$ is a filling of $c$ in this case.
If $k$ is odd, a filling of $c$ is given by

$$\bar{c} := \phi_# K(c) + \sum_{i=1}^{m} g_i x_0^{k+1}.$$ 

For all these fillings is by (12.3.2)

$$\text{diam}(\bar{c}) = \text{diam}(\phi_# K(c)) \leq 2\gamma_x \text{diam}(c).$$

Note moreover that, if $c$ is a Lipschitz chain, so is $\bar{c}$.

Now let $X$ be complete. If $T \in \mathbf{I}_k(X)$ has $\text{spt}(T) \subset B_{r_x}(x)$ and $\partial T = 0$, we get a filling $\bar{T} := \bar{\phi}_# ([0,1] \times T)$ in $\mathbf{I}_{k+1}(X)$ with $\text{spt}(\bar{T}) \subset \text{im}(\phi)$, i.e. $\text{diam}(\bar{T}) \leq 2\gamma \text{diam}(\text{spt}(T))$. If in addition $T \in \mathbf{I}_c^{\circ}(X)$ then $[0,1] \times T \in \mathbf{I}_{c+1}^{\circ}([0,1] \times X)$ and therefore $\text{spt}(\bar{T})$ is compact.

Concluding we see that if $X$ is locally Lipschitz contractible then $X$ admits local cone inequalities for $\mathbf{I}_c^k(X)$, $k \geq 1$, as well as for $\mathbf{C}_j^{(L)}(X;G)$, $j \geq 0$, with $F_x(t) := 2\gamma_x t$. \qed
Chapter 13

Cones and homologies

Here we give the proofs of the Corollaries 11.1.1, 11.1.2 and 11.1.3. This is done by showing more general results, e.g., Proposition 13.1.1 yields for a given integral current $T$ an integral Lipschitz chain $\sum c_i$ which is homologous to $T$ and 'lives close to $T$'.

13.1 $HIC^n(X, A)$ and $H^L_n(X, A)$

Proposition 13.1.1. Let $X$ be a complete metric space. Then for $T \in I_0(X)$ there exists $c \in C^{(L)}_0(X)$ with $T = [c]$.

Suppose the complete metric space $X$ admits local cone inequalities for $I_j(X)$ and $C^L_k(X)$ for $j = 1,\ldots,n$ and $k = 0,\ldots,n-1$; let $\epsilon > 0$. Given $T \in I_n(X)$ with $\partial T = [c]$ for $c \in C^L_{n-1}(X)$ and $bc = 0$, there exist $N \in \mathbb{N} \cup \{0\}$, $T_1,\ldots,T_N \in I_n(X)$, $c_1,\ldots,c_N \in C^L_n(X)$ and $V_1,\ldots,V_N \in I_{n+1}(X)$ such that

(i) $\sum_{i=1}^N T_i = T$ and $\partial V_i = T_i - [c_i]$

(ii) $\exists m \in \mathbb{N} \cup \{0\}$: $b(\sum_{i=1}^N c_i) = sd^m(c)$.

(iii) $\text{spt}(V_i) \cup \text{im}(c_i) \subset U_\epsilon(\text{spt}(T) \cup \text{im}(c))$ and $\text{diam}(V_i) < \epsilon$.

Note that by (i) holds: $\partial \sum_{i=1}^N V_i = T - [\sum_{i=1}^N c_i]$.

Proof of Proposition 13.1.1. We argue inductively on the dimension of the current; for the induction step we use another induction on the number of balls needed to cover the compact set $\text{spt}(T) \cup \text{im}(c)$.  

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Let $n = 0$. Then $T = \sum_{i=1}^{m} n_i[q_i]$ for some $m \in \mathbb{N}$, $n_i \in \mathbb{Z}$, $q_i \in X$ and $c := \sum_{i=1}^{m} n_i q_i \in C_0^L(X)$.

Now let $n > 0$. For $x \in X$ we find $r_x > 0$ and a function $F_x \in \mathcal{E}$ such that all the cone inequalities in the proposition hold around $x$ for $r_x$ and $F_x$ (i.e., let $r_x > 0$ denote the minimum of all radii in the (finitely many) cone inequalities for $x$ and let $F_x$ the maximum function of all those diameter functions for $x$). We can assume that $F_x(r) \geq r$. Choose $0 < R_x$ such that

$$R_x + F_x(2R_x + 2F_x(2R_x)) < \min\{\varepsilon/2, r_x/2\}.$$  

Cover $\text{spt}(T) \cup \text{im}(c)$ by balls $U_{R_i}(x)$, $x \in X$. We get a minimal finite subcover with centers $x_1, \ldots, x_M$; set $R_i := R_{x_i}$ and $F_i := F_{x_i}$. We show inductively that there are $V_i$, $c_i$ and $T_i$, $i = 1, \ldots, M$, with properties (i) and (ii) of Proposition 13.1.1 that fulfill moreover

$$\text{spt}(V_i) \cup \text{im}(c_i) \subset B_{\frac{R}{2}}(x_i); \quad (13.1.1)$$

this clearly implies (iii).

If $M = 1$ the cone inequality for $C_{n-1}^L(X)$ gives a filling $c_1 \in C_n^L(X)$ of $c$ (for $n = 1$, $C_0(X) \ni c = \sum n_i q_i$ necessarily has $\chi(c) = \sum n_i = T(1,1) = 0$). We have now $\text{im}(c_1) \subset B_{R_1+F_1(2R_1)}(x_1)$. Set $T_1 := T$; now the cone inequality for $\mathbf{I}_n^c(X)$ gives a filling $V_1 \in \mathbf{I}_{n+1}^c(X)$ of $T - [c_1]$ with support in $B_{\frac{R}{2}}(x_1)$.

For $M > 1$ let $0 < R < R_M$ be such that

$$\text{spt}(T) \cup \text{im}(c) \subset U_{R}(x_M) \cup \bigcup_{i=1}^{M-1} U_{R_i}(x_i).$$

We find $a, r > 0$ such that $R < r - a < r + a < R_M$. We slice $T$ by $d(x) := d(x_M, x)$ at $r$. We can assume that $\langle T, d, r \rangle \in \mathbf{I}_n^c(X)$ is defined, and that $\partial \langle T, d, r \rangle = -\langle \partial T, d, r \rangle \in \mathbf{I}_{n-2}^c(X)$ for $n > 1$, respectively that $\|\partial T\| (d^{-1}(r)) = 0$ if $n = 1$. Moreover, we can assume that $T[\{d > r\}] \in \mathbf{I}_n^c(X)$.

For $n = 1$ set directly $S := T[\{d > r\}]$; this is an integral 1-current with compact support. As above we see that $\partial S = \sum n_i[q_i]$ with $\sum n_i = 0$. By induction there are $c_1, \ldots, c_{M-1}, T_1, \ldots, T_{M-1}$ and $V_1, \ldots, V_{M-1}$ with $b \sum c_i = \sum n_i q_i$, property (i) of Proposition 13.1.1 for $S$ instead of $T$ and with (13.1.1). Now, $T_M := T - S$ has $\text{spt}(T_M) \subset B_{R_M}(x_M)$ and boundary $\partial(T_M) = \partial T - \partial S =: \sum n'_j[p_j]$. Again, $\sum n'_j = 0$, so there exists

$$c_M \in C_1^L(B_{R_M+F_M(2R_M)}(x_M))$$
and a filling $V_M$ of $T_M - [c_M]$ with
\[ \text{spt}(V_M) \cup \text{im}(c_M) \subset B_{\frac{\epsilon}{2}}(x_M) \]
(as $R_M + F_M(2R_M + 2F_M(2R_M)) \leq \epsilon/2$), proving the case $n = 1$ (clearly, also (ii) is satisfied).

If $n > 1$, choose $\epsilon' \in (0, \alpha/2)$. We can apply Remark 12.2.3 for $c$, $\epsilon'$ and the slice by $d$ at $r$ to get $\bar{c} \in C_{n-1}^L(B_{r+\epsilon'}(x_M))$ and $m_1 \in \mathbb{N}$ with
\[ \text{im}(sd^{m_1}(c) - \bar{c}) \subset X \setminus B_r(x_M) \] (13.1.2)
(due to the slicing we may have to vary $r$ a little bit, but we easily find a good $r' \in (\bar{R} + \alpha, \bar{R} + 3\alpha)$).

