DISS. ETH NO. 19871

DEFORMATION AND DUALITY FROM THE SYMPLECTIC POINT OF VIEW

A dissertation submitted to
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for the degree of
Doctor of Sciences

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2012

Acknowledgements

I would like to express my gratitude to my supervisor, Professor Paul Biran for his guidance, support and encouragement during the years, for sharing with me his inspiration and enthusiasm to the subject, and for setting for me a model and an example for mathematical work in general.

I would also like to thank Professor Dietmar Salamon and Professor Felix Schlenk for their interest in this work.

Abstract

In this dissertation we study applications of methods of symplectic topology, mainly the theories of Floer and quantum homology, to the following two subjects in complex projective geometry: (1) Morse-Bott degenerations of Kähler Manifolds and (2) manifolds with small dual.

(1) Morse-Bott degenerations: A Morse-Bott degeneration is a holomorphic map $\pi : X \to \mathbb{D}$ from a Kähler manifold (X, Ω) to the unit disc $\mathbb{D} \subset \mathbb{C}$ such that $Crit(\pi) \subset \pi^{-1}(0)$ and the holomorphic Hessian of π is a a non-degenerate quadratic form when restricted to the normal bundle of $Crit(\pi)$ in X. Our first result on the subject is showing that Lagrangian submanifolds in the singularity locus of the degeneration $L \subset Crit(\pi) \subset \Sigma_0$ give rise to Lagrangian submanifolds $N_z(L)$ in $\Sigma_z = \pi^{-1}(z)$ in the smooth fibers $z \neq 0$. Moreover, $N_z(L) \subset \Sigma_z$ is topologically a sphere bundle over L. This could be seen as a generalization of the well known result stating that fibers of a Lefschetz fibration admit Lagrangian spheres corresponding to the singularities of the fibration. We then show, based on the construction of $N_z(L)$, that Morse-Bott degenerations admit non-trivial restrictions as an application of the theory of Lagrangian Floer homology in various cases.

(2) Manifolds with small dual from the symplectic viewpoint: Let $X \subset \mathbb{C}P^N$ be a projectively embedded manifold and let $X^* \subset (\mathbb{C}P^N)^*$ be its dual. The manifold X is said to be of small dual if X^* is not a hypersurface. Let Σ be a smooth hyperplane section of X. Our first result is showing the following relation

$$ind(X \setminus \Sigma) \leq dim_{\mathbb{C}}(X) - def(X)$$

where $ind(X \setminus \Sigma)$ is the sub-criticallity index of the Stein manifold $X \setminus \Sigma$ studied in symplectic topology and $def(X) = codim(X^*) - 1$. In particular, the Stein manifold $X \setminus \Sigma$ is subcritical whenever X is a manifold with small dual.

We then turn to study the properties of the quantum cohomology of $QH^*(\Sigma)$ under the additional assumption that X is spherically monotone. Let ω_Σ be the restriction of the Fubini-Study form of $\mathbb{C}P^N$ to Σ. We show that when X is a spherically monotone manifold with small dual the class $[\omega_\Sigma] \in QH^*(\Sigma)$ is invertible. As an application we obtain new topological restrictions on both X and Σ. We prove the inevitability of $[\omega_\Sigma]$ by showing that it could be expressed as

the Seidel element $S(\gamma) \in QH^*(X)$ of a non-trivial element $\gamma \in Ham(\pi_1(\Sigma, \omega_\Sigma))$ naturally associated to Σ.

Abstract

Diese Dissertation befasst sich mit der Anwendung von Methoden der symplektischen Topologie, in erster Linie Floer-Theorie und Quanten-Homologie, auf die folgenden beiden Bereiche der komplexen projektiven Geometrie: (1) Morse-Bott-Degenerationen von Kähler-Mannigfaltigkeiten, sowie (2) Mannigfaltigkeiten mit kleinem Dual.

(1) Morse-Bott-Degenerationen: Eine Morse-Bott-Degeneration ist eine holomorphe Abbildung $\pi : X \to \mathbb{D}$, wobei (X, Ω) eine Kähler-Mannigfaltigkeit und $\mathbb{D} \subset \mathbb{C}$ die Einheitskreisscheibe bezeichnen, so dass $Crit(\pi) \subset \pi^{-1}(0)$ gilt und die Einschränkung der holomorphen Hesse-Form von π auf das Normalenbündel von $Crit(\pi) \subset X$ nicht-ausgeartet ist. Als erstes Resultat zeigen wir, dass Lagrange Untermannigfaltigkeiten $L \subset Crit(\pi) \subset \Sigma_0$ in der singulären Faser der Degeneration Lagrange-Untermannigfaltigkeiten $N_z(L) \subset \Sigma_z = \pi^{-1}(z)$ in den glatten Fasern ($z \neq 0$) induzieren, wobei $N_z(L) \subset \Sigma_z$ topologisch ein Sphärenbündel über L ist. Dies kann als Verallgemeinerung des bekannten Resultats aufgefasst werden, dass in den Fasern einer Lefschetz-Faserung Lagrange-Sphären existieren, die den Singularitäten der Faserung entsprechen. Wir zeigen dann, basierend auf der Konstruktion von $N_z(L)$ und unter Verwendung von Lagrange-Schnitt-Floer-Homologie, dass Morse-Bott-Degenerationen nicht-triviale Einschränkungen zulassen.

(2) Mannigfaltigkeiten mit kleinem Dual aus der symplektischen Sicht: Sei $X \subset \mathbb{C}P^N$ eine projektiv eingebettete Mannigfaltigkeit und sei $X^* \subset (\mathbb{C}P^N)^*$ ihr Dual. Wir bezeichen X als Mannigfaltigkeit mit kleinem Dual, falls X^* keine Hyperfläche ist. Sei Σ ein glatter Hyperebenenschnitt von X. Unser erstes Resultat zeigt die folgende Beziehung:

$$ind(X \setminus \Sigma) \leq dim_{\mathbb{C}}(X) - def(X),$$

wobei $ind(X \setminus \Sigma)$ der in der Symplektischen Topologie betrachtete Subkritikalitätsindex der Stein-Mannigfaltigkeit $X \setminus \Sigma$ ist, und $def(X) := codim(X^*) - 1$.

Danach studieren wir die Eigenschaften der Quanten-Kohomologie $QH^*(\Sigma)$ unter der zusätzlichen Annahme, dass X sphärisch monoton ist. Sei ω_Σ die Einschränkung der Fubini-Study Form von $\mathbb{C}P^N$ auf Σ. Wir zeigen, dass für eine sphärisch monotone Mannigfaltigkeit X mit kleinem Dual die Klasse $[\omega_\Sigma] \in$

$QH^*(\Sigma)$ invertierbar ist. Als Anwendung davon erhalten wir neue topologische Restriktionen sowohl an X als auch an Σ. Wir beweisen die Invertierbarkeit der Klasse $[\omega_\Sigma]$, indem wir zeigen, dass sie sich als Seidel-Element $S(\gamma) \in QH^*(X)$ eines nicht-trivialen Element $\gamma \in Ham(\pi_1(\Sigma, \omega_\Sigma))$ ausdrücken lässt, das sich Σ auf natürliche Art und Weise zuordnen lässt.

Contents

Chapter 1

Introduction

1.1 Introduction

Since their introduction in (22) methods of pseudo-holomorphic curves have been widely used in the field of symplectic topology. One of the main applications is that, considered in the proper homological setting, pseudo-holomorphic curves lead to the definition of powerful invariants such as Floer and quantum cohomology which have been used to solve many questions in the field of symplectic topology. For instance, regarding intersection properties of Lagrangian submanifolds and the dynamical properties of the group of Hamiltonian diffeomorphisms. both being central objects of study in symplectic topology.

In this work we would study applications of methods of pseudo-hlomorphic curves to the following two settings of complex *projective* geometry:

- Degenerations of Kähler Manifolds of Morse-Bott type.

- Manifolds with small dual.

In both situations we show that the algebraic geometric setting admits non-trivial restrictions due to properties which are essentially *symplectic*.

1.2 Morse-Bott Degenerations and Lagrangian Sphere Bundles

A classical way to construct a Lagrangian sphere L in a symplectic manifold Σ, which in a sense is considered as part of the "folklore" of the field, is to let Σ participate as a smooth fibre in a Lefschetz fibration. If this is possible, the singularities of the fibration induce Lagrangian spheres in Σ and these spheres, in turn, are representatives of the corresponding vanishing cycles in the homology of Σ , see for instance (3; 34). In particular, the Lagrangian sphere L could be seen to exist from a study of the dynamical properties of the local model of the fibration, which is known as a *degeneration*.

A specific feature of Lefschetz fibrations is that their singularities are *isolated*. In this part of the work we ask what could be said in the *non-isolated* case. First, we consider the non-isolated local model, to which we refer as a *Morse-Bott degeneration* and show that Lagrangian submanifolds also arise in this case. However, in the non-isolated case the Lagrangian submanifolds which arise are sphere bundles rather than just spheres. Moreover, the base space of the bundle depends on the geometry of the corresponding singularity set.

Concretely, let X be a Kähler manifold and let Ω be a Kähler form on X which we will assume from now on to be fixed. Our setting is:

Definition 1.2.A *A proper holomorphic map $\pi : X \to \mathbb{D}$ map from a Kähler manifold (X, Ω) to the unit disc $\mathbb{D} \subset \mathbb{C}$ is called a Morse-Bott degeneration if:*

(1) The only critical value of π is $0 \in \mathbb{D}$ and Crit(π) is a Kähler submanifold of X.

(2) The holomorphic Hessian of π is a non-degenerate quadratic form when restricted to the normal bundle of Crit(π) in X.

Note that if $Crit(\pi)$ is a point the above definition reduces to give a description of the local model of a classical Lefschetz fibration around a singularity. Our first result is hence the following:

Theorem A *Let $\pi : X \to \mathbb{D}$ be a Morse-Bott degeneration and let $L \subset Crit(\pi)$ be a compact Lagrangian submanifold. Every smooth fiber Σ of π contains a Lagrangian submanifold $N_\pi(L)$ diffeomorphic to a sphere bundle over L.*

We shall refer to the Lagrangian submanifold $N_\pi(L)$ as the Lagrangian "vanishing necklace" over L induced by the degeneration π.

As mentioned, degenerations with isolated singularities exist in abundance due to the fact that they arise in the natural projective geometric construction of Lefschetz fibrations. However, it turns that examples of non-isolated degenerations are drastically harder to find. In section §2.4 we discuss the difference between the isolated and non-isolated case from the point of view of projective geometry. On the other hand, consider the following example:

Example: Let $\pi : X \to \mathbb{D}$ be a Morse degeneration i.e having an isolated singularity and $p : E \to X$ be a vector bundle on X. The map $\widetilde{\pi} : \mathbb{P}(E) \to \mathbb{D}$ given by $\widetilde{\pi} = \pi \circ p$ is a Morse-Bott degeneration with $Crit(\widetilde{\pi}) = \mathbb{C}P^k$ where $k = \mathrm{rank}(E) - 1$. Note that, in this case, the fiber satisfies $b_2(\Sigma) \geq 2$.

Recall that the minimal Chern number of a symplectic manifold Σ is given by

$$C_\Sigma = \min\left\{c_1(A) > 0 \mid A \in \mathrm{image}\left(\pi_2(\Sigma) \to H_2(\Sigma;\mathbb{Z})\right)\right\} \in \mathbb{N}$$

A fundamental invariant in the study of Lagrangian intersection properties is Floer homology theory. the theory was originally introduced by Floer in (19). The theory was later extended by Oh in (30) for the class of so called monotone Lagrangian submanifolds. The Floer homology ring associated to a monotone Lagrangian submanifold L of a symplectic manifold (M, ω) is denoted $HF(L)$. As an application of methods of Floer homology for Lagrangian intersections we obtain the following restrictions:

Theorem B *Let Σ be a Fano manifold of $dim_{\mathbb{C}}\Sigma = n$ with $h^{1,1}(\Sigma) = 1$ and minimal Chern number C_Σ. Let $1 \leq k \leq n-2$. Suppose there exists a Morse-Bott degeneration $\pi : X \to \mathbb{D}$ of Σ with $Crit(\pi) \simeq \mathbb{C}P^k$ then:*

(1) If $n \neq 3k+1$ then one of the following holds:

(a) $C_\Sigma \mid k+1$.

(b) $2C_\Sigma \mid n - k + 1$.

(c) $2C_\Sigma \mid n + k + 2$. *Furthermore* $C_\Sigma \mid 2k + 1$ *if* $n > 3k + 1$ *and* $C_\Sigma \mid 2k + 2$ *if* $n < 3k + 1$.

(2) If $n = 3k + 1$ *then* $2C_\Sigma \mid n - k + 1$.

Let us note that the proof of Theorem B is based on showing that the degeneration π gives rise to a displaceable Lagrangian sphere bundle $N_{\widetilde{\pi}}(L) \subset \Sigma \times A$ over S^{2k+1}, where A is a certain $(k + 1)$-dimensional variety. The Lagrangian $N_{\widetilde{\pi}}(L)$ is obtained via Theorem A applied to a modification $\widetilde{\pi} : X \times A \to \mathbb{D}$ of the original degeneration π. The restrictions of Theorem B follow from an application of methods of Floer homology to the Lagrangian $N_{\widetilde{\pi}}(L)$. Theorem B is proved in section §2.3. A fundamental aspect of our approach is a spectral sequence on Floer homology due to Biran and Cornea (11), which is a variant of the spectral sequence introduced by Oh in (31).

1.3 Manifolds with Small Dual.

In the second part of the work we study a special class of complex algebraic manifolds called *projective manifolds with small dual.* Recall that the dual variety X^* of a projectively embedded algebraic manifold $X \subset \mathbb{C}P^N$ is by definition the space of all hyperplanes $H \subset \mathbb{C}P^N$ that are not transverse to X, i.e.

$$X^* = \{H \in (\mathbb{C}P^N)^* \mid H \text{ is somewhere tangent to } X\}.$$

A projectively embedded algebraic manifold $X \subset \mathbb{C}P^N$ is said to have small dual if the dual variety $X^* \subset (\mathbb{C}P^N)^*$ has (complex) codimension ≥ 2. Let us mention that for "most" manifolds the codimension of X^* is 1, however, in special situations the codimension might be larger. To measure to which extent X deviates from the typical case one defines the defect number of an algebraic manifold $X \subset \mathbb{C}P^N$ by

$$\operatorname{def}(X) = codim(X^*) - 1.$$

See (37). Thus we will refer to manifolds with small dual also as *manifolds with positive defect.* Note that this is not an intrinsic property of X, but rather of a given projective embedding of X.

The class of algebraic manifolds with $\mathrm{def}(X) > 0$ was studied by many authors, for instance see (14; 15; 21; 24; 36). These studies show that manifolds with small dual have very special geometry. We will show that these manifolds exhibit various unique properties from the point of view of symplectic topology.

