Continuation maps in Morse theory

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Author(s):
Komani, Driton

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CONTINUATION MAPS IN MORSE THEORY

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Driton KOMANI

Dipl. Math. ETH Zürich

born 30.12.1982

citizen of Luzern LU, Switzerland

accepted on the recommendation of

Prof. Dr. Dietmar SALAMON, examiner
Prof. Dr. Michael HUTCHINGS, co-examiner
Prof. Dr. Paul BIRAN, co-examiner

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To Gjeni

"jeta s'osht kurgjo budall
nese s'bone sen
nese s'ki veprimtari
nese kerkush nuk t'man n'men..."

Viktor Paloka

Abstract

Dynamical processes in nature may be described by differential equations. These differential equations depend on parameters that are not known precisely. One only captures qualities of differential equations that remain visible under small perturbations of the equation. Mathematically this means that one is interested in geometric invariants of dynamical systems. Non-degenerate critical points of the dynamical system are such examples. They stay non-degenerated while degenerated critical points may be perturbed away or produce two non-degenerated critical points. So degeneracies can be perturbed away. That is true if one considers a single differential equation. The situation changes if one considers a *family* of differential equations. In a one parameter family of dynamical systems it might well be that the degeneracies can't be perturbed away. In this thesis we consider generic one parameter families of gradients and three different continuation maps determined by a given family. We prove that all continuation maps are equal.

More precisely: Let $f : M \to \mathbb{R}$ be a Morse function on a not necessarily compact manifold M and g a metric on M such that the flow φ^t induced by $\dot{x} = -\nabla f(x)$ is Morse-Smale on a compact isolated invariant set S. The Morse homology (with integer coefficients) defined by the Morse-Smale triple (S, f, g) is denoted by $HM_*(S, f, g)$ and its homological Conley index is denoted by $HC_*(S, f, g)$. A continuation $(S, f, g) := \{(S_\lambda, f_\lambda, g_\lambda)\}_{0\leqslant\lambda\leqslant 1}$ connecting two Morse-Smale triples (S_0, f_0, g_0) and (S_1, f_1, g_1) determines three different continuation maps

(1) $\Phi_{\mathbf{con}}(S, f, g) : HC_*(S_0, f_0, g_0) \to HC_*(S_1, f_1, g_1)$

(2) $\Phi_{\mathbf{flo}}(S, f, g) : HM_*(S_0, f_0, g_0) \to HM_*(S_1, f_1, g_1)$

(3) $\Phi_{\mathbf{bif}}(S, f, g) : HM_*(S_0, f_0, g_0) \to HM_*(S_1, f_1, g_1)$.

The map $\Phi_{\mathbf{con}}(S, f, g)$ is the *Conley continuation map* [1]. The map $\Phi_{\mathbf{flo}}(S, f, g)$ is the *Floer continuation map* defined by counting orbits of $\dot{x} = -\nabla f_t(x)$, where λ is replaced by the time parameter t. The map

$\Phi_{\mathbf{bif}}(S, f, g)$ is the *Floer bifurcational continuation map* and is defined by studying the change of the orbit structure of $\dot{x} = -\nabla_\lambda f_\lambda(x)$ at bifurcations. The Conley index of a Morse-Smale triple is isomorphic to its Morse homology

$$\alpha : HC_*(S, f, g) \to HM_*(S, f, g)$$

see Theorem A. In Theorem C we prove that the continuation map $\Phi_{\mathbf{flo}}(S, f, g)$ is isomorphic to the Conley continuation map $\Phi_{\mathbf{con}}(S, f, g)$ meaning

$$\Phi_{\mathbf{con}}(S, f, g) = \alpha \circ \Phi_{\mathbf{flo}}(S, f, g) \circ \alpha^{-1}.$$

If $f = \{f_\lambda\}_{\lambda \in [0,1]}$ is a family of Morse functions f_λ we prove in Theorem D that

$$\Phi_{\mathbf{con}}(S, f, g) = \alpha \circ \Phi_{\mathbf{bif}}(S, f, g) \circ \alpha^{-1}.$$

We think that the assumption that f_λ is Morse for all $\lambda \in [0, 1]$ can be removed.

Zusammenfassung

Dynamische Prozesse in der Natur können durch Differentialgleichungen beschrieben werden. Diese Differentialgleichungen sind abhängig von Parametern, die nicht genau bekannt sind. Man erfasst nur Qualitäten von Differentialgleichungen, die sichtbar unter kleinen Störungen der Gleichung bleiben. Mathematisch bedeutet dies, dass ein Interesse an geometrischen Invarianten von dynamischen Systemen besteht. Nicht-entartete Singularitäten eines dynamischen Systems sind solche Beispiele. Diese bleiben nicht-entartet bezüglich Störungen, während entartete Singularitäten weggestört werden können oder in neue nicht-entarteten Singularitäten gestört werden können. Entartungen kann man also wegstören. Das ist wahr, wenn man eine einzige Differentialgleichung anschaut. Die Situation ändert sich, wenn man eine *Familie* von Differentialgleichungen betrachtet. In einer einparametrigen Familie von dynamischen Systemen kann es gut sein, dass die Entartung nicht weggestört werden kann. In dieser Doktorarbeit betrachten wir einparametrige Familien von dynamischen Systemen, welche drei verschiedene Fortsetzungsabbildungen zwischen den Invarianten bestimmen. Wir beweisen, dass die drei Fortsetzungsabbildungen gleich sind.

Genauer: Sei $f : M \to \mathbb{R}$ eine Morse Funktion auf einer möglicherweise nicht kompakten Mannifaltigkeit M und g eine glatte Metrik auf M, so dass der lokale Fluss φ^t induziert durch $\dot{x} = -\nabla f(x)$ auf einer kompakten isoliert invarianten Menge S die Morse-Smale Bedingung erfüllt. Die Morse Homologie (über den ganzen Zahlen) des Morse-Smale Tripels (S, f, g) gegeben durch $HM_*(S, f, g)$ und der homologische Conley Index ist gekennzeichnet durch $HC_*(S, f, g)$. Eine Fortsetzung $(S, f, g) := \{(S_\lambda, f_\lambda, g_\lambda)\}_{0 \leqslant \lambda \leqslant 1}$ zwischen zwei Morse-Smale Tripel (S_0, f_0, g_0) und (S_1, f_1, g_1) induziert drei verschiedene Fortsetzungsabbildungen:

(1) $\Phi_{\mathbf{con}}(S, f, g) : HC_*(S_0, f_0, g_0) \to HC_*(S_1, f_1, g_1)$

(2) $\Phi_{\mathbf{flo}}(S, f, g) : HM_*(S_0, f_0, g_0) \to HM_*(S_1, f_1, g_1)$

(3) $\Phi_{\mathbf{bif}}(S,f,g) : HM_*(S_0,f_0,g_0) \to HM_*(S_1,f_1,g_1)$.

Die Abbildung $\Phi_{\mathbf{con}}(S,f,g)$ its die *Conley Fortsetzungsabbildung* [1]. Die Abbildung $\Phi_{\mathbf{flo}}(S,f,g)$ ist die *Floer Fortsetzungsabbildung* definiert durch das Zählen von verbindenden Bahnen der Gleichung $\dot{x} = -\nabla f_t(x)$, wobei λ durch die Zeit t ersetzt wurde. Die Abbildung $\Phi_{\mathbf{bif}}(S,f,g)$ nennt man die *Floer Verzweigungs Fortsetzungsabbildung* und ist definiert durch das Studium der Verzweigungen der Familie von Gleichungen $\dot{x} = -\nabla_\lambda f_\lambda(x)$. Der homologische Conley Index ist isomorph zur Morse Homologie, d.h. es existiert ein Isomorphismus

$$\alpha : HC_*(S,f,g) \to HM_*(S,f,g)$$

siehe Theorem A. Im Theorem C beweisen wir, dass die Fortsetzungsabbildung $\Phi_{\mathbf{flo}}(S,f,g)$ isomorph zur Conley Fortsetzungsabbildung $\Phi_{\mathbf{con}}(S,f,g)$ ist, d.h.

$$\Phi_{\mathbf{con}}(S,f,g) = \alpha \circ \Phi_{\mathbf{flo}}(S,f,g) \circ \alpha^{-1}.$$

Falls $f = \{f_\lambda\}_{\lambda\in[0,1]}$ eine Familie bestehend aus Morse Funktionen f_λ ist beweisen wir dass

$$\Phi_{\mathbf{con}}(S,f,g) = \alpha \circ \Phi_{\mathbf{bif}}(S,f,g) \circ \alpha^{-1}.$$

Wir denken, dass Theorem D wahr bleibt auch wenn man die Bedingung f_λ ist Morse für alle $\lambda \in [0,1]$ weglässt.

Danksagung

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Contents

Chapter 1

Introduction

Continuation maps were first introduced by Conley in [1] to establish the invariance of the Conley index. Based on these ideas Floer gave two different continuation maps for proving invariance of Floer homology: In [3] by studying bifurcation analysis and in [4] by studying a time dependent equation. Conleys as well as Floer's continuation maps may be defined in the finite dimensional model case of Morse theory. The Conley index of an isolated invariant set of a gradient flow is isomorphic to its Morse homology, see Theorem A. Using this identification we prove that the Conley continuation map and Floer's continuation maps in Morse theory are the same, see Theorem C and Theorem D.

1.1 Conley index and Morse homology

Let M be a smooth finite dimensional manifold without boundary which is *not* necessarily compact and $X : M \to TM$ a smooth vector field. Denote by φ^t the *local* flow generated by the vector field via

$$\dot{x} = X(x) \tag{1.1}$$

Since the manifold M is not necessarily compact a solution curve $t \mapsto \varphi^t(x)$ may fail to exist for all times $t \in \mathbb{R}$. A subset $S \subset M$ is called *invariant* if $\varphi^t(S) = S$ for all times t, where the local flow φ^t is defined. The *maximal invariant set* of of a subset $N \subset M$ is defined by $I(N) := \{x \in N \mid \varphi^t(x) \in N \ \forall t\}$. An invariant set $S \subset M$ is called **isolated invariant set** if there exists a *compact* neighborhood N of S such that

$$I(N) = S.$$

Such a compact neighborhood N is called an *isolating neighborhood*. It follows that S is compact. Every isolated invariant set S has a Conley index, which essentially means that there exists an *index pair* (N, L) of compact sets in M such that $N\backslash L$ is a neighborhood of S with $S = I(\mathrm{cl}(N\backslash L))$ and $L \subset N$ being a *positively invariant exit set* in N, meaning that a trajectory starting in N and leaving N in positive time must go through L first. In any neighborhood of $S \subset M$ there is an index pair, see Proposition 2.3. Applying the singular homology to the quotient space N/L with distinguished point $[L]$ we define the **homological Conley index** by

$$HC_*(S) := H_*^{\mathrm{sing}}(N/L, [L])$$

which is independent of the index pair (N, L). Details are given in Chapter 2. Examples of invariant sets are displayed in the next figure, not all of them are isolated invariant sets.