Now, $T' := \langle T, d, r \rangle - [\bar{c}]_{\{d>r\}}$ has by (12.2.4)
\[ \partial T' = -\langle \partial T, d, r \rangle - \partial([\bar{c}]_{\{d>r\}}) = -[b\bar{c}] ; \]
in particular is $T' \in I_{n-1}^c(X)$. By induction assumption (for the dimension), for $T'$ and $\epsilon'$ and $c' := -b\bar{c}$ there exist $T'_1, \ldots, T'_K \in I_{n-1}^c(X)$, $V'_1, \ldots, V'_K \in I_n^c(X)$, $c'_1, \ldots, c'_K \in C_{n-1}^L(X)$ and $m_2 \in \mathbb{N} \cup \{0\}$ with
\[ \partial \sum_{i=1}^K V'_i = T' - \sum_{i=1}^K [c'_i], \quad sd^{m_2}(-b\bar{c}) = b \sum_{i=1}^K c'_i \] (13.1.3)
and
\[ \text{im} \left( \sum_{i=1}^K c'_i \right) \cup \text{spt} \left( \sum_{i=1}^K V'_i \right) \subset B_{\epsilon'}(\text{spt}(T') \cup \text{im}(b\bar{c})) \subset B_{R_M}(x_M) \setminus B_{\bar{R}}(x_M). \]

So we have a compactly supported integral $n$-current
\[ S := T|_{\{d>r\}} + \sum_{i=1}^K V'_i \]
with $\text{spt}(S) \subset \bigcup_{i=1}^{M-1} B_{R_i}(x_i)$. The boundary is by (13.1.2)
\[ \partial S = (\partial T)|_{\{d>r\}} - [\bar{c}]_{\{d>r\}} - \sum_{i=1}^K [c'_i] = [c - \bar{c}]_{\{d>r\}} - \sum_{i=1}^K [c'_i] \]
\[ = \left[ sd^{m_1}(c) - \bar{c} - \sum_{i=1}^K c'_i \right]. \]
By construction are $\text{im}(sd^{m_1}(c) - \bar{c})$ and $\text{im}(\sum c'_i)$ subsets of $\bigcup_{i=1}^{M-1} B_{R_i}(x_i)$, and for

$$z := sd^{m_2}(sd^{m_1}(c) - \bar{c}) - \sum_{i=1}^{K} c'_i$$

we have by (13.1.3) that $bz = sd^{m_1+m_2}(bc) = 0$.

Inductively (on the number of balls), for $S$ exist $T_1, \ldots, T_{M-1}, c_1, \ldots, c_{M-1}, V_1, \ldots, V_{M-1}$ and $m_3$ with properties (i) and (ii) of Proposition 13.1.1 (for $S$ instead of $T$) and (13.1.1). Set $T_M := T - S$, then

$$\partial T_M = [c] - [z] = \left[ sd^{m_1+m_2+m_3}(c) - sd^{m_3}\left(sd^{m_1}(c) - \bar{c}\right) - \sum_{i=1}^{K} c'_i \right]$$

$$= \left[ sd^{m_2+m_3}(\bar{c}) + sd^{m_3}\left(\sum_{i=1}^{K} c'_i \right) \right].$$

With

$$c'' := sd^{m_2+m_3}(\bar{c}) + sd^{m_3}\left(\sum_{i=1}^{K} c'_i \right),$$

we have $\text{im}(c'') \subset B_{R_M}(x_M)$ and $bc'' = sd^{m_3}(sd^{m_2}(bc) + b\sum_{i=1}^{K} c'_i) = 0$ (by (13.1.3)). So by the cone inequalities there exist fillings $c_M \in C^L_n(X)$ of $c''$ and $V_M \in I^c_{n+1}(X)$ of $T - S - [c_M]$ with image and support in $B_{\frac{1}{2}}(x_M)$. Finally,

$$b\left(\sum_{i=1}^{M} c_i \right) = sd^{m_3}(z) + c'' = sd^{m}(c),$$

i.e. (ii) for $m = m_1 + m_2 + m_3$.

\section*{13.1.1 Proof of Corollary 11.1.1}

By Proposition 12.3.3 we can use Proposition 13.1.1 to prove Corollary 11.1.1. We show that the chain homomorphism $[\cdot] : C^L_n(X) \to I^c_n(X)$ induces an isomorphism $[\cdot] : H^L_n(X, A) \to H^I_n(X, A)$ for all $n \geq 0$. Note that $[\cdot]$ sends $Z^L_n(X, A)$ to $Z^I_n(X, A)$ and $B^L_n(X, A)$ to $B^I_n(X, A)$, so $[\cdot]$ induces a homomorphism from $H^L_n(X, A)$ to $H^I_n(X, A)$.

The case $n = 0$ is easier, so assume that $n \geq 1$.

Let $T \in Z^I_n(X, A)$, as $\partial T \in I^c_{n-1}(A)$ and $\partial(\partial T) = [0]$ there exists $c \in C^L_n(A)$ with $bc = sd^n(0) = 0$ (respectively $\chi(c) = 0$ for $n = 1$) and $V \in I^c_n(A)$
with $\partial V = \partial T - [c]$. Now, $T - V$ has boundary $\partial(T - V) = [c]$. So there exists $\bar{c} \in C^L_n(X)$ with $b\bar{c} = sd^m(c)$ (thus, $\bar{c} \in Z^L_n(X, A)$) and there exists $V \in I^c_{n+1}(X)$ with $\partial V = (T - V) - [\bar{c}]$. Hence $V + \partial V \in B^L_{n+1}(X, A)$ and

$$[\bar{c}] + V + \partial V = T,$$

showing the surjectivity of the homomorphism.

Let $c, \bar{c} \in Z^L_n(X, A)$ with $[c] + B^L_{n+1}(X, A) = [\bar{c}] + B^L_{n+1}(X, A)$, i.e. there exists $R + \partial S \in B^L_{n+1}(X, A)$:

$$[c] + R + \partial S = [\bar{c}]. \quad (13.1.4)$$

Then $\partial R = [b(\bar{c} - c)]$ and $b(\bar{c} - c) \in C^L_{n-1}(A)$ with $b(b(\bar{c} - c)) = 0$. So there exists $c_1 \in C^L_n(A)$, $V \in I^c_{n+1}(A)$ such that

$$\partial V = R - [c_1] \quad \text{and} \quad bc_1 = sd^m(b(\bar{c} - c)). \quad (13.1.5)$$

Now, by (12.2.2) we get

$$b(D_{m_1, X}(\bar{c} - c)) + D_{m_1, X}(b(\bar{c} - c)) = sd^m(\bar{c} - c) - \bar{c} + c. \quad (13.1.6)$$

Set $c_2 := D_{m_1, X}(\bar{c} - c) \in C^L_{n+1}(X)$.

Note that by naturality

$$c_3 := D_{m_1, X}(b(\bar{c} - c)) = D_{m_1, A}(b(\bar{c} - c)) \in C^L_n(A). \quad (13.1.7)$$

On the other hand, by (13.1.4) and (13.1.5)

$$\partial(S + V) = [\bar{c} - c] - [c_1] = [sd^m(\bar{c} - c) - c_1].$$

Set $c_4 := sd^m(\bar{c} - c) - c_1 \in C^L_n(X)$; then $bc_4 = 0$. So there is a filling $c_5 \in C^L_{n+1}(X)$ with $bc_5 = sd^m(c_4)$. Thus, $c_6 := D_{m_2, X}(c_4) \in C^L_{n+1}(X)$ has

$$bc_6 = sd^m(c_4) - c_4 = bc_5 - sd^m(\bar{c} - c) + c_1, \quad (13.1.7)$$

and so $sd^m(\bar{c} - c) = c_1 + b(c_5 - c_6)$. Now, (13.1.6) and the definitions give

$$\bar{c} - c = c_1 - c_3 + b(-c_2 + c_5 - c_6).$$

As $c_1 - c_3 \in C^L_n(A)$ and $c_5 - c_2 - c_6 \in C^L_{n+1}(X)$, we see that the homomorphism is injective. This ends the proof.
13.2 \( H^L_n(X, A; G) \) and \( H_n(X, A; G) \)