1.3.1 Small Dual and Sub-crticality.

In symplectic topology one has the class of *sub-critical Stein manifolds* which has been studied in many contexts (9; 10; 16; 17). We will first show that the class of projective manifolds with positive defect is intimately related to the class of sub-critical Stein manifolds. Let Y be a Stein manifold and set:

$$\mathrm{ind}(Y) = \min \{\mathrm{ind}_{\max}(\varphi) \mid \varphi : Y \to \mathbb{R} \text{ exhausting p.s.h Morse function}\},$$

where $\mathrm{ind}_{\max}(\varphi)$ is the maximal Morse index of φ. By exhausting we mean that the function is proper and bounded from below and p.s.h. stands for *pluri-subharmonic*. We refer to the number $\mathrm{ind}(Y)$ as the sub-criticality index of Y. It is well known (16; 17) that $\mathrm{ind}(Y) \leq \dim_{\mathbb{C}} Y$. A Stein manifold Y is called sub-critical if $\mathrm{ind}(Y) < \dim_{\mathbb{C}} Y$, i.e. it admits an exhausting Morse pluri-subharmonic function $\varphi : Y \to \mathbb{R}$ with $\mathrm{ind}_z(\varphi) < \dim_{\mathbb{C}} Y$ for every $z \in \mathrm{Crit}(\varphi)$. (We refer the reader to (16; 17) for the symplectic theory of Stein manifolds.) Our first result is:

Theorem C *Let $X \subset \mathbb{C}P^N$ be a projective manifold with small dual and let $\Sigma \subset X$ be a smooth hyperplane section of X. Then the Stein manifold $X \setminus \Sigma$ is sub-critical. In fact:*

$$\mathrm{ind}(X \setminus \Sigma) \leq dim_{\mathbb{C}}(X) - \mathrm{def}(X).$$

1.3.2 Hyperplane Sections and the Seidel Element.

Our next results are concerned with geometric properties of a smooth hyperplane section $\Sigma \subset X$ of a manifold $X \subset \mathbb{C}P^N$ with small dual, under the additional assumption that $b_2(X) = 1$. (Here and in what follows we denote by $b_j(X) = \dim H^j(X; \mathbb{R})$ the j'th Betti-number of X.) By a well known result of Ein (15) the assumption $b_2(X) = 1$ implies that both X and Σ are Fano manifolds.

For a space Y we will denote from now on by

$$H^*(Y) := H^*(Y;\mathbb{Z})/\text{torsion}$$

the torsion-free part of the integral cohomology $H^*(Y;\mathbb{Z})$. (We will sometime denote this also by $H^*(Y;\mathbb{Z})_{fr}$.) Denote by

$$QH^*(\Sigma;\Lambda) = (H^\bullet(\Sigma)\otimes\Lambda)^*$$

the quantum cohomology ring of Σ with coefficients in the Novikov ring $\Lambda = \mathbb{Z}[q,q^{-1}]$ (see below for our grading conventions), and endowed with the quantum product $*$. We prove:

Theorem D *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with small dual, $b_2(X) = 1$ and $\dim_{\mathbb{C}}(X) \geq 2$. Let Σ be a smooth hyperplane section of X. Let ω be the restriction of the Fubini-Study Kähler form of $\mathbb{C}P^N$ to Σ. Then*

$$[\omega] \in QH^2(\Sigma;\Lambda)$$

is an invertible element with respect to the quantum product.

We will actually prove a slightly stronger result in §3.4, see Theorem 3.4.0.B and the discussion after it.

A classical result of Lanteri and Struppa (25) on the topology of projective manifolds with positive defect states that if $X \subset \mathbb{C}P^N$ is a projective manifold with $dim_{\mathbb{C}}X = n$ and $\text{def}(X) = k > 0$ then:

$$b_j(X) = b_{j+2}(X) \;\; \forall\, n-(k-1) \leq j \leq n+k-1.$$

(In §3.2 we will reprove this fact using Morse theory). As we will see in Corollary 3.4.0.D below, Theorem D implies stronger topological restrictions in the case $b_2(X) = 1$.

As mentioned, under the assumption $b_2(X) = 1$ the manifold Σ is Fano and in this case the quantum cohomology $QH^*(\Sigma;\Lambda) = (H^\bullet(\Sigma)\otimes\Lambda)^*$ admits a grading induced from both factors $H^\bullet(\Sigma)$ and Λ. Here we grade Λ by taking $deg(q) = 2C_\Sigma$, where

$$C_\Sigma = \min\left\{c_1^\Sigma(A) > 0 \mid A \in \text{image}\left(\pi_2(\Sigma) \to H_2(\Sigma;\mathbb{Z})\right)\right\} \in \mathbb{N}$$

is the minimal Chern number of Σ. Here we have denoted by $c_1^{\Sigma} \in H^2(\Sigma;\mathbb{Z})$ the first Chern class of the tangent bundle $T\Sigma$ of Σ. Theorem D implies that the map

$$*[\omega] : QH^*(\Sigma;\Lambda) \longrightarrow QH^{*+2}(\Sigma;\Lambda), \quad a \longmapsto a*[\omega]$$

is an isomorphism. In our case, a computation of Ein (15) gives:

$$2C_X = n+k+2, \quad 2C_{\Sigma} = n+k.$$

(It is well known, by a result of Landman, that n and k must have the same parity. See §3.1). Define now the cohomology of X graded cyclically as follows:

$$\widetilde{H}^i(X) = \bigoplus_{l\in\mathbb{Z}} H^{i+2C_X l}(X), \quad \widetilde{b}_i(X) = \operatorname{rank} H^i(X). \tag{1.1}$$

Similarly, define $\widetilde{H}^i(\Sigma)$ and $\widetilde{b}_i(\Sigma)$ in a similar way (note that in the definition of $\widetilde{H}^i(\Sigma)$ one has to replace also C_X by C_{Σ}). Theorem D together with a simple application of the Lefschetz hyperplane section theorem give the following result:

Corollary 1.3.A *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with small dual and $b_2(X) = 1$. Then $\widetilde{b}_j(X) = \widetilde{b}_{j+2}(X)$, $\forall\, j \in \mathbb{Z}$. Moreover, if $\Sigma \subset X$ is a smooth hyperplane section then similarly to X we have $\widetilde{b}_j(\Sigma) = \widetilde{b}_{j+2}(\Sigma)$, $\forall\, j \in \mathbb{Z}$.*

Note that if $dim_{\mathbb{C}}(X) = n$ and $\operatorname{def}(X) = k$ the above implies the following relations among the Betti numbers of X:

$$\mathrm{b}_j(X) + b_{j+n+k+2}(X) = b_{j+2}(X) + b_{j+n+k+4}(X), \ \ \forall\, 0 \le j \le n+k-1,$$
$$b_{n+k}(X) = b_{n+k+2}(X) + 1, \quad b_{n+k+1}(X) = b_{n+k+3}(X),$$

and the following ones for those of Σ:

$$\mathrm{b}_j(\Sigma) + b_{j+n+k}(\Sigma) = b_{j+2}(\Sigma) + b_{j+n+k+2}(\Sigma), \ \ \forall\, 0 \le j \le n+k-3,$$
$$b_{n+k-2}(\Sigma) = b_{n+k}(\Sigma) + 1, \quad b_{n+k-1}(\Sigma) = b_{n+k+1}(\Sigma).$$

We will prove a slightly stronger result in §3.4, see Corollary 3.4.0.D.

Example: Consider the complex Grassmannian $X = Gr(5,2) \subset \mathbb{C}P^9$ of 2-dimensional subspaces in $\mathbb{C}^5$ embedded in projective space by the Plücker embedding. It is known that $\operatorname{def}(X) = 2$, see (37). We have $\dim_{\mathbb{C}}(X) = 6$ and $2C_X = 10$. The table of Betti numbers of X is given as follows:

q	0	1	2	3	4	5	6	7	8	9	10	11	12
$b_q(X)$	1	0	1	0	2	0	2	0	2	0	1	0	1

Further implications of Theorem D are obtained by studying the algebraic properties of the inverse $[\omega]^{-1}$. First note that due to degree reasons the inverse element should be of the form

$$[\omega]^{-1} = \alpha_{n+k-2} \otimes q^{-1} \in QH^{-2}(\Sigma; \Lambda)$$

where $\alpha_{n+k-2} \in H^{n+k-2}(\Sigma)$ is a nontrivial element. Moreover, this element needs to satisfy the following conditions:

$$[\omega] \cup \alpha_{n+k-2} = 0, \quad ([\omega] * \alpha_{n+k-2})_1 = 1,$$

where $([\omega] * \alpha_{n+k-2})_1 \in H^0(\Sigma)$ is determined by the condition that

$$\langle ([\omega] * \alpha_{n+k-2})_1, - \rangle = GW_1^{\Sigma}(PD[\omega], PD(\alpha_{n+k-2}), -).$$

Here PD stands for Poincaré duality, and for $a \in QH^l(\Sigma; \Lambda)$ and $i \in \mathbb{Z}$ we denote by $(a)_i \in H^{l-2iC_\Sigma}(\Sigma)$ the coefficient of q^i in a. The notation $GW_j^{\Sigma}(A, B, C)$ stands for the Gromov-Witten invariant counting the number of rational curves $u : \mathbb{C}P^1 \to \Sigma$ passing through three cycles representing the homology classes A, B, C with $c_1(u_*[\mathbb{C}P^1]) = jC_\Sigma$.

Let us mention that the fact that the class $([\omega] * \alpha_{n+k-2})_1$ is non-trivial implies that Σ is uniruled. The uniruldness of Σ was previously known and the variety of rational curves on it was studied by Ein in (15).

The method of proof of Theorem D is an application of the theory of Hamiltonian S^2-fibrations and, in particular, the Seidel representation, see (35). In (35) Seidel constructed a representation of $\pi_1(Ham(\Sigma, \omega))$ on $QH(\Sigma; \Lambda)$ given by a group homomorphism

$$S : \pi_1(Ham(\Sigma, \omega)) \longrightarrow QH(\Sigma; \Lambda)^{\times},$$

where $QH(\Sigma; \Lambda)^{\times}$ is the group of invertible elements of the quantum cohomology algebra.

Theorem D follows from:

Theorem F *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with small dual and $b_2(X) = 1$. Let $\Sigma \subset X$ be a smooth hyperplane section of X and denote by ω the symplectic* *structure induced on* Σ *from* $\mathbb{C}P^N$. *There exists a* nontrivial *element* $1 \neq \lambda \in \pi_1(Ham(\Sigma, \omega))$ *whose Seidel element is given by*

$$S(\lambda) = [\omega] \in QH^2(\Sigma; \Lambda).$$

Before continuing to examples, let us mention that by results of (6), based on Mori theory, the classification of manifolds with small dual is reduced to the case $b_2(X) = 1$. Here is a list of examples of manifolds with small dual and $b_2(X) = 1$ (see (37) for more details):

Example:

1. $X = \mathbb{C}P^n \subset \mathbb{C}P^{n+1}$ has $\mathrm{def}(X) = n$.
2. $X = Gr(2l+1, 2)$ embedded via the Plucker embedding has $\mathrm{def}(X) = 2$.
3. $X = \mathbb{S}_5 \subset \mathbb{C}P^{15}$ the 10–dimensional spinor variety has $\mathrm{def}(X) = 4$.
4. In any of the examples (1)–(3) one can take iterated hyperplane sections and still get manifolds with def > 0 and $b_2 = 1$, provided that the number of iterations does not exceed the defect-1. (See §3.1.)

The manifolds in (1) $-$ (3) together with the corresponding hyperplane sections (4) are the only known examples of projective manifolds with small dual and $b_2(X) = 1$, see (7; 36). On the basis of these examples, it is conjectured in (7) that all non-linear algebraic manifolds with $b_2(X) = 1$ have $\mathrm{def}(X) \leq 4$.

The rest of this work is organized as follows: Morse-Bott degenerations are considered in §2: In §2.1 we prove Theorem A. In §2.2 we review relevant facts from Lagrangian Floer theory and Lagrangian quantum cohomology theory. Section §2.3 is devoted to the proof of Theorem B. In section §2.4 we consider Morse-Bott degenerations in projective algebriac geometry.
In section §3 we study the symplectic topology of manifolds with positive defect: In §3.1 we review basic facts on projective manifolds with positive defect. In §3.2 we prove Theorem C. In §3.3 we review relevant results from the theory of quantum cohomology and the Seidel representation. In section §3.4 and §3.5 we prove Theorem D. In section §3.7 we prove additional results on the topology of manifolds with small dual using methods of Lagrangian (non)-intersections. In §3.8 we discuss further possible directions of study.

Chapter 2

Morse-Bott Degenerations and Lagrangian Subamnifolds

2.1 Morse-Bott Degenerations and Lagrangian Necklaces

Consider the following setting of Morse-Bott degenerations:

Definition 2.1.B *A proper holomorphic map $\pi : X \to \mathbb{D}$ map from a Kähler manifold (X, Ω) to the unit disc $\mathbb{D} \subset \mathbb{C}$ is called a Morse-Bott degeneration if:*

(1) The only critical value of π is $0 \in \mathbb{D}$ and Crit(π) is a Kähler submanifold of X.

(2) The holomorphic Hessian of π is a non-degenerate quadratic form when restricted to the normal bundle of Crit(π) in X.

This section is devoted to the proof of the following theorem:

Theorem 2.1.C *Let $\pi : X \to \mathbb{D}$ be a Morse-Bott degeneration and let $L \subset$ Crit(π) be a Lagrangian submanifold. Every smooth fiber $\Sigma_z = \pi^{-1}(z)$ for $0 \neq z \in \mathbb{D}$ contains a Lagrangian submanifold $N_z(L)$ diffeomorphic to a sphere bundle over L.*

Proof: We follow the lines of Donaldson's proof for the Morse case which appears in (34). We refer the reader to (5) for relevant facts on Morse-Bott theory. Set $F = Re(\pi)$ and $H = Im(\pi)$ and denote by J the complex structure of X. Since π is holomorphic the negative gradient flow of F with respect to the metric $g_J(\cdot,\cdot) = \Omega(\cdot, J\cdot)$ is the same as the Hamiltonian flow of H with respect to the Kähler form Ω. Denote this flow by ϕ_t and let $S(L)$ be the stable submanifold of L under this flow, see (23).

Lemma 2.1.D *$S(L)$ is a lagrangian submanifold in (X, Ω).*

Proof: Let $\Theta : S(L) \to L$ be the end point map given by $\Theta(x) := lim_{t\to\infty}\phi_t(x)$. In a small neighborhood $U \subset S(L)$ of L the map Θ is a locally trivial fibration over L, see (23). Let us show that U is isotropic. Indeed, for every $x \in U$ and $v_1, v_2 \in T_x(U)$ we have

$$\Omega_x(v_1, v_2) = \Omega_{\pi(x)}(d\Theta(v_1), d\Theta(v_2)) = 0$$

because ϕ_t is Hamiltonian with respect to Ω and L is Lagrangian. Moreover, the fact that U is isotropic implies that $S(L)$ is isotropic since ϕ_t is Hamiltonian and every point $x \in S(L)$ is sent to U by ϕ_t for t large enough.

Finally recall that

$$dim_{\mathbb{R}}S(L) = dim_{\mathbb{R}}L + index(F)$$

Where $index(F)$ is the Morse-Bott index, see (5). A simple computation shows that

$$indexF = \frac{1}{2}\left(dim_{\mathbb{R}}X - dim_{\mathbb{R}}Crit(\pi)\right) = \frac{1}{2}dim_{\mathbb{R}}X - dim_{\mathbb{R}}L$$

Thus $dim_{\mathbb{R}}S(L) = \frac{1}{2}dim_{\mathbb{R}}X$ and hence $S(L)$ is Lagrangian. $\square$

Continuation of the proof of Theorem 2.1.C: Denote $N_\epsilon(L) = F^{-1}(\epsilon) \cap S(L)$ for $\epsilon > 0$ small enough. By Morse-Bott theory $N_\epsilon(L)$ is a sphere bundle over L, of dimension $dim_{\mathbb{R}}N_\epsilon(L) = dim_{\mathbb{R}}S(L) - 1$. As ϕ_t is the Hamiltonian flow of H, the function H is constant along the flow lines of ϕ_t. We deduce that $S(L) \subset H^{-1}(0)$. Consequently

$$N_\epsilon(L) \subset F^{-1}(\epsilon) \cap H^{-1}(0) = \pi^{-1}(\epsilon) = \Sigma_\epsilon$$

This proves that the fibre Σ_ϵ contains the Lagrangian submanifold $N_\epsilon(L)$ which is a sphere bundle over L. Finally, by Moser argument all fibers $\Sigma_z = \pi^{-1}(z)$, $z \neq 0$, are symplectomorphic, hence they all contain such Lagrangians. $\square$

Definition 2.1.E *Let $\pi : X \to \mathbb{D}$ be a Morse-Bott degeneration and $L \subset Crit(\pi)$ be a Lagrangian submanifold. We refer to $N_z(L) \subset \Sigma_z$ as the Lagrangian necklace over L in the fibre Σ_z.*

Remark: In case $Crit(\pi)$ is a finite number of points and L coincides with this set of points then $N_z(L)$ is just the set of Lagrangian spheres representing the vanishing cycles as appears in (3; 34).

