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The Multiple Access Channel with Correlated Sources and Cribbing Encoders

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I. NOTATION

A discrete memoryless multiple access channel (MAC) is a quadruple $\{\mathcal{X}_1, \mathcal{X}_2, P_{Y|X_1, X_2}, \mathcal{Y}\}$ where $\mathcal{X}_1, \mathcal{X}_2$ are the input alphabets, \mathcal{Y} the output alphabet, and $P_{Y|X_1, X_2}$ a transition probability matrix from $\mathcal{X}_1 \times \mathcal{X}_2$ to \mathcal{Y} . For shorthand notation, we will refer to the MAC by $P_{Y|X_1, X_2}$. A source pair is a triplet $\{\mathcal{U}, \mathcal{V}, P_{U, V}\}$ where $P_{U, V}$ is a distribution on $\mathcal{U} \times \mathcal{V}$. We use the following notation: random variables are denoted by uppercase letters. Sequence of n letters from alphabet \mathcal{X} is denoted by x^n , and the substring x_i, x_{i+1}, \dots, x_j as x_i^j . When the dimension n is understood from the context, n -vectors are denoted by boldface letters: $x^n = \mathbf{x}$. Similar notation holds for random variables and random vectors, e.g., V, \mathbf{U}, U^n , etc. The channel and source are assumed memoryless, thus probabilities of n -sequences are given by

$$P_{Y|X_1, X_2}(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2) = \prod_{i=1}^n P_{Y|X_1, X_2}(y_i|x_{1,i}, x_{2,i})$$

$$P_{U, V}(\mathbf{u}, \mathbf{v}) = \prod_{i=1}^n P_{U, V}(u_i, v_i)$$

II. PROBLEM DEFINITION AND PREVIOUS RESULTS

We investigate a joint source-channel coding schemes for the discrete memoryless MAC with cribbing encoders. The sources, U and V , deliver their output to two separate encoders 1 and 2. An (n, m, ϵ) joint source channel code for the source $P_{U, V}$ and the MAC $P_{Y|X_1, X_2}$ with strictly causal cribbing encoder consists of $m+1$ encoding functions

$$f_1 : \mathcal{U}^n \rightarrow \mathcal{X}_1^m \quad (1a)$$

$$f_{2,i} : \mathcal{V}^n \times \mathcal{X}_1^{i-1} \rightarrow \mathcal{X}_{2,i} \quad i = 1, \dots, m \quad (1b)$$

and a decoding function

$$d^m : \mathcal{Y}^m \rightarrow \mathcal{U}^n \times \mathcal{V}^n$$

such that the probability of error is bounded from above by ϵ :

$$P_e = Pr \{(u^n, v^n) \neq d^m(y^m)\}$$

$$= \sum_{(\mathbf{u}, \mathbf{v}) \in \mathcal{U}^n \times \mathcal{V}^n} P_{U, V}(\mathbf{u}, \mathbf{v}) \cdot Pr \{d^m(y^m) \neq (\mathbf{u}, \mathbf{v})\} \leq \epsilon$$

We say that encoder 2 is the cribbing encoder, as reflected in (1b). The rate of the code is defined as $\rho = n/m$. A source pair (U, V) is said to be transmissible via the MAC $P_{Y|X_1, X_2}$ with strictly causal cribbing at rate ρ if for every $\epsilon > 0$, $\delta > 0$, and sufficiently large

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n there exists an $(n, n/(\rho - \delta), \epsilon)$ code for the source pair and the channel.

We will examine also the model of causal cribbing, where the cribbing encoder sees the output of the other encoder without the delay of (1b), that is, when the sequence of m encoders at channel input 2 is of the form

$$f_{2,i} : \mathcal{V}^n \times \mathcal{X}_1^i \rightarrow \mathcal{X}_{2,i} \quad i = 1, \dots, m \quad (2)$$

The definition of transmissibility of a source pair via the channel remains as in the case of strictly causal cribbing, with the only difference that (2) replaces (1b). The joint probability mass function for strictly causal cribbing is given by

$$P(\mathbf{u}, \mathbf{v}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \prod_{i=1}^n P_{U, V}(u_i, v_i) \prod_{i=1}^n P_{Y|X_1, X_2}(y_i|x_{1,i}(u^n), x_{2,i}(v^n, x_1^{i-1}(u^n)))$$

and for causal cribbing by

$$P(\mathbf{u}, \mathbf{v}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \prod_{i=1}^n P_{U, V}(u_i, v_i) \prod_{i=1}^n P_{Y|X_1, X_2}(y_i|x_{1,i}(u^n), x_{2,i}(v^n, x_1^i(u^n)))$$

We are interested in finding conditions for transmissibility of a source via the MAC with causal and strictly causal cribbing. We start by stating previous relevant results, on capacity regions of MAC with cribbing, and on distributed source coding.