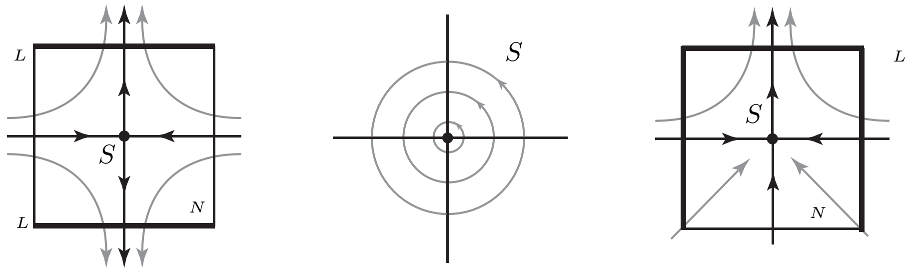


Figure 1. On the left $S = \{0\}$ is an isolated invariant set. In the middle there is *no* isolated invariant set. On the right $S = \{0\}$ is an isolated invariant set.

The Conley index of a hyperbolic singularity is a sphere.

Example 1.2. Let x be a hyperbolic singularity of a vector field X on M. Then for $0 \leqslant k \leqslant n$ one may chose an index pair (N, L) for x diffeomorphic to $(B^k \times B^{n-k}, \partial B^k \times B^{n-k})$. The Conley index has the homotopy type of the pointed sphere $h(S) = [(S^k, *)]$ where k is the dimension of the unstable manifold.

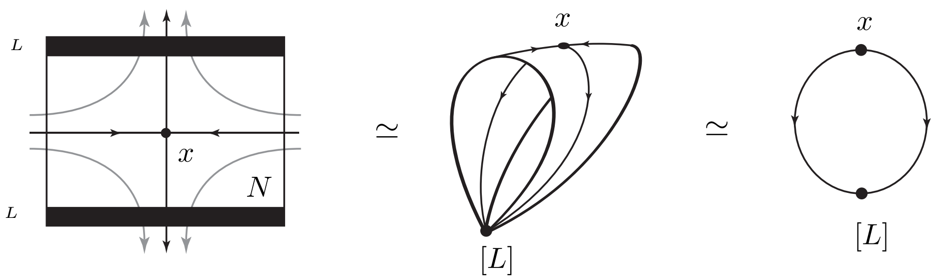


Figure 2. If we collaps the exit set of the index pair (N, L) of an hyperbolic singularity we get a homotopy sphere.

The homological Conley index $H_*(N/L, [L])$ is that of a pointed sphere of dimension k. A generator is determined by the unstable manifold in N and its orientation

$$W^u(x, N) := \{p \in M \mid \lim_{t \to -\infty} \varphi^t(p) = x, \ \varphi^{(-\infty,0]}(p) \subset N\}.$$

On the other hand a generator $[W^u(x, N)] \in H_k(N, L)$ determines an orientation of $W^u(x, N)$.

We specialize to gradient flows. Let $f : M \to \mathbb{R}$ be a smooth function and g a smooth metric on M. We consider the gradient flow φ^t generated by the *gradient pair* (f, g) via

$$\dot{x} = -\nabla f(x) \tag{1.3}$$

where $-\nabla f$ is the gradient of f with respect to the metric g. If S is an isolated invariant set of φ^t generated by (f, g) we call (S, f, g) a *gradient triple.* A gradient triple (S, f, g) is called **Morse-Smale triple** if all its critical points $\mathrm{Crit}(S, f) = \{x \in S \mid df(x) = 0\}$ are non-degenerate and if the stable and unstable manifolds of $\mathrm{Crit}(S, f)$ intersect transversally whenever they intersect in S, see Definition 3.1. Critical points of f together with orientations of their unstable manifolds $W^u(x)$ generate a graded free abelian group $CM_*(S, f)$ defined by $CM_k(S, f) := \bigoplus_x \mathbb{Z}_x$ where the sum runs over critical points of Morse index $|x| := \dim W^u(x) = k$. For a Morse-Smale triple (S, f, g) it follows that $W^u(y) \pitchfork W^s(x) \cap S$ consists of finitely many signed trajectories if $|y| - |x| = 1$. Denote by $n_S(y, x)$ the number of connecting trajectories in S going from y to x counted with sign. Then the boundary operator $\partial^S : CM_k(S, f) \to CM_{k-1}(S, f)$ on generators is given by

$$\partial^S(y) = \sum_{|x|=k} n_S(y, x)x.$$

The homology $HM_*(S, f) := H_*(CM_*(S, f), \partial^S)$ is called the **local Morse homology** of the Mors-Smale triple (S, f, g). The local Morse homology of the Morse-Smale triple (S, f, g) is isomorphic to the homological Conley index of S. That is the assertion of the next theorem.

Theorem A (Thom, Smale, Milnor, Conley, Witten, Floer, McCord)**.** *Let M be a smooth manifold without boundary, possibly non-compact, and let (S, f, g) be a Morse-Smale triple. Then there exists an isomorphism*

(i) $\alpha : HC_*(S) \overset{\cong}{\to} HM_*\big(S, f, g\big)$.

(ii) *For compact M the isomorphism becomes* $\alpha : H_*^{sing}(M) \overset{\cong}{\to} HM_*(M, f, g)$.

Proof. See Section 3.3. □

For the sake of completeness we will give a proof of Theorem A following Salamons proof of Theorem A part (ii) in [11], see Chapter 3 on Morse theory. If M is compact it is a classical theorem that the Morse homology is isomorphic to the singular homology of M already formulated by Thom [14] and Smale [13]. The formulation in terms of chain complexes generated by critical points is due to Witten [16]. If one removes the compactness assumption on M, then the isomorphism in Theorem A part (i) was first proved by Milnor [8] in the case of self-indexing Morse functions, without using the notion of the Conley index. Floer [5] proved Theorem A for Alexander-Spanier cohomology with field coefficients. McCord [6] gives a proof of the isomorphism (i) in the case of $\mathbb{Z}_2$ coefficients. As it is the case in Milnor [8], Floer [5], McCord [6] and in Salamon [11] the proof is based on the existence of an filtration $N_{-1} \subset \cdots N_n = M$ with (N_k, N_{k-1}) being an index pair for $\mathrm{Crit}_k(S, f) = \{x \in \mathrm{Crit}(S, f) \mid |x| = k\}$. For any Morse-Smale triple (S, f, g) there exsists an index filtration

$$L = N_{-1} \subset N_0 \subset \cdots \subset N_n = N$$

such that any (N_k, N_{k-1}) is an index pair of $\mathrm{Crit}_k(f)$ while (N, L) is an index pair of S, see Conley and Zehner [2] for the construction of an index filtration. After orienting the unstable manifolds the cellular complex of this filtration turns out to be isomorphic to the Morse complex, see Lemma 3.14, which is the essential part of the proof of Theorem A.

Example 1.4. If M is compact and (f, g) is Morse-Smale then (M, f, g) is an Morse-Smale triple with $N = M = S$ and $L = \varnothing$ being an index pair for M. The isomorphism of Theorem A part (i) asserts then $H_*^{\mathrm{sing}}(M) \cong HM_*(M, f, g)$.

Let $S \subset M$ be invariant set of φ^t generated by (1.1). An invariant subset $A^+ \subset S$ is called *attractor* if there exists an neighborhood U of A^+ relative to S such that $\omega^+(U) = A^+$, where $\omega^+(U)$ is the ω^+-limit set of U under forward time, see Chapter 2. The *complementary reppeler* of A^+ is defined via $A^- := \{x \in S \mid \omega^+(x) \cap A = \emptyset\}$. The pair (A^+, A^-) is called **attractor-repeller pair** of S. If S is an isolated invariant set, then so are A^- and A^+. In that case there exists a filtration $N_0 \subset N_1 \subset N_2$ such that (N_2, N_0) is an index pair for S, (N_2, N_1) is an index pair for A^- and (N_1, N_0) is an index pair for A^+. The long exact sequence of the triple (N_2, N_1, N_0)

$$\cdots \xrightarrow{\partial} H_*(N_1, N_0) \to H_*(N_2, N_0) \to H_*(N_2, N_1) \xrightarrow{\partial} H_*(N_1, N_0) \to \ldots$$

gives the exact sequence of homological Conley indices

$$\cdots \longrightarrow HC_*(A^+) \longrightarrow HC_*(S) \longrightarrow HC_*(A^-) \xrightarrow{\partial} HC_{*-1}(A^+) \longrightarrow \cdots \tag{1.5}$$

Let (S, f, g) be a Morse-Smale triple admitting an attractor-repeller pair (A^+, A^-) of S. Then (A^-, f, g) and (A^+, f, g) are itself Morse-Smale triples giving the short exact sequence of Morse complexes

$$0 \to CM_*(A^+, f) \to CM_*(S, f) \to CM_*(S, f)/CM_*(A^+, f) \to 0$$

whose long exact sequence is

$$\longrightarrow HM_*(A^+, f) \longrightarrow HM_*(S, f, g) \longrightarrow HM_*(A^-, f) \longrightarrow HM_*(A^+, f) \longrightarrow \tag{1.6}$$

It follows by definition that the boundary operator of (1.6) counts connecting trajectories from A^- to A^+. Next theorem asserts that the boundary operator in (1.5) also counts connecting trajectories from A^- to A^+ meaning that it is isomorphic to (1.6).

Theorem B. *Let M be a smooth manifold without boundary, possibly non-compact, and let (S, f, g) be a Morse-Smale triple admitting an attractor-repeller pair (A^+, A^-). Then the sequences (1.5) and (1.6) are isomorphic, i.e. the following diagram commutes*

$$\begin{tikzcd}
{} \arrow[r] & HC_*(A^+) \arrow[r] \arrow[d, "\alpha_+", "\cong"'] & HC_*(S) \arrow[r] \arrow[d, "\alpha", "\cong"'] & HC_*(A^-) \arrow[r, "\partial"] \arrow[d, "\alpha_-", "\cong"'] & HC_{*-1}(A^+) \arrow[r] \arrow[d, "\alpha_+", "\cong"'] & {} \\
{} \arrow[r] & HM_*(A^+, f) \arrow[r] & HM_*(S, f) \arrow[r] & HM_*(A^-, f) \arrow[r, "\partial"] & HM_{*-1}(A^+, f) \arrow[r] & {}
\end{tikzcd}$$

The vertical maps are the isomorphisms of Theorem A.

