Stability of Ricci-flat spaces and singularities in 4d Ricci flow

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Author(s):
Haslhofer, Robert

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Stability of Ricci-flat Spaces and Singularities in 4d Ricci Flow

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presented by
Robert Haslhofer
MSc ETH Mathematics
born January 17, 1984
citizen of Austria

accepted on the recommendation of
Prof. Dr. Tom Ilmanen, examiner
Prof. Dr. Michael Struwe, co-examiner
Prof. Dr. Gerhard Huisken, co-examiner

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Abstract

In this thesis, we describe some closely related results on Ricci curvature and Ricci flow that we obtained during the last couple of years.

In Chapter 1, we discuss the formation of singularities in higher-dimensional Ricci flow without pointwise curvature assumptions. We prove that the space of singularity models with bounded entropy and locally bounded energy is orbifold-compact in arbitrary dimensions. In dimension four, a delicate localized Gauss-Bonnet estimate even allows us to drop the assumption on energy in favor of (essentially) an upper bound for the Euler characteristic. These results form one of the highlights of this thesis and have been obtained in a joint work with Reto Müller.

In Chapter 2, we investigate the stability of compact Ricci-flat metrics (under the technical assumption that all infinitesimal Ricci-flat deformations are integrable). We prove a Łojasiewicz-Simon inequality for Perelman's λ-functional and establish a transversality estimate that shows that the Ricci flow does not move excessively in gauge directions. As consequences, we obtain a rigidiy result, a new proof of Sesum's dynamical stability theorem for the Ricci flow and, as a sharp complement, a dynamical instability theorem.

In Chapter 3, in order to be able to study stability also in the noncompact case, we introduce a renormalized $\mathcal{F}$-functional for perturbations of noncompact steady Ricci solitons. This functional motivates a stability inequality for Ricci-flat spaces, in particular for Ricci-flat cones. We also introduce a geometric invariant λ_{AF} for asymptotically flat manifolds with nonnegative scalar curvature. This invariant gives a quantitative lower bound for the ADM-mass from general relativity, motivates a proof of the rigidity statement in the positive mass theorem using the Ricci flow, and eventually leads to the discovery of a mass-decreasing flow in dimension three.

In Chapter 4, we thoroughly investigate the above-motivated stability inequality for Ricci-flat cones. We prove that the Ricci-flat cone over $\mathbb{C}P^2$ is stable, showing that the first stable nonflat Ricci-flat cone occurs in the smallest possible dimension in sharp contrast with the case of stable minimal hypersurfaces. On the other hand, we prove that many further examples of Ricci-flat cones over 4-manifolds are unstable, and that Ricci-flat cones over products of Einstein manifolds and over Kähler-Einstein manifolds with $h^{1,1} > 1$ are unstable in dimension less than 10. We also give plenty of motiva-

tions and partly confirm a conjecture of Tom Ilmanen relating the λ-functional, the positive mass theorem and the nonuniqueness of Ricci flows with conical initial data. The results from this chapter have been obtained in a joint work with Stuart Hall and Michael Siepmann.

In Chapter 5, we finally investigate the above-mentioned mass-decreasing flow for asymptotically flat 3-manifolds with nonnegative scalar curvature. Our flow is defined by iterating a suitable Ricci flow with surgery and conformal rescalings. It exists for all times, preserves the class of asymptotically flat metrics with nonnegative scalar curvature, and decreases the mass. Wormholes pinch off and nontrivial spherical space forms bubble off in finite time. Moreover, we show that our quantity λ_{AF} is monotone along the flow. Assuming a certain inequality between λ_{AF} and the mass a priori, we can prove that the flow squeezes out all the initial mass.

Almost everything said in this thesis is based on the following five articles: A compactness theorem for complete Ricci shrinkers (joint with Reto Müller; Geom. Funct. Anal. 21(5):1091–1116, 2011), Perelman's lambda-functional and the stability of Ricci-flat metrics (Calc. Var. Partial Differential Equations, online first, DOI 10.1007/s00526-011-0468-x), A renormalized Perelman-functional and a lower bound for the ADM-mass (J. Geom. Phys. 61(11):2162–2167, 2011), The stability inequality for Ricci-flat cones (joint with Stuart Hall and Michael Siepmann; arXiv:1111.4981, submitted), A mass-decreasing flow in dimension three (arXiv:1107.3220, submitted).

Zusammenfassung

In dieser Arbeit beschreiben wir einige eng verwandte Resultate über Ricci-Krümmung und Ricci-Fluss, die wir im Verlauf der letzten Jahre gefunden haben.

In Kapitel 1 diskutieren wir die Entwicklung von Singularitäten in höherdimensionalem Ricci-Fluss ohne punktweise Krümmungsannahmen. Wir beweisen, dass der Raum der Singularitätsmodelle mit beschränkter Entropie und lokal beschränkter Energie in beliebigen Dimensionen Orbifaltigkeit-kompakt ist. In Dimension vier ermöglicht uns ein delikates Gauss-Bonnet-Argument sogar die Energieannahme (im wesentlichen) durch eine obere Schranke an die Eulercharakteristik zu ersetzen. Diese Resultate bilden einen der Höhepunkte dieser Arbeit und stammen aus einem gemeinsamen Projekt mit Reto Müller.

In Kapitel 2 untersuchen wir die Stabilität von kompakten Ricci-flachen Metriken (unter der technischen Annahme, dass alle infinitesimalen Ricci-flachen Deformationen integrierbar sind). Wir beweisen eine Łojasiewicz-Simon-Ungleichung für Perelman's λ-Funktional und erarbeiten eine Transversalitätsabschätzung, die zeigt, dass sich der Ricci-Fluss nicht zu stark in Eichrichtungen bewegt. Als Folgerungen erhalten wir ein Starrheitsresultat, einen neuen Beweis von Sesum's dynamischen Stabilitätsresultat für den Ricci-Fluss und, als scharfes Gegenstück, ein dynamisches Instabilitätsresultat.

In Kapitel 3 führen ein renormiertes $\mathcal{F}$-Funktional für Störungen von Ricci-Solitonen ein, damit wir Stabilität auch im nichtkompakten Fall untersuchen können. Dieses Funktional motiviert eine Stabilitätsungleichung für Ricci-flache Räume, insbesondere für Ricci-flache Kegel. Wir führen auch eine geometrische Invariante λ_{AF} für asymptotisch flache Mannigfaltigkeiten mit nichtnegativer Skalarkrümmung ein. Diese Invariante gibt eine quantitative untere Schranke für die ADM-Masse aus der allgemeinen Relativitätstheorie, motiviert einen Ricci-Fluss-Beweis der Starrheitsaussage im Positve-Masse-Theorem, und führt schliesslich zur Entdeckung eines massenveringernden Flusses in Dimension drei.

In Kapitel 4 untersuchen wir die oben erwähnte Stabilitätsungleichung für Ricci-flache Kegel. Wir beweisen, dass der Ricci-flache Kegel über $\mathbb{C}P^2$ stabil ist, was zeigt, dass der erste stabile nichtflache Ricci-flache Kegel in der niedrigstmöglichen Dimension auftritt im scharfen Gegensatz zum Fall von minimalen Hyperflächen. Andererseits beweisen wir, dass viele weitere Beispiele von Ricci-flachen Kegeln über 4-Mannigfaltigkeiten in-

stabil sind und, dass Ricci-flache Kegel über Produkten von Einstein-Mannigfaltigkeiten und über Kähler-Einstein-Mannigfaltigkeiten mit $h^{1,1} > 1$ in Dimension kleiner 10 instabil sind. Wir motivieren das Problem umfassend und bestätigen Teile einer Vermutung von Tom Ilmanen, die das λ-Funktional, das Positive-Masse-Theorem und die Nichteindeutigkeit von Ricci-Flüssen mit konischen Anfangsdaten verbindet. Die Resultate dieses Kapitels stammen aus einer gemeinsamen Arbeit mit Stuart Hall und Michael Siepmann.

In Kapitel 5 untersuchen wir schliesslich den oben erwähnten massenveringernden Fluss für asymptotisch flache 3-Mannigfaltigkeiten mit nichtnegativer Skalarkrümmung. Unser Fluss ist als Iteration eines geeigneten Ricci-Flusses mit Chirurgie und konformen Reskalierungen definiert. Er existiert für alle Zeiten, erhält die Klasse der asymptotisch flachen Metriken mit nichtnegativer Skalarkrümmung, und verringert die Masse. Wurmlöcher schnüren ab und sphärische Raumformen lösen sich ab. Desweiteren zeigen wir, dass unser Funktional λ_{AF} monoton entlang des Flusses ist. Unter der a-priori Annahme einer gewissen Ungleichung zwischen λ_{AF} und der Masse könnne wir beweisen, dass der Fluss die gesamte Masse herausquetscht.

Fast alles was in dieser Arbeit gesagt wird beruht auf den folgenden fünf Artikeln: A compactness theorem for complete Ricci shrinkers (zusammen mit Reto Müller; Geom. Funct. Anal. 21(5):1091–1116, 2011), Perelman's lambda-functional and the stability of Ricci-flat metrics (Calc. Var. Partial Differential Equations, online first, DOI 10.1007/s00526-011-0468-x), A renormalized Perelman-functional and a lower bound for the ADM-mass (J. Geom. Phys. 61(11):2162–2167, 2011), The stability inequality for Ricci-flat cones (zusammen mit Stuart Hall und Michael Siepmann; arXiv: 1111.4981, eingereicht), A mass-decreasing flow in dimension three (arXiv:1107.3220, eingereicht).

Contents

Overview and main results

Content of this thesis

In this thesis, we describe some closely related results on Ricci curvature and Ricci flow that we obtained during the last couple of years. Almost everything said here is based on the five articles [69, 67, 66, 61, 68], i.e. on

1. *A compactness theorem for complete Ricci shrinkers* (joint with Reto Müller), Geom. Funct. Anal. 21(5):1091–1116, 2011.[1]

2. *Perelman's lambda-functional and the stability of Ricci-flat metrics*, Calc. Var. Partial Differential Equations, online first, DOI 10.1007/s00526-011-0468-x.[1]

3. *A renormalized Perelman-functional and a lower bound for the ADM-mass*, J. Geom. Phys. 61(11):2162–2167, 2011.

4. *The stability inequality for Ricci-flat cones* (joint with Stuart Hall and Michael Siepmann), arXiv:1111.4981, submitted.

5. *A mass-decreasing flow in dimension three*, arXiv:1107.3220, submitted.

We will first give a short overview about some of the most important developments in the study of Ricci curvature and Ricci flow. We will then describe the main questions that we investigated and summarize the main results that we obtained. For much more comprehensive introductions, motivations, explanations and all proofs we refer the reader to the remaining five chapters, which constitute the actual bulk of this thesis.

Ricci curvature and Ricci flow

A Ricci-flat manifold is a Riemannian manifold with vanishing Ricci curvature, and more generally an Einstein manifold is a Riemannian manifold with constant Ricci curvature. Einstein manifolds can be thought of as optimal geometries, and are of great interest in analysis, geometry, topology and physics, see [11, 73, 81] for extensive

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[1]The original publication is available at www.springerlink.com.

background.

To quickly remind the reader about the meaning of Ricci curvature, let (M, g) be a Riemannian manifold of dimension n, ∇ the Levi-Civita connection, and X, Y, Z, W vectorfields on M. The Riemann tensor captures the noncommutativity of second derivatives due to curvature,

$$g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W) = \mathrm{Rm}(X, Y, W, Z). \tag{0.1}$$

Let $e_1, \ldots, e_n$ be a local orthonormal frame. The Ricci tensor is the trace of the Riemann tensor,

$$\mathrm{Rc}(X, Y) = \sum_{k=1}^{n} \mathrm{Rm}(X, e_k, Y, e_k). \tag{0.2}$$

Geometrically, $\mathrm{Rc}(e_i, e_i)$ is $(n-1)$ times the average sectional curvature of the two-planes spanned by e_i and an orthonormal vector. Computationally, in harmonic coordinates there is an expression of the form,

$$-2R_{ij} = \sum_k \partial_k \partial_k g_{ij} + Q_{ij}(g, \partial g), \tag{0.3}$$

and thus, analytically, Ricci curvature can be thought of as nonlinear Laplacian of the metric.

Landmark results about Ricci curvature in general include Yau's solution of the Calabi Conjecture [131], the construction of compact manifolds with special holonomy by Joyce [73], the theory about limits of manifolds with Ricci bounded below due to Cheeger, Colding, Naber and Tian (see [28] for a nice survey), and the recent notion of Ricci bounded below for metric measure spaces due to Lott, Villani, and Sturm [86, 114]. Some counterparts of Ricci-flat metrics in neighboring fields are harmonic maps, minimal surfaces, and Yang-Mills fields.

A very successful method is to deform the metric in time according to Hamilton's Ricci flow equation [63],

$$\partial_t g = -2 \, \mathrm{Rc} \, . \tag{0.4}$$

Due to (0.3), the Ricci flow can be thought of as geometric heat equation, and indeed the heat equation does a great job in evolving a given metric towards a nicer one. We refer the reader to [34, 90, 123] for some introductions to the Ricci flow, and to [65, 115] for some surveys. Hamilton already proved in [63], that on a closed 3-manifold the (volume-normalized) Ricci flow deforms any metric with positive Ricci curvature towards a round one, proving in particular that the underlying manifold must be a quotient of the 3-sphere. This is a landmark result that initiated a long line of research, culminating in Perelman's spectacular proof of the geometrization conjecture [97, 99, 98], see [76, 89, 25, 12] for detailed expositions of Perelman's work. Other highlights include the Brendle-Schoen proof of the differentiable sphere theorem [16],

and the analytic minimal model program via the Kähler-Ricci flow initiated by Tian, Song and others. Of course, these works build upon a wealth of results on the Ricci flow due to Hamilton and others, and there are many further interesting results on the Ricci flow, far to numerous to survey them here. Some counterparts of the Ricci flow in neighboring fields are the harmonic map flow, the mean curvature flow and the Yang-Mills flow.

Main results in this thesis

Despite the great success of the Ricci flow mentioned above, relatively little is known in the general higher-dimensional case without strong positivity assumptions for the curvatures. In this general case very complex singularities can form and the key problem is to study the nature of these singularities. In a joint work with Reto Müller, we investigated the compactness properties of the space of singularity models.

Background on singularities in the Ricci flow. The ideal singularity models for the Ricci flow are shrinkers, given by a manifold M, a metric g and a function f such that the following equation holds:

$$\mathrm{Rc} + \nabla^2 f = \tfrac{1}{2} g. \tag{0.5}$$

Solutions of (0.5) correspond to selfsimilar solutions of the Ricci flow, moving only by homotheties and diffeomorphisms. Most interesting singularity models are noncompact, the cylinder being the most basic example. Shrinkers always have a point $p \in M$, where the potential f attains its minimum. After imposing the normalization $\int_M (4\pi)^{-n/2} e^{-f} dV = 1$, they also have a well-defined Perelman entropy,

$$\mu(g) = \int_M \left(R + |\nabla f|^2 + f - n \right) (4\pi)^{-n/2} e^{-f} dV. \tag{0.6}$$

Investigating the compactness properties of the space of shrinkers amounts to considering sequences of shrinkers satisfying certain assumptions and trying to find a subsequence that converges in a suitable sense. Smooth limits of sequences of Riemannian manifolds are taken in the sense of Cheeger-Gromov. An important feature about sequences of noncollapsed Einstein manifolds in dimension n with $L^{n/2}$-bounds on curvatures is that orbifold singularities can occur, i.e. singularities modelled on $\mathbb{R}^n / \Gamma$ for some finite subgroup $\Gamma \subset \mathrm{O}(n)$. The simplest example for this is the blowdown sequence of the Eguchi-Hanson metric g_{EH} [47], yielding an orbifold Cheeger-Gromov convergence, $(TS^2, \frac{1}{i} g_{\mathrm{EH}}) \to \mathbb{R}^4 / \mathbb{Z}_2$.

Main results on singularities in the Ricci flow. We prove the following theorem, showing that the space of singularity models for the Ricci flow with bounded entropy and locally bounded energy is orbifold-compact in arbitrary dimensions.

Theorem 0.1 (joint with Reto Müller)
For every sequence of shrinkers (M_i^n, g_i, f_i, p_i) satisfying the entropy and local energy assumptions,

$$\mu(g_i) \geq \underline{\mu} > -\infty, \qquad \qquad \int_{B_r(p_i)} |\mathrm{Rm}|^{n/2} dV \leq C(r) < \infty, \qquad (0.7)$$

there exists a subsequence that converges to an orbifold shrinker in the pointed orbifold Cheeger-Gromov sense.

Assuming a lower bound for the entropy is very natural, since it is nondecreasing along the Ricci flow by Perelman's celebrated monotonicity formula [97]. In dimension four, a delicate localized Gauss-Bonnet argument even allows us to drop the assumption on energy in favor of (essentially) an upper bound for the Euler characteristic.

Theorem 0.2 (joint with Reto Müller)
For four-dimensional shrinkers (M^4, g, f, p) we have the weighted L^2-estimate

$$\int_M |\mathrm{Rm}|^2 e^{-f} dV \leq C(\underline{\mu}, \overline{\chi}, C_{tech}) < \infty, \qquad (0.8)$$

depending only on a lower bound $\underline{\mu}$ for the entropy, an upper bound $\overline{\chi}$ for the Euler characteristic, and a technical constant C_{tech} such that

$$|\nabla f|(x) \geq 1/C_{tech} \qquad whenever \qquad d(x, p) \geq C_{tech}. \qquad (0.9)$$

To get across the flavor of our localized Gauss-Bonnet argument, one key step is the following new and surprising estimate for shrinkers:

$$\int_M |\mathrm{Rc}|^3 e^{-f} dV \leq \varepsilon \int_M |\mathrm{Rm}|^2 e^{-f} dV + C(\varepsilon, \underline{\mu}). \qquad (0.10)$$

The strength of Theorem 0.1 compared to previous compactness-theorems in the literature is, that we can do it for noncompact manifolds, that we can do it without volume and diameter bounds, and that we do not need any pointwise curvature assumptions.

Assuming Ricci-curvature bounds Ovidiu Munteanu and Mu-Tao Wang obtained very interesting energy-estimates in dimension $n \geq 6$, see [92]. They used these estimates in combination with Theorem 0.1 to obtain an interesting variant of our compactness-theorem in higher dimensions, see Corollary 3 in their paper.

This completes a first study about the formation of singularities in higherdimensional Ricci flow. It also leads to many further interesting questions about the formation of singularities in geometric flows. Some of them are the object of study in ongoing joint projects with Reto Müller and with Jeff Cheeger and Aaron Naber. These projects will not be part of this thesis, however. Instead, we now move on pursuing a different

line of ideas, showing an intriguing relationship between Perelman's λ-functional, the stability of Ricci-flat spaces and the ADM-mass from general relativity.

Background on stability of Ricci-flat spaces and ADM-mass. Perelman made the remarkable discovery that the Ricci flow can be interpreted as gradient flow of the energy-functional

$$\lambda(g) = \inf_{f:\int e^{-f}dV=1} \mathcal{F}(g,f), \text{ where } \mathcal{F}(g,f) = \int \left(R + |\nabla f|^2\right) e^{-f} dV. \qquad (0.11)$$

On compact manifolds, the critical points of λ are precisely the Ricci-flat metrics. A compact Ricci-flat metric can be dynamically stable in the sense that the Ricci flow starting nearby exists for all times and converges to a nearby Ricci flat metric, and linearly stable in the sense that the second variation of λ is nonpositive. Finally, the ADM-mass of an asymptotically flat 3-manifold with nonnegative integrable scalar curvature is defined as

$$m(g) = \lim_{r \to \infty} \int_{S_r} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) dA^i, \qquad (0.12)$$

and is always nonnegative by the celebrated positive mass theorem [104, 130, 70].

Main results on stability of Ricci-flat spaces and ADM-mass. We first consider *compact* Ricci-flat manifolds (M, g_0), satisfying the technical assumption that all infinitesimal Ricci-flat deformations are integrable. Regarding their dynamical stability and instability properties under Ricci flow we prove:

Theorem 0.3
If g_0 is a local maximizer of λ, then the Ricci flow starting near g_0 exists for all times and converges to a nearby Ricci-flat metric. If g_0 is not a local maximizer of λ, then there exists a nontrivial ancient Ricci flow emerging from g_0, i.e. a Ricci flow $g(t) \neq g_0$ with $\lim_{t \to -\infty} g(t) = g_0$.

The first part of this theorem has been proved before by Natasa Sesum using different techniques [107]. To obtain Theorem 0.3, we prove the following Lojasiewicz-Simon gradient inequality and transversality estimate.

Theorem 0.4
For metrics g in a neighborhood of the Ricci-flat metric g_0 we have the estimates

$$\|\mathrm{Rc} + \nabla^2 f\|_{L^2}^2 \geq c\lambda(g) \qquad and \qquad \|\mathrm{Rc} + \nabla^2 f\|_{L^2} \geq c\|\mathrm{Rc}\|_{L^2}, \qquad (0.13)$$

where $f = f(g)$ is the minimizer in (0.11).

These estimates are of independent interest and reflect the gradient-like nature of the Ricci flow and the role of the gauge group. The proofs of the above results are quite

involved and technical. To get the flavor across, some key ingredients - among many others - are the Ebin-Palais slice theorem and the cubic estimate

$$\left| \tfrac{d^3}{d\varepsilon^3}|_0 \lambda(g + \varepsilon h) \right| \le C \|h\|_{C^{2,\alpha}} \|h\|_{H^1}^2 \tag{0.14}$$

for the error term in the Taylor expansion of λ.

Having obtained a quite comprehensive picture in the compact case, we start to observe new and interesting phenomena in the case of *noncompact* manifolds. Note that $\mathcal{F} = \infty$ in many noncompact examples, e.g. on the Bryant soliton. To overcome this difficulty, we introduce a renormalized Perelman-functional.

Definition 0.5
Let (M, g_s, f_s) be a steady soliton. For metrics g and functions f close to (g_s, f_s) we define the renormalized energy,

$$\mathcal{F}^{(g_s, f_s)}(g, f) := \int \left(R + |\nabla f|^2 - c_s \right) e^{-f} dV, \tag{0.15}$$

where $c_s := R(g_s) + |\nabla f_s|^2$ is the central charge of the steady soliton.

Another closely related problem is that λ vanishes identically in the setting of asymptotically flat manifolds with nonnegative scalar curvature. To also overcome this difficulty, we introduce a geometric invariant λ_{AF}.

Definition 0.6
Let (M^n, g) be an asymptotically flat manifold of order $\tau > \frac{n-2}{2}$ with nonnegative scalar curvature. We define

$$\lambda_{AF}(g) := \inf \int_M \left(4|\nabla w|^2 + Rw^2 \right) dV, \tag{0.16}$$

where the infimum is taken over all $w \in C^\infty(M)$ such that $w = 1 + O(r^{-\tau})$ at infinity.

The formula for the second variation of $\mathcal{F}^{(g_s, f_s)}$ gives a stability inequality for steady Ricci solitons, which in the case of Ricci-flat spaces takes the form

$$\int_M 2 \, \mathrm{Rm}(h, h) \le \int_M |\nabla h|^2 \qquad (\text{assuming } \mathrm{div} h = 0). \tag{0.17}$$

In a joint work with Stuart Hall and Michael Siepmann, we thoroughly investigated this stability inequality in the setting of Ricci-flat cones.

Theorem 0.7 (joint with Stuart Hall and Michael Siepmann)
The Ricci-flat cone over $\mathbb{C}P^2$ is stable. In particular, the first stable nonflat Ricci-flat cone occurs in the smallest possible dimension.

This is in sharp contrast with the case of minimal hypersurfaces, where the first non-trivial stable example is the Simons cone of dimension 7. Complementing the stability theorem, we prove that many other Ricci-flat cones in low dimensions are unstable.

Theorem 0.8 (joint with Stuart Hall and Michael Siepmann)
Ricci-flat cones over products of Einstein manifolds and over Kähler-Einstein manifolds with $h^{1,1} > 1$ are unstable in dimension less than 10. Moreover, the following Ricci-flat cones over 4-manifolds are unstable: $C(S^2 \times S^2)$, $C(\mathbb{C}P^2 \sharp p \overline{\mathbb{C}P^2})$ $(3 \leq p \leq 8)$ and $C(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2})$.

As result of independent interest, our computations show that $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with the Page metric is an unstable shrinker. Furthermore, we have numerical evidence for the instability of $\mathbb{C}P^2 \sharp 2 \overline{\mathbb{C}P^2}$ with the Chen-LeBrun-Weber metric and for the Ricci-flat cone over it. We also partly confirm a conjecture of Ilmanen about the nonuniqueness of Ricci flow with conical initial data, but on the other hand also give examples of unstable Ricci-flat cones that have no evolution that becomes smooth instantaneously.

The geometric invariant λ_{AF} gives a quantitative lower bound for the ADM-mass and motivates a proof of the rigidity statement in the positive mass theorem using the Ricci flow. Pursuing these ideas further, we discovered a mass-decreasing flow in dimension three. Some versions of this flow have been discovered independently by Hugh Bray and Lars Andersson. Our flow is defined for asymptotically flat three-manifolds with nonnegative scalar curvature and consists of an iteration of the Ricci flow with surgery and conformal rescalings. It is a discrete iteration with some choice of a time-step parameter $\varepsilon > 0$. The basic idea, going back to the fundamental work of Schoen and Yau, is that conformal rescalings to a scalar flat metric decrease the mass, and that the scalar curvature can be increased again by the Ricci flow, thus allowing even more mass-decrease by another conformal rescaling, etc. Thanks to the recent precise geometric-analytic understanding of the Ricci flow in dimension three, mainly due to Perelman, this process can be iterated forever. In fact we use a nice variant of the Ricci-flow with surgery for noncompact manifolds due to Bessières-Besson-Maillot [13].

Theorem 0.9
The mass-decreasing flow $(M(t), g(t))$ exists for all times, and preserves asymptotic flatness and nonnegative scalar curvature. The mass is constant in the time intervals $t \in ((k-1)\varepsilon, k\varepsilon)$ and jumps down by

$$\delta m_k = -\int_M (8|\nabla w_k|^2 + R w_k^2) dV \tag{0.18}$$

at the conformal rescaling times $t_k = k\varepsilon$, where w_k is the solution of

$$\left(-8 \triangle_{g(t_k)} + R_{g(t_k)}\right) w_k = 0, \qquad w_k \to 1 \quad at \quad \infty. \tag{0.19}$$

The monotonicity of the mass is strict as long as the metric is nonflat.

We also remark that the (formal) limiting equation for $\varepsilon \to 0$ is

$$\partial_t g = -2\operatorname{Rc} + \triangle^{-1}|\operatorname{Rc}|^2 g \ . \tag{0.20}$$

Two month after we posted the initial version of our article on arXiv [68], Peng Lu, Jie Qing and Yu Zheng posted a very interesting note [87], where they prove short time existence for this nonlocal flow. However, since our long time existence result relies on the theory of Ricci flow with surgery, we prefer to work with the discrete ε-iteration.

Finally, regarding the topological and geometric aspects of the long time behavior of the mass-decreasing flow we have:

Theorem 0.10
Wormholes pinch off and nontrivial spherical space-forms bubble off in finite time, i.e.
$M(t) \cong \mathbb{R}^3$ *for t large enough. Moreover, under the a-priori assumption that there is a constant $c > 0$, such that $\lambda_{AF}(g(t_k)) \geq cm(g(t_k))^2$ for all positive integers k, the mass-decreasing flow squeezes out all the initial mass, i.e. $\lim_{t\to\infty} m(g(t)) = 0$.*

The a-priori assumption is partly motivated by a monotonicity formula for our renormalized functional λ_{AF}. However, it is a very interesting (and difficult) question, whether the flow squeezes out all the initial mass even without a-priori assumption.

A compactness theorem for complete Ricci shrinkers

In this chapter, we prove precompactness in an orbifold Cheeger-Gromov sense of complete gradient Ricci shrinkers with a lower bound on their entropy and a local integral Riemann bound. We do not need any pointwise curvature assumptions, volume or diameter bounds. In dimension four, under a technical assumption, we can even replace the local integral Riemann bound by an upper bound for the Euler characteristic. The proof relies on a Gauss-Bonnet with cutoff argument.

1.1 Introduction

Let us start with some background: The classical Cheeger-Gromov theorem says that every sequence of closed Riemannian manifolds with uniformly bounded curvatures, volume bounded below, and diameter bounded above has a $C^{1,\alpha}$-convergent subsequence [27, 55, 54]. The convergence is in the sense of Cheeger-Gromov, meaning $C^{1,\alpha}$-convergence of the Riemannian metrics after pulling back by suitable diffeomorphisms. Without diameter bounds, the global volume bound should be replaced by a local volume noncollapsing assumption [29], and the appropriate notion of convergence is convergence in the pointed Cheeger-Gromov sense. If one can also control all the derivatives of the curvatures, e.g. in the presence of an elliptic or parabolic equation, the convergence is smooth [64]. To remind the reader about the precise definition, a sequence of complete smooth Riemannian manifolds with basepoints (M_i^n, g_i, p_i) converges to $(M_\infty^n, g_\infty, p_\infty)$ in the *pointed smooth Cheeger-Gromov sense* if there exist an exhaustion of M_∞ by open sets U_i containing p_∞ and smooth embeddings $\phi_i : U_i \to M_i$ with $\phi_i(p_\infty) = p_i$ such that the pulled back metrics $\phi_i^* g_i$ converge to g_∞ in C_{loc}^∞.

Now, let us describe the problem under consideration: Hamilton's Ricci flow in higher dimensions without curvature assumptions leads to the formation of intriguingly complex singularities [63, 65]. The specific question we are concerned with is about the compactness properties of the corresponding space of singularity models. Namely, given a sequence of *gradient shrinkers*, i.e. a sequence of smooth, connected, complete Riemannian manifolds (M_i^n, g_i) satisfying

$$\mathrm{Rc}_{g_i} + \mathrm{Hess}_{g_i} f_i = \tfrac{1}{2} g_i \tag{1.1}$$

for some smooth function $f_i : M \to \mathbb{R}$ (called the potential), under what assumptions can we find a convergent subsequence? In the *compact case*, this problem was first studied by Cao-Sesum [22], see also Zhang [136], and Weber succeeded in removing their pointwise Ricci bounds [128]. We have profited from these previous works and the papers by Anderson, Bando, Kasue, Nakajima and Tian about the Einstein case [5, 94, 9, 117], as well as from the papers [6, 118, 119, 120, 124].

In this chapter, we generalize the shrinker orbifold compactness result to the case of *noncompact manifolds*. The obvious motivation for doing this is the fact that most interesting singularity models are noncompact, the cylinder being the most basic example. We manage to remove all volume and diameter assumptions, and we do not need any positivity assumptions for the curvatures nor pointwise curvature bounds (as the blow-down shrinker shows [48], even the Ricci curvature can have both signs). In fact, if the curvature is uniformly bounded below, it is easy to pass to a smooth limit (Theorem 1.7). The general case without positivity assumptions is much harder.

Having removed all other assumptions, we prove a precompactness theorem for complete Ricci shrinkers, assuming only a lower bound for the Perelman entropy and local $L^{n/2}$ bounds for the Riemann tensor (Theorem 1.1). The assumptions allow orbifold singularities to occur (these are isolated singularities modelled on $\mathbb{R}^n/\Gamma$ for some finite subgroup $\Gamma \subset \mathrm{O}(n)$), and the convergence is in the *pointed orbifold Cheeger-Gromov sense*. In particular, this means that the sequence converges in the pointed Gromov-Hausdorff sense (this is the natural notion of convergence for complete metric spaces), and that the convergence is in the smooth Cheeger-Gromov sense away from the isolated point singularities (see Section 1.3 for the precise definitions).

Our results are most striking in dimension four. In this case, the local L^2 Riemann bound is *not* an a priori assumption, but we prove it modulo a technical assumption on the soliton potential (Theorem 1.2). Our proof is based on a 4d-Chern-Gauss-Bonnet with cutoff argument (see Section 1.4). In particular, the key estimate for the cubic boundary term (Lemma 1.14) is based on a delicate use of partial integrations and soliton identities.

Before stating our main results, let us explain a few facts about gradient shrinkers, see Section 1.2 and the appendix for proofs and further references. Associated to every gradient shrinker (M^n, g, f), there is a family of Riemannian metrics $g(t)$, $t \in (-\infty, 1)$, evolving by Hamilton's Ricci flow $\frac{\partial}{\partial t} g(t) = -2 \operatorname{Rc}_{g(t)}$ with $g(0) = g$, which is self-similarly shrinking, i.e. $g(t) = (1-t)\phi_t^* g$ for the family of diffeomorphisms ϕ_t generated by $\frac{1}{1-t} \nabla f$, see [34, 137]. In this chapter however, we focus on the elliptic point of view. Gradient shrinkers always come with a natural basepoint, a point $p \in M$ where the potential f attains its minimum (such a minimum always exists and the distance between two minimum points is bounded by a constant depending only on the dimension). The potential grows like one-quarter distance squared, so $2\sqrt{f}$ can be thought of as distance from the basepoint. Moreover, the volume growth is at most

Euclidean, hence it is always possible to normalize f (by adding a constant) such that

$$\int_M (4\pi)^{-n/2} e^{-f} dV_g = 1. \tag{1.2}$$

Then the gradient shrinker has a well defined *entropy,*

$$\mu(g) = \mathcal{W}(g, f) = \int_M \left(|\nabla f|_g^2 + R_g + f - n \right) (4\pi)^{-n/2} e^{-f} dV_g > -\infty. \tag{1.3}$$

The entropy was introduced by Perelman in his famous paper [97] to solve the long standing problem of ruling out collapsing with bounded curvature (see [12, 25, 76, 89] for detailed expositions of Perelman's work). For general Ricci flows, the entropy is time-dependent, but on gradient shrinkers it is constant and finite (even without curvature assumptions). Assuming a lower bound for the entropy is natural, because it is nondecreasing along the Ricci flow in the compact case or under some mild technical assumptions. Under a local scalar curvature bound, a lower bound on the entropy gives a local volume noncollapsing bound.

The main results of this chapter are the following two theorems.

Theorem 1.1
Let (M_i^n, g_i, f_i) be a sequence of gradient shrinkers (with normalization and basepoint p_i as above) with entropy uniformly bounded below, $\mu(g_i) \geq \underline{\mu} > -\infty$, and uniform local energy bounds,

$$\int_{B_r(p_i)} |\mathrm{Rm}_{g_i}|_{g_i}^{n/2} dV_{g_i} \leq E(r) < \infty, \quad \forall i, r. \tag{1.4}$$

Then a subsequence of (M_i^n, g_i, f_i, p_i) converges to an orbifold gradient shrinker in the pointed orbifold Cheeger-Gromov sense.

Here is a cute way to rephrase this theorem: *The space of Ricci flow singularity models with bounded entropy and locally bounded energy is orbifold compact.*

In the case $n = 4$, we obtain a particularly strong compactness result under a technical assumption on the potential.

Theorem 1.2
Let (M_i^4, g_i, f_i) be a sequence of four-dimensional gradient shrinkers (with normalization and basepoint p_i as above) with entropy uniformly bounded below, $\mu(g_i) \geq \underline{\mu} > -\infty$, Euler characteristic bounded above, $\chi(M_i) \leq \overline{\chi} < \infty$, and the technical assumption that the potentials do not have critical points at large distances, more precisely

$$|\nabla f_i|(x) \geq c > 0 \qquad \text{if } d(x, p_i) \geq r_0, \tag{1.5}$$

for some constant $r_0 < \infty$. Then we have the weighted L^2 estimate

$$\int_{M_i} |\mathrm{Rm}_{g_i}|^2_{g_i} e^{-f_i} dV_{g_i} \leq C(\underline{\mu}, \overline{\chi}, c, r_0) < \infty. \tag{1.6}$$

In particular, the energy condition (1.4) is satisfied and by Theorem 1.1 a subsequence converges in the pointed orbifold Cheeger-Gromov sense.

As explained above, to appreciate our theorems it is most important to think about the assumptions that we do *not* make.

Remark. The technical assumption (1.5) is satisfied in particular if the scalar curvature satisfies

$$R_{g_i}(x) \leq \alpha d(x, p_i)^2 + C \tag{1.7}$$

for some $\alpha < \frac{1}{4}$. The scalar curvature grows at most like one-quarter distance squared and the average scalar curvature on $2\sqrt{f}$-balls is bounded by $n/2$ (see Section 1.2 and appendix), so the technical assumption is rather mild. However, it would be very desirable to remove (or prove) it. Of course, (1.5) would also follow from a diameter bound.

In this chapter, the potentials of the gradient shrinkers play a central role in many proofs. In particular, we can view (a perturbation of) f as a Morse function, use e^{-f} as weight or cutoff function and use balls defined by the distance $2\sqrt{f}$ instead of the Riemannian distance. This has the great advantage, that we have a formula for the second fundamental form in the Gauss-Bonnet with boundary argument.

There are very deep and interesting other methods that yield comparable results, in particular the techniques developed by Cheeger-Colding-Tian in their work on the structure of spaces with Ricci curvature bounded below (see [28] for a nice survey) and the nested blowup and contradiction arguments of Chen-Wang [32]. Finally, let us mention the very interesting recent paper by Song-Weinkove [113].

This chapter is organized as follows. In Section 1.2, we collect and prove some properties of gradient shrinkers. In Section 1.3, we prove Theorem 1.1. Finally, we prove Theorem 1.2 in Section 1.4 using the Chern-Gauss-Bonnet theorem for manifolds with boundary and carefully estimating the boundary terms. We would like to point out that the Sections 1.3 and 1.4 are completely independent of each other and can be read in any order.

1.2   Some properties of gradient shrinkers

Let us start by collecting some basic facts about gradient shrinkers (for a recent survey about Ricci solitons, see [20]). Tracing the soliton equation,

$$R_{ij} + \nabla_i \nabla_j f = \tfrac{1}{2} g_{ij}, \tag{1.8}$$

gives

$$R + \triangle f = \tfrac{n}{2}. \tag{1.9}$$

Using the contracted second Bianchi identity, inserting the soliton equation (1.8), and commuting the derivatives, we compute

$$\tfrac{1}{2}\nabla_i R = \nabla_i R - \nabla_j R_{ij} = -\nabla_i \nabla_j \nabla_j f + \nabla_j \nabla_i \nabla_j f = R_{ik}\nabla_k f. \tag{1.10}$$

As observed by Hamilton, from this formula and equation (1.8), it follows that

$$C_1(g) := R + |\nabla f|^2 - f \tag{1.11}$$

is constant (note that we always assume that our manifold is connected). By (1.8), the Hessian of f is uniquely determined by g. Thus, the potential has the form $f(x,y) = \tilde{f}(x) + \tfrac{1}{4}|y - y_0|^2$ after splitting $M \cong \tilde{M} \times \mathbb{R}^k$ isometrically. Note that the constant $C_1(g)$ and also the normalization (1.2) do not depend on the point $y_0 \in \mathbb{R}^k$. It follows that $\tilde{f}$ is completely determined by fixing the normalization (1.2), and that $C_1(g)$ is independent of f after fixing this normalization. Gradient shrinkers always have nonnegative scalar curvature,

$$R \geq 0. \tag{1.12}$$

This follows from the elliptic equation

$$R + \langle \nabla f, \nabla R \rangle = \triangle R + 2|\mathrm{Rc}|^2 \tag{1.13}$$

by the maximum principle, see [137] for a proof in the noncompact case without curvature assumptions. Equation (1.13) is the shrinker version of the evolution equation $\frac{\partial}{\partial t} R = \triangle R + 2|\mathrm{Rc}|^2$ under Ricci flow.

