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Factoring linear trellises

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Abstract—Koetter/Vardy proved in [9] the remarkable result that any linear trellis can be factored into elementary trellises. We prove that the spans (and their multiplicities) of the elementary factors of a linear trellis are uniquely determined, extending a known result for minimal trellises ([4], [8], [11]). In doing so, we give a graphical method to compute all the spans and their multiplicities. We show also how to determine all the possible edge-labelings of the factors of a linear trellis, and give exact conditions for their uniqueness. These results can help to compute and classify all trellises for a given code.

I. INTRODUCTION

Linear trellises are amongst the most important graph representations of linear codes, both for theoretical and practical reasons. Towards the end of last millennium, due to the observation of striking low complexity achieved by nonconventional trellises ([1]), the interest expanded to the whole class of tail-biting trellises (of which conventional trellises can be seen as a particular case). As a consequence, in the recent years, linear tail-biting trellises has been the subject of major research within trellis theory ([9], [8], [4], [5], [10], [7]). The most important achievement is probably the factorization theorem proven in [9], which states that every linear trellis is a product of elementary trellises. This fundamental theorem opened the road to other central works concerning minimal linear trellis representations of codes ([8], [4]).

In this paper we investigate further linear trellis factorizations and give graphical/combinatorial methods to compute such factorizations. We first prove that if we forget about edge-labels then unique factorization holds, which means that all the spans of factors and their multiplicities are uniquely determined. Our proof yields at the same time a method for computing all the spans and multiplicities. This result was proven before for minimal trellises ([11]), and the striking fact is that this holds true for any linear trellis and that edge-labels play no role. We then show how to find all the possible labelings of the elementary factors of a linear trellis, once the spans' multiplicities are known. As a corollary, we get precise conditions under which such labelings are uniquely determined. Finally, we mention how these results help us to compute and classify all trellises representing a fixed code, which is important for the search of trellis representations that perform well under iterative or LP decoding.

A. Basics on Trellises

All trellises considered will be over a fixed finite field $\mathbb{F} \equiv \mathbb{F}_q$ and have a fixed length n . Also, we want to do modular arithmetic on the coordinate indices of \mathbb{F}^n , so $\mathbb{Z}_n \equiv \mathbb{Z}/n\mathbb{Z}$ will be the index set of coordinates. A *trellis* (over \mathbb{F} , and of length n) is a directed graph $T = (V, E)$ together with a partition $V = \sqcup_{i \in \mathbb{Z}_n} V_i$ indexed over \mathbb{Z}_n , such that the edges are \mathbb{F} -labeled and can only go from V_i to V_{i+1} , so that E has also a partition $E = \sqcup_{i \in \mathbb{Z}_n} E_i$, where $E_i \subseteq V_i \times \mathbb{F} \times V_{i+1}$. For formal reasons we also allow multiple edges in a trellis, i.e. each edge $e \in E$ also has a multiplicity $m(e) \geq 1$. A trellis is *conventional* if $|V_0| = 1$.

Two trellises $T = (V, E)$, $T' = (V', E')$ are *isomorphic* if there exists a bijection $f : V \rightarrow V'$ such that $e = (v, a, w) \in E$ if and only if $f(e) := (f(v), a, f(w)) \in E'$, $m(e) = m(f(e))$ for all $e \in E$, and $f(V_0) = V'_0$. If by forgetting their labels T and T' are isomorphic as directed multigraphs, then we say that T and T' are *structurally isomorphic* (again, we require V_0 to be sent to V'_0) and write $\bar{T} = \bar{T}'$.