Proof. See Section 3.3. $\square$

1.2 Conley continuation and Floer's cobordism construction

To define the Conley index $HC_*(S) = H_*(N/L, [L])$ respectively the Morse homology $HM_*(S, f, g)$ we used the flow of a vector field X respectively the gradient flow of the gradient pair (f, g). Certain *global* deformations of the data X and (f, g) leave the invariants $HC_*(S)$ and $HM_*(S, f, g)$ unchanged. These global deformations will be called *continuations.*

Definition 1.7. Let (S_a, f_a, g_a) and (S_b, f_b, g_b) be two Morse-Smale triples on M. A **continuation** relating (S_a, f_a, g_a) to (S_b, f_b, g_b) is a family of gradient triples

$$(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [a, b]\}$$

such that

(i) the map $[a, b] \times M \to \mathbb{R}, \ (\lambda, x) \to f_\lambda(x)$ is smooth and $\{g_\lambda\}_{\lambda \in [a,b]}$ is a smooth family of Riemannian metrics on M.

(ii) the set

$$S := \{(\lambda, x) \mid x \in S_\lambda\} \subset [a, b] \times M$$

is an *isolated invariant set* for the product flow $\Phi^t(\lambda, x) = (\lambda, \varphi_\lambda^t(x))$, where φ_λ^t is the local gradient flow of (f_λ, g_λ).

(iii) there exists a constant $\delta > 0$ such that

$$(S_\lambda, f_\lambda, g_\lambda) = \begin{cases} (S_a, f_a, g_a) & \text{for } a \leqslant \lambda \leqslant a + \delta \\ (S_b, f_b, g_b) & \text{for } b - \delta \leqslant \lambda \leqslant b \end{cases}$$

Two Morse-Smale triples (S_a, f_a, g_a) and (S_b, f_b, g_b) are said to be **related by continuation** if there exists a continuation (S, f, g) relating (S_a, f_a, g_a) to (S_b, g_b, f_b). The fundamental observation of Conley is that the Conley index is invariant under continuations, that is any continuation (S, f, g) relating (S_a, f_a, g_a) to (S_b, f_b, g_b) determines an isomorphism

$$\Phi_{\mathbf{con}}(S, f, g) : HC_*(S_a) \to HC_*(S_b)$$

called the **Conley continuation map**, see Definition 2.16.

Floer's cobordism construction

We consider a more general situation: Let (S_a, f_a, g_a) and (S_b, f_b, g_b) be two Morse-Smale triples on M. Pick a family of gradient pairs $(f, g) = \{(f_\lambda, g_\lambda) \mid \lambda \in [a, b]\}$ such that

$$(f_\lambda, g_\lambda) = \begin{cases} (f_a, g_a) & \text{for } a \leqslant \lambda \leqslant a + \delta \\ (f_b, g_b) & \text{for } b - \delta \leqslant \lambda \leqslant b. \end{cases}$$

Denote by $(f, g)_{\mathbb{R}} := \{(f_\lambda, g_\lambda) \mid \lambda \in \mathbb{R}\}$ the extended family parametrized by $\lambda \in \mathbb{R}$ such that (f_λ, g_λ) is constant outside $(a + \delta, b - \delta)$. Replacing the parameter λ by t we get a *time-dependent gradient equation*

$$\dot{x}(t) = -\nabla f_t(x(t)) \tag{1.8}$$

with local flow φ^{t,t_0}. The set of initial points $x \in M$ for which the zero flow $t \mapsto \varphi^{t,0}(x)$ is globally defined with asymptotics in S_a and S_b is defined by

$$C := \left\{ x \in M \;\middle|\; \lim_{t \to -\infty} \varphi^{t,0}(x) \in S_a \text{ and } \lim_{t \to \infty} \varphi^{t,0}(x) \in S_b \right\}$$

and is called the *connecting set* of $(f, g)_{\mathbb{R}}$. For isolating neighborhoods N_a, N_b of S_a, S_b we define the unstable and stable sets by

$$W^u(S_a, f_a, N_a) := \left\{ p \in M \mid \varphi_{f_a}^{(-\infty,0]}(p) \subset N_a \right\}$$

and

$$W^s(S_b, f_b, N_b) := \left\{ p \in M \mid \varphi_{f_b}^{[0,\infty)}(p) \subset N_b \right\}.$$

Definition 1.9. Let (S_a, f_a, g_a) and (S_b, f_b, g_b) be two Morse-Smale triples and (f, g) be a smooth family of gradient pairs such that $(f_\lambda, g_\lambda) = (f_a, g_a)$ for $\lambda \leqslant a + \delta$ and $(f_\lambda, g_\lambda) = (f_b, g_b)$ for $\lambda \geqslant b - \delta$. A subset $K \subset C$ is called an **isolated connecting set** from S_0 to S_1 if there exists

(i) a compact isolating neighborhood N_a of S_a

(ii) a compact isolating neighborhood N_b of S_b

(iii) a *compact* neighborhood W of K

(iv) a time $T > 0$ such that for every $p \in W$ and all $t \in [-T, T]$

$$t \mapsto \varphi_f^{t,0}(p) \quad \text{is defined}$$

and

$$K = \varphi_f^{0,-T}\big(W^u(S_a, f_a, N_a)\big) \cap \varphi_f^{0,T}\big(W^s(S_b, f_b, N_b)\big) \cap W. \tag{1.10}$$

In this case $\Gamma = (N_a, N_b, W, T)$ is called an **isolating quadruple.**

Being an isolated connecting set is an open condition meaning that they remain isolated for nearby families of gradient pairs. Associated to every triple (K, f, g) consisting of an isolating quadruple and a family of gradient pairs is a *Floer type homomorphism*

$$\Phi_{\mathbf{flo}}(K, f, g) : HM_*(S_a, f_a, g_a) \to HM_*(S_b, f_b, g_b)$$

on local Morse homology, see Section 4.2. If the intersection in (1.10) is transverse it is defined by counting solutions of (1.8) going through K at time zero. If K is non-transverse it is still defined in terms of intersection numbers. The map $\Phi_{\mathbf{flo}}(K, f, g)$ is well defined after a small perturbation of the family (f, g), see Theorem 4.10. The map $\Phi_{\mathbf{flo}}(K, f, g)$ does *not* depends on the perburbation. Any continuation (S, f, g) relating (S_0, f_0, g_0) to (S_1, f_1, g_1) determines a triple $(K_\varepsilon, f_\varepsilon, g_\varepsilon)$ in the following way: Choose an isolating neighborhood

$$N := \bigcup_{\lambda \in [0,1]} \{\lambda\} \times N_\lambda$$

of S with respect to the product flow Φ^t. For $\varepsilon > 0$ we define the time-dependent gradient equation

$$\dot{x}(t) = -\nabla f_{\varepsilon t}(x(t)) \tag{1.11}$$

with local flow $\varphi_\varepsilon^{t,t_0}$. The initial points $x \in M$ for which the flow is globally defined, satisfies $\varphi_\varepsilon^{t,0}(x) \in N_{\varepsilon t}$ for all $t \in \mathbb{R}$ and has asymptotics in S_0, S_1 are denoted by

$$K_\varepsilon := \left\{ x \in M \;\middle|\; \begin{array}{c} \lim_{t \to -\infty} \varphi_\varepsilon^{t,0}(x) \in S_0, \lim_{t \to \infty} \varphi_\varepsilon^{t,0}(x) \in S_1 \\ \varphi_\varepsilon^{t,0}(x) \in N_{\varepsilon t} \;\forall t \in [\delta/\varepsilon, (1-\delta)/\varepsilon] \end{array} \right\}.$$

The set K_ε is an *isolated connected set* for the family $(f_\varepsilon, g_\varepsilon) := \{(f_{\varepsilon t}, g_{\varepsilon t}) \mid t \in [0, 1/\varepsilon]\}$, see Theorem 4.15. The triple $(K_\varepsilon, f_\varepsilon, g_\varepsilon)$ determines the **Floer continuation map**

$$\Phi_{\mathbf{flo}}(K_\varepsilon, f_\varepsilon, g_\varepsilon) : HM_*(S_0, f_0, g_0) \to HM_*(S_1, f_1, g_1).$$

For small $\varepsilon > 0$ the map $\Phi_{\mathbf{flo}}(K_\varepsilon, f_\varepsilon, g_\varepsilon)$ is related to the Conley continuation map $\Phi_{\mathbf{con}}(S, f, g)$ in the following way.

Theorem C. *Let M be a manifold without boundary, possibly non-compact. Let (S, f, g) be a continuation relating the Morse-Smale triple (S_0, f_0, g_0) to the Morse-Smale triple (S_1, f_1, g_1). Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following diagramm commutes*

$$\begin{tikzcd} HC_*(S_0) \arrow[r, "\Phi_{\mathbf{con}}(S,f,g)"] \arrow[d, "\alpha", "\cong"'] & HC_*(S_1) \arrow[d, "\alpha", "\cong"'] \\ HM_*(S_0, f_0) \arrow[r, "\Phi_{\mathbf{flo}}(K_\varepsilon, f_\varepsilon, g_\varepsilon)"] & HM_*(S_1, f_1) \end{tikzcd}$$

The vertical maps are the isomorphisms of Theorem A.

1.3 Conley continuation and Floer's bifurcational map

Assume that (S, f, g) is a continuation relating two Morse-Smale triples (S_0, f_0, g_0) and (S_1, f_1, g_1) such that

$$\boxed{f_\lambda \quad \text{is Morse} \quad \forall\, \lambda \in [0, 1].}$$

Floer [3] defined another continuation map by studying the change of the orbit structure of the flow φ_λ^t generated by the family

$$\dot{x} = -\nabla f_\lambda(x). \tag{1.12}$$

Since f_λ is Morse for all $\lambda \in [0, 1]$ the critical points come in smooth families, i.e. there exist smooth maps $x_1, \dots, x_\ell : [0, 1] \to M$ called *critical families* such that

$$\mathrm{Crit}(S_\lambda, f_\lambda) = \{x_1(\lambda), \dots, x_\ell(\lambda)\} \quad \text{for all } \lambda \in [0, 1].$$

Let $N = \cup_{\lambda\in[0,1]}\{\lambda\} \times N_\lambda$ be an isolating neighborhood for S. For a critical family $x : [0, 1] \to M$, $x_\lambda := x(\lambda) \in \mathrm{Crit}(S_\lambda, f_\lambda)$, the unstable manifold of φ_λ^t relative to N_λ is denoted by

$$W^u(x_\lambda, f_\lambda, N_\lambda) := \left\{ p \in M \;\Big|\; \lim_{t\to-\infty} \varphi_\lambda^t(p) = x(\lambda),\ \varphi_\lambda^{(-\infty,0]}(p) \subset N_\lambda \right\}.$$

The stable manifolds $W^u(x_\lambda, f_\lambda, N_\lambda)$ and $\mathcal{W}^u(x, f, N)$ are defined similarly. The parametrized unstable manifold of a critical family x is then

$$\mathcal{W}^u(x, f, N) := \{(\lambda, p) \mid p \in W^u(x_\lambda, f_\lambda, N_\lambda)\}.$$

Floer gave an explicit isomorphism

$$\Phi_{\mathbf{bif}}(S,f,g) : HM_*(S_0,f_0,g_0) \to HM_*(S_1,f_1,g_1)$$

called **bifurcational continuation map** for continuations (S,f,g) satisfying the parametrized Morse-Smale condition in the following sense.