2.2 Floer Homology and Lagrangian Quantum Homology

An essential ingredient in our approach are two homology theories associated to monotone Lagrangian submanifolds : Floer homology and Lagrangian quantum homology.

Floer homology theory was originally introduced by Floer, for some class of Lagrangian submanifolds, in order to study problems of Lagrangian intersections, see e.g (19). The theory was later extended by Oh (30) for the class of so called monotone Lagrangian submanifolds. The Floer homology ring associated to a monotone Lagrangian submanifold L of a symplectic manifold (M, ω) is denoted by $HF(L)$.

Another homological invariant of Lagrangian submanifolds is Lagrangian quantum homology, introduced by Biran and Cornea in (11; 12; 13) . The quantum homology ring of a Lagrangian L is denoted by $QH_*(L)$. It is shown in (11) that the quantum homology ring $QH_*(L)$ admits many further non-trivial algebraic properties. One of these properties, which is essentially used in this work, is the existence of a spectral sequence relating $QH_*(L)$ to the singular homology $H_*(L)$.

Let us note that, although the quantum homology ring is defined in essentially different terms than the Floer homology, it is shown in (11) that, as rings, $QH_*(L)$ is isomorphic to $HF(L)$. This isomorphism allows one to relate the rich algebraic structures of $QH_*(L)$ to the geometric intersection properties underlying Floer homology $HF(L)$.

Here is a more detailed description. let (M, ω) be a connected tame symplectic manifold, see (4). Denote by $\mathcal{J}$ the space of ω-compatible almost complex structures J on M for which (M, ω, g_J) is geometrically bounded, where g_J is the associated riemannian metric defined by

$$g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$$

Let $L \subset M$ be a closed connected Lagrangian manifold. Denote by $H_2^D(M,L) \subset H_2(M,L)$ the image of the Hurewicz homomorphism $\pi_2(M,L) \to H_2(M,L)$. One has two homomorphisms

$$\omega : H_2^D(M,L) \to \mathbb{Z} \quad , \quad \mu : H_2^D(M,L) \to \mathbb{R}$$

given respectively by integration of ω and by the Maslov index. One has the following essential definition:

Definition 2.2.F *The Lagrangian $L \subset M$ is said to be monotone if*

$$\omega(A) = \lambda \cdot \mu(A) \textit{ for all } A \in H_2^D(M,L)$$

for some $\lambda > 0$.

We refer to

$$N_L := min\left\{\mu(A) > 0 \mid A \in H_2^D(M,L)\right\} \in \mathbb{N}$$

as the minimal Maslov number of the Lagrangian submanifold $L \subset M$. From now on we assume L is monotone with minimal Maslov number $N_L \geq 2$.

Let us describe the Floer complex associated to the Lagrangian submanifold L. Let $H : M \times [0,1] \to \mathbb{R}$ be a time dependent Hamiltonian. Consider the path space

$$\mathcal{P}_0(L) = \{\gamma \in C^\infty([0,1], M) \mid \gamma(0), \gamma(1) \in L\ ,\ [\gamma] = 0 \in \pi_2(M,L)\}$$

Inside this space one has $\mathcal{O}_H \subset \mathcal{P}_0(L)$, the set of contractible chords of the Hamiltonian flow X_H, that is, solutions of the Hamiltonian differential equation

$$\dot{\gamma}(t) = -X_H^t(\gamma(t))$$

Note that if H is assumed to be generic then the space of contractible chords $\mathcal{O}_H$ is a finite set.

One has a natural epimorphism $p : \pi_1(\mathcal{P}_0(L)) \to H_2^D(M,L)$. Let $\widetilde{\mathcal{P}}_0(L)$ be the regular, Abelian cover associated to $ker(p)$ so that $H_2^D(M,L)$ is the group of deck transformations for this covering. Define $\widetilde{\mathcal{O}}_H$ to be the set of all lifts of orbits in $\mathcal{O}_H$ to $\widetilde{\mathcal{P}}_0(L)$. Fixing a base point $\eta_0 \in \widetilde{\mathcal{P}}_0(L)$ one defines the degree of $\widetilde{x} \in \widetilde{\mathcal{O}}_H$ by $| \widetilde{x} |= \mu(\widetilde{x}, \eta_0)$ where μ is the Viterbo-Maslov index, see (38). Fix a generic almost complex structure $J \in \mathcal{J}$. The Floer complex is given by:

$$CF_*(L; H, J) = \mathbb{Z}_2 \left\langle \widetilde{\mathcal{O}}_H \right\rangle$$

The differential is given by $d\widetilde{x} = \sum \sharp \mathcal{M}(\widetilde{x}, \widetilde{y}) \cdot \widetilde{y}$ where $\mathcal{M}(\widetilde{x}, \widetilde{y})$ is the moduli space of solutions $u : \mathbb{R} \times [0, 1] \to M$ of Floer's equation

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = -\nabla H_t(u(s, t))$$

which verify $u(s, 0), u(s, 1) \in L$ and lifting in $\widetilde{P}_0(L)$ to paths relating $\widetilde{x}$ and $\widetilde{y}$. Moreover, the sum is subject to the condition $\mu(\widetilde{x}, \widetilde{y}) = 1$. The homology of this complex is the Floer homology of L and is independent of the choices of H and J. It is denoted by $HF(L)$.

Call a Lagrangian submanifold $L \subset M$ *displaceable* if there exists a Hamitonian diffeomorphism $\varphi \in Ham(M, \omega)$ such that $\varphi(L) \cap L = \phi$. Note that, in particular, one has the following important vanishing property of HF : *If L is displaceable then $HF(L) = 0$.*

On the other hand, let $f : L \to \mathbb{R}$ be a Morse function and ρ a Riemannian metric on L. Denote by $Crit(f)$ the set of critical points of f and denote by $| x |$ the Morse index of $x \in Crit(f)$. Let $\Lambda = \mathbb{Z}_2[t, t^{-1}]$ be the ring of Laurent polynomials graded so that $deg(t) = -N_L$. As before, fix a generic almost complex structure $J \in \mathcal{J}$. The quantum pearl complex is defined by

$$\mathcal{C}_*(f, \rho, J) = \mathbb{Z}_2 \langle Crit(f) \rangle \otimes \Lambda$$

where the grading is the one induced by the Morse indices on $Crit(f)$ and the grading of Λ mentioned above. The differential $d : \mathcal{C}_*(f, \rho, J) \to \mathcal{C}_{*-1}(f, \rho, J)$ is defined by counting the dimensions of moduli spaces of objects called pearl trajectories. We refer the reader to (11; 13) for the definition. The cohomology of the quantum pearl complex is the quantum Lagrangian homology of L denoted by $QH_*(L)$.

Consider the following decreasing $\mathbb{Z}$-filtration on the pearl complex given by

$$F_p\mathcal{C}_*(f, \rho, J) = \bigoplus_{j \geq -p} \mathcal{C}_{*+jN_L}(f, \rho, J) \otimes t^j$$

where $C_*(f, \rho)$ is the corresponding Morse complex. Note that the pearl differential respects this filtration. By a well known construction in homological algebra, see (26) , to such a filtered complex one can associate a spectral sequence denoted by $(E^r_{p,q}, d_r)_{r \geq 0}$. The main properties of this sequence that are relevant for our applications are listed in the following theorem

Theorem 2.2.G *(see Oh (31), Biran-Cornea (11)). There exists a spectral sequence $\left\{E^r_{p,q}, d^r\right\}_{r\geq 1}$ with the following properties:*

(1) $E^1_{p,q} \simeq H_{p+q-pN_L}(L; \mathbb{Z}_2) \cdot t^{-p}$ *for every* p, q.

(2) $E^r_{p+1,q} \simeq E^r_{p,q+1-N_L} t^{-1}$ *for every* p, q, r.

(3) $\left\{E^r_{p,q}, d^r\right\}$ *collapses after a finite number of pages and converges to* $QH_*(L)$.

In particualr, due to the isomorphism $HF(L) \simeq QH_*(L)$ of (11), if L is displaceable then $E^\infty_{p,q} = 0$ for every p, q.

2.3 Restrictions on Degenerations with $Crit(\pi) = \mathbb{C}P^k$

Let Σ be a manifold of $dim_{\mathbb{C}}\Sigma = n$. In this section we consider the special case of Morse-Bott degenerations $\pi : X \to \mathbb{D}$ of Σ with $Crit(\pi) \simeq \mathbb{C}P^k$. Let $B^{2n}(r) \subset \mathbb{C}^n$ be the standard symplectic ball of radius $r > 0$ with symplectic form ω_0 induced by the standard symplectic form on $\mathbb{C}^n$ and let $S^{2n-1}(r) = \partial B^{2n}(r)$ be the corresponding sphere. In (4) it was observed that, for any $r > 1$, the manifold $\mathbb{C}P^k \times B^{2k+2}(r)$ contains the embedded Lagrangian sphere

$$L = \left\{(h(z), \bar{z}) \mid z \in S^{2k+1}(1)\right\} \subset (\mathbb{C}P^k \times B^{2k+2}(r), \omega_{FS} \oplus \omega_0)$$

where $h : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{C}P^k$ is the Hopf map given by $z \mapsto \mathbb{C} \cdot z$ and ω_{FS} is the Fubini-Study form on $\mathbb{C}P^k$ normalized such that $\int_{\mathbb{C}P^1} \omega_{FS} = \pi$.

Let (A, ω) be a compact closed symplectic manifold of $dim_{\mathbb{C}}A = k+1$ such that there exists a symplectic embedding of the standard symplectic ball $(B^{2k+2}(r), \omega_0)$ into A for $r >> 0$. Consider the stabilization

$$\widetilde{\pi} : X \times A \to \mathbb{D}$$

of the map π given by $(z,u) \mapsto \pi(z)$. This induces a Morse-Bott degeneration of $\Sigma \times A$ with $Crit(\widetilde{\pi}) = \mathbb{C}P^k \times A$. Thus, for any Kähler form Ω on X such that $\Omega \mid_{\mathbb{C}P^k} = \omega_{FS}$ we get by Theorem 2.1.C a Lagrangian Necklace $N(L) \subset (\Sigma \times A, \Omega \mid_{\Sigma} \times \omega)$. We have:

Lemma 2.3.H *The Lagrangian necklace $N(L) \subset \Sigma \times A$ is a Hamiltonianly displaceable, simply connected, Lagrangian sub-manifold which is topologically a sphere bundle over S^{2k+1}.*

Proof: Note that the Lagrangian sphere $L \subset \mathbb{C}P^k \times B^{2k+2}(4)$ can be displaced from itself by Hamiltonian isotopy φ which is the identity on the $\mathbb{C}P^k$ factor and acting as a Hamitonian isotopy in the $B^{2k+2}(4)$ factor which displaces $B^{2k+2}(1) \subset B^{2k+2}(4)$ from itself. Denote by $L' = \varphi(L)$. In particular, for $\epsilon > 0$ small enough, $N_\epsilon(L) \cap N_\epsilon(L') = \phi$ since $L \cap L' = \phi$. $\square$

The following lemma gives information on the homology of $N(L)$:

Lemma 2.3.I *Let $1 \leq k \leq n-2$.*

(1) If $n \neq 3k, 3k+1$ then

$$H_i(N(L);\mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2 & i = 0, 2k+1, n-k, n+k+1 \\ 0 & \textit{otherwise} \end{cases}$$

(2) If $n = 3k+1$ then

$$H_i(N(L);\mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2 & i = 0, n+k+1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i = 2k+1 \\ 0 & \textit{otherwise} \end{cases}$$

(3) If $n = 3k$ then

$$H_i(N(L);\mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2 & i = 0, n+k+1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof: As $N(L)$ is compact $H_{n+k+1-j}(N(L);\mathbb{Z}_2) \simeq H^j(N(L);\mathbb{Z}_2)$. Consider the $\mathbb{Z}_2$-coefficients Gysin sequence for the sphere bundle $N(L)$ given by

$$... \to H^j(S^{2k+1};\mathbb{Z}_2) \to H^j(N(L);\mathbb{Z}_2) \to H^{j-(n-k)}(S^{2k+1};\mathbb{Z}_2) \to H^{j+1}(S^{2k+1};\mathbb{Z}_2) \to ...$$

which gives in all cases

$$j \neq 0, 2k+1, n-k, n+k+1 \Rightarrow H^j(N(L);\mathbb{Z}_2) = 0$$

If $n \neq 3k, 3k+1$ the sequence gives

$$0 \to \mathbb{Z}_2 \to H^{2k+1}(N(L);\mathbb{Z}_2) \to 0 \quad ; \quad 0 \to H^{n-k}(N(L);\mathbb{Z}_2) \to \mathbb{Z}_2 \to 0$$

for $j = 2k+1, n-k$, which gives (1). If $n = 3k+1$ one has

$$0 \to \mathbb{Z}_2 \to H^{2k+1}(N(L);\mathbb{Z}_2) \to \mathbb{Z}_2 \to 0$$

which gives (2), note that in this case $2k+1 = n-k$. If $n = 3k$ the sequence gives

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to H^{2k+1}(N(L);\mathbb{Z}_2) \to 0$$

which implies (3). □

On the other hand for the Floer homology we have:

Lemma 2.3.J *Let Σ be a Fano manifold which satisfies $h^{1,1}(\Sigma) = 1$. Assume Σ appears as a fibre in a degeneration $\pi : X \to \mathbb{D}$ with $Crit(\pi) = \mathbb{C}P^k$. If $1 \leq k \leq n-2$ then $HF(N(L)) = 0$.*

Proof: The condition $h^{1,1}(\Sigma) = 1$ assures that the form on Σ is monotone. In particular, since under the above conditions $N(L)$ is simply connected, it would be a monotone Lagrangian submanifold of Σ. Hence $N(L)$ has a well defined Floer homology. By Lemma 2.3.H $N(L)$ is Hamiltonianly displaceable and thus $HF(N(L)) = 0$. □

We are now in position to prove the following theorem:

Theorem 2.3.K *Let Σ be a Fano manifold of $dim_{\mathbb{C}}\Sigma = n$ with $h^{1,1}(\Sigma) = 1$ and minimal Chern number C_Σ. Let $1 \leq k \leq n-2$. Suppose there exists a Morse-Bott degeneration $\pi : X \to \mathbb{D}$ of Σ with $Crit(\pi) \simeq \mathbb{C}P^k$ then:*

(1) If $n \neq 3k+1$ then one of the following holds:

(a) $C_\Sigma \mid k+1$.

(b) $2C_\Sigma \mid n-k+1$.

(c) $2C_\Sigma \mid n+k+2$. Furthermore $C_\Sigma \mid 2k+1$ if $n > 3k+1$ and $C_\Sigma \mid 2k+2$ if $n < 3k+1$.

(2) If $n = 3k+1$ then $2C_\Sigma \mid n-k+1$.