By a *cycle* of a trellis T we mean a closed directed path in T of length n starting and ending at V_0 . The subset $C(T) := \{\text{edge-label sequences of cycles of } T\} \subseteq \mathbb{F}^n$ is the *code represented by } T. A trellis is *reduced* if every edge and every vertex belongs to a cycle. A trellis is *linear* if it is reduced, the multiplicity function $m : E \rightarrow \mathbb{N}$ on each E_i is a constant of the form q^j for some $j \geq 0$, and all V_i 's have an \mathbb{F} -vector space structure such that E_i is an \mathbb{F} -subspace of $V_i \times \mathbb{F} \times V_{i+1}$ for all i . If T and T' are isomorphic then T is linear if and only if T' is linear. Also, if T is linear then so is $C(T)$. In this paper we will consider only linear trellises, so, to simplify the terminology, from now on **by a trellis we will always mean a linear trellis**.*

We will denote by \mathbb{T} the set of trellises up to isomorphism, and by $\bar{\mathbb{T}}$ the set of *unlabeled trellises*, i.e. trellises up to structural isomorphism. Clearly $\bar{\mathbb{T}}$ can be identified with the subset of all trellises having all edge-labels equal to 0, so, we see $\bar{\mathbb{T}} \subseteq \mathbb{T}$. We have a projection map $\mathbb{T} \rightarrow \bar{\mathbb{T}}$, which to each trellis T associates its underlying unlabeled trellis \bar{T} (equivalently, putting all edge-labels of T equal to 0).

For $0 \leq h \leq n$ and $i \in \mathbb{Z}_n$, $(i, i+h] := \{i+1, i+2, \dots, i+h\} \subseteq \mathbb{Z}_n$ is a *span of length } h* of $v \in \mathbb{F}^n$ if $\text{supp}(v) \subseteq [i, i+h] := \{i\} \cup (i, i+h]$. If $h < n$ we say that the span $(i, i+h]$ is *proper*, and that it *starts* at i and *ends* at $j = i+h$.

The *degenerate span* \mathbb{Z}_n is a span of any $v \in \mathbb{F}^n$. Now, let \mathfrak{s} be a span of $v \in \mathbb{F}^n$. Let $V_t := \mathbb{F}$ for all $t \in \mathfrak{s}$ and else $V_t := 0$. Choose $w_t \in V_t \setminus \{0\}$ for all $t \in \mathfrak{s}$, while for all $t \notin \mathfrak{s}$ put $w_t := 0$. Let then $E_t := \mathbb{F}(w_t, v_t, w_{t+1})$, i.e. E_t is the subspace of $V_t \times \mathbb{F} \times V_{t+1}$ generated by the vector (w_t, v_t, w_{t+1}) . We also put $m(e) := 1$ for all $e \in E$, except in the case $v|\mathfrak{s} = 0|(i, i)$ where we put $m(e) = q$ for the (only) edge $e = (0, 0, 0) \in E_i$ (i.e. e is repeated q times). This gives the *elementary trellis* $v|\mathfrak{s}$ (up to trellis isomorphism, it doesn't depend on the choice of the w_t 's). It is clearly a (linear) trellis, and it satisfies $v|\mathfrak{s} = \lambda v|\mathfrak{s}$ for $\lambda \neq 0$, $C(v|\mathfrak{s}) = \mathbb{F}v$. Notice also that $v|\mathfrak{s} = v'|\mathfrak{s}'$ if and only if $\mathfrak{s} = \mathfrak{s}'$, and that since \mathbb{F} is finite, the set \mathbb{T}_{El} of elementary trellises is finite too.

B. Trellis products

\mathbb{T} benefits of a natural monoid structure (\mathbb{T}, \otimes) given as follows. Let $T, T' \in \mathbb{T}$ be two trellises. For $e = (v_i, a, v_{i+1}) \in E_i$, $e' = (v'_i, a', v'_{i+1}) \in E'_i$ define $e \otimes e' := ((v_i, v'_i), a + a', (v_{i+1}, v'_{i+1}))$. For each i let $V_i'' := V_i \times V'_i$ and let $E_i'' := \{e \otimes e' | e \in E_i, e' \in E'_i\}$. Then the trellis $T \otimes T' := (\sqcup V_i'', \sqcup E_i'')$ is called the *trellis product* of T and T' . This is the usual definition of trellis product as given for example in [8], [9]. Since we also allow multiple edges in a trellis, we naturally define the multiplicity of $e'' \in E''$ as $m(e'') := \sum_{e \otimes e' = e''} m(e)m(e')$. It's easy to see that if T is a product of elementary trellises then T has no multiple edges if and only if no factor of span length 0 is repeated and no factor is of the form $0|(i, i)$.