Definition 1.13. A continuation (S,f,g) from (S_0,f_0,g_0) to (S_1,f_1,g_1) satisfies the **parametrized Morse-Smale condition** if

(i) (S_0,f_0,g_0) and (S_1,f_1,g_1) are Morse-Smale triples.

(ii) f_λ is Morse $\forall \lambda \in [0,1]$.

(iii) for all critical families $x,y : [0,1] \to M$ the parametrized stable and unstable manifolds intersect transversally

$$\mathcal{W}^u(y,f,N) \pitchfork \mathcal{W}^s(x,f,N).$$

(iv) For every $\lambda \in [0,1]$ there is at most one ordered pair y,x of critical families with equal Morse index $|y| = |x|$ such that

$$W^u(y_\lambda,f_\lambda,N_\lambda) \cap W^s(x_\lambda,f_\lambda,N_\lambda) \neq \emptyset.$$

In a continuation (S,f,g) satisfying the parametrized Morse-Smale condition, the algebraic count of the number of *index zero* flow lines in (iv) defines a chain isomorphism

$$\Phi_{\mathbf{bif}}(S,f,g) : CM_*(S_0,f_0) \to CM_*(S_1,f_1)$$

of Morse complexes, see Definition 4.23. The next theorem asserts that the induced map on homology is isomorphic to the Conley continuation map.

Theorem D. *Let M be a manifold without boundary, possibly non-compact. Let (S,f,g) be a continuation that satisfies the parametrized Morse-Smale condition. Then the following diagram commutes*

$$\begin{tikzcd}
HC_*(S_0) \arrow[r, "\Phi_{\mathbf{con}}(S,f,g)"] \arrow[d, "\alpha", "\cong"'] & HC_*(S_1) \arrow[d, "\alpha", "\cong"'] \\
HM_*(S_0,f_0) \arrow[r, "\Phi_{\mathbf{bif}}(S,f,g)"] & HM_*(S_1,f_1)
\end{tikzcd}$$

The vertical maps are the isomorphisms of Theorem A.

Proof. See Chapter 5. □

There are two notions of transversality assumptions on (f,g) according to each map $\Phi_{\textbf{flo}}(K_\varepsilon, f_\varepsilon, g_\varepsilon)$ and $\Phi_{\textbf{bif}}(S,f,g)$. See Chapter 4. Theorem C and Theorem D show that the maps $\Phi_{\textbf{flo}}(S,f,g)$ and $\Phi_{\textbf{bif}}(S,f,g)$ determined by a continuation (S,f,g) with f_λ Morse for all $\lambda \in [0,1]$ are equal.

Corollary E. *Let M be a manifold without boundary, possibly non-compact. Let (S,f,g) be a continuation that satisfies the parametrized Morse-Smale condition. Then*

$$\Phi_{\textbf{flo}}(S,f,g) = \Phi_{\textbf{bif}}(S,f,g).$$

Proof. Theorem C and Theorem D. □

1.4 An application to Picard-Lefschetz theory

Let M be a Kähler manifold of real dimension $2m$ with Kähler structure (ω, g, J) and let $F = f + \mathbf{i}h : M \to \mathbb{C}$ be a holomorphic map. The Cauchy-Riemann equations imply that the negative gradient of the real part f equals the hamiltonian vector field of the imaginary part h, i.e.

$$-\nabla f = J\nabla h.$$

A critical point of $p \in \mathrm{Crit}(F) = \{p \in M \mid dF(p) = 0\}$ is called *non-degenerate* if there exists a holomorphic chart (U,φ) around p such that

$$F \circ \varphi^{-1}(z) = f(p) + z_1^2 + \cdots + z_m^2.$$

The map $F : M \to \mathbb{R}$ is called **Lefschetz fibration** if all its critical points are non-degenerate. We make the following hypotheses on F :

(a) F is proper.

(b) F has only finitely many critical points $\mathrm{Crit}(F) = \{p_1, \ldots, p_\ell\}$ with distinct critical values $F(p_1) \neq F(p_2) \neq \cdots \neq F(p_\ell)$.

(c) $\mathrm{Im}(F(p_1)) < \mathrm{Im}(F(p_2)) < \cdots < \mathrm{Im}(F(p_\ell))$ and no three critical values lie on the same line.

Denote by φ^t the local flow of the gradient equation (1.3) generated by the real part of F via

$$\dot{x} = -\nabla f(x).$$

Since F is a Lefschetz fibration the critical points of f are Morse with index m, i.e.

$$\mathrm{Crit}(F) = \mathrm{Crit}(f) = \mathrm{Crit}_m(f).$$

A solution curve $t \mapsto \varphi^t(x)$ is also a solution of $\dot{x} = J\nabla h(x)$ hence composing with F we get

$$t \longmapsto F(\varphi^t(p)) = f(\varphi^t(p)) + \mathbf{i}h(\varphi^t(p))$$

has decreasing real part and constant imaginary part. The projected dynamics is illustrated in the next figure

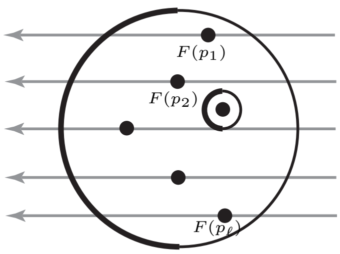


Figure 3. There is no trajectory connecting critical points since they have different imaginary parts.

Let $\mathbb{D} \subset \mathbb{C}$ be a disc around zero containing all critical points $\mathrm{Crit}(F) = \{p_1, \ldots, p_\ell\}$ in its interior and denote its left half circle by $\partial\mathbb{D}^- := \{z \in \partial\mathbb{D} \mid \mathrm{Re}(z) \leqslant 0\}$. It follows that $(N, L) := (F^{-1}(\mathbb{D}), F^{-1}(\partial\mathbb{D}^-))$ is an index pair for $S = \mathrm{Crit}(f) = \mathrm{Crit}_m(f)$. Choose orientations of the unstable manifolds $W^u(p_i, f, N)$. Then by (c) the relative homology classes

$$[W^u(p_i, f, N)] \in H_m(N, L) =: HC_*(S) \quad i = 1, \ldots, \ell$$

generate the Conley index of $S = \mathrm{Crit}_m(f)$. Denote the real part of the rotated map F by

$$f_\lambda := \mathrm{Re}(e^{-2\mathbf{i}\pi\lambda} \cdot F) = \cos(2\pi\lambda)f + \sin(2\pi\lambda)h.$$

and observe the dynamics of

$$\dot{x} = -\nabla f_\lambda(x).$$

Denote by $(N, L_\lambda) := \big(F^{-1}(N), F^{-1}(e^{2\mathbf{i}\pi\lambda}L)\big)$ the index pair for

$$S_\lambda = I(N, \varphi^t_\lambda).$$

This gives a continuation

$$(S, f, g) := \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [0, 1]\}$$

relating $S_0 = (\mathrm{Crit}_m(f), f, g)$ to itself, i.e. $S_0 = S_1$. The only parameter values $\lambda_{ij} \in (0,1)$ where connecting trajectories from p_i to p_j may occur are given by

$$e^{2\pi\mathbf{i}\lambda_{ij}} := \frac{F(p_i) - F(p_j)}{|F(p_i) - F(p_j)|}.$$

These numbers are pairwise distinct by condition (b) and can be ordered as

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{\ell^2-\ell} < 1.$$

For $\lambda_{ij} = \lambda_\nu$ define the elementary matrix

$$E_\nu = \left(\delta_{rs} - n_{ij}\delta_{rj}\delta_{is}\right)_{r,s=1}^{\ell}$$

having units on the diagonal and exactly one non-zero entry $-n_{ij}$ in the (j,i) slot. The number $-n_{ij}$ is the intersection number $\langle L_i, L_j\rangle$ of the vanishing cycles L_i and L_j along the straight line from $F(p_i)$ to $F(p_j)$ with appropriate choices of orientations.

Corollary F. *Let $F : M \to \mathbb{C}$ be a Lefschetz map satisfying (a)-(c) with continuation (S, f, g). Choose orientations of the unstable manifolds $W^u(p_i) = W^u(p_i, f, N)$ as above. Then the Conley continuation map $\Phi_{\mathbf{con}}(S, f, g) : HC_m(S_0) \to HC_m(S_0)$ acts on generators by*

$$\Phi_{\mathbf{con}}(S, f, g)[W^u(p_j)] = \sum_{i=1}^{\ell} a_{ji}[W^u(p_i)]$$

with representation matrix $A = (a_{ij})_{i,j=1}^{\ell}$ being the product

$$A = (-1)^m E_{\ell^2-\ell} \cdot \cdots \cdot E_2 \cdot E_1$$

of elementary matrices.

Proof. Theorem D. □

The thesis is organized as follows: In Chapter 2 we give some background on the Conley index and define the Conley continuation map $\Phi_{\mathbf{con}}(S, f, g)$. In Section 3.3 of Chapter 3 we give a proof of Theorem A and use it to proof Theorem B. In Section 4 we define the homomorphism $\Phi_{\mathbf{flo}}(S, f, g)$ for families that are more general than continuations. Section 4.5 is devoted to Floer's bifurcational map $\Phi_{\mathbf{bif}}(S, f, g)$. The proofs of Theorem C and Theorem D are given in Section 4.4 and Chapter 5 respectively.

Chapter 2

Conley Index theory

In this chapter we give some background on Conley index theory. The Conley index of an compact isolated invariant set is defined in two steps: First one shows that every compact isolated invariant set admits an index pair consisting of an isolating neighborhood and a so called exist set. Secondly one collapses the exit set and shows that there exists flow induced homotopy equivalences between any to collapsed spaces. The homotopy type of the collapsed space is the *homotopy Conley index* whereas its homology is called *homological Conley index* of the isolated invariant set. The fundamental property of the Conley index is that it is invariant under continuations of isolated invariant sets: A continuation relating two isolated invariant sets yields an isomorphism between their Conley indices, Theorem 2.14. The crucial observation is that there are different continuations relating two isolated invariant sets for the *same* family of flows, see Example 2.3.