Proof: The Maslov index of $N(L)$ satisfies $N_L = 2C_\Sigma$. Assume $n \neq 3k+1$ then by Lemma 2.3.I we have that the singular homology $H_i(N(L), \mathbb{Z}_2)$ can only be nonzero if

$$i = 0, 2k+1, n-k, n+k+1$$

Let us consider the spectral sequence described in section 2.2. Taking $p = q = 0$, the differential of the spectral sequence has the following form

$$... \to E^r_{r,-r+1} \to E^r_{0,0} \to E^r_{-r,r-1} \to ...$$

for $r \geq 0$ Moreover, for degree reasons we have $E^r_{-r,r+1} = 0$ for all $r \geq 0$.
Thus, since $E^1_{0,0} \simeq \mathbb{Z}_2$, in order for $E^{r+1}_{0,0}$ to vanish for some $r \geq 1$ we must have $E^r_{-r,r-1} \neq 0$. But this can only happen in one of the following cases:

(i) $2rC_\Sigma - 1 = 2k+1$ for some $r \geq 0$.

(ii) $2rC_\Sigma - 1 = n-k$ for some $r \geq 0$.

(iii)

(iii.1) $2r_1C_\Sigma - 1 = n+k+1$ for some $r_1 \geq 0$.

(iii.2.1) $(2k+1) + 2r_2C_\Sigma - 1 = n-k$ for some $r_2 \geq 0$ if $3k+1 < n$.

(iii.2.2) $(n-k) + 2r_2C_\Sigma - 1 = 2k+1$ for some $r_2 \geq 0$ if $n < 3k+1$.

Finally, note that (iii) can hold only if $n \neq 3k+1$ and in the complementary case (i) and (ii) coincide. This proves (1) and (2). □

2.4 Remarks on Degenerations and Lefschetz Pencils.

A common way to obtain a degeneration with isolated singular points is by considering Lefschetz fibrations which in turn arise from Lefschetz pencils. In the non-isolated case, however, the transition from pencils to fibrations is not as straight forward. In this section we would like to describe the difference between the isolated and non-isolated cases. We refer the reader to (39) for a treatment on the geometry of Lefschetz pencils.

Let $X \subset \mathbb{C}P^N$ be a manifold and denote by $(\mathbb{C}P^N)^*$ the dual projective space parametrizing hyperplanes in $\mathbb{C}P^N$. The discriminant variety of X is given by

$$X^* = \{H \mid H \text{ is somewhere tangent to } X\} \subset (\mathbb{C}P^N)^*$$

where $\Sigma_H = X \cap H$ is the hyperplane section corresponding to H. A *pencil* on X is a line $\ell \subset (\mathbb{C}P^N)^*$. For a pencile ℓ on X define the variety

$$\widetilde{X}_\ell = \{(x, H) \mid x \in \Sigma_H\} \subset X \times \ell$$

and denote by $\pi : \widetilde{X}_\ell \to \ell \simeq \mathbb{C}P^1$ the map given by projection on the ℓ factor. Let

$$B(\ell) = \bigcap_{H\in\ell} \Sigma_H \quad ; \quad S(\ell) = \bigcup_{H\in\ell\cap X^*} Sing(\Sigma_H)$$

be the base locus and singular locus of ℓ respectively. We have:

Lemma 2.4.L *$Sing(\widetilde{X}_\ell) = (B(\ell) \cap S(\ell)) \times (\ell \cap X^*)$.*

Proof: Let (z, t) be local coordinates in a neighborhood $U \times \mathbb{C} \subset X \times \ell$ around (x, H_0) such that:

$$\widetilde{X}_\ell \cap (U \times \mathbb{C}) = \{(z, t) \mid f_0(z) + tf_1(z) = 0\}$$

where

$$\Sigma_{H_0} \cap U = \{f_0(z) = 0\} \quad , \quad \Sigma_{H_1} \cap U = \{f_1(z) = 0\}$$

with $H_1 \in \ell$ a hyperplane different from H_0. The origin is thus a singular point if and only if $f_0(0) = f_1(0) = 0$ and $df_0(0) = 0$. □

In particular we have:

Corollary 2.4.M *$\widetilde{X}_\ell$ is singular if $dim(S(\ell)) \geq 1$.*

Recall that a pencil ℓ is said to be a *Lefschetz pencil* on X if it intersects the discriminant variety X^* transversally. It is well known that if ℓ is a Lefschetz pencil the map $\pi : \widetilde{X}_\ell \to \ell$, referred to as a Lefschetz pencil, has only isolated singular points and satisfies the Morse(-Bott) conditions locally around these point. In particular, a Lefschetz pencil exists on any projective manifold X. Furthermore, the pencil would have singular fibers if and only if X^* is a hypersurface.
On the other hand, by Corollary 2.4.M if $dim(S(\ell)) \geq 1$ the map $\pi : \widetilde{X}_\ell \to \ell$ cannot give rise to a Morse-Bott degeneration simply because $\widetilde{X}_\ell$ will be singular. One can, however, attempt to resolve the singularities of $\widetilde{X}_\ell$. The following is an example where such a resolution can be readily described:

Example: Let $z = [z_0 : ... : z_n]$ be homogenous coordinates on $X = \mathbb{C}P^n$ and consider the pencil ℓ of quadrics given by

$$Q_\lambda = \{\lambda_0 F_0(z) + \lambda_1 F_1(z) = 0\} \subset \mathbb{C}P^n$$

Where

$$F_0(z) = z_0^2 + ... + z_n^2 \quad ; \quad F_1(z) = -(2z_2^2 + 3z_3^2 + ... + nz_n^2)$$

In particular, the quadric Q_1 is singular with

$$Sing(Q_1) = \{z_2 = ... = z_n = 0\}$$

Moreover, since the rest of the singular fibers are isolated and disjoint of the base locus we have

$$B(\ell) \cap S(\ell) = B(\ell) \cap Sing(Q_1) = \left\{ \begin{array}{c} z_0^2 + z_1^2 = 0 \\ z_2 = ... = z_n = 0 \end{array} \right\}$$

which constitutes of the two points $z_\pm = [1, \pm i, 0, ..., 0]$. If one would blow up $\widetilde{X}_\ell$ along the two points $(z_\pm, 0)$ one would obtain a degeneration. The fibres of this degeneration, however, would be quadrics blown up at two points and not the original quadrics we began with.

Chapter 3

Manifolds with Small Dual from the Symplectic Viewpoint

3.1 Projective Duality and Manifolds With Small Dual.

Let $X \subset \mathbb{C}P^N$ be a *smooth* algebraic manifold of $dim_{\mathbb{C}} X = n$. Denote by $(\mathbb{C}P^N)^*$ the dual projective space parametrizing hyperplanes $H \subset \mathbb{C}P^N$. To X one associates the *dual variety* $X^* \subset (\mathbb{C}P^N)^*$, defined as

$$X^* := \{H \mid H \text{ is somewhere tangent to } X\} \subset (\mathbb{C}P^N)^*$$

which is also referred to as the *discriminant* of X in $\mathbb{C}P^N$. We refer the reader to (37) and references therein, for a detailed overview of the geometry of projective duality. Let us note that discriminants are usually viewed as a geometric generalization of the algebraic notions of : determinants, minors, and, of course, linear duality.

Although in this work we are interested in discriminants of smooth manifolds (and hence, the definition above is sufficient) the definition of the discriminant X^* naturally extends to any algebraic variety $X \subset \mathbb{C}P^N$. In this general setting discriminants satisfy the following fundamental bi-duality property $X^{**} \simeq X$, which justifies the term *dual variety* for X^*, see (37)

Considering a naive dimension count, one excepts the (complex) dimension of the dual variety X^* to be $N-1$ i.e a hypersurface in $(\mathbb{C}P^N)^*$. However, this is not always the case and the manifolds for which X^* is not a hypersurface are referred to as manifolds with small dual, or, manifolds with positive defect.

In this section we will review basic properties of projective manifolds with positive defect. First, define the defect number of X to be

$$\mathrm{def}(X) = \mathrm{codim}_{(\mathbb{C}P^N)^*} X^* - 1$$

In particular, when X^* is a hypersurface the defect number of X is zero. The defect number has the following properties:

• If $\mathrm{def}(X) = k$ then for $H \in X^*_{sm}$, a smooth point of the dual X^*, the singular part $sing(X \cap H)$ of $\Sigma_H = X \cap H$ is a projective subspace of dimension k in $\mathbb{C}P^N$. In particular, X is ruled by projective subspaces of dimension k.

• If $\Sigma_H \subset X$ is a smooth hyperplane section of X the defect of Σ_H is expressed by the following formula

$$\mathrm{def}(\Sigma_H) = \max\{\mathrm{def}(X) - 1, 0\}$$

• by a well known (unpublished) result of Landman the defect and the dimension always have the same parity i.e

$$\dim_{\mathbb{C}} X \equiv \mathrm{def}(X) \pmod 2,$$

see (37).

It is important to note that the defect number is an invariant of the given projective embedding, rather than of the manifold X itself in the sense that X could have different defect number in different projective embeddings.

Denote by $Pic(X)$ the Picard group of X, classifying isomorphism classes of holomorphic line bundles on X. Recall that by the Kodaira embedding theorem projective embeddings are classified by very ample line bundles in $Pic(X)$, see (20). In this sense, the defect number is an invariant of the pair (X, L) where $L \in Pic(X)$ is a very ample line bundle. In particular, when different embeddings are considered we denote by $\mathrm{def}(X, L)$ the defect number of X in the embedding determined by the very ample line bundle L. Let X be a projective manifold and let $L \in Pic(X)$ be a very ample line bundle. As an invariant of the embedding the defect number satisfies the following property:

- If $\mathrm{def}(X, L) > 0$ then $\mathrm{def}(X, mL) = 0$ for $m \geq 2$.

The class of projective manifolds with small dual has been widely studied in algebraic geometry. In particular, these manifolds have been shown to have various unique properties. One has the following examples:

Example:

1. $X = \mathbb{C}P^n \subset \mathbb{C}P^{n+1}$ has $\mathrm{def}(X) = n$.
2. $X = Gr(2l+1, 2)$ embedded via the Plücker embedding has $\mathrm{def}(X) = 2$.
3. $X = \mathbb{S}_5 \subset \mathbb{C}P^{15}$ the 10–dimensional spinor variety has $\mathrm{def}(X) = 4$.
4. $X \subset \mathbb{CP}^N$ is a projective bundle with base Y such that $dim_{\mathbb{C}}Y < \frac{1}{2}dim_{\mathbb{C}}X$ and the fibers are embedded linearly has $def(X) = dim_{\mathbb{C}}X - 2dim_{\mathbb{C}}Y$.
5. $X \subset \mathbb{CP}^N$ is a F-bundle with base Y has $def(X) = max\{def(F) - dim_{\mathbb{C}}Y, 0\}$.
6. Given X with small dual, iterated hyperplane sections of X give manifolds with small dual, provided that the number of iterations does not exceed $def(X) - 1$.

Note that examples $(1) - (3)$, and (6) arising from them, are of $b_2(X) = 1$ while examples $(4) - (5)$, and (6) resulting from them, necessarily has $b_2(X) > 2$. However, results of (6; 7) using Mori theory show that the classification of projective manifolds with small dual can be reduced to the classification of the $b_2(X) = 1$ case, which are to some extent seen as "building blocks" for the general case.

In particular, the case $b_2(X) = 1$ was extensively studied by Ein in (15). Amongst other things, Ein proves in (15) that for $X \subset \mathbb{C}P^N$ with $dim_{\mathbb{C}}X = n$ and $\mathrm{def}(X) = k > 0$ there exists a projective line $S \subset X$ such that

$$c_1^X(S) = \frac{n+k+2}{2}$$

In particular, if $b_2(X) = 1$ then

$$c_1^X = \left(\frac{n+k}{2} + 1\right) \cdot h$$

where $h \in H^2(X; \mathbb{Z}) \simeq \mathbb{Z}$ is the positive generator. Note that, in this case, both X and Σ are Fano manifolds.

3.2 Sub-criticality and Manifolds with Small Dual.

Let $Y \subset \mathbb{C}^N$ be a smooth Stein manifold. The study of Morse theory on Stein manifolds was initiated in the classical paper (2) of Andreotti and Frankel. Further aspects of Morse theory on Stein manifolds were studied by various authors (9; 10; 16; 17).

Let $\varphi : Y \to \mathbb{R}$ be a smooth function. A point $z \in Y$ is said to be a critical point for the function φ if $d\varphi_z(\cdot) = 0$. Given a critical point $z \in Y$ one defines the Hessian of φ at z, which is the quadratic form given by

$$\mathrm{Hess}_z(\varphi)\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \frac{\partial^2 \varphi}{\partial z_i \partial z_j}(z)$$

The critical point z is non-degenerate if the $\mathrm{Hess}_z(\varphi) \in S^2 T_z Y$ is non-degenerate. The index of a non-degenerate critical point is given by $\mathrm{ind}_z(\varphi) = \mathrm{rankHess}_z(\varphi)$. In particular, the function φ is said to be Morse if all of its critical points are non-degenerate.

A function $\varphi : Y \to \mathbb{R}$ is called pluri-subharmonic (p.s.h in short) if the form $\Omega = -dd^{\mathbb{C}}\varphi$ is a Kähler form on Y. Here $d^{\mathbb{C}}\varphi = d\varphi \circ J$, where J is the complex structure of Y. For a pluri-subharmonic Morse function $\varphi : Y \to \mathbb{R}$ consider

$$\mathrm{ind}_{\max}(\varphi) = \max\{\mathrm{ind}_z(\varphi) \mid z \in \mathrm{Crit}(\varphi)\}$$

Where $\mathrm{ind}_z(\varphi)$ is the Morse index of the critical point $z \in \mathrm{Crit}(\varphi)$. The sub-ciritcality index of the Stein manifold Y is defined by

$$\mathrm{ind}(Y) := \min\{\mathrm{ind}_{\max}(\varphi) \mid \varphi : Y \to \mathbb{R} \text{ p.s.h exhausting Morse function}\}$$

where a function is said to be exhausting if it is proper and bounded from below. A fundamental property of the index is

$$0 \leq \mathrm{ind}(Y) \leq dim_{\mathbb{C}}(Y)$$

for any affine manifold $Y \subset \mathbb{C}^N$, see (2) and also (17). We have:

Theorem 3.2.A *Let $X \subset \mathbb{C}P^N$ be a smooth projective manifold and Σ a smooth hyperplane section. Then*

$$\mathrm{ind}(X \setminus \Sigma) \leq dim_{\mathbb{C}}(X) - def(X).$$

Proof: Assume def$(X) = k$ and set $n = dim_{\mathbb{C}} X$. Denote by $Y = X \setminus \Sigma$. Fix a Hermitian metric $h(\cdot, \cdot)$ and denote by $< , >$ the Riemannian metric on $\mathbb{C}^N$ associated to h. Take a point $w \in \mathbb{C}^N$ and define $\varphi_w : Y \to \mathbb{R}$ to be the function given by $\varphi_w(z) := | z - w |^2$ for $z \in Y$. Due to standard arguments, for a generic point $w \in \mathbb{C}^N$, this function is a pluri-subharmonic Morse function. Without loss of generality we assume this is the case for $w = 0$ and denote $\varphi = \varphi_0$.

The function φ is defined in terms of the intrinsic geometry of Y in $\mathbb{C}^N$. In particular, its Morse theoretic properties could be expressed in terms of the differential geometric properties of Y.