The following two lemmas show, that the shrinker potential f grows like one-quarter distance squared and that gradient shrinkers have at most Euclidean volume growth.

Lemma 1.3 (Growth of the potential)
Let (M^n, g, f) be a gradient shrinker with $C_1 = C_1(g)$ as in (1.11). Then there exists a point $p \in M$ where f attains its infimum and f satisfies the quadratic growth estimate

$$\tfrac{1}{4}\big(d(x,p) - 5n\big)_+^2 \leq f(x) + C_1 \leq \tfrac{1}{4}\big(d(x,p) + \sqrt{2n}\big)^2 \tag{1.14}$$

for all $x \in M$, where $a_+ := \max\{0, a\}$. If p_1 and p_2 are two minimum points, then their distance is bounded by

$$d(p_1, p_2) \leq 5n + \sqrt{2n}. \tag{1.15}$$

Lemma 1.4 (Volume growth)
There exists a constant $C_2 = C_2(n) < \infty$ such that every gradient shrinker (M^n, g, f) with $p \in M$ as in Lemma 1.3 satisfies the volume growth estimate

$$\mathrm{Vol}\, B_r(p) \leq C_2 r^n. \tag{1.16}$$

The proofs are small but crucial improvements of the proofs by Cao-Zhou and Munteanu [23, 91]. In fact, their results are not strong enough for our purpose for which it is necessary, in particular, to remove the dependence on the geometry on a unit ball in Theorem 1.1 of [23] and to show that the constant in the volume growth estimate can be chosen uniformly for a sequence of shrinkers. In order to keep this section compact, we moved the proofs of both lemmas to the appendix.

From now on, we fix a point $p \in M$ where f attains its minimum.

By Lemma 1.3 and Lemma 1.4, any function φ that satisfies the growth estimate

$$|\varphi(x)| \leq Ce^{\alpha d(x,p)^2} \qquad \text{for some } \alpha < \tfrac{1}{4} \tag{1.17}$$

is integrable with respect to the measure $e^{-f}dV$. In particular, the integral $\int_M e^{-f}dV$ is finite and f can be normalized (by adding a constant if necessary) to satisfy the normalization constraint (1.2).

From now on, we will fix the normalization (1.2).

Let us now explain the logarithmic Sobolev inequality, compare with Carrillo-Ni [26]. Any polynomial in $R, f, |\nabla f|, \triangle f$ is integrable with respect to the measure $e^{-f}dV$. Indeed, using one after another (1.12), $0 \leq |\nabla f|^2$, (1.11), and Lemma 1.3, we compute

$$0 \leq R(x) \leq R(x) + |\nabla f|^2(x) = f(x) + C_1 \leq \tfrac{1}{4}\big(d(x,p) + \sqrt{2n}\big)^2, \tag{1.18}$$

and using furthermore (1.9) this implies

$$-\tfrac{n}{2} \leq -\triangle f(x) \leq -\tfrac{n}{2} + \tfrac{1}{4}\big(d(x,p) + \sqrt{2n}\big)^2. \tag{1.19}$$

So, any polynomial in $R, f, |\nabla f|, \triangle f$ has at most polynomial growth and thus in particular satisfies the growth estimate (1.17). It follows that the entropy

$$\mu(g) := \mathcal{W}(g, f) = \int_M \big(|\nabla f|^2 + R + f - n\big)(4\pi)^{-n/2}e^{-f}dV \tag{1.20}$$

is well defined. To obtain another expression for the entropy, we will use the partial integration formula

$$\int_M \triangle f e^{-f}dV = \int_M |\nabla f|^2 e^{-f}dV, \tag{1.21}$$

which is justified as follows: Let $\eta_r(x) := \eta(d(x,p)/r)$, where $0 \leq \eta \leq 1$ is a cutoff function such that $\eta(s) = 1$ for $s \leq 1/2$ and $\eta(s) = 0$ for $s \geq 1$. Then

$$\int_M \eta_r \triangle f e^{-f}dV = \int_M \eta_r |\nabla f|^2 e^{-f}dV - \int_M \langle \nabla \eta_r, \nabla f\rangle e^{-f}dV.$$

Now, using the estimates for $f, |\nabla f|$ and the volume growth, we see that

$$\int_M |\nabla \eta_r||\nabla f|e^{-f}dV \leq C\int_{B_r(p)\backslash B_{r/2}(p)} \tfrac{1}{r}d(x,p)e^{-\frac{(d(x,p)-5n)^2}{4}}dV \leq Cr^n e^{-\frac{(r/2-5n)^2}{4}}$$

converges to zero for $r \to \infty$, and (1.21) follows from the dominated convergence theorem. Moreover, note that (1.9) and (1.11) imply the formula

$$2\triangle f - |\nabla f|^2 + R + f - n = -C_1. \tag{1.22}$$

Putting everything together, we conclude that

$$\mu(g) = \int_M \left(2\triangle f - |\nabla f|^2 + R + f - n\right)(4\pi)^{-n/2}e^{-f}dV = -C_1(g), \tag{1.23}$$

where we also used the normalization (1.2) in the last step. In other words, the *auxiliary constant* $C_1(g)$ of the gradient shrinker is minus the *Perelman entropy*. Carrillo-Ni made the wonderful observation that Perelman's logarithmic Sobolev inequality holds even for noncompact shrinkers without curvature assumptions [26, Thm. 1.1], i.e.

$$\inf \mathcal{W}(g, \tilde{f}) \geq \mu(g), \tag{1.24}$$

where the infimum is taken over all $\tilde{f} : M \to \mathbb{R} \cup \{+\infty\}$ such that $\tilde{u} = e^{-\tilde{f}/2}$ is smooth with compact support and $\int_M \tilde{u}^2 dV = (4\pi)^{n/2}$. Essentially, this follows from $\mathrm{Rc}_f = \mathrm{Rc} + \mathrm{Hess}\, f \geq 1/2$ and the Bakry-Emery theorem [125, Thm. 21.2].

Remark. In fact, equality holds in (1.24), which can be seen as follows. First observe that, as a function of g and $\tilde{u}$,

$$\mathcal{W}(g, \tilde{u}) = (4\pi)^{-n/2} \int_M \left(4|\nabla \tilde{u}|^2 + (R - n)\tilde{u}^2 - \tilde{u}^2 \log \tilde{u}^2\right) dV, \tag{1.25}$$

and that one can take the infimum over all properly normalized Lipschitz functions $\tilde{u}$ with compact support. Now, the equality follows by approximating $u = e^{-f/2}$ by $\tilde{u}_r := C_r \eta_r u$, with η_r as above and with constants

$$C_r = \sqrt{\frac{(4\pi)^{n/2}}{\int_M \eta_r^2 u^2 dV}} \searrow 1 \tag{1.26}$$

to preserve the normalization. Indeed, arguing as before we see that

$$\int_M (R - n)\eta_r^2 u^2 \to \int_M (R - n)u^2, \qquad \int_M C_r^2 \eta_r^2 u^2 \log u^2 \to \int_M u^2 \log u^2,$$
$$\int_M |\nabla(\eta_r u)|^2 \to \int_M |\nabla u|^2, \qquad \int_M C_r^2 \eta_r^2 \log(C_r^2 \eta_r^2)u^2 \to 0, \tag{1.27}$$

which together yields $\mathcal{W}(g, \tilde{u}_r) \to \mathcal{W}(g, u)$.

From Perelman's logarithmic Sobolev inequality (1.24) and the local bounds for the scalar curvature (1.18), we get the following noncollapsing lemma.

Lemma 1.5 (Noncollapsing)
There exists a function $\kappa(r) = \kappa(r, n, \underline{\mu}) > 0$ such that for every gradient shrinker (M^n, g, f) (with basepoint p and normalization as before) with entropy bounded below, $\mu(g) \geq \underline{\mu}$, we have the lower volume bound $\operatorname{Vol} B_\delta(x) \geq \kappa(r)\delta^n$ for every ball $B_\delta(x) \subset B_r(p)$, $0 < \delta \leq 1$.

The proof is strongly related to Perelman's proof for finite-time Ricci flow singularities (see Kleiner-Lott [76, Sec. 13] for a nice and detailed exposition), and can be found in the appendix. Given a lower bound $\mu(g) \geq \underline{\mu}$, we also get an upper bound $\mu(g) \leq \overline{\mu} = \overline{\mu}(\underline{\mu}, n)$ using $\tilde{u} = c^{-1/2}\eta(d(x, p))$ as test function. Of course, the conjecture is $\mu(g) \leq 0$ even for noncompact shrinkers without curvature assumptions.[1]

Equipped with the above lemmas, we can now easily prove the noncollapsed pointed Gromov-Hausdorff convergence in the general case, and the pointed smooth Cheeger-Gromov convergence in the case where the curvature is uniformly bounded below.

Theorem 1.6 (Noncollapsed Gromov-Hausdorff convergence)
Let (M_i^n, g_i, f_i) be a sequence of gradient shrinkers with entropy uniformly bounded below, $\mu(g_i) \geq \underline{\mu} > -\infty$, and with basepoint p_i and normalization as before. Then the sequence is volume noncollapsed at finite distances from the basepoint and a subsequence (M_i, d_i, p_i) converges to a complete metric space in the pointed Gromov-Hausdorff sense.

Proof. The first part is Lemma 1.5. For the second part, to find a subsequence that converges in the pointed Gromov-Hausdorff sense, it suffices to find uniform bounds $N(\delta, r)$ for the number of disjoint δ-balls that fit within an r-ball centered at the basepoint [55, Prop. 5.2]. Assume $\delta \leq 1$ without loss of generality. By Lemma 1.4 the ball $B_r(p)$ has volume at most $C_2 r^n$, while by Lemma 1.5 each ball $B_\delta(x) \subset B_r(p)$ has volume at least $\kappa\delta^n$. Thus, there can be at most $N(\delta, r) = C_2 r^n / \kappa\delta^n$ disjoint δ-balls in $B_r(p)$. $\qquad\square$

Remark. Alternatively, the Gromov-Hausdorff convergence also follows from the volume comparison theorem of Wei-Wylie for the Bakry-Emery Ricci tensor [129], using the estimates for the soliton potential from this section. This holds even without entropy and energy bounds, but in that case the limit can be collapsed and very singular.

Theorem 1.7 (Smooth convergence in the curvature bounded below case)
Let (M_i^n, g_i, f_i) be a sequence of gradient shrinkers (with basepoint p_i and normalization as before) with entropy uniformly bounded below, $\mu(g_i) \geq \underline{\mu} > -\infty$, and curvature uniformly bounded below, $\operatorname{Rm}_{g_i} \geq \underline{K} > -\infty$. Then a subsequence (M_i, g_i, f_i, p_i) converges to a gradient shrinker $(M_\infty, g_\infty, f_\infty, p_\infty)$ in the pointed smooth Cheeger-Gromov sense (i.e. there exist an exhaustion of M_∞ by open sets U_i containing p_∞ and smooth embeddings $\phi_i : U_i \to M_i$ with $\phi_i(p_\infty) = p_i$ such that $(\phi_i^* g_i, \phi_i^* f_i)$ converges to (g_∞, f_∞) in C_{loc}^∞).

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[1]Indeed this has been proved recently by Yokota [133].

Proof. Recall the following Cheeger-Gromov compactness theorem from the very beginning of the introduction: For every sequence (M_i^n, g_i, p_i) of complete Riemannian manifolds with uniform local bounds for the curvatures and all its derivatives,

$$\sup_{B_r(p_i)} |\nabla^k \mathrm{Rm}_{g_i}| \leq C_k(r), \tag{1.28}$$

and with a uniform local volume noncollapsing bound around the basepoint,

$$\mathrm{Vol}_{g_i} B_1(p_i) \geq \kappa, \tag{1.29}$$

we can find a subsequence that converges to a limit $(M_\infty, g_\infty, p_\infty)$ in the pointed smooth Cheeger-Gromov sense.

Moreover, if we have also uniform local bounds for the shrinker potential and all its derivatives,

$$\sup_{B_r(p_i)} |\nabla^k f_i| \leq C_k(r), \tag{1.30}$$

then by passing to another subsequence if necessary the functions $\phi_i^* f_i$ will also converge to some function f_∞ in C_{loc}^∞ (the embeddings $\phi_i : U_i \to M_i$ come from the pointed Cheeger-Gromov convergence). From the very definition of smooth convergence it is clear that the shrinker equation will pass to the limit, i.e. that $(M_\infty, g_\infty, f_\infty)$ will be a gradient shrinker. Thus, it remains to verify (1.28), (1.29), and (1.30) for our sequence of shrinkers.

By Lemma 1.5, we have uniform local volume noncollapsing, in particular condition (1.29) is satisfied. From (1.18) we have uniform local bounds for the scalar curvature, and together with the assumption $\mathrm{Rm}_{g_i} \geq \underline{K}$ this gives uniform local Riemann bounds,

$$\sup_{B_r(p_i)} |\mathrm{Rm}_{g_i}| \leq C_0(r). \tag{1.31}$$

From (1.18) and the bounds $\underline{\mu} \leq -C_1(g_i) \leq \overline{\mu}$, we get local C^1 bounds for f_i,

$$\sup_{B_r(p_i)} |f_i| \leq C_0(r), \qquad \sup_{B_r(p_i)} |\nabla f_i| \leq C_1(r). \tag{1.32}$$

Finally, by some very well known arguments, we can bootstrap the elliptic system

$$\begin{aligned} \triangle \mathrm{Rm} &= \nabla f * \nabla \mathrm{Rm} + \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}, \\ \triangle f &= \tfrac{n}{2} - R, \end{aligned} \tag{1.33}$$

starting from (1.31) and (1.32) to arrive at (1.28) and (1.30). Here, the second equation is just the traced soliton equation (1.9), while the first equation is obtained from the soliton equation (1.8) and the Bianchi identity as follows:

$$\begin{aligned} \nabla_p \nabla_p R_{ijk\ell} &= -\nabla_p \nabla_k R_{ij\ell p} - \nabla_p \nabla_\ell R_{ijpk} \\ &= -\nabla_k \nabla_p R_{ij\ell p} - \nabla_\ell \nabla_p R_{ijpk} + (\mathrm{Rm} * \mathrm{Rm})_{ijk\ell} \\ &= \nabla_k (\nabla_i R_{j\ell} - \nabla_j R_{i\ell}) + \nabla_\ell (\nabla_j R_{ik} - \nabla_i R_{jk}) + (\mathrm{Rm} * \mathrm{Rm})_{ijk\ell} \\ &= \nabla_k (R_{ji\ell p} \nabla_p f) + \nabla_\ell (R_{ijkp} \nabla_p f) + (\mathrm{Rm} * \mathrm{Rm})_{ijk\ell} \\ &= (\nabla f * \nabla \mathrm{Rm} + \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm})_{ijk\ell}. \end{aligned} \tag{1.34}$$

Here, we used the Bianchi identity and the commutator rule in the first three lines and in the fourth and fifth line we used the soliton equation. This finishes the proof of the theorem. $\square$

Remark. The more interesting case without positivity assumptions is treated in Section 1.3. A related simple and well known example for singularity formation is the following. Consider the Eguchi-Hanson metric g_{EH} [47], a Ricci-flat metric on TS^2 which is asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$ (remember that the unit tangent bundle of the 2-sphere is homeomorphic to $S^3/\mathbb{Z}_2$). Then $g_i := \frac{1}{i}g_{\text{EH}}$ is a sequence of Ricci-flat metrics, that converges to $\mathbb{R}^4/\mathbb{Z}_2$ in the orbifold Cheeger-Gromov sense. In particular, an orbifold singularity develops as the central 2-sphere (i.e. the zero section) shrinks to a point. For the positive Kähler-Einstein case, see Tian [117], in particular Theorem 7.1.

Remark. For a sequence of gradient shrinkers with entropy uniformly bounded below, by Lemma 1.3 and Lemma 1.4, $(4\pi)^{-n/2}e^{-f_i}dV_{g_i}$ is a sequence of uniformly tight probability measures. Thus, a subsequence of $(M_i, d_i, e^{-f_i}dV_{g_i}, p_i)$ converges to a pointed measured complete metric space $(M_\infty, d_\infty, \nu_\infty, p_\infty)$ in the pointed *measured* Gromov-Hausdorff sense. By (1.18), Lemma 1.3 and a Gromov-Hausdorff version of the Arzela-Ascoli theorem, there exists a continuous limit function $f_\infty : M_\infty \to \mathbb{R}$. It is an interesting question, if ν_∞ equals e^{-f_∞} times the Hausdorff measure.

Remark. It follows from the recent work of Lott-Villani and Sturm that the condition $\text{Rc}_f \geq 1/2$ is preserved in a weak sense [125].

1.3   Proof of orbifold Cheeger-Gromov convergence

In this section, we prove Theorem 1.1. For convenience of the reader, we will also explain some steps that are based on well known compactness techniques.

The structure of the proof is the following: First, we show that we have a uniform estimate for the local Sobolev constant (Lemma 1.9). Using this, we prove the ε-regulartiy Lemma 1.10, which says that we get uniform bounds for the curvatures on balls with small energy. We can then pass to a smooth limit away from locally finitely many singular points using in particular the energy assumption (1.4) and the ε-regulartiy lemma. This limit can be completed as a metric space by adding locally finitely many points. Finally, we prove that the singular points are of smooth orbifold type.

We start by giving a precise definition of the convergence.

Definition 1.8 (Orbifold Cheeger-Gromov convergence)
A sequence of gradient shrinkers (M_i^n, g_i, f_i, p_i) converges to an orbifold gradient shrinker $(M_\infty^n, g_\infty, f_\infty, p_\infty)$ in the pointed orbifold Cheeger-Gromov sense, if the following properties hold.

1. *There exist a locally finite set $S \subset M_\infty$, an exhaustion of $M_\infty \setminus S$ by open sets U_i and smooth embeddings $\phi_i : U_i \to M_i$, such that $(\phi_i^* g_i, \phi_i^* f_i)$ converges to (g_∞, f_∞)*

in C^∞_{loc} on $M_\infty \setminus S$.

2. *The maps ϕ_i can be extended to pointed Gromov-Hausdorff approximations yielding a convergence $(M_i, d_i, p_i) \to (M_\infty, d_\infty, p_\infty)$ in the pointed Gromov-Hausdorff sense.*

Here, an orbifold gradient shrinker is a complete metric space that is a smooth gradient shrinker away from locally finitely many singular points. At a singular point q, M_∞ is modeled on $\mathbb{R}^n / \Gamma$ for some finite subgroup $\Gamma \subset O(n)$ and there is an associated covering $\mathbb{R}^n \supset B_\varrho(0) \setminus \{0\} \overset{\pi}{\to} U \setminus \{q\}$ of some neighborhood $U \subset M_\infty$ of q such that $(\pi^* g_\infty, \pi^* f_\infty)$ can be extended smoothly to a gradient shrinker over the origin.

For the arguments that follow, it will be very important to have a uniform local Sobolev constant that works simultaneously for all shrinkers in our sequence.

Lemma 1.9 (Estimate for the local Sobolev constant)
There exist $C_S(r) = C_S(r, n, \underline{\mu}) < \infty$ and $\delta_0(r) = \delta_0(r, n, \underline{\mu}) > 0$ such that for every gradient shrinker (M^n, g, f) (with basepoint p and normalization as before) with $\mu(g) \geq \underline{\mu}$, we have

$$\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq C_S(r) \|\nabla\varphi\|_{L^2} \tag{1.35}$$

for all balls $B_\delta(x) \subset B_r(p)$, $0 < \delta \leq \delta_0(r)$ and all functions $\varphi \in C^1_c(B_\delta(x))$.

Proof. The main point is that the estimate for the local Sobolev constant will follow from the noncollapsing and the volume comparison for the Bakry-Emery Ricci tensor. The detailed argument goes as follows:
The first reduction is that it suffices to control the optimal constant $C_1(B)$ in the L^1-Sobolov inequality,

$$\|\psi\|_{L^{\frac{n}{n-1}}(B)} \leq C_1(B) \|\nabla\psi\|_{L^1(B)}, \tag{1.36}$$

for all $\psi \in C^1_c(B)$, where in our case B will always be an open ball in a Riemannian manifold. Indeed, applying (1.36) for $\psi = \varphi^{(2n-2)/(n-2)}$ and using Hölder's inequality we can compute

$$\left(\|\varphi\|_{L^{\frac{2n}{n-2}}(B)} \right)^{\frac{2n-2}{n-2}} \leq \tfrac{2n-2}{n-2} C_1(B) \|\varphi^{n/(n-2)} \nabla\varphi\|_{L^1(B)}$$

$$\leq \tfrac{2n-2}{n-2} C_1(B) \left(\|\varphi\|_{L^{\frac{2n}{n-2}}(B)} \right)^{\frac{n}{n-2}} \|\nabla\varphi\|_{L^2(B)},$$

so the L^1-Sobolov inequality (1.36) implies the L^2-Sobolov inequality

$$\|\varphi\|_{L^{\frac{2n}{n-2}}(B)} \leq C_2(B) \|\nabla\varphi\|_{L^2(B)}, \tag{1.37}$$

with $C_2(B) = \tfrac{2n-2}{n-2} C_1(B)$. Next, it is a classical fact, known under the name Federer-Fleming theorem, that the optimal constant $C_1(B)$ in (1.36) is equal to the optimal

constant $C_I(B)$ in the isoperimetric inequality,

$$|\Omega|^{\frac{n-1}{n}} \leq C_I(B)|\partial\Omega|, \tag{1.38}$$

for all regions $\Omega \Subset B$ with C^1-boundary. Third, by a theorem of Croke [39, Thm. 11], the isoperimetric constant can be estimated by

$$C_I(B) \leq C(n)\omega(B)^{-\frac{n+1}{n}}, \tag{1.39}$$

where $C(n) < \infty$ is an explicit constant whose value we do not need and $\omega(B)$ is the visibility angle defined as

$$\omega(B) = \inf_{y \in B} |U_y|/|S^{n-1}|, \tag{1.40}$$

where $U_y = \{v \in T_y B \ ; \ |v| = 1, \text{the geodesic } \gamma_v \text{ is minimizing up to } \partial B\}$.

Putting everything together, to finish the proof of our lemma it suffices to find a lower bound for the visibility angle (1.40) for $B = B_\delta(x) \subset B_r(p)$ inside a shrinker for $\delta \leq \delta_0(r)$, where $\delta_0(r)$ will be chosen sufficiently small later. To find such a lower bound, we will use the volume comparison theorem for the Bakry-Emery Ricci tensor due to Wei-Wylie [129] which we now explain:

Fix $y \in M$, use exponential polar coordinates around y and write $dV = \mathcal{A}(r,\theta)dr \wedge d\theta$ for the volume element, where $d\theta$ is the standard volume element on the unit sphere S^{n-1}. Let $\mathcal{A}_f(r,\theta) = \mathcal{A}(r,\theta)e^{-f}$. Note that $\mathrm{Rc}_f = \mathrm{Rc} + \nabla^2 f \geq 0$ by the soliton equation. The angular version of the volume comparison theorem for the Bakry-Emery Ricci tensor [129, Thm 1.2a] says that if in addition $|\nabla f| \leq a$ on $B_R(y)$ then

$$\frac{\int_0^R \mathcal{A}_f(s,\theta)ds}{\int_0^r \mathcal{A}_f(s,\theta)ds} \leq e^{aR}\left(\frac{R}{r}\right)^n. \tag{1.41}$$

for $0 < r \leq R$. If we have moreover $\max_{B_R(y)} f \leq \min_{B_R(y)} f + b$ then this implies

$$\int_0^R \mathcal{A}(s,\theta)ds \leq e^{aR+b}\left(\frac{R}{r}\right)^n \int_0^r \mathcal{A}(s,\theta)ds, \tag{1.42}$$

and finally by sending r to zero we obtain the form of the volume comparison estimate that we will actually use, namely

$$\int_0^R \mathcal{A}(s,\theta)ds \leq \tfrac{1}{n}e^{aR+b}R^n. \tag{1.43}$$

In our application everything will stay inside a ball $B_{r+1}(p)$ around the basepoint of the soliton, so by (1.18) we can take $a := \frac{1}{2}(r + 1 + \sqrt{2n})$ and $b := a^2$.

Now, coming back to actually estimating the visibility angle of $B_\delta(x) \subset B_r(p)$, we let $y \in B_\delta(x)$ and apply the above ideas. Since the volume is computed using exponential polar coordinates around y, we have the estimate

$$|B_1(x)| - |B_\delta(x)| \leq \int_{U_y} \int_0^{1+\delta} \mathcal{A}(s,\theta)dsd\theta, \tag{1.44}$$

where U_y denotes the set of unit tangent vectors in whose direction the geodesics are minimizing up to the boundary of $B_\delta(x)$. Using (1.43), we can estimate this by

$$|B_1(x)| - |B_\delta(x)| \leq \tfrac{1}{n} e^{2a+b} |U_y| (1+\delta)^n, \tag{1.45}$$

and minimizing over $y \in B_\delta(x)$ we obtain the inequality

$$|B_1(x)| - |B_\delta(x)| \leq C\omega(B_\delta(x))(1+\delta)^n \tag{1.46}$$

with $C = C(r,n) = \tfrac{1}{n} e^{2a+b} |S^{n-1}|$. Moreover, using (1.43) again, we obtain the upper bound

$$|B_\delta(x)| \leq \int_{S^{n-1}} \int_0^\delta \mathcal{A}(s,\theta) \, ds \, d\theta \leq C\delta^n. \tag{1.47}$$

Finally, we have the lower volume bound

$$|B_1(x)| \geq \kappa, \tag{1.48}$$

for $\kappa = \kappa(r+1, n, \underline{\mu})$ from Lemma 1.5. If we now choose $\delta_0 = \delta_0(r, n, \underline{\mu}) = (\kappa/2C)^{1/n}$, then putting together (1.46), (1.47) and (1.48) gives the lower bound

$$\omega(B_\delta(x)) \geq \frac{\kappa}{2^{n+1} C} \tag{1.49}$$

for the visibility angle for $\delta \leq \delta_0$, and this finishes the proof of the lemma. $\square$

Using the uniform estimate for the local Sobolev constant, we obtain the following ε-regularity lemma.

Lemma 1.10 (ε-regularity)
There exist $\varepsilon_1(r) = \varepsilon_1(r, n, \underline{\mu}) > 0$ and $K_\ell(r) = K_\ell(r, n, \underline{\mu}) < \infty$ such that for every gradient shrinker (M^n, g, f) (with basepoint p and normalization as before) with $\mu(g) \geq \underline{\mu}$ and for every ball $B_\delta(x) \subset B_r(p), 0 < \delta \leq \delta_0(r)$, we have the implication

$$\|\mathrm{Rm}\|_{L^{n/2}(B_\delta(x))} \leq \varepsilon_1(r) \Rightarrow \sup_{B_{\delta/4}(x)} |\nabla^\ell \mathrm{Rm}| \leq \frac{K_\ell(r)}{\delta^{2+\ell}} \|\mathrm{Rm}\|_{L^{n/2}(B_\delta(x))}. \tag{1.50}$$

Proof. The gradient shrinker version of the evolution equation of the Riemann tensor under Ricci flow, $\frac{\partial}{\partial t} \mathrm{Rm} = \triangle \mathrm{Rm} + Q(\mathrm{Rm})$, is the elliptic equation

$$\triangle \mathrm{Rm} = \nabla f * \nabla \mathrm{Rm} + \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}. \tag{1.51}$$

Here, we used (1.8) to eliminate $\nabla^2 f$ in $L_{\nabla f} \mathrm{Rm} = \nabla f * \nabla \mathrm{Rm} + \nabla^2 f * \mathrm{Rm}$. Alternatively, we have given another derivation of the elliptic equation (1.51) in (1.34). Now, we set $u := |\mathrm{Rm}|$ and compute

$$\begin{aligned}
-u\triangle u &= -\tfrac{1}{2} \triangle u^2 + |\nabla u|^2 \\
&= -\tfrac{1}{2} \triangle |\mathrm{Rm}|^2 + |\nabla |\mathrm{Rm}||^2 \\
&= -\langle \mathrm{Rm}, \triangle \mathrm{Rm} \rangle - |\nabla \mathrm{Rm}|^2 + |\nabla |\mathrm{Rm}||^2.
\end{aligned} \tag{1.52}$$

By equation (1.51) and Young's inequality, we can estimate

$$
\begin{aligned}
-\langle \mathrm{Rm}, \triangle \mathrm{Rm} \rangle &\leq C_3 \left(|\mathrm{Rm}||\nabla f||\nabla \mathrm{Rm}| + |\mathrm{Rm}|^2 + |\mathrm{Rm}|^3 \right) \\
&\leq \tfrac{1}{10}|\nabla \mathrm{Rm}|^2 + \left(1 + \tfrac{10}{4}C_3|\nabla f|^2\right) C_3|\mathrm{Rm}|^2 + C_3|\mathrm{Rm}|^3,
\end{aligned}
\tag{1.53}
$$

for some constant $C_3 = C_3(n) < \infty$ depending only on the dimension. Finally we use Kato's inequality $|\nabla|\mathrm{Rm}|| \leq |\nabla \mathrm{Rm}|$, and the estimate (1.18) for $|\nabla f|$. Putting everything together, we obtain the elliptic inequality

$$
-u\triangle u \leq \tfrac{1}{10}|\nabla u|^2 + C_4 u^2 + C_3 u^3
\tag{1.54}
$$

on $B_r(p)$, where $C_4 = C_4(r, n) := (1 + \tfrac{5}{8}C_3(r + \sqrt{2n})^2)C_3$. Given an elliptic inequality like (1.54) it is well known to PDE-experts that if the $L^{n/2}$-norm of u is sufficiently small on a ball, then one gets L^∞-bounds on a smaller ball, more precisely

$$
\|u\|_{L^{n/2}(B_\delta(x))} \leq \varepsilon \Rightarrow \sup_{B_{\delta/2}(x)} |u| \leq \frac{K}{\delta^2}\|u\|_{L^{n/2}(B_\delta(x))},
\tag{1.55}
$$

for some constants $\varepsilon > 0$ and $K < \infty$. For convenience of the reader, we sketch the necessary Moser-iteration argument here: To keep the notation reasonably concise let us assume $\delta = 1$ and $n = 4$, the general case works similarly. Choose a cutoff-function $0 \leq \eta \leq 1$ that is 1 on $B_{3/4}(x)$, has support in $B_1(x)$, and satisfies $|\nabla \eta| \leq 8$. Multiplying (1.54) by η^2 and integrating by parts we obtain

$$
\tfrac{9}{10}\int_M \eta^2|\nabla u|^2 dV \leq 2\int_M \eta|\nabla \eta||u||\nabla u|dV + \int_M \left(C_4\eta^2 u^2 + C_3\eta^2 u^3\right)dV.
\tag{1.56}
$$

Dealing with the first term on the right hand side by Young's inequality and absorption, this gives the estimate

$$
\tfrac{1}{2}\int_M \eta^2|\nabla u|^2 dV \leq (C_4 + 160)\int_{B_1} u^2 dV + C_3\int_M \eta^2 u^3 dV.
\tag{1.57}
$$

For the last term, using Hölder's inequality, the assumption that the energy on B_1 is less than ε, and the Sobolev-inequality, we get

$$
\begin{aligned}
\int_M \eta^2 u^3 dV &\leq \left(\int_{B_1} u^2 dV\right)^{1/2}\left(\int_M (\eta u)^4 dV\right)^{1/2} \\
&\leq \varepsilon C_S^2 \int_M |\nabla(\eta u)|^2 dV \\
&\leq 2\varepsilon C_S^2 \int_M \eta^2|\nabla u|^2 dV + 50\varepsilon C_S^2 \int_{B_1} u^2 dV,
\end{aligned}
\tag{1.58}
$$

where $C_S < \infty$ is the local Sobolev constant on B_1. The main idea is that if we choose ε so small that $2\varepsilon C_S^2 C_3 \leq \tfrac{1}{4}$ then the $\int \eta^2|\nabla u|^2$ term can be absorbed, giving

$$
\tfrac{1}{4}\int_M \eta^2|\nabla u|^2 dV \leq (C_4 + 200)\int_{B_1} u^2 dV,
\tag{1.59}
$$

and using the Sobolev inequality we arrive at the L^4-estimate

$$\|u\|_{L^4(B_{3/4})} \leq 2C_S\sqrt{C_4 + 200}\|u\|_{L^2(B_1)}. \tag{1.60}$$

Now we choose a sequence of radii $r_k = \frac{1}{2} + \frac{1}{2^k}$ interpolating between $r_1 = 1$ and $r_\infty = \frac{1}{2}$. We multiply (1.54) by $\eta_k^2 u^{p_k}$, where $p_k = 2^k - 2$, and $0 \leq \eta_k \leq 1$ is a cutoff function that equals 1 on $B_{r_{k+1}}$, has support in B_{r_k}, and satisfies $|\nabla\eta_k| \leq 2/(r_k - r_{k+1})$. Carrying out similar steps as above we obtain the iterative estimates

$$\|u\|_{L^{2^{k+1}}(B_{r_{k+1}})} \leq C_k\|u\|_{L^{2^k}(B_{r_k})}. \tag{1.61}$$

The product of the constants C_k converges and sending $k \to \infty$ gives the desired estimate

$$\|u\|_{L^\infty(B_{1/2})} \leq K\|u\|_{L^2(B_1)}. \tag{1.62}$$

Note that the estimate (1.55) is of course only useful if we can get uniform constants $\varepsilon > 0$ and $K < \infty$ for our sequence of shrinkers. This crucial point is taken care of by Lemma 1.9, so we indeed get constants $\varepsilon_1(r) = \varepsilon_1(r, n, \underline{\mu}) > 0$ and $K_0(r) = K_0(r, n, \underline{\mu}) < \infty$, such that

$$\|\mathrm{Rm}\|_{L^{n/2}(B_\delta(x))} \leq \varepsilon_1(r) \Rightarrow \sup_{B_{\delta/2}(x)}|\mathrm{Rm}| \leq \frac{K_0(r)}{\delta^2}\|\mathrm{Rm}\|_{L^{n/2}(B_\delta(x))}, \tag{1.63}$$

for every ball $B_\delta(x) \subset B_r(p), 0 < \delta \leq \delta_0(r)$. Once one has L^∞ control, the hard work is done and it is standard to bootstrap the elliptic equation (1.51) to get C^∞ bounds on the ball $B_{\delta/4}(x)$. The only slightly subtle point is that higher derivatives of f appear when differentiating (1.51), but one can get rid of them again immediately using the soliton equation (1.8). This finishes the proof of the ε-regularity lemma. $\qquad\square$

Let us now explain how to finish the proof of Theorem 1.1. Let (M_i^n, g_i, f_i) be a sequence of gradient shrinkers satisfying the assumptions of Theorem 1.1. By Theorem 1.6, we can assume (after passing to a subsequence) that the sequence converges in the pointed Gromov-Hausdorff sense. By passing to another subsequence, we can also assume that the auxiliary constants converge.

Let $r < \infty$ large and $0 < \delta \leq \delta_1(r, E(r), n, \underline{\mu})$ small enough. The assumption (1.4) gives a uniform bound $E(r)$ for the energy contained in $B_r(p_i)$. So there can be at most $\frac{E_1(r)}{\varepsilon_1(r)}$ disjoint δ-balls in $B_r(p_i)$ that contain energy more than $\varepsilon_1(r)$, and away from those bad balls we get C^∞-estimates for the curvatures using the ε-regularity lemma. Recall that we also have volume-noncollapsing by Lemma 1.5, and that we get C^∞_{loc} bounds for f_i in regions with bounded curvature, using the elliptic equation

$$\triangle f = \tfrac{n}{2} - R. \tag{1.64}$$

Thus, putting everything together (and playing around with the parameters r and δ a bit), we can find on any (M_i, g_i) suitable balls $B_\delta(x_i^k(\delta))$, $1 \leq k \leq L_i(r) \leq L(r) = L(r, E(r), n, \underline{\mu})$, such that on

$$X_i := B_r(p_i) \setminus \bigcup_{k=1}^{L_i(r)} B_\delta(x_i^k(\delta)) \subset M_i, \tag{1.65}$$

we have the estimates

$$\sup_{X_i}|\nabla^\ell \operatorname{Rm}_{g_i}| \le C_\ell(\delta, r, n, \underline{\mu})$$

$$\sup_{X_i}|\nabla^\ell f_i| \le C_\ell(\delta, r, n, \underline{\mu}). \tag{1.66}$$

Together with the volume-noncollapsing, this is exactly what we need to pass to a smooth limit. Thus, sending $r \to \infty$ and $\delta \to 0$ suitably and passing to a diagonal subsequence, we obtain a (possibly incomplete) smooth limit gradient shrinker. Since we already know, that the manifolds M_i converge in the pointed Gromov-Hausdorff sense, this limit can be completed as a metric space by adding locally finitely many points and the convergence is in the sense of Definition 1.8. We have thus proved Theorem 1.1 up to the statement that the isolated singular points are of orbifold shrinker type.

This claimed orbifold structure at the singular points is a local statement, so we can essentially refer to [22, 136]. Nevertheless, let us sketch the main steps, following Tian [117, Sec. 3 and 4] closely (see also [5, 9] for similar proofs).

Step 1 (C^0-multifold): The idea is that blowing up around a singular point will show that the tangent cone is a union of finitely many flat cones over spherical space forms. Improving this a bit, one also gets C^0-control over g, and thus the structure of a so called C^0-multifold ("multi" and not yet "orbi", since we have to wait until the next step to see that "a union of finitely many flat cones over spherical space forms" can actually be replaced by "a single flat cone over a spherical space form"). The precise argument goes as follows: Near an added point $q \in S \subset M_\infty$, we have

$$|\nabla^\ell \operatorname{Rm}_{g_\infty}|_{g_\infty}(x) \le \frac{\varepsilon(\varrho(x))}{\varrho(x)^{2+\ell}}, \tag{1.67}$$

where $\varrho(x) = d_\infty(x, q)$. Here and in the following, $\varepsilon(\varrho)$ denotes a quantity that tends to zero for $\varrho \to 0$ and we always assume that ϱ is small enough. With $\varepsilon(\varrho) \to 0$ in (1.67), together with the Bakry-Emery volume comparison and the noncollapsing, it follows that the tangent cone at q is a finite union of flat cones over spherical space forms S^{n-1}/Γ_β. The tangent cone is unique and by a simple volume argument we get an explicit bound (depending on $r, n, \underline{\mu}$) for the order of the orbifold groups Γ_β and the number of components $\sharp\{\beta\}$. As in [117, Lemma 3.6, Eq. (4.1)] there exist a neighborhood $U \subset M_\infty$ of q and for every component U_β of $U \setminus \{q\}$ an associated covering $\pi_\beta : B_\varrho^* = B_\varrho(0) \setminus \{0\} \to U_\beta$ such that $g^\beta := \pi_\beta^* g_\infty$ can be extended to a C^0-metric over the origin with the estimates

$$\sup_{B_\varrho^*}|g^\beta - g_E|_{g_E} \le \varepsilon(\varrho),$$

$$|D^I g^\beta|_{g_E}(x) \le \frac{\varepsilon(\varrho(x))}{\varrho(x)^{|I|}}, \quad x \in B_\varrho^*, \ 1 \le |I| \le 100, \tag{1.68}$$

where g_E is the Euclidean metric, D the Euclidean derivative and I a multiindex.