**Proposition 13.2.1.** Let \( X \) be a metric space. Then \( C_0(X; G) = C^L_0(X; G) \).

Suppose the metric space \( X \) admits local cone inequalities for \( C_k(X; G) \) and \( C^L_j(X; G) \) for \( k = 0, \ldots, n \) and \( j = 0, \ldots, n - 1 \); let \( \epsilon > 0 \). Given \( c \in C_n(X; G) \) with \( bc \in C^L_{n-1}(X; G) \) there exist \( N \in \mathbb{N} \), \( m \in \mathbb{N} \cup \{0\} \), \( c_1, \ldots, c_N \in C_n(X; G) \), \( c^L_1, \ldots, c^L_N \in C^L_n(X; G) \) and \( \bar{c}_1, \ldots, \bar{c}_N \in C_{n+1}(X; G) \) such that

\[
\begin{align*}
(i) & \quad \sum_{i=1}^N c_i = sd^m(c) \text{ and } bc_i = c_i - c^L_i \\
(ii) & \quad \text{im}(\bar{c}_i) \subset B_\epsilon(\text{im}(c)) \text{ and } \text{diam}(\bar{c}_i) < \epsilon.
\end{align*}
\]

**Proof.** The idea of the proof of this proposition is essentially the same in the proof of Proposition 13.1.1, when slices are exchanged by boundaries of appropriate parts of a chain, compare Remark 12.2.3:

Clearly, \( C_0(X; G) = C^L_0(X; G) \). Let now \( n > 0 \) and assume that the proposition holds for \( n - 1 \); we give the proof below for \( n > 1 \), the case \( n = 1 \) is analogous.

As in the proof of Proposition 13.1.1, we argue by induction on the number (say \( M \)) of small balls \( U_{r_i}(x_i) \) needed to cover \( \text{im}(c) \) where \( r_i > 0 \) are small enough: We show by induction that there are \( c_i, c^L_i \) and \( \bar{c}_i, i = 1, \ldots, M \), with property (i) of Proposition 13.2.1 and the property

\[
\text{im}(\bar{c}_i) \subset B_{\frac{\epsilon}{2}}(x_i). \tag{13.2.1}
\]

If \( M = 1 \), the claim follows directly from the cone inequalities. If \( M > 1 \), let \( 0 < \tilde{R} < R_M \) such that \( \text{im}(c) \subset U_{\tilde{R}}(x_M) \cup \bigcup_{i=1}^{M-1} U_{R_i}(x_i) \). We find \( \alpha, r > 0 \) such that \( \tilde{R} < r - \alpha < r + \alpha < R_M \). Now let \( m_1 \geq 0 \) be such that each simplex of \( sd^{m_1}(c) \) has image with diameter less than \( \alpha/2 \).

Let \( c^+ \) be the part of \( sd^{m_1}(c) \) consisting of all simplices whose image has non-empty intersection with \( X \setminus B_r(x_M) \) and let \( c^- := sd^{m_1}(c) - c^+ \).

Let \( c^{nL} \) denote the part of \( bc^+ \) that is not Lipschitz (note that also the negative of the non-Lipschitzian part of \( bc^- \) since \( bc \) is Lipschitz). The Lipschitzian part we denote by

\[
c^L := bc^+ - c^{nL} \in C^L_{n-1}(X; G). \tag{13.2.2}
\]

Now, \( bc^{nL} = b(bc^+ - c^L) = -bc^L \in C^L_{n-2}(X; G) \) and by construction is \( \text{im}(c^{nL}) \subset B_r(x_M) \setminus U_{r - \alpha/2}(x_M) \). By induction for \( c^{nL} \) with \( \epsilon' := \alpha/2 \), there are \( c'_1, \ldots, c'_K \in C_{n-1}(X; G) \), \( c'^L_1, \ldots, c'^L_K \in C^L_{n-1}(X; G) \), \( \bar{c}'_1, \ldots, \bar{c}'_K \in C_n(X; G) \)
and \( m_2 \in \mathbb{N} \cup \{0\} \) such that

\[
\sum_{i=1}^{K} c'_i = sd^{m_2}(c^{nL}), \quad b\bar{c}'_i = c'_i - c'_i^{L}
\]  \hspace{1cm} (13.2.3)

and \( \text{im}(\sum_{i=1}^{K} \bar{c}'_i) \subset B_{\epsilon'}(\text{im}(c^{nL})) \subset B_{r+\alpha}(x_M) \setminus B_{r-\alpha}(x_M) \). Set

\[
z := sd^{m_2}(c^+) - \sum_{i=1}^{K} \bar{c}'_i \in C_n(X;G).
\]

Then, \( \text{im}(z) \subset \bigcup_{i=1}^{M-1} U_{R_i} (x_i) \) and by (13.2.2), (13.2.3)

\[
bz = sd^{m_2}(c^L + c^{nL}) + \sum_{i=1}^{K} c_i^{L} - \sum_{i=1}^{K} c'_i = sd^{m_2}(c^L) + \sum_{i=1}^{K} c'_i \in C'_n(X;G).
\]

Thus, by induction for \( z \) there exist \( c_1, \ldots, c_{M-1} \in C_n(X;G) \), \( c_i^{L}, \ldots, c_{M-1}^{L} \in C'_n(X;G) \), \( \bar{c}_1, \ldots, \bar{c}_{M-1} \in C_{n+1}(X;G) \) and \( m_3 \in \mathbb{N} \cup \{0\} \) such that

\[
\sum_{i=1}^{M-1} c_i = sd^{m_3}(z), \quad b\bar{c}_i = c_i - c_i^{L} \quad \text{and} \quad \text{im}(\bar{c}_i) \subset B_{\frac{\epsilon}{2}}(x_i).
\]

Now, let \( m := m_1 + m_2 + m_3 \). Then, by the definition of \( c^- \),

\[
c_M := sd^m(c) - sd^{m_3}(z) = sd^m(c) - sd^{m_2+m_3}(c^+) + sd^{m_3} \left( \sum_{i=1}^{K} \bar{c}'_i \right)
\]

\[
= sd^{m_2+m_3}(c^-) + sd^{m_3} \left( \sum_{i=1}^{K} c'_i \right)
\]

and therefore \( \text{im}(c_M) \subset B_{R_M}(x_M) \). Set \( v := bc^+ + c^{nL} \); recall that \( c^{nL} \) is the non-Lipschitzian part of \(-bc^-\), so \( v \in C'_n(X;G) \). By (13.2.3) is

\[
b_{c_M} = sd^{m_2+m_3}(v - c^{nL}) + sd^{m_3} \left( \sum_{i=1}^{K} c'_i - \sum_{i=1}^{K} c'_i^{L} \right)
\]

\[
= sd^{m_2+m_3}(v) - sd^{m_3} \left( \sum_{i=1}^{K} c'_i^{L} \right),
\]
i.e. $bc_M \in C^{L}_{n-1}(X; G)$. The cone inequalities give now fillings $c^L_M \in C^L_n(X; G)$ of $bc_M$ and $\bar{c}_M \in C_{n+1}(X; G)$ of $c_M - c^L_M$ such that (13.2.1) is satisfied, and
\[
\sum_{i=1}^{M} c_i = s d^{m_3}(z) + s d^m(c) - s d^{m_3}(z) = s d^m(c).
\]

Proof of Corollary 11.1.2. Using Proposition 13.2.1 instead of 13.1.1, Corollary 11.1.2 follows analogously as Corollary 11.1.1.

13.3 $H^F_n(X, A; \bar{G}), H^L_n(X, A; \bar{G})$ and $H_n(X, A; \bar{G})$

Again, $[\sum g_i c_i] := \sum g_i [c_i]$ defines a homomorphism $[\cdot] : C^L_n(X; \bar{G}) \to R^c_n(X, G)$ and this is also a chain map. We define local cone inequalities for $R^c_n(X, G)$ analogously as above.

**Proposition 13.3.1.** Let $n \geq 0$ and let $X$ be a complete metric space; let $\epsilon > 0$. If $n = 0$, let $X$ admit a local cone inequality for $C^L_0(X; \bar{G})$; if $n > 0$, let $X$ admit local cone inequalities for $R^c_j(X, G)$ and $C^L_k(X; \bar{G})$ for $j = 1, \ldots, n$ and $k = 0, \ldots, n - 1$.

Let $R \in R^c_n(X, G)$, assume that if $n > 0$ then $\partial R = [c]$ for $c \in C^L_{n-1}(X; \bar{G})$ and $bc = 0$. Then there exist $N \in \mathbb{N}$, $R_1, \ldots, R_N \in R^c_n(X, G)$, $c_1, \ldots, c_N \in C^L_n(X; \bar{G})$ and $V_1, \ldots, V_N \in R^c_{n+1}(X, G)$ such that

(i) $\sum_{i=1}^{N} R_i = R$ and $\partial V_i = R_i - [c_i]$

(ii) if $n > 0$, there exists $m \in \mathbb{N} \cup \{0\}$: $b(\sum_{i=1}^{N} c_i) = s d^m(c)$.