Let $(z_1, ..., z_n)$ be local coordinates around $z_0 \in Y$. The embedding of Y in $\mathbb{C}^N$ is given as

$$(z_1, ..., z_n) \mapsto (f_1(z_1, ..., z_n), ..., f_N(z_1, ..., z_n))$$

where $\{f_i\}_{i=1}^N$ is a certain collection of analytic functions. Moreover, the (complex) tangent space $T_{z_0}Y$ is given in $T_{z_0}\mathbb{C}^N \simeq \mathbb{C}^N$ as the span of the following vectors

$$v_i = \left(\frac{\partial f_1}{\partial z_i}(0), ..., \frac{\partial f_N}{\partial z_i}(0) \right) \quad for \;\; i = 1, ..., n$$

and

$$\varphi(z_1, ..., z_n) = | f_1 |^2 + ... + | f_N |^2$$

and thus one sees that

$$d_{z_0}\varphi(v_i) = \frac{\partial \varphi}{\partial z_i}(0) = 2 \sum_{j=1}^{n} \frac{\partial f_j}{\partial z_i}(0) \cdot \overline{f_j}(0) = 2h(v_i, \overrightarrow{0z_0})$$

Thus, $d_{z_0}\varphi(v) = 0$ for $v \in T_{z_0}Y$ if and only if $v \perp z_0$ (both viewed as vectors of $\mathbb{C}^N$). In particular, $z_0 \in Y$ is a critical point of φ if and only if $z_0 \perp Y$.

In order to compute the Hessian of φ at a crticial point z_0 we need the second fundamental form. We will follow here the conventions from (39). Denote by $\gamma : Y \longrightarrow \mathrm{Gr}(n, N)$ the Gauss map, $\gamma(x) = T_x Y$. Consider the differential of this map

$$D\gamma_x : T_x Y \longrightarrow T_{T_x Y}\mathbb{C}^N = \hom(T_x Y, \mathbb{C}^N / T_x Y)$$

This map induces a symmetric bilinear form:

$$\Phi : S^2 T_x Y \longrightarrow \mathbb{C}^N / T_x Y$$

which is called the second fundamental form.

As $h(v, \overrightarrow{0z_0}) = 0$ for every $v \in T_{z_0}Y$ we can define a symmetric complex bilinear form:

$$H : S^2 T_{z_0} Y \longrightarrow \mathbb{C}, \quad H(u,v) = h(\Phi(u,v), \overrightarrow{0z_0}).$$

A standard computation (see e.g. (39)) shows that the Hessian of φ is given by:

$$\mathrm{Hess}_{z_0}\varphi(u,v) = 2((u,v) + \mathrm{Re}H(u,v)), \ \forall u, v \in T_{z_0}Y.$$

Next, by a result of Katz we have: $\mathrm{rank}_{\mathbb{C}} H \leq n-k$. (See (37) and the references therein, e.g. exposé XVII by N. Katz in (1). See also (21).) It follows that $\dim_{\mathbb{R}} \ker(\mathrm{Re}H) \geq 2k$.

Denote the non-zero eigenvalues of ReH (in some orthonormal basis) by λ_i, $i = 1, \ldots, r$, with $r \leq 2n - 2k$. It is well known that for real symmetric bilinear forms that appear as the real part of complex ones (e.g. ReH) the following holds: λ is an eigenvalue if and only if $-\lambda$ is an eigenvalue (see (39)), moreover this correspondence holds also for the multiplicities of the corresponding eigenvalues. (See e.g. (39) for a proof.) It follows that the number of negative λ_i's can be at most $n-k$.

Note that the eigenvalues of Hess_{z_0} are of the form $1+\lambda$ with λ and eigenvalue of ReH. It follows that the number of negative eigenvalues of $\mathrm{Hess}_{z_0}\varphi$ is at most $n-k$. This shows that $\mathrm{ind}_{z_0}(\varphi) \leq n-k$ for every $z_0 \in \mathrm{Crit}(\varphi)$. In particular, $\mathrm{ind}(Y) \leq n-k$. $\square$

We have the following topological corollary (compare (25)):

Corollary 3.2.B *Let $X \subset \mathbb{C}P^N$ be a smooth algebraic manifold with $\dim_{\mathbb{C}} X = n$ and def$(X) = k$ and let $\Sigma \subset X$ be a smooth hyperplane section. Denote by $i : \Sigma \to X$ the inclusion map. The map $i^* : H^j(X;\mathbb{Z}) \to H^j(\Sigma;\mathbb{Z})$ is*

(1) An isomorphism for $j < n+k-1$.

(2) Injective for $j = n+k-1$.

Proof: Note that Theorem 3.2.A implies that $Y = X \setminus \Sigma$ is homotopically equivalent to a CW-complex of dimension $\leq n-k$. In particular, $H_j(Y;\mathbb{Z}) = 0$ for $n-k+1 \leq j$. On the other hand, consider the relative long exact sequence for (X, Σ) given by

$$\ldots \to H^j(X,\Sigma;\mathbb{Z}) \to H^j(X;\mathbb{Z}) \to H^j(\Sigma;\mathbb{Z}) \to H^{j+1}(X,\Sigma;\mathbb{Z}) \to \ldots$$

Finally, the corollary follows from the fact that $H^j(X, \Sigma; \mathbb{Z}) \simeq H_{2n-j}(Y; \mathbb{Z})$. □

Furthermore, we have:

Corollary 3.2.C *Let $X \subset \mathbb{C}P^N$ be as in Corollary ??. Denote by ω the Kähler form on X induced from the standard Kähler form of $\mathbb{C}P^N$. Then the map*

$$L : H^j(X; \mathbb{R}) \longrightarrow H^{j+2}(X; \mathbb{R}), \quad L(a) = a \cup [\omega],$$

is an isomorphism for every $n - k - 1 \leq j \leq n + k - 1$.

Proof: This follows from Corollary 3.2.B together with the Hard Lefschetz theorem applied both to Σ and X.

3.3 Quantum Cohomology and the Seidel Element.

3.3.1 Quantum Cohomology

In this section we recall notations and construction of the quantum cohomology ring of a *spherically monotone* symplectic manifold, we refer the reader to (29) for a complete treatment.

Let (Σ, ω_Σ) be a closed symplectic manifold. Denote by $H_2^S \subset H_2(\Sigma; \mathbb{Z})$ the image of the Hurewicz homomorphism $\pi_2(\Sigma) \longrightarrow H_2(\Sigma; \mathbb{Z})$. Denote by $c_1^\Sigma \in H^2(\Sigma; \mathbb{Z})$ the first Chern class of the tangent bundle $(T\Sigma, J_\Sigma)$, where J_Σ is any almost complex structure compatible with ω_Σ. The symplectic manifold (Σ, ω_Σ) is called *spherically monotone* if there exists a constant $\lambda > 0$ such that for every $A \in H_2^S$ we have $\omega_\Sigma(A) = \lambda c_1^\Sigma(A)$. Denote by $C_\Sigma \in \mathbb{N}$ the minimal Chern number, i.e.

$$C_\Sigma = \min\{c_1^\Sigma(A) \mid A \in H_2^S, c_1^\Sigma(A) > 0\}.$$

Denote by $l : H_2^S \to \mathbb{Z}$ the homomorphism given by $A \mapsto c_1(A)/C_\Sigma$.

First, one defines the quantum product $a*b$ of two cohomology classes $a \in H^{k_1}(\Sigma)$ and $b \in H^{k_2}(\Sigma)$ by

$$a * b = \sum_{A \in H_2^S} (a * b)_A \, q^{l(A)}$$

where q is a formal variable of degree $2C_\Sigma$ and the cohomology class $(a * b)_A \in H^{k_1+k_2-2c_1(A)}(\Sigma)$ is given by:

$$\langle (a * b)_A, c\rangle = GW^{\Sigma}_{A,3}(a, b, c), \quad c \in H^*(\Sigma)$$

where $GW^{\Sigma}_{A,3}(\cdot,\cdot,\cdot)$ is the 3-point Gromov-Witten invariant of A - see (29). As in order for the Gromov-Witten invariants to be non-trivial the classes a, b, c must satisfy the dimension count

$$2n + 2c_1(A) = deg(a) + deg(b) + deg(c)$$

one sees that the only classes $A \in H_2^S$ such that $0 \leq c_1(A) \leq 2n$. In particular, due to the monotonicity assumptions $a * b$ involves only a finite number of summands.

Denote by $\Lambda = \mathbb{Z}[q^{-1}, q]$ the ring of Laurent polynomials. We endow Λ with a grading by setting $\deg(q) = 2C_\Sigma$. Set $QH^*(\Sigma;\Lambda) = (H(\Sigma) \otimes \Lambda)^*$, where the grading is induced from both factors $H^*(\Sigma)$ and Λ. It turns that the product $*$ extends by linearity to give a well defined, distributive, commutative and associative product

$$* : QH^{k_1}(\Sigma;\Lambda) \otimes QH^{k_2}(\Sigma;\Lambda) \to QH^{k_1+k_2}(\Sigma;\Lambda)$$

to which we refer as the quantum product of $QH^*(\Sigma;\Lambda)$. We endow $QH(\Sigma;\Lambda)$ with the quantum product $*$ and refer to $(QH^*(X;\Lambda), *)$ as the quantum cohomology ring of Σ. The unity will be denoted as usual by $1 \in QH^0(\Sigma;\Lambda)$. Let us note that the distributivity and commutativity of the product follow readily from the definition. However, the associativity of the quantum product is a more subtle property which requires the splitting axiom theorem for Gromov-Witten invariants. We refer the reader to Chapter 11 of (29) for the definitions and foundations of quantum cohomology. Note however that our grading conventions are slightly different than the ones in (29). With our grading conventions we have:

$$QH^j(\Sigma;\Lambda) = \bigoplus_{l\in\mathbb{Z}} H^{j-2lC_\Sigma}(\Sigma)q^l$$

3.3.2 Hamiltonian Fibrations and the Seidel Element

In what follows we will use the theory of symplectic and Hamiltonian fibrations and their invariants. We refer the reader to (28; 29) for the foundations.

Let $\pi : \widetilde{X} \to B$ be a smooth locally trivial fibration with fibre Σ and base B which are both closed manifolds. We will assume in addition that B is a *simply connected* manifold. Further, let Ω be a closed 2-form on $\widetilde{X}$ such that the restriction $\omega_b = \Omega|_{\Sigma_b}$ to each fibre $\Sigma_b = \pi^{-1}(b)$, $b \in B$, is a symplectic form. Fix $b_0 \in B$, and let ω_{Σ} be a symplectic form on Σ such that $(\Sigma, \omega_{\Sigma})$ is symplectomorphic to $(\Sigma_{b_0}, \omega_{b_0})$. This structure is a special case of a so called Hamiltonian fibration. It is well known that under these assumptions all fibres (Σ_b, ω_b) are symplectomorphic and in fact the structure group of π can be reduced to $\mathrm{Ham}(\Sigma, \omega_{\Sigma})$.

Denote by $T^V\widetilde{X} = \ker(D\pi)$ the vertical part of the tangent bundle of $\widetilde{X}$. The 2-form Ω gives rise to a connection on $\widetilde{X}$ defined by the following horizontal subbundle:

$$T_x^H\widetilde{X} = \{v \in T_x\widetilde{X} \mid \Omega(v, \xi) = 0 \ \forall \xi \in T_x^V\widetilde{X}\}.$$

We will denote this connection by Γ_Ω or by Γ when the choice of the form Ω is clear. Note that if σ is a 2-form on B and $\Omega' = \Omega + \pi^*\sigma$ then $\Gamma_{\Omega'} = \Gamma_\Omega$.

We will assume from now on that $B = S^2$. We identify $S^2 \cong \mathbb{C}P^1$ in a standard way and view S^2 as a Riemann surface with complex structure j.

3.3.3 Holomorphic curves in Hamiltonian fibrations

Let $\pi : (\widetilde{X}, \Omega) \to S^2$ be a Hamiltonian fibration as above. We now introduce almost complex structures compatible with the connection induced by Ω. These are by definition almost complex structures $\widetilde{J}$ on $\widetilde{X}$ with the following properties:

1. The projection π is $(\widetilde{J}, j)$–holomorphic.
2. The horizontal distribution $T^H\widetilde{X}$ is $\widetilde{J}$ invariant.
3. For every $z \in S^2$ the restriction J_z of $\widetilde{J}$ to Σ_z is compatible with ω_z, i.e. $\omega_z(J_z\xi, J_z\eta) = \omega(\xi, \eta)$ for every $\xi, \eta \in T_z^V\widetilde{X}$ and $\omega_z(\xi, J_z\xi) > 0$ for every $0 \neq \xi \in T_z^V\widetilde{X}$.

We denote the space of such almost complex structures by $\widetilde{\mathcal{J}}(\Omega)$. Clearly every $\widetilde{J} \in \widetilde{\mathcal{J}}$ is uniquely determined by its vertical part J, i.e. its restriction $T^V\widetilde{X}$. Conversely, every vertical almost complex structure J which is fibrewise compatible with Ω can be extended in a unique way to an almost complex structure $\widetilde{J}$ compatible with the fibration. We denote by $\mathcal{J}^V(\Omega)$ the space of vertical fibrewise

Ω-compatible almost complex structures. Note that $\mathcal{J}^V(\Omega)$ depends only on the vertical part of Ω, but the correspondence $J \to \widetilde{J}$ depends on the connection Γ_Ω.

Denote by $H_2^\pi \subset H_2(\widetilde{X};\mathbb{Z})$ the set of classes $\widetilde{A}$ such that $\pi_*(\widetilde{A}) = [S^2]$. Given $\widetilde{A}$ and $\widetilde{J} \in \widetilde{\mathcal{J}}$ denote by $\mathcal{M}(\widetilde{A},\widetilde{J})$ the space of $\widetilde{J}$–holomorphic sections in the class $\widetilde{A}$, i.e. the space of maps $\widetilde{u} : S^2 \longrightarrow \widetilde{X}$ with the following properties:

1. $\widetilde{u}$ is $(j,\widetilde{J})$–holomorphic.
2. $\widetilde{u}$ is a section, i.e. $\pi \circ \widetilde{u} = \mathrm{id}$.
3. $\widetilde{u}_*[S^2] = \widetilde{A}$.

Fix $z_0 \in S^2$ and fix an identification $(\Sigma,\omega_\Sigma) \cong (\Sigma_{z_0},\omega_{z_0})$. The space of sections comes with an evaluation map:

$$ev : \mathcal{M}(\widetilde{A},\widetilde{J}) \longrightarrow \Sigma, \quad ev(\widetilde{u}) = u(z_0).$$

3.3.3.1 Transversality

In order to obtain regularity and transversality properties for the moduli spaces of holomorphic sections and their evaluation maps we will need now to work with so called *regular* almost complex structures. More specifically, we have to overcome two difficulties. Moreover, since the moduli spaces of holomorphic sections are usually not compact they do not carry fundamental classes and so the evaluation maps do not induce in a straighforward way homology classes in their target (Σ in this case). The reason for non-comactness of these moduli spaces is that a sequence of holomorphic sections might develop bubbles in one of the fibres (see e.g. (29)). The simplest way to overcome this difficulty is to make some positivity assumptions on the fiber Σ (called *monotonicity*). Under such conditions the moduli spaces of holomorphic sections admit a nice compactification which makes it possible to define homology classes induced by the evaluation maps. Here is the relevant definition.