2.1 The Conley index

Let M be a smooth manifold of dimension m without boundary, possibly non-compact and $X : M \to TM$ be a vector field with associated local flow φ^t generated by the equation

$$\dot{x} = X(x).$$

Solutions $t \mapsto \varphi^t(x)$ may not exist globally. The *trajectory* $\mathcal{O}(x) \subset M$ of an element $x \in M$ is the image of $t \mapsto \varphi^t(x)$ on the maximal existence interval, i.e.

$$\mathcal{O}(x) := \left\{\varphi^t(x) \mid \forall t \text{ where } \varphi^t(x) \text{ is defined}\right\}.$$

A set $S \subset M$ is called *invariant* if every trajectory in S lies in S again, i.e.

$$x \in S \Rightarrow \mathcal{O}(x) \subset S.$$

The *maximal invariant set* of N is defined as the union of elements that have their trajectories in N and is denoted by

$$I(N, \varphi^t) := \{x \in N \mid \mathcal{O}(x) \subset N\}.$$

A subset $S \subset M$ is called **isolated invariant set** if there exists a *compact* neighorhood N of S such that

$$S = I(N, \varphi^t) \subset \operatorname{int}(N).$$

The compact set N is called *isolating neighborhood* of $S = (S, \varphi^t)$. It follows from definition that S is closed and hence compact. So isolated invariant sets are by definition compact. The crucial property of isolating neighborhoods N is that they are robust with respect to small perturbations.

Lemma 2.1 (Conley)**.** *Let N be an isolating neighborhood of S for the flow φ^t. The set of flows having N as an isolating neighborhood is open with respect to the C^0-topology in the space of flows on M.*

Proof. If the assertion wasn't true then there exists a sequence of flows φ_k^t converging in the C^0-topology to φ^t such that N is an isolating neighborhood for φ^t but is *not* an isolating neighborhood for φ_k^t. Hence there exists a sequence $x_k \in \partial N$ with

$$\varphi_k^{\mathbb{R}}(x_k) \subset N.$$

By continuity any limit point $\lim_{k\to\infty} x_k = x \in \partial N$ satisfies $\varphi^{\mathbb{R}}(x) \subset N$ contradicting the fact that N is an isolating neighborhood for φ^t. □

If S is an isolated invariant set and $x \in S$ then the solution curve $t \mapsto \varphi^t(x)$ is defined globally. In that case

$$\omega^+(x) := \bigcap_{t>0} \operatorname{cl}\left(\varphi^{[t,\infty)}(x)\right) \quad \text{and} \quad \omega^-(x) := \bigcap_{t>0} \operatorname{cl}\left(\varphi^{(-\infty,-t]}(x)\right)$$

are called the $\omega^{\pm}-$*limit set* of $x \in S$.

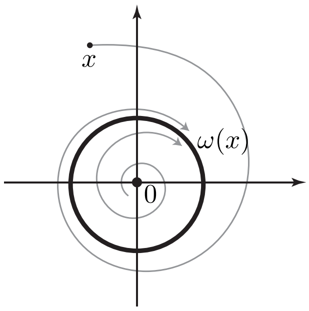


Figure 4. This is the phase space of the flow generated by the equaitons $\dot{\theta} = -1,\ \dot{r} = r(1 - r^2)$ in polar coordinates. The limit sets are $\omega^+(x) = S^1$ for $x \in \mathbb{R}^2\backslash\{0\}$ and $\omega^-(x) = \{0\}$ for $|x| < 1$.

In the same way we define the $\omega^{\pm}-$*limit set* of of a subset $U \subset S$ by

$$\omega^{+}(U) := \bigcap_{t>0} \mathrm{cl}\left(\varphi^{[t,\infty)}(U)\right) \quad \text{and} \quad \omega^{-}(U) := \bigcap_{t>0} \mathrm{cl}\left(\varphi^{(-\infty,-t]}(U)\right).$$

A compact invariant subset $A \subset S$ of an isolated invariant set S is said to be an **attractor** in S if there exists a neighborhood U of A in S such that $\omega^{+}(U) = A$. A compact invariant set $A \subset S$ is said to be an **repeller** if there exists a neighborhood U of A in S such that $\omega^{-}(U) = A$. Let $A^{+} \subset S$ be an attractor; its *complementary repeller* is defined by

$$A^{-} := \left\{x \in S \mid \omega^{+}(x) \cap A = \emptyset\right\}.$$

A pair (A^{-}, A^{+}) consisting of an attractor A^{+} and its complementary repeller A^{-} is called **attractor-reppeller decomposition** of S. For an attractor-repeller par (A^{-}, A^{+}) of S define the set of *connecting trajectories* with $\omega^{\pm}-$limits in $A^{\pm}$ by

$$C(A^{-}, A^{+}) := \left\{x \in S \backslash (A^{-} \cup A^{+}) \mid \omega^{\pm}(x) \in A^{\pm}\right\}.$$

Example 2.2. The flow associated to the negative gradient of the height function on the sphere S gives rise to a attractor-repeller pair (A^{-}, A^{+}) consisting of the maximum and the minimum of the height function. Assume that a solution curve $\gamma : \mathbb{R} \to M$ has the rest points of the flow as asymptotics $\lim_{t\to\pm} \gamma(t) = x^{\pm}$ then the pair (x^{-}, x^{+}) is an attractor repeller decomposition of the set $S = \{x^{-}, x^{+}\} \cup \gamma(\mathbb{R})$.

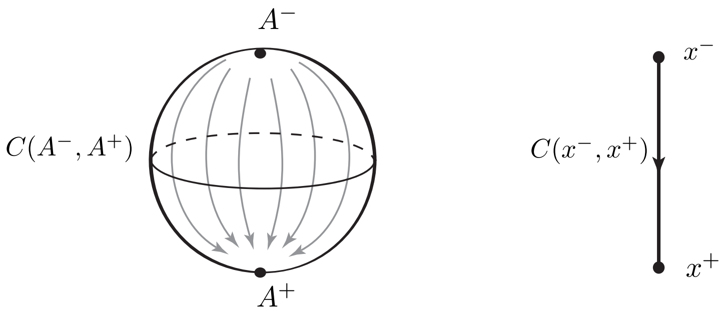


Figure 5. Examples of an attractor repeller pair.

Every isolated invariant set S has a Conley index. This essentially means that there exists a so called index pair. For an isolated invariant set S a pair (N, L) of compact sets in M is called an **index pair** for S if $L \subset N$ and if it satisfies the following three conditions.

(i) $N \backslash L$ is a neighborhood of S and $S = I(\mathrm{cl}(N \backslash L))$.

(ii) L is positively invariant in N, i.e. if $x \in L$ with $\varphi^{[0,t]}(x) \subset N$ then $\varphi^t(x) \in L$.

(iii) If $x \in N$ and $\varphi^{[0,t]}(x) \not\subset N$ then there exists a time $t^* \in [0,t]$ with $\varphi^{[0,t^*]}(x) \subset N$ and $\varphi^{t^*}(x) \in L$.

Condition (iii) requires that every trajectory, which leaves N in forward time has to go through L first. Hence L is called *exit set.* In any neighborhood of an isolated invariant set $S \subset M$ there is an index pair, see Conley-Zehnder [2]. Moreover this index pair is robust with respect to small perturbations as asserted in the following theorem.

Proposition 2.3 (Robbin-Salamon [9]). *Let N' be an isolating neighborhood for the isolated invariant set S of the flow φ^t_X generated by the vector field X. Then the following holds*

(i) *There exists an index pair (N, L) for S in N' such that N/L has the homotopy type of a compact CW-complex.*

(ii) *There exists a neighborhood $\mathcal{U}$ of X in the C^0-topology, on the space of vector fields, such that for all $Y \in \mathcal{U}$*

$$(N, L) \quad \textit{is an index pair for} \quad I(N', \varphi^t_Y).$$

Proof. For a detailed proof see Robbin-Salamon [9]. We sketch the arguments therein.
(i). One can construct a Lyapunov function for $S = (S, \varphi^t)$ on an open neighborhood U' of N' which by definition is a smooth function $f : U' \to \mathbb{R}$ such that $f(x) = 0$ for $x \in S$ and

$$f(\varphi^t(x)) < f(x) \quad \text{for } \varphi^{[0,t]}(x) \in N' \backslash S.$$

Define

$$N := \{x \in N' \mid -\varepsilon \leqslant f(x) \leqslant f(\varphi^{-t_0}(x)) \leqslant \varepsilon\}$$

where $t_0 > 0$ is so small that $\varphi^{[-t_0,t_0]}(x) \subset U'$ for all $x \in N'$ and $\varepsilon > 0$ is so small that

$$\partial N \subset f^{-1}(-\varepsilon) \cup \varphi^{t_0}(f^{-1}(\varepsilon)).$$

and $L := f^{-1}(-\varepsilon) \cap N$. By Sard's theorem we may choose ε such that $-\varepsilon$ is a regular value of f and ε is a regular value of $\tilde{f} := f \circ \varphi^{-t_0}$. In a next step perturb f and $\tilde{f}$ such that $f^{-1}(-\varepsilon)$ and $\tilde{f}^{-1}(\varepsilon)$ intersect transversely. It follows that

$$N = f^{-1}\big([-\varepsilon, \infty)\big) \cap \tilde{f}^{-1}\big((-\infty, \varepsilon]\big)$$

is a manifold with corners with exit set $L = N \cap f^{-1}(-\varepsilon)$. Using Morse theory one finds a cellular decomposition of L which can be extended to N. The pair (N, L) is an index pair: By $S \in \text{int}(N') \cap f^{-1}(0)$ we get $S \subset \text{int}(N) \subset N'$. Form $S \subset \text{int}(N)$ it follows that $S \subset \text{int}(\varphi^{-t_0}(N))$ and therfore $S \subset N \backslash L \subset N'$. It follows that $N \backslash L$ is an isolating neighborhood for S, which proves statement (i).
For statement (ii) we pick a neighborhood $\mathcal{U}$ of X in the C^0-topology on the space of vector fields such that

$$df(x)Y(x) < 0 \quad \text{and} \quad d(f \circ \varphi^{-t_0}(x))Y(x) < 0$$

for $x \in \partial N$. This is possible since the equality is true for $X = Y$ by definition of f. Denote the flow of Y by φ_Y^t. If $x \in L$ then $\varphi_Y^t(x) \notin N$ for small positive $t > 0$. On the other hand if $x \in \partial N \backslash L$ then $\varphi_Y^t(x) \in \text{int}(N)$ for small $t > 0$. Hence (N, L) is an index pair for $I(N, \varphi_Y^t)$. □

Proposition 2.3 tells us that there are lot of index pairs for the same isolated invariant set. If we collaps the exit set L in N we get a pointed space N/L with $[L]$ being the distinguished point. The collapsed space N/L is unique up to homotopy.