Step 2 (C^0-orbifold): The idea is that if $U \setminus \{q\}$ had two or more components, than all geodesics in an approximating sequence would pass through a very small neck, but this yields a contradiction to the volume comparison theorem. For the precise argument, let $q \in S \subset M_\infty$ be an added point and choose points $x_i \in M_i$ converging to q. By the noncollapsing and the Bakry-Emery volume comparison with the bounds for f_i and $|\nabla f_i|$, there exists a constant $C < \infty$ such that for ϱ small enough any two points in $\partial B_\varrho(x_i)$ can be connected by a curve in $B_\varrho(x_i) \setminus B_{\varrho/C}(x_i)$ of length less than $C\varrho$. This follows by slightly modifying the proof of [6, Lemma 1.2] and [1, Lemma 1.4]. Since the convergence is smooth away from q, it follows that $U \setminus \{q\}$ is connected. In particular, the tangent cone at q consists of a single flat cone over a spherical space form S^{n-1}/Γ.

Step 3 (C^∞-orbifold): The final step is to get C^∞ bounds in suitable coordinates. Let g^1 be the metric on B_ϱ^* from Step 1 and 2. In the case $n = 4$, using Uhlenbeck's method for removing finite energy point singularities in the Yang-Mills field [124], we get an improved curvature decay

$$|\mathrm{Rm}_{g^1}|_{g^1}(x) \leq \frac{1}{|x|^\delta} \tag{1.69}$$

for $\delta > 0$ as small as we want on a small enough punctured ball B_ϱ^*. The proof goes through almost verbatim as in [117, Lemma 4.3]. The only difference is that instead of the Yang-Mills equation we use

$$\nabla_i R_{ijk\ell} = \nabla_k R_{\ell j} - \nabla_\ell R_{kj} = -\nabla_k \nabla_\ell \nabla_j f + \nabla_\ell \nabla_k \nabla_j f = R_{\ell kjp} \nabla_p f \tag{1.70}$$

and the estimates for $|\nabla f|$ from Section 1.2. The point is that the Yang-Mills equation $\nabla_i R_{ijk\ell} = 0$ is satisfied up to some lower order term that can be dealt with easily. The case $n \geq 5$ is more elementary, and Sibner's test function [108] gives L^∞ bounds for the curvature, in particular (1.69) is also satisfied in this case.

Due to the improved curvature decay, there exists a diffeomorphism $\psi : B_{\varrho/2}^* \to \psi(B_{\varrho/2}^*) \subset B_\varrho^*$ that extends to a homeomorphism over the origin such that $\psi^* g^1$ extends to a $C^{1,\alpha}$ metric over the origin (for any $\alpha < 1 - \delta$). By composing with another diffeomorphism (denoting the composition by φ), we can assume that the standard coordinates on $B_{\varrho/4}$ are harmonic coordinates for $\varphi^* g^1$. Finally, let $\pi := \pi_1 \circ \varphi$, $g := \pi^* g_\infty$ and $f := \pi^* f_\infty$. Then, for (g, f) we have the elliptic system

$$\begin{aligned} \triangle_g f &= |\nabla f|_g^2 - f + \tfrac{n}{2} - C, \\ \mathrm{Rc}_g &= \tfrac{1}{2}g - \mathrm{Hess}_g f. \end{aligned} \tag{1.71}$$

This is indeed elliptic, since $R_{ij}(g) = -\frac{1}{2}\sum_k \partial_k \partial_k g_{ij} + Q_{ij}(g, \partial g)$ in harmonic coordinates. It is now standard to bootstrap (1.71) starting with the $C^{1,\alpha}$-bound for g and the $C^{0,1}$-bound for f to obtain C^∞-bounds for (g, f) and to conclude that (g, f) can be extended to a smooth gradient shrinker over the origin. This finishes the proof of Theorem 1.1.

Remark. Every added point is a singular point. Indeed, suppose Γ is trivial and $K_i = |\mathrm{Rm}_{g_i}|_{g_i}(x_i) = \max_{B_\varrho(x_i)}|\mathrm{Rm}_{g_i}|_{g_i} \to \infty$, $x_i \to q$ for some subsequence. Then,

a subsequence of $(M_i, K_i g_i, x_i)$ converges to a nonflat, Ricci-flat manifold with the same volume ratios as in Euclidean space, a contradiction.

Remark. As discovered by Anderson [5], one can use the following two observations to rule out or limit the formation of singularities a priori: For n odd, $\mathbb{R}P^{n-1}$ is the only nontrivial spherical space form and it does not bound a smooth compact manifold. For $n = 4$, every nontrivial orientable Ricci-flat ALE manifold has nonzero second Betti number.

1.4   A local Chern-Gauss-Bonnet argument

In this section, we prove Theorem 1.2. To explain and motivate the Gauss-Bonnet with cutoff argument, we will first prove a weaker version (Proposition 1.13).

Recall from Section 1.2 that f grows like one-quarter distance squared, that R and $|\nabla f|^2$ grow at most quadratically, and that the volume growth is at most Euclidean. These growth estimates will be used frequently in the following.

The next lemma, first observed by Munteanu-Sesum [93], will be very useful in the following. To keep this section self-contained, we give a quick proof.

Lemma 1.11 (Weighted L^2 estimate for Ricci)
For $\lambda > 0$ and $\underline{\mu} > -\infty$ there exist constants $C(\lambda) = C(\lambda, \underline{\mu}, n) < \infty$ such that for every gradient shrinker (M^n, g, f) with $\mu(g) \geq \underline{\mu}$ and normalization as before,

$$\int_M |\mathrm{Rc}|^2 e^{-\lambda f} dV \leq C(\lambda) < \infty. \tag{1.72}$$

Proof. Take a cutoff function η as in Section 1.2 and set $\eta_r(x) = \eta(d(x, p)/r)$. Note that $\mathrm{div}(e^{-f} \mathrm{Rc}) = 0$ by (1.10). Using this, the soliton equation, a partial integration and the inequality $ab \leq a^2/4 + b^2$, we compute

$$\int_M \eta_r^2 |\mathrm{Rc}|^2 e^{-\lambda f} dV = \int_M \eta_r^2 \langle \tfrac{1}{2} g - \nabla^2 f, \mathrm{Rc} \rangle e^{-\lambda f} dV$$

$$= \int_M \left(\tfrac{1}{2} \eta_r^2 R + (1 - \lambda) \eta_r^2 \mathrm{Rc}(\nabla f, \nabla f) + 2 \eta_r \mathrm{Rc}(\nabla \eta_r, \nabla f) \right) e^{-\lambda f} dV$$

$$\leq \tfrac{1}{2} \int_M \eta_r^2 |\mathrm{Rc}|^2 e^{-\lambda f} dV + \int_M \eta_r^2 \left(\tfrac{1}{2} R + (1 - \lambda)^2 |\nabla f|^4 \right) e^{-\lambda f} dV$$

$$+ 4 \int_M |\nabla \eta_r|^2 |\nabla f|^2 e^{-\lambda f} dV.$$

The first term can be absorbed. The second term is uniformly bounded and the last term converges to zero as $r \to \infty$ by the growth estimates from Section 1.2. $\square$

As a consequence of Lemma 1.11, we can replace the Riemann energy bound in Theorem 1.1 by a Weyl energy bound in dimension four.

Corollary 1.12 (Weyl implies Riemann energy condition)
Every sequence of 4-dimensional gradient shrinkers (M_i, g_i, f_i) (with normalization and basepoint as usual) with entropy bounded below, $\mu(g_i) \geq \underline{\mu}$, and a local Weyl energy bound

$$\int_{B_r(p_i)} |W_{g_i}|^2_{g_i} dV_{g_i} \leq C(r) < \infty, \quad \forall i, r \tag{1.73}$$

satisfies the energy condition (1.4).

Remark. As a consistency check, note that in dimension $n = 3$, Rm is determined by Rc and thus only a lower bound for the entropy is needed and the limit is smooth. Of course, the existence of a smooth limit also follows from Theorem 1.7 and the fact that Rm ≥ 0 on gradient shrinkers for $n = 3$. All this is not surprising, since the only 3-dimensional gradient shrinkers are the Gaussian soliton, the cylinder, the sphere and quotients thereof [20].

In the following, the goal is to get local energy bounds from 4d-Gauss-Bonnet with boundary. For a 4-manifold N with boundary ∂N, the Chern-Gauss-Bonnet formula says (see e.g. [53])

$$32\pi^2 \chi(N) = \int_N \left(|\mathrm{Rm}|^2 - 4|\mathrm{Rc}|^2 + R^2 \right) dV$$
$$+ 16 \int_{\partial N} k_1 k_2 k_3 \, dA + 8 \int_{\partial N} \left(k_1 K_{23} + k_2 K_{13} + k_3 K_{12} \right) dA, \tag{1.74}$$

where the $k_i = \mathrm{II}(e_i, e_i)$ are the principal curvatures of ∂N (here e_1, e_2, e_3 is an orthonormal basis of $T\partial N$ diagonalizing the second fundamental form) and the $K_{ij} = \mathrm{Rm}(e_i, e_j, e_i, e_j)$ are sectional curvatures of N.

In a first step, we prove Theorem 1.2 under an extra assumption which ensures in particular that the cubic boundary term has the good sign.

Proposition 1.13 (Convexity implies Riemann energy condition)
Every sequence of 4-dimensional gradient shrinkers (M_i, g_i, f_i) (with normalization and basepoint as usual) with entropy bounded below, $\mu(g_i) \geq \underline{\mu}$, Euler characteristic bounded above, $\chi(M_i) \leq \overline{\chi}$, and convex potential at large distances,

$$\mathrm{Hess}_{g_i} f_i(x) \geq 0 \qquad \text{if } d(x, p_i) \geq r_0, \tag{1.75}$$

satisfies the energy condition (1.4).

Proof. Let us introduce some notation first. We suppress the index i and write $F(x) = e^{-f(x)}$ and define the level and superlevel sets

$$\Sigma_u = \{x \in M \mid F(x) = u\}, \quad M_u = \{x \in M \mid F(x) \geq u\}. \tag{1.76}$$

Note that $M_0 = M$ and $M_{u_2} \subset M_{u_1}$ if $u_2 \geq u_1$.

By the traced soliton equation (1.9) and assumption (1.75), we have $R \leq \frac{n}{2}$ at large distances. Using this, the auxiliary equation (1.11), Lemma 1.3, and the bounds $\underline{\mu} \leq -C_1(g) \leq \overline{\mu}$, we see that f does not have critical points at large distances. In fact, there is a constant $u_0 = u_0(r_0, \underline{\mu}) > 0$ such that $|\nabla f| \geq 1$ and $\nabla^2 f \geq 0$ if $F(x) \leq u_0$. Moreover, for $0 < u \leq u_0$ the Σ_u are smooth compact hypersurfaces, they are all diffeomorphic and we have $\partial M_u = \Sigma_u$ and $\chi(M_u) = \chi(M)$.

Define a cutoff function $\vartheta(x) := \min\{u_0, F(x)\}$, then

$$\int_M |\mathrm{Rm}|^2 \vartheta \, dV = \int_M |\mathrm{Rm}|^2 \int_0^{u_0} 1_{\{u \leq F\}} \, du \, dV = \int_0^{u_0} \int_{M_u} |\mathrm{Rm}|^2 dV \, du. \qquad (1.77)$$

Now, we can apply (1.74) for $N = M_u$. Note that $\chi(M_u) \leq \overline{\chi}$, and that the scalar curvature term and the cubic boundary term are nonnegative. Indeed,

$$\mathrm{II} = -\nabla_\perp^2 F / |\nabla F| = \tfrac{1}{|\nabla f|} \left(\nabla^2 f - \nabla f \otimes \nabla f \right)_\perp = \tfrac{1}{|\nabla f|} \nabla_\perp^2 f \geq 0, \qquad (1.78)$$

where $\perp$ denotes the restriction of the Hessian to $T\Sigma_u$. Thus, we obtain

$$\begin{aligned} \int_{M_u} |\mathrm{Rm}|^2 dV \leq{}& 32\pi^2 \overline{\chi} + 4 \int_{M_u} |\mathrm{Rc}|^2 dV \\ &- 8 \int_{\Sigma_u} (k_1 K_{23} + k_2 K_{13} + k_3 K_{12}) dA \end{aligned} \qquad (1.79)$$

and undoing (1.77), using $\vartheta \leq e^{-f}$, $|k_i| \leq |\mathrm{II}|$ and $|K_{ij}| \leq |\mathrm{Rm}|$, this implies

$$\int_M |\mathrm{Rm}|^2 \vartheta \, dV \leq 32\pi^2 \overline{\chi} u_0 + 4 \int_M |\mathrm{Rc}|^2 e^{-f} dV + 24 \int_0^{u_0} \int_{\Sigma_u} |\mathrm{II}||\mathrm{Rm}| dA \, du. \qquad (1.80)$$

The Ricci term can be estimated as in (1.72). For the last term we use the coarea formula (observe the cancelation):

$$\begin{aligned} \int_0^{u_0} \int_{\Sigma_u} |\mathrm{II}||\mathrm{Rm}| dA \, du \leq{}& \int_{M \setminus M_{u_0}} \frac{|\nabla^2 f|}{|\nabla f|} |\mathrm{Rm}||\nabla f| \vartheta \, dV \\ \leq{}& \tfrac{1}{48} \int_M |\mathrm{Rm}|^2 \vartheta \, dV + 12 \int_M |\nabla^2 f|^2 e^{-f} dV. \end{aligned} \qquad (1.81)$$

The first term can be absorbed, the second one can be dealt with as in (1.72),

$$\begin{aligned} \int_M |\nabla^2 f|^2 e^{-f} dV &= \int_M \langle \nabla^2 f, \tfrac{1}{2} g - \mathrm{Rc} \rangle e^{-f} dV \\ &= \tfrac{1}{2} \int_M \triangle f e^{-f} dV + \int_M \langle \nabla f, \mathrm{div}(e^{-f} \mathrm{Rc}) \rangle dV \qquad (1.82) \\ &= \tfrac{1}{2} \int_M \left(\tfrac{n}{2} - R \right) e^{-f} dV, \end{aligned}$$

where we used the traced soliton equation and $\text{div}(e^{-f}\,\text{Rc}) = 0$ in the last step. Putting everything together, we obtain a uniform bound for $\int_M |\text{Rm}|^2 \vartheta\, dV$, and (1.4) follows. $\square$

Let us now replace the (unnatural) assumption (1.75) by the weaker assumption (1.5). Let $u_0 = u_0(r_0, \underline{\mu}) > 0$ such that $|\nabla f| \geq c$ if $F(x) \leq u_0$ and $\vartheta(x) := \min\{u_0, F(x)\}$ a cutoff function as before. The proof is essentially identical, except that in addition we have to estimate (the negative part of) the cubic boundary term in the Gauss-Bonnet formula. By the coarea formula

$$\left| \int_0^{u_0} \int_{\Sigma_u} \det \text{II}\, dA du \right| \leq \int_{M \setminus M_{u_0}} \frac{|\nabla_\perp^2 f|^3}{|\nabla f|^2} e^{-f} dV \leq \frac{1}{c^2} \int_M |\text{Rc} - \tfrac{1}{2}g|^3 e^{-f} dV. \tag{1.83}$$

Note that the only difficult term is $\int_M |\text{Rc}|^3 e^{-f} dV$, since all other terms can be uniformly bounded using Lemma 1.11. Fortunately, we can bound this weighted L^3-norm of Ricci by uniformly controlled terms and a weighted L^2 Riemann term that can be absorbed in the Gauss-Bonnet argument. Exploiting the algebraic structure of the equations for gradient shrinkers and the full strength of Lemma 1.11, we obtain the following key estimate.

Lemma 1.14 (Weighted L^3 estimate for Ricci)
For $\varepsilon > 0$ and $\underline{\mu} > -\infty$ there exist constants $C(\varepsilon) = C(\varepsilon, \underline{\mu}, n) < \infty$ such that for every gradient shrinker (M^n, g, f) with our usual normalization and $\mu(g) \geq \underline{\mu}$ we have the estimate

$$\int_M |\text{Rc}|^3 e^{-f} dV \leq \varepsilon \int_M |Rm|^2 e^{-f} dV + C(\varepsilon). \tag{1.84}$$

Proof. Analogous to (1.10), we have

$$\nabla_k R_{ij} - \nabla_i R_{kj} = -\nabla_k \nabla_i \nabla_j f + \nabla_i \nabla_k \nabla_j f = R_{ikj\ell} \nabla_\ell f \tag{1.85}$$

and as a direct consequence $\text{div}(e^{-f}\,\text{Rm}) = 0$. Moreover, analogous to (1.13), the shrinker version of the evolution equation for the Ricci tensor is

$$R_{ij} + \langle \nabla f, \nabla R_{ij} \rangle = \triangle R_{ij} + 2R_{ikj\ell} R_{k\ell}. \tag{1.86}$$

Now, for a cutoff function η_r as in the proof of Lemma 1.11, we compute

$$\int_M \eta_r |\text{Rc}|^3 e^{-f} dV = \int_M \eta_r |\text{Rc}| \langle \tfrac{1}{2}g - \nabla^2 f, \text{Rc} \rangle e^{-f} dV$$

$$= \int_M \left(\tfrac{1}{2}\eta_r |\text{Rc}|R + |\text{Rc}|\,\text{Rc}(\nabla f, \nabla \eta_r) + \eta_r \,\text{Rc}(\nabla f, \nabla |\text{Rc}|) \right) e^{-f} dV$$

$$\leq \int_M \left(\tfrac{1}{2}\eta_r |\text{Rc}|R + |\nabla \eta_r||\text{Rc}|^2 |\nabla f| \right) e^{-f} dV$$

$$+ \delta \int_M \eta_r |\nabla \text{Rc}|^2 e^{-\frac{3}{2}f} dV + \tfrac{1}{4\delta} \int_M \eta_r |\text{Rc}|^2 |\nabla f|^2 e^{-\frac{1}{2}f} dV$$

$$\leq \delta \int_M \eta_r |\nabla \text{Rc}|^2 e^{-\frac{3}{2}f} dV + C(\delta),$$

for $\delta > 0$ to be chosen later. Here, we used Young's inequality, Kato's inequality, the growth estimates from Section 1.2 and Lemma 1.11 (note that $|\nabla f|^2 e^{-f/2} \leq C e^{-f/4}$ etc.). Note that the constant $C(\delta)$ does not depend on the scaling factor r of the cutoff function η_r, so by sending $r \to \infty$, we obtain

$$\int_M |\mathrm{Rc}|^3 e^{-f} dV \leq \delta \int_M |\nabla \mathrm{Rc}|^2 e^{-\frac{3}{2}f} dV + C(\delta). \tag{1.87}$$

Next, we estimate the weighted L^2-norm of $\nabla \mathrm{Rc}$ with a partial integration, equation (1.86), and Young's inequality,

$$\int_M \eta_r^2 |\nabla \mathrm{Rc}|^2 e^{-\frac{3}{2}f} dV = -\int_M \eta_r^2 \big(\triangle R_{ij} - \tfrac{3}{2}\langle \nabla f, \nabla R_{ij}\rangle\big) R_{ij} e^{-\frac{3}{2}f} dV$$

$$- \int_M 2\eta_r \langle \nabla \eta_r, \nabla R_{ij}\rangle R_{ij} e^{-\frac{3}{2}f} dV$$

$$\leq -\int_M \eta_r^2 \big(R_{ij} - 2R_{ikj\ell}R_{k\ell}\big) R_{ij} e^{-\frac{3}{2}f} dV$$

$$+ \int_M \big(\tfrac{1}{2}\eta_r^2|\nabla \mathrm{Rc}|^2 + \tfrac{1}{4}\eta_r^2|\mathrm{Rc}|^2|\nabla f|^2 + 4|\nabla \eta_r|^2|\mathrm{Rc}|^2\big) e^{-\frac{3}{2}f} dV.$$

By absorption, the growth estimates from Section 1.2, Lemma 1.11 and the soliton equation, we obtain

$$\int_M \eta_r^2 |\nabla \mathrm{Rc}|^2 e^{-\frac{3}{2}f} dV \leq C - 2\int_M \eta_r^2 \big(R_{ij} - 2R_{ikj\ell}R_{k\ell}\big) R_{ij} e^{-\frac{3}{2}f} dV$$

$$= C - 4\int_M \eta_r^2 R_{ikj\ell} R_{ij} \nabla_k \nabla_\ell f\, e^{-\frac{3}{2}f} dV.$$

Finally, with another partial integration, $\mathrm{div}(e^{-f}\,\mathrm{Rm}) = 0$, and with

$$2R_{ikj\ell}\nabla_k R_{ij}\nabla_\ell f = R_{ikj\ell}\nabla_\ell f(\nabla_k R_{ij} - \nabla_i R_{kj}) = |R_{ikj\ell}\nabla_\ell f|^2, \tag{1.88}$$

which follows from (1.85) and which is the identity that makes the proof work, we get

$$\int_M \eta_r^2 |\nabla \mathrm{Rc}|^2 e^{-\frac{3}{2}f} dV \leq C + 2\int_M \eta_r^2 |R_{ikj\ell}\nabla_\ell f|^2 e^{-\frac{3}{2}f} dV$$

$$+ \int_M \big(8\eta_r \nabla_k \eta_r R_{ikj\ell} R_{ij}\nabla_\ell f - 2\eta_r^2 R_{ikj\ell} R_{ij}\nabla_k f\nabla_\ell f\big) e^{-\frac{3}{2}f} dV$$

$$\leq C + C\int_M \eta_r^2 |\mathrm{Rm}|^2 e^{-f} dV. \tag{1.89}$$

In the last step, we used again Young's inequality, the growth estimates from Section 1.2, Lemma 1.11 and $|\nabla f|^2 e^{-\frac{3}{2}f} \leq C e^{-f}$. The claim now follows by sending $r \to \infty$, plugging into (1.87), and choosing $\delta > 0$ such that $C\delta \leq \varepsilon$ for the constant C in (1.89). $\qquad\square$

Now Theorem 1.2 is an immediate consequence.

Proof of Theorem 1.2. Picking $\varepsilon = \varepsilon(c, r_0, \underline{\mu}) > 0$ so small that $\varepsilon e^{-f} \leq \vartheta c^2/100$ and applying Lemma 1.14, the theorem follows as explained in the discussion after Proposition 1.13. $\qquad\square$

Perelman's lambda-functional and the stability of Ricci-flat metrics

In this chapter, we introduce a new method (based on Perelman's λ-functional) to study the stability of compact Ricci-flat metrics. Under the assumption that all infinitesimal Ricci-flat deformations are integrable we prove: (1) a Ricci-flat metric is a local maximizer of λ in a $C^{2,\alpha}$-sense if and only if its Lichnerowicz Laplacian is nonpositive, (2) λ satisfies a Łojasiewicz-Simon gradient inequality, (3) the Ricci flow does not move excessively in gauge directions. As consequences, we obtain a rigidity result, a new proof of Sesum's dynamical stability theorem, and a dynamical instability theorem.

2.1 Introduction

A Ricci-flat manifold is a Riemannian manifold with vanishing Ricci curvature. Compact Ricci-flat manifolds are fairly hard to find, and their properties are of great interest (see [11, 73] for extensive information). They are the critical points of the Einstein-Hilbert functional and the fixed points of Hamilton's Ricci flow [63],

$$\partial_t g(t) = -2\operatorname{Rc}_{g(t)}, \qquad g(0) = g_0. \tag{2.1}$$

Historically, since Ricci-flat metrics are saddle points (but not extrema!) of the Einstein-Hilbert functional and since the Ricci flow is not a gradient flow in the strict sense, the variational interpretation of Ricci-flatness was rather obscure. However, Perelman made the remarkable discovery that the Ricci flow can be interpreted as gradient flow of the functional

$$\lambda(g) = \inf_{\substack{f \in C^\infty(M) \\ \int_M e^{-f} dV_g = 1}} \int_M \left(R_g + |Df|_g^2\right) e^{-f} dV_g \tag{2.2}$$

on the space of metrics modulo diffeomorphisms. In particular, λ is nondecreasing under the Ricci flow and the stationary points of λ are precisely the Ricci-flat metrics [97, 76]. The second variation of λ is given in terms of the Lichnerowicz Laplacian [21].

We will be concerned with the *stability* of compact Ricci-flat metrics g_{RF}. To discuss this properly, let us consider the following notions of stability:

i. *(Dynamical stability)* For every neighborhood $\mathcal{V}$ of g_{RF} in the space of metrics there exists a smaller neighborhood $\mathcal{U} \subset \mathcal{V}$ such that the Ricci flow starting in $\mathcal{U}$ exists and stays in $\mathcal{V}$ for all $t \geq 0$ and converges to a Ricci-flat metric in $\mathcal{V}$.[1]

ii. *(Local maximum of λ)* There exists a neighborhood $\mathcal{U}$ of g_{RF} such that $\lambda(g) \leq 0$ for all $g \in \mathcal{U}$ with equality if and only if g is Ricci-flat.

iii. *(Linear stability)* All eigenvalues of the Lichnerowicz Laplacian $\triangle_{g_{\mathrm{RF}}}^{L} = \triangle_{g_{\mathrm{RF}}} + 2\operatorname{Rm}_{g_{\mathrm{RF}}}$ are nonpositive.

It is easy to see that i⇒ii⇒iii. Conversely, Natasa Sesum proved that linear stability implies dynamical stability if all infinitesimal Ricci flat deformations are integrable [107]. This was not straightforward, but using the integrability condition she succeeded in finding a good sequence of new reference metrics for the Ricci-DeTurck flow. In particular, she proved the dynamical stability of the K3 surface, which already had been conjectured and partly proven by Guenther-Isenberg-Knopf [58]. Additional interesting results about the dynamical stability of the Ricci flow can be found in [8, 77, 78, 83, 95, 102, 103, 132], and the Kähler case has also been studied by various authors. The proofs are mainly based on the Ricci-DeTurck flow, its linearization and parabolic estimates.

In this chapter, we introduce a *new method* inspired by the work of Leon Simon [110] and the work of Dai-Wang-Wei [41], see also [33, 21, 100, 121]. We use the λ-functional to study stability and instability. With this method we obtain a new proof of Sesum's dynamical stability result (Theorem 2.5) and a number of new results: In particular, we prove a Łojasiewicz-Simon gradient inequality for the λ-functional (Theorem 2.2), and a transversality estimate (Theorem 2.3). Moreover, we prove a local analogue of the positive mass theorem for some compact Ricci-flat metrics (Theorem 2.1), and the corresponding rigidity result (Corollary 2.4). We also prove that unstable Ricci-flat metrics give rise to nontrivial ancient Ricci flows emerging from them (Theorem 2.6). In addition to these results, which we hope are of independent interest, the focus is on methods and proofs. We believe that it is important to understand stability in terms of the λ-functional and not just in terms of PDEs and that our proofs shed new light on the variational structure of Ricci-flat metrics and the role of the gauge group.

The logical structure is that we have three general theorems (Theorem 2.1,Theorem 2.2, Theorem 2.3) and three consequences (Corollary 2.4, Theorem 2.5, Theorem 2.6). To state them, let us fix the following assumption for the whole chapter.

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[1]This notion of stability was called weak dynamical stability in [107]. However, it is the strongest possible notion of stability for dynamical systems with nonisolated critical points where one can only hope to prove convergence to *some* critical point close to the specified one. Since Ricci-flat metrics are always nonisolated critical points in the space of metrics modulo diffeomorphisms, we decided to simply drop the word weak.

Assumption. Let (M, g_{RF}) be a compact, Ricci-flat manifold and assume that all infinitesimal Ricci-flat deformations of g_{RF} are integrable.

The integrability condition means that for every symmetric 2-tensor h in the kernel of the linearization of Ricci, we can find a curve of Ricci-flat metrics with initial velocity h (see [11, Sec. 12] and Section 2.3 for details, and the discussion after Theorem 2.3 for applicability and context).

To set the stage for Theorem 2.1, recall that at a Ricci-flat metric $\lambda(g_{\mathrm{RF}}) = 0$, $D\lambda(g_{\mathrm{RF}}) = 0$ and [21]

$$D^2\lambda(g_{\mathrm{RF}})[h, h] = \tfrac{1}{2}\oint_M \langle h, \triangle^L_{g_{\mathrm{RF}}} h\rangle_{g_{\mathrm{RF}}} dV_{g_{\mathrm{RF}}}, \quad h \in \ker \mathrm{div}_{g_{\mathrm{RF}}}, \tag{2.3}$$

where $\triangle^L_{g_{\mathrm{RF}}} h_{ij} = \triangle h_{ij} + 2R_{ipjq}h_{pq}$. Thus, as mentioned above, local maxima of λ are linearly stable. In the *integrable* case, we can prove the converse implication:

Theorem 2.1 (Local maxima of λ)
If the Lichnerowicz Laplacian $\triangle^L_{g_{\mathrm{RF}}} = \triangle_{g_{\mathrm{RF}}} + 2\,\mathrm{Rm}_{g_{\mathrm{RF}}}$ is nonpositive, then there exists a $C^{2,\alpha}$-neighborhood $\mathcal{U} \subset \mathcal{M}(M)$ of g_{RF} in the space of metrics on M, such that $\lambda(g) \leq 0$ for all $g \in \mathcal{U}$. Moreover, equality holds if and only if g is Ricci-flat.

Theorem 2.1 is nontrivial for the following three reasons: $D^2\lambda$ vanishes on Lie-derivatives, $\triangle^L$ always has a kernel, and it is difficult to estimate the error term in the Taylor-expansion coming from the third variation of λ.

Next, let us state our Łojasiewicz-Simon gradient inequality for the λ-functional (with optimal Łojasiewicz exponent $1/2$ due to integrability):

Theorem 2.2 (Łojasiewicz inequality for λ)
There exists a $C^{2,\alpha}$-neighborhood $\mathcal{U} \subset \mathcal{M}(M)$ of g_{RF} and a constant $c = c(M, g_{\mathrm{RF}}) > 0$ such that

$$\|\mathrm{Rc}_g + \mathrm{Hess}_g\, f_g\|_{L^2} \geq c|\lambda(g)|^{1/2} \tag{2.4}$$

for all $g \in \mathcal{U}$, where f_g is the minimizer in (2.2).

For the interpretation of (2.4) as a gradient inequality note that $\mathrm{Rc}_g + \mathrm{Hess}_g\, f_g$ is the (negative) $L^2(M, e^{-f_g}dV_g)$-gradient of λ by Perelman's first variation formula

$$D\lambda(g)[h] = -\int_M \langle h, \mathrm{Rc}_g + \mathrm{Hess}_g\, f_g\rangle_g e^{-f_g}dV_g. \tag{2.5}$$

Theorem 2.2 is interesting, since it can be used as a general tool to prove convergence and to draw further dynamical conclusions. More precisely, we will always apply it in combination with the following theorem:

Theorem 2.3 (Transversality)
There exists a $C^{2,\alpha}$-neighborhood $\mathcal{U} \subset \mathcal{M}(M)$ of g_{RF} and a constant $c = c(M, g_{\mathrm{RF}}) > 0$ such that

$$\|\mathrm{Rc}_g + \mathrm{Hess}_g\, f_g\|_{L^2} \geq c\|\mathrm{Rc}_g\|_{L^2} \tag{2.6}$$

for all $g \in \mathcal{U}$, where f_g is the minimizer in (2.2).

Theorem 2.3 is a quantitative generalization of the fact that compact steady solitons are Ricci-flat. Since $\mathrm{div}_f(\mathrm{Rc} + \mathrm{Hess}\, f) = 0$ [76, Eq. 10.11], it shows that the Ricci flow does not move excessively in gauge directions.

Before turning to the applications, let us discuss what is currently known and unknown: All *known* compact Ricci-flat manifolds satisfy the integrability assumption and $\triangle^L \leq 0$ (they have special holonomy, so this follows from the results in [41, 73, 116, 122, 126, 131]), but it is a major open question what the true landscape of all compact Ricci-flat manifolds looks like. One main open problem is to construct a compact Ricci-flat manifold with holonomy full SO_n (see e.g. [11, Sec. 0.I]). Another related open question (called the positive mass problem for Ricci flat manifolds in [21]) is if unstable *compact* Ricci-flat metrics exist. Finally, one can ask if there is a Ricci-flat metric with nonintegrable deformations.

Given the difficulty of the above questions and that our picture of compact Ricci-flat metrics already has been drastically changed twice due to Yau and Joyce, we find it very interesting to discuss all cases. First, as an immediate consequence of Theorem 2.1 we obtain (compare with [41]):

Corollary 2.4 (Rigidity of Ricci-flat metrics)
If $\triangle^L_{g_{\mathrm{RF}}} \leq 0$, then every small deformation of g_{RF} with nonnegative scalar curvature is Ricci-flat.

The reader might wish to compare Corollary 2.4 with the following rigidity case of the positive mass theorem [104, 130]: Every compact deformation of the flat metric on $\mathbb{R}^n$ with nonnegative scalar curvature is flat. More generally, Theorem 2.1 can be thought of as the positive mass theorem for linearly stable, integrable, compact Ricci-flat metrics.

Second, as mentioned before, we have a new proof of Sesum's dynamical stability theorem:

Theorem 2.5 (Dynamical stability)
If $k \geq 3$ and $\triangle^L_{g_{\mathrm{RF}}} \leq 0$, then for every C^k-neighborhood $\mathcal{V}$ of g_{RF} there exists a C^{k+2}-neighborhood $\mathcal{U} \subset \mathcal{V}$ of g_{RF} such that the Ricci flow starting in $\mathcal{U}$ exists and stays in $\mathcal{V}$ for all $t \geq 0$ and converges exponentially to a Ricci-flat metric in $\mathcal{V}$.

Our proof is based on Theorem 2.1, 2.2 and 2.3, and shows that the energy controls

the distance (Lemma 2.20).

Third, using Theorem 2.2 and 2.3 we obtain:

Theorem 2.6 (Dynamical instability)
If $\triangle_{g_{\mathrm{RF}}}^L \not\leq 0$, then there exists a nontrivial ancient solution emerging from g_{RF}, i.e. a nontrivial Ricci flow $g(t), t \in (-\infty, T)$ with $\lim_{t \to -\infty} g(t) = g_{\mathrm{RF}}$.

Note that the statement of Theorem 2.6 is much sharper than just some sort of instability/nonconvergence of flows starting nearby. Together with Theorem 2.5 it gives a quite complete picture of what could happen in the compact integrable case.

It should be straightforward to generalize Theorem 2.6 and our proof based on the Łojasiewicz inequality to other flows. A generalization to the noncompact case would be very interesting, since for example the Riemannian Schwarzschild metric is linearly unstable [57, Sec. 5].

There are various further applications of Theorem 2.2 and 2.3. For example, the reader might wish to prove a dichotomy theorem in the case where λ is not a local maximum, i.e. the flow starting near such a Ricci-flat metric either converges or runs away (compare with [110, Thm. 2]). Finally, the nonintegrable case is discussed in the remark at the end of this chapter.

Remark. It suffices to check the condition $\triangle^L \leq 0$ on TT, i.e. on transverse traceless symmetric 2-tensors, since $\triangle^L$ is always nonpositive on the other components [58].

Technical aspects of the proofs. We take care of the gauge directions using the Ebin-Palais slice theorem and of the kernel of $\triangle^L$ using the integrability assumption. The main technical step in the proof of Theorem 2.1, is the estimate

$$\left| \tfrac{d^3}{d\varepsilon^3} \big|_0 \lambda(g + \varepsilon h) \right| \leq C \|h\|_{C^{2,\alpha}} \|h\|_{H^1}^2 \tag{2.7}$$

uniformly in a $C^{2,\alpha}$-neighborhood of g_{RF} (Proposition 2.8). This allows us to conclude that λ is indeed maximal, since

$$D^2 \lambda(g_{\mathrm{RF}})[h, h] \leq -c \|h\|_{H^1}^2 \tag{2.8}$$

on the space normal to the flat directions.
Regarding Theorem 2.2, 2.3 and 2.5, let us just emphasize that it was not at all straightforward to adapt Leon Simon's methods to the Ricci flow. For the numerous technical problems and their solutions we refer the reader to Section 2.4 and Section 2.5. In particular, with Theorem 2.3 we find a way to handle the $\mathrm{Hess}_g \, f_g$-term, a term that is the source of many difficulties. The technical heart consists of Lemmas 2.18 and 2.19. Finally, the Ricci flow in Theorem 2.6 is constructed by a suitable limiting process, and the main step is to prove that this limit is *nontrivial*.

This chapter is organized as follows: In Section 2.2, we analyze the variational structure of λ, in particular, we prove (2.7). In Section 2.3, we recall some facts about the Ebin-Palais slice theorem and integrability. In Section 2.4, we prove Theorem 2.1, 2.2, 2.3 and Corollary 2.4. Finally, as a consequence, we obtain the stability and instabilty results (Theorem 2.5 and 2.6) in Section 2.5.

2.2   The variational structure

We will analyze the variational structure of λ using eigenvalue perturbation theory [101, Sec. XII].
Let (M, g) be a compact Riemannian manifold. Substituting $w = e^{-f/2}$ in (2.2), we see that $\lambda(g)$ is the smallest eigenvalue of the Schrödinger operator $H_g = -4\triangle_g + R_g$. The spectrum of H_g consists only of real eigenvalues of finite multiplicity $\lambda(g) = \lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \ldots$ tending to infinity and the smallest eigenvalue is simple. From the minimax characterization

$$\lambda_k(g) = \min_{\substack{W \subset C^\infty(M) \\ \dim W = k}} \max_{\substack{w \in W \\ w \neq 0}} \frac{\int_M \left(4|Dw|_g^2 + R_g w^2\right) dV_g}{\int_M w^2 dV_g}, \qquad (2.9)$$

we see that $\lambda_k : \mathcal{M}(M) \to \mathbb{R}$ is continuous with respect to the C^2-topology on the space of metrics on M. Along a variation, $g(\varepsilon) = g + \varepsilon h$, the smallest eigenvalue $\lambda(g(\varepsilon))$ depends analytically on ε [76, Sec. 7.I.2.2]. To analyze this ε-dependence, it is convenient to study the resolvent $(\lambda - H_{g(\varepsilon)})^{-1}$, defined for complex λ outside the spectrum. Observe that

$$P_{g(\varepsilon)} = \tfrac{1}{2\pi i} \oint_{|\lambda - \lambda(g)| = r} (\lambda - H_{g(\varepsilon)})^{-1} d\lambda \qquad (2.10)$$

is the projection to the one-dimensional $\lambda(g(\varepsilon))$-eigenspace of $H_{g(\varepsilon)}$. Here, r is assumed to be large enough to encircle $\lambda(g(\varepsilon))$, but small enough to stay away from the other eigenvalues. Thus

$$H_{g(\varepsilon)} P_{g(\varepsilon)} w_g = \lambda(g(\varepsilon)) P_{g(\varepsilon)} w_g, \qquad (2.11)$$

where w_g, called the ground state in the following, is the unique positive $L^2(M, dV_g)$-normalized eigenfunction of H_g with eigenvalue $\lambda(g)$. Thus, for small ε, we obtain

$$\lambda(g(\varepsilon)) = \lambda(g) + \frac{\langle w_g, (H_{g(\varepsilon)} - H_g) P_{g(\varepsilon)} w_g \rangle_{L^2(M, dV_g)}}{\langle w_g, P_{g(\varepsilon)} w_g \rangle_{L^2(M, dV_g)}}. \qquad (2.12)$$

Lemma 2.7
Let (M, g) be a compact Riemannian manifold and h a symmetric 2-tensor. Then the smallest eigenvalue $\lambda(g + \varepsilon h)$ of the operator $H_{g+\varepsilon h} = -4\triangle_{g+\varepsilon h} + R_{g+\varepsilon h}$ depends

analytically on ε and the first three derivatives are given by the following formulas:

$$\frac{d}{d\varepsilon}|_0\lambda(g+\varepsilon h) = \langle w, H'[h]w\rangle, \tag{2.13}$$

$$\frac{d^2}{d\varepsilon^2}|_0\lambda(g+\varepsilon h)$$
$$= \langle w, H''[h,h]w\rangle + \frac{2}{2\pi i}\oint\langle w, H'[h](\lambda-H)^{-1}H'[h]w\rangle\frac{d\lambda}{\lambda-\lambda(g)}, \tag{2.14}$$

$$\frac{d^3}{d\varepsilon^3}|_0\lambda(g+\varepsilon h)$$
$$= \langle w, H'''[h,h,h]w\rangle$$
$$+\frac{6}{2\pi i}\oint\langle w, H'[h](\lambda-H)^{-1}H'[h](\lambda-H)^{-1}H'[h]w\rangle\frac{d\lambda}{\lambda-\lambda(g)}$$
$$+\frac{3}{2\pi i}\oint\langle w, H'[h](\lambda-H)^{-1}H''[h,h]w\rangle\frac{d\lambda}{\lambda-\lambda(g)} \tag{2.15}$$
$$+\frac{3}{2\pi i}\oint\langle w, H''[h,h](\lambda-H)^{-1}H'[h]w\rangle\frac{d\lambda}{\lambda-\lambda(g)}$$
$$-\langle w, H'[h]w\rangle\frac{6}{2\pi i}\oint\langle w, H'[h](\lambda-H)^{-1}H'[h]w\rangle\frac{d\lambda}{(\lambda-\lambda(g))^2}.$$

Here $w = w_g$ is the ground state of $H = H_g = -4\triangle_g + R_g$ and $H^{(k)}[h,\ldots,h] = \frac{d^k}{d\varepsilon^k}|_0(-4\triangle_{g+\varepsilon h}+R_{g+\varepsilon h})$. The complex integrals are over a small circle around $\lambda(g)$ and $\langle\,,\,\rangle$ denotes the $L^2(M, dV_g)$ inner product.