Theorem 1. *F. Willems & E. C. van der Meulen [4]: The capacity region for the strictly causal cribbing communication model is the collection of all pairs (R_1, R_2) such that*

$$R_1 \leq H(X_1 | W) \quad (3)$$

$$R_2 \leq I(X_2; Y | X_1, W) \quad (4)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y) \quad (5)$$

for some distribution of the form

$$P_{W, X_1, X_2, Y}(w, x_1, x_2, y) = P_W(w) P_{X_1|W}(x_1 | w) \cdot P_{X_2|W}(x_2 | w) P_{Y|X_1, X_2}(y | x_1, x_2)$$

and $|\mathcal{W}| \leq \min\{|\mathcal{X}_1| \cdot |\mathcal{X}_2| + 1, |\mathcal{Y}| + 2\}$ (6)

The capacity region for the causal cribbing communication model is given by the collection of all rate pairs (R_1, R_2) such that

$$R_1 \leq H(X_1) \quad (7)$$

$$R_1 \leq I(X_2; Y | X_1) \quad (8)$$

$$R_1 + R_2 \leq I(X_1, X_2; Y) \quad (9)$$

for

$$\begin{aligned} P_{X_1, X_2, Y}(x_1, x_2, y) = \\ P_{X_1, X_2}(x_1, x_2) P_{Y|X_1, X_2}(y | x_1, x_2) \end{aligned} \quad (10)$$

Theorem 2. Slepian-Wolf [3]: For the distributed source coding problem with source (U, V) drawn i.i.d $\sim P_{U, V}$, the achievable rate region is given by:

$$H(U | V) \leq R_1 \quad (11)$$

$$H(V | U) \leq R_2 \quad (12)$$

$$H(U, V) \leq R_1 + R_2 \quad (13)$$

III. STATEMENT OF RESULTS

Theorem 3. A source $(U, V) \sim P_{U, V}$ is transmissible via the MAC $P_{Y|X_1, X_2}$ with strictly causal cribbing at rate $\rho = 1$ if

$$H(U | V) \leq H(X_1 | W, V) \quad (14)$$

$$H(V | U) \leq I(Y; X_2 | W, U, X_1) \quad (15)$$

$$H(U, V) \leq I(Y; X_1, X_2 | U) \quad (16)$$

for some

$$\begin{aligned} P_{W, U, V, X_1, X_2, Y}(w, u, v, x_1, x_2, y) = \\ P_W(w) P_{U, V}(u, v) P_{X_1|W, U}(x_1 | w, u) \cdot \\ P_{X_2|W, V}(x_2 | w, v) P_{Y|X_1, X_2}(y | x_1, x_2) \end{aligned}$$

where $P_W(\alpha) = P_U(\alpha)$ and $W = U$.

The proof of Theorem 3 is omitted due to space considerations.

Theorem 4. A source $(U, V) \sim P_{U, V}$ is transmissible via the MAC $P_{Y|X_1, X_2}$ with causal cribbing at rate ρ if and only if there exists a joint input distribution P_{X_1, X_2} such that:

$$\rho H(U | V) \leq H(X_1) \quad (17)$$

$$\rho H(V | U) \leq I(Y; X_2 | X_1) \quad (18)$$

$$\rho H(U, V) \leq I(Y; X_1, X_2) \quad (19)$$

The proof of the converse part of Theorem 4 is omitted. Two proofs of the achievability part of Theorem 4, based on Theorem 3 and based on a separation argument, are given in Section V.

Further results on transmission with distortion, and different models of cribbing, are reported elsewhere.

IV. DISCUSSION

As shown in Section V, Theorem 4 implies that a source $P_{U, V}$ is transmissible via the MAC $P_{Y|X_1, X_2}$ with causal cribbing at rate ρ if and only if the Slepian-Wolf region scaled by ρ intersects the capacity region of the MAC with causal cribbing. This implies that with causal cribbing, a separation strategy, where we use a Slepian-Wolf code to compress the source and a channel code to transmit the source codewords via the MAC, is optimal. In view of this, let us examine the achievability result for strictly causal cribbing. By Theorem 3, the following inequalities are sufficient conditions for the source $P_{U, V}$ to be transmissible via the channel $P_{Y|X_1, X_2}$ with strictly causal cribbing at rate $\rho = 1$:

$$H(U | V) \leq H(X_1 | W) \quad (20)$$

$$H(V | U) \leq I(Y; X_2 | W, X_1) \quad (21)$$

$$H(U, V) \leq I(Y; X_1, X_2) \quad (22)$$

for some

$$\begin{aligned} P_{W, U, V, X_1, X_2, Y}(w, u, v, x_1, x_2, y) = \\ P_W(w) P_{U, V}(u, v) P_{X_1|W}(x_1 | w) \cdot \\ P_{X_2|W}(x_2 | w) P_{Y|X_1, X_2}(y | x_1, x_2) \end{aligned} \quad (23)$$

where $P_U(\alpha) = P_W(\alpha)$ and $U = W$. Note that the right hand sides of (20)-(22) do not depend on the source, and coincide with the right hand sides of (3)-(5). The left hand sides of (20)-(22) coincide with the left hand sides of (11)-(13). This observation may suggest that if $P_{U, V}$ and $P_{Y|X_1, X_2}$ satisfy (20)-(22), then the source can be transmitted via the channel with a separation strategy, namely, by concatenating a Slepian-Wolf code that compresses the source, with a channel code for the MAC with strictly causal cribbing. Unfortunately, this is not the case - conditions (20)-(22) do not imply separation. This is demonstrated in the following example.

Example 1. Consider the joint source (U, V) where $U \sim \text{Bernoulli}(1/2)$, $V = U$, and the binary MAC given by:

$$P_{Y|X_1, X_2}(y|x_1, x_2) = \begin{cases} \mathbb{1}[y = x_1] & \text{if } x_1 = x_2 \\ \text{Bernoulli}(1/2) & \text{otherwise} \end{cases}$$

where $\mathbb{1}[\cdot]$ is the indicator function. Thus, if $x_1 = x_2$ the inputs are connected directly to the output. If $x_1 \neq x_2$ the output is uniformly distributed on $\{0, 1\}$, independently of the inputs.

The Slepian-Wolf rate region for this source is given by

$$R_1 + R_2 \geq 1 \quad (24)$$

We claim that the capacity region of this MAC with strictly causal cribbing does not contain the line $R_1 + R_2 = 1$. Indeed, by (5)

$$R_1 + R_2 \leq I(X_1, X_2; Y) \leq 1 - H(Y|X_1, X_2) \leq 1$$

where the last inequality holds with equality only if Y is a deterministic function of X_1 and X_2 . Since by (6) we have the Markov relations $X_1 \rightarrow W \rightarrow X_2$, $H(Y|X_1, X_2) = 0$ only if X_1 and X_2 are determined by W . But this, in turn, implies that $R_1 = R_2 = 0$, by (3) and (4). Therefore the capacity region of this MAC with strictly causal cribbing does not contain the line $R_1 + R_2 = 1$.

It is easy to verify that the source (U, V) is transmissible via the channel at rate $\rho = 1$. We can employ single letter code: connect U and V directly to X_1 and X_2 , respectively. Moreover, transmissibility can be verified directly from conditions (20)-(22): the conditional entropies on the left hand side there are 0, so we can choose $P_W = P_U$, $X_1 = X_2 = W$, and the transmissibility conditions are satisfied. Hence, (20)-(22) (and therefore also Theorem 3) predict that the source can be transmitted via the MAC with strictly causal cribbing, yet it cannot be transmitted by a separation strategy.

V. PROOF OF THE ACHIEVABILITY PART OF THEOREM 4

Proof: The achievability part of Theorem 4 can be proved based on Theorem 3, or based on a separation argument. We first give a proof based on separation.

Given three nonempty intervals $\mathcal{I}_1 = [a, b]$, $\mathcal{I}_2 = [c, d]$, and $\mathcal{I}_3 = [e, f]$, a necessary and sufficient condition for the existence of a pair (R_1, R_2) such that $R_1 \in \mathcal{I}_1$, $R_2 \in \mathcal{I}_2$, and $R_1 + R_2 \in \mathcal{I}_3$, is that the endpoints satisfy

$$a + c \leq f \quad \text{and} \quad b + d \geq e \quad (25)$$

If the source $P_{U,V}$ and the channel and input distribution $P_{Y|X_1,X_2}$, P_{X_1,X_2} satisfy (17), (18), and (19), then necessarily the intervals

$$\mathcal{I}_1 = [\rho H(U|V), H(X_1)] \quad (26)$$

$$\mathcal{I}_2 = [\rho H(V|U), I(Y; X_2 | X_1)] \quad (27)$$

$$\mathcal{I}_3 = [\rho H(U, V), I(Y; X_1, X_2)] \quad (28)$$

are nonempty. By basic properties of information functions we have

$$\begin{aligned} \rho H(U|V) + \rho H(V|U) &\leq \rho H(U, V) \\ &\leq I(X_1, X_2; Y) \\ &= I(X_1; Y) + I(X_2; Y|X_1) \\ &\leq H(X_1) + I(X_2; Y|X_1) \end{aligned} \quad (29)$$