Lemma 2.4 (Conley)**.** *Let (N_α, L_α) and (N_β, L_β) be index pairs for S. Then there exists a flow induced homotopy equivalence*

$$\phi_{\beta\alpha} : N_\alpha/L_\alpha \to N_\beta/L_\beta.$$

Proof. Conley [1] and Salamon [10]. For $T_{\beta\alpha} \geqslant 0$ sufficiently large, define the family of maps $\phi_{\beta\alpha}^t : N_\alpha/L_\alpha \longrightarrow N_\beta/L_\beta$ by

$$\phi_{\beta\alpha}^t([x]) := \begin{cases} \left[\varphi^{3t}(x)\right] & \text{if } \varphi^{[0,2t]}(x) \subset N_\alpha \backslash L_\alpha \text{ and } \varphi^{[t,3t]}(x) \subset N_\beta \backslash L_\beta \\ \\ \left[L_\beta\right] & \text{otherwise.} \end{cases} \tag{2.5}$$

This map is continuous if $t > T_{\beta\alpha} \geqslant 0$ where $T_{\beta\alpha} > 0$ has to be chosen such that for all $t > T_{\beta\alpha}$

$$\varphi^{[-t,t]}(x) \subset N_\alpha \backslash L_\alpha \Rightarrow x \in N_\beta \backslash L_\beta, \quad \varphi^{[-t,t]}(x) \subset N_\beta \backslash L_\beta \Rightarrow x \in N_\alpha \backslash L_\alpha.$$

See Salamon [10]. The flow induced map $\phi_{\alpha\alpha}^t : N_\alpha/L_\alpha \to N_\alpha/L_\alpha$ is continuous for all $t > T_{\alpha\alpha} = 0$. For another index pair (N_γ, L_γ) we have $T_{\alpha\gamma} \leqslant T_{\alpha\beta} + T_{\beta\gamma}$ and

$$\phi_{\gamma\beta}^t \circ \phi_{\beta\alpha}^s = \phi_{\gamma\alpha}^{t+s}, \quad \phi_{\alpha\alpha}^0 = \text{id}$$

for $t > T_{\beta\gamma}$ and $s > T_{\alpha\beta}$. It follows that $\phi_{\alpha\beta}^t$ is the homotopy inverse of $\phi_{\beta\alpha}^t$. □

It follows from the proof that the homotopy class of the homotopy equivalence $\phi^t_{\beta\alpha} : N_\alpha/L_\alpha \to N_\beta/L_\beta$ for any two index pairs (N_α, L_α) and (N_β, L_β) for S is canonically determined by the flow. The collection of collapsed index pairs together with homotopy classes of flow induced maps determines a *small* category with unique isomorphisms between any two objects. Such a category is called *connected simple system*.

Definition 2.6. Let $S \subset M$ be an isolated invariant set of the flow φ^t and (N, L) be an index pair for S.

(i) The **homotopy Conley index** $h(S)$ of S is the homotopy type of a collapsed index pair (N, L) of S, and is denoted by $h(S) := [N/L]$. Here $[N/L]$ denotes the homotopy class of N/L.

(ii) It follows from Proposition 2.3 that the index pair (N, L) can be chosen such that L is an neighborhood deformation retract of N. Applying the homology functor to N/L we get the **homological Conley index**

$$H_*(S) := H_*(N/L, [L]) = H_*(N, L).$$

Often we write $H_*(N, L)$ for the Conley index $H_*(S)$.

For isolated singularities of a vector field the Conley index can be computed directely.

Example 2.7. Let x be a hyperbolic singularity of a vector field X on a manifold M with $\dim W^u(x) = k$. Then in a chart one may chose an index pair (N, L) for x diffeomorphic to $(B^k \times B^{m-k}, \partial B^k \times B^{m-k})$. The Conley index has the homotopy type of the pointed sphere $h(S) = [(S^k, *)]$ where k is the dimension of the unstable manifold.

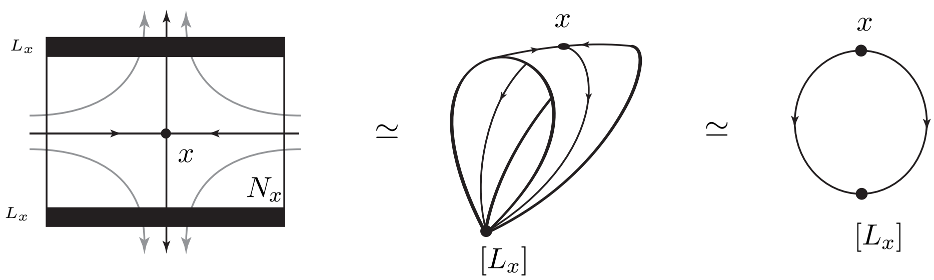


Figure 6. If we collaps the exit set of the index pair (N_x, L_x) of an hyperbolic singularity we get a homotopy sphere.

The homology Conley index $H_*(N, L)$ is that of a pointed sphere of dimension k.

Remark 2.8. Let (N, L) be an index pair for a hyperbolic singularity x. Picking a generator of $H_*(N, L)$ determines uniquely an orientation of the unstable manifold relative to N

$$W^u(x, N) := \{p \in M \mid \lim_{t \to -\infty} \varphi^t(p) = x,\ \varphi^{(-\infty,0]}(p) \subset N\}.$$

and vise versa. We denote any generator of $H_*(N, L)$ by $[W^u(x, N)]$. Let (N_α, L_α) and (N_β, L_β) be two index pairs for x. Then the flow induced homotopy equivalence $\phi^t_{\beta\alpha}$ sends the canonacical generator relative to N_α to the canonical generator relative to N_β i.e. $(\phi_{\beta\alpha})_*[W^u(x, N_\alpha)] = [W^u(x, N_\beta)]$. Sometimes we suppress relative to which isolating neighborhood the generator is and write $[W^u(x)]$.

Example 2.9. Let $S = y \cup x$ be the *disjoint* union of two hyperbolic singularities x and y as in the next figure

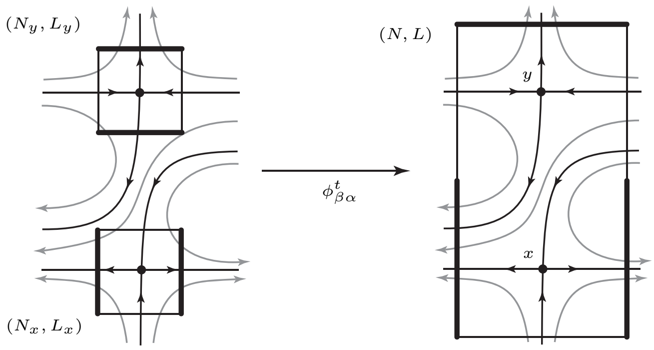


Figure 7. Pick small index pairs $(N_x, L_x), (N_y, L_y)$ for x, y and a big index pair (N, L) for $y \cup x$. Then $(N_\alpha, L_\alpha) = (N_y \cup N_x, L_y \cup L_x)$ and $(N_\beta, L_\beta) = (N, L)$ are two index pairs for $S = y \cup x$.

The flow map $\phi^t_{\beta\alpha}$ sends any basis $\{[W^u_{\rm loc}(y)], [W^u_{\rm loc}(y)]\}$ of $H_1(N_y, L_y) \oplus H_1(N_x, L_x)$ to a basis $\{[W^u(y)], [W^u(x)]\}$ of $H_1(N, L)$ via

$$\phi^t_{\beta\alpha} : H_1(N_y, L_y) \oplus H_1(N_x, L_x) \to H_1(N, L).$$

Hence choosing a base of the small index pairs determines a basis of the big index pair via flow maps.

If some invariant set is the disjoint union of invariant sets or is the product of invariant sets one may write down formulas:

Lemma 2.10 (Conley). *Let M and N be smooth finite dimensional manifolds.*

(i) *Let S_α and S_β be disjoint isolated invariant sets of the same flow φ^t on M. Then the homotopical Conley index of the disjoint union is the wedge product of the Conley indices, i.e.*

$$h(S_\alpha \cup S_\beta) = h(S_\alpha) \vee h(S_\beta).$$

(ii) *Let $S_\alpha \subset M$ and $S_\beta \subset N$ be isolated invariant sets with respect to φ^t_α and φ^t_β. Then $S_\alpha \times S_\beta \subset M \times N$ is an isolated invariant set of $\varphi^t_\alpha \times \varphi^t_\beta$. Then the Conley index of the product is the smash product of the Conley indices, i.e.*

$$h(S_\alpha \times S_\beta) = h(S_\alpha) \wedge h(S_\beta).$$

Proof. (i). Let (N_α, L_α) and (N_β, L_β) be index pairs for S_α and S_β. Then $(N, L) := (N_\alpha \cup N_\beta, L_\alpha \cup L_\beta)$ is an index pair of $S_\alpha \cup S_\beta$ and it follows by Lemma 2.4 that

$$N/L = N_\alpha/L_\alpha \vee N_\beta/L_\beta.$$

Applying the homology functor the assertion follows.
(ii). Let (N_α, L_α) and (N_β, L_β) be index pairs for S_α and S_β. Then $(N, L) := (N_\alpha \times N_\beta, N_\alpha \times L_\alpha \cup N_\beta \times L_\beta)$ is an index pair for $S_\alpha \times S_\beta$ and

$$N/L = N_\alpha/L_\alpha \wedge N_\beta/L_\beta.$$

This shows the Lemma. □

Example 2.11. Assume a flow on $[-\delta, 1+\delta] \times M$ with isolated invariant set

$$\mathscr{S} = [0,1] \times S$$

such that $h([0,1]) = (*, *)$ is the one point pointed space. Then $h([0,1] \times S) = h([0,1]) \wedge h(S) = (*, *)$. It follows that $\mathscr{S}$ has trivial Conley index, i.e. $h(\mathscr{S}) = (*, *)$ and the Conley index $H_*(\mathscr{S}) = 0$.

2.2 Conley continuation

Let $X : [0,1] \times M \to TM$ with $X_\lambda(x) := X(\lambda, x)$ be a family of vector fields parametrized by $\lambda \in [0,1]$ with associated local flows φ^t_λ.

Definition 2.12. A **continuation** of isolated invariant sets on M is a collection of pairs

$$(S, \Phi^t) := \left\{ (S_\lambda, \varphi^t_\lambda) \mid \lambda \in [0,1] \right\}$$

such that the set

$$S := \{(\lambda, x) \mid x \in S_\lambda\} \subset [0,1] \times M$$

is an isolated invariant set of the product flow Φ^t on $[0,1] \times M$ defined by $\Phi^t(\lambda, x) := (\lambda, \varphi_\lambda^t(x))$.

The fibers $S_0 = (S_0, \varphi_0^t)$ and $S_1 = (S_1, \varphi_1^t)$ in a continuation are said to be **related by continuation** via (S, Φ^t). Sometimes we write (S, X) for the continuation to emphasise the family of vector fields X. Two isolated invariant sets $S_0 = (S_0, \varphi_0^t)$ and $S_1 = (S_1, \varphi_1^t)$ may be related by different continuations for the same family of flows.