Definition 3.3.3.A *Let (Σ,ω_Σ) be a symplectic manifold. Denote by $H_2^S \subset H_2(\Sigma;\mathbb{Z})$ the image of the Hurewicz homomorphism $\pi_2(\Sigma) \longrightarrow H_2(\Sigma;\mathbb{Z})$. Denote by $c_1^\Sigma \in H^2(\Sigma;\mathbb{Z})$ the first Chern class of the tangent bundle $(T\Sigma, J_\Sigma)$, where J_Σ is any almost complex structure compatible with ω_Σ. The symplectic manifold (Σ,ω_Σ) is called spherically monotone if there exists a constant $\lambda > 0$ such that for every $A \in H_2^S$ we have $\omega_\Sigma(A) = \lambda c_1^\Sigma(A)$. For example, if Σ is a Fano manifold* *and* ω_Σ *is a symplectic form with* $[\omega_\Sigma] = c_1^\Sigma$ *then obviously* (Σ, ω_Σ) *is spherically monotone.*

From now on we assume that the fibre (Σ, ω_Σ) of $\pi : (\widetilde{X}, \widetilde{\Omega}) \longrightarrow S^2$ is spherically monotone. Denote by $c_1^v = c_1(T^v\widetilde{X}) \in H^2(\widetilde{X})$ the vertical Chern class, i.e. the first Chern class of the vertical tangent bundle of $\widetilde{X}$. The following is proved in (29; 35). There exists a *dense subset* $\widetilde{\mathfrak{J}}_{\text{reg}}(\pi, \widetilde{\Omega}) \subset \widetilde{\mathfrak{J}}(\pi, \widetilde{\Omega})$ such that for every $\widetilde{J} \in \widetilde{\mathfrak{J}}_{\text{reg}}(\pi, \widetilde{\Omega})$ and every $\widetilde{A} \in H_2^\pi$ the following holds:

1. For every $\widetilde{A} \in H_2^\pi$, the moduli space $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J})$ of $\widetilde{J}$-holomorphic sections in the class $\widetilde{A}$ is either empty or a smooth manifold of dimension $\dim_{\mathbb{R}} \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}) = \dim_{\mathbb{R}} \Sigma + 2c_1^v(\widetilde{A})$. Moreover, $\mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J})$ has a canonical orientation.

2. The evaluation map $ev_{\widetilde{J}, z_0} : \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J}) \longrightarrow \Sigma$ is a pseudo-cycle (see (29) for the definition). In particular, its Poincaré dual gives a cohomology class $\mathcal{S}(\widetilde{A}; \widetilde{J}) \in H^d(\Sigma; \mathbb{Z})_{fr} = H^d(\Sigma; \mathbb{Z})/\text{torsion}$, where $d = -2c_1^v(\widetilde{A})$. Moreover, the class $\mathcal{S}(\widetilde{A})$ is independent of the regular $\widetilde{J}$ used to define it. Therefore we will denote it from now on by $\mathcal{S}(\widetilde{A})$.

We refer the reader to (29; 35) for more general results on transversality.
The definition of regularity for $\widetilde{J} \in \widetilde{\mathfrak{J}}(\pi, \widetilde{\Omega})$ involves three ingredients. The first is that the restriction J_{z_0} of $\widetilde{J}$ to $\Sigma = \Sigma_{z_0}$ is regular in the sense of Chapter 3 of (29), namely that the linearization of the $\overline{\partial}_{J_{z_0}}$-operator at every J_{z_0}-holomorphic curve in Σ is surjective. (In addition one has to require that certain evaluation maps for tuples of such curves are mutually transverse.) The second ingredient is that (the vertical part of) the $\overline{\partial}_{\widetilde{J}}$-operator at every $\widetilde{J}$-holomorphic section is surjective. The third one is that $ev_{\widetilde{J}, z_0}$ is transverse to all J_{z_0}-holomorphic bubble trees in Σ.
In practice, we will have to compute cohomology classes of the type $\mathcal{S}(\widetilde{A}) = \mathcal{S}(\widetilde{A}; \widetilde{J})$ using a specific choice of $\widetilde{J}$ that naturally appears in our context. It is not an easy task to decide whether a given almost complex structure $\widetilde{J}$ is regular or not. However, in some situations it is possible to compute some of the classes $\mathcal{S}(\widetilde{A})$ by using almost complex structures $\widetilde{J}$ that satisfy weaker conditions than regularity. Criteria for verification of these conditions have been developed in (35) (see Proposition 7.11 there) and in (29) (see Section 3.3 and 3.4 there). Below we will actually not appeal to such criteria and use simpler arguments.

3.3.3.2 Monotonicity assumptions

Let (Σ, ω_Σ) be a symplectic manifold as in 3.3.3.A. Under such monotonicity assumptions, the evaluation maps have nice behaviour too. The following is also proved in (29). Given $\widetilde{A} \in H_2^\pi$ there exists a second category subset $\mathcal{JH}_{\text{reg}}(\widetilde{A}) \subset \mathcal{J}^V(\Omega) \times \mathcal{H}$ such that for every $(J, H) \in \mathcal{JH}_{\text{reg}}$ the following holds:

1. $H \in \mathcal{H}_{\text{reg}}(\widetilde{A}, J)$. In particular $\mathcal{M}(\widetilde{A}, \widetilde{J}_H)$ is a smooth oriented manifold with dimension as in eq:dim-M(A,J).
2. The evaluation map $ev : \mathcal{M}(\widetilde{A}, \widetilde{J}_H) \longrightarrow \Sigma$ is a pesudo cycle (see (29) for the definition). In particular, its Poincaré dual gives a cohomology class $\mathcal{S}(\mathcal{A}; J, H) \in H^d(\Sigma; \mathbb{Z})_{fr} = H^d(\Sigma; \mathbb{Z})/\text{torsion}$, where $d = -2c_1^V(\widetilde{A})$.
3. The class $\mathcal{S}(\widetilde{A})$ is independent J, H. Therefore we will denote it from now on by $\mathcal{S}(\widetilde{A})$.

3.3.4 The Seidel representation

In (35) Seidel associated to a Hamiltonian fibration $\pi : \widetilde{X} \longrightarrow S^2$ with fiber Σ an invertible element $S(\pi) \in QH(\Sigma; \Lambda)$. We refer the reader to (29; 35) for a detailed account of this theory. Here is a brief review of the main construction.

Fix a reference class $\widetilde{A}_0 \in H_2^\pi$ and set $c_0(\pi) = c_1^v(\widetilde{A}_0)$. Since any two classes in H_2^π differ by a class in $H_2^S(\Sigma)$, there exists a uniquely defined function $\nu : H_2^\pi \to \mathbb{Z}$ such that

$$c_1^v(\widetilde{A}) = c_0(\pi) + \nu(\widetilde{A})C_\Sigma, \quad \forall \widetilde{A} \in H_2^\pi.$$

Pick a regular almost complex structure $\widetilde{J} \in \widetilde{\mathcal{J}}_{\text{reg}}(\pi, \widetilde{\Omega})$. Define a class:

$$S(\pi) := \sum_{\widetilde{A} \in H_2^\pi} \mathcal{S}(\widetilde{A}; \widetilde{J}) \otimes q^{\nu(\widetilde{A})} \in QH^{-2c_0(\pi)}(\Sigma, \Lambda). \tag{3.1}$$

Note that since the degree of $\mathcal{S}(\widetilde{A}, \widetilde{J})$ is $-2c_1^v(\widetilde{A})$, a class $\widetilde{A} \in H_2^\pi$ contributes to the sum in eq:seidle-element only if

$$2 - 2n \leq 2c_1^v(\widetilde{A}) \leq 0. \tag{3.2}$$

The class $S(\pi)$ is called the Seidel element of the fibration $\pi : (\widetilde{X}, \widetilde{\Omega}) \longrightarrow S^2$. Of course the element $S(\pi)$ (as well as its degree) depends on the choice of the reference section $\widetilde{A}_0$, however different reference sections $\widetilde{A}_0$ will result in elements that differe by a factor of the type q^r for some $r \in \mathbb{Z}$. In particular, many algebraic properties of $S(\pi)$ (such as invertibility) do not depend on this choice. We will therefore ignore this ambiguity from now on.

3.3.4.1 Relations to Hamiltonian loops

An important feature of the theory is the connection between Hamiltonian fibrations over S^2 with fibre (Σ, ω_Σ) and $\pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma))$. To a loop based at the identity $\lambda = \{\varphi_t\}_{t \in S^1}$ in $Ham(\Sigma, \omega_\Sigma)$ one can associate a Hamiltonian fibration $\pi_\lambda : \widetilde{M}_\lambda \to S^2$ as follows. Let D_+ and D_- be two copies of the unit disk in $\mathbb{C}$, where the orientation on D_- is reversed. Define:

$$\widetilde{M}_\lambda = \Big((\Sigma \times D_+)\coprod(\Sigma \times D_-)\Big)/\sim, \quad \text{where } (x, e_+^{2\pi i t}) \sim (\varphi_t(x), e_-^{2\pi i t}). \tag{3.3}$$

Identifying $S^2 \approx D_+ \cup_\partial D_-$ we obtain a fibration $\pi : \widetilde{M}_\lambda \longrightarrow S^2$. As the elements of λ are symplectic diffeomorphisms, the form ω_Σ gives rise to a family of symplectic forms $\{\Omega_z\}_{z \in S^2}$ on the fibres $\Sigma_z = \pi^{-1}(z)$ of π. Moreover, $\pi : (\widetilde{M}_\lambda, \{\Omega_z\}_{z \in S^2}) \longrightarrow S^2$ is locally trivial. Moreover, since the elements of λ are in fact Hamiltonian diffeomorphisms it follows that the family of fibrewise forms $\{\omega_z\}_{z \in S^2}$ can be extended to a closed 2-form $\widetilde{\Omega}$ on $\widetilde{M}_\lambda$, i.e. $\widetilde{\Omega}|_{\Sigma_z} = \Omega_z$ for every z. See (29; 35) for the proofs. We therefore obtain from this construction a Hamiltonian fibration $\pi : (\widetilde{M}_\lambda, \widetilde{\Omega}) \longrightarrow S^2$.

From the construction one can see that homotopic loops in $\mathrm{Ham}(\Sigma, \omega_\Sigma)$ give rise to isomorphic fibrations. We denote the isomoprhism class of fibrations corresponding to an element $\gamma \in \pi_1(Ham(\Sigma, \omega_\Sigma))$ by π_γ.

Conversely, if $\pi : (\widetilde{M}, \widetilde{\Omega}) \longrightarrow S^2$ is a Hamiltonian fibration with fiber (Σ, ω_Σ) one can express $\widetilde{M}$ as a gluing of two trivial bundles over the two hemispheres in S^2. The gluing map is a loop of Hamiltonian diffeomorphisms of (Σ, ω_Σ). Different trivializations lead to homotopic loops. Thus the fibration π determines a class $\gamma_\pi \in \pi_1(Ham(\Sigma, \omega_\Sigma))$.

This correspondence has the following properties in relation to the Seidel elements (see (35) for the proofs):

$$S(\pi_{[\gamma_1 \cdot \gamma_2]}) = S(\pi_{[\gamma_1]}) * S(\pi_{[\gamma_2]}), \quad \forall\, \gamma_1, \gamma_2 \in \pi_1(Ham(\Sigma, \omega_\Sigma)).$$

Here $*$ stands for the quantum product. Moreover, the unit element $e \in \pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma))$ corresponds to the trivial fibration $\pi_e : \Sigma \times S^2 \longrightarrow S^2$ and we have $S(\pi_e) = 1 \in$

$QH(\Sigma;\Lambda)$. It follows that $S(\pi)$ is an invertible element in $QH(\Sigma;\Lambda)$ for every π. The corresponding homomorphism

$$S : \pi_1(Ham(\Sigma, \omega_\Sigma)) \longrightarrow QH(\Sigma, \Lambda)^\times, \quad \gamma \longmapsto S(\pi_{[\gamma]})$$

(which by abuse of notation we also denote by S), where $QH(\Sigma, \Lambda)^\times$ is the group of invertible elements in $QH(\Sigma, \Lambda)$, is called the Seidel representation.

3.4 From manifolds with small dual to Hamiltonian fibrations

Let $X \subset \mathbb{C}P^N$ be a projective manifold with small dual. Put $n = \dim_{\mathbb{C}} X$ and $k = \mathrm{def}(X) > 0$. Since $X^* \subset (\mathbb{C}P^N)^*$ has codimension $k+1 \geq 2$ we can find a pencil of hyperplanes $\ell \subset (\mathbb{C}P^N)^*$ such that ℓ does not intersect X^*. Consider the manifold

$$\widetilde{X} = \{(x, H) \mid H \in \ell, \ x \in H\} \subset X \times \ell.$$

Identify $\ell \cong \mathbb{C}P^1 \cong S^2$ in an obvious way. Denote by

$$p : \widetilde{X} \longrightarrow X, \quad \pi_\ell : \widetilde{X} \longrightarrow \ell \cong S^2$$

the obvious projections. The map p can be considered as the blow-up of X along the base locus of the pencil ℓ. The map π_ℓ is a honest holomorphic fibration (without singularities) over $\ell \cong \mathbb{C}P^1$ with fibers $\pi_\ell^{-1}(H) = \Sigma_H = X \cap H$.

Denote by ω_X the symplectic form on X induced from the Fubini-Study Kähler form of $\mathbb{C}P^N$. Let ω_{S^2} be an area form on S^2 with $\int_{S^2} \omega_{S^2} = 1$. Endow $X \times S^2$ with $\omega_X \oplus \omega_{S^2}$ and denote by $\widetilde{\Omega}$ the restriction of $\omega_X \oplus \omega_{S^2}$ to $\widetilde{X} \subset X \times S^2$. The restriction of $\widetilde{\Omega}$ to the fibres $\widetilde{\Omega}|_{\pi_\ell^{-1}(H)}$, $H \in \ell$, coincides with the symplectic forms $\omega_X|_{\Sigma_H}$. Thus $\pi_\ell : \widetilde{X} \longrightarrow S^2$ is a Hamiltonian fibration. Fix a point $H_0 \in \ell$, and set $(\Sigma, \omega_\Sigma) = (\pi_\ell^{-1}(H_0), \omega_X|_{\Sigma_{H_0}})$.

Remark 3.4.0.A *Different pencils $\ell \subset (\mathbb{C}P^N)^*$ with $\ell \cap X^* = \emptyset$ give rise to isomorphic Hamiltonian fibrations. This is so because the* real *codimension of X^* is at least* 4 *hence any two pencils ℓ, ℓ' which do not intersect X^* can be connected by a* real *path of pencils in the complement of X^*. Thus the isomorphism class of the Hamiltonian fibration π_ℓ, the element $\gamma_{\pi_\ell} \in \pi_1(\mathrm{Ham}(\Sigma, \omega_\Sigma))$, as well as the corresponding Seidel element $S(\pi_\ell)$ can all be viewed as invariants of the projective embedding $X \subset \mathbb{C}P^N$.*

Theorem 3.4.0.B *Let $X \subset \mathbb{C}P^N$ be an algebraic manifold with $\dim_{\mathbb{C}}(X) = n \geq 2$ and def$(X) = k > 0$. Denote by $H_2^S = \text{image}\,(\pi_2(X) \longrightarrow H_2(X;\mathbb{Z})) \subset H_2(X;\mathbb{Z})$ the image of the Hurewicz homomorphism. Denote by $h \in H^2(X)$ the class dual to the hyperplane section Σ. Assume that there exists $0 < \lambda \in \mathbb{Q}$ such that $\langle c_1^X, A\rangle = \lambda\langle h, A\rangle$ for every $A \in H_2^S$. Then the Seidel element of the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$ is:*

$$S(\pi_\ell) = [\omega_\Sigma] \in QH^2(\Sigma;\Lambda).$$

The degree of the variable $q \in \Lambda$ is $deg(q) = \frac{n+k}{2}$.

The proof of this Theorem is given in §3.5.