where the second inequality above holds since \mathcal{I}_3 is nonempty. It is easy to see that (29) implies (25). Therefore, we can find a pair (R_1, R_2) satisfying

$$\rho H(U|V) \leq R_1 \leq H(X_1) \quad (30)$$

$$\rho H(V|U) \leq R_2 \leq I(Y; X_2 | X_1) \quad (31)$$

$$\rho H(U, V) \leq R_1 + R_2 \leq I(Y; X_1, X_2) \quad (32)$$

In other words, the scaled (by ρ) Slepian-Wolf rate region and the capacity region of the MAC with causal cribbing intersect. We can now construct a source-channel code by concatenating a Slepian-Wolf source code and a channel code for the MAC with causal cribbing. The details are omitted.

We give now a proof of the direct part of Theorem 4 for the special case of $\rho = 1$, based on Theorem 3. We will use Shannon strategies, as in [4]. Consider all different strategies $t \in \mathcal{T} \triangleq |\mathcal{X}_1|^{|\mathcal{X}_2|}$ that map inputs $x_1 \in \mathcal{X}_1$ into inputs $x_2 \in \mathcal{X}_2$. A distribution on this set induces a random variable, which we denote by T . For the DMMAC $(\mathcal{X}_1, \mathcal{X}_2, P_{Y|X_1,X_2}, \mathcal{Y})$, the DM derived MAC is denoted by $(\mathcal{X}_1, \mathcal{T}, P_{Y|X_1,\mathcal{T}}, \mathcal{Y})$, where $P_{Y|X_1,\mathcal{T}}^*(y | x_1, t) \triangleq P_{Y|X_1,X_2}(y | x_1, t(x_1))$.

Define a set of conditions as follows:

$$\begin{aligned} H(U|V) &\leq H(X_1|W, V) \\ H(V|U) &\leq I(T; Y|X_1, W, U) \\ H(U, V) &\leq I(X_1, T; Y | U) \end{aligned}$$

for

$$\begin{aligned} &P_{W,U,V,X_1,T,Y}(w, u, v, x_1, t, y) \\ &= P_W(w) P_{U,V}(u, v) P_{X_1|W,U}(x_1 | w, u) \\ &\quad \cdot P_{T|W,V}(t | w, v) P_{Y|X_1,T}^*(y | x_1, t) \end{aligned} \quad (33)$$

By Theorem 3, a source $P_{UV}(u, v)$ for which the conditions above hold, is transmissible by the derived MAC with strictly causal cribbing. If we now restrict the distribution in (33) to satisfy

$$\begin{aligned} P_{W,U,V,X_1,T,Y}(w, u, v, x_1, t, y) &= P_W(w) P_{U,V}(u, v) P_{X_1}(x_1) \\ &\quad P_T(t) P_{Y|X_1,T}^*(y | x_1, t) \end{aligned}$$

then

$$\begin{aligned} H(X_1|W, V) &= H(X_1) \\ I(T; Y|X_1, W, U) &= I(T; Y|X_1) \\ &= I(T, X_2; Y|X_1) \\ &= I(X_2; Y|X_1) + I(T; Y|X_1, X_2) \\ &= I(X_2; Y|X_1) \\ I(X_1, T; Y | U) &= I(X_1, X_2; Y) \end{aligned}$$

and

$$P_{X_1,X_2,Y}(x_1, x_2, y) = P_{X_1}(x_1) \cdot \sum_{t:t(x_1)=x_2} P_T(t) \cdot P_{Y|X_1,X_2}(y | x_1, x_2) \quad (34)$$

where for (34) we used the same method presented in [4, (44), (45)], expressing $P_{X_2|X_1}(x_2 | x_1)$ as a function of $P_T(t)$:

$$\begin{aligned} P_{X_2|X_1}(x_2 | x_1) &= \sum_{t:t(x_1)=x_2} P_T(t) \\ P_T(t) &= \prod_{x_1} \frac{P_{X_1,X_2}(x_1, x_2 = t(x_1))}{P_{X_1}(x_1)} \end{aligned}$$

Note that any joint distribution P_{X_1,X_2} can be expressed with strategies, as indicated in (10) above, and in [4, (44), (45)]. Therefore the joint distribution of X_1 and X_2 can be arbitrary. This completes the proof of achievability, based on Theorem 3. ■

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