The proof of Lemma 2.7 can be found in the appendix, but let us illustrate here, where (2.14) comes from. We differentiate (2.12) twice. To get a nonzero contribution when evaluated at $\varepsilon = 0$ the derivative has to hit $H_{g(\varepsilon)}$ at least once, thus

$$\frac{d^2}{d\varepsilon^2}|_0\lambda(g+\varepsilon h)$$
$$= \langle w, H''[h,h]w\rangle + 2\langle w, H'[h]P'[h]w\rangle - 2\langle w, H'[h]w\rangle\langle w, P'[h]w\rangle, \quad (2.16)$$

where we also used $P_g w_g = w_g$ and $\langle w_g, w_g\rangle_{L^2(M,dV_g)} = 1$. Differentiating (2.10) and taking care of the operator ordering, we obtain

$$P'[h] = \frac{1}{2\pi i}\oint_{|\lambda-\lambda(g)|=r}(\lambda-H)^{-1}H'[h](\lambda-H)^{-1}d\lambda. \tag{2.17}$$

Using $(\lambda-H)^{-1}w = (\lambda-\lambda(g))^{-1}w$ and the fact that H is symmetric with respect to the L^2 inner product, Equation (2.14) follows. In particular, observe that $\langle w, P'[h]w\rangle$ vanishes, since $\oint(\lambda-\lambda(g))^{-2}d\lambda = 0$. The computation for (2.13) and (2.15) is similar and the differentiability and convergence can be justified, see appendix for details.

From the usual formulas for the variation of the Laplacian and the scalar curvature (see e.g. [11, Sec. 1.K]), we obtain

$$H'[h] = 4h\!:\!D^2 + 4\mathrm{div}h\!:\!D - 2D\mathrm{tr}h\!:\!D - \langle h, \mathrm{Rc}\rangle + \mathrm{div}\mathrm{div}h - \triangle\mathrm{tr}h. \tag{2.18}$$

Inserting this in (2.13), substituting $w = e^{-f/2}$ and using partial integration gives Perelman's first variation formula

$$D\lambda(g)[h] = -\int_M\langle h, \mathrm{Rc}_g + \mathrm{Hess}_g\,f_g\rangle_g e^{-f_g}dV_g, \tag{2.19}$$

where f_g is the minimizer in (2.2). Due to diffeomorphism invariance, $D\lambda(g)$ vanishes on Lie-derivatives, in particular

$$\int_M \langle \mathrm{Hess}_g\, f_g, \mathrm{Rc}_g + \mathrm{Hess}_g\, f_g \rangle_g e^{-f_g} dV_g = 0. \tag{2.20}$$

The stationary points of λ are precisely the Ricci-flat metrics (compactness is crucial here). At a Ricci-flat metric g_{RF} we have $\lambda(g_{\mathrm{RF}}) = 0, D\lambda(g_{\mathrm{RF}}) = 0$ and (compare with [21])

$$D^2\lambda(g_{\mathrm{RF}})[h,h] = \begin{cases} \frac{1}{2\mathrm{Vol}_{g_{\mathrm{RF}}}(M)} \int_M \langle h, \triangle_{g_{\mathrm{RF}}}^L h \rangle_{g_{\mathrm{RF}}} dV_{g_{\mathrm{RF}}} & h \in \ker \mathrm{div}_{g_{\mathrm{RF}}}, \\ 0 & h \in \mathrm{im}\, \mathrm{div}_{g_{\mathrm{RF}}}^*. \end{cases} \tag{2.21}$$

This can also be computed using (2.14), see appendix.

Proposition 2.8 (Third variation of λ)
Let (M, g_0) be a compact Riemannian manifold. Then there exists a $C^{2,\alpha}$-neighborhood $\mathfrak{U}_{g_0} \subset \mathcal{M}(M)$ of g_0 in the space of metrics on M and a constant $C < \infty$ such that

$$\left| \frac{d^3}{d\varepsilon^3}\big|_0 \lambda(g + \varepsilon h) \right| \leq C\|h\|_{C^{2,\alpha}} \|h\|_{H^1}^2 \tag{2.22}$$

for all $g \in \mathfrak{U}_{g_0}$ and all $h \in C^\infty(S^2 T^* M)$.

Proof. We have $C^{2,\alpha}$-bounds for the ground-state w_g of H_g, which we will often use in the following. Let us estimate (2.15) term by term. The first term has the schematic form

$$\langle w, H'''[h,h,h]w \rangle \tag{2.23}$$
$$= \langle w, (\mathrm{Rm}\, hhh + hhDhD + hDhDh + hhhD^2 + hhD^2h)w \rangle.$$

Since M is compact, we get the estimate

$$|\langle w_g, H_g'''[h,h,h]w_g \rangle| \leq C\|h\|_{C^2}\|h\|_{H^1}^2. \tag{2.24}$$

Let us continue with the second term,

$$\oint_{|\lambda-\lambda(g)|=r} \langle w_g, H_g'[h](\lambda - H_g)^{-1} H_g'[h](\lambda - H_g)^{-1} H_g'[h]w_g \rangle \frac{d\lambda}{\lambda-\lambda(g)}. \tag{2.25}$$

Recall that for $|\lambda - \lambda(g)| = r$ the operator $\lambda - H_g : C^\infty(M) \to C^\infty(M)$ is indeed invertible and that H_g is symmetric with respect to the $L^2(M, dV_g)$-inner product. Let us insert the left and right

$$H'[h] = \mathrm{Rc}\, h + DhD + hD^2 + D^2h \tag{2.26}$$

in (2.25). By partial integration, it can be brought into the form

$$\oint_{|\lambda-\lambda(g)|=r} \langle v_{\bar\lambda}[h], H'[h]v_\lambda[h] \rangle \frac{d\lambda}{\lambda-\lambda(g)}, \tag{2.27}$$

where

$$v_\lambda[h] = (\lambda - H_g)^{-1} \left(\text{Rc } wh + DwDh + D^2wh + wD^2h \right). \tag{2.28}$$

We have the elliptic estimate

$$\|v_\lambda[h]\|_{L^2} \le C\|h\|_{L^2}. \tag{2.29}$$

Indeed $(\lambda - H_g)^{-1} \colon H^{-2}(M) \to L^2(M)$ is well defined and continuous, since it is the dual of the continuous map $(\lambda - H_g)^{-1} \colon L^2(M) \to H^2(M)$. The constant in (2.29) can be chosen uniformly for all λ on the circle around $\lambda(g)$, since we have a lower bound for the distance between this circle and the spectrum of H_g. Finally, we insert the middle $H'[h]$ and take care of hD^2 by partial integration. Putting everything together, we obtain

$$\left| \oint \langle w, H'[h](\lambda - H)^{-1} H'[h](\lambda - H)^{-1} H'[h]w \rangle \frac{d\lambda}{\lambda - \lambda(g)} \right| \le C\|h\|_{C^2}\|h\|_{H^1}^2. \tag{2.30}$$

To continue, inserting (2.26) and

$$H''[h, h] = \text{Rm } hh + hhD^2 + hDhD + hD^2h + DhDh \tag{2.31}$$

and using partial integration, the third and the fourth term can be brought into the form

$$\oint \langle \text{Rm } whh + D^2whh + DwhDh + whD^2h + whDhD, v_{\lambda \text{ or } \bar\lambda}[h] \rangle \frac{d\lambda}{\lambda - \lambda(g)}. \tag{2.32}$$

With the L^2-estimates, this can be bounded by $C\|h\|_{C^2}\|h\|_{H^1}^2$.

For the last term, note that

$$|\langle w, H'[h]w \rangle| \le C\|h\|_{C^2}. \tag{2.33}$$

Inserting $H'[h]$ and taking care of $hD^2 + D^2h$ by partial integration the last integral can be estimated by

$$\left| \oint \langle w, H'[h]v_\lambda[h] \rangle \frac{d\lambda}{(\lambda - \lambda(g))^2} \right| \le C\|h\|_{H^1}^2. \tag{2.34}$$

Finally, by Lemma 2.9 below, we have $\|w_g\|_{C^{2,\alpha}} \le C$ uniformly for all g in a $C^{2,\alpha}$-neighborhood $\mathcal{U}_{g_0} \subset \mathcal{M}(M)$ of g_0. This uniform bound and the continuity of eigenvalues discussed at the beginning of Section 2.2 show that all the above estimates go through uniformly in a small enough neighborhood $\mathcal{U}_{g_0}$ of g_0. $\qquad\square$

Lemma 2.9
Let (M, g_0) be a compact Riemannian manifold. Then there exists a $C^{2,\alpha}$-neighborhood $\mathcal{U}_{g_0} \subset \mathcal{M}(M)$ of g_0 and a constant $C < \infty$ such that

$$\|w_g\|_{C^{2,\alpha}} \le C \tag{2.35}$$

for all $g \in \mathcal{U}_{g_0}$, where w_g denotes the ground state of $H_g = -4\triangle_g + R_g$.

Proof. By definition of the ground state,

$$(-4\triangle_g + R_g - \lambda(g))w_g = 0, \qquad \|w_g\|_{L^2(M,dV_g)} = 1. \tag{2.36}$$

By DeGiorgi-Nash-Moser and Schauder estimates [52, Thm. 8.17, Thm. 6.2]

$$\|w_g\|_{C^{2,\alpha}} \le C\|w_g\|_{L^2} \le C. \tag{2.37}$$

Here, for definiteness, we define the norms using the background metric g_0. The constants $\lambda(g)$ are uniformly bounded by the continuity of eigenvalues and we also have a uniform $C^{0,\alpha}$-bound for the coefficient R_g and good control over $\triangle_g$. Thus, the estimates are uniform in a $C^{2,\alpha}$-neighborhood of g_0. $\qquad\square$

2.3   The Ebin-Palais slice theorem and integrability

Let us recall some facts from [46]. Fix a compact manifold M. The group of diffeomorphisms $\mathcal{D}(M)$ acts on the space of metrics $\mathcal{M}(M) \subset C^\infty(S^2 T^* M)$ by pullback. Fix $g_0 \in \mathcal{M}(M)$. Since $\operatorname{div}_{g_0}^*$ is overdetermined elliptic we have the L^2-orthogonal decomposition:

$$C^\infty(S^2 T^* M) = \ker \operatorname{div}_{g_0} \oplus \operatorname{im} \operatorname{div}_{g_0}^*. \tag{2.38}$$

Let $\mathcal{O}_{g_0} \subset \mathcal{M}(M)$ be the orbit of g_0 under the action of $\mathcal{D}(M)$. By the Ebin-Palais slice theorem, there exists a slice $\mathcal{S}_{g_0}$ for the action of $\mathcal{D}(M)$ on $\mathcal{M}(M)$. In particular, $\mathcal{M}(M) \cong_{\mathrm{loc}} \mathcal{S}_{g_0} \times \mathcal{O}_{g_0}$ is locally a product near g_0 (in the sense of inverse limit Banach manifolds) and the induced decomposition of $T_{g_0}\mathcal{M}(M) = C^\infty(S^2 T^* M)$ is given by (2.38). We will only use the following part of the theorem.

Theorem 2.10 (Ebin-Palais [46])
Let M be a compact manifold and $\mathcal{M}(M)$ the space of metrics on M.
Then for every metric $g_0 \in \mathcal{M}(M)$, there exists a $C^{2,\alpha}$-neighborhood $\mathcal{U}_{g_0} \subset \mathcal{M}(M)$ of g_0, such that every metric $g \in \mathcal{U}_{g_0}$ can be written as $g = \varphi^* \hat{g}$ for some diffeomorphism $\varphi \in \mathcal{D}(M)$ and some metric $\hat{g} \in \mathcal{S}_{g_0} = (g_0 + \ker \operatorname{div}_{g_0}) \cap \mathcal{U}_{g_0}$.

Remark. Ebin uses Sobolev spaces, Palais uses Hölder spaces. Moreover, Ebin uses the exponential map Exp_g of the L^2-metric on $\mathcal{M}(M)$ to construct his slice. Palais uses the map $E_g(h) = g + h$ and thus gets an *affine* slice $\mathcal{S}_{g_0} \subset (g_0 + \ker \operatorname{div}_{g_0}) \cap \mathcal{M}(M)$ (the crucial property for constructing the slice is that the exponential map is equivariant, i.e. $\varphi^*(\operatorname{Exp}_g h) = \operatorname{Exp}_{\varphi^* g} \varphi^* h$, which is true for the 'exponential map' E, since the action is linear).

Let us now discuss our integrability assumption, following [11, Sec. 12].

Definition 2.11
Let M be compact and $g_{\mathrm{RF}} \in \mathcal{M}(M)$ Ricci-flat. We call

$$I_{g_{\mathrm{RF}}} = \{h \in \ker \operatorname{div}_{g_{\mathrm{RF}}};\ D\operatorname{Rc}(g_{\mathrm{RF}})[h] = 0\} \tag{2.39}$$

the space of infinitesimal Ricci-flat deformations of g_{RF} and

$$\mathcal{P}_{g_{\mathrm{RF}}} = \{\bar{g} \in \mathcal{S}_{g_{\mathrm{RF}}}; \ \mathrm{Rc}_{\bar{g}} = 0\} \tag{2.40}$$

the premoduli space of Ricci-flat metrics near g_{RF} (the true moduli space is modeled on $\mathcal{P}_{g_{\mathrm{RF}}}/\mathrm{Isom}_{g_{\mathrm{RF}}}$).

Lemma 2.12
Let (M, g_{RF}) compact, Ricci-flat. Then $I_{g_{\mathrm{RF}}} = \mathbb{R}g_{\mathrm{RF}} \oplus K_{g_{\mathrm{RF}}}$, where

$$K_{g_{\mathrm{RF}}} = \{h \in C^\infty(S^2 T^* M); \ \mathrm{div}_{g_{\mathrm{RF}}} h = 0, \ \mathrm{tr}_{g_{\mathrm{RF}}} h = 0, \ \triangle^L_{g_{\mathrm{RF}}} h = 0\}. \tag{2.41}$$

Proof. On transverse symmetric 2-tensors, the linearization of Ricci is proportional to $\triangle^L + D^2 \circ \mathrm{tr}$. Thus for $h \in I_{g_{\mathrm{RF}}}$, we have $\triangle^L h + D^2 \mathrm{tr}\ h = 0$. Taking the trace, we get $\triangle \mathrm{tr}\ h = 0$, thus $\mathrm{tr}\ h = c$ and $\triangle^L h = 0$. Therefore

$$h = \tfrac{c}{n} g_{\mathrm{RF}} + (h - \tfrac{c}{n} g_{\mathrm{RF}}) \in \mathbb{R}g_{\mathrm{RF}} \oplus K_{g_{\mathrm{RF}}}. \tag{2.42}$$

The converse inclusion is clear. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2.13 (Integrability)
Let M be compact and $g_{\mathrm{RF}} \in \mathcal{M}(M)$ be Ricci-flat. We say that all infinitesimal Ricci-flat deformations of g_{RF} are integrable if there is a smooth familiy $g_h(t) \in \mathcal{M}(M)$ of Ricci-flat metrics with $g_h(0) = g_{\mathrm{RF}}$ and $\dot{g}_h(0) = h$, defined for all $h \in I_{g_{\mathrm{RF}}}$ with norm less then one and all $t \in (-\varepsilon, \varepsilon)$.

Proposition 2.14
Let M be compact and $g_{\mathrm{RF}} \in \mathcal{M}(M)$ be Ricci-flat. If all infinitesimal Ricci-flat deformations of g_{RF} are integrable, then $\mathcal{P}_{g_{\mathrm{RF}}}$ is a manifold near g_{RF} with $T_{g_{\mathrm{RF}}}\mathcal{P}_{g_{\mathrm{RF}}} = I_{g_{\mathrm{RF}}}$.

Proof. As in the proof of Koiso's theorem, we construct a manifold $\mathcal{Z}_{g_{\mathrm{RF}}} \subset \mathcal{S}_{g_{\mathrm{RF}}}$ near g_{RF} that contains $\mathcal{P}_{g_{\mathrm{RF}}}$ and satisfies $T_{g_{\mathrm{RF}}}\mathcal{Z}_{g_{\mathrm{RF}}} = I_{g_{\mathrm{RF}}}$. Possibly after passing to smaller neighborhoods, we have $\mathcal{P}_{g_{\mathrm{RF}}} = \mathcal{Z}_{g_{\mathrm{RF}}}$ due to integrability (see [11, Thm. 12.49] for details). $\qquad\qquad\square$

2.4 Local maxima, gradient inequality and transversality

In this section, we prove Theorem 2.1, 2.2, 2.3 and Corollary 2.4.

Proof of Theorem 2.1. Let $\mathcal{U}_{g_{\mathrm{RF}}} \supset \mathcal{S}_{g_{\mathrm{RF}}} \supset \mathcal{P}_{g_{\mathrm{RF}}}$ be as in Section 2.3. We divide the proof of the theorem into the following three steps, whose detailed proofs can be found below:

i For $\ker \operatorname{div}_{g_{\mathrm{RF}}} = T_{g_{\mathrm{RF}}} \mathcal{P}_{g_{\mathrm{RF}}} \oplus N_{g_{\mathrm{RF}}}$, where

$$N_{g_{\mathrm{RF}}} = \{ h \in \ker \operatorname{div}_{g_{\mathrm{RF}}}; \ \langle h, k \rangle_{L^2_{g_{\mathrm{RF}}}} = 0 \text{ for all } k \in T_{g_{\mathrm{RF}}} \mathcal{P}_{g_{\mathrm{RF}}} \}, \qquad (2.43)$$

the second variation $D^2\lambda(g_{\mathrm{RF}})$ vanishes on the first summand and is strictly negative on the second one.

ii By Taylor expansion with careful estimate of the error term, possibly after passing to smaller neighborhoods, λ is nonpositive on $\mathcal{S}_{g_{\mathrm{RF}}}$ and vanishes only on $\mathcal{P}_{g_{\mathrm{RF}}}$.

iii The assertion of the theorem follows from the Ebin-Palais slice theorem and the diffeomorphism invariance of λ.

Proof of i. Since g_{RF} is Ricci-flat, we have the L^2-orthogonal, $\triangle^L_{g_{\mathrm{RF}}}$-invariant decomposition [58, Sec. 4],

$$\ker \operatorname{div}_{g_{\mathrm{RF}}} = \mathbb{R} g_{\mathrm{RF}} \oplus \operatorname{im}(C_{g_{\mathrm{RF}}}) \oplus TT_{g_{\mathrm{RF}}}, \qquad (2.44)$$

where $\mathbb{R} g_{\mathrm{RF}}$ describes scaling, $C_{g_{\mathrm{RF}}} u = (\triangle_{g_{\mathrm{RF}}} u) g_{\mathrm{RF}} - \operatorname{Hess}_{g_{\mathrm{RF}}} u$ describes the other conformal transformations (projected on $\ker \operatorname{div}_{g_{\mathrm{RF}}}$) and

$$TT_{g_{\mathrm{RF}}} = \{ h \in C^\infty(S^2 T^* M); \ \operatorname{div}_{g_{\mathrm{RF}}} h = 0, \operatorname{tr}_{g_{\mathrm{RF}}} h = 0 \} \qquad (2.45)$$

denotes the space of transverse, traceless, symmetric 2-tensors.
Let us analyse the spectrum. The Lichnerowicz Laplacian $\triangle^L_{g_{\mathrm{RF}}}$ vanishes on $\mathbb{R} g_{\mathrm{RF}}$. It is strictly negative on $\operatorname{im}(C)$, since $\triangle^L C u = C \triangle u$. Indeed, taking the trace shows that the elements of the kernel of C are harmonic and thus constant functions (the theorem is trivial in one dimension, where every metric is flat and λ vanishes identically). So, given the eigenvalue equation,

$$\triangle^L C u = \alpha C u, \qquad C u \neq 0, \qquad (2.46)$$

by adding a constant, we can assume without loss of generality $\int_M u = 0$. Now

$$C(\triangle u - \alpha u) = \triangle^L C u - \alpha C u = 0, \qquad (2.47)$$

so $\triangle u - \alpha u$ is constant and by integration this constant is seen to be zero. Thus $\alpha \leq 0$. If α were zero, then u would be constant and $Cu = 0$, a contradiction. Finally, $\triangle^L$ is nonpositive on TT by the hypothesis of the theorem (more precisely, by the weaker hypothesis $\triangle^L \leq 0$ on TT). The kernel

$$K_{g_{\mathrm{RF}}} = \{ h \in TT_{g_{\mathrm{RF}}}; \ \triangle^L_{g_{\mathrm{RF}}} h = 0 \} \qquad (2.48)$$

is finite dimensional and $\triangle^L$ is strictly negative on $TT_{g_{\mathrm{RF}}} \ominus K_{g_{\mathrm{RF}}}$.
By Lemma 2.12 and Proposition 2.14, $T_{g_{\mathrm{RF}}} \mathcal{P}_{g_{\mathrm{RF}}} = \mathbb{R} g_{\mathrm{RF}} \oplus K_{g_{\mathrm{RF}}}$. Now claim i. follows from (2.21). More precisely, there exists a constant $c > 0$, such that

$$\langle h, \triangle^L_{g_{\mathrm{RF}}} h \rangle_{L^2_{g_{\mathrm{RF}}}} \leq -c \langle h, h \rangle_{L^2_{g_{\mathrm{RF}}}} \qquad \text{for all } h \in N_{g_{\mathrm{RF}}}. \qquad (2.49)$$

$\square$

Proof of ii. For small $\varepsilon > 0$, by continuity,

$$\langle h, \triangle_{\bar{g}}^L h \rangle_{L_{\bar{g}}^2} = -\varepsilon \langle Dh, Dh \rangle_{L_{\bar{g}}^2} + (1-\varepsilon) \langle h, \triangle_{\bar{g}} h + \tfrac{2}{1-\varepsilon} \operatorname{Rm}_{\bar{g}} : h \rangle_{L_{\bar{g}}^2}$$

$$\leq -c \|h\|_{H^1}^2 \qquad \text{for all } \bar{g} \in \mathcal{P}_{g_{\mathrm{RF}}},\ h \in N_{g_{\mathrm{RF}}} \qquad (2.50)$$

for some new constant $c > 0$, possibly after passing to smaller neighborhoods. Now $\bar{g} \in \mathcal{P}_{g_{\mathrm{RF}}}$ is Ricci-flat, so $\lambda(\bar{g}) = 0$ and $D\lambda(\bar{g}) = 0$. Thus

$$\lambda(\bar{g} + h) \leq -c\|h\|_{H^1}^2 + |R(\bar{g}, h)|. \qquad (2.51)$$

Here we used the formula

$$\lambda(\bar{g} + h) = \lambda(\bar{g}) + \tfrac{d}{dt}|_0 \lambda(\bar{g} + th) + \tfrac{1}{2}\tfrac{d^2}{dt^2}|_0 \lambda(\bar{g} + th) + R(\bar{g}, h), \qquad (2.52)$$

$$R(\bar{g}, h) = \int_0^1 \left(\tfrac{1}{2} - t + \tfrac{1}{2}t^2 \right) \tfrac{d^3}{dt^3} \lambda(\bar{g} + th) dt. \qquad (2.53)$$

By Proposition 2.8 we have the uniform estimate

$$|R(\bar{g}, h)| \leq C\|h\|_{C^{2,\alpha}} \|h\|_{H^1}^2 \qquad (2.54)$$

for the remainder, if $\bar{g} - g_{\mathrm{RF}}$ and h are $C^{2,\alpha}$-small. For sufficiently small $C^{2,\alpha}$-norm, the negative term in (2.51) dominates. Finally, the 'exponential map'

$$E : \mathcal{P}_{g_{\mathrm{RF}}} \times N_{g_{\mathrm{RF}}} \to g_{\mathrm{RF}} + \ker \operatorname{div}_{g_{\mathrm{RF}}}, \qquad E(\bar{g}, h) = \bar{g} + h \qquad (2.55)$$

maps a $C^{2,\alpha}$-neighborhood of $(g_{\mathrm{RF}}, 0)$ onto a $C^{2,\alpha}$-neighborhood of g_{RF}. Here, to apply the inverse function theorem, we temporarily enlarge the involved spaces to $C^{2,\alpha}$-spaces. Since the kernel of $\triangle_{g_{\mathrm{RF}}}^L$ is smooth by elliptic regularity, the proof of Proposition 2.14 shows that $\mathcal{P}_{g_{\mathrm{RF}}}$ only consists of smooth elements also after passing to $C^{2,\alpha}$-spaces. Thus $E(\bar{g}, h)$ is smooth if and only if h is smooth. This finishes the proof of Claim ii. $\square$

Proof of iii. By the Ebin-Palais slice theorem, every $g \in \mathcal{U}_{g_{\mathrm{RF}}}$ can be written as $g = \varphi^* \hat{g}$ for some $\varphi \in \mathcal{D}(M), \hat{g} \in \mathcal{S}_{g_{\mathrm{RF}}}$. Since λ is diffeomorphism invariant

$$\lambda(g) = \lambda(\hat{g}) \leq 0 \qquad (2.56)$$

by step ii. If $\lambda(g) = 0$, then $\hat{g} \in \mathcal{P}_{g_{\mathrm{RF}}}$, so $\operatorname{Rc}_{\hat{g}} = 0$ and thus $\operatorname{Rc}_g = 0$. $\square$

This finishes the proof of Theorem 2.1. $\square$

Proof of Corollary 2.4. Let $\mathcal{U}_{g_{\mathrm{RF}}}$ be as in Theorem 2.1 and $g \in \mathcal{U}_{g_{\mathrm{RF}}}$. If $R_g \geq 0$, then $\lambda(g) \geq 0$. Thus $\lambda(g) = 0$ and $\operatorname{Rc}_g = 0$ by the equality case of Theorem 2.1. $\square$

We will now estimate the motion in the gauge directions. Namely, we have to deal with the minimizer f_g from (2.2) appearing in $e^{-f_g} dV_g$ and more importantly in $\operatorname{Rc}_g + \operatorname{Hess}_g f_g$ in (2.19). We start with the following refinement of Lemma 2.9.

Lemma 2.15
Let (M, g_{RF}) be compact, Ricci-flat and $\varepsilon > 0$. Then there exists a $C^{2,\alpha}$-neighborhood $\mathcal{U}_{g_{\mathrm{RF}}}$ of g_{RF} such that

$$\|f_g - \log \mathrm{Vol}_{g_{\mathrm{RF}}}(M)\|_{C^{2,\alpha}} < \varepsilon \tag{2.57}$$

for all $g \in \mathcal{U}_{g_{\mathrm{RF}}}$, where f_g is the minimizer in (2.2).

Proof. Assume the volume is normalized, then

$$f_{g_{\mathrm{RF}}} = \log \mathrm{Vol}_{g_{\mathrm{RF}}}(M) = 0. \tag{2.58}$$

Write $w_g = e^{-f_g/2}$. There is some $\tilde{\varepsilon} > 0$, such that

$$\|w_g - 1\|_{C^{2,\alpha}} < \tilde{\varepsilon} \Rightarrow \|f_g\|_{C^{2,\alpha}} < \varepsilon \tag{2.59}$$

We will prove $\|w_g - 1\|_{C^{2,\alpha}} < \tilde{\varepsilon}$ for g near g_{RF} using the implicit function theorem. Let

$$X = \{g \in C^{2,\alpha}(S^2 T^* M);\ g \text{ positive definite}\}, \tag{2.60}$$

$$Y = \{u \in C^{2,\alpha}(M);\ \int_M u\, dV_{g_{\mathrm{RF}}} = 0\}, \tag{2.61}$$

$$Z = \{l \in C^{0,\alpha}(M);\ \int_M l\, dV_{g_{\mathrm{RF}}} = 0\}. \tag{2.62}$$

Define $F : X \times Y \to Z$ by

$$F(g, u) = (-4\triangle_g + R_g - \lambda(g))(1 + u)$$
$$- \int_M (-4\triangle_g + R_g - \lambda(g))(1 + u)\, dV_{g_{\mathrm{RF}}}. \tag{2.63}$$

From Section 2.2, we know that F is C^1. Observe that $F(g_{\mathrm{RF}}, 0) = 0$ and

$$F(g, u) = 0 \Leftrightarrow (-4\triangle_g + R_g)(1 + u) = \lambda(g)(1 + u). \tag{2.64}$$

Indeed, $F(g, u) = 0$ implies $(-4\triangle_g + R_g - \lambda(g))(1 + u) = c$ and by the Fredholm alternative $\int_M c w_g dV_g = 0$. Thus $c = 0$, since w_g is positive. Now

$$DF(g_{\mathrm{RF}}, 0)|_Y = -4\triangle_{g_{\mathrm{RF}}} : Y \to Z \tag{2.65}$$

is indeed an isomorphism. By the implicit function theorem there exists a $C^{2,\alpha}$-neighborhood of g_{RF} such that (2.64) can be solved for $u = u(g)$ with the estimate $\|u(g)\|_{C^{2,\alpha}} < \tilde{\varepsilon}/100$. Since

$$w_g = \left(\int_M (1 + u(g))^2 dV_g \right)^{-\frac{1}{2}} (1 + u(g)) \tag{2.66}$$

we obtain $\|w_g - 1\|_{C^{2,\alpha}} < \tilde{\varepsilon}$ in a small enough $C^{2,\alpha}$-neighborhood. $\square$

Let $g \in g_{\mathrm{RF}} + \ker \mathrm{div}_{g_{\mathrm{RF}}}, g = \bar{g} + h, \bar{g} \in \mathcal{P}_{g_{\mathrm{RF}}}, h \in N_{g_{\mathrm{RF}}}$ as in the proof of Theorem 2.1. In the following four lemmas, we will show

$$\mathrm{Rc}_g + \mathrm{Hess}_g\, f_g = -\tfrac{1}{2}\triangle^L_{g_{\mathrm{RF}}} h + O_1(h * h) + O_2((\bar{g} - g_{\mathrm{RF}}) * h) \qquad (2.67)$$

in a $C^{2,\alpha}$-neighborhood of g_{RF} with estimates for O_1 and O_2.

Lemma 2.16
Let (M, g_{RF}) be compact Ricci-flat and $h \in \ker \mathrm{div}_{g_{\mathrm{RF}}}$. Then

$$\tfrac{d}{dt}|_0 f_{g_{\mathrm{RF}}+th} = \tfrac{1}{2} \mathrm{tr}_{g_{\mathrm{RF}}} h \qquad (2.68)$$

$$\tfrac{d}{dt}|_0 \left(\mathrm{Rc}_{g_{\mathrm{RF}}+th} + \mathrm{Hess}_{g_{\mathrm{RF}}+th}\, f_{g_{\mathrm{RF}}+th} \right) = -\tfrac{1}{2}\triangle^L_{g_{\mathrm{RF}}} h. \qquad (2.69)$$

Proof. Since $f_{g_{\mathrm{RF}}} = \log \mathrm{Vol}_{g_{\mathrm{RF}}}(M)$ is a constant function, many terms will drop out in the following computation. From Section 2.2, we know that $t \mapsto f_{g_{\mathrm{RF}}+th}$ is analytic. Differentiating the equations

$$\left(-4\triangle_{g_{\mathrm{RF}}+th} + R_{g_{\mathrm{RF}}+th} - \lambda(g_{\mathrm{RF}} + th)\right) e^{-\frac{1}{2} f_{g_{\mathrm{RF}}+th}} = 0, \qquad (2.70)$$

$$\int_M e^{-f_{g_{\mathrm{RF}}+th}} dV_{g_{\mathrm{RF}}+th} = 1 \qquad (2.71)$$

at $t = 0$, we obtain

$$\triangle_{g_{\mathrm{RF}}} \left(\tfrac{d}{dt}|_0 f_{g_{\mathrm{RF}}+th} - \tfrac{1}{2} \mathrm{tr}_{g_{\mathrm{RF}}} h \right) = 0, \qquad (2.72)$$

$$\int_M \left(\tfrac{d}{dt}|_0 f_{g_{\mathrm{RF}}+th} - \tfrac{1}{2} \mathrm{tr}_{g_{\mathrm{RF}}} h \right) dV_{g_{\mathrm{RF}}} = 0, \qquad (2.73)$$

and Equation (2.68) follows. Equation (2.69) follows from

$$\tfrac{d}{dt}|_0 \mathrm{Rc}_{g_{\mathrm{RF}}+th} = -\tfrac{1}{2} \left(\triangle^L_{g_{\mathrm{RF}}} h + \mathrm{Hess}_{g_{\mathrm{RF}}} \mathrm{tr}_{g_{\mathrm{RF}}} h \right) \qquad (2.74)$$

and $\tfrac{d}{dt}|_0 \left(\mathrm{Hess}_{g_{\mathrm{RF}}+th}\, f_{g_{\mathrm{RF}}+th} \right) = \mathrm{Hess}_{g_{\mathrm{RF}}} \tfrac{d}{dt}|_0 f_{g_{\mathrm{RF}}+th}$. $\qquad\square$

Lemma 2.17
Let $F(s,t)$ be a C^2-function on $0 \le s, t \le 1$ with values in a Frechet-space. Then

$$F(1,1) = F(1,0) + \tfrac{d}{dt}|_0 F(0,t) + \int_0^1 (1-t)\tfrac{d^2}{dt^2} F(0,t)dt$$
$$+ \int_0^1 \int_0^1 \tfrac{\partial^2}{\partial s \partial t} F(s,t)dsdt. \qquad (2.75)$$

Proof. By the Hahn-Banach theorem, it suffices to prove the lemma for real valued F and this follows from

$$\int_0^1 (1-t)\tfrac{d^2}{dt^2} F(0,t)dt = -\tfrac{d}{dt}|_0 F(0,t) + \underbrace{\int_0^1 \tfrac{d}{dt} F(0,t)dt}_{=F(0,1)-F(0,0)}, \qquad (2.76)$$

$$\int_0^1 \int_0^1 \tfrac{\partial^2}{\partial s \partial t} F(s,t)dsdt = F(1,1) + F(0,0) - F(1,0) - F(0,1). \qquad (2.77)$$

$\square$

Lemma 2.18

Let $g \in g_{\mathrm{RF}} + \ker \mathrm{div}_{g_{\mathrm{RF}}}, g = \bar{g} + h, \bar{g} \in \mathcal{P}_{g_{\mathrm{RF}}}, h \in N_{g_{\mathrm{RF}}}$ as in the proof of Theorem 2.1. Then, in a $C^{2,\alpha}$-neighborhood of g_{RF} in $g_{\mathrm{RF}} + \ker \mathrm{div}_{g_{\mathrm{RF}}}$, we have the equality

$$\mathrm{Rc}_g + \mathrm{Hess}_g\, f_g = -\tfrac{1}{2}\triangle^L_{g_{\mathrm{RF}}} h + O_1 + O_2 \tag{2.78}$$

with

$$O_1 = \int_0^1 (1-t)\tfrac{d^2}{dt^2}\left(\mathrm{Rc}_{g_{\mathrm{RF}}+th} + \mathrm{Hess}_{g_{\mathrm{RF}}+th}\, f_{g_{\mathrm{RF}}+th}\right) dt, \tag{2.79}$$

$$O_2 = \int_0^1 \int_0^1 \tfrac{\partial^2}{\partial s \partial t}\Big(\mathrm{Rc}_{g_{\mathrm{RF}}+s(\bar{g}-g_{\mathrm{RF}})+th}$$
$$+ \mathrm{Hess}_{g_{\mathrm{RF}}+s(\bar{g}-g_{\mathrm{RF}})+th}\, f_{g_{\mathrm{RF}}+s(\bar{g}-g_{\mathrm{RF}})+th}\Big)ds\,dt. \tag{2.80}$$

Proof. Use Lemma 2.17 with

$$F(s,t) = \mathrm{Rc}_{g_{\mathrm{RF}}+s(\bar{g}-g_{\mathrm{RF}})+th} + \mathrm{Hess}_{g_{\mathrm{RF}}+s(\bar{g}-g_{\mathrm{RF}})+th}\, f_{g_{\mathrm{RF}}+s(\bar{g}-g_{\mathrm{RF}})+th}. \tag{2.81}$$

Note that $F(1,0) = \mathrm{Rc}_{\bar{g}} + \mathrm{Hess}_{\bar{g}}\, f_{\bar{g}} = 0$ and use (2.69). $\square$

Lemma 2.19

Let $g \in g_{\mathrm{RF}} + \ker \mathrm{div}_{g_{\mathrm{RF}}}, g = \bar{g} + h, \bar{g} \in \mathcal{P}_{g_{\mathrm{RF}}}, h \in N_{g_{\mathrm{RF}}}$ as in the proof of Theorem 2.1. Then, there exists a $C^{2,\alpha}$-neighborhood of g_{RF} in $g_{\mathrm{RF}} + \ker \mathrm{div}_{g_{\mathrm{RF}}}$ and a constant $C < \infty$ such that the inequalities

$$\|O_1\|_{L^2} \leq C\|h\|_{C^{2,\alpha}}\|h\|_{H^2}, \tag{2.82}$$
$$\|O_2\|_{L^2} \leq C\|\bar{g} - g_{\mathrm{RF}}\|_{C^{2,\alpha}}\|h\|_{H^2} \tag{2.83}$$

hold in this neighborhood.