Example 2.13. The complex polynomial $p = f + \mathbf{i}h : \mathbb{C} \to \mathbb{C}$ defined by $p(z) = \frac{1}{3}z^3 + z$. The real parts of the rotated polynomial $e^{2\pi \mathbf{i}\lambda}p(z)$ give rise to a family of Morse functions on $\mathbb{R}^2$ defined by

$$f_\lambda := \mathrm{Re}(e^{-2\pi\mathbf{i}\lambda}p(z))$$

with negative gradients $X_\lambda := -\nabla f_\lambda$. Now let $M = \mathbb{R}^2$ and $X_\lambda = -\nabla f_\lambda$ a family of (gradient) vector fields parametrized by $\lambda \in [0,1]$ with associated flows φ_λ^t. Since $\varphi_0^t = \varphi_1^t$ we have a loop of flows. For all parameter values there are two fixed critical points $\mathrm{Crit}(f_\lambda) = \{\mathbf{i}, -\mathbf{i}\}$ both of Morse index one. For $\lambda = 0.25$ there exists a gradient flow line $\gamma_1 : \mathbb{R} \to \mathbb{R}^2$ with $\lim_{t\to\pm\infty} \gamma_1(t) = \mp\mathbf{i}$. For $\lambda = 0.75$ there exists an other gradient flow line $\gamma_2 : \mathbb{R} \to \mathbb{R}^2$ with $\lim_{t\to\pm\infty} \gamma_2(t) = \pm\mathbf{i}$.

Figure 8. The set of all isolated invariant sets of the dynamical system generated by $\dot{x} = -\nabla f_\lambda(x)$.

Define $S_\alpha = [0,1] \times \mathrm{Crit}(f_\lambda)$ which defines a continuation relating S_0 to S_1. Another continuation is defined by

$$S_\beta := S_\alpha \cup \left(\left\{\frac{1}{4}\right\} \times \gamma_1(\mathbb{R})\right) \cup \left(\left\{\frac{3}{4}\right\} \times \gamma_2(\mathbb{R})\right).$$

We see that (S_α, Φ^t) and (S_β, Φ^t) are two different continuations relating $S_0 = \{\mathbf{i}, -\mathbf{i}\}$ and $S_1 = \{\mathbf{i}, -\mathbf{i}\}$ for the *same* family of gradient flows.

Being related by continuation is an equivalence relation. The Conley index distinguishes equivalence classes.

Theorem 2.14 (Conley Continuation Theorem). *Let $S_0 = (S_0, \varphi_0^t)$ and $S_1 = (S_1, \varphi_1^t)$ be related by continuation via (S, Φ^t). Let (N_0, L_0) and (N_1, L_1) be index pairs for S_0 and S_1 respectively. Then S determines a flow induced homotopy equivalence*

$$F_{10}(S, \Phi^t) : N_0/L_0 \to N_1/L_1$$

called the **homotopical Conley continuation map.**

Proof. There exists an open covering I_i of the unit interval $[0,1]$ and a collection of index pairs (N^i, L^i) with $i = 1, \dots, n$, such that for $\lambda \in I_i$

$$(N^i, L^i) \quad \text{is an index pair for } S_\lambda = (S_\lambda, \varphi_\lambda^t).$$

We prove this inductively: First pick a constant index pair (N_0, L_0) for S_0 as in Proposition 2.3. Define the first parameter for which (N_0, L_0) fails to be an index pair

$$\lambda_1 := \inf_{\lambda \in [0,1]} \left\{ \lambda \in [0,1] \mid (N_0, L_0) \text{ is } \textit{not} \text{ an index pair for } S_\lambda \right\}.$$

Now pick an constant index pair (N^1, L^1) for S_{λ_1} and define λ_2 to be the first parameter for which (N^1, L^1) fails to be an index pair as above. By induction there exists a collection of parameters $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n = 1$ and a collection index pairs

$$(N_0, L_0) = (N^0, L^0), (N^1, L^1), \dots, (N^n, L^n) = (N_1, L_1)$$

and a $\delta > 0$ such that for $i = 1, \dots, n$

$$(N^{i-1}, L^{i-1}) \text{ and } (N^i, L^i) \quad \text{are index pairs for } S_{\lambda_i - \delta}.$$

Then by Lemma 2.4 for any $i = 1, \dots, n$ there exists a flow induced homotopy equivalence

$$\phi_{i,i-1}^t : N^{i-1}/L^{i-1} \to N^i/L^i$$

induced by the flow $\varphi_{\lambda_i - \delta}^t$ via

$$\phi_{i,i-1}^t([x]) := \begin{cases} \left[\varphi_{\lambda_i - \delta}^{3t}(x)\right] & \text{if } \varphi^{[0,2t]}(x) \subset N^{i-1} \backslash L^{i-1} \text{ and } \varphi^{[t,3t]}(x) \subset N^i \backslash L^i \\ \left[L^i\right] & \text{otherwise.} \end{cases}$$

For big enough $t > 0$ the Conley continuation map $F_{10}(S, \Phi^t) : N_0/L_0 \to N_1/L_1$ is then defined by composition

$$F_{10}(S) := \phi_{n,n-1}^t \circ \cdots \circ \phi_{1,0}^t. \tag{2.15}$$

□

We define the homological Conley continuation map.

Definition 2.16. Let M be a smooth manifold without boundary, not necessarily compact. Let $S_0 = (S_0, \varphi_0^t)$ and $S_1 = (S_1, \varphi_1^t)$ be related by continuation via (S, Φ^t). Applying the homology functor to the homotopical Conley continuation map $F_{10}(S) : N_0/L_0 \to N_1/L_1$ defined in (2.15) we get the **homological Conley continuation map** denoted by

$$\Phi_{\mathbf{con}}(S, \Phi^t) : H_*(S_0) \to H_*(S_1).$$

In what follows we will use the homological Conley continuation map which is denoted by $\Phi_{\mathbf{con}}(S, \Phi^t)$ refer to it as the Conley continuation map. Note that the Conley continuation map depends on S as is explored in the next example.

2.3 Continuation in the model example

Let $M = \mathbb{R}^2$ and define the family of vector fields X_λ defined by

$$X_\lambda(x, y) := \begin{pmatrix} \cos(\pi y)x + \lambda \sin(\pi y) \\ y(y-1) \end{pmatrix} \quad \text{for } \lambda \in [-\varepsilon, \varepsilon].$$

The vector fields X_λ have singularities $\{y, x\} = \{(0,1), (0,0)\}$ for all $\lambda \in [-\varepsilon, \varepsilon]$ and for $\lambda = 0$ there exists a flow line $\gamma_0 : \mathbb{R} \to \mathbb{R}^2$ of $\dot{x} = X_0(x)$ with aymtotics

$$\lim_{t\to\infty} \gamma_0(t) = x \quad \text{and} \quad \lim_{t\to-\infty} \gamma_0(t) = y.$$

Here we see the phase space of the flows φ_λ^t for special paratmeters:

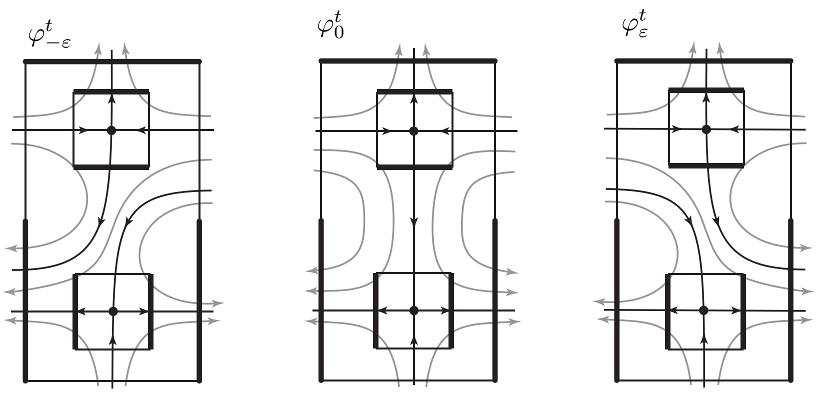


Figure 9. For small $\varepsilon > 0$ we may pick a constant index pair (N, L).

We pick a big index pair (N, L) and two small index pairs (N_y, L_y) of (N_x, L_x) such that $N_y \cap N_x = \emptyset$. Define isolated invariant sets

$$S_\lambda(\alpha) := I(N_y \cup N_x, \varphi_\lambda^t) \quad \text{and} \quad S_\lambda(\beta) := I(N, \varphi_\lambda^t).$$

Then it follows that the isolated invariant sets of the product flow Φ^t in $[-\varepsilon, \varepsilon] \times M$.

$$S_\alpha := \{(\lambda, x) \mid x \in S_\lambda(\alpha)\} \quad \text{and} \quad S_\beta := \{(\lambda, x) \mid x \in S_\lambda(\beta)\}$$

define *different* continuations between $S_{-\varepsilon} = \{y_{-\varepsilon}, x_{-\varepsilon}\}$ and $S_\varepsilon = \{y_\varepsilon, x_\varepsilon\}$. Now define the homological indices

$$C\Delta_1(S_{-\varepsilon}) := HC_1(y_{-\varepsilon}) \oplus HC_1(x_{-\varepsilon})$$

and

$$C\Delta_1(S_\varepsilon) := HC_1(y_\varepsilon) \oplus HC_1(x_\varepsilon).$$

(i) *The continuation* $\Phi_{\mathbf{con}}(S_\alpha)$. The isolated invaraint set is the disjoint union of continuations of the critical points, i.e.

$$S_\alpha = [-\varepsilon, \varepsilon] \times y \sqcup [-\varepsilon, \varepsilon] \times x.$$

It follows immediately that the continuation map

$$\Phi_{\mathbf{con}}^\Delta(S_\alpha) : C\Delta_1(S_{-\varepsilon}) \to C\Delta_1(S_\varepsilon)$$

is given by

$$\Phi_{\mathbf{con}}^\Delta(S_\alpha)[W_{-\varepsilon}^u(y, N_y)] = [W_\varepsilon^u(y, N_y)]$$
$$\Phi_{\mathbf{con}}^\Delta(S_\alpha)[W_{-\varepsilon}^u(x, N_x)] = [W_\varepsilon^u(x, N_x)].$$

(ii) *The continuation* S_β. The isolated invariant set S_β is the union of S_α and the trajectory connecting y to x, i.e.

$$S_\beta := S_\alpha \cup \{0\} \times \gamma_0(\mathbb{R}).$$

If $\lambda \neq 0$ Lemma 2.10 part **(i)** implies that there exists an flow induced isomorphism (2.5)

$$\phi_\lambda^t : C\Delta_1(S_\lambda) \longrightarrow H_1(N, L).$$

Using this flow induced isomorphism, choosing a basis in $C\Delta_1(S_\lambda)$ corresponds to choosing a basis in $H_1(N, L)$. The Conley continuation map $\Phi_{\mathbf{con}}^\Delta(S_\beta) := (\phi_\varepsilon^t)^{-1} \circ \phi_{-\varepsilon}^t$ is the unique map that makes the diagram

$$\begin{tikzcd} C\Delta_1(S_{-\varepsilon}) \arrow[rr, "\Phi_{\mathbf{con}}^\Delta(S_\beta)"] \arrow[dr, "\phi_{-\varepsilon}^t"'] & & C\Delta_1(S_{-\varepsilon}) \arrow[dl, "\phi_\varepsilon^t"] \\ & H_1(N, L) & \end{tikzcd}$$

comute. We orient $W^u_\lambda(y,N_y)$ from "up to down" and $W^u(x,N_x)$ from "left to right". It follows from the figure that

$$\begin{aligned}\big[W^u_{-\varepsilon}(y,N)\big] &= \big[W^u_{\varepsilon}(y,N)\big] &&-\big[W^u_{\varepsilon}(x,N)\big]\\ \big[W^u_{-\varepsilon}(x,N)\big] &= 0 &&+\big[W^u_{\varepsilon}(x,N)\big]\end{aligned}$$

