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Coding Theory and Compressed Sensing through Convex Relaxations

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Abstract—We survey recent connections between convex relaxations used in coding theory and in sparse approximation theory. One important conclusion arising from our results is that girth can be used to certify that sparse compressed sensing matrices have good sparse approximation guarantees. This allows us to present the first deterministic measurement matrix constructions that have an optimal number of measurements for \( l_2/l_1 \) sparse approximation.

I. INTRODUCTION

Sparse approximation theory and compressed sensing have recently received significant attention in signal processing and statistics. We focus on two developments that connect this area with coding theory and allow the translation of results and techniques. The first involves mathematical links [1]–[3] between two linear programming relaxations: basis pursuit for CS reconstruction and the fundamental polytope relaxation for the decoding of binary linear codes [4], [5]. In coding theory, the linear-program (LP) relaxation was introduced by Feldman, Wainwright and Karger [4] and is closely related to the work by Koetter and Vontobel [5]. This work defined the so-called fundamental polytope and structures called pseudocodewords that cause problems to channel decoders (see also the work of Wiberg [6] and Forney et al. [7].

Our recent results [1], [8] connect this theory to the basis pursuit LP relaxation used in compressed sensing. The high-level message is that ‘good’ channel codes can be used over the reals as provably good CS measurement matrices under basis pursuit. This connection allowed us to solve a well-known open problem in compressed sensing, the deterministic construction of measurement matrices with an order-optimal number of rows [1], [8].

The second connection involves the nature of iterative message-passing algorithms used both for channel coding and compressed sensing. A number of iterative algorithms (e.g. [9]–[13]) have been proposed for compressed sensing recovery, with the benefit of faster decoding complexity and comparable performance to basis pursuit. Very similar iterative message-passing algorithms have been developed and used for the decoding of error-correcting codes [14]. More interestingly, the techniques for their analysis have recently found applicability for compressed sensing. Channel-coding-inspired message-passing decoders for compressed sensing problems have been also discussed in [15]–[17].

The fact that both iterative message-passing algorithms and LP relaxations are mathematically connected for these two problems is not an accident: For channel coding, the LP relaxation is currently understood as an approximation to iterative message-passing [18] and the theory of pseudocodewords allows a rigorous non-asymptotic theory for decoding and design of error-correcting codes under polynomial time decoding algorithms.

II. RECENT RESULTS

Assume we observe \( m \) linear measurements of an unknown vector \( e \in \mathbb{R}^n \):

\[
H \cdot e = s,
\]

where \( H \) is a real-valued matrix of size \( m \times n \), called the measurement matrix. When \( m < n \), this is an underdetermined system of linear equations and one fundamental compressed sensing problem involves recovering \( e \) assuming that it is also \( k \)-sparse, i.e. it has \( k \) or less non-zero entries.

The sparse approximation problem goes beyond exactly sparse vectors and requires the recovery of a \( k \)-sparse vector \( \hat{e} \) that is close to \( e \), even if \( e \) is not exactly \( k \)-sparse itself. Recent breakthrough results [19], [20] showed that it is possible to construct measurement matrices with \( m = O(k \log(n/k)) \) rows that recover \( k \)-sparse signals exactly in polynomial time. These results rely on randomized matrix constructions and establish that the optimal number of measurements will be sufficient with high probability over the choice of the matrix and/or the signal. Unfortunately, the required properties of Restricted Isometry Property (RIP) [20], Nullspace [21] and high expansion (expansion quality \( \epsilon < 1/6 \)) have no known ways to be deterministically constructed or efficiently checked.

There are several explicit constructions of measurement matrices (e.g. [22], [23]) which, however, require a slightly suboptimal number of measurements \( m \) growing super-linearly as a function of \( n \), for \( k = p \cdot n \).

We focus on the linear sparsity regime where \( k \) is a fraction of \( n \) and optimal number of measurements will also be a fraction of \( n \). The explicit construction of measurement matrices with an optimal number of rows is a well-known open problem in compressed sensing theory (see e.g. [24] and references therein). A closely related issue is that of checking or certifying in polynomial time that a given candidate matrix has good recovery guarantees or satisfies the RIP.

Consider a sparse matrix \( H \in \{0, 1\}^{m \times n} \) that has \( d_c \) ones per row and \( d_v \) ones per column. If the bipartite graph
corresponding to $H$ has $\Omega(\log n)$ girth, then for $k = p \cdot n$ and an optimal number of measurements $m = c_2 \cdot n$, we show that $H$ offers $\ell_1/\ell_1$ sparse approximation under basis pursuit decoding. Our technical requirement of girth, unlike expansion or RIP, is easy to check and several deterministic constructions of matrices with $m = c_1 \cdot n$ and $\Omega(\log n)$ exist, starting with the early construction in Gallager’s thesis [25], and progressive edge-growth Tanner graphs.

Our result is a weak bound, also known as a “for-every signal” guarantee [24]. This means that we have a fixed deterministic matrix and show the $\ell_1/\ell_1$ sparse approximation guarantee with high probability over the support of the signal. To the best of our knowledge, this is the first deterministic construction of matrices with an optimal number of measurements.

Our proof relies on the discussed connections of channel decoding LP and compressed sensing. We rely on a primal-based density evolution technique initiated by Koetter and Vontobel [26] and analytically strengthened in the break-through paper of Arora et al. [27] that established the best known finite-length threshold results for LDPC codes under LP decoding. In our presentation we will survey this area, present recent results and discuss open problems.

REFERENCES