Proof. The estimate is clear for the part of O_i coming from Rc (since it contains at most second derivatives). The part coming from Hess f is more tricky. Let h, k be symmetric 2-tensors. We will show

$$\|\tfrac{\partial^2}{\partial s \partial t}|_{(0,0)} \mathrm{Hess}_{g+sk+th}\, f_{g+sk+th}\|_{L^2} \leq C\|k\|_{C^{2,\alpha}}\|h\|_{H^2} \tag{2.84}$$

uniformly for all g in a $C^{2,\alpha}$-neighborhood. We differentiate:

$$\tfrac{\partial^2}{\partial s \partial t}|_{(0,0)} \mathrm{Hess}_{g+sk+th}\, f_{g+sk+th} = \dot{\mathrm{Hess}}'f + \mathrm{Hess}'\dot{f} + \dot{\mathrm{Hess}}f' + \mathrm{Hess}\,\dot{f}'. \tag{2.85}$$

The first term has the schematic form

$$\dot{\mathrm{Hess}}'f = kDhDf + hDkDf, \tag{2.86}$$

thus

$$\|\dot{\mathrm{Hess}}'f\|_{L^2} \leq C\|k\|_{C^1}\|h\|_{H^1} \leq C\|k\|_{C^{2,\alpha}}\|h\|_{H^2} \tag{2.87}$$

by Lemma 2.15 (we will use Lemma 2.15 frequently below without mentioning it again). To control the other three terms, we will differentiate the equation

$$2\triangle_{g+sk+th}f_{g+sk+th} - |Df_{g+sk+th}|^2_{g+sk+th} + R_{g+sk+th} = \lambda(g+sk+th) \tag{2.88}$$

and use elliptic estimates. Differentiating with respect to t gives the linear elliptic equation

$$P_g\dot{f} = F_g[h], \tag{2.89}$$

where $P_g = \triangle_g - g^{ij}D_i f_g D_j$ and F has the schematic form

$$F_g[h] = \dot{\lambda} + D^2 h + DhDf + hD^2 f + hDfDf + h\,\mathrm{Rc}. \tag{2.90}$$

By the maximum principle, only constant functions are in the kernel of P. Thus $\dot{f} - \bar{\dot{f}}$ is L^2-orthogonal to $\ker P$ (the bar denotes the average). Since it also solves the equation

$$P_g(\dot{f} - \bar{\dot{f}}) = F_g[h], \tag{2.91}$$

we get the estimate

$$\|\dot{f} - \bar{\dot{f}}\|_{H^2} \leq C\|F_g[h]\|_{L^2} \leq C\|h\|_{H^2}. \tag{2.92}$$

In the last step, we used the estimate (cf. Section 2.2),

$$|\dot{\lambda}| \leq C\|h\|_{H^2}. \tag{2.93}$$

Thus

$$\|\dot{\mathrm{Hess}}' \dot{f}\|_{L^2} \leq C\|DkD\dot{f}\|_{L^2} \leq C\|k\|_{C^1}\|\dot{f} - \bar{\dot{f}}\|_{H^1} \leq C\|k\|_{C^{2,\alpha}}\|h\|_{H^2}. \tag{2.94}$$

Next, we will estimate $\dot{\mathrm{Hess}}f'$. Similar as above, we obtain:

$$P_g(f' - \bar{f}') = F_g[k] \tag{2.95}$$

$$\|f' - \bar{f}'\|_{H^2} \leq C\|k\|_{H^2}. \tag{2.96}$$

From (2.95), by DeGiorgi-Nash-Moser and Schauder estimates we get

$$\|f' - \bar{f}'\|_{C^{2,\alpha}} \leq C\left(\|F_g[k]\|_{C^{0,\alpha}} + \|f' - \bar{f}'\|_{L^2}\right) \leq C\|k\|_{C^{2,\alpha}}, \tag{2.97}$$

where we used (2.96) and $|\lambda'| \leq C\|k\|_{C^{2,\alpha}}$. Thus

$$\|\dot{\mathrm{Hess}}f'\|_{L^2} \leq C\|DhDf'\|_{L^2} \leq C\|k\|_{C^{2,\alpha}}\|h\|_{H^2}. \tag{2.98}$$

Finally, let us estimate $\mathrm{Hess}\,\dot{f}'$. Differentiating (2.88) twice gives the linear elliptic equation

$$P_g\dot{f}' = G_g[h,k], \tag{2.99}$$

where G has the schematic form,

$$\begin{aligned}
G_g[h, k] &= \dot{\lambda}' + D\dot{f}Df' + \left(hD^2f' + DhDf' + hDf'Df\right) \\
&\quad + \left(kD^2\dot{f} + DkD\dot{f} + kD\dot{f}Df\right) \\
&\quad + \Big(kD^2h + hD^2k + DhDk + hDkDf + kDhDf \\
&\qquad\qquad + hkDfDf + hkD^2f + hk\,\mathrm{Rm}\Big).
\end{aligned} \tag{2.100}$$

Similar as before, we get the estimate

$$\|\mathrm{Hess}\,\dot{f}'\|_{H^2} \leq C\|\dot{f}' - \bar{\dot{f}}'\|_{H^2} \leq C\|G_g[h,k]\|_{L^2} \leq C\|k\|_{C^{2,\alpha}}\|h\|_{H^2}, \tag{2.101}$$

where the last inequality is obtained as follows: The expression (2.100) for G consists of five terms. The inequality is clear for the fifth term, for the fourth term it follows from (2.92), for the third term from (2.97) and for the second term from (2.92) and (2.97). Finally, from Section 2.2, we know

$$|\dot{\lambda}'| \leq C\|k\|_{C^{2,\alpha}}\|h\|_{H^2}, \tag{2.102}$$

and this yields the inequality for the first term. Indeed, from (2.14) by polarization

$$\begin{aligned}
\tfrac{\partial^2}{\partial s\partial t}|_{(0,0)}\lambda(g + sk + th) &= \langle w, \dot{H}'[h,k]w\rangle \tag{2.103} \\
&\quad + \tfrac{1}{2\pi i}\oint\langle w, \dot{H}[h](\lambda - H)^{-1}H'[k]w\rangle\tfrac{d\lambda}{\lambda - \lambda(g)} \\
&\quad + \tfrac{1}{2\pi i}\oint\langle w, H'[k](\lambda - H)^{-1}\dot{H}[h]w\rangle\tfrac{d\lambda}{\lambda - \lambda(g)}
\end{aligned}$$

and this can be estimated using the same methods as in the proof of Proposition 2.8. All the above estimates are uniform in a $C^{2,\alpha}$-neighborhood. This finishes the proof of the lemma. $\qquad\square$

Proof of Theorem 2.3. We can assume $g \in g_{\mathrm{RF}} + \ker\mathrm{div}_{g_{\mathrm{RF}}}, g = \bar{g} + h, \bar{g} \in \mathcal{P}_{g_{\mathrm{RF}}}, h \in N_{g_{\mathrm{RF}}}$. This reduction is justified using the Ebin-Palais slice theorem and integrability as in the proof of Theorem 2.1. In particular, note that $\varphi^*f_g = f_{\varphi^*g}$ and that the different L^2-norms are uniformly equivalent.
Since

$$\|\mathrm{Rc}_g\|_{L^2} \leq C\|h\|_{H^2}, \tag{2.104}$$

it suffices to show

$$\|\mathrm{Rc}_g + \mathrm{Hess}_g\,f_g\|_{L^2}^2 \geq c\|h\|_{H^2}^2 \tag{2.105}$$

for some $c > 0$. To see this, using Lemma 2.18, note that

$$\begin{aligned}
\|\mathrm{Rc}_g + \mathrm{Hess}_g\,f_g\|_{L^2}^2 &= \tfrac{1}{4}\|\triangle_{g_{\mathrm{RF}}}^L h\|_{L^2}^2 - \langle O_1 + O_2, \triangle_{g_{\mathrm{RF}}}^L h\rangle + \|O_1 + O_2\|_{L^2}^2 \\
&\geq 2c\|h\|_{H^2}^2 - C(\|O_1\|_{L^2} + \|O_2\|_{L^2})\|h\|_{H^2} \tag{2.106}
\end{aligned}$$

for some $c > 0$, since $\triangle_{g_{\mathrm{RF}}}^L|_{N_{g_{\mathrm{RF}}}}$ is injective. Together with Lemma 2.19, this proves (2.105) in a $C^{2,\alpha}$-neighborhood and the theorem follows. $\qquad\square$

Remark. The reverse inequality,

$$\|\text{Rc}_g + \text{Hess}_g\, f_g\|_{L^2(M, e^{-f_g} dV_g)} \leq \|\text{Rc}_g\|_{L^2(M, e^{-f_g} dV_g)} \tag{2.107}$$

follows immediatly from the $L^2(M, e^{-f_g} dV_g)$-orthogonality of $\text{Rc} + \text{Hess}\, f$ and $\text{Hess}\, f$ (see (2.20)).

Proof of Theorem 2.2. We can assume $g \in g_{\text{RF}} + \ker \text{div}_{g_{\text{RF}}}, g = \bar{g} + h, \bar{g} \in \mathcal{P}_{g_{\text{RF}}}, h \in N_{g_{\text{RF}}}$, arguing as in the proof of Theorem 2.3. Then, always working in a small enough $C^{2,\alpha}$-neighborhood,

$$|\lambda(g)| \leq C\|h\|_{H^2}^2. \tag{2.108}$$

This estimate follows from $\lambda(\bar{g}) = 0, D\lambda(\bar{g}) = 0$ and (2.14). Together with (2.105), the theorem follows. $\square$

Remark. To show convergence of a parabolic gradient flow, $\frac{d}{dt} g = \nabla\lambda(g)$, starting near a local maximizer $g_{\max}$ of its energy λ, an inequality of the form $\|\nabla\lambda(g)\|_{L^2} \geq c|\lambda(g) - \lambda(g_{\max})|^{1-\theta}$ for some $\theta \in (0, \frac{1}{2}]$ is sufficient [110]. Let us also remark that from Perelman's evolution inequality $\frac{d\lambda}{dt} \geq \frac{2}{n}\lambda^2$ we only get the inequality for $\theta = 0$.

2.5   Stability and instability under Ricci flow

Let (M^n, g_{RF}) be compact Ricci-flat. Assume all infinitesimal Ricci-flat deformations of g_{RF} are integrable and $\triangle_{g_{\text{RF}}}^L \leq 0$ on TT. Let $k \geq 3$.
By the Theorems 2.1, 2.2, 2.3 and Lemma 2.15, there exist constants $\varepsilon_0 > 0$ and $C_1, C_2 < \infty$ such that for all g with $\|g - g_{\text{RF}}\|_{C_{g_{\text{RF}}}^k} < \varepsilon_0$:

$$\lambda(g) \leq 0 \quad \text{and} \quad \lambda(g) = 0 \Leftrightarrow \text{Rc}_g = 0 \tag{2.109}$$

$$|\lambda(g)|^{1/2} \leq C_1 \|\text{Rc}_g + \text{Hess}_g\, f_g\|_{L_{f_g}^2} \tag{2.110}$$

$$\|\text{Rc}_g\|_{L_g^2} \leq C_2 \|\text{Rc}_g + \text{Hess}_g\, f_g\|_{L_{f_g}^2} \tag{2.111}$$

Here, we define the C^k-norm using g_{RF} and the L_f^2-norm using the metric g and the measure $e^{-f_g} dV_g$.

Lemma 2.20 (Energy controls the distance)
Let (M^n, g_{RF}) and $k, \varepsilon_0, C_1, C_2$ as above. Let $0 \leq t_1 < t_2 < T$ and $g(t)$ a Ricci flow (2.1) with $\|g(t) - g_{\text{RF}}\|_{C_{g_{\text{RF}}}^k} < \varepsilon_0$ for all $t \in [0, T)$. Then

$$\int_{t_1}^{t_2} \|\text{Rc}_{g(t)}\|_{L_{g(t)}^2} dt \leq C_1 C_2 \left(|\lambda(g(t_1))|^{\frac{1}{2}} - |\lambda(g(t_2))|^{\frac{1}{2}}\right). \tag{2.112}$$

Proof. Without loss of generality, we can assume the inequality in (2.109) is strict, i.e. $\lambda(g(t)) < 0$ for all $t \in [0, T)$. By Perelman's monotonicity formula $\lambda = -|\lambda|$ is

increasing along the flow, more precisely,

$$
\begin{aligned}
-\tfrac{d}{dt}|\lambda(g(t))|^{1/2} &= \tfrac{1}{2}|\lambda(g(t))|^{-1/2}\tfrac{d}{dt}\lambda(g(t)) \\
&= |\lambda(g(t))|^{-1/2}\langle \mathrm{Rc}_{g(t)} + \mathrm{Hess}_{g(t)}\, f_{g(t)}, \mathrm{Rc}_{g(t)}\rangle_{L^2_{f_{g(t)}}} \\
&= |\lambda(g(t))|^{-1/2}\|\mathrm{Rc}_{g(t)} + \mathrm{Hess}_{g(t)}\, f_{g(t)}\|^2_{L^2_{f_{g(t)}}} \\
&\geq \tfrac{1}{C_1 C_2}\|\mathrm{Rc}_{g(t)}\|_{L^2_{g(t)}}
\end{aligned}
\tag{2.113}
$$

where we used (2.19), (2.20), (2.110) and (2.111). This proves the lemma. $\qquad\square$

Lemma 2.21 (Estimates for $t \leq 1$)
Let (M^n, g_{RF}) be compact, Ricci-flat, $k \geq 3, \varepsilon > 0$. Then there exists a constant $\delta_1 = \delta_1(M^n, g_{\mathrm{RF}}, \varepsilon, k) > 0$ such that: If $\|g_0 - g_{\mathrm{RF}}\|_{C^{k+2}_{g_{\mathrm{RF}}}} < \delta_1$ then the Ricci flow starting at g_0 exists on $[0,1]$ and satisfies

$$
\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} < \varepsilon \quad \forall t \in [0,1]. \tag{2.114}
$$

Proof. From $\partial_t \mathrm{Rm} = \triangle \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}$ and $\partial_t \mathrm{Rc} = \triangle \mathrm{Rc} + \mathrm{Rm} * \mathrm{Rc}$, we get the evolution inequalities

$$
\partial_t |D^i \mathrm{Rm}|^2 \leq \triangle |D^i \mathrm{Rm}|^2 + \sum_{j=0}^{i} C_{ij} |D^{i-j}\mathrm{Rm}||D^j\mathrm{Rm}||D^i\mathrm{Rm}|, \tag{2.115}
$$

$$
\partial_t |D^i \mathrm{Rc}|^2 \leq \triangle |D^i \mathrm{Rc}|^2 + \sum_{j=0}^{i} C_{ij} |D^{i-j}\mathrm{Rm}||D^j\mathrm{Rc}||D^i\mathrm{Rc}|. \tag{2.116}
$$

From (2.115), by the maximum principle, there exists a $\tilde{K} = \tilde{K}(K, n, k) < \infty$ such that if $g(t)$ is a Ricci flow on [0,T] with $T \leq 1$ and

$$
|\mathrm{Rm}(x,t)| \leq K, \quad |D^i \mathrm{Rm}(x,0)| \leq K, \quad \forall x \in M, t \in [0,T], i \leq k \tag{2.117}
$$

then

$$
|D^i \mathrm{Rm}(x,t)| \leq \tilde{K} \quad \forall x \in M, t \in [0,T], i \leq k. \tag{2.118}
$$

From (2.116), by the maximum principle, for every $\tilde{\varepsilon} > 0$, there exists a $\tilde{\delta} = \tilde{\delta}(\tilde{K}, \tilde{\varepsilon}, n, k) > 0$ such that for $g(t)$ as above:

$$
\begin{aligned}
&|D^i \mathrm{Rc}(x,0)| \leq \tilde{\delta} \quad \forall x \in M, i \leq k \\
\Rightarrow \quad &|D^i \mathrm{Rc}(x,t)| \leq \tilde{\varepsilon} \quad \forall x \in M, t \in [0,T], i \leq k.
\end{aligned}
\tag{2.119}
$$

Finally, as long as the C^k-norms defined via g_{RF} and $g(t)$ differ at most by a factor 2,

$$
\tfrac{d}{dt}\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} \leq \|2\,\mathrm{Rc}_{g(t)}\|_{C^k_{g_{\mathrm{RF}}}} \leq 4 \sum_{i=0}^{k} \sup_{x \in M} |D^i \mathrm{Rc}(x,t)|. \tag{2.120}
$$

Now, we put the above facts together: Without loss of generality, assume $\varepsilon > 0$ is small enough that the C^k-norms defined via g_{RF} and via g with $\|g - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} \leq \varepsilon$ differ at most by a factor 2. Pick some small enough $\bar{\delta} > 0$. Define

$$K := \sup\{|\mathrm{Rm}_g(x)|;\ \|g - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} \leq \varepsilon, x \in M\}$$
$$+ \sup\{|D^i \mathrm{Rm}_g(x)|;\ \|g - g_{\mathrm{RF}}\|_{C^{k+2}_{g_{\mathrm{RF}}}} \leq \bar{\delta}, x \in M, i \leq k\} < \infty. \quad (2.121)$$

Let $\tilde{K} := \tilde{K}(K, n, k), \tilde{\delta} := \tilde{\delta}(\tilde{K}, \frac{\varepsilon}{16(k+1)}, n, k)$ and let $\delta_1 < \bar{\delta}$ be so small that

$$\|g - g_{\mathrm{RF}}\|_{C^{k+2}_{g_{\mathrm{RF}}}} \leq \delta_1 \Rightarrow \sup_{x \in M, i \leq k} |D^i \mathrm{Rc}_g(x)| \leq \tilde{\delta}, \quad \|g - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} \leq \tfrac{\varepsilon}{4}. \quad (2.122)$$

Let $\|g_0 - g_{\mathrm{RF}}\|_{C^{k+2}_{g_{\mathrm{RF}}}} < \delta_1$. Let $T \in (0, \infty]$ be the maximal time such that the Ricci flow starting at g_0 exists on [0,T) and satisfies

$$\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} < \varepsilon \quad \forall t \in [0, T). \quad (2.123)$$

Suppose, towards a contradiction, $T \leq 1$. Then

$$\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} \leq \|g_0 - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} + 4(k+1) \sup_{\substack{x \in M, i \leq k \\ t \in [0,T]}} |D^i \mathrm{Rc}(x, t)| \leq \tfrac{\varepsilon}{2}. \quad (2.124)$$

for all $t \in [0, T]$. This contradicts the maximality in the definition of T and proves the lemma. $\qquad \square$

Lemma 2.22 (Estimates for $t \geq 1$)
Let $(M^n, \bar{g})$ be compact and $\varepsilon > 0$ small enough. Then there exist constants $C_i = C_i(M^n, \bar{g}, \varepsilon, i) < \infty$ such that if $g(t)$ is a Ricci flow with $\|g(t) - \bar{g}\|_{C^2_{\bar{g}}} < \varepsilon$ for all $t \in [0, T)$ then

$$\|\mathrm{Rc}_{g(t)}\|_{C^i_{g(t)}} \leq C_i \|\mathrm{Rc}_{g(t-1/2)}\|_{L^2_{g(t-1/2)}} \quad \forall t \in [1, T). \quad (2.125)$$

Proof. Since ε is small enough, we have uniform curvature bounds and a uniform bound for the Sobolev constant. Thus, from the evolution inequality

$$\partial_t |\mathrm{Rc}|^2 \leq \triangle |\mathrm{Rc}|^2 + CK |\mathrm{Rc}|^2, \quad (2.126)$$

by Moser iteration (see e.g. [132]), there exists $\tilde{K} = \tilde{K}(M, \bar{g}, \varepsilon) < \infty$ such that

$$\sup_{x \in M} |\mathrm{Rc}(x, t)| \leq \tilde{K} \|\mathrm{Rc}_{g(t-1/4)}\|_{L^2_{g(t-1/4)}}. \quad (2.127)$$

Note that usually a spacetime integral appears on the right hand side, however one can get rid of the time integral using

$$\frac{d}{dt} \int_M |\mathrm{Rc}|^2 dV \leq \tilde{C} K \int_M |\mathrm{Rc}|^2 dV. \quad (2.128)$$

From the evolution equation of Rm, by Bando-Bernstein-Shi estimates (see e.g. [65]), one gets uniform bounds for the derivatives of Rm let's say on $[\frac{3}{4}, T]$. Using these bounds in the evolution equation of Rc, again by Bando-Bernstein-Shi type estimates, we get constants $\bar{K}_i = \bar{K}_i(M, \bar{g}, \varepsilon, i) < \infty$ such that

$$\sup_{x \in M, j \leq i} |D^j \operatorname{Rc}(x, t)| \leq \bar{K}_i \sup_{x \in M} |\operatorname{Rc}(x, t - \tfrac{1}{4})| \qquad \forall t \in [1, T). \tag{2.129}$$

This proves the lemma. $\qquad\square$

Let us restate Theorem 2.5 in the following equivalent form.

Theorem (Dynamical stability)
Let (M^n, g_{RF}) be compact, Ricci-flat and $k \geq 3$. Assume that all infinitesimal Ricci-flat deformations of g_{RF} are integrable and $\triangle^L_{g_{\mathrm{RF}}} \leq 0$ on TT.
Then for every $\varepsilon > 0$ there exists a $\delta = \delta(M^n, g_{\mathrm{RF}}, \varepsilon, k) > 0$ such that if $\|g_0 - g_{\mathrm{RF}}\|_{C^{k+2}_{g_{\mathrm{RF}}}} < \delta$, then the Ricci flow starting at g_0 exists on $[0, \infty)$, satisfies $\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} < \varepsilon$ for all $t \in [0, \infty)$ and $g(t) \to g_\infty$ exponentially as $t \to \infty$, with $\operatorname{Rc}_{g_\infty} = 0$ and $\|g_\infty - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} < \varepsilon$.

Proof of Theorem 2.5. Without loss of generality, assume ε is small enough that the previous lemmas apply and that the C^k-norms defined via g and g_{RF} with $\|g - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} < \varepsilon$ differ at most by a factor 2. Let $\delta := \min\{\delta_1, \delta_2\}$, where the constant $\delta_1 = \delta_1(M, g_{\mathrm{RF}}, \frac{\varepsilon}{4}, k) > 0$ is from Lemma 2.21 and $\delta_2 = \delta_2(M, g_{\mathrm{RF}}, \varepsilon, k) > 0$ is such that $\|g_0 - g_{\mathrm{RF}}\|_{C^{k+2}_{g_{\mathrm{RF}}}} < \delta_2$ implies

$$4C_1 C_2 C_k |\lambda(g_0)|^{1/2} \leq \tfrac{\varepsilon}{4} \tag{2.130}$$

where $C_k = C_k(M^n, g_{\mathrm{RF}}, \varepsilon, k)$ is from Lemma 2.22 and C_1, C_2 are from the beginning of Section 2.5. Let $\|g_0 - g_{\mathrm{RF}}\|_{C^{k+2}_{g_{\mathrm{RF}}}} < \delta$ and $T \in (1, \infty]$ be the maximal time such that the Ricci flow starting at g_0 satisfies

$$\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} < \varepsilon \quad \forall t \in [0, T). \tag{2.131}$$

Without loss of generality, assume the inequality $\lambda(g(t)) \leq 0$ is strict for all $t \in [0, T)$. Suppose towards a contradiction $T < \infty$. Then for all $t \in [1, T)$

$$\tfrac{d}{dt} \|g(t) - g(1)\|_{C^k_{g_{\mathrm{RF}}}} \leq 4\|\operatorname{Rc}_{g(t)}\|_{C^k_{g(t)}} \leq 4C_k \|\operatorname{Rc}_{g(t-1/2)}\|_{L^2_{g(t-1/2)}} \tag{2.132}$$

by Lemma 2.22, and thus by Lemma 2.21 and Lemma 2.20

$$\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} \leq \|g(1) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} + 4C_1 C_2 C_k |\lambda(g_0)|^{1/2} \leq \tfrac{\varepsilon}{2} \tag{2.133}$$

for all $t \in [1, T)$. This contradicts the maximality in the definition of T, thus $T = \infty$ and

$$\|g(t) - g_{\mathrm{RF}}\|_{C^k_{g_{\mathrm{RF}}}} < \varepsilon, \qquad t \in [0, \infty) \tag{2.134}$$

$$\int_0^\infty \|\dot{g}(t)\|_{C^k_{g_{\mathrm{RF}}}} dt < \infty. \tag{2.135}$$

Thus $g(t) \to g_\infty$ in $C^k_{g_{\mathrm{RF}}}$ for $t \to \infty$ (since $g(t)$ is a Ricci flow with smooth initial metric, the convergence is in fact smooth). Along the flow, we have

$$- \tfrac{d}{dt}|\lambda| = 2\|\mathrm{Rc} + \mathrm{Hess}\, f\|_{L^2_f}^2 \geq \tfrac{2}{C_1^2}|\lambda| \qquad (2.136)$$

$$\Rightarrow |\lambda(g(t_2))| \leq e^{-2(t_2-t_1)/C_1^2}|\lambda(g(t_1))|. \qquad (2.137)$$

Thus $\lambda(g_\infty) = 0$, $\mathrm{Rc}_{g_\infty} = 0$ and, using in particular Lemma 2.20 and Lemma 2.22, we see that the convergence is exponential (the exponential convergence is a consequence of the optimal Łojasiewicz exponent $\tfrac{1}{2}$). This proves the theorem. $\qquad \square$

Remark. Since the Ricci flow is not strictly parabolic, we mostly worked with the evolution equations of the curvatures. This is the reason for the loss of two derivatives in Theorem 2.5. For the Ricci-DeTurck flow one of course gets optimal regularity. However, when translating back to the Ricci flow, one also loses two derivatives.

Proof of Theorem 2.6. Pick a sequence of metrics $g_i^0 \to g_{\mathrm{RF}}$ in C^∞ with $\lambda(g_i^0) > 0$. Let $\tilde{g}_i(t)$ be the Ricci flows starting at g_i^0. Since $\lambda(g_i^0) > 0$, by Perelman's evolution inequality $\tfrac{d\lambda}{dt} \geq \tfrac{2}{n}\lambda^2$, the flows become singular in finite time. Since $g_i^0 \to g_{\mathrm{RF}}$ in C^∞, the flows exist and stay inside a small ball for longer and longer times. Let $\varepsilon > 0$ be small enough. Let t_i be the first time when $d_{C^\infty}(\tilde{g}_i(t), g_{\mathrm{RF}}) = \varepsilon$. Then $t_i \to \infty$ and, always assuming i is large enough,

$$\tfrac{\varepsilon}{2} \leq d_{C^\infty}(\tilde{g}_i(t_i), \tilde{g}_i(1)) \leq C\lambda(\tilde{g}_i(t_i))^{1/2}, \qquad (2.138)$$

by the Łojasiewicz inequality, the gauge estimate and parabolic estimates. Thus $\lambda(\tilde{g}_i(t_i)) \geq c > 0$, which will be used to exclude trivial solutions.
Shifting time, we obtain a family of Ricci flows $g_i(t) := \tilde{g}_i(t + t_i), t \in [-t_i, T), -t_i \to -\infty, T > 0$ with

$$d_{C^\infty}(g_i(t), g_{\mathrm{RF}}) \leq 2\varepsilon \quad \forall t \in [-t_i, T), \qquad (2.139)$$

$$\lambda(g_i(0)) \geq c > 0, \qquad (2.140)$$

$$g_i(-t_i) = g_i^0 \to g_{\mathrm{RF}} \quad \text{in } C^\infty. \qquad (2.141)$$

From (2.139) and the Ricci flow equation, we have C^∞ space-time bounds. Thus, after passing to a subsequence, g_i converges to an ancient Ricci flow g in $C^\infty_{\mathrm{loc}}(M \times (-\infty, T))$ with $\lambda(g(0)) \geq c > 0$. In particular, this implies that g is nontrivial and becomes singular in finite time. Moreover, $\lambda(g(t)) \geq 0$ for all $t \in (-\infty, T)$. Finally, for $-t_i \leq t$,

$$d_{C^\infty}(g_{\mathrm{RF}}, g(t)) \leq d_{C^\infty}(g_{\mathrm{RF}}, g_i^0) + d_{C^\infty}(g_i(-t_i), g_i(t)) + d_{C^\infty}(g_i(t), g(t))$$

$$\leq d_{C^\infty}(g_{\mathrm{RF}}, g_i^0) + C\lambda(g_i(t))^{1/2} + d_{C^\infty}(g_i(t), g(t)) \qquad (2.142)$$

by the Łojasiewicz inequality, the gauge estimate and parabolic estimates. Since $\lambda(g_i(t))$ is bounded up to time zero and $\tfrac{d\lambda}{dt} \geq \tfrac{2}{n}\lambda^2$, we see that $\lambda(g_i(t))$ is very small for very negative t. Thus $g(t) \to g_{\mathrm{RF}}$ in C^∞ as $t \to -\infty$ and this finishes the proof of the theorem. $\qquad \square$

Remark. In general, Łojasiewicz type inequalities find their truest applications in the nonintegrable case. In fact, the conclusions of Theorem 2.5 and 2.6 hold under the slightly weaker assumption that λ is maximal respectively nonmaximal and g_{RF} satisfies the Łojasiewicz type inequalities

$$\|\mathrm{Rc}_g + \mathrm{Hess}_g\, f_g\|_{L^2} \geq c|\lambda(g)|^{1-\theta_1} \tag{2.143}$$

$$\|\mathrm{Rc}_g + \mathrm{Hess}_g\, f_g\|_{L^2}^{\theta_2} \geq c\|\mathrm{Rc}_g\|_{L^2} \tag{2.144}$$

for $\theta_1 \in (0, \frac{1}{2}], \theta_2 \in (0, 1]$ with

$$2\theta_1 + \theta_2 - \theta_1\theta_2 > 1. \tag{2.145}$$

The estimates (2.143) and (2.144) should be the natural generalizations of Theorem 2.2 and 2.3 to the nonintegrable case. It is an interesting problem to prove them using Lyapunov-Schmidt reduction, the finite-dimensional Łojasiewicz inequalities and the estimates for the error terms developed in this chapter. Note however, that the condition (2.145) is essentially uncheckable, so a new way of dealing with the gauge problem should be found.

A renormalized Perelman-functional and the ADM-mass

In the first part of this short chapter, we define a renormalized $\mathcal{F}$-functional for perturbations of noncompact steady Ricci solitons. This functional motivates a stability inequality which plays an important role in questions concerning the regularity of Ricci-flat spaces and the nonuniqueness of the Ricci flow with conical initial data. In the second part, we define a geometric invariant λ_{AF} for asymptotically flat manifolds with nonnegative scalar curvature. This invariant gives a quantitative lower bound for the ADM-mass from general relativity, motivates a proof of the rigidity statement in the positive mass theorem using the Ricci flow, and eventually leads to the discovery of a mass decreasing flow in dimension three.

3.1   Introduction

The purpose of this short chapter is to introduce some concepts and ideas showing an intriguing relationship between Perelman's energy-functional, the stability of Ricci-flat spaces, and the ADM-mass from general relativity. These ideas will be exploited further in the two subsequent chapters.

To start with, in his famous paper [97], Perelman introduced the energy-functional

$$\mathcal{F}(g, f) = \int_M \left(R + |\nabla f|^2 \right) e^{-f} dV, \tag{3.1}$$

for a metric g and a function f on a closed manifold M. If g evolves by Hamilton's Ricci flow and e^{-f} by the adjoint heat equation, we have the fundamental monotonicity formula

$$\partial_t \mathcal{F} = 2 \int_M |\mathrm{Rc} + \nabla^2 f|^2 e^{-f} dV \geq 0, \tag{3.2}$$

with equality precisely on steady gradient Ricci solitons. Ricci solitons (for a recent survey see [20]) are the fixed points of the Ricci flow up to diffeomorphism and scaling, and play a crucial role in understanding the formation of singularities.

In fact, all steady solitons on a *closed* manifold are Ricci-flat, but there exist nontrivial gradient steadies on *complete* manifolds. A nice example is the rotationally symmetric steady gradient Ricci soliton on $\mathbb{R}^n (n \geq 3)$ discovered by Bryant [17]. The Bryant soliton has positive curvature and looks like a paraboloid, where the spheres of geodesic radius r have diameter of order $\sqrt{r}$. The scalar curvature decays like $\frac{1}{r}$ and the potential f behaves asymptotically like $-r$. From these asymptotics it is immediate that $\mathcal{F} = \infty$ on the Bryant soliton.

Nevertheless, we manage to make sense of the $\mathcal{F}$-functional in the situation where (M, g_s, f_s) is a noncompact steady gradient Ricci soliton (e.g. the Bryant soliton) and g and f are a nearby metric and function on M. The idea is to consider the relative energy $\mathcal{F}^{(g_s, f_s)}(g, f) := \mathcal{F}(g, f) - \mathcal{F}(g_s, f_s)$, where $\infty - \infty$ gives a finite quantity when suitably interpreted. Actually, we find it technically more convenient to give a slightly different definition, and we call the resulting functional the *renormalized energy* $\mathcal{F}^{(g_s, f_s)}$ (Definition 3.1).

It turns out that (g_s, f_s) is a critical point of this functional if the possibly infinite total measure $\int_M e^{-f} dV$ is kept fixed (Theorem 3.2). The second variation at (g_s, f_s) is of fundamental importance (Theorem 3.3). It motivates us to introduce the *stability inequality* for steady gradient Ricci solitons,

$$\int_M \left[-\tfrac{1}{2}|\nabla h|^2 + \mathrm{Rm}(h, h) \right] e^{-f_s} dV \leq 0 \tag{3.3}$$

for all $h \in \ker \mathrm{div}_{f_s}$ with compact support. Here $\mathrm{Rm}(h, h) = R_{ijkl} h_{ik} h_{jl}$ and we restrict to variations h satisfying $\mathrm{div}(e^{-f_s} h) = 0$. Applications of inequality (3.3) appear in Chapter 4. In particular, we will show that many Ricci-flat cones in small dimensions are unstable, and discuss a conjecture of Tom Ilmanen relating the stability of Ricci-flat cones, the existence of positive scalar curvature deformations, and the nonuniqueness of Ricci flow with conical initial data (see also Chapter 2 for ancient Ricci flows coming out of unstable closed Ricci-flat manifolds, and [106] for Ricci flows coming out of positively curved cones).

For asymptotically flat manifolds (M, g_{ij}) with nonnegative scalar curvature, we also define a geometric invariant $\lambda_{AF}(g)$ (Definition 3.4), motivated by Perelman's λ-functional for closed manifolds. This invariant is closely related with the positive mass theorem in general relativity. Recall that the ADM-mass [7, 10] is defined as

$$m_{\mathrm{ADM}}(g) := \lim_{r \to \infty} \int_{S_r} \left(\partial_j g_{ij} - \partial_i g_{jj} \right) dA^i. \tag{3.4}$$

Physical arguments suggest that the mass is always nonnegative. Schoen and Yau proved that this is indeed the case in dimension $n \leq 7$ using minimal surface techniques [104]. Huisken and Ilmanen found a different proof for $n = 3$ based on the inverse mean curvature flow [70]. Witten discovered a proof for spin manifolds of arbitrary dimension using Dirac spinors [130]. Recently, Lohkamp announced a proof of

the positive mass theorem valid in all dimensions and without spin assumption based on singular minimal surface techniques [85].

For the sake of exposition, the following discussion contains some facts well known to experts. Indeed, it is well known that conformal deformations can decrease the mass and that deformations in direction of the Ricci curvature can be used to prove the rigidity statement in the positive mass theorem [104, Thm. 2]. Our main motivation is to explain the relationship with the Ricci flow and Perelman's λ-functional.

This said, it is an interesting problem to find quantitative lower bounds for the ADM-mass in terms of other geometric quantities. In fact, the Penrose inequality gives such a quantitative lower bound in terms of the area of the outermost minimal surface, at least for $n \leq 7$ [70, 14, 15]. Our geometric invariant $\lambda_{\mathrm{AF}}(g)$ also gives a quantitative lower bound for the ADM-mass, i.e.

$$m_{\mathrm{ADM}}(g) \geq \lambda_{\mathrm{AF}}(g), \tag{3.5}$$

at least for spin-manifolds or in dimension $n \leq 7$. For spin-manifolds this directly follows from Witten's formula (Theorem 3.5), and in fact the positive mass theorem combined with a conformal transformation implies a slightly stronger lower bound (Theorem 3.6). The discrepancy comes from the difference between the operator $-4\triangle + R$ and the conformal Laplacian $-\frac{4(n-1)}{n-2}\triangle + R$ and disappears in the limit $n \to \infty$. Our motivation for stating the inequality in the weaker form (3.5) is that this is better adapted to Ricci flow techniques. As an application, we give a Ricci flow proof of the rigidity statement in the positive mass theorem (Theorem 3.7).

Guided by the above ideas, we can revisit the question if there exists a geometric flow that decreases the ADM-mass. Indeed, we discovered such a flow in dimension three, and this flow conjecturally squeezes out all the mass of an asymptotically flat 3-manifold with nonnegative scalar curvature. Our flow is based on conformal rescalings and the Ricci flow with surgery and will be discussed in Chapter 5.

This chapter is organized as follows: Section 3.2 is about $\mathcal{F}^{(g_s, f_s)}$ and the stability inequality. Section 3.3 is about λ_{AF} and the positive mass theorem.

3.2   The renormalized $\mathcal{F}$-functional

Let (M, g_s, f_s) be a *steady gradient Ricci soliton*. This means that (M, g_s) is a smooth, connected, complete Riemannian manifold and $f_s : M \to \mathbb{R}$ is a smooth function such that the following equation holds:

$$\mathrm{Rc}(g_s) + \mathrm{Hess}_{g_s}(f_s) = 0. \tag{3.6}$$

Steady gradient Ricci solitons always have nonnegative scalar curvature and correspond to eternal Ricci flows moving only by a diffeomorphism [137].

Definition 3.1
Let $c_s := R(g_s) + |\nabla f_s|^2_{g_s} \geq 0$ be the central charge *(or auxiliary constant) of the steady soliton. We define the* renormalized energy

$$\mathcal{F}^{(g_s, f_s)}(g, f) := \int_M \left(R(g) + |\nabla f|^2_g - c_s \right) e^{-f} dV_g. \tag{3.7}$$

By construction, this is well defined and finite if $(g - g_s, f - f_s)$ has compact support (or decays sufficiently fast), in particular, $\mathcal{F}^{(g_s, f_s)}(g_s, f_s) = 0$.

The variational structure of compact Ricci-flat metrics has been discussed in [21] and in Chapter 2. We will now carry out a similar discussion for noncompact gradient steady Ricci solitons using our renormalized functional $\mathcal{F}^{(g_s, f_s)}$. Let g be a metric, h a symmetric 2-tensor, and f and l functions on M.

Theorem 3.2
If $(g - g_s, f - f_s)$ and (h, l) have compact support (or decay sufficiently fast), then the first variation of $\mathcal{F}^{(g_s, f_s)}$ at (g, f) is well defined and given by the following formula:

$$\frac{d}{d\varepsilon}|_0 \mathcal{F}^{(g_s, f_s)}(g + \varepsilon h, f + \varepsilon l) \tag{3.8}$$

$$= \int_M \left[-\langle h, \mathrm{Rc} + \nabla^2 f \rangle + \left(\tfrac{1}{2} \mathrm{tr}\, h - l \right) \left(2 \triangle f - |\nabla f|^2 + R - c_s \right) \right] e^{-f} dV.$$

In particular, (g_s, f_s) is a critical point, if the (possibly infinite) total measure $\int_M e^{-f} dV$ is kept fixed in the sense that $\int_M \left(\tfrac{1}{2} \mathrm{tr}\, h - l \right) e^{-f} dV = 0$.