Remark 3.4.0.C *The condition $\langle h, A\rangle = \lambda\langle c_1^X, A\rangle$, $\forall A \in H_2^S$, implies that $\lambda = n+k+22$. Indeed, as explained in §3.1, manifolds X with small dual contain projective lines $S \subset X$ (embedded linearly in $\mathbb{C}P^N$) with $c_1^X(S) = n+k+22$. As $h(S) = 1$ it follows that $\lambda = \frac{n+k+2}{2}$.*

Theorem 3.4.0.B applies for example to algebraic manifolds $X \subset \mathbb{C}P^N$ with small dual that satisfy one of the following conditions:

1. $b_2(X) = 1$.

2. More generally, the free part of H_2^S has rank 1.

This is so because in both of these cases we must have $h = \lambda c_1^X$ for some $\lambda \in \mathbb{Q}$. The fact that $\lambda > 0$ follows from the existence of rational curves $S \subset X$ with $c_1^X(S) = \frac{n+k+2}{2}$ as explained in 3.1.

Corollary 3.4.0.D *Under the assumptions of Theorem 3.4.0.B we have:*

$$\widetilde{b}_j(X) = \widetilde{b}_{j+2}(X), \quad \widetilde{b}_j(\Sigma) = \widetilde{b}_{j+2}(\Sigma) \;\; \forall\; j \in \mathbb{Z},$$

where the definition of $\widetilde{b}_j$ is given in (1.1) in §1.1. Or, put in an un-wrapped way, we have the following identities for X:

$$b_j(X) + b_{j+n+k+2}(X) = b_{j+2}(X) + b_{j+n+k+4}(X), \;\; \forall\; 0 \leq j \leq n+k-1,$$
$$b_{n+k}(X) = b_{n+k+2}(X) + 1, \quad b_{n+k+1}(X) = b_{n+k+3}(X) + b_1(X),$$

and the following ones for Σ:

$$b_j(\Sigma) + b_{j+n+k}(\Sigma) = b_{j+2}(\Sigma) + b_{j+n+k+2}(\Sigma), \;\; \forall\; 0 \leq j \leq n+k-3,$$
$$b_{n+k-2}(\Sigma) = b_{n+k}(\Sigma) + 1, \quad b_{n+k-1}(\Sigma) = b_{n+k+1}(\Sigma) + b_1(\Sigma).$$

The proof is given in §3.6

3.5 Proofs of theorem 3.4.0.B and Theorems D and F

As noted in the discussion after the statement of Theorem 3.4.0.B, Theorems D, F from §1.1 are immediate consequences of Theorem 3.4.0.B. Therefore we will concentrate in this section in proving the latter. We will make throughout this section the same assumptions as in Theorem 3.4.0.B and use here the construction and notation of §3.4.

Denote by $B = \Sigma_{H_0} \cap \Sigma_{H_1} \subset X$, $(H_0, H_1 \in \ell)$, the base locus of the pencil ℓ. Recall that $p : \widetilde{X} \longrightarrow X$ can be viewed as the blow-up of X along B. Denote by $E \subset \widetilde{X}$ the exceptional divisor of this blow-up. The restriction $p|_E : E \longrightarrow B$ is a holomorphic fibration with fiber $\mathbb{C}P^1$. Denote the homology class of this fiber by $F \in H_2(\widetilde{X}; \mathbb{Z})$. Since $\dim_{\mathbb{R}} B = 2n - 4$, the map induced by inclusion $H_2(X \setminus B; \mathbb{Z}) \longrightarrow H_2(X; \mathbb{Z})$ is an isomorphism, hence we obtain an obvious injection $j : H_2(X; \mathbb{Z}) \longrightarrow H_2(\widetilde{X}; \mathbb{Z})$. The 2'nd homology of $\widetilde{X}$ is then given by

$$H_2(\widetilde{X}; \mathbb{Z}) = j(H_2(X; \mathbb{Z})) \oplus \mathbb{Z}F.$$

The $(2n-2)$'th homology of $\widetilde{X}$ fits into the following exact sequence:

$$0 \longrightarrow \mathbb{Z}[E] \longrightarrow H_{2n-2}(\widetilde{X}; \mathbb{Z}) \xrightarrow{p_*} H_{2n-2}(X; \mathbb{Z}) \longrightarrow 0,$$

where the first map is induced by the inclusion. Denote by $\widetilde{\Sigma} \subset \widetilde{X}$ the proper transform of Σ (with respect to p) in $\widetilde{X}$. The intersection pairing between H_{2n-2} and H_2 in $\widetilde{X}$ is related to the one in X as follows:

$$V \cdot j(A) = p_*(V) \cdot A, \quad \forall V \in H_{2n-2}(\widetilde{X}; \mathbb{Z}),\ A \in H_2(X; \mathbb{Z}),$$

$$[\widetilde{\Sigma}] \cdot F = 1, \quad [E] \cdot F = -1, \quad [E] \cdot j(A) = 0, \quad \forall A \in H_2(X; \mathbb{Z}).$$

Consider now the fibration $\pi_\ell : \widetilde{X} \longrightarrow \ell$. The fibre over $H_0 \in \ell$ is precisely $\Sigma = \Sigma_{H_0}$. It follows from (3.5) that the set of classes $H_2^{\pi_\ell}$ that represent sections of π satisfies:

$$H_2^{\pi_\ell} \subset \{j(A) + dF \mid [\Sigma] \cdot A = 1 - d\}. \tag{3.4}$$

Denote by J_0 the complex structure of X. Denote by $\mathcal{R}(X) \subset H_2(X; \mathbb{Z})$ the positive cone generated by classes that represent J_0-holomorphic rational curves in X, i.e.

$$\mathcal{R}(X) = \Big\{ \sum a_i [C_i] \mid a_i \in \mathbb{Z}_{\geq 0},\ C_i \subset X \text{ is a rational } J_0\text{-holomorphic curve} \Big\}.$$

Lemma 3.5.A *Let* $\widetilde{A} = j(A) + dF \in H_2^{\pi_\ell}$*, with* $A \in H_2(X;\mathbb{Z})$*,* $d \in \mathbb{Z}$*. If* $\mathcal{S}(\widetilde{A}) \neq 0$ *then* $A \in \mathcal{R}(X)$ *and* $d \leq 1$*, with equality if and only if* $A = 0$*.*

Proof: Denote by $\widetilde{J_0}$ the standard complex structure on $\widetilde{X} \subset X \times \ell$, namely the complex structure induced from the standard complex structure $J_0 \oplus i$ on $X \times \ell$. Let $\widetilde{J_n}$ be a sequence of regular almost complex structure on $\widetilde{X}$ with $\widetilde{J_n} \longrightarrow \widetilde{J_0}$. Since $\mathcal{S}(\widetilde{A}, \widetilde{J_n}) \neq 0$, there exists $\widetilde{J_n}$-holomorphic sections $u_n \in \mathcal{M}^{\mathfrak{s}}(\widetilde{A}, \widetilde{J_n})$. After passing to the limit $n \longrightarrow \infty$ we obtain by Gromov compactness theorem a (possibly reducible) $\widetilde{J_0}$-holomorphic curve $D \subset \widetilde{X}$ in the class $\widetilde{A}$. As $p : \widetilde{X} \longrightarrow X$ is $(\widetilde{J_0}, J_0)$-holomorphic it follows that $p(D)$ is a J_0-holomorphic rational curve, hence $A = p_*([D]) \in \mathcal{R}(X)$.
Next, recall that $[\Sigma] \cdot A = 1 - d$. But $\Sigma \subset X$ is ample, hence $[\Sigma] \cdot A = 1 - d \geq 0$ with equality if and only if $A = 0$. □

The next lemma shows that when $d < 1$ the sections in the class $\widetilde{A}$ do not contribute to the Seidel element in 3.1.

Lemma 3.5.B *Let* $\widetilde{A} = j(A) + dF \in H_2^{\pi_\ell}$ *with* $A \in H_2(X;\mathbb{Z})$ *and* $d < 1$*. Then* $c_1^v(\widetilde{A}) > 0$*. In particular, in view of 3.2,* $\widetilde{A}$ *does not contribute to* $S(\pi_\ell)$*.*

proof: Denote by $c_1^{\widetilde{X}}$ the first Chern class of (the tangent bundle of) $\widetilde{X}$ and by c_1^X that of X. Since $\widetilde{X}$ is the blow-up of X along B, the relation between these Chern classes is given by:

$$c_1^{\widetilde{X}} = p^* c_1^X - PD([E]), \tag{3.5}$$

where $PD([E]) \in H^2(\widetilde{X})$ stands for the Poincaré dual of $[E]$. (See e.g. (20).)
Denote by c_1^ℓ the first Chern class of $\ell \cong \mathbb{C}P^1$. Since $\widetilde{A}$ is represented be sections of π_ℓ we have:

$$c_1^v(\widetilde{A}) = c_1^{\widetilde{X}}(\widetilde{A}) - \pi_\ell^*(c_1^\ell)(\widetilde{A}) = c_1^{\widetilde{X}}(\widetilde{A}) - 2.$$

Together with 3.5 and 3.5 this implies:

$$c_1^v(\widetilde{A}) = p^*(c_1^X)(\widetilde{A}) - [E] \cdot \widetilde{A} - 2 = c_1^X(A) + d - 2. \tag{3.6}$$

By Lemma 3.5.A, $A \in \mathcal{R}(X) \subset H_2^S$, hence by Remark 3.4.0.C we have

$$c_1^X(A) = \frac{n+k+2}{2} h(A) = \frac{n+k+2}{2}([\Sigma] \cdot A) = \frac{n+k+2}{2}(1-d).$$

Together with 3.6 we obtain:

$$c_1^v(\widetilde{A}) = \frac{n+k}{2}(1-d) - 1 \geq \frac{n+k}{2} - 1 > 0,$$

because $d < 1$ and $n \geq 2$. □

We now turn to the case $\widetilde{A} = F$. Let $b \in B$. Define

$$\widetilde{u}_b : \ell \longrightarrow \widetilde{X}, \quad \widetilde{u}_b(z) = (b, z).$$

It is easy to see that $\widetilde{u}_b$ is a $\widetilde{J}_0$-holomorphic section of π_ℓ representing the class F.

Lemma 3.5.C *The sections $\widetilde{u}_b$, $b \in B$, are the only $\widetilde{J}_0$-holomorphic sections in the class F, hence $\mathcal{M}^s(F, \widetilde{J}_0) = \{\widetilde{u}_b \mid b \in B\}$. The evaluation map is given by*

$$ev_{\widetilde{J}_0, z_0}(\widetilde{u}_b) = b \in \Sigma$$

and is an orientation preserving diffeomorphism between $\mathcal{M}^s(F, \widetilde{J}_0)$ and the base locus B.

proof: Let $\widetilde{u} : \ell \longrightarrow \widetilde{X}$ be a $\widetilde{J}_0$-holomorphic section in the class F. Write $\widetilde{u}(z) = (v(z), z) \in X \times \ell$. Due to our choice of $\widetilde{J}_0$, v is a J_0-holomorphic map. Since $p_*(F) = 0$ the map $v = p \circ \widetilde{u} : \ell \longrightarrow X$ must be constant, say $v(z) \equiv b$, $b \in X$. But $v(z) \in \Sigma_z$ for every $z \in \ell$. It follows that $b \in \cap_{z \in \ell} \Sigma_z = B$. The rest of the statements in the lemma are immediate.

We are now ready for the

proof: [Proof of Theorem 3.4.0.B] In view of 3.4 and Lemmas 3.5.A, 3.5.B, the only class that contributes to the Seidel element $S(\pi_\ell)$ is F, hence:

$$S(\pi_\ell) = \mathcal{S}(F) \in QH^2(\Sigma; \Lambda).$$

(We take F to be the reference class of sections and note that $c_1^v(F) = -1$.)
In order to evaluate $\mathcal{S}(F)$ we need to compute $\mathcal{S}(F, \widetilde{J})$ for a regular $\widetilde{J}$. We first claim that there exists a neighborhood $\mathcal{U}$ of $\widetilde{J}_0$ inside $\widetilde{\mathcal{J}}(\pi_\ell, \widetilde{\Omega})$ such that for every $\widetilde{J} \in \mathcal{U}$ the space $\mathcal{M}^s(F, \widetilde{J})$ is compact.
To see this, first note that $\widetilde{\Omega}$ is a genuine symplectic form on $\widetilde{X}$ and that $\widetilde{J}_0$ is tamed by $\widetilde{\Omega}$ (i.e. $\Omega(v, \widetilde{J}_0 v) > 0$ for all non-zero vectors $v \in T\widetilde{X}$ be they vertical or

not). Hence there is a neighbourhood $\mathcal{U}$ of $\widetilde{J}_0$ in $\widetilde{\mathcal{J}}(\pi_\ell, \widetilde{\Omega})$ such that every $\widetilde{J} \in \mathcal{U}$ is tamed by $\widetilde{\Omega}$. Next note that $\widetilde{\Omega}$ defines an integral (modulo torsion) cohomology class $[\widetilde{\Omega}] \in H^2(\widetilde{X})_{fr}$ and that $\widetilde{\Omega}(F) = 1$ (see §3.4). It follows that F is a class of minimal positive area for $\widetilde{\Sigma}$. Therefore, for $\widetilde{J}$ tamed by $\widetilde{\Omega}$, a sequence of $\widetilde{J}$-holomorphic rational curves in the class F cannot develop bubbles. By Gromov compactness $\mathcal{M}^{\mathfrak{s}}(F, \widetilde{J})$ is compact.

Next we claim that $\widetilde{J}_0$ is a regular almost complex structure in the sense of the general theory of pseudo-holomorphic curves (see Chapter 3 in (29)). To see this recall that by Lemma 3.3.1 of (29): *let* (M, ω) *be a symplectic manifold and* J *an* integrable *almost complex structure. Then* J *is regular for a* J*-holomorphic curve* $u : \mathbb{C}P^1 \longrightarrow M$ *if every summand of the holomorphic bundle* $u^*TM \to \mathbb{C}P^1$ *(in its splitting to a direct sum of line bundles) has Chern number* ≥ -1. Applying this to our case, a simple computation shows that for every $\widetilde{u}_b \in \mathcal{M}^{\mathfrak{s}}(F, \widetilde{J}_0)$ we have

$$\widetilde{u}_b^* T\widetilde{X} = \mathcal{O}_\ell(2) \oplus \mathcal{O}_\ell^{\oplus(n-2)} \oplus \mathcal{O}_\ell(-1),$$

hence $\widetilde{J}_0$ is regular for all $\widetilde{u} \in \mathcal{M}^{\mathfrak{s}}(F, \widetilde{J}_0)$.

Pick a regular almost complex structure $\widetilde{J} \in \widetilde{\mathcal{J}}_{\mathrm{reg}}(\pi, \widetilde{\Omega}) \cap \mathcal{U}$ which is close enough to $\widetilde{J}_0$. By the standard theory of pseudo-holomorphic curves (29) the evaluation maps $ev_{\widetilde{J}, z_0}$ and $ev_{\widetilde{J_0}, z_0}$ are cobordant, hence give rise to cobordant pseudo-cycles. Moreover by what we have seen before this cobordism can be assumed to be compact (and the pseudo-cycles are in fact cycles). It follows that the homology class $(ev_{\widetilde{J}, z_0})_*[\mathcal{M}^{\mathfrak{s}}(F, \widetilde{J})]$ equals to $(ev_{\widetilde{J_0}, z_0})_*[\mathcal{M}^{\mathfrak{s}}(F, \widetilde{J}_0)] = [B]$. Putting everything together we obtain:

$$S(\pi_\ell) = \mathcal{S}(F, \widetilde{J}) = PD([B]) = [\omega_{\Sigma}].$$

In particular, we have:

Corollary 3.5.D *Let* $\Sigma \subset X \subset \mathbb{CP}^N$ *be as in Theorem 3.4.0.B . Then*

$$\pi_1(Ham(\Sigma, \omega_\Sigma)) \neq 0$$

.