The operation \otimes is associative, commutative, and has an identity, the *zero trellis* $\bullet \xrightarrow{0} \bullet \xrightarrow{0} \dots \xrightarrow{0} \bullet \xrightarrow{0} \bullet$, i.e. (\mathbb{T}, \otimes) is a commutative monoid. Note that the only invertible element of (\mathbb{T}, \otimes) is the zero trellis, that the subset $\overline{\mathbb{T}}$ is a submonoid of (\mathbb{T}, \otimes) , and $\overline{T} \otimes \overline{T'} = \overline{T \otimes T'}$.

Also, a fundamental property of the trellis product is the identity $C(T \otimes T') = C(T) + C(T')$, from which follows that any linear code can be represented by a product of elementary trellises. That any trellis is a product of elementary trellises is the following famous factorization theorem (see [9]):

Theorem 1 (KV). *Any (linear) trellis is a product of elementary trellises.*

II. UNIQUE FACTORIZATION IN $\overline{\mathbb{T}}$

Knowing that every trellis can be factored into elementary trellises, a natural following question is whether the factorization must be unique or not. The answer is that in general it does not. For example, $111|(0, 2] \otimes 010|(1, 1] = 101|(0, 2] \otimes 010|(1, 1]$, but the two factorizations are different, as $111|(0, 2] \neq 101|(0, 2]$. Nevertheless, one sees here that the two factorizations have precisely the same set of spans, i.e. up to labels the two factorizations are equal. This is no coincidence, as it is a well known result that different factorizations of the minimal conventional trellis for a linear code must have the same set of spans (for details on minimal and conventional trellises see [11], [8]), a result that is proved only as a byproduct of the minimality assumption. More recently

the same result was also proven for minimal nonconventional trellises (see [8], [4]), again by exploiting the minimality assumption. We show that this holds actually for any trellis, and that the edge-labels really play no role in determining the spans of factors. The structure alone determines the spans. Here is our main result of this section:

Theorem 2. *In $(\overline{\mathbb{T}}, \otimes)$ unique factorization holds, that is, if $\overline{T}_1 \otimes \overline{T}_2 \otimes \dots \otimes \overline{T}_r = \overline{T}'_1 \otimes \overline{T}'_2 \otimes \dots \otimes \overline{T}'_{r'}$, with $T_i, T'_j \in \mathbb{T}_{El}$, then $r = r'$ and, up to order, $\overline{T}_i = \overline{T}'_i$ for all i . In other words, the spans (along with their multiplicities) appearing in a factorization of a trellis T are uniquely determined.*

It is easy to see that the multiplicity of the degenerate span \mathbb{Z}_n is uniquely determined and is equal to the number of connected components of T , so, the real problem is understanding what happens with proper spans. In what follows we will tackle this problem by a graph-theoretical approach, which yields also a method to compute the structural factorization. Before going on to prove Theorem 2, we also inform the reader that we were recently told by Heide Gluesing-Luerssen that her PhD student Elizabeth Weaver has independently obtained uniqueness of spans of factors in the particular case of *one-to-one* trellises.

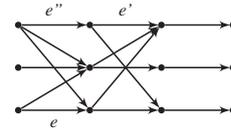
A. Graph-theoretical approach

The graph-theoretical approach involves studying the earliest intersections of paths starting along different edges from a fixed vertex. Not only does this enable to prove unique factorization in $(\overline{\mathbb{T}}, \otimes)$, but from the same intersection data one is also able to determine precisely all the spans of the factors.