Proof. Equation (3.8) follows from a computation as in [97]. For the second part, note that $\mathrm{Rc}(g_s) + \nabla^2 f_s = 0$ per definition of the soliton (M, g_s, f_s). Moreover, we have the traced soliton equation,

$$R(g_s) + \triangle f_s = 0, \tag{3.9}$$

and the auxiliary equation,

$$R(g_s) + |\nabla f_s|^2 = c_s. \tag{3.10}$$

Putting everything together we obtain

$$\frac{d}{d\varepsilon}|_0 \mathcal{F}^{(g_s, f_s)}(g_s + \varepsilon h, f_s + \varepsilon l) = -2c_s \int_M \left(\tfrac{1}{2} \mathrm{tr}\, h - l \right) e^{-f} dV = 0, \tag{3.11}$$

if the total measure is kept fixed in the sense that $\int_M \left(\tfrac{1}{2} \mathrm{tr}\, h - l \right) e^{-f} dV = 0$. $\qquad \square$

In the next theorem, we will use the notation $\mathrm{div}_f(\cdot) = e^f \mathrm{div}(e^{-f} \cdot)$.

Theorem 3.3

If (h, l) has compact support (or decays sufficiently fast), then:

$$\frac{d^2}{d\varepsilon^2}|_0 \mathcal{F}^{(g_s, f_s)}(g_s + \varepsilon h, f_s + \varepsilon l) \tag{3.12}$$

$$= \int_M \left[-\tfrac{1}{2}|\nabla h|^2 + \mathrm{Rm}(h, h) + |\mathrm{div}_{f_s} h|^2 \right] e^{-f_s} dV$$

$$+ 2 \int_M \left[|\nabla \left(\tfrac{1}{2} \operatorname{tr} h - l \right)|^2 + \left(\tfrac{1}{2} \operatorname{tr} h - l \right) \mathrm{div}_{f_s} \mathrm{div}_{f_s} h - c_s \left(\tfrac{1}{2} \operatorname{tr} h - l \right)^2 \right] e^{-f_s} dV.$$

Proof. We write $(g, f) = (g_s, f_s)$ and $(g_\varepsilon, f_\varepsilon) = (g + \varepsilon h, f + \varepsilon l)$. Similar as in [24, Lemma 2.3], using the soliton equation we obtain

$$\frac{d}{d\varepsilon}|_0 \left(\mathrm{Rc}_{g_\varepsilon} + \mathrm{Hess}_{g_\varepsilon} f_\varepsilon \right) = -\tfrac{1}{2}\triangle_f h - \mathrm{Rm}(h, .) - \mathrm{div}_f^* \mathrm{div}_f h + \nabla^2 (l - \tfrac{1}{2} \operatorname{tr} h), \tag{3.13}$$

where $\triangle_f = \triangle - \nabla f \cdot \nabla$, and div_f^* is the formal $L^2(M, e^{-f} dV)$-adjoint of div_f. Another computation using the soliton equation shows

$$\frac{d}{d\varepsilon}|_0 \left(2\triangle_{g_\varepsilon} f_\varepsilon - |\nabla f|^2_{g_\varepsilon} + R_{g_\varepsilon} \right) = 2\triangle_f (l - \tfrac{1}{2} \operatorname{tr} h) + \mathrm{div}_f \mathrm{div}_f h. \tag{3.14}$$

Using this, Theorem 3.2, the soliton equation, and the auxiliary equation we obtain

$$\frac{d^2}{d\varepsilon^2}|_0 \mathcal{F}^{(g, f)}(g_\varepsilon, f_\varepsilon) \tag{3.15}$$

$$= \int_M \left[\tfrac{1}{2}\langle h, \triangle_f h \rangle + \mathrm{Rm}(h, h) + \langle h, \mathrm{div}_f^* \mathrm{div}_f h \rangle - \langle h, \nabla^2 (l - \tfrac{1}{2} \operatorname{tr} h) \rangle \right] e^{-f} dV$$

$$+ \int_M \left(\tfrac{1}{2} \operatorname{tr} h - l \right) \left[\left(2\triangle_f \left(l - \tfrac{1}{2} \operatorname{tr} h \right) + \mathrm{div}_f \mathrm{div}_f h \right) - 2c_s \left(\tfrac{1}{2} \operatorname{tr} h - l \right) \right] e^{-f} dV,$$

and the claim follows from partial integration. $\square$

Restricting to $h \in \ker \mathrm{div}_{f_s}$ (this corresponds to choosing a slice for the action of the diffeomorphism group) and setting $l = \tfrac{1}{2} \operatorname{tr} h$ (this corresponds to keeping the measure $e^{-f} dV$ fixed), Theorem 3.3 motivates the stability inequality (3.3).

Remark. In particular, the *stability inequality for Ricci-flat manifolds or cones* is

$$\int_M 2 \mathrm{Rm}(h, h) dV \leq \int_M |\nabla h|^2 dV \tag{3.16}$$

for all $h \in \ker \mathrm{div}$ with compact support. As mentioned in the introduction, important consequences of this inequality are discussed in Chapter 4.

3.3 $\quad \lambda_{\mathrm{AF}}$ and the positive mass theorem

For perturbations of $(M, g_s, f_s) = (\mathbb{R}^n, \delta, 0)$ with nonnegative scalar curvature, or more generally for asymptotically flat manifolds with nonnegative scalar curvature we define a functional λ_{AF} as follows:

Definition 3.4
Assume (M^n, g_{ij}) is a complete asymptotically flat manifold of order $\tau > \frac{n-2}{2}$ with nonnegative scalar curvature. We define

$$\lambda_{\mathrm{AF}}(g) := \inf \int_M \left(4|\nabla w|^2 + Rw^2 \right) dV, \tag{3.17}$$

where the infimum is taken over all $w \in C^\infty(M)$ such that $w = 1 + O(r^{-\tau})$ at infinity (here the O notation includes the condition that the derivatives also decay appropriately).

Remark. We could also define renormalized λ-functionals for perturbations of nonflat steadies (M, g_s, f_s). However, in that case we find it technically more convenient to work with $\mathcal{F}^{(g_s, f_s)}$ and to fix the measure $e^{-f} dV$.

Theorem 3.5
Assume (M^n, g_{ij}) is a complete asymptotically flat spin manifold of order $\tau > \frac{n-2}{2}$ with nonnegative scalar curvature. Then

$$m_{\mathrm{ADM}}(g) \geq \lambda_{\mathrm{AF}}(g). \tag{3.18}$$

Proof. By Witten's formula [130, 10, 82] for the mass of spin manifolds,

$$m_{\mathrm{ADM}}(g) = \int_M \left(4|\nabla \psi|^2 + R|\psi|^2 \right) dV, \tag{3.19}$$

where ψ is a Dirac spinor with asymptotically unit norm. Thus, recalling Definiton 3.4, using $w = |\psi|$ as a test function, and using Kato's inequality, the claim immediately follows. $\qquad\square$

Remark. Note that λ_{AF} and m_{ADM} are finite if and only if the scalar curvature is integrable.

Remark. The physical interpretation of Witten's formula (3.19) is that the Dirac spinor with asymptotic boundary conditions is a test field that measures the mass of the gravitational field. Similarly, one of our motivations for Definiton 3.4 is to minimize over all test fields $w = e^{-f/2}$ with the asymptotic boundary conditions coming from the trivial potential $f_s = 0$. Instead of the Dirac field, we use a Klein-Gordon field. This test field interpretation of the mass also motivates (3.5).

As mentioned in the introduction the inequality (3.5) is well adapted to Ricci flow techniques, but not sharp in general. From the refined Kato inequality for Dirac spinors, $|\nabla|\psi|| \leq \sqrt{1 - 1/n}|\nabla\psi|$, one gets a slightly better but still not sharp bound. To find the sharp inequality, let us consider the example of the spatial Schwarzschild metric $g_{ij} = (1 + cr^{2-n})^{4/(n-2)} \delta_{ij}$ on $M = \mathbb{R}^n \setminus \{0\}$, where $c = \frac{m}{4(n-1)|S^{n-1}|}$. Since this manifold has two ends, what we really should do is impose the boundary conditions

$w \to 0$ for $r \to 0$ and $w \to 1$ for $r \to \infty$. The rotationally symmetric solution of $\triangle w = 0$ with the given asymptotic boundary conditions is $w(r) = (1 + cr^{2-n})^{-1}$. Now, a straightforward computation shows that

$$4\tfrac{n-1}{n-2}\int_M |\nabla w|^2 dV = 4(n-1)(n-2)|S^{n-1}|c^2 \int_0^\infty (1 + cr^{2-n})^{-2} r^{1-n} dr = m \quad (3.20)$$

equals the ADM-mass of the end in consideration. This example motivates the following theorem.

Theorem 3.6
Assume (M^n, g_{ij}) is a complete asymptotically flat manifold of order $\tau > \frac{n-2}{2}$ with non-negative scalar curvature. Then there exists a unique positive solution of the equation

$$\left(-\tfrac{4(n-1)}{n-2}\triangle + R\right) w = 0, \quad (3.21)$$

with $w \to 1$ at infinity. If moreover M is spin or $n \leq 7$, then we have the lower bound

$$m_{\mathrm{ADM}}(g) \geq \int_M \left(\tfrac{4(n-1)}{(n-2)}|\nabla w|^2 + Rw^2\right) dV. \quad (3.22)$$

Proof. The proof is closely related with the article by Zhang-Zhang [135]. For the first part, writing $w = 1 + u$, we have to solve

$$\left(-\tfrac{4(n-1)}{n-2}\triangle + R\right) u = -R, \qquad u \to 0 \quad \text{at infinity.} \quad (3.23)$$

Since $R \geq 0$ by assumption, the operator $\left(-\tfrac{4(n-1)}{n-2}\triangle + R\right)$ is positive and thus invertible (viewed as operator between suitable weighted function spaces). In fact, we can solve (3.23) with the estimate $u = O(r^{-\tau})$ at infinity. For the second part, consider the conformal metric $\tilde{g} = w^{4/(n-2)}g$. Note that $R(\tilde{g}) = 0$ and that $(M, \tilde{g})$ is asymptotically flat of order τ. Using the definition of the ADM-mass and the asymptotics $w - 1 = O(r^{-\tau})$ and $g_{ij} - \delta_{ij} = O(r^{-\tau})$ we obtain

$$m_{ADM}(\tilde{g}) = \lim_{r\to\infty} \int_{S_r} w^{\frac{4}{n-2}} \left[(\partial_j g_{ij} - \partial_i g_{jj}) + \tfrac{4}{(n-2)w}(g_{ij}\partial_j w - g_{jj}\partial_i w)\right] dA^i \quad (3.24)$$

$$= \lim_{r\to\infty} \int_{S_r} \left[(\partial_j g_{ij} - \partial_i g_{jj}) + \tfrac{4}{(n-2)}(\delta_{ij}\partial_j w - \delta_{jj}\partial_i w)\right] dA^i \quad (3.25)$$

$$= m_{\mathrm{ADM}}(g) - \tfrac{4(n-1)}{(n-2)}\lim_{r\to\infty} \int_{S_r} \partial_r w\, dA. \quad (3.26)$$

Furthermore, using partial integration and the asymptotics from above we compute

$$\int_M \left(\tfrac{4(n-1)}{(n-2)}|\nabla w|^2 + Rw^2\right) dV = \lim_{r\to\infty} \int_{B_r} \left(\tfrac{4(n-1)}{(n-2)}|\nabla w|^2 + Rw^2\right) dV \quad (3.27)$$

$$= \lim_{r\to\infty} \int_{B_r} \left(\tfrac{4(n-1)}{(n-2)}w(-\triangle w) + Rw^2\right) dV + \lim_{r\to\infty} \int_{S_r} \tfrac{4(n-1)}{(n-2)}w\partial_r w\, dA \quad (3.28)$$

$$= \tfrac{4(n-1)}{(n-2)}\lim_{r\to\infty} \int_{S_r} \partial_r w\, dA, \quad (3.29)$$

where we also used equation (3.21) in the last step. Putting everything together this implies

$$m_{\text{ADM}}(\tilde{g}) = m_{\text{ADM}}(g) - \int_M \left(\tfrac{4(n-1)}{(n-2)} |\nabla w|^2 + R w^2 \right) dV, \tag{3.30}$$

and the claim follows from the positive mass theorem applied to $\tilde{g}$. □

Remark. It is interesting to observe that Kato's inequality for Witten's spinor becomes sharp in L^2 in the limit $n \to \infty$.

Remark. Per definition, the asymptotically flat manifold in Theorem 3.6 has only a single end. In the case of multiple ends, we can solve equation (3.21) with the modified boundary conditions $w \to 1$ at the end E in consideration and $w \to 0$ at the other ends. Then

$$m_{\text{ADM}}^{\text{E}}(g) \geq \int_M \left(\tfrac{4(n-1)}{(n-2)} |\nabla w|^2 + R w^2 \right) dV \tag{3.31}$$

gives a lower bound for the mass of the end E. In particular, this improves Bray's lower bound [14, Thm. 8]

$$m_{\text{ADM}}^{\text{E}}(g) \geq \inf \int_M 8 |\nabla w|^2 dV \qquad (n = 3), \tag{3.32}$$

where the infimum is taken over all smooth functions w with $w \to 1$ at the end E and $w \to 0$ at the other ends. In the one-ended case the lower bound in (3.32) is zero (the constant function $w \equiv 1$ is the minimizer), while the lower bound in (3.31) is strictly positive if the scalar curvature does not vanish identically.

As an application of Theorem 3.6 we give a Ricci flow proof of the following rigidity statement in the positive mass theorem:

Theorem 3.7
Assume (M^n, g_{ij}) is a complete asymptotically flat manifold of order $n - 2$ with non-negative scalar curvature, and that M is spin or $n \leq 7$. Then $m_{\text{ADM}}(g) = 0$ implies that (M, g) is isometric to $\mathbb{R}^n$.

Proof. Suppose towards a contradiction that $\text{Rc} \neq 0$ somewhere. Consider the Ricci flow $g(t)$ starting at g. The flow exists for a short time and preserves $R \geq 0$, the asymptotic flatness, and m_{ADM} [40, 95]. In fact, from the evolution equation

$$\partial_t R = \triangle R + 2 |\text{Rc}|^2 \tag{3.33}$$

we see that R becomes strictly positive. Thus, together with Theorem 3.6 we obtain

$$m_{\text{ADM}}(g) = m_{\text{ADM}}(g(t)) > 0, \tag{3.34}$$

a contradiction. So (M, g) is Ricci-flat, and it is easy to conclude that it is in fact isometric to $\mathbb{R}^n$ (see e.g. [82, Prop. 10.2]). □

Remark. As mentioned in the introduction, the ideas discussed in this section motivate our mass-decreasing flow in dimension three, see Chapter 5.

The stability inequality for Ricci-flat cones

In this chapter, we thoroughly investigate the stability inequality for Ricci-flat cones. Perhaps most importantly, we prove that the Ricci-flat cone over $\mathbb{C}P^2$ is stable, showing that the first stable nonflat Ricci-flat cone occurs in the smallest possible dimension. On the other hand, we prove that many other examples of Ricci-flat cones over 4-manifolds are unstable, and that Ricci-flat cones over products of Einstein manifolds and over Kähler-Einstein manifolds with $h^{1,1} > 1$ are unstable in dimension less than 10. As results of independent interest, our computations indicate that the Page metric and the Chen-LeBrun-Weber metric are unstable Ricci shrinkers. As a final bonus, we give plenty of motivations, and partly confirm a conjecture of Tom Ilmanen relating the λ-functional, the positive mass theorem and the nonuniqueness of Ricci flow with conical initial data.

4.1 Introduction

The purpose of this chapter is to thoroughly investigate the stability of Ricci-flat cones. To start with, we call a Ricci-flat manifold or cone *stable*, if

$$\int_M 2 \operatorname{Rm}(h, h) \leq \int_M |\nabla h|^2 \tag{4.1}$$

for all transverse symmetric 2-tensors h with compact support. In this definition, $\operatorname{Rm}(h,h) = R_{ijkl}h_{ik}h_{jl}$ and transverse means $\nabla_i h_{ij} = 0$.

Historical motivations for this notion of stability come from considering the linearization of the Ricci curvature and from computing the second variation of the Einstein-Hilbert functional restricted to transverse-traceless tensors [11]. A more modern motivation comes from the second variation of Perelman's λ-functional [97]. In the case of closed manifolds this second variation is computed in [21]. In the case of noncompact spaces one can use a suitable variant of Perelman's energy functional as explained in Chapter 3.

Stability plays an important role in gravitational physics, see e.g. [57], and also is of great numerical relevance. Complementing that, our main motivations for the problem

under consideration come from questions concerning the regularity of Ricci-flat spaces, from questions about generic Ricci flow singularities, and from a conjecture of Tom Ilmanen relating the λ-functional, the positive mass theorem and the nonuniqueness of Ricci flow with conical initial data. They are all inspired by analogies and disanalogies between minimal surfaces (respectively the mean curvature flow) and Ricci-flat spaces (respectively the Ricci flow), and will be described now.

Motivation I: Regularity of Ricci-flat spaces

To start with the mentioned analogy, the stability inequality for minimal hypersurfaces,

$$\int_\Sigma \left(|A|^2 + \mathrm{Rc}(\nu, \nu)\right) \psi^2 \leq \int_\Sigma |\nabla \psi|^2, \tag{4.2}$$

has a long and successful history. In particular, it plays a prominent role in the work of Schoen-Yau on the structure of manifolds with positive scalar curvature and in their proof of the positive mass theorem [105, 104]. By Almgren's big regularity theorem the singular set of a general area-minimizing rectifiable current has codimension at least 2 [4]. In the case of *stable* minimal hypersurfaces there is a much better (and in fact much easier to prove) conclusion: the singular set has codimension at least 7 [111]. The number 7 ultimately comes from the Simons cones [112], cones with vanishing mean curvature over a product of spheres. In analogy with the Simons cones, as computed some time ago by Tom Ilmanen [72] (see also [51]), Ricci-flat cones over a product of spheres are unstable in dimension less than 10.

In the general case of Ricci curvature, there is a deep regularity theory developed by Cheeger, Colding and Tian, see [28] for a very nice survey (see also the recent work of Cheeger-Naber [30] for very interesting quantitative results). In particular (focusing on the Ricci-flat case here), they proved that the singular set of noncollapsed Gromov-Hausdorff limits of Ricci-flat manifolds with special holonomy has codimension at least 4 (it is conjectured that this also holds without the special holonomy assumption). Given this general regularity theory, it is an intriguing question whether or not an improved regularity occurs in the case of *stable* Ricci-flat spaces.

Motivation II: Generic Ricci flow singularities

Recently, Colding and Minicozzi proposed a generic mean curvature flow in dimension two [38]. The idea, going back to Huisken, is that very complicated singularities can form in two-dimensional mean curvature flow, but all except the cylinders and spheres are unstable and can be perturbed away. Singularities are modeled on shrinkers, and recently Kapouleas-Kleene-Møller [75] constructed examples of shrinkers with arbitrarily high genus.

Two-dimensional mean curvature flow shares many similarities with four-dimensional Ricci flow. One reason for this is that the integrals $\int |A|^2 d\mu$ and $\int |\mathrm{Rm}|^2 dV$ are scale

invariant in dimension two and four, respectively. These integrals show up naturally in the compactness theorems for shrinkers in [37] and in Chapter 1 and can be estimated using the Gauss-Bonnet formula in the respective dimension. As hinted at in [21], there is hope that actually all but very few singularity models for four-dimensional Ricci flow are unstable (the known stable examples are $S^4, S^3 \times \mathbb{R}, S^2 \times \mathbb{R}^2, \mathbb{C}P^2$ and their quotients, and presumably the blow-down shrinker of Feldman-Ilmanen-Knopf [48] is also stable). This might lead to a theory of generic Ricci flow in dimension four.

Motivation III: Ilmanen's conjecture

In July 2008, Tom Ilmanen conjectured an intriguing relationship for ALE-spaces and Ricci-flat cones between:

1. lambda not a local maximum

2. failure of positive mass

3. nonuniqueness of Ricci flow

He also told us that part of the problem is to find an appropriate definition of Perelman's λ-functional in the noncompact case. We will explain all this later. For the moment, let us just mention that the positive mass theorem [104, 130] does not generalize to ALE-spaces [79], that Ricci flows coming out of cones have been constructed in [48, 49, 106], and that nonuniqueness caused by unstable cones has been observed for various other geometric heat flows [71, 50].

Heuristics about proving instability

To prove instability of a Ricci-flat cone, we have to find an almost parallel test variation h (in the sense that the right hand side of (4.1) is small), such that the left hand side is large. To discuss the left hand side, let us diagonalize $h = \text{diag}(\lambda_1, \ldots, \lambda_n)$ in an orthonormal basis $e_1, \ldots, e_n$ and introduce the following curvature condition: The *sectional curvature matrix* K is the matrix with the entries (no sum!)

$$K_{ij} = \text{Rm}(e_i, e_j, e_i, e_j). \tag{4.3}$$

Observe that, $\text{Rm}(h, h) = \lambda^T K \lambda$, so to make the left hand side large, we have to find a large eigenvalue of the sectional curvature matrix. Note that K is a symmetric matrix with vanishing diagonal. Thus, unless $K = 0$, there exists at least one positive eigenvalue. Taking into account the interaction with the Hardy-inequality, we thus expect that many Ricci-flat cones in small dimensions are unstable.

The results

Obviously, flat implies stable. Next, in analogy with the Simons cones, we consider Ilmanen's example of Ricci-flat cones over products of spheres or more generally Ricci-flat cones over products of Einstein manifolds:

Theorem 4.1
A Ricci-flat cone over a product of Einstein manifolds is unstable in dimension less than 10.

We also prove:

Theorem 4.2
A Ricci-flat cone over a Kähler-Einstein manifold with Hodge number $h^{1,1} > 1$ is unstable in dimension less than 10.

Given Theorem 4.1 and Theorem 4.2, one might guess for a moment that the singular set of stable Ricci-flat spaces has codimension at least 10. However, we manage to show that this naive guess is very wrong. In fact, we prove that the first stable non-flat Ricci-flat cone occurs in the smallest possible dimension (note that every at most 4-dimensional Ricci-flat cone is flat, since every at most 3-dimensional Einstein metric is automatically a space-form):

Theorem 4.3
The Ricci-flat cone over $\mathbb{C}P^2$ is stable.

Note that no weak definition of Ricci curvature is available (see however the recent week definition of Ricci bounded below due to Lott-Villani and Sturm [86, 114]), but the main point of Theorem 4.3 is that the analogy with stable minimal hypersurfaces breaks down already at the level of cones. To rescue the analogy between minimal surfaces and Ricci-flat spaces, one can of course reformulate this last statement, and conclude that Ricci-flat spaces behave more like minimal surfaces of general codimension.

The above theorems answer the question posed in Motivation I. Related to Motivation II, and also to obtain a more detailed understanding of the problem from Motivation I, we examine the case of Ricci-flat cones over 4-manifolds in greater detail: Besides the one over $\mathbb{C}P^2$, the other fundamental examples of Ricci-flat cones over 4-manifolds are the one over $S^2 \times S^2$, the ones over $(\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2})$ $(3 \leq p \leq 8)$ with Tian's Kähler-Einstein metrics [117], the one over $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with the Page metric [96], and the one over $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$ with the Chen-LeBrun-Weber metric [31]. We prove:

Theorem 4.4
The Ricci-flat cones $C(S^2 \times S^2)$, $C(\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2})$ $(3 \leq p \leq 8)$ and $C(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2})$, are all unstable.

For the cone over the Chen-LeBrun-Weber metric we carried out a computation that strongly indicates that it is also unstable. However, there is one particular step in our computation that relies on a numerical estimate for extremal Kähler metrics using the algorithm of Donaldson-Bunch [18]. Therefore, we state the following as a conjecture

and not as a theorem:

Conjecture 4.5
The Ricci-flat cone $C(\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2})$ is unstable.

As results of independent interest, our computations have applications for Ricci shrinkers. A Ricci shrinker is given by a complete Riemannian manifold (M, g) and a smooth function f such that

$$\text{Rc} + \text{Hess } f = \tfrac{1}{2\tau}g, \qquad (4.4)$$

for some $\tau > 0$. Solutions of equation (4.4) correspond to self-similarly shrinking solutions of Hamilton's Ricci flow [63], and they model the formation of singularities in the Ricci flow (see [20] for a recent survey on Ricci solitons). By the formula for the second variation of Perelman's shrinker entropy [97, 21, 24, 62], the stability inequality for shrinkers is

$$\int_M 2 \, \text{Rm}(h, h)e^{-f} \leq \int_M |\nabla h|^2 e^{-f} \qquad (4.5)$$

for all symmetric 2-tensors h with compact support satisfying $\text{div}(e^{-f}h) = 0$ and $\int_M \text{tr } h \, e^{-f} = 0$. In particular, for positive Einstein metrics this is formally the same as (4.1) with the additional requirement that $\int_M \text{tr } h = 0$. Our computations prove:

Theorem 4.6
$\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ *with the Page metric is an unstable Ricci shrinker.*

Numerical evidence for this result was already given by Roberta Young in 1983 [134]. Modulo the estimate for extremal Kähler metrics mentioned above, we can also prove:

Conjecture 4.7
$\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$ *with the Chen-LeBrun-Weber metric is an unstable Ricci shrinker.*

Theorem 4.6 and Conjecture 4.7 are relevant for developing a theory of generic Ricci flow in dimension four (the instability of $S^2 \times S^2$ and $\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2}$ for $3 \leq p \leq 8$ has already been observed in [21]).

Finally, regarding Motivation III we prove a theorem which we informally state as follows:

Theorem 4.8
The implications (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (1) in Ilmanen's conjecture hold. Regarding (1) $\overset{?}{\Rightarrow}$ (3), for some unstable Ricci-flat cones over 4-manifolds there do not exist instantaneously smooth Ricci flows coming out of them, but possibly there do exist many singular solutions.

To answer (1) $\overset{?}{\Rightarrow}$ (3) in full generality, one essentially would have to develop a theory of weak Ricci flow solutions first. In Theorem 4.8 we use a suitable noncompact variant

of Perelman's energy functional introduced in Chapter 3. For the detailed statement and further explanations we refer to Section 4.8.

Finally, it would be very interesting to obtain a better picture about the *dynamical stability* and about the *dynamical instability* of noncompact Ricci-flat spaces under the Ricci flow. On the one hand, one could try to do some explicit computations for Ricci-flat cones with enough symmetry. On the other hand, one could try to generalize the techniques introduced in [107] and in Chapter 1 to the noncompact setting.

This chapter is organized as follows: In Section 4.2 we collect some basic facts about Ricci-flat cones. In Section 4.3 and Section 4.4 we prove Theorem 4.1 and Theorem 4.2 respectively. In Section 4.5 we carefully investigate the important example of the Ricci-flat cone over $\mathbb{C}P^2$ and prove Theorem 4.3. In Section 4.6 we analyse the other fundamental examples of Ricci-flat cones over four manifolds, in particular the ones over the manifolds with the Page metric and the Chen-LeBrun-Weber metric. The first part of the proof of Theorem 4.4 and Theorem 4.6 is in that section, while the second part of the proof is based on estimates for extremal Kähler metrics which we carry out in Section 4.7. In the latter two sections, we also give evidence for Conjecture 4.5 and Conjecture 4.7. Finally, in Section 4.8 we discuss Ilmanen's conjecture and prove Theorem 4.8.

4.2   Basic facts about Ricci-flat cones

Let $(M, g) = C(\Sigma, \gamma) = (\mathbb{R}_+ \times \Sigma, dr^2 + r^2\gamma)$ be the Riemannian cone over a closed $(n-1)$-dimensional manifold (Σ, γ). We write $x^0 = r$, and let $(x^1, ..., x^{n-1})$ be coordinates on Σ. The non-vanishing Christoffel symbols are:

$$\Gamma(g)^k_{ij} = \Gamma(\gamma)^k_{ij} \qquad \Gamma(g)^0_{ij} = -r\gamma_{ij} \qquad \Gamma(g)^k_{i0} = \Gamma(g)^k_{0i} = \tfrac{1}{r}\delta^k_i. \qquad (4.6)$$

Thus, the non-vanishing components of the Riemann-tensor are

$$R(g)_{ijkl} = r^2(R(\gamma)_{ijkl} - \gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk}). \qquad (4.7)$$

In particular, the cone is Ricci-flat if and only if

$$\mathrm{Rc}(\gamma) = (n-2)\gamma, \qquad (4.8)$$

which we will always assume in the following. The basic facts about cones collected here will be used frequently and without further reference in the following parts of this chapter. Also, we will always assume that all variations are supported away from the tip of the cone (however, this assumption could be relaxed by a standard approximation argument).

4.3   Ricci-flat cones over product spaces

Suppose (Σ_1, γ_1) and (Σ_2, γ_2) are two positive Einstein manifolds of dimension n_1 and n_2 respectively. Then the associated cone $(M, g) = C(\Sigma_1 \times \Sigma_2)$ of dimension $n = n_1 + n_2 + 1$ is indeed Ricci-flat if normalize the Einstein metrics to have Einstein constant $n - 2$. We will now prove:

Theorem 4.9
The Ricci-flat cone $C(\Sigma_1 \times \Sigma_2)$ is unstable for $n < 10$.

Proof. Consider the variation (geometrically, this corresponds to making one factor larger and the other factor smaller):

$$h = f(r)r^2 \left(\tfrac{\gamma_1}{n_1} - \tfrac{\gamma_2}{n_2} \right). \tag{4.9}$$

Note that

$$\nabla_0 h_{ij} = \tfrac{f'}{f} h_{ij} \quad \nabla_k h_{0j} = \nabla_k h_{j0} = -\tfrac{1}{r} h_{kj}, \tag{4.10}$$

while the other components vanish. Thus h is transverse-traceless (TT), and

$$|\nabla h|^2 = \left(\tfrac{1}{n_1} + \tfrac{1}{n_2} \right) \left(f'^2 + 2\tfrac{f^2}{r^2} \right). \tag{4.11}$$

Furthermore, using the notation $(\gamma \odot \gamma)_{ijkl} = \gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}$, we compute

$$\begin{aligned} \mathrm{Rm}_g(h, h) &= \tfrac{f^2}{r^2} \left(\mathrm{Rm}_\gamma - \gamma \odot \gamma \right) \left(\tfrac{\gamma_1}{n_1} - \tfrac{\gamma_2}{n_2}, \tfrac{\gamma_1}{n_1} - \tfrac{\gamma_2}{n_2} \right) \\ &= \tfrac{f^2}{r^2} \Big[\tfrac{1}{n_1^2} (\mathrm{Rm}_{\gamma_1} - \gamma_1 \odot \gamma_1)(\gamma_1, \gamma_1) + \tfrac{2}{n_1 n_2} \mathrm{tr}_{\gamma_1}(\gamma_1) \mathrm{tr}_{\gamma_2}(\gamma_2) \\ &\qquad + \tfrac{1}{n_2^2} (\mathrm{Rm}_{\gamma_2} - \gamma_2 \odot \gamma_2)(\gamma_2, \gamma_2) \Big] \\ &= \tfrac{f^2}{r^2} \left[\tfrac{n_2}{n_1} + 2 + \tfrac{n_1}{n_2} \right]. \end{aligned} \tag{4.12}$$

Putting everything together, we obtain

$$|\nabla h|^2 - 2\mathrm{Rm}(h, h) = \left(\tfrac{1}{n_1} + \tfrac{1}{n_2} \right) \left(f'^2 - 2(n-2)\tfrac{f^2}{r^2} \right), \tag{4.13}$$

and thus

$$\begin{aligned} \int_{C(\Sigma_1 \times \Sigma_2)} & \left(|\nabla h|^2 - 2\mathrm{Rm}(h, h) \right) dV \\ &= \left(\tfrac{1}{n_1} + \tfrac{1}{n_2} \right) \mathrm{vol}(\Sigma_1 \times \Sigma_2) \int_0^\infty \left(f'^2 - 2(n-2)\tfrac{f^2}{r^2} \right) r^{n-1} dr. \end{aligned} \tag{4.14}$$

Recall that $C_H = 4/(n-2)^2$ is the optimal constant in the Hardy-inequality

$$\int_0^\infty \tfrac{f^2}{r^2} r^{n-1} dr \le C_H \int_0^\infty f'^2 r^{n-1} dr. \tag{4.15}$$

Thus, the expression in (4.14) can become negative if and only if

$$2(n-2)C_H = \tfrac{8}{n-2} > 1, \tag{4.16}$$

i.e. if and only if $n < 10$. This proves the theorem. $\qquad \square$

4.4 Ricci-flat cones over Kähler-Einstein manifolds

Suppose (Σ, γ) is Kähler-Einstein with Einstein constant $n - 2$ and, and let $(M, g) = C(\Sigma)$ be the corresponding Ricci-flat cone of real dimension n.

Theorem 4.10
If $h^{1,1}(\Sigma) > 1$, then $C(\Sigma)$ is unstable for $n < 10$.

Proof. The assumptions of the theorem imply that there exists a transverse-traceless symmetric 2-tensor $k \neq 0$ on (Σ, γ) with

$$(\triangle + 2\operatorname{Rm}_\gamma)k = 2(n - 2)k. \tag{4.17}$$

Consider the variation (this is quite related to the previous section, and geometrically corresponds to making one $(1, 1)$-cycle larger and another one smaller):

$$h = f(r)r^2 k. \tag{4.18}$$

This variation is TT, and we compute

$$|\nabla h|^2 = \left(f'^2 + 2\tfrac{f^2}{r^2}\right)|k|^2 + \tfrac{f^2}{r^2}|\nabla k|^2 \tag{4.19}$$

$$2\operatorname{Rm}(h, h) = 2\tfrac{f^2}{r^2}\left(\operatorname{Rm}_\gamma(k, k) + |k|^2\right). \tag{4.20}$$

Thus

$$\int_{C(\Sigma)} \left(|\nabla h|^2 - 2\operatorname{Rm}(h, h)\right) dV$$

$$= \int_\Sigma |k|^2 dV_\gamma \int_0^\infty \left(f'^2 - 2(n - 2)\tfrac{f^2}{r^2}\right) r^{n-1} dr, \tag{4.21}$$

and this expression can become negative if and only if $n < 10$. $\qquad\square$

4.5 The Ricci-flat cone over $\mathbb{C}P^2$

In this section we prove the following theorem:

Theorem 4.11
The Ricci-flat cone over $\mathbb{C}P^2$ is stable. In particular, five is the smallest dimension of a stable Ricci-flat cone that is not flat.

For the proof of Theorem 4.11 we will use the following inequality:

Theorem 4.12 (Warner [127])
On $\mathbb{C}P^2$ with the standard metric we have:

$$\int_{\mathbb{C}P^2} \operatorname{tr} k \, dV = 0 \quad \Rightarrow \quad \int_{\mathbb{C}P^2} \left[|\nabla k|^2 - 2\operatorname{Rm}(k, k)\right] dV \geq 0. \tag{4.22}$$

Proof of Theorem 4.11. Write $g = dr^2 + r^2\gamma$ as usual. A general variation h can be expanded as follows:

$$h = A dr^2 + r B_i(dr \otimes dx^i + dx^i \otimes dr) + r^2 C_{ij} dx^i \otimes dx^j. \tag{4.23}$$

We assume that h is transverse, i.e. $\mathrm{div}_g(h) = 0$. With respect to the γ-metric this transversality is expressed by the following equations:

$$0 = r\partial_r A + 4A + \mathrm{div}(B) - \mathrm{tr}(C), \tag{4.24}$$
$$0 = r\partial_r B + 5B + \mathrm{div}(C). \tag{4.25}$$

These relations between A, B and C will be used later. Next, we note that

$$\mathrm{Rm}_g(h, h) = \tfrac{1}{r^2}\left[\mathrm{Rm}_\gamma(C, C) + |C|^2 - \mathrm{tr}(C)^2\right], \tag{4.26}$$

where the right hand side is computed with respect to the metric γ. Furthermore, a somewhat cumbersome computation yields:

$$|\nabla^g h|_g^2 = (\partial_r A)^2 + 2|\partial_r B|^2 + |\partial_r C|^2 \tag{4.27}$$
$$+ \tfrac{1}{r^2}\left[|\nabla A - 2B|^2 + 2|A\gamma + \nabla B - C|^2 + |\nabla_i C_{jk} + \gamma_{ij} B_k + \gamma_{ik} B_j|^2\right].$$

Squaring this out, and using also integration by parts and (4.26) we obtain

$$\int_{C(\mathbb{C}P^2)} \left(|\nabla^g h|_g^2 - 2\,\mathrm{Rm}_g(h, h)\right) dV_g \tag{4.28}$$
$$= \int_0^\infty \int_{\mathbb{C}P^2} \left((\partial_r A)^2 + 2|\partial_r B|^2 + |\partial_r C|^2 + \tfrac{1}{r^2}\left[|\nabla C|^2 - 2\,\mathrm{Rm}_\gamma(C, C) + 2\mathrm{tr}(C)^2 \right.\right.$$
$$+ |\nabla A|^2 + 2|\nabla B|^2 + 8A^2 + 14|B|^2 - 4A\mathrm{tr}(C) + (8 - \alpha)A\mathrm{div}(B)$$
$$\left.\left. + (8 - \beta)\langle B, \mathrm{div}(C)\rangle - \alpha\langle\nabla A, B\rangle - \beta\langle\nabla B, C\rangle\right]\right) dV\, r^4 dr,$$

where α and β are parameters that will be chosen later. Now, let us estimate the quantities in (4.28) in five steps. First, by Kato's and Hardy's inequality we get

$$\int_0^\infty \int_{\mathbb{C}P^2} \left[(\partial_r A)^2 + 2|\partial_r B|^2 + |\partial_r C|^2\right] dV\, r^4 dr$$
$$\geq \int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{r^2}\left[\tfrac{9}{4}A^2 + \tfrac{9}{2}|B|^2 + \tfrac{9}{4}|C|^2\right] dV\, r^4 dr. \tag{4.29}$$

Second, by Theorem 4.12 applied to $C - \left(\frac{\int_{\mathbb{C}P^2} \mathrm{tr}C\, dV}{4\int_{\mathbb{C}P^2} dV}\right)\gamma$ and Hölder's inequality we obtain

$$\int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{r^2}\left[|\nabla C|^2 - 2\,\mathrm{Rm}_\gamma(C, C) + \tfrac{3}{2}\mathrm{tr}(C)^2\right] dV\, r^4 dr \geq 0. \tag{4.30}$$

Third, there is one term that we keep as it stands:

$$\int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{r^2}\left[\tfrac{1}{2}\mathrm{tr}(C)^2 + |\nabla A|^2 + 2|\nabla B|^2 + 8A^2 + 14|B|^2 - 4A\mathrm{tr}(C)\right] dV\, r^4 dr. \tag{4.31}$$

Fourth, using (4.24), (4.25), and integration by parts we obtain

$$\int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{r^2}\big[(8-\alpha)A\mathrm{div}(B) + (8-\beta)\langle B, \mathrm{div}(C)\rangle\big]dV\, r^4 dr \tag{4.32}$$
$$\geq \int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{r^2}\big[-\tfrac{5}{2}(8-\alpha)A^2 - \tfrac{7}{2}(8-\beta)|B|^2 + (8-\alpha)A\mathrm{tr}(C)\big]dV\, r^4 dr.$$

Fifth and finally, using the Cauchy-Schwarz inequality and Young's inequality we get

$$\int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{r^2}\big[-\alpha\langle\nabla A, B\rangle - \beta\langle\nabla B, C\rangle\big]dV\, r^4 dr \tag{4.33}$$
$$\geq \int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{r^2}\big[-|\nabla A|^2 - \tfrac{\alpha^2}{4}|B|^2 - 2|\nabla B|^2 - \tfrac{\beta^2}{8}|C|^2\big]dV\, r^4 dr.$$

Putting everything together, noting also that the sum of the cross-terms in (4.31) and (4.32) can be estimated by Young's inequality,

$$(4-\alpha)A\mathrm{tr}C \geq -\varepsilon|4-\alpha|\mathrm{tr}(C)^2 - \tfrac{1}{4\varepsilon}|4-\alpha|A^2, \tag{4.34}$$

we obtain the estimate

$$\int_{C(\mathbb{C}P^2)} \big(|\nabla^g h|_g^2 - 2\,\mathrm{Rm}_g(h,h)\big)\, dV_g \tag{4.35}$$
$$\geq \int_0^\infty \int_{\mathbb{C}P^2} \tfrac{1}{4r^2}\big[(10\alpha - 39 - |4-\alpha|/\varepsilon)A^2 + (-38 - \alpha^2 + 14\beta)|B|^2$$
$$+ (9 - \beta^2/2)|C|^2 + (2 - 4\varepsilon|4-\alpha|)\mathrm{tr}(C)^2\big]dV\, r^4 dr.$$

Choosing α, β, and ε suitably (e.g. $\alpha = 21/5$, $\beta = 4$, $\varepsilon = 1$ does the job), we conclude that

$$\int_{C(\mathbb{C}P^2)} \big(|\nabla^g h|_g^2 - 2\,\mathrm{Rm}_g(h,h)\big)\, dV_g \geq 0, \tag{4.36}$$

for all transverse variations h with compact support, i.e. the Ricci-flat cone over $\mathbb{C}P^2$ is stable. $\qquad\square$

4.6 Ricci-flat cones over 4-manifolds

In the last section we have seen that the cone over $\mathbb{C}P^2$ is stable. Besides this cone, we have the following fundamental examples of Ricci-flat cones over 4-manifolds:

$$C(S^2 \times S^2),\ C(\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2})_{3 \leq p \leq 8},\ C(\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}),\ C(\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}).$$

The cone over $S^2 \times S^2$ is unstable by Theorem 4.9. Concerning the next family of examples, Tian proved the existence of Kähler-Einstein metrics on $\mathbb{C}P^2 \sharp p\overline{\mathbb{C}P^2}$, that is on the blowup of $\mathbb{C}P^2$ at $3 \leq p \leq 8$ points in general position [117]. The cones over them are unstable by Theorem 4.10. Finally, there exists no Kähler-Einstein metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$, but an Einstein metric conformal to an extremal Kähler metric:

Theorem 4.13 (Page [96], Chen-LeBrun-Weber [31])
There exists a positive Einstein metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$. This Einstein metric g is conformal to an extremal Kähler metric k with positive scalar curvature s_k, in fact $g = s_k^{-2} k$.