(iii) $\text{spt}(V_i) \cup \text{im}(c_i) \subset U_\epsilon(\text{spt}(R) \cup \text{im}(c))$ and $\text{diam}(V_i) < \epsilon$.

**Proof.** For $n = 0$ we divide $\text{spt} R$ into finitely many pieces of small diameter; now we can contract these to single points (note that the local cone inequality for $C^L_0(X; \bar{G})$ implies a sort of weak quasiconvexity (unless $G$ is trivial)). This settles the case $n = 0$. The rest of the proof is as the proof of Proposition 13.1.1.

Proof of Corollary 11.1.3. By Remark 9.0.5 we can consider $H^R_n(X, A; G)$ instead of $H^F_n(X, A; G)$. The proof is now as the one of Corollary 11.1.1; this gives the first isomorphism. The second comes from Corollary 11.1.2.
Chapter 14

Proof of Theorem 11.1.4

The goal of this chapter is to show that the maximal divisible subgroup of $H_1^{IC}(\mathbb{H})$ is trivial (see Section 14.1) whereas the ones of $H_1^L(\mathbb{H})$ and $H_1(H)$ are non-trivial (Section 14.2). This implies that $H_1^{IC}(\mathbb{H})$ is not isomorphic to either one of these groups.

Recall that the Hawaiian Earring $\mathbb{H} \subset \mathbb{R}^2$ is the countable union of the circles

$$L_n = \left\{ x \in \mathbb{R}^2 \mid |x - (1/n, 0)| = 1/n \right\},$$

with metric given by

$$d(x, y) := \begin{cases} |x - y|, & \text{if } \exists n \in \mathbb{N} : x, y \in L_n, \\ |x| + |y|, & \text{otherwise}. \end{cases}$$

An abelian group $G$ is called divisible if and only if for every $g \in G$ and every positive integer $n$ there exists $h \in G$ so that $nh = g$, i.e., every element of $G$ is divisible by every positive integer.

14.1 Metric currents and the Hawaiian Earring

In the proof we use the fact that the first homology group of the complex of integral currents on $S^1$ is isomorphic to $\mathbb{Z}$. This follows from Corollaries 11.1.1 and 11.1.2 since $H_1(S^1) \cong \mathbb{Z}$.

By definition, we have $I_2(X) = 0$ for any $\mathcal{H}^2$ null set $X$. It follows that $H_1^{IC}(\mathbb{H})$ and $H_1^{IC}(L_n)$ are simply the kernels of the maps $\partial: I_1(\mathbb{H}) \to I_0(\mathbb{H})$ and $\partial: I_1(L_n) \to I_0(L_n)$ respectively. Thus, showing that an element of
CHAPTER 14. PROOF OF THEOREM 11.1.4

$H^1_{IC}(\mathbb{H})$ is zero is equivalent to showing that the integral current representing it is zero.

**Proposition 14.1.1.** The maximal divisible subgroup of $H^1_{IC}(\mathbb{H})$ is trivial.

**Proof.** Let $T \in I^1_{ic}(\mathbb{H}) = I_1(\mathbb{H})$ be an element of the maximal divisible subgroup of $H^1_{IC}(\mathbb{H})$. Let $n \in \mathbb{N}$. We write $p_n : \mathbb{H} \to L_n$ for the map which sends $x \in \mathbb{H}$ to $(0,0)$ if $x \notin L_n$ and to itself otherwise. We denote the inclusion $L_n \to \mathbb{H}$ by $i_n$. The above remark shows that $H^1_{IC}(L_n) \cong \mathbb{Z}$. We claim that $H^1_{IC}(p_n)(T) = (p_n)\#(T)$ is zero. To see this, let $k \in \mathbb{Z}$ be an arbitrary integer. By assumption there exists an element $T' \in H^1_{IC}(L_n)$ such that $T = k \cdot T'$. It follows that $(p_n)\#(T) = k \cdot (p_n)\#(T') \in \mathbb{Z}$, i.e. every integer divides $(p_n)\#(T)$, which shows that $(p_n)\#(T) = 0$. We find in particular that $(i_n p_n)\#(T) = 0$.

On the other hand, we have $(i_n p_n)\#(T) = T\mid_{L_n}$. This follows since both currents have support $L_n$ and since for any function $f : \mathbb{H} \to \mathbb{R}$, the restriction of $f$ to $L_n$ agrees with the restriction of $f \circ i_n \circ p_n$ to $L_n$. These two facts imply the desired equality (by [Lan08, Lemma 3.2 and Theorem 4.4]).

Together this implies that

$$\|T\|(L_n) = M_{AK}(T\mid_{L_n}) = M_{AK}((i_n p_n)\#(T)) = M_{AK}(0) = 0,$$

and therefore that

$$M_{AK}(T) = \|T\|(\mathbb{H}) \leq \sum_{n=1}^{\infty} \|T\|(L_n) = 0,$$

which follows by countable subadditivity of $\|T\|$. We find that $T = 0$, i.e. that the maximal divisible subgroup of $H^1_{IC}(\mathbb{H})$ is indeed trivial. \qed

14.2 The maximal divisible subgroup of $H^L_1(\mathbb{H})$

14.2.1 Overview

In [EK00] it is shown that the maximal divisible subgroup of $H_1(\mathbb{H})$ is non-trivial. In order to construct a nontrivial element of the maximal divisible subgroup of $H^L_1(\mathbb{H})$ we follow the construction given in the proof of Theorem 4.14 in [Eda92]. Since we do not have a concise description of the Lipschitz homotopy group $\pi^L_1(\mathbb{H})$ we have to translate the algebraic definition given in loc. cit. to an explicit construction of certain Lipschitz maps $\sigma_n : [0, 2\lambda(n)] \to \mathbb{H}$, $n \in \mathbb{N}$ (where $\lambda(n) \leq 1$ is a real number). The $\sigma_n$ have the following properties:
14.2. THE MAXIMAL DIVISIBLE SUBGROUP OF $H^L_1(H)$

(i) For $n \geq 2$, $[\sigma_{n-1}] = n \cdot [\sigma_n]$ in $H^L_1(H)$.

(ii) The element $[\sigma_1] \in H^L_1(H)$ maps to a nonzero element under the homomorphism $H^L_1(H) \to H_1(H)$.

The idea behind the construction is the following. We first choose a sequence of maps $c_n : [0, \lambda(n)] \to H$, $n \in \mathbb{N}$, which represent commutators of certain standard loops in $\pi_1(H)$. We construct the maps $\sigma_n$ in such a way that for $n \geq 2$ the equation

$$\sigma_{n-1} = c_{n-1} \cdot \sigma_1 \cdot \ldots \cdot \sigma_n$$

holds. This could be depicted as follows:

Condition (i) is satisfied since $c_n$ is Lipschitz homotopic to a constant map. The commutators $c_n$ are inserted to ensure that the element $[\sigma_1] \in H^L_1(H)$ does not vanish, which follows from the stronger fact that its image under the comparison map $H^L_1(H) \to H_1(H)$ is a non-zero element. This is equivalent to proving that the corresponding element $[\sigma_1]_{\pi_1} \in \pi_1(H)$ does not lie in the commutator subgroup. We prove this in Proposition 14.3.3 by reducing the problem to a question about commutator subgroups of free groups.

14.2.2 Preliminaries

Let $S$ be the set of finite sequences of (non-zero) natural numbers, i.e. of maps $\{1, \ldots, n\} \to \mathbb{N}$ for $n \in \mathbb{N}$. We write $s = \langle s_1, \ldots, s_n \rangle$ for the sequence with $s(i) = s_i$. We call $n$ the length of $s$ and denote it by $\ell(s)$. For $k, m \in \mathbb{N}$ let $k \cdot \langle m \rangle$ denote the sequence $s \in S$ of length $k$ with $s(i) = m$, $1 \leq i \leq k$. The concatenation of sequences $+ : S \times S \to S$ is given by the map which sends $(s, s') \in S \times S$ to the sequence $s''$ given by $s''(i) = s(i)$, $1 \leq i \leq \ell(s)$ and $s''(j) = s'(j - \ell(s))$ for $\ell(s) + 1 \leq j \leq \ell(s) + \ell(s')$.