Now, let us give some notation. Given a multiset S , we write $m(x, S)$ for the multiplicity of x in S (in particular $m(x, S) > 0$ if and only if $x \in S$). If T is a trellis and $e = (v, a, v')$ is an edge of T , then $h(e) := v'$ is the *head* of e , and $t(e) := v$ is its *tail*. A *path* $p \equiv p_0 p_1 \dots p_r$ in T is a sequence of edges p_0, p_1, \dots, p_r in T such that $h(p_i) = t(p_{i+1})$, $i = 0, \dots, r-1$.

Now, let $e \neq e'$ be two different edges of T such that $t(e) = t(e')$. Define $l(e, e')$ to be the smallest $r \geq 0$ such there exist two directed paths $p = p_0 \dots p_r$, $p' = p'_0 \dots p'_r$ in T satisfying $p_0 = e$, $p'_0 = e'$, and $h(p_r) = h(p'_r)$. If there exist no directed paths satisfying those conditions then we put $l(e, e') = \infty$. Define then the multiset $I(e) := \{\{l(e, e') | e' \neq e, t(e') = t(e)\}\}$.

Example 1. *Consider the unlabeled trellis*



Then $I(e) = \{\{1\}\}$, $I(e') = \{\{2\}\}$, and $I(e'') = \{\{1, 1\}\}$.

Now, let T be a trellis and let e, e' be two edges with same tail. A priori, to compute $l(e, e')$ one needs to compute the set of all possible couple of paths p, p' starting along

e, e' , and look for the earliest possible intersection. But in fact, by linearity (we are only considering trellises with linear structure) one can fix p and let only p' vary:

Observation 1. Let T be a trellis and let e, e' be two edges such that $t(e) = t(e')$. Fix a path p such that $p_0 = e$. Then $l(e, e') = \min\{r \mid \exists p' = p'_0 \dots p'_r \text{ such that } p'_0 = e', h(p'_r) = h(p_r)\}$.

The highly symmetrical structure of (linear) trellises is further reflected in the following fundamental lemma:

Lemma 1. Let T be a trellis of length n . Then:

- $l(e, e') \leq n - 1$ for any edges e, e'
- $I(e) = I(e')$ if $t(e), t(e') \in V_i$ for some i

In sight of this result, for a trellis T and $i \in \mathbb{Z}_n$ it is legitimate to define $I_i(T) := I(e)$, where e is any edge such that $t(e) \in V_i$.

The next lemma tells us that $I_i(T)$ is determined only by those factors of T whose spans start at i .

Lemma 2. Let T be a trellis, and let $T' \in \mathbb{T}_{El}$ be an elementary trellis with span not starting at i or degenerate span. Then $I_i(T \otimes T') = I_i(T)$.

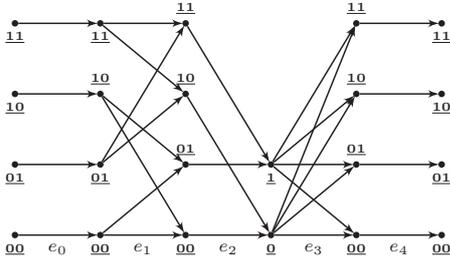
The above lemma is crucial for the following theorem, which is the key result:

Theorem 3. Let $T = T_1 \otimes \dots \otimes T_r$, with $T_1, \dots, T_r \in \mathbb{T}_{El}$. Fix $i \in \mathbb{Z}_n$, and let $L_i(T)$ be the multiset of span lengths of those T_j 's with (proper) span starting at i . Then for $l = 0, \dots, n-1$

$$\log_q \left(1 + \sum_{j=0}^l m(j, I_i(T)) \right) = \sum_{j=1}^l m(0, L_i(T))$$

The above equations can be recursively solved for $m(j, I_i(T))$, $j = 0, \dots, n-1$, and the solution is unique. Therefore, since the left-hand side depends only on T , we conclude that Theorem 2 holds. Moreover, by computing all $I_i(T)$'s we are able to determine all the (proper) spans and their multiplicities in any factorization of T .