We will now investigate the stability of the Ricci-flat cone over the Page metric and the Chen-LeBrun-Weber metric. As a first step we express our stability integrand in terms of the conformally related extremal Kähler-metric, namely we have the following lemma:

Lemma 4.14
For every traceless symmetric 2-tensor h we have the pointwise identity

$$
\begin{aligned}
\left(-|\nabla^g h|_g^2 + 2\operatorname{Rm}_g(h,h)\right) dV_g = \quad\quad\quad\quad (4.37) \\
\left(-|\nabla h|^2 + 2\operatorname{Rm}(h,h) - 4|h|^2|\nabla\log s|^2 - 2\langle\nabla|h|^2,\nabla\log s\rangle - 12|h(\nabla\log s,\cdot)|^2\right. \\
\left. - 4\langle h^2,\nabla^2\log s\rangle - 4\langle\operatorname{div}h^2,\nabla\log s\rangle + 8h(\operatorname{div}h,\nabla\log s)\right)s^2 dV,
\end{aligned}
$$

where the quantities on the right hand side are computed with respect to the extremal Kähler-metric k.

Proof. Since $g = s_k^{-2} k$, by the usual formulas for the conformal transformation of geometric quantities (see e.g. [11, Thm. 1.159]) we obtain

$$
\begin{aligned}
\operatorname{Rm}_g(h,h)s_k^{-2}\tfrac{dV_g}{dV_k} \quad\quad\quad\quad (4.38) \\
= \left(\operatorname{Rm}_k + k \wedge \left(\operatorname{Hess}_k \log s + \nabla\log s \otimes \nabla\log s - \tfrac{1}{2}|\nabla\log s|_k^2 k\right)\right)(h,h) \\
= \operatorname{Rm}(h,h) - 2\langle h^2,\nabla^2\log s\rangle - 2\langle h^2,\nabla\log s \otimes \nabla\log s\rangle + |h|^2|\nabla\log s|^2,
\end{aligned}
$$

where the last line is computed with respect to the metric k. Furthermore

$$
\begin{aligned}
|\nabla^g h|_g^2 s_k^{-2}\tfrac{dV_g}{dV_k} = |\nabla^g h|_k^2 \quad\quad\quad\quad (4.39) \\
= |\nabla_i h_{jk} + 2h_{jk}\nabla_i\log s + h_{ik}\nabla_j\log s \\
\quad + h_{ij}\nabla_k\log s - h_{pk}k_{ij}\nabla_p\log s - h_{pj}k_{ik}\nabla_p\log s|^2 \\
= |\nabla h|^2 + 6|h|^2|\nabla\log s|^2 + 8|h(\nabla\log s,\cdot)|^2 + 4\nabla_i h_{jk}h_{jk}\nabla_i\log s \\
\quad + 4\nabla_i h_{jk}h_{ik}\nabla_j\log s - 4\nabla_j h_{jk}h_{pk}\nabla_p\log s,
\end{aligned}
$$

and the claim follows. $\qquad\square$

Let $\Sigma = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ respectively $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$, and write $\tilde{h} = s^{-2}h$. From now on, we assume in addition $\operatorname{div}_g(h) = 0$, that is $\operatorname{div}(s^{-2}h) = 0$ with respect to the metric k.

Using this, the lemma, and integration by parts we obtain

$$\int_\Sigma \left(-|\nabla^g h|_g^2 + 2\operatorname{Rm}_g(h,h) \right) dV_g \tag{4.40}$$
$$= \int_\Sigma \left(-|\nabla h|^2 + 2\operatorname{Rm}(h,h) + 2|h|^2 \triangle \log s + 12|h(\nabla \log s, \cdot)|^2 \right) s^2 dV$$
$$= \int_\Sigma \left(-|\nabla \tilde{h}|^2 + 2\operatorname{Rm}(\tilde{h},\tilde{h}) + 4|\tilde{h}|^2 \triangle \log s + 8|\tilde{h}|^2 |\nabla \log s|^2 + 12|\tilde{h}(\nabla \log s, \cdot)|^2 \right) s^6 dV.$$

where again the right hand side is computed with respect to the metric k. Since $h^{(1,1)}(\Sigma) > 1$, we can find a traceless test variation $\tilde{h}$ that comes from a harmonic $(1,1)$-form on the Kähler manifold (Σ, k). Then the Bochner formula gives

$$0 = \triangle \tilde{h} + 2\operatorname{Rm}(\tilde{h}, \cdot) - \operatorname{Rc}.\tilde{h} - \tilde{h}.\operatorname{Rc}. \tag{4.41}$$

Furthermore, the conformal transformation law for the Ricci-tensor yields

$$\operatorname{Rc} = \left(3s^{-2} + 2|\nabla \log s|^2 - \triangle \log s \right) k - 2 \left(\nabla^2 \log s + \nabla \log s \otimes \nabla \log s \right). \tag{4.42}$$

Finally, a pointwise computation gives the following formulas:

$$4|\tilde{h}(\nabla \log s, \cdot)|^2 = |\tilde{h}|^2 |\nabla \log s|^2, \tag{4.43}$$
$$4\operatorname{div}(\tilde{h}^2) = \nabla |\tilde{h}|^2. \tag{4.44}$$

Putting everything together we obtain the following theorem (see also [60] for an alternative proof):

Theorem 4.15
If h is a test-variation as above, then:

$$\int_\Sigma \left(-|\nabla^g h|_g^2 + 2\operatorname{Rm}_g(h,h) \right) dV_g = \int_\Sigma \left(6 - \triangle_k s^2 \right) |h|_g^2 dV_g. \tag{4.45}$$

Proof. Putting together the above facts we compute

$$\int_\Sigma \left(-|\nabla^g h|_g^2 + 2\operatorname{Rm}_g(h,h) \right) dV_g \tag{4.46}$$
$$= \int_\Sigma \left(\langle \tilde{h}, \triangle \tilde{h} \rangle + 2\operatorname{Rm}(\tilde{h}, \tilde{h}) + |\tilde{h}|^2 \triangle \log s - 7|\tilde{h}|^2 |\nabla \log s|^2 \right) s^6 dV$$
$$= \int_\Sigma \left(6s^{-2}|\tilde{h}|^2 - |\tilde{h}|^2 \triangle \log s - 4\langle \tilde{h}^2, \nabla^2 \log s \rangle - 4|\tilde{h}|^2 |\nabla \log s|^2 \right) s^6 dV$$
$$= \int_\Sigma \left(6s^{-2}|\tilde{h}|^2 - 2|\tilde{h}|^2 \triangle \log s - 4|\tilde{h}|^2 |\nabla \log s|^2 \right) s^6 dV,$$

and the claim follows. $\qquad \square$

To finish the proof of Theorem 4.6 and Conjecture 4.7 it would be sufficient to prove the pointwise inequality $\triangle s^2 < 6$ for the extremal Kähler metrics conformal to the Page-metric respectively to the Chen-LeBrun-Weber metric (normalized such that the Einstein constant equals 3). To finish the proof of Theorem 4.4 and Conjecture 4.5 an estimate $\triangle s^2 < 15/4$ would be sufficient, as shown by the following lemma.

Lemma 4.16
Let $C(\Sigma, \gamma)$ be a Ricci-flat cone over a 4-manifold and assume there exists a transverse-traceless symmetric 2-tensor h on Σ with

$$\int_\Sigma \left(-|\nabla h|^2 + 2\operatorname{Rm}(h, h) - \tfrac{9}{4}|h|^2 \right) dV > 0. \tag{4.47}$$

Then the Ricci-flat cone $C(\Sigma, \gamma)$ is unstable.

Proof. Consider the variation

$$H = f(r)r^2 h. \tag{4.48}$$

This variation is TT, and we compute

$$|\nabla H|^2 = \left(f'^2 + 2\tfrac{f^2}{r^2} \right) |h|^2 + \tfrac{f^2}{r^2}|\nabla h|^2 \tag{4.49}$$

$$2\operatorname{Rm}(H, H) = 2\tfrac{f^2}{r^2} \left(\operatorname{Rm}_\gamma(h, h) + |h|^2 \right). \tag{4.50}$$

By compactness the assumption of the lemma is also satisfied for some λ strictly greater than $\tfrac{9}{4}$. Thus

$$\int_{C(\Sigma)} \left(|\nabla H|^2 - 2\operatorname{Rm}(H, H) \right) dV < \int_\Sigma |h|^2 dV_\gamma \int_0^\infty \left(f'^2 - \lambda\tfrac{f^2}{r^2} \right) r^{n-1} dr. \tag{4.51}$$

Since the Hardy-constant is $C_H = \tfrac{4}{9}$ for $n = 5$, we can choose f such that this expression becomes negative. $\qquad\square$

4.7 Estimates for extremal Kähler metrics

The purpose of this section is to estimate $\triangle s^2$ for the extremal metric corresponding to the Page metric respectively to the Chen-LeBrun-Weber metric. Some of these estimates also appeared in [60]. We will prove:

Theorem 4.17
Let k be the extremal Kähler metric on $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$, such that $s^{-2}k$ is the Page metric with Einstein-constant equal to 3. Then we have the pointwise estimate

$$\triangle s^2 < 15/4. \tag{4.52}$$

Furthermore, we will give strong numerical evidence for:

Conjecture 4.18
Let k be the extremal Kähler metric on $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$, such that $s^{-2}k$ is the Chen-LeBrun-Weber metric with Einstein-constant equal to 3. Then we have the pointwise estimate

$$\triangle s^2 < 15/4. \tag{4.53}$$

Note that, using the results from the previous sections, Theorem 4.17 finishes the proofs of Theorem 4.4 and Theorem 4.6, and the statement of Conjecture 4.18 implies the statements of Conjecture 4.5 and Conjecture 4.7.

To prove Theorem 4.17 and to give evidence for Conjecture 4.18 we will employ the fact that the extremal metrics on both $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$ are toric-Kähler metrics. The theory of toric-Kähler metrics is well developed and documented [3]. The main aspect we will use is that a toric-Kähler manifold Σ^{2m} (in our case $m = 2$) admits a dense open set Σ° on which the action of the torus is free. Furthermore one may pick a special coordinate system called *symplectic coordinates* such that

$$\Sigma^\circ = P^\circ \times \mathbb{T}^m$$

where P° is the interior of a convex polytope $P \subset \mathbb{R}^m$ known as the *moment polytope*. In these coordinates the metric is encoded by a convex function $u : P^\circ \to \mathbb{R}$ (known as the *symplectic potential*) in the following way:

$$k = u_{ij}dx^i dx^j + u^{ij}d\theta^i d\theta^j.$$

Here the x^i are coordinates on the polytope P, θ^i are coordinates on the torus, u_{ij} is the Euclidean Hessian of u and u^{ij} is the matrix inverse of u_{ij}.

In general, an r-sided polytope (our polytope P is in fact a *Delzant* polytope [42]), can be described by r inequalities $l_i(x) > 0$ where the $l_i(x)$ are affine functions of x. As the scalar curvature is invariant under the torus action it is a function of x only. The equations for an extremal metric in these coordinates can be shown to be equivalent to requiring that the scalar curvature is an affine function of the polytope coordinates, i.e.

$$s(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_i x_i + b.$$

The constants a_i and b can all be worked out *a priori* from the elementary geometry of the polytope. In order to describe how, for each l_i we define a one-form $d\sigma_i$ by requiring,

$$dl_i \wedge d\sigma_i = \pm dx,$$

where the l_i are the functions from above and dx denotes the Euclidean volume form. The one-form $d\sigma_i$ then defines a measure on the edge defined by the l_i. We denote the measure obtained on the whole boundary ∂P in this way by $d\sigma$. Donaldson proves the following integration by parts formula:

Lemma 4.19 (Donaldson [43])
Let (M, k) be a toric-Kähler manifold with polytope P, symplectic potential u and let s_k be the scalar curvature of k. Then for all $f \in C^\infty(P)$

$$\int_P u^{ij} f_{ij} dx = \int_{\partial P} f d\sigma - \int_P s_k f dx, \qquad (4.54)$$

where $d\sigma$ is the measure defined above.

Clearly the left-hand-side of (4.54) vanishes for all affine functions f. Hence if $s_k = a_1 x_1 + ... a_m x_m + b$ then by successively taking $f = x_1$ to $f = x_m$ and $f = 1$ we obtain $m+1$ linear constraints for the $m+1$ unknowns and we can find the a_i and b explicitly. A good reference for this is the survey by Donaldson [44].

Another point to note is that we do not really need to compute two derivatives of the scalar curvature. If $g = s^{-2}k$ has constant scalar curvature κ, then the conformal transformation law for the scalar curvature in dimension four, gives the equation

$$s^3 + 6s\Delta s - 12|\nabla s|^2 = \kappa. \qquad (4.55)$$

Now as $\Delta s^2 = 2s\Delta s + 2|\nabla s|^2$ we have

$$\Delta s^2 = \tfrac{\kappa}{3} + 6|\nabla s|^2 - \frac{s^3}{3}. \qquad (4.56)$$

Finally, a word about scaling. The extremal Kähler metrics below will show up with some specific normalization. Let $K := \sup \triangle s^2$ in this normalization. A rescaling of the Kähler metric $k \to c^2 k$ gives a rescaling $s \to c^{-2}s$ of the scalar curvature and hence a rescaling $g \to c^6 g$ of the Einstein metric. Now

$$\mathrm{Rc}(c^6 g) = \mathrm{Rc}(g) = \tfrac{\kappa}{4}g = \frac{\kappa}{4c^6}c^6 g,$$

so we rescale the Kähler metric by c^2 where $c = \left(\tfrac{\kappa}{12}\right)^{1/6}$ to ensure that the Einstein constant is 3. Since $\triangle s^2 \to c^{-6}\triangle s^2$, for this correctly normalized metric we have

$$\sup \triangle s^2 = \tfrac{12}{\kappa}K, \qquad (4.57)$$

i.e. what we want to show is the inequality $\tfrac{12}{\kappa}K < \tfrac{15}{4}$.

The Page Metric

The moment polytope of the manifold $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$ is a trapezium T. We shall parameterise it as the set of points $x = (x_1, x_2) \in \mathbb{R}^2$ satisfying the inequalities $l_i(x) > 0$ where:

$$l_1(x) = x_1, l_2(x) = x_2, l_3(x) = 1 - x_1 - x_2 \text{ and } l_4(x) = x_1 + x_2 - a.$$

Here a is a parameter that effectively determines the Kähler class by varying the volume of the exceptional divisor. Abreu [2] following Calabi [19] gave an explicit description of the symplectic potential u of the extremal metric k. It has the form:

$$u(x_1, x_2) = \sum_i l_i(x) \log(l_i(x)) + f(x_1 + x_2)$$

where l_i are the lines defining the trapezium and the function f satisfies

$$f''(t) = \frac{2a(1-a)}{2at^2 + (1 + 2a - a^2)t + 2a^2} - \frac{1}{t}.$$

The scalar curvature is then an affine function of $x_1 + x_2$ and is given by $s(x_1, x_2) = c_1(x_1 + x_2) + c_2$ where

$$c_1 = \frac{24a}{(1-a)(1+4a+a^2)} \quad \text{and} \quad c_2 = \frac{6(1 - 3a^2)}{(1-a)(1+4a+a^2)}.$$

A computation aided by Mathematica (we emphasize that this and all the following computations involve just polynomials and in principle could be carried out to arbitrary precision without computer aid) shows that the scalar curvature κ of the Page metric $g = s^{-2}k$, that is $\kappa = s^3 + 6s\Delta s - 12|\nabla s|^2$, equals

$$\frac{864a^2(1 - 6a^2 - 16a^3 + 9a^4) + 216(3a^6 - 24a^5 + 53a^4 + 32a^3 - 15a^2 - 8a - 1)(x_1 + x_2)}{(a-1)^3(1+4a+a^2)^3(x_1 + x_2)}.$$

Hence the extremal metric occurs in the class where $0 < a < 1$ and

$$1 - 6a^2 - 16a^3 + 9a^4 = 0.$$

This value is $a \approx 0.31408$. Plugging this in yields the scalar curvature of the Page metric $\kappa \approx 182.219$.

As an aside, LeBrun [80] calculates that the critical Kähler class is the one for which the area of a projective line is 3.1839 times the area of the exceptional divisor. In our toric description the area of a projective line has been scaled to be 1 and the area of the exceptional divisor is a, hence $a \approx (3.1839)^{-1} \approx 0.31408$. Another neat verification one can do is to compute the volume of the Page metric using the toric description. This is given by

$$(4\pi)^2 \int \int_T (c_1(x_1 + x_2) + c_2)^{-4} dx_1 dx_2 \approx 0.072699.$$

The factor $(4\pi)^2$ appears above as the S^1 factors have volume 4π in this description. In Page's original paper [96] he gives the volume V of the metric as $V = 150.862\Lambda^{-2}$ where Λ is the Einstein constant. Hence in our case $\Lambda \approx 45.554$ and so $\kappa \approx 182.219$.

Getting back to the original problem, a calculation (aided by Mathematica) shows that

$$\triangle s^2 = \frac{\kappa}{3} + 6|\nabla s|^2 - \frac{s^3}{3} = \frac{\kappa}{3} + \frac{1}{(a-1)^3(1+4a+a^2)^3 t} \sum_{i=0}^{4} \alpha_i t^i, \qquad (4.58)$$

with $t = x_1 + x_2$ and the coefficients α_i given by:

$$\alpha_0 = 6912a^5$$
$$\alpha_1 = 72 - 648a^2 + 3456a^3 + 1944a^4 - 10368a^5 - 1944a^6$$
$$\alpha_2 = 864a - 3456a^2 - 15552a^3 + 10368a^4 + 11232a^5$$
$$\alpha_3 = 6912a^2 - 20736a^5$$
$$\alpha_4 = 11520a^3.$$

Denote by K the supremum of $\triangle s^2$ over T. It can be checked that $\frac{12}{\kappa}K < \frac{15}{4}$. In fact, a calculation (aided by Mathematica) shows that $\frac{12}{\kappa}K < 2.65$. This proves Theorem 4.17.

The Chen-LeBrun-Weber metric

The moment polytope for $\Sigma = \mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P}^2$ is a pentagon P with vertices at $(0,0)$, $(a,0)$, $(a,1)$, $(1,a)$ and $(0,a)$. If we view $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P}^2$ as $(\mathbb{C}P^1 \times \mathbb{C}P^1)\sharp\overline{\mathbb{C}P}^2$, then a effectively determines the volume of the exceptional divisor. Chen, LeBrun and Weber calculate that their extremal metric k occurs for $a \approx 1.958$ [31]. In order to calculate the scalar curvature using Lemma 4.19 we compute the integrals

$$A = \int_P x_1^2 dx = \int_P x_2^2 dx = \frac{1}{12}(a^4 + 4a^3 - 1),$$

$$B = \int_P x_1 x_2 dx = \frac{1}{24}(a^4 + 4a^3 + 6a^2 - 4a - 1),$$

$$C = \int_P x_1 dx = \int_P x_2 dx = \frac{1}{6}(a^3 + 3a^2 - 1),$$

$$D = \int_P dx = \frac{1}{2}(a^2 + 2a - 1),$$

and

$$E_0 = \int_{\partial P} d\sigma = 1 + 3a, \qquad E_1 = \int_{\partial P} x_1 d\sigma = \int_{\partial P} x_2 d\sigma = a^2 + a.$$

Hence, if $s_k = a_1 x_1 + a_2 x_2 + b$ then we can find a_i and b by solving

$$\begin{pmatrix} A & B & C \\ B & A & C \\ C & C & D \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b \end{pmatrix} = \begin{pmatrix} E_1 \\ E_1 \\ E_0 \end{pmatrix}.$$

Solving this system the value of $a = 1.958$ yields the scalar curvature

$$s_k(x_1, x_2) = -0.423(x_1 + x_2) + 2.790. \tag{4.59}$$

Let $g = s_k^{-2}k$ be the Chen-LeBrun-Weber metric. We will now determine the normalization. Since $\mathrm{Rc}(g) = \Lambda g$, the Gauss-Bonnet formula gives

$$\chi(\Sigma) = \frac{1}{8\pi^2} \int_\Sigma \left(|W_g|^2 - \frac{|\mathrm{Rc}_g^\circ|}{2} + \frac{R_g^2}{24} \right) dV_g = \frac{1}{8\pi^2} \int_\Sigma |W_g|^2 dV_g + \frac{\Lambda^2}{12\pi^2} \mathrm{Vol}_g(\Sigma).$$

The integral $\int_\Sigma |W_g|^2 dV_g$ is conformally invariant, and thus can be computed with respect to the Kähler metric k. The Hirzebruch signature formula reads

$$\sigma(\Sigma) = \frac{1}{12\pi^2} \int_\Sigma \left(|W_+|^2 - |W_-|^2 \right) dV,$$

and it is a standard fact for Kähler surfaces that $|W_+|^2 = s^2/24$. Putting things together we obtain

$$\int_\Sigma |W|^2 dV = 2\int_\Sigma |W_+|^2 dV - 12\pi^2\sigma(\Sigma) = \frac{1}{12} \int_\Sigma s^2 dV - 12\pi^2\sigma(\Sigma),$$

and conclude that
$$\Lambda^2 = \frac{96\pi^2\chi(\Sigma) + 144\pi^2\sigma(\Sigma) - \int_\Sigma s^2 dV}{8\mathrm{Vol}_g(\Sigma)}.$$

As $\mathrm{Vol}_g(\Sigma) = 16\pi^2 \int_P s(x)^{-4} dx$ we can calculate $\Lambda \approx 1.886$, $\kappa \approx 7.54$.

Our task is now to estimate $K := \sup \triangle s^2$. To do this, we will use Donaldson's method of numerically approximating extremal metrics by 'balanced' metrics [45], implemented by Bunch and Donaldson in [18]. Donaldson's algorithm only works for rational Kähler classes (and is computationally unfeasible for rational numbers with large denominators). It is a wonderful serendipity that the value $a \approx 1.958$ is actually very close to being integral and determining an integral Kähler class (unlike for the Page metric). The Kähler class corresponding to $a = 2$ is in fact the anticanonical class $c_1(\Sigma)$. Using this approximation, Donaldson's algorithm gives the estimate $K < 1.363$,[1] and thus $\frac{12}{\kappa}K < 2.17$. This gives evidence for Conjecture 4.18.

4.8 Ilmanen's conjecture

Let us first explain the relevant background about Perelman's energy functional, and our new variant for the noncompact case: As Ilmanen pointed out, the functional

$$\lambda(g) = \inf_{w: \int w^2 = 1} \int (4|\nabla w|^2 + Rw^2) dV \tag{4.60}$$

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[1]The C++ files are available on http://www.buckingham.ac.uk/directory/dr-stuart-hall/

does not detect counterexamples to the positive mass theorem for ALE-spaces, i.e. asymptotically locally euclidean metrics with nonnegative scalar curvature and negative mass. Here, the mass of an ALE-space of order $\tau > (n-2)/2$ is defined as

$$m(g) = \lim_{r \to \infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) \, dA^i. \tag{4.61}$$

Minimizing sequences w_i for λ escape to infinity, giving $\lambda(g) = 0$. The solution is to consider instead a noncompact variant of Perelman's energy functional that we introduced in Chapter 3. For ALE-spaces, Ricci-flat cones or manifolds asymptotic to Ricci-flat cones it takes the form

$$\lambda_{\mathrm{nc}}(g) = \inf_{w:\, w \to 1} \int (4|\nabla w|^2 + R w^2) dV, \tag{4.62}$$

where the infimum is now taken over all smooth functions w approaching 1 at infinity in the sense that $w - 1 = O(r^{-\tau})$ (here, the O-notation includes the condition that the derivatives decay appropriately), where $\tau > (n-2)/2$. In particular, $\lambda_{\mathrm{nc}}(g)$ is strictly positive if the scalar curvature is nonnegative and positive at some point.

We can now formulate precisely and prove the first part of Ilmanen's conjecture:

Theorem 4.20
Let M^n $(n \geq 3)$ be a complete manifold with one end, and assume the end is diffeomorphic to $S^{n-1}/\Gamma \times \mathbb{R}$. Then the following are equivalent:

1. *(lambda not a local maximum) There exists a metric g on M that agrees with the flat conical metric outside a compact set such that $\lambda_{nc}(g) > 0$.*

2. *(failure of positive mass) There exists an asymptotically locally euclidean metric g on M such that $R_g \geq 0$ and $m(g) < 0$.*

Proof. (1) $\Rightarrow$ (2): Let g be a metric on M hat agrees with the flat conical metric outside a compact set such that $\lambda_{\mathrm{nc}}(g) > 0$. Note that

$$\inf_{w:\, w \to 1} \int \left(\tfrac{4(n-1)}{(n-2)} |\nabla w|^2 + R w^2 \right) dV \geq \lambda_{\mathrm{nc}}(g) > 0. \tag{4.63}$$

For each metric g we have an elliptic index-zero operator

$$-\tfrac{4(n-1)}{(n-2)} \triangle_g + R_g : C_\tau^{2,\alpha}(M) \to C_{\tau+2}^{0,\alpha}(M), \tag{4.64}$$

between Hölder-spaces with weight $\tau = n - 2 - \varepsilon$, $\varepsilon > 0$ small. By perturbing the metric a bit we can assume that this operator is invertible. Choosing the perturbation small enough we can also assume that (4.63) still holds. Writing $w = 1 + u$ we can now solve the equation

$$\left(-\tfrac{4(n-1)}{(n-2)} \triangle_g + R_g \right) w = 0, \quad w \to 1 \text{ at } \infty. \tag{4.65}$$

The conformally related metric $\tilde{g} := w^{4/n-2}g$ has vanishing scalar curvature and is asymptotically locally euclidean of order τ. Moreover, by a computation as in Chapter 3, the mass drops down under the conformal rescaling, in fact

$$m(\tilde{g}) = m(g) - \int \left(\tfrac{4(n-1)}{(n-2)}|\nabla w|^2 + Rw^2 \right) dV. \tag{4.66}$$

Using $m(g) = 0$ and the inequality (4.63) this implies $m(\tilde{g}) < 0$.

$(2) \Rightarrow (1)$: Let g be an asymptotically locally euclidean metric on M such that $R_g \geq 0$ and $m(g) < 0$. By an ALE-version of Lohkamp's method [84, Prop. 6.1], we can find a metric $\tilde{g}$ that agrees with the flat conical metric outside a compact set, such that $R_{\tilde{g}} \geq 0$ and $R_{\tilde{g}} > 0$ somewhere. In particular, $\lambda_{\mathrm{nc}}(\tilde{g}) > 0$. $\qquad\square$

To discuss the second part of Ilmanen's conjecture, let (C, g_C) be a Ricci-flat cone (of dimension at least 3) over a closed manifold. In particular, this is a static (nonsmooth) solution of the Ricci flow. We say that an instantaneously smooth Ricci flow is coming out of the cone, if there exists a smooth Ricci flow $(M, g(t))_{t \in (0,T)}$ such that the following conditions are satisfied:

- $(M, g(t))$ is complete with bounded curvature for each $t \in (0, T)$.

- $(M, g(t))$ converges to (C, g_C) for $t \to 0$ in the Gromov-Hausdorff sense everywhere and in the Cheeger-Gromov away from the tip.

- There is no negative L^1-curvature concentration in the tip, in the sense that $\liminf_{t \to 0} \lambda_{\mathrm{nc}}(g(t)) \geq 0$.

Having the static solution and another solution coming out of the cone in particular gives nonuniqueness of the Ricci flow with conical initial data. With the above definition we have:

Theorem 4.21
If there is an instantaneously smooth Ricci flow coming out of (C, g_C) then there exists a complete smooth manifold (M, g) that exponentially approaches (C, g_C) at infinity such that $\lambda_{nc}(g) > 0$. However, there exist unstable Ricci-flat cones with no instantaneously smooth Ricci flow coming out of them.

Proof. Let $(M, g(t))_{t \in (0,T)}$ be the Ricci flow coming out of the cone. Using Perelman's pseudolocality theorem we get a rough decay estimate, and using maximum principle estimates the decay rate can be improved. This is explained in detail in [109], and the conclusion is that $g(t)$ approaches g_C exponentially at infinity. Using this decay estimate, by a similar computation as the one that will be carried out in more detail in Chapter 5, we obtain the monotonicity formula,

$$\frac{d}{dt}\lambda_{\mathrm{nc}}(g(t)) = 2 \int_M |\mathrm{Rc} + \nabla^2 f|^2 e^{-f} dV \geq 0, \tag{4.67}$$

where $w = e^{-f/2}$ is the minimizer in the definition of λ_{nc}. By the definition from above and the monotonicity of λ_{nc} we obtain $\lambda_{\mathrm{nc}}(g(t)) > 0$ for $t > 0$ and $(M, g(t))$ has the desired properties.

For the final part, let $C = C(\Sigma)$ be an unstable Ricci-flat cone over closed 4-manifold Σ that is not the boundary of a smooth compact 5-manifold. From Section 4.6 we know that there are many such examples. Since Σ does not bound, one cannot even find topologically a manifold M smoothing out the cone. Thus there is no smooth Ricci flow coming out of the cone. $\qquad\square$

In full generality, it is an interesting open problem if there exist singular Ricci flows coming out of unstable Ricci-flat cones. To answer it, one has to come up with a good notion of singular Ricci flow solutions first.

A mass-decreasing flow in dimension three

In this final chapter, we introduce a mass-decreasing flow for asymptotically flat three-manifolds with nonnegative scalar curvature. This flow is defined by iterating a suitable Ricci flow with surgery and conformal rescalings and has a number of nice properties. In particular, wormholes pinch off and nontrivial spherical space forms bubble off in finite time. Moreover, a noncompact variant of the Perelman-energy is monotone along the flow. Assuming a certain inequality between the mass and this Perelman-energy a priori, we can prove that the flow squeezes out all the initial mass.

5.1 Introduction

Let (M, g_{ij}) be an asymptotically flat three-manifold with nonnegative integrable scalar curvature. The ADM-mass [7, 10] from general relativity is defined as

$$m(g) := \lim_{r \to \infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) \, dA^i. \tag{5.1}$$

By the positve mass theorem, the mass is always nonnegative and vanishes only for flat space. Beautiful proofs employing a variety of techniques have been discovered [104, 130, 70]. The first one, due to Schoen and Yau, is based on a very nice argument by contradiction using the stability inequality for minimal surfaces. The second one, discovered by Witten, utilizes beautiful identities for Dirac spinors. A third remarkable proof of the positive mass theorem (and in fact of the Penrose inequality), due to Huisken and Ilmanen, is based on the inverse mean curvature flow.

It is of great analytic, geometric, and physical interest to investigate how Hamilton's Ricci flow [63] interacts with the positive mass theorem. This relationship has been studied in [59], [40], [95], and in Chapter 3. In particular, asymptotic flatness and nonnegative integrable scalar curvature are preserved, and the Ricci flow can be used to prove the rigidity statement in the positive mass theorem. However, the mass (although not the quasilocal mass) is constant along the Ricci flow and this leads to Bray's intriguing question whether there exists a geometric flow that decreases the mass. Some hope that such a flow might exist comes from the deep relationship between the mass

and geometric flows in the proofs of the Penrose inequality by Huisken-Ilmanen, Bray, and Bray-Lee [70, 14, 15].

The purpose of this chapter is to confirm this hope, and to introduce and investigate a mass-decreasing flow in dimension three. We have announced the discovery of this flow in Chapter 3, where we also sketched some concepts and ideas showing an intriguing relationship between the mass, the Perelman-energy, and the stability of Ricci-flat spaces. Motivated by that, our mass-decreasing flow is defined by iterating a suitable Ricci flow with surgery and conformal rescalings. The point is, that conformal rescalings to scalar flat metrics squeeze out of the manifold as much mass as possible. However, unless the manifold is flat, the scalar curvature becomes strictly positive again under the Ricci flow and thus the mass can be decreased even more by another conformal rescaling. This process can be iterated forever.

The idea that conformal transformations can be used to decrease the mass and that deformations in direction of the Ricci curvature can be used to increase the scalar curvature again goes back to the fundamental work of Schoen and Yau [104]. What is more recent, is the precise geometric-analytic understanding of the Ricci flow in dimension three due to the revolutionary work of Perelman [97, 99, 98], see [76, 89, 25, 12] for detailed expositions. In particular, we use Perelman's existence theorem for the Ricci flow with surgery, or more precisely a very nice variant for noncompact manifolds due to Bessières-Besson-Maillot [13].

In Section 5.2, we give the precise definition of our flow. We prove that it exists for all times, and preserves asymptotic flatness and nonnegative integrable scalar curvature (Theorem 5.3). Most importantly, the mass $m(g(t))$ is strictly decreasing along the flow. Since it is also bounded below by zero, it has a nonnegative limit for $t \to \infty$. We conjecture that the flow always squeezes out all the initial mass, i.e. $\lim_{t\to\infty} m(g(t)) = 0$. Support for this conjecture comes from the analysis of the long-time behavior of the mass-decreasing flow, which we carry out in the following two sections.

In Section 5.3, we treat the toplogical aspects of the long-time behavior. We prove that nontrivial topology becomes extinct in finite time, analogous to the extinction theorem for closed manifolds admitting a metric of positive scalar curvature due to Perelman [98] and Colding-Minicozzi [35, 36]. The manifolds in consideration are diffeomorphic to $\mathbb{R}^3$ with finitely many $S^1 \times S^2$ and S^3/Γ pieces attached (see e.g. Corollary 5.5). Thus, the extinction result can be rephrased in more physical words saying that wormholes pinch off and nontrivial spherical space forms bubble off in finite time. In fact, this happens in time at most $T = A_0/4\pi$, where A_0 is the area of the largest outermost minimal two-sphere in the initial manifold (Theorem 5.4).

In Section 5.4, we make partial progress towards understanding the geometric-analytic aspects of the long-time behavior. Following the general principle that monotonicity formulas are a crucial tool, we encounter the problem that Perelman's λ-energy is in

fact identically zero in the case of asymptotically flat manifolds with nonnegative scalar curvature. We overcome this difficulty by using a suitable noncompact variant of the Perelman-energy, an energy-functional λ_{AF} that we introduced in Chapter 3. This energy λ_{AF} is nontrivial in the asymptotically flat setting, and we prove that it satisfies a Perelman-like monotonicity formula (Theorem 5.6). Assuming a certain inequality between the mass and λ_{AF} a priori, we can prove that the mass-decreasing flow indeed squeezes out all the initial mass (Theorem 5.7).

In Section 5.5, we derive the limiting equations that formally arise when our iteration parameter ε is sent to zero.[1] However, we actually prefer to *avoid* to really take the limit $\varepsilon \to 0$, since we want to use our long-time existence result that relies on the theory of Ricci flow with surgery. Nevertheless, taking ε small enough one is still close to the limiting equations and therefore the limiting equations might be useful to analyze the long time behavior in the general case without a priori assumptions.

Finally, in Section 5.6, we collect some open problems and questions.

Remark. After giving a talk at the conference 'Geometric flows in mathematics and physics' at BIRS Banff, I learned that Lars Andersson and Hugh Bray had been thinking about related issues. I greatly thank both of them for sharing with me their proposals, questions and ideas.

5.2 Definition of the flow and long-time existence

We will use the following existence theorem for surgical Ricci flow solutions due to Bessières-Besson-Maillot [13] which relies heavily on the work of Perelman [97, 99].

Theorem 5.1 (Bessières-Besson-Maillot [13, Theorem 5.5])
For every $\iota > 0$, $K < \infty$ and $T < \infty$, there exist $r, \delta, \kappa > 0$ such that for any oriented complete Riemannian 3-manifold (M_0, g_0) with $|Rm| \leq K$ and injectivity radius at least ι, there exists an (r, δ, κ)-surgical Ricci flow solution defined on $[0, T]$ with initial conditions (M_0, g_0).

For full details about surgical Ricci flow, please see [13], but let us give a quick overview here and collect some facts that we will use later.

A surgical Ricci flow solution is a sequence of Ricci flows $(M_i, g_i(t))_{t \in [t_i, t_{i+1}]}$ with $0 = t_0 < t_1 < \ldots$, such that M_{i+1} is obtained from M_i by splitting along embedded two-spheres, gluing in standard caps, and throwing away connected components

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[1]Shortly after the author posted the first version of this chapter on arXiv, Peng Lu, Jie Qing and Yu Zheng posted a very interesting note where they proved short time existence for the nonlocal limiting equation [87]. We have now added Section 5.5 in this new version to clarify the relationship with their paper and also to indicate the possible relevance for the analysis of the long-time behavior.

covered entirely by canonical neighborhoods.

The main difference with Perelman's original surgery procedure is that this one is done *before* the singular time, namely when the supremum of the scalar curvature reaches a certain threshold Θ.

Since the curvature is pinched towards positive and $R \leq \Theta$ by construction, we have uniform curvature bounds along the flow. Moreover, the surgeries are done in such a way that the supremum of the scalar curvature drops by at least a factor $1/2$. This ensures that the surgery times don't accumulate. Also, the infimum of the scalar curvature is nondecreasing.