Let $B := \{s \in S \mid s(i) \leq i \text{ and } \ell(s) > 0\}$. We write $S_n$ for the subset of $S$ consisting of all sequences of length $n$ and $B_n$ for $B \cap S_n$. There is a linear
order relation \( \preceq \) on \( S \) such that for \( s, t \in S \), \( s \preceq t \) if and only if one of the following holds:

(i) there exists \( k \leq \min\{\ell(s), \ell(t)\} \) such that \( s(k) \neq t(k) \) and \( s(j) < t(j) \) for the minimal \( j \) with \( s(j) \neq t(j) \)

(ii) \( \ell(s) \leq \ell(t) \) and \( s(j) = t(j) \) for all \( 1 \leq j \leq \ell \).

The ordered set \( B \) can be embedded in the unit interval \([0, 1]\). For \( n \in \mathbb{N} \) let

\[
\lambda(n) := \frac{1}{2^n n!}
\]

and for \( s \in B \) let

\[
\tau(s) := \sum_{t \preceq s} \lambda(\ell(t)),
\]

where the sum is taken over any enumeration \((t_n)_{n \in \mathbb{N}}\) of \( \{t \in B \mid t \prec s\} \) if this set is infinite. Note that the sum does not depend on the chosen enumeration and that for all \( s \in B \), \( \tau(s) \leq 1 \). This follows since the cardinality of \( B_n \) is \( n! \). Indeed, let \( m \in \mathbb{N} \) and write \( N(m) \) for the maximum of \( \{\ell(t_1), \ldots, \ell(t_m)\} \).

Then the equation

\[
\sum_{k=1}^{m} \lambda(\ell(t_k)) \leq \sum_{n=1}^{N(m)} \left( \sum_{t \in B_n, \, t \prec s} \lambda(n) \right) \leq \sum_{n=1}^{\infty} \lambda(n)n! = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1
\]

holds, which shows that the series is absolutely convergent. Using a suitable enumeration we find that for \( s \preceq s' \) the useful formula

\[
\tau(s') - \tau(s) = \sum_{s \preceq t \prec s'} \lambda(\ell(t)) \quad (14.2.1)
\]

holds.

The next lemma summarizes the basic properties of the map \( \tau : B \to [0, 1] \).

**Lemma 14.2.1.** Let \( n, m \in \mathbb{N} \), \( s, s' \in B_n \), and let \( s'' \in S \) be such that \( s + s'' \) and \( s' + s'' \) belong to \( B \).

(i) The map \( \tau \) is strictly order preserving.

(ii) If \( s \preceq t \preceq s + \langle 1 \rangle \), then \( t \in \{s, s + \langle 1 \rangle\} \). Moreover \( \tau(s + \langle 1 \rangle) - \tau(s) = \lambda(n) \).

(iii) Let \( 1 \leq m \leq n \). Then \( s + \langle m \rangle \preceq t \preceq s + \langle m + 1 \rangle \) if and only if \( t(i) = s(i) \) for \( 1 \leq i \leq n \) and \( t(n + 1) = m \).
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(iv) For $1 \leq m \leq n$, $\tau(s + \langle m + 1 \rangle) - \tau(s + \langle m \rangle) = 2\lambda(n + 1)$.

(v) The equality $\tau(s + s'') - \tau(s' + s'') = \tau(s) - \tau(s')$ holds.

Proof. Assume $s \prec \tilde{s}$. Then we have by definition

$$
\tau(\tilde{s}) = \sum_{t \prec \tilde{s}} \lambda(\ell(t)) = \sum_{t \prec s} \lambda(\ell(t)) + \sum_{s \leq t \prec \tilde{s}} \lambda(\ell(t))
\geq \sum_{t \prec s} \lambda(\ell(t)) + \lambda(\ell(s)) > \sum_{t \prec s} \lambda(\ell(t)) = \tau(s)
$$

since $\lambda(\ell(s)) > 0$. This shows that $\tau$ is indeed strictly order preserving.

Note that the second part of (ii) follows immediately from the first and equation (14.2.1). Now let $s \leq t \leq s + \langle 1 \rangle$ and let $1 \leq j \leq n$. Then $s(j) < t(j)$ would imply that $s + \langle 1 \rangle \prec t$, but this contradicts the hypothesis $t \leq s + \langle 1 \rangle$. Conversely, $s(j) > t(j)$ contradicts $s \leq t$, so we find that $\ell(t) \geq n$ and $t(j) = s(j)$ for $1 \leq j \leq n$. If $\ell(t) = n$ we have $t = s$, so we can assume that $\ell(t) > n$ and hence that $t(n + 1) \geq 1$. If $t(n + 1)$ were greater than 1 we would have $s + \langle 1 \rangle \prec t$, so we can conclude that $t(n + 1) = 1$. But $\ell(t) > n + 1$ then implies $s + \langle 1 \rangle \prec t$ (by case (ii) of the definition of $\prec$) which is again a contradiction. So we finally find that $t = s + \langle 1 \rangle$, which establishes the second claim.

To see (iii) we have to consider two cases. First note that if $t = s + \langle m \rangle$, the conclusion holds trivially, i.e. we can assume that $s + \langle m \rangle \prec t$. Then by definition we have to consider the two cases

1. there exists $k \leq \min(n + 1, \ell(t))$ such that $s + \langle m \rangle(k) \neq t(k)$ and $s + \langle m \rangle(j) < t(j)$ for the minimal $j$ with $s + \langle m \rangle(j) \neq t(j)$, or

2. $n + 1 \leq \ell(t)$ and $s + \langle m \rangle(j) = t(j)$ for all $1 \leq j \leq n + 1$.

In case (2) we are obviously finished, so we can assume that we are in the situation of case (1). If $j \leq n$ we have $s + \langle m + 1 \rangle \prec t$ which contradicts our assumption. It follows that $s(i) = t(i)$ for $1 \leq i \leq n$ and that $m < t(n + 1)$. But this shows that $s + \langle m + 1 \rangle \leq t$. Since we also have $t \prec s + \langle m + 1 \rangle$ we find that case (1) cannot occur under the assumptions of (iii).
The proof of (iv) follows directly from (iii) and equation (14.2.1). We have
\[
\tau(s + \langle m + 1 \rangle) - \tau(s + \langle m \rangle) = \sum_{t < s + \langle m + 1 \rangle} \lambda(\ell(t)) - \sum_{t < s + \langle m \rangle} \lambda(\ell(t)) = \sum_{s + \langle m \rangle \leq t < s + \langle m + 1 \rangle} \lambda(\ell(t)) = \lambda(\ell(s + \langle m \rangle)) + \sum_{k > n+1} \left( \sum_{t \in C_k} \lambda(k) \right)
\]
where \(C_k := \{ t \in B_k \mid t(j) = s(j) \text{ for } 1 \leq j \leq n \text{ and } t(n + 1) = m \} \). Since the cardinality of \(C_k\) is \(\frac{k!}{(n+1)!}\) we find that
\[
\tau(s + \langle m + 1 \rangle) - \tau(s + \langle m \rangle) = \lambda(n + 1) + \sum_{k > n+1} \frac{k!}{(n+1)!} 2^{k/k!} = \lambda(n + 1) + \frac{1}{2^{n+1}(n+1)!} \sum_{k=1}^{\infty} \frac{1}{2^k} = 2\lambda(n + 1).
\]

It remains to show (v). There are natural numbers \(m_1, \ldots, m_d\) such that \(s'' = \langle m_1 \rangle + \ldots + \langle m_d \rangle\). By induction we can reduce the problem to the case \(s'' = \langle m \rangle\). Using (iv) we can further reduce this to the case \(m = 1\). Indeed, for \(m > 1\) we have
\[
\tau(s + \langle m \rangle) = \tau(s + \langle m - 1 \rangle) + \tau(s + \langle m - 1 \rangle) + \tau(s + \langle m - 1 \rangle) + 2\lambda(n + 1)
\]
which shows that
\[
\tau(s + \langle m \rangle) - \tau(s' + \langle m \rangle) = \tau(s + \langle 1 \rangle) - \tau(s' + \langle 1 \rangle).
\]
With (ii) and equation (14.2.1) we find that \(\tau(s + \langle 1 \rangle) = \tau(s) + \lambda(n + 1)\) and therefore that the equality
\[
\tau(s + s'') - \tau(s' + s'') = \tau(s + \langle 1 \rangle) - \tau(s' + \langle 1 \rangle) = \tau(s) - \tau(s')
\]
holds, as claimed. \(\square\)
Lemma 14.2.2. The set

\[ I' := \bigcup_{s \in \mathcal{B}} [\tau(s), \tau(s + \langle 1 \rangle)] \subseteq [0, 1] \]

is dense in \([0, 1]\).