Example 2. Let T be the unlabeled connected trellis



This trellis is \mathbb{F}_2 -linear, its linear structure given by the shown \mathbb{F}_2 -labels of vertices. Then one easily computes

- $I_0(T) = I(e_0) = \emptyset$
- $I_1(T) = I(e_1) = \{\{2\}\}$
- $I_2(T) = I(e_2) = \emptyset$
- $I_3(T) = I(e_3) = \{\{3, 4, 4\}\}$

- $I_4(T) = I(e_4) = \emptyset$

So, by Theorem 3, we get that $L_0(T) = L_2(T) = L_4(T) = \emptyset$, $L_1(T) = \{\{3\}\}$, $L_3(T) = \{\{4, 5\}\}$. Therefore $T = (1, 3) \otimes (3, 1) \otimes (3, 2)$, as one can check.

III. FACTORIZATION IN \mathbb{T}

We have already pointed out that in (\mathbb{T}, \otimes) unique factorization doesn't hold, since the labels are not necessarily uniquely determined. Nevertheless, we give a method to find a legitimate labeling of a structural factorization of a trellis. We will also deduce precise conditions for a trellis T under which unique factorization in (\mathbb{T}, \otimes) holds.

Let $T \in \mathbb{T}$, and let $\mathcal{S}(T)$ be the span multiset of T , which by Theorem 2 is uniquely determined. If $(i, j) \in \mathcal{S}(T)$, i.e. $m((i, j), \mathcal{S}(T)) \geq 1$, we say that (i, j) is a span of T . Also, we put $\tilde{s} := [i, j]$ for a proper span $s = (i, j)$, and $\tilde{\mathbb{Z}}_n = \mathbb{Z}_n$.

By linearity there exists a zero vertex $z_i \in V_i$ for all i . Let (i, j) be a proper span of T , and let $C_{(i,j)}(T)$ denote the code of all label-sequences of paths starting at z_i and ending at z_j . This is a code of length equal to the span length of (i, j) , but it is naturally a code in \mathbb{F}^n by setting the coordinates outside the span (i, j) equal to 0. It is not hard to see that it is linear, since T is so. Also, for the degenerate span \mathbb{Z}_n we put $C_{\mathbb{Z}_n}(T) := C(T)$. Then we have the following:

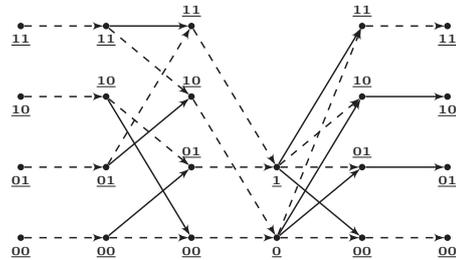
Theorem 4. Let $T \in \mathbb{T}$, and let s be a span of T of multiplicity $r = m(s, \mathcal{S}(T))$. Then $v_1|_s \otimes \dots \otimes v_r|_s$ divides T if and only if

$$\mathbb{F}v_1 + \dots + \mathbb{F}v_r + \sum_{s' \in \mathcal{S}(T) \mid \tilde{s}' \subsetneq \tilde{s}} C_{s'}(T) = C_s(T)$$

The following corollary extends the well known result that if there are no containments between different atomic spans of a code then the associated minimal conventional trellis factorizes uniquely, since all atomic generators are uniquely determined (up to scalar multiplication).

Corollary 1. Let $T = v_1|(i_1, j_1) \otimes \dots \otimes v_r|(i_r, j_r)$. Then v_1, \dots, v_r are uniquely determined up to scalar multiplication and ordering (i.e. the factorization of T is unique) if and only if there are no containments between different spans of T and all their multiplicities are equal to 1.