The most important parameters in the construction are the canonical neighborhood scale r, the noncollapsing parameter κ, the surgery parameter δ, and the threshold Θ. These parameters have the following significance: First, if $R(x,t) \geq r^{-2}$ at some point (x,t) in a surgical solution, then there exists a canonical neighborhood of (x,t). Second, the solution is κ-noncollapsed at scales less than one. Third, the surgeries are performed inside very small δ-necks. Fourth and finally, $R \leq \Theta$ along the flow and the surgeries are done when the scalar curvature reaches the treshold Θ.

Having completed this very quick overview, we can now define a new geometric flow as follows:

Definition 5.2 (mass-decreasing flow)
Let (M, g_0) be an (oriented, smooth, complete, connected) asymptotically flat 3-manifold of order one, with nonnegative integrable scalar curvature, and fix a parameter $\varepsilon > 0$.

- *Let $(M(t), g(t))_{t \in [0,\varepsilon]}$ be the surgical Ricci flow solution of Theorem 5.1 starting at g_0, with all connected components except the one containing the asymptotically flat end thrown away.*

- *As a second step, solve the elliptic equation*

$$\left(-8\triangle_{g(\varepsilon)} + R_{g(\varepsilon)}\right) w_1 = 0, \qquad w_1 \to 1 \quad at \quad \infty, \tag{5.2}$$

and set $g_1 := w_1^4 g(\varepsilon)$.

- *Finally, let $(M(\varepsilon), g_1)$ be the new initial condition and iterate the above procedure. The concatenation 'flow, conformal rescaling, flow, conformal rescaling, ...' gives an evolution $(M(t), g(t))_{t \in [0,\infty)}$ which we call the* mass-decreasing flow.

Essentially, the first part of the definition means that we run a suitable Ricci flow with surgery for one unit of time (throwing away the pieces that bubble off). The second part of the definition means that we conformally rescale to a scalar flat metric. Finally,

this process is iterated forever.

For definiteness, if t is a surgery or rescaling time we denote by $(M(t), g(t))$ the *pre*surgery, *pre*rescaling manifold.

The following theorem shows that the mass-decreasing flow exists for all times and that it has the desired properties.

Theorem 5.3
The mass-decreasing flow exists for all times, and preserves the asymptotic flatness and the nonnegative integrable scalar curvature. The mass is constant in the time intervals $t \in ((k-1)\varepsilon, k\varepsilon)$ and jumps down by

$$\delta m_k = -\int_M (8|\nabla w_k|^2 + Rw_k^2)dV \tag{5.3}$$

at the conformal rescaling times $t_k = k\varepsilon$, where w_k is the solution of

$$\left(-8\triangle_{g(t_k)} + R_{g(t_k)}\right) w_k = 0, \qquad w_k \to 1 \quad at \quad \infty. \tag{5.4}$$

The monotonicity of the mass is strict as long as the metric is nonflat.

Proof. The surgical Ricci flow exists by Theorem 5.1. As we recalled above, there are only finitely many surgeries in finite time intervals and nonnegative scalar curvature is preserved.
The asymptotic flatness, the mass, and the integrable scalar curvature are all preserved along a nonsurgical Ricci flow with bounded curvature, see [40, 95]. Since $R \leq \Theta$ and since the surgeries only occur in regions with high curvature (i.e. in particular inside a compact region), these properties are also preserved along the surgical Ricci flow.
For the conformal rescaling part, writing $w_k = 1 + u_k$, we have to solve

$$\left(-8\triangle_{g(t_k)} + R_{g(t_k)}\right) u_k = -R_{g(t_k)}, \qquad u_k \to 0 \quad at \quad \infty. \tag{5.5}$$

Since $R \geq 0$, the operator $(-8\triangle + R)$ is positive and thus invertible (viewed as operator between suitable weighted function spaces). In fact, we can solve (5.5) with the estimate $u_k = O(r^{-1})$ at infinity. Consider the conformal metric $g_k = w_k^4 g(t_k)$. Note that $(M(t_k), g_k)$ is an asymptotically flat manifold of order one with vanishing scalar curvature. Using the definition of the mass, the asymptotics $w_k - 1 = O(r^{-1})$ and $g_{ij} - \delta_{ij} = O(r^{-1})$, and writing $g_{ij} = g(t_k)_{ij}$ we compute

$$m(g_k) = \lim_{r\to\infty} \int_{S_r} w_k^4 \left[(\partial_j g_{ij} - \partial_i g_{jj}) + \tfrac{4}{w_k}(g_{ij}\partial_j w_k - g_{jj}\partial_i w_k)\right] dA^i \tag{5.6}$$

$$= \lim_{r\to\infty} \int_{S_r} [(\partial_j g_{ij} - \partial_i g_{jj}) + 4(\delta_{ij}\partial_j w_k - \delta_{jj}\partial_i w_k)] dA^i \tag{5.7}$$

$$= m(g(t_k)) - 8 \lim_{r\to\infty} \int_{S_r} \partial_r w_k dA. \tag{5.8}$$

Furthermore, using partial integration and the asymptotics from above we compute

$$\int_M \left(8|\nabla w_k|^2 + Rw_k^2\right) dV = \lim_{r\to\infty} \int_{B_r} \left(8|\nabla w_k|^2 + Rw_k^2\right) dV \tag{5.9}$$

$$= \lim_{r\to\infty} \int_{B_r} \left(8w_k(-\triangle w_k) + Rw_k^2\right) dV + \lim_{r\to\infty} \int_{S_r} 8w_k\partial_r w_k dA \tag{5.10}$$

$$= 8 \lim_{r\to\infty} \int_{S_r} \partial_r w_k dA, \tag{5.11}$$

where we also used equation (5.4) in the last step. Putting everything together this implies

$$\delta m_k = m(g_k) - m(g(t_k)) = -\int_M \left(8|\nabla w_k|^2 + Rw_k^2\right) dV. \tag{5.12}$$

Finally, if g_{k-1} is nonflat, then the scalar curvature becomes strictly positive under the Ricci flow (with surgery), and thus we conclude that $\delta m_k < 0$. $\qquad\square$

Remark. In fact, it is not really necessary to assume that the scalar curvature of the initial metric is integrable. If $\int_M RdV = \infty$ initially, then the mass is infinite initially, but it becomes finite after one conformal rescaling.

5.3   Long-time behavior I

Recall that the Ricci flow with surgery on a closed manifold that admits a metric with positive scalar curvature becomes extinct in finite time [98, 35, 36]. In a similar spirit, along the mass-decreasing flow wormholes pinch off and nontrivial spherical space forms bubble off in finite time.

Theorem 5.4
There exists a $T < \infty$, such that $M(t) \cong \mathbb{R}^3$ for $t > T$. In fact, one can take $T = \frac{A_0}{4\pi}$, where A_0 is the area of the largest outermost minimal two-sphere in (M, g_0).

Proof. The idea is that minimal two-spheres shrink at rate at least 4π. So for $t > T = \frac{A_0}{4\pi}$ the manifold $M(t)$ is diffeomorphic to $\mathbb{R}^3$.
Let us now go through the details. Outermost minimal two-spheres exist by a result of Meeks-Simon-Yau [88]. Instead of the area of the largest outermost two-sphere we actually consider the slightly different function

$$A(t) := \inf_\Omega \sup_{S_i} |S_i|, \tag{5.13}$$

where $\Omega \subset M(t)$ is a manifold with boundary $\partial\Omega = S_1 \cup \ldots \cup S_k$ that contains infinity and has the topology of a large disk with finitely many small disks removed. Along the mass-decreasing flow $A(t)$ satisfies the differential inequality,

$$\tfrac{d}{dt} A \leq -4\pi \tag{5.14}$$

in the sense of the limsup of forward difference quotients, compare with [35, Lemma 2.1]. Indeed, this follows from a nice computation using the Gauss-Codazzi equation and the Gauss-Bonnet formula (note that the computation and the result simplify since $R \geq 0$). The surgeries and the conformal rescalings only help (since $w_k \leq 1$ by the maximum principle). Note that $A(0) \leq A_0$. The result follows. $\square$

Corollary 5.5
The initial manifold had the diffeomorphism type

$$M \cong \mathbb{R}^3 \# S^3/\Gamma_1 \# \ldots \# S^3/\Gamma_k \#(S^1 \times S^2)\# \ldots \#(S^1 \times S^2). \qquad (5.15)$$

Conversely, any such manifold admits an asymptotically flat metric of order one with nonnegative integrable scalar curvature (in fact there exists an asymptotically flat metric on M with vanishing scalar curvature).

Proof. The proof is along the lines of Perelman [99] and Bessières-Besson-Maillot [13]. When flowing from $M(0)$ to $M(t)$, the topology can only change for the following two reasons. First, it can happen that compact components with positive scalar curvature are removed. These components are diffeomorphic to a connected sum of spherical space forms and $S^1 \times S^2$ pieces [99]. Second, the surgery can change the topology by pinching off wormholes. This can be seen by moving from the center of the δ-neck to the left and to the right until arriving in regions with lower curvature, the swept out manifold being diffeomorphic to $S^2 \times \mathbb{R}$. Therefore, the initial manifold has the topology as stated in (5.15).
Conversely, as proved by Schoen-Yau [105] and Gromov-Lawson [56], one can construct a metric of positive scalar curvature on

$$S^3/\Gamma_1 \# \ldots \# S^3/\Gamma_k \#(S^1 \times S^2)\# \ldots \#(S^1 \times S^2). \qquad (5.16)$$

One can then obtain an asymptotically flat metric of order one with vanishing scalar curvature on

$$\mathbb{R}^3 \# S^3/\Gamma_1 \# \ldots \# S^3/\Gamma_k \#(S^1 \times S^2)\# \ldots \#(S^1 \times S^2), \qquad (5.17)$$

by 'stereographic projection' using the Greens-function of the conformal Laplacian, $-8\triangle + R$, see e.g. Lee-Parker [82]. $\square$

Remark. There are two other ways how the assertion of Corollary 5.5 can be proved. The first one is to compactify the initial manifold to a closed manifold with positive scalar curvature and to use Perelman's result for closed manifolds with positive scalar curvature [97, 99]. The second one is to combine the result of Schoen-Yau [105] with a lot of three-manifold topology. One needs the spherical space form conjecture proved by Perelman [97, 99], and also the solution of the surface subgroup conjecture. The proof of the surface subgroup conjecture is obtained by combining Perelman's solution of the geometrization conjecture [97, 99], and the recent work of Kahn-Markovic [74].

5.4   Long-time behavior II

To investigate the geometric-analytic aspects of the long-time behavior we will follow the general principle that monotonicity formulas are a very useful tool. However, when trying to follow this principle one encounters the fundamental problem that Perelman's λ-energy [97] adapted as it stands,

$$\lambda(g) := \inf_{w:\int w^2=1} \int_M \left(4|\nabla w|^2 + Rw^2\right) dV, \tag{5.18}$$

is monotone only in a very trivial way. Namely, minimizing sequences w_i escape to infinity and the value of λ is identically zero along the flow. We overcome this difficulty by considering instead the following variant of Perelman's λ-functional,

$$\lambda_{\mathrm{AF}}(g) := \inf_{w:w\to 1} \int_M \left(4|\nabla w|^2 + Rw^2\right) dV, \tag{5.19}$$

where the infimum is now taken over all $w \in C^\infty(M)$ such that $w = 1 + O(r^{-1})$ at infinity. Unless the scalar curvature vanishes identically, the value of λ_{AF} is strictly positive in our setting of asymptotically flat manifolds with nonnegative scalar curvature.

We have introduced the energy-functional λ_{AF} in Chapter 3, where we also observed that λ_{AF} gives a lower bound for the mass, i.e. we have the inequality

$$m(g) \geq \lambda_{\mathrm{AF}}(g). \tag{5.20}$$

Furthermore, the (renormalized) Perelman-energy also plays an important role in questions concerning the stability of Ricci-flat spaces, see e.g. [21, 107, 102] and Chapter 2, 3 and 4 for more information on this aspect. Finally, the Perelman-energy and its variant $\bar{\lambda} = \sup \lambda V^{2/3}$ numerically characterize the long-time behavior of the Ricci flow with surgery on closed three-manifolds [99].

Having completed this short overview and motivation, we will now prove that λ_{AF} satisfies a Perelman-type monotonicity formula.

Theorem 5.6
Away from the conformal rescaling and surgery times, we have the monotonicity formula

$$\tfrac{d}{dt}\lambda_{\mathrm{AF}}(g(t)) = 2 \int_M |\mathrm{Rc} + \nabla^2 f|^2 e^{-f} dV \geq 0, \tag{5.21}$$

where f is the unique solution of

$$\left(-4\triangle + R\right) e^{-f/2} = 0, \qquad f \to 0 \quad at \quad \infty. \tag{5.22}$$

Remark. To avoid problems with modifying the Ricci flow by a family of diffeomorphisms in the noncompact setting, we will prove our monotonicity formula by a direct computation. This computation is quite different than Perelman's original computation in the compact setting (note however, that the Bianchi identity used below is of course just another manifestation of diffeomorphism invariance).

Proof. Substituting $w = e^{-f/2}$, the definition of λ_{AF} can be rewritten as

$$\lambda_{\mathrm{AF}}(g) = \inf_{f:f \to 0} \int_M \left(R_g + |\nabla f|^2_g \right) e^{-f} dV_g, \tag{5.23}$$

where the infimum is taken over all $f \in C^\infty(M)$ such that $f = O(r^{-1})$ at infinity. By a similar argument as in the proof of Theorem 5.3, there exists a unique minimizer. The time derivative of λ_{AF} along the Ricci flow equals

$$\tfrac{d}{dt}\lambda_{\mathrm{AF}}(g(t)) = \int_M \left[\triangle R + 2|\mathrm{Rc}|^2 + 2\,\mathrm{Rc}(\nabla f, \nabla f) - (R + |\nabla f|^2)R \right] e^{-f} dV, \tag{5.24}$$

where f is the minimizer (at the time in consideration). Here, the first two terms come from the evolution of the scalar curvature, the third term comes from the evolution of the inverse metric, and the last term comes from the evolution of the volume element. The monotonicity formula will now follow from a computation using partial integrations, the Bianchi identity, and the equation (5.22) for the minimizer, which can be rewritten as

$$2\triangle f - |\nabla f|^2 + R = 0. \tag{5.25}$$

The partial integrations can be justified using the decay estimates $g_{ij} - \delta_{ij} = O(r^{-1})$, $f = O(r^{-1})$, and the computation consists of the following pieces. First, we have the Bochner-type identity

$$\int_M |\nabla^2 f|^2 e^{-f} dV =$$
$$\int_M \left[-\langle \nabla f, \nabla \triangle f \rangle - \mathrm{Rc}(\nabla f, \nabla f) + \nabla^2 f(\nabla f, \nabla f) \right] e^{-f} dV. \tag{5.26}$$

Second, using the Bianchi identity we obtain

$$\int_M \langle \mathrm{Rc}, \nabla^2 f \rangle e^{-f} dV = \int_M \left[\mathrm{Rc}(\nabla f, \nabla f) - \tfrac{1}{2}\langle \nabla R, \nabla f \rangle \right] e^{-f} dV. \tag{5.27}$$

Third, using (5.25) we get the pointwise identity

$$\nabla^2 f(\nabla f, \nabla f) - \langle \nabla f, \nabla \triangle f \rangle = \tfrac{1}{2}\langle \nabla f, \nabla R \rangle. \tag{5.28}$$

Fourth, we have the partial integration formula

$$\int_M \triangle R e^{-f} dV = \int_M \langle \nabla R, \nabla f \rangle e^{-f} dV = \int_M (|\nabla f|^2 - \triangle f) R e^{-f} dV. \tag{5.29}$$

Fifth and finally, equation (5.25) can be rewritten as

$$2(|\nabla f|^2 - \triangle f) = R + |\nabla f|^2. \tag{5.30}$$

Putting everything together (starting from the right hand side of (5.21) for convenience), the monotonicity formula follows. $\qquad\square$

Remark. Choosing the surgery parameter δ small enough, λ_{AF} can be made almost monotone at the surgery times (compare with [99, 76]).

Remark. Immediately after the conformal rescaling we have $f = 0$ for the minimizer, and thus

$$\tfrac{d}{dt}|_{t_k+}\lambda_{\text{AF}}(g(t)) = 2\int_M |\text{Rc}|^2 dV. \tag{5.31}$$

Moreover, looking at the expression for a Schwarzschild-end suggests that

$$\int_M |\text{Rc}|^2 dV \sim m^2, \tag{5.32}$$

and thus $\lambda_{\text{AF}} \sim \varepsilon m^2$ after time ε, i.e. we expect that λ_{AF} is proportional to the square of the mass at the time $t_{k+1} = (k+1)\varepsilon$.

Under an a priori assumption, an inequality between λ_{AF} and the mass that is motivated by the above remark and complements the inequality (5.20), we can prove that the flow indeed squeezes out all the initial mass.

Theorem 5.7
Let $(M(t), g(t))_{t\in[0,\infty)}$ be a solution of the mass-decreasing flow and assume a priori there exist a constant $c > 0$, such that $\lambda_{\text{AF}}(g(t_k)) \geq cm(g(t_k))^2$ for all positive integers k. Then there exists a constant $C < \infty$ such that $m(g(t)) \leq C/t$. In particular, the mass-decreasing flow squeezes out all the initial mass, i.e. $\lim_{t\to\infty} m(g(t)) = 0$.

Proof. Using Theorem 5.3, the definition of λ_{AF}, the inequality $4 \leq 8$, and the a priori assumption, a computation gives

$$\begin{aligned}
m(g(t_{k+1})) - m(g(t_k)) &= -\int_M (8|\nabla w_k|^2 + Rw_k^2)dV \\
&\leq -\lambda_{\text{AF}}(g(t_k)) \leq -cm(g(t_k))^2.
\end{aligned} \tag{5.33}$$

This implies that there exists a constant $C < \infty$ such that $m(g(t)) \leq C/t$. Using this and the positive mass theorem, we conclude that the flow indeed squeezes out all the initial mass. $\qquad\square$

5.5   The continuum limit

We now derive the limiting equations that formally arise when the iteration parameter ε is sent to zero. In the following the symbol $\cong$ denotes equality modulo terms of order

ε^2 and higher (assuming curvature bounds a priori, the error terms could be estimated explicitly).

From the evolution equation $\partial_t R = \triangle R + 2|\mathrm{Rc}|^2$ and $R_{g_{k-1}} = 0$ we get

$$R_{g(t_k)} \cong 2\varepsilon |\mathrm{Rc}_{g(t_{k-1})}|^2. \tag{5.34}$$

Then, solving $(-8\triangle_{g(t_k)} + R_{g(t_k)})w_k = 0$ with $w_k \to 1$ at infinity gives

$$w_k \cong 1 + \frac{\varepsilon}{4}\triangle^{-1}_{g(t_k)}|\mathrm{Rc}_{g(t_{k-1})}|^2. \tag{5.35}$$

To first order the metric $g_k = w_k^4 g(t_k)$ equals

$$g_k \cong \left(1 + \varepsilon\triangle^{-1}_{g(t_k)}|\mathrm{Rc}_{g(t_{k-1})}|^2\right) g(t_k), \tag{5.36}$$

and using also $g(t_k) \cong g_{k-1} - 2\varepsilon \, \mathrm{Rc}_{g_{k-1}}$ this becomes

$$g_k \cong g_{k-1} - 2\varepsilon \, \mathrm{Rc}_{g_{k-1}} + \varepsilon\triangle^{-1}_{g_{k-1}}|\mathrm{Rc}_{g_{k-1}}|^2 g_{k-1}, \tag{5.37}$$

where we also approximated the inverse Laplacian and the Ricci curvature dropping terms of higher order. Thus, the limiting evolution equation is

$$\partial_t g = -2\,\mathrm{Rc} + \triangle^{-1}|\mathrm{Rc}|^2 g, \tag{5.38}$$

which is the Ricci flow modified by a nonlocal conformal factor which has the effect of projecting to the space of scalar flat metrics. Short time existence for this nonlocal flow was proved in [87, Thm 1.3].

Furthermore, note that the quantity $m - \lambda_{\mathrm{AF}}$ is monotone at all times (i.e. at the conformal rescaling times and also in between). In the formal limit $\varepsilon \to 0$ the quantity λ_{AF} vanishes identically and our monotonicity formulas boil down to the formula

$$\partial_t m = -2 \int_M |\mathrm{Rc}|^2 dV. \tag{5.39}$$

This monotonicity formula also appear in [87, Thm 1.4].

5.6 Problems and questions

We conclude this chapter with a list of open problems and questions. Some of them came up in discussions with Lars Andersson, Hugh Bray, Gerhard Huisken, and Tom Ilmanen.

- Can one get rid of the a priori assumption relating the mass and the Perelman-energy?

- Can the mass-decreasing flow be used to give an independent proof of the positive mass theorem?

- Is there some clever argument in higher dimensions?

- Is there some relationship with the Penrose inequality?

Let us comment on the first to questions. Assuming the positive mass theorem instead of proving it for the moment, motivated by (5.39) we expect a space-time integral bound

$$\int_0^\infty \int_M |\mathrm{Rc}|^2 dV\, dt \le C \tag{5.40}$$

for the mass-decreasing flow, at least when ε is chosen small enough. Thus the limit for $t \to \infty$ is flat in some integral sense. To answer the first question, one has to improve this into a convergence sufficiently strong to conclude that the mass limits to zero. To answer the second question, one could try to couple this argument with the derivation of (5.40).

Appendix

Proofs of some lemmas from Chapter 1

Proof of Lemma 1.3. From (1.11) and (1.12), we obtain

$$0 \leq |\nabla f|^2 \leq f + C_1, \tag{A.1}$$

i.e. $|\nabla \sqrt{f + C_1}| \leq \frac{1}{2}$ whenever $f + C_1 > 0$. Hence $\sqrt{f + C_1}$ is $\frac{1}{2}$-Lipschitz and thus

$$\sqrt{f(x) + C_1} \leq \tfrac{1}{2}\big(d(x, y) + 2\sqrt{f(y) + C_1}\big), \tag{A.2}$$

for all $x, y \in M$, which will give the upper bound in (1.14). The idea to prove the lower bound is the same as in the theorem of Myers (which would give a diameter bound if the shrinker potential was constant). Consider a minimizing geodesic $\gamma(s)$, $0 \leq s \leq s_0 := d(x, y)$, joining $x = \gamma(0)$ with $y = \gamma(s_0)$. Assume $s_0 > 2$ and let

$$\phi(s) = \begin{cases} s, & s \in [0, 1] \\ 1, & s \in [1, s_0 - 1] \\ s_0 - s, & s \in [s_0 - 1, s_0]. \end{cases}$$

By the second variation formula for the energy of γ,

$$\int_0^{s_0} \phi^2 \operatorname{Rc}(\gamma', \gamma') ds \leq (n - 1) \int_0^{s_0} \phi'^2 ds = 2n - 2,$$

where $\gamma'(s) = \frac{\partial}{\partial s}\gamma(s)$. Note that by the soliton equation (1.8)

$$\operatorname{Rc}(\gamma', \gamma') = \tfrac{1}{2} - \nabla_{\gamma'}\nabla_{\gamma'}f,$$

which implies

$$\begin{aligned}
\tfrac{d(x,y)}{2} + \tfrac{4}{3} - 2n &\leq \int_0^{s_0} \phi^2 \nabla_{\gamma'}\nabla_{\gamma'}f \, ds \\
&= -2\int_0^1 \phi\nabla_{\gamma'}f \, ds + 2\int_{s_0-1}^{s_0} \phi\nabla_{\gamma'}f \, ds \\
&\leq \sup_{s\in[0,1]}|\nabla_{\gamma'}f| + \sup_{s\in[s_0-1,s_0]}|\nabla_{\gamma'}f| \\
&\leq \sqrt{f(x) + C_1} + \tfrac{1}{2} + \sqrt{f(y) + C_1} + \tfrac{1}{2},
\end{aligned} \tag{A.3}$$

where we used (A.1) and the fact that $\sqrt{f + C_1}$ is $\frac{1}{2}$-Lipschitz in the last step. By (A.3), every minimizing sequence is bounded and f attains its infimum at a point p. Since $\triangle f(p) \geq 0$, (1.9) and (1.12) imply

$$0 \leq R(p) \leq \tfrac{n}{2}. \tag{A.4}$$

Using this and $\nabla f(p) = 0$, equation (1.11) implies

$$0 \leq f(p) + C_1 \leq \tfrac{n}{2}. \tag{A.5}$$

Now the quadratic growth estimate (1.14) follows from (A.2), (A.3) and (A.5) by setting $y = p$. Finally, if $d(x, p) > 5n + \sqrt{2n}$, then

$$f(x) + C_1 \geq \tfrac{1}{4}(d(x, p) - 5n)^2 > \tfrac{n}{2} \geq f(p) + C_1,$$

which implies the last statement of the lemma. $\qquad\square$

Proof of Lemma 1.4. Let $\varrho(x) = 2\sqrt{f(x) + C_1}$. This grows linearly, since (1.14) implies

$$d(x, p) - 5n \leq \varrho(x) \leq d(x, p) + 5n. \tag{A.6}$$

Define ϱ-discs by $D(r) := \{x \in M \mid \varrho(x) < r\}$, let $V(r)$ be their volume and

$$S(r) := \int_{D(r)} R \, dV \tag{A.7}$$

their total scalar curvature. Since $\int_{D(r)} \triangle f \, dV = \int_{\partial D(r)} |\nabla f| \, dA \geq 0$, integrating (1.9) gives

$$S(r) \leq \frac{n}{2} V(r), \tag{A.8}$$

i.e. the average scalar curvature is bounded by $\frac{n}{2}$. Moreover, (1.9) and (1.11) imply

$$(r^{-n} V(r))' = 4r^{-(n+2)} S'(r) - 2r^{-(n+1)} S(r),$$

which yields the following estimate by integration

$$V(r) \leq \frac{V(r_0)}{r_0^n} r^n + \frac{4}{r^2} S(r) \tag{A.9}$$

for $r \geq r_0 := \sqrt{2n + 4}$, see [91, Eq. (3)] or [23, Eq. (3.6)] for details. Hence, if $r \geq \sqrt{4n}$, we get by absorption

$$V(r) \leq \frac{2V(r_0)}{r_0^n} r^n.$$

Thus, for every $r \geq 5n$ we obtain

$$\operatorname{Vol} B_r(p) \leq V(r + 5n) \leq V(2r) \leq \tfrac{2^{n+1}}{r_0^n} V(r_0) r^n \leq \tfrac{2^{n+1}}{r_0^n} \operatorname{Vol} B_{r_0 + 5n}(p) r^n.$$

This proves the lemma up to the statement that C_2 depends only on the dimension and that (1.16) also holds for balls with $r < 5n$. To get this, note that $|\nabla f(x)| \leq \frac{1}{2} r_0 + 5n =:$

a for $d(x,p) \leq r_0 + 5n =: R_0$. Now, using the fact that the Bakry-Emery Ricci tensor $\mathrm{Rc}_f = \mathrm{Rc} + \mathrm{Hess}\, f$ of the manifold with density $(M, g, e^{-(f+C_1)}dV)$ is nonnegative by the soliton equation, we obtain, see [129, Thm. 1.2a],

$$\frac{\int_{B_R(p)} e^{-(f+C_1)}dV}{\int_{B_\varepsilon(p)} e^{-(f+C_1)}dV} \leq e^{aR}\frac{R^n}{\varepsilon^n}. \tag{A.10}$$

for $0 < \varepsilon < R \leq R_0$. Since $|f + C_1| \leq a^2$ on $B_{R_0}(p)$, this implies

$$\mathrm{Vol}\, B_R(p) \leq e^{2a^2+aR}\frac{R^n}{\varepsilon^n}\,\mathrm{Vol}\, B_\varepsilon(p) \tag{A.11}$$

and by sending ε to zero the claim follows. $\qquad\square$

Proof of Lemma 1.5. Suppose towards a contradiction that there exist a sequence of gradient shrinkers (M_i, g_i, f_i) with $\mu(g_i) \geq \underline{\mu}$ and some balls $B_{\delta_i}(x_i) \subset B_r(p_i)$ with $\delta_i^{-n}\,\mathrm{Vol}\, B_{\delta_i}(x_i) \to 0$. We will not directly use $B_{\delta_i}(x_i)$ but consider the sequence of unit balls $B_1(x_i) \subset B_{r+1}(p_i)$ instead, which allows to work with the shrinker entropy as defined above rather than with a version that explicitly involves a scaling or time parameter τ as it is necessary for the argument in [76]. Set $a := \frac{1}{2}(r + 1 + 5n)$, then $|\nabla f_i| \leq a$, $|f_i + C_1(g_i)| \leq a^2$ and $R_{g_i} \leq a^2$ on $B_{r+1}(p_i)$. The volume comparison theorem for the Bakry-Emery Ricci tensor implies, as in (A.11),

$$\mathrm{Vol}\, B_1(x_i) \leq e^{2a^2+a}\delta_i^{-n}\,\mathrm{Vol}\, B_{\delta_i}(x_i) \to 0 \tag{A.12}$$

for $i \to \infty$, as well as

$$\mathrm{Vol}\, B_1(x_i) \leq 2^n e^{2a^2+a}\,\mathrm{Vol}\, B_{1/2}(x_i), \quad \forall i \in \mathbb{N}. \tag{A.13}$$

Define the test functions $\tilde{u}_i = c_i^{-1/2}\eta_i$ with $\eta_i(x) = \eta(d(x, x_i))$ for a cutoff function η as in Section 1.2 and with $\int_{M_i} \tilde{u}_i^2 dV = (4\pi)^{n/2}$, i.e.

$$c_i = (4\pi)^{-n/2}\int_{M_i} \eta_i^2 dV \leq (4\pi)^{-n/2}\,\mathrm{Vol}\, B_1(x_i) \to 0.$$

Let C be an upper bound for $4\eta'^2 - \eta^2\log\eta^2$. Using (A.13) and $c_i^{-1}\,\mathrm{Vol}\, B_{1/2}(x_i) \leq c_i^{-1}\int_{M_i}\eta_i^2 = (4\pi)^{n/2}$, we obtain

$$c_i^{-1}\int_{M_i}\Big(4|\nabla\eta_i|^2 - \eta_i^2\log\eta_i^2\Big)dV \leq c_i^{-1}\,\mathrm{Vol}\, B_1(x_i)C \leq (4\pi)^{n/2}2^n e^{2a^2+a}C.$$

Hence

$$\mathcal{W}(g_i, \tilde{u}_i) = (4\pi)^{-n/2}c_i^{-1}\int_{M_i}\Big(4|\nabla\eta_i|^2 - \eta_i^2\log\eta_i^2\Big)dV$$
$$+ (4\pi)^{-n/2}\int_{M_i}(R - n + \log c_i)\tilde{u}_i^2 dV$$
$$\leq 2^n e^{2a^2+a}C + a^2 - n + \log c_i,$$

which tends to $-\infty$ as c_i tends to zero. This contradicts the lower entropy bound $\mathcal{W}(g_i, \tilde{u}_i) \geq \mu(g_i) \geq \underline{\mu} > -\infty$. $\qquad\square$

Proofs of some lemmas from Chapter 2

Proof of Lemma 2.7. We expand

$$H_{g(\varepsilon)} = H + \varepsilon H'[h] + \tfrac{\varepsilon^2}{2} H''[h,h] + \tfrac{\varepsilon^3}{6} H'''[h,h,h] + O(\varepsilon^4), \qquad (A.14)$$

and

$$(\lambda - H_{g(\varepsilon)})^{-1} = (\lambda - H)^{-1} + \varepsilon(\lambda - H)^{-1} H'[h](\lambda - H)^{-1} \qquad (A.15)$$
$$+ \tfrac{\varepsilon^2}{2} \left((\lambda - H)^{-1} H''[h,h] + 2\left((\lambda - H)^{-1} H'[h] \right)^2 \right) (\lambda - H)^{-1} + O(\varepsilon^3).$$

We insert this in (2.10), use $(\lambda - H)^{-1} w = (\lambda - \lambda(g))^{-1} w$, use that H is symmetric with respect to the L^2-inner product and that w is normalized. Thus

$$\frac{1}{\langle w, P_{g(\varepsilon)} w \rangle} = 1 - \varepsilon^2 \tfrac{1}{2\pi i} \oint \langle w, H'[h](\lambda - H)^{-1} H'[h] w \rangle \tfrac{d\lambda}{(\lambda - \lambda(g))^2} + O(\varepsilon^3), \qquad (A.16)$$

$$\begin{aligned}
\langle w, &(H_{g(\varepsilon)} - H) P_{g(\varepsilon)} w \rangle \\
&= \varepsilon \langle w, H'[h] w \rangle \\
&\quad + \tfrac{\varepsilon^2}{2} \left(\langle w, H''[h,h] w \rangle + \tfrac{2}{2\pi i} \oint \langle w, H'[h](\lambda - H)^{-1} H'[h] w \rangle \tfrac{d\lambda}{\lambda - \lambda(g)} \right) \\
&\quad + \tfrac{\varepsilon^3}{6} \langle w, H'''[h,h,h] w \rangle \\
&\quad + \tfrac{\varepsilon^3}{6} \tfrac{6}{2\pi i} \oint \langle w, H'[h](\lambda - H)^{-1} H'[h](\lambda - H)^{-1} H'[h] w \rangle \tfrac{d\lambda}{\lambda - \lambda(g)} \\
&\quad + \tfrac{\varepsilon^3}{6} \tfrac{3}{2\pi i} \oint \langle w, H'[h](\lambda - H)^{-1} H''[h,h] w \rangle \tfrac{d\lambda}{\lambda - \lambda(g)} \\
&\quad + \tfrac{\varepsilon^3}{6} \tfrac{3}{2\pi i} \oint \langle w, H''[h](\lambda - H)^{-1} H'[h] w \rangle \tfrac{d\lambda}{\lambda - \lambda(g)} + O(\varepsilon^4),
\end{aligned} \qquad (A.17)$$

and the formulas in the lemma follow from (2.12).

Let us now justify convergence and analyticity. We have a family of closed operators

$$H(\varepsilon) = -4\triangle_{g+\varepsilon h} + R_{g+\varepsilon h} : H^2(M) \subset L^2(M, dV_g) \to L^2(M, dV_g). \qquad (A.18)$$

For every $u \in L^2(M, dV_g)$ and $v \in H^2(M)$, the $L^2(M, dV_g)$-inner product $\langle u, H(\varepsilon) v \rangle$ depends analytically on ε. Since every weakly analytic function is strongly analytic, for every $v \in H^2(M)$, $\varepsilon \mapsto H(\varepsilon) v$ is an $L^2(M, dV_g)$-valued analytic function. By the above, $H(\varepsilon)$ is an analytic family of type (A) and thus an analytic family in the sense of Kato [101, Sec. XII.2]. Therefore, the smallest eigenvalue $\lambda(g(\varepsilon))$ is an analytic function of ε by the Kato-Rellich theorem [101, Thm. XII.8]. Finally, by [101, Thm. XII.7] the function

$$(\lambda, \varepsilon) \mapsto (\lambda - H(\varepsilon))^{-1} \qquad (A.19)$$

is an $\mathcal{L}(L^2(M, dV_g))$-valued analytic function of two variables defined on an open set, say on $\{(\lambda, \varepsilon) \in \mathbb{C}^2 : r - \delta < |\lambda - \lambda(g)| < r + \delta, |\varepsilon| < \delta\}$, and this justifies the above computations. $\qquad \square$

Proof of (2.21). Let (M, g_{RF}) be compact, Ricci-flat, $h \in \ker \mathrm{div}_{g_{\mathrm{RF}}}$. We would like to evaluate (2.14). Since w is constant, there are some simplifications. To get $H''[h, h]$ we compute

$$\tfrac{d^2}{d\varepsilon^2}|_0 R_{g(\varepsilon)} = \tfrac{d}{d\varepsilon}|_0 g^{ij} g^{kl}(-h_{ik}R_{jl} + D_iD_kh_{jl} - D_iD_jh_{kl}). \tag{A.20}$$

There are contributions from the derivative of g^{-1} (first line), Rc (second line) and D (third line) respectively. Using $\mathrm{div}h = 0, \mathrm{Rc} = 0$, which implies in particular $-D_iD_kh_{il} = R_{kplq}h_{pq}$, we obtain

$$\begin{aligned}
\tfrac{d^2}{d\varepsilon^2}&|_0 R_{g(\varepsilon)} \\
&= h_{ij}\triangle h_{ij} + R_{ipjq}h_{ij}h_{pq} + h_{ij}D_iD_j \,\mathrm{tr}\, h \\
&\quad + \tfrac{1}{2}h_{ij}\triangle h_{ij} + R_{ipjq}h_{ij}h_{pq} + \tfrac{1}{2}h_{ij}D_iD_j \,\mathrm{tr}\, h \\
&\quad + |Dh|^2 - \tfrac{1}{2}|D\,\mathrm{tr}\, h|^2 - D_ih_{jk}D_kh_{ij} + \tfrac{1}{2}D_i(h_{jk}D_ih_{jk} + h_{ij}D_j \,\mathrm{tr}\, h) \\
&= 2\langle h, \triangle h\rangle + \tfrac{3}{2}|Dh|^2 - \tfrac{1}{2}|D\,\mathrm{tr}\, h|^2 + 2\langle h, D^2 \,\mathrm{tr}\, h\rangle \\
&\quad + 2R_{ipjq}h_{ij}h_{pq} - D_ih_{jk}D_kh_{ij}. \tag{A.21}
\end{aligned}$$

Together with $\tfrac{d^2}{d\varepsilon^2}|_0 \triangle_{g(\varepsilon)}1 = 0$, after partial integration, commuting the derivatives in $D_iD_kh_{ij}$ and using $\mathrm{div}h = 0$ again, we get

$$\langle 1, H''[h, h]1\rangle = \tfrac{1}{2}\int_M \left(\langle h, \triangle^L h\rangle + \mathrm{tr}\, h \triangle \,\mathrm{tr}\, h\right) dV. \tag{A.22}$$

The other term contributing to the second variation is proportional to

$$\frac{2}{2\pi i}\oint_{|\lambda|=r}\langle 1, H'[h](\lambda - H)^{-1}H'[h]1\rangle\frac{d\lambda}{\lambda}. \tag{A.23}$$

Now, we insert $H'[h]$ from (2.18). Since $\mathrm{Rc} = 0, \mathrm{div}h = 0$ and $D1{=}0$, many terms vanish. After a partial integration, even more terms vanish and we obtain

$$\begin{aligned}
\frac{2}{2\pi i}&\oint_{|\lambda|=r}\langle 1, H'[h](\lambda - H)^{-1}H'[h]1\rangle\frac{d\lambda}{\lambda} \\
&= -\frac{2}{2\pi i}\oint_{|\lambda|=r}\langle \mathrm{tr}\, h, \triangle(\lambda + 4\triangle)^{-1}\triangle \,\mathrm{tr}\, h\rangle\frac{d\lambda}{\lambda} = -\tfrac{1}{2}\int_M \mathrm{tr}\, h \triangle \,\mathrm{tr}\, h \, dV. \tag{A.24}
\end{aligned}$$

To justify the last step, note that $\triangle(\lambda+4\triangle)^{-1}$ converges to $\tfrac{1}{4}$ as λ tends to zero. Finally, $w = \mathrm{Vol}(M)^{-1/2}$, g_{RF} is a critical point of λ, and λ is invariant under diffeomorphism. This proves (2.21). $\qquad\square$

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