Proof. Let \(x \in [0, 1]\) and assume that for all \(s \in \mathcal{B}, x \notin [\tau(s), \tau(s + \langle 1 \rangle)]\). We construct a sequence \((s_i)_{i \in \mathbb{N}}\) with \(\tau(s_i) < x\) as follows. Let \(s_1 = \langle 1 \rangle\) (which trivially satisfies \(\tau(s_1) = 0 < x\)), and if \(s_k\) is defined for all \(k < i\), let \(s_i\) be the maximal element of \(\{s \in \mathcal{B}_i \mid \tau(s) < x\}\). This set is not empty since it contains \(s_{i-1} + \langle 1 \rangle\). Indeed, by induction hypothesis we have \(\tau(s_{i-1}) < x\) and \(x \notin I'\) implies in particular \(x \notin [\tau(s_{i-1}), \tau(s_{i-1} + \langle 1 \rangle)]\), so \(\tau(s_{i-1} + \langle 1 \rangle) < x\). Moreover, for any \(i \in \mathbb{N}\) there exists \(m \in \mathbb{N}\) such that

\[ s_{i+1} = s_i + \langle m \rangle. \quad (14.2.2) \]

This follows since \(s_i < s_i + \langle 1 \rangle \leq s_{i+1}\) and therefore one of the following must hold:

(a) there is a \(k \leq i\) with \(s_i(k) \neq s_{i+1}(k)\) and \(s_i(j) < s_{i+1}(j)\) for the minimal \(j\) with \(s_i(j) \neq s_{i+1}(j)\), or

(b) the sequence \(s_{i+1}\) extends \(s_i\), i.e. \(s_{i+1} = s_i + \langle m \rangle\).

But case (a) would contradict the maximality of \(s_i\): \(s' = \langle s_{i+1}(1), \ldots, s_{i+1}(i) \rangle\) is an element of \(\mathcal{B}_i\), the equation

\[ \tau(s') \leq \tau(s' + \langle s_{i+1}(i + 1) \rangle) = \tau(s_{i+1}) < x \]

holds, and \(s_i < s'\) since \(s_i(j) < s_{i+1}(j) = s'(j)\).

This implies in particular that \((\tau(s_i))_{i \in \mathbb{N}}\) is a strictly increasing sequence and that \(\sup \{\tau(s_i) \mid i \in \mathbb{N}\}\) equals \(\lim_{i \to \infty} \tau(s_i) \leq x\). We will show that this is an equality and therefore that \(x\) lies in the closure of \(I'\).

First note that the set

\[ C := \{i \in \mathbb{N} \mid s_i(i) < i\} \]

is either empty or infinite. If \(C\) were finite and non-empty, it would contain a maximal element \(n \in \mathbb{N}\). We write \(s'\) for the sequence \(\langle s_n(1), s_n(2), \ldots, s_n(n - 1) \rangle\) and let \(m = s_n(n)\). Since \(n \in C\) we have \(m < n\). By maximality of \(n\) and with equation (14.2.2) it follows that for \(i > n\) we must have \(s_i = s_n + \langle n + 1, n + 2, \ldots, i \rangle\). Now let \(B \ni t < s' + \langle m + 1 \rangle\). Then we have either
that we have $N > n$ and there exists $i$ such that

\[ |t' - s| \leq \lambda |t' - s| \]

contradiction, so assume now that $\lim_{i \to \infty} \tau(s_i) \leq x$ and hence, because $x \notin I'$, that $\tau(s' + \langle m + 1 \rangle) < x$. But this contradicts the maximality of $s_n$ in $\{ s \in B_n \mid \tau(s) < x \}$, so $C$ must be empty or infinite.

We first consider the case $C = \emptyset$. By construction we must have $s_i = (1, 2, \ldots, i)$ and it follows that

\[ \tau(s_i) \geq \sum_{k=1}^{i-1} \left( \sum_{t \in B_k} \lambda(k) \right) = \sum_{k=1}^{i-1} \frac{1}{2^k} = 1 - \frac{1}{2^{i-1}}, \]

which shows that $\lim_{i \to \infty} \tau(s_i) = 1$. But this implies that $x = 1$ and that $x$ is the limit of $(\tau(s_i))_{i \in \mathbb{N}}$.

It remains to show that the same holds if $C$ is infinite. We prove this by contradiction, so assume now that $\lim_{i \to \infty} \tau(s_i) < x$. Choose $n \in \mathbb{N}$ such that $2\lambda(n) < \varepsilon$, where $\varepsilon = x - \lim_{i \to \infty} \tau(s_i)$. Since the set $C$ is infinite it follows that there exists $N > n$ such that $N \in C$. But now Lemma 14.2.1 (iv) implies that we have

\[ \tau(\langle s_N(1), s_N(2), \ldots, s_N(N - 1) \rangle + \langle s_N(N + 1) \rangle) = \tau(s_N) + 2\lambda(N) < \tau(s_N) + \varepsilon \leq x, \]

which contradicts the maximality of $s_N$ in $\{ s \in B_N \mid \tau(s) < x \}$. So we have indeed found a sequence $(\tau(s_i))_{i \in \mathbb{N}}$ in $I'$ with $\lim_{i \to \infty} \tau(s_i) = x$. 

This fact enables us to specify Lipschitz functions from $[0, 1]$ to a complete metric space $X$ (e.g. $H$) by specifying their restriction to $I'$. This is quite simple since for $s \prec s' \in B$, the intersection of the two intervals $[\tau(s), \tau(s + \langle 1 \rangle)] \cap [\tau(s'), \tau(s' + \langle 1 \rangle)]$ is either empty or the singleton $\{ \tau(s') \}$. So if we have a $1$-Lipschitz function

\[ \varphi_s : [\tau(s), \tau(s + \langle 1 \rangle)] \to H \]
for each \( s \in B \) with \( \varphi_s(\tau(s)) = \varphi_s(\tau(s + \langle 1 \rangle)) = x_0 \), there is a unique 1-Lipschitz function \( \varphi: [0, 1] \to X \) with

\[
\varphi|_{[\tau(s), \tau(s + \langle 1 \rangle)]} = \varphi_s.
\]

We use this construction to define \( \sigma_1: [0, 1] \to \mathbb{H} \).

The following construction is useful for the computation of certain finite sums in \( H_L^1(X) \). Given two Lipschitz maps \( \sigma: [0, a] \to X \) and \( \sigma': [0, b] \to X \), \( a, b \in \mathbb{R} \) with \( \sigma(a) = \sigma'(0) \), their concatenation \( \sigma \cdot \sigma': [0, a + b] \to X \) is given by

\[
\sigma \cdot \sigma'(t) := \begin{cases} 
\sigma(t) & \text{if } t \leq a \\
\sigma'(t - a) & \text{if } a \leq t \leq b.
\end{cases}
\]

If \( \sigma \) and \( \sigma' \) represent elements of \( H_L^1(X) \), so does \( \sigma \cdot \sigma' \) and moreover the equation

\[
[\sigma \cdot \sigma'] = [\sigma] + [\sigma']
\]

holds. This follows as in the continuous case.

### 14.3 Construction of the \( \sigma_n \)

We write \( \varphi_n \) for the Lipschitz function \([0, 1] \to \mathbb{H} \) which traverses the \( n \)-th loop \( L_n \) of \( \mathbb{H} \) with constant speed. Explicitly, for \( t \in [0, 1] \) the equation

\[
\varphi_n(t) = \left( \frac{- \cos(2\pi t)}{n} + \frac{1}{n} \cdot \frac{\sin(2\pi t)}{n} \right)
\]

(14.3.1)

holds. The Lipschitz constant of \( \varphi_n \) is bounded by \( \mu_n := 2\pi/n \). Choose a sequence \( (n_k)_{k \in \mathbb{N}} \) such that

\[
4\mu_{n_k} \leq \lambda(k)
\]

and \( n_{k+1} > n_k + 1 \). We write

\[
c_k := \varphi_{n_k} \cdot \varphi_{n_k+1} \cdot \varphi_{n_k}^{-1} \cdot \varphi_{n_k+1}^{-1} \circ \psi: [0, \lambda(k)] \to \mathbb{H}
\]

(14.3.2)

for the composition of (a representative of) the commutator of \( \varphi_{n_k} \) and \( \varphi_{n_k+1} \) with the reparametrization \( \psi: [0, \lambda(k)] \to [0, 4] \) which sends \( t \) to \( 4t/\lambda(k) \). By choice of the sequence \( (n_k)_{k \in \mathbb{N}} \) we find that \( c_k \) is a 1-Lipschitz function.