Example 3. Label the edges of T from example 2 as follows (by using dashed lines for 0's and full lines for 1's):



Thus labeled, T is again \mathbb{F}_2 -linear. Then one gets:

- $C_{(1,3)}(T) = \{00000, 01010\}$

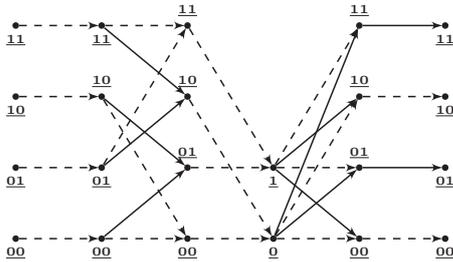
- $C_{(3,1)}(T) = \{00000, 01011\}$
- $C_{(3,2)}(T) = \{00000, 01011\}$

Therefore by Theorem 4 and its corollary T has two (and only two) distinct factorizations,

$$T = 01010|(1, 3) \otimes 01011|(3, 1) \otimes 01011|(3, 2)$$

$$T = 01010|(1, 3) \otimes 01011|(3, 1) \otimes 00000|(3, 2)$$

Note also that $T' = 01010|(1, 3) \otimes 00000|(3, 1) \otimes 01011|(3, 2)$ (depicted below) is a different trellis from T , according to Theorem 4. In fact, in T' it is possible to go out from $\underline{0} \in V_3$ along two paths with all edge-labels equal to 0, and that meet at V_2 , while this is not possible in T .



IV. APPLICATIONS

The complexity of trellis-based decoding is directly proportional to the size of the chosen representing trellis, so, to achieve low complexity it is important to search for the smallest trellis representations of codes. If trellis size is all what one is interested in, then obviously the way labels are arranged on a trellis doesn't matter (as long as the trellis represents the wanted code), which implies that it is sufficient to classify all the trellis representations up to structural isomorphism. In [8] Koetter and Vardy show that up to structural isomorphism all the minimal trellises for a given code are encoded into a single matrix (*characteristic matrix*), and show how to efficiently compute that matrix (see also [4]).

On the other hand, the performance of iterative or LP decoding algorithms on a trellis is affected by so-called *pseudocodewords*, which makes it important to sort trellises also with respect to their pseudocodewords, as one wants to find trellises that yield few bad pseudocodewords ([3], [6]). In [2] we have given an example which shows that by changing labels in a trellis it is possible to change the set of yielded pseudocodewords and yet represent the same code. This means that to sort all the trellis representations of a given code with respect to pseudocodewords it is not sufficient to do a classification of trellises up to structural isomorphism. We need a complete classification of trellises. Theorem 2 and Theorem 4 help in that direction. Indeed they yield an exact method to check whether two trellises are isomorphic or not.

To see this recall that (by Theorem 1) to define a trellis we need to specify a span multiset and a list of vectors that is compatible with that span multiset. Theorem 2 tells us first of all that two different span multisets always yield two different trellises (since they are not even structurally isomorphic!). Once a span multiset (and so by Theorem 2 the unlabeled

graph structure of the trellis) is chosen, Theorem 4 tells us the precise relations that hold among all the possible lists of vectors that give the same trellis. More precisely, if the chosen spans are $\mathfrak{s}_1, \dots, \mathfrak{s}_r$ with respective multiplicities m_1, \dots, m_r , and $\{v_{ij}\}_{1 \leq i \leq r, 1 \leq j \leq m_i}$, $\{w_{ij}\}_{1 \leq i \leq r, 1 \leq j \leq m_i}$ are list of vectors associated to the spans, then it is easy to see from Theorem 4 that the corresponding trellises are isomorphic if and only if for all $i = 1, \dots, r$ the equation

$$\sum_{h|\mathfrak{s}_h \subseteq \mathfrak{s}_i} \sum_{1 \leq j \leq m_h} \mathbb{F}v_{hj} = \sum_{h|\mathfrak{s}_h \subseteq \mathfrak{s}_i} \sum_{1 \leq j \leq m_h} \mathbb{F}w_{hj}$$

holds.

Remark 1. It is easy to show that if a trellis T for a given code is optimal with respect to iterative or LP decoding (in the sense that it yields few bad pseudocodewords) then it must be one-to-one, i.e. the list of generating vectors of T must be a basis of the code. So, for iterative or LP decoding purposes, it is actually sufficient to do a complete classification of one-to-one trellises.

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