3.6 Proof of Corollary 3.4.0.D

The quantum cohomology of Σ can be written as a vector space as

$$QH^j(\Sigma;\Lambda) \cong \bigoplus_{l\in\mathbb{Z}} H^{j+2C_\Sigma l}(\Sigma).$$

By Theorem 3.4.0.B, $[\omega_\Sigma] \in QH^2(\Sigma;\Lambda)$ is invertible with respect to the quantum product $*$, hence the map

$$(-) * [\omega_\Sigma] : QH^j(\Sigma;\Lambda) \longrightarrow QH^{j+2}(\Sigma;\Lambda), \quad a \longmapsto a * [\omega_\Sigma]$$

is an isomorphism for every $j \in \mathbb{Z}$. The statement about $\widetilde{b}_j(\Sigma)$ follows immediately.

We now turn to the proof of the statement about $\widetilde{b}_j(X)$. First recall that $2C_\Sigma = n+k$ and $2C_X = n+k+2$. We will show now that for every $0 \leq j \leq n+k+1$ we have $\widetilde{b}_j(X) = \widetilde{b}_{j+2}(X)$.

Step 1. Assume $j \leq n+k-4$.

By Corollary 3.2.B, $b_j(\Sigma) = b_j(X)$ and $b_{j+2}(\Sigma) = b_{j+2}(X)$. We claim that

$$b_{j+n+k}(\Sigma) = b_{j+n+k+2}(X), \quad b_{j+n+k+2}(\Sigma) = b_{j+n+k+4}(X). \tag{3.7}$$

Indeed, by Corollary 3.2.B, $b_{n-j-k-2}(\Sigma) = b_{n-j-k-2}(X)$, hence the first equation follows from Poincaré duality for Σ and X. The proof of the second equality is similar. It follows that $\widetilde{b}_j(X) = b_j(X)+b_{j+n+k+2}(X) = b_j(\Sigma)+b_{j+n+k}(\Sigma) = \widetilde{b}_j(\Sigma)$ $= \widetilde{b}_{j+2}(\Sigma) = b_{j+2}(\Sigma) + b_{j+n+k+2}(\Sigma) = b_{j+2}(X) + b_{j+n+k+4}(X) = \widetilde{b}_{j+2}(X)$.

Step 2. Assume $n+k-3 \leq j \leq n+k-1$.

In this case we have $\widetilde{b}_j(X) = b_j(X)$ and $\widetilde{b}_{j+2}(X) = b_{j+2}(X)$ and the equality between the two follows from Corollary 3.2.C.

Step 3. Assume $j = n+k$.

We have to prove that

$$b_{n+k}(X) = b_0(X) + b_{n+k+2}(X).$$

By Poincaré duality this is equivalent to showing that $b_{n-k}(X) = b_0(X) + b_{n-k-2}(X)$. The last equivalent is, by Corollary 3.2.B, to

$$b_{n-k}(\Sigma) = b_0(\Sigma) + b_{n-k-2}(\Sigma).$$

Applying Poincaré duality on Σ the latter becomes equivalent to

$$b_{n+k-2}(\Sigma) = b_0(\Sigma) + b_{n+k}(\Sigma).$$

But this has already been proved since $b_{n+k-2}(\Sigma) = \widetilde{b}_{n+k-2}(\Sigma) = \widetilde{b}_{n+k}(\Sigma) = b_0(\Sigma) + b_{n+k}(\Sigma)$.

Step 4. Assume $j = n + k + 1$.

The proof in this case is very similar to the case $j = n + k$. We omit the details.

3.7 Further Applications of Sub-Criticality.

In this section we present two further applications of the sub-criticality property of section 3.2 using methods of Floer homology.

3.7.1 Positive Defect and Lagrangian Spheres.

We show the following result regarding the (non) existence of Lagrangian spheres in hyperplane sections of spherically monotone manifolds with positive defect:

Theorem 3.7.A *Let $\Sigma \subset X \subset \mathbb{C}P^N$ be as in Theorem 3.4.0.B and assume in addition that $\dim_{\mathbb{C}}(\Sigma) \geq 3$. Let ω_{Σ} be the restriction of the symplectic form of $\mathbb{C}P^N$ to Σ. Then the symplectic manifold $(\Sigma, \omega_{\Sigma})$ contains no Lagrangian spheres.*

proof: Suppose by contradiction that $L \subset (\Sigma, \omega_{\Sigma})$ is a Lagrangian sphere. We apply Lagrangian Floer cohomology theory (refer to section 2.2) in order to arrive at a contradiction. We take $\mathbb{Z}_2$ as the ground ring and work with the self Floer cohomology $HF(L)$ of L, with coefficients in the Novikov ring $\Lambda_{\mathbb{Z}_2} = \mathbb{Z}_2[q, q^{-1}]$. We set the grading so that the variable q has degree $\deg(q) = N_L$, where N_L is the minimal Maslov number of L.

Since L is simply connected, the conditions on Σ and X imply that $L \subset \Sigma$ is a monotone Lagrangian submanifold and its minimal Maslov number is $N_L =$

$2C_\Sigma = n + k$. (Here, as in Theorem 3.4.0.B $k = \mathrm{def}(X) \geq 1$.) Under these circumstances, as in 2.2, it is well known in Floer theory that the self Floer homology of L, $HF(L)$, is well defined and moreover we have an isomorphism of graded $\Lambda_{\mathbb{Z}_2}$-modules:

$$HF^*(L) \cong (H^\bullet(L; \mathbb{Z}_2) \otimes \Lambda_{\mathbb{Z}_2})^*.$$

Moreover, since L is a sphere of dimension $\dim_{\mathbb{R}}(L) \geq 3$ this implies, in particular, that

$$HF^0(L, L) \cong \mathbb{Z}_2, \quad HF^2(L, L) \cong H^2(L; \mathbb{Z}_2) = 0. \tag{3.8}$$

Denote by $QH(\Sigma; \Lambda_{\mathbb{Z}_2})$ the modulo-2 reduction of $QH(\Sigma; \Lambda)$ (obtained by reducing the ground ring $\mathbb{Z}$ to $\mathbb{Z}_2$). By Theorem 3.4.0.B, $[\omega_\Sigma] \in QH^2(\Sigma; \Lambda)$ is an invertible element, hence its modulo-2 reduction, say $\alpha \in QH^2(\Sigma; \Lambda_{\mathbb{Z}_2})$ is invertible too.

We now appeal to the quantum module structure of $HF(L)$ introduced in (11; 13). By this construction, $HF(L)$ has a structure of a graded module over the ring $QH(\Sigma; \Lambda_{\mathbb{Z}_2})$ where the latter is endowed with the quantum product. We denote the module action of $QH^*(\Sigma; \Lambda_{\mathbb{Z}_2})$ on $HF^*(L)$ by

$$QH^i(\Sigma; \Lambda_{\mathbb{Z}_2}) \otimes_{\Lambda_{\mathbb{Z}_2}} HF^j(L) \longrightarrow HF^{i+j}(L), \quad \alpha \otimes x \longmapsto \alpha \circledast x, \quad i, j \in \mathbb{Z}.$$

Since $\alpha \in QH^2(\Sigma; \Lambda_{\mathbb{Z}_2})$, α induces an isomorphism $\alpha \circledast (-) : HF^*(L) \longrightarrow HF^{*+2}(L, L)$. This however, is impossible (e.g for $* = 0$) in view of 3.8. Contradiction.

Corollary 3.7.B *Let Σ be an algebraic manifold with $b_2(\Sigma) = 1$. Suppose that Σ can be realized as a hyperplane section of a projective manifold $X \subset \mathbb{C}P^N$ with small dual. Then in any other realization of Σ as a hyperplane section of a projective manifold $X' \subset \mathbb{C}P^{N'}$ we have $\mathrm{def}(X') > 0$. In fact, $\mathrm{def}(X') = \mathrm{def}(X)$.*

3.7.2 An additional proof of 2-periodicity of the singular cohomology.

Set

$$\widetilde{H}^i(X; \mathbb{Z}_2) = \bigoplus_{l \in \mathbb{Z}} H^{i+2C_X l}(X; \mathbb{Z}_2)$$

In this section we show a further proof of Corollary 3.4.0.D with $\mathbb{Z}_2$-coefficients using methods of the theory of Lagrangian (non)-intersections.

Theorem 3.7.A *Let $X \subset \mathbb{C}P^N$ be a spherically monotone manifold with def$(X) = k > 0$ and let $\Sigma \subset X$ be a smooth hyperplane section. Then*

$$\widetilde{H}^q(X;\mathbb{Z}_2) = \widetilde{H}^{q+2}(X;\mathbb{Z}_2) \quad ; \quad \widetilde{H}^q(\Sigma;\mathbb{Z}_2) = \widetilde{H}^{q+2}(\Sigma;\mathbb{Z}_2)$$

for any $q \in \mathbb{Z}$.

Proof: Let $X \subset \mathbb{C}P^N$ be a projectively embedded manifold and let $\Sigma = X \cap H$ be a smooth hyperplane section. Denote by $Y = X \setminus \Sigma \subset \mathbb{C}^N$ the corresponding affine manifold. Let ω_X, ω_Σ be the restrictions of the Fubini-Study form of $\mathbb{C}P^N$ on X and Σ respectively. Let $\pi : N \to \Sigma$ be the normal bundle of Σ in X endowed with some Hermitian metric $|\cdot|$. On N one has the following canonical symplectic form

$$\omega_0 = \pi^*(\omega_\Sigma) + d(r^2\alpha)$$

where r is the radial coordinate on the fibers given by the metric and α is a connection 1-form on $N \setminus \Sigma$ with curvature $d(\alpha) = -\pi^*(\omega_\Sigma)$. Set

$$E_\Sigma = \{z \in N \mid\mid z \mid\leq 1\}$$

the corresponding disc bundle over Σ. According to results of (8) there exists a CW-complex $\Delta \subset Y$ of dimension at most $n = dim_{\mathbb{C}}X$, called the skeleton of T, such that $(X \setminus \Delta, \omega_X)$ is symplectomorphic to (E_Σ, ω_0). Fix $0 < \epsilon < 1$ and consider the circle bundle over Σ

$$P = \{z \in N \mid\mid z \mid= \epsilon\} \subset Y$$

where the inclusion to Y is given in terms of the identification $(E_\Sigma, \omega_0) \simeq (X \setminus \Delta, \omega_X)$. Consider the Lagrangian embedding $diag : \Sigma \to (\Sigma \times \Sigma, \omega_\Sigma \oplus (-\omega_\Sigma))$ as the diagonal and consider the map $\pi \times id : P \times \Sigma \to \Sigma \times \Sigma$. It is shown in (8) that

$$\Gamma := (\pi \times id)^{-1}(diag(\Sigma)) \subset P \times \Sigma \subset Y \times \Sigma$$

is a Lagrangian submanifold of $Y \times \Sigma$ with the symplectic form $\omega_X \oplus (\epsilon^2 - 1)\omega_\Sigma$.

Under the assumptions of the theorem, by Ein's formula, we have $2C_\Sigma = n + k$ and thus

$$\Gamma \subset (Y \times \Sigma, \omega_X \oplus (\epsilon^2 - 1)\omega_\Sigma)$$

is a monotone Lagrangian submanifold with $N_\Gamma = 2C_\Sigma = n + k$. It is shown in (8) that $HF(\Gamma)$, when computed in $Y \times \Sigma$ with the form $\Omega_Y \oplus (\epsilon^2 - 1)\omega_\Sigma)$

is isomorphic to $HF(\Gamma)$ when computed with the form $\omega_X \oplus (\epsilon^2 - 1)\omega_\Sigma$. Since, due to Theorem 3.2.A. , Y is a subcritical Stein manifold the Lagrangian Γ is displaceable w.r.t the form $\omega_X \oplus (\epsilon^2 - 1)\omega_\Sigma$. In particular, $HF(\Gamma) = 0$. Consider the corresponding spectral sequence $\{E_r^{p,q}, d^r\}$ of 2.2. Since $N_L = n + k$ the spectral sequence degenerates at the first step. Thus, the Floer differential map

$$d^1 : H^{q+n+k-1}(\Gamma; \mathbb{Q}) \to H^q(\Gamma; \mathbb{Q})$$

is an isomorphism. On the other hand, consider the Gysin sequence for the circle bundle $P_\Sigma \to \Sigma$ and note that Γ is diffeomorphic to P. Due to the Lefschetz hyperplane theorem, for $0 \le q \le n + k - 1$ the sequence splits as follows

$$0 \to H^{q-2}(\Sigma; \mathbb{Q}) \to H^q(\Sigma; \mathbb{Q}) \to H^q(P_\Sigma; \mathbb{Q}) \to 0$$

Finally, since in this case

$$\widetilde{H}^j(\Sigma; \mathbb{Z}_2) = H^j(\Sigma; \mathbb{Z}_2) \oplus H^{j+n+k}(\Sigma; \mathbb{Z}_2)$$

we get the required result. □

3.8 Further Discussion

We conclude with a discussion of further implications and possible directions of study arising from the results described in this work.

3.8.1 Further properties of $\pi_1(Ham(\Sigma, \omega_\Sigma))$:

The structure of the fundamental group $\pi_1(Ham(M, \omega_M))$ of the group of Hamiltonian diffeomorphisms of general symplectic manifolds (M, ω_M) has been the subject of much study in the field of symplectic topology, we refer the reader to (18; 32; 33; 35). It is interesting to ask what the methods of the general theory imply in this special case here. More concretely, let $X \subset \mathbb{C}P^N$ be an algebraic manifold with $\mathrm{def}(X) = k > 0$ and $b_2(X) = 1$. Denote by $\lambda_\pi \in \pi_1(Ham(\Sigma, \omega_\Sigma))$ the non-trivial element constructed in section 4.

- What can be said about the minimal Hofer length of loops in $\mathrm{Ham}(\Sigma, \omega_\Sigma)$ in the homotopy class λ?

• Can the homotopy class of λ be represented by a Hamiltonian circle action ? (Note that at least in some examples this indeed turns out to be the case.)

• Is λ an element of finite or infinite order?

• In case λ has infinite order, what can be said about the value of the Calabi homomorphism $\widetilde{cal} : \pi_1(Ham(\Sigma), \omega_\Sigma)) \to \mathbb{R}$ of the element λ, see (18).

3.8.2 The algebraic structure of $QH^*(\Sigma; \Lambda)$:

Let $X \subset \mathbb{C}P^N$ be a manifold with $\mathrm{def}(X) = k > 0$ and $b_2(X) = 1$ as discussed in this work. As a corollary of Theorem D we have obtained that $QH^*(\Sigma; \Lambda)$ satisfies the relation $[\omega] * \alpha = q$ for some $\alpha \in H^{n+k-2}(\Sigma; \mathbb{Z})$. However, the examples seem to indicate that actually

$$QH^*(\Sigma; \Lambda) \simeq \frac{H^*(\Sigma; \mathbb{Z}) \otimes \Lambda}{\langle [\omega] * \alpha = q \rangle}$$

where $\simeq$ stands for ring isomorphism. In a similar context, it is interesting to note that the algebraic structure of quantum cohomology of uniruled manifolds has been studied in a recent paper of McDuff (27). In particular, in (27) McDuff proves a general existence result for non-trivial invertible elements of the quantum cohomology of uniruled manifolds using purely algebraic methods. One should note that the inevitability of $[\omega_\Sigma] \in QH^2(\Sigma, \Lambda)$ of Theorem D is a direct computation of such an element in the case of manifolds with positive defect and $b_2(X) = 1$.

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