For \( s \in B \) and \( t \in [\tau(s), \tau(s + \langle 1 \rangle)] \) let \( (\sigma_1)_s(t) := c_{\ell(s)}(t - \tau(s)) \) and note that \( (\sigma_1)_s(\tau(s)) = (\sigma_1)_s(\tau(s + \langle 1 \rangle)) = (0, 0) \). By the comment succeeding Lemma 14.2.2 there is a unique 1-Lipschitz function \( \sigma_1: [0, 1] \to \mathbb{H} \) such that

\[
\sigma_1|_{[\tau(s), \tau(s + \langle 1 \rangle)]}(t) = c_{\ell(s)}(t - \tau(s)).
\]

(14.3.3)
For $n > 1$ let $\sigma_n : [0, 2\lambda(n)] \to \mathbb{H}$ be the function
\[
\sigma_n(t) := \sigma_1[\tau(n \cdot (1)), \tau((n-1) \cdot (1) + (2))] (t + \tau(n \cdot (1))).
\] (14.3.4)

**Proposition 14.3.1.** For $n > 1$ the equation
\[
\sigma_{n-1} = c_{n-1} \cdot \overbrace{\sigma_n \cdot \ldots \cdot \sigma_n}^{n \text{ times}}
\]
holds, and consequently $[\sigma_{n-1}] = n \cdot [\sigma_n]$ in $H^L_1(\mathbb{H})$.

**Proof.** Let $a := \tau((n-1) \cdot (1))$, let $t \in [0, 2\lambda(n-1)]$ and let $x := t + a$. If $t \leq \lambda(n-1)$ or equivalently $x \in [\tau((n-1) \cdot (1)), \tau(n \cdot (1))]$ we find that
\[
\sigma_{n-1}(t) = \sigma_1(x) = \sigma_1[\tau((n-1) \cdot (1)), \tau(n \cdot (1))](x)
= c_{n-1}(x-a) = c_{n-1}(t) = c_{n-1} \cdot \overbrace{\sigma_n \cdot \ldots \cdot \sigma_n}^{n \text{ times}}(t)
\]
holds. We can therefore restrict our attention to those points $t \in [\lambda(n-1), 2\lambda(n-1)]$ with the property that $t + a$ lies in $I'$, so we can assume that there is a sequence $s_0 \in B$ such that $x = t + a$ lies in the interior of $[\tau(s_0), \tau(s_0 + \langle 1 \rangle)]$. But $t \geq \lambda(n-1)$ implies that $x \geq \tau(n \cdot \langle 1 \rangle)$, and it follows that $n \cdot \langle 1 \rangle \leq s_0$ since $s_0$ is maximal with $\tau(s_0) \leq x$ (see Lemma 14.2.1 (ii)). On the other hand, we have
\[
\tau(s_0) < x \leq \tau((n - 1) \cdot \langle 1 \rangle) + 2\lambda(n-1) = \tau((n - 2) \cdot \langle 1 \rangle + \langle 2 \rangle)
\]
and therefore $(n - 1) \cdot \langle 1 \rangle \leq s_0 < (n - 2) \cdot \langle 1 \rangle + \langle 2 \rangle$. By Lemma 14.2.1 (iii) it follows that $s_0 = (n - 1) \cdot \langle 1 \rangle + s'$ for some sequence $s' \in S$. But $n \cdot \langle 1 \rangle \leq (n - 1) \cdot \langle 1 \rangle + s'$ implies that $s'$ is of the form $\langle m \rangle + s$ for some $m \in \mathbb{N}$ and a (possibly empty) sequence $s \in S$. Hence $s_0 = (n - 1) \cdot \langle 1 \rangle + \langle m \rangle + s$ and we find that
\[
\sigma_{n-1}(t) = \sigma_1(x) = c_{n+\ell(s)}(x - \tau((n - 1) \cdot \langle 1 \rangle + \langle m \rangle + s))
\]
\[
= c_{n+\ell(s)}(x - \tau((n - 1) \cdot \langle 1 \rangle + \langle m \rangle + s) + \tau(n \cdot \langle 1 \rangle + s) - \tau(n \cdot \langle 1 \rangle + s))
\]
\[
= c_{n+\ell(s)}(x - \tau((n - 1) \cdot \langle 1 \rangle + \langle m \rangle)) + \tau(n \cdot \langle 1 \rangle) - \tau(n \cdot \langle 1 \rangle + s))
\]
\[
= \sigma_1(x - \tau((n - 1) \cdot \langle 1 \rangle + \langle m \rangle)) + \tau(n \cdot \langle 1 \rangle))
\]
\[
= \sigma_n(x - \tau((n - 1) \cdot \langle 1 \rangle + \langle m \rangle))
\]
\[
= \sigma_n(t + \tau((n - 1) \cdot \langle 1 \rangle) - \tau((n - 1) \cdot \langle 1 \rangle + \langle m \rangle))
\]
\[
= \sigma_n(t - \lambda(n-1) - (m-1) \cdot 2\lambda(n))
\]
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holds, where the crucial step is an application of Lemma 14.2.1 (v) and the last equality follows from Lemma 14.2.1 (ii) and (iii). On the other hand, since we have

$$\tau((n-1)\langle 1 \rangle + \langle m \rangle) \leq x \leq \tau((n-1)\langle 1 \rangle + \langle m+1 \rangle)$$

it follows that $\lambda(n-1) + (m-1) \cdot 2\lambda(n) \leq t \leq \lambda(n-1) + m \cdot 2\lambda(n)$, again from Lemma 14.2.1 (ii) and (iii). So in this case we have

$$c_{n-1} \cdot \sigma_{\ell \ldots \ell} (t) = \sigma_{\ell \ldots \ell} (t - \lambda(n-1))$$

$$= \sigma_{\ell \ldots \ell} (t - \lambda(n-1) - 2\lambda(n))$$

$$= \sigma_{\ell \ldots \ell} (t - \lambda(n-1) - 2 \cdot 2\lambda(n))$$

$$= \sigma_{\ell \ldots \ell} (t - \lambda(n-1) - (m-1) \cdot 2\lambda(n))$$

and we find that the claimed equality holds for a dense subset.

The second statement follows since we have in general $[\sigma \cdot \sigma'] = [\sigma] + [\sigma']$. This implies in particular that the commutator $[c_{n-1}]$ vanishes (since $H_1^L(H)$ is abelian) and therefore that $[\sigma_{n-1}] = n \cdot [\sigma_n]$. \hfill $\Box$

**Corollary 14.3.2.** The element $[\sigma_1]$ lies in the maximal divisible subgroup of $H_1^L(H)$.

**Proof.** The set

$$D := \{ m \cdot [\sigma_n] \mid m \in \mathbb{Z}, n \in \mathbb{N} \}$$

is a divisible subgroup of $H_1^L(H)$. To see this, let $m, m' \in \mathbb{Z}$, $n, n' \in \mathbb{N}$ with $n \leq n'$. Then $[\sigma_n] = \frac{n'}{n!} [\sigma_{n'}]$ by the above proposition. It follows that

$$m \cdot [\sigma_n] - m' \cdot [\sigma_{n'}] = \left( m \cdot \frac{n'!}{n!} - m' \right) [\sigma_{n'}],$$

so $D$ is indeed a subgroup. But it is also divisible, for if $l \in \mathbb{N}$ we have

$$m \cdot [\sigma_n] = m \cdot \frac{(l \cdot n)!}{n!} [\sigma_{l \cdot n}] = l \cdot \left( m \cdot \frac{(l \cdot n-1)!}{(n-1)!} \right) [\sigma_{l \cdot n}]$$

and $m \cdot \frac{(l \cdot n-1)!}{(n-1)!} [\sigma_{l \cdot n}] \in D$. \hfill $\Box$
It remains to show that $[\sigma_1] \neq 0$. We will show that the image $[\sigma_1]_S \in H_1(H)$ of $[\sigma_1]$ under the comparison map $H^k_1(H) \to H_1(H)$ is non-zero. Using the isomorphism

$$H_1(H) \cong \pi_1(H)/[\pi_1(H), \pi_1(H)]$$

we find that this is equivalent to showing that the loop $\sigma_1$ does not represent an element of the commutator subgroup of $\pi_1(H)$.

Let $p_k : H \to L_{n_k} \cup L_{n_k+1}$ be the map which is the identity on $L_{n_k} \cup L_{n_k+1}$ and which sends an element $x \in H \setminus L_{n_k} \cup L_{n_k+1}$ to $(0,0)$. This is a (Lipschitz) continuous map and therefore induces a group homomorphism $\pi_1(p_k) : \pi_1(H) \to \pi_1(L_{n_k} \cup L_{n_k+1})$. Since $L_{n_k} \cup L_{n_k+1}$ is homeomorphic to $S^1 \vee S^1$, $\pi_1(L_{n_k} \cup L_{n_k+1})$ is isomorphic to the free group on two generators $(a,b)$. Equation (14.3.1) implies that $\pi_1(p_k)([\varphi_{n_k}]_{\pi_1}) = [p_k \circ \varphi_{n_k}]_{\pi_1} = a$ and $\pi_1(p_k)([\varphi_{n_k+1}]_{\pi_1}) = [p_k \circ \varphi_{n_k+1}]_{\pi_1} = b$ under this isomorphism. We find in particular that

$$\pi_1(p_k)([c_k]) = aba^{-1}b^{-1}. \quad (14.3.5)$$

**Proposition 14.3.3.** The element $[\sigma_1]_{\pi_1} \in \pi_1(H)$ does not lie in the commutator subgroup $[\pi_1(H), \pi_1(H)]$.

**Proof.** From now on we identify $\pi_1(L_{n_k} \cup L_{n_k+1})$ with $(a,b)$. We will first show that $\pi_1(p_k)$ maps $[\sigma_1]_{\pi_1}$ to $(aba^{-1}b^{-1})^k$. By Lemma 14.2.2 the map $p_k \circ \sigma_1$ is uniquely determined by its restriction to the intervals $[\tau(s), \tau(s+1))$. By equation (14.3.2) it follows that

$$p_k \circ \sigma_1|_{[\tau(s), \tau(s+1))}(t) = p_k \circ c_{\ell(s)}(t - \tau(s)) = \begin{cases} c_k(t - \tau(s)) & \text{if } \ell(s) = k \\ (0,0) & \text{else}. \end{cases}$$

Since there are precisely $k!$ sequences $s \in B$ with $\ell(s) = k$ we find that $p_k \circ \sigma_1$ is homotopic (relative endpoints) to

$$\underbrace{c_k \cdot \cdots \cdot c_k}_{k! \text{ times}}$$

and together with equation (14.3.5) that $\pi_1(p_k)([\sigma_1]_{\pi_1}) = (aba^{-1}b^{-1})^k$.

Assume now that $[\sigma_1]_{\pi_1}$ does lie in the commutator subgroup, i.e. that $[\sigma_1]_{\pi_1} = [x_1, y_1] \cdot \cdots \cdot [x_n, y_n]$ for some $x_i, y_i \in \pi_1(H)$. Since $\pi_1(p_k)$ is a group homomorphism, it follows that for all $k \in \mathbb{N}$ the element $(aba^{-1}b^{-1})^k$ of $(a,b)$ can be written as a product of $n$ elementary commutators. On the other hand, in [Cul81, Example 2.6] it was shown that $(aba^{-1}b^{-1})^k$ cannot be written as a product of less than $[k!/2] + 1$ elementary commutators. Since $k \in \mathbb{N}$ was arbitrary this is clearly a contradiction. \qed
Appendix
Appendix A

Lipschitz chains versus integral currents

Let $X$ be an injective metric space. De Pauw and Hardt use for the definition of virtual chains as starting set singular Lipschitz chains, that is, the group $C_{L}^{n}(X;G)$ is considered. We started with $\tilde{P}_{n}(X,G)$. Thus, in a sum like $\sum_{i=1}^{m} g_{i}T_{i}$, we have that $T_{i}$ is either an integral Lipschitz chain or an integral current. Both approaches are equivalent. This is easily seen from the lemma below together with the fact that for $F \in F_{n}(X,G)$ there exists a sequence $P_{j} \rightarrow F$ such that $M(P_{j}) \rightarrow M(F)$ and $M(\partial P_{j}) \rightarrow M(\partial F)$ (this follows similarly as in [Fle66]): When $\text{spt} F$ is bounded one shows first that for $F \in P_{n-1}(X,G)$ there exist $P_{j} \rightarrow F$ with $\partial P_{j} = \partial F$ and $M(P_{j}) \rightarrow M(F)$; then one argues inductively. In the other case, assume that $X = l^{\infty}(B)$ and that $M(F)$ (or $M(\partial F)$) is finite. Then for every $\epsilon > 0$ there exists $r > 0$ and a $1$-Lipschitz retraction $l_{r} : l^{\infty}(B) \rightarrow B_{r}(0) \subset l^{\infty}(B)$ (as such balls are injective as well) such that $M(F - l_{r}\#F) < \epsilon$ (respectively $M(\partial F - l_{r}\#\partial F) < \epsilon$). Now we are back to the bounded case.

Lemma A.0.1. Let $X$ be an injective metric space. For $P \in \tilde{P}_{n}(X,G)$ and $\epsilon > 0$ there exists $c \in C_{L}^{n}(X;G)$ such that $N(P - [c]) < \epsilon$.

Proof. Let $X = l^{\infty}(B)$. Assume first that $\partial P = [c']$ for $c' \in C_{L}^{n-1}(X;G)$. We can assume that $bc' = 0$ for $n \geq 2$ (otherwise we replace $c'$ by the boundary of a 'cone' $z \in C_{L}^{n}(X;G)$ over $c'$; note that then $[c'] = [bz]$). For $n = 1$ we have $\chi(c') = \chi([c']) = \chi(\partial P) = 0$; the case $n = 0$ is trivial. We easily find $c_{1} \in C_{L}^{n}(X;G)$ such that $M(P - [c_{1}]) < \epsilon/3$ (looking at a representation of
Then we can embed isometrically the support of $P$ and the images of $c'$ and $c_1$ in $A := (C([0,1]), \| \cdot \|_{\infty})$ (compare the proof of Theorem 4.4.1); so let $P \in \tilde{P}_n(A,G)$, $c' \in C^L_{n-1}(A;G)$ and $c_1 \in C^L_n(A;G)$. Now there exists a linear 1-Lipschitz map $B : A \to A$ with finite dimensional image (say $W$) such that the affine filling $v \in C^L_n(A;G)$ of $B#c_1 - bc_1$ has mass bounded by $\epsilon/3$.

In $W$ we find $c_2 \in C^L_n(W;G)$ with $bc_2 = B#c' - B#bc_1$ and $M([c_2]) < \epsilon/3$ (as $M(B#P - B#[c_1]) < \epsilon/3$ and since such a result holds for Euclidean space for every $\epsilon' > 0$. Possibly we have to take subdivisions of the chains). Now, $c := c_1 + v + c_2 \in C^L_n(A;G)$ has

$$M(P - [c]) < \epsilon \quad \text{and} \quad bc = c', \quad \text{so} \quad \partial P = [bc].$$

Note that we can assume that $\text{im } c \subset B_\epsilon(\text{spt } P)$. Then the above holds for $X$ too.

For general $P$ we argue as follows: By the above argument we know for $\partial P$ that there exists $R > 0$ and $c_3 \in C^L_{n-1}(X;G)$ with

$$\text{spt } P \cup \text{im } c_3 \subset B_R(x), \quad M(\partial P - [c_0]) < \epsilon/(3\tilde{C}_nR)$$

(recall (4.8.1) for $\tilde{C}_n$) and we can assume that $bc_3 = 0$ (respectively $\chi(c_3) = 0$ for $n = 1$). Let $V \in \tilde{P}_n(X,G)$ be a cone filling of $\partial P - [c_3]$, so $M(V) < \epsilon/3$. Now let $c_4$ be such that $M(P - V - [c_4]) < \epsilon/3$ and $bc_4 = c_3$. Then,

$$N(P - [c_4]) \leq M(P - V - [c_4]) + M(V) + M(\partial P - [c_3]) < \epsilon.$$

\[\square\]
Bibliography
Bibliography


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