Using $\phi_{\pm\varepsilon}([W^u_{\pm\varepsilon}(y,N_y)) = W^u_{\pm\varepsilon}(y,N)$ we get

$$\begin{aligned}\Phi^{\Delta}_{\mathbf{con}}(S_\beta)[W^u_{-\varepsilon}(y,N_y)] &= [W^u_{\varepsilon}(y,N_y)] &&-[W^u_{\varepsilon}(x,N_x)]\\ \Phi^{\Delta}_{\mathbf{con}}(S_\beta)[W^u_{-\varepsilon}(x,N_x)] &= 0 &&+[W^u_{\varepsilon}(x,N_x)]\end{aligned}$$

So the continuation map $\Phi_{\mathbf{con}}(S_\beta) : C\Delta_1(S_{-\varepsilon}) \to C\Delta_1(S_\varepsilon)$ has the matrix representation

$$\Phi_{\mathbf{con}}(S_\beta) = \begin{bmatrix} 1 & 0\\ -1 & 1\end{bmatrix}$$

with respect to the bases $[W^u_{-\varepsilon}(y,N_y)], [W^u_{-\varepsilon}(x,N_x)]$ and $[W^u_{\varepsilon}(y,N_y)], [W^u_{\varepsilon}(x,N_x)]$.

Example 2.17. Compute continuationm maps with different isolated invariant sets.

2.4 The dynamical cone

Let M be a possibly non-compact manifold without boundary and $X : [0,1]\times M \to TM$ with $X_\lambda(x) := X(\lambda,x)$ be a family of vector fields parametrized by $\lambda \in [-\delta, 1+\delta]$ with associated local flows φ^t_λ. We assume that the family X_λ is constant outside the interval $(\delta, 1-\delta)$, i.e.

$$X_\lambda = \begin{cases} X_0 & \text{for } -\delta \leqslant \lambda \leqslant \delta\\ X_1 & \text{for } 1-\delta \leqslant \lambda \leqslant 1+\delta.\end{cases} \tag{2.18}$$

We define the space of families of vector fields being constant outside $(\delta, 1-\delta)$ by

$$\mathscr{P}(\delta) := \left\{ X : [-\delta, 1+\delta]\times M \to TM \;\middle|\; X_\lambda \text{ satisfies (2.18)}\right\}$$

equipped with the compact-open topology. Moreover we define the space of continuous functions

$$\mathscr{B}(\delta) := \left\{ \beta : [-\delta, 1+\delta] \to [0,1] \;\middle|\; \begin{array}{ll} \beta(0)=0, & \beta([0,1]) \geqslant 0\\ \beta(1)=0, & \beta([0,1]^c) < 0\end{array}\right\}$$

with supremum norm $\|\beta\| = \sup_{\lambda\in[0,1]}|\beta(\lambda)|$ and $\|\beta\|_{[0,1]} := \sup_{\lambda\in[0,1]}|\beta(\lambda)|$. The parameter space over which we are applying Conley continuation is

$$\mathscr{P}(\delta) \times \mathscr{B}(\delta).$$

Any pair $(X, \beta) \in \mathscr{P}(\delta) \times \mathscr{B}(\delta)$ determines a vectorfield $[-\delta, 1+\delta] \times M$ by

$$(\lambda, x) \longmapsto \big(\beta(\lambda), X_\lambda(x)\big).$$

We denote by $\Psi^t_{(X,\beta)}(\lambda, x)$ the flow generated by (X, β) via equation

$$\left\{ \begin{array}{rcl} \dot\lambda(t) &=& \beta(\lambda(t)) \\ \dot x(t) &=& X_{\lambda(t)}(x(t)) \end{array} \right. \tag{2.19}$$

Remark 2.20. There are special elements in $\mathscr{P}(\delta) \times \mathscr{B}(\delta)$:

(i) Let (S_0, φ_0^t) be an isolated invariant set of the flow φ_0^t generated by a vector field $X_0 : M \to TM$. Pick a constant family of vector fields $X = X_0$ in $\mathscr{P}(\delta)$ then for any $\beta \in \mathscr{B}(\delta)$ the pair $(X, \beta) \in \mathscr{P}(\delta) \times \mathscr{B}(\delta)$ gives an uncoupled system

$$\begin{array}{rcl} \dot\lambda(t) &=& \beta(\lambda(t)) \\ \dot x(t) &=& X_0(x(t)). \end{array}$$

The set $[0,1] \times S_0$ is an isolated invariant set of the flow $\Psi^t_{(X,\beta)}(x, \lambda)$ generated via equation (2.19).

(ii) Let (S, Φ^t) be a *continuation* relating (S_0, φ_0^t) to (S_1, φ_0^t). If β restricted to the unit interval vanishes, i.e. $\beta_{|[0,1]} \equiv 0$ we get a fibre preserving flow generated by the equation

$$\begin{array}{rcl} \dot\lambda(t) &=& 0 \\ \dot x(t) &=& X_{\lambda(t)}(x(t)) \end{array}$$

Then $S = \{(\lambda, x) \mid x \in S_\lambda\} \subset [0,1] \times M$ is an isolated invariant set of the flow generated by (X, β) via equation (2.19).

Let $X \in \mathscr{P}(\delta)$ and denote by φ_λ^t the family of flows associated to X_λ. Let (S, X) be a continuation relating S_0 to S_1. It follows from Remark 2.20 that $S \subset [0,1] \times M$ is an isolated invariant set of the flow $\Psi^t_{(X,\beta_0)}$ if β_0 vanishes on $[0,1]$. Pick an isolating neighborhood $N \subset [-\delta, 1+\delta] \times M$ of S such that $N_\lambda = N_0$ for $\lambda \in [-\delta, \delta]$ and $N_\lambda = N_1$ for $\lambda \in [1-\delta, 1+\delta]$ and define

$$\mathscr{S}(S, X, \beta) := I\left(N, \Psi^t_{(X,\beta)}\right). \tag{2.21}$$

Lemma 2.22. *Let (S, X) be a continuation relating S_0 to S_1. Then there exists an $\varepsilon_0 > 0$ such that*

$$\mathscr{S}(S, X, \beta) \text{ is an isolated invariant set for } \|\beta\|_{[0,1]} < \varepsilon_0.$$

Proof. This follows directly from Lemma 2.1. □

We want to restrict to functions β that are positive on $(0, 1)$ hence we define

$$\mathscr{B}^+(\delta) := \left\{\beta \in \mathscr{B}(\delta) \mid \beta^{-1}(0) = \{0, 1\}\right\}. \tag{2.23}$$

Definition 2.24. Let (S, X) be a continuation relating S_0 to S_1. Then for any $\beta^+ \in \mathscr{B}(\delta)$ with $\|\beta^+\|_{[0,1]} < \varepsilon_0$ as in Lemma 2.22 the isolated invariant set

$$\mathscr{S}(S, X, \beta^+) \subset [0, 1] \times M$$

with respect to the flow $\Psi^t_{(X,\beta^+)}$ of (2.19) is called **the dynamical cone** of (S, X).

Remark 2.25. The dynamical cone is not unique and admits an attractor-repeller decomposition:

(i) The dynamical cone $\mathscr{S}(S, X, \beta^+)$ depends on $\beta^+ \in \mathscr{B}(\delta)$. However it is related by continuation to $S \subset [0, 1] \times M$.

(ii) For all β^+ with $\|\beta\|_{[0,1]} < \varepsilon_0$ as in Lemma 2.22 it follows that

$$\big(\{0\} \times S_0, \{1\} \times S_1\big)$$

is an attractor-repeller pair of $\mathscr{S}(S, X, \beta^+)$.

Lemma 2.26. *Let (S, X) be a continuation relating S_0 to S_1. Then there exists suspension isomorphisms*

$$\sigma_0 : HC_*(S_0) \xrightarrow{\cong} HC_{*+1}(\{0\} \times S_0)$$

and

$$\sigma_1 : HC_*(S_1) \xrightarrow{\cong} HC_*(\{1\} \times S_1)$$

between the Conley indicies of S_0, S_1 and $\{0\} \times S_0, \{1\} \times S_1$ with respect to the correspondig flows.

Proof. Let $(N_0, L_0), (N_1, L_1)$ be index pairs for S_0, S_1. Then

$$(\tilde{N}_0, \tilde{L}_0) := ([-\delta, \delta] \times N_0, [-\delta, \delta] \times L_0 \cup \{\pm\delta\} \times N_0)$$

is an index pair for $\{0\} \times S_0$ with respect to the flow $\Psi^t_{(X,\beta^+)}$ and

$$(\tilde{N}_1, \tilde{L}_1) := ([1-\delta, 1+\delta] \times N_1, [1-\delta, 1+\delta] \times L_1)$$

is an index pair for $\{1\} \times S_1$. Note that $\tilde{N}_0/\tilde{L}_0$ is the smash product of N_0/L_0 and the index of the invariant set $\{0\}$ with respect to the flow of $\dot{\lambda}(t) = \beta(\lambda(t))$ being the pointed one sphere. Hence on homology we get

$$H_*(\{0\} \times S_0) \cong H_1(\{0\}) \otimes H_*(S_0)$$

and analogously

$$HC_*(\{1\} \times S_1) \cong HC_0(\{1\}) \otimes HC_*(S_1).$$

The suspension isomorphisms are defined as

$$\sigma_0 : HC_*(S_0) \xrightarrow{\Delta_0 \otimes} HC_1(\{0\}) \otimes HC_*(S_0) \cong HC_{*+1}(\{0\} \times S_0)$$

and

$$\sigma_1 : HC_*(S_1) \xrightarrow{\Delta_1 \otimes} HC_0(\{1\}) \otimes HC_*(S_1) \cong HC_*(\{1\} \times S_1).$$

Here Δ_0, Δ_1 are generators of the Conley index of $\{0\}, \{1\}$ w.r.t. the flow of $\dot{\lambda}(t) = \beta(\lambda(t))$. $\square$

The boundary operator of of the attractor-repeller pair $(\{0\} \times S_0, \{1\} \times S_1)$ essentially is the "same" as the Conley continuation map $\Phi_{\mathbf{con}}(S, X) : HC_*(S_0) \to HC_*(S_1)$.

Proposition 2.27 (McCord-Mischaikov [7])**.** *Let (S, X) be a continuation relating S_0 to S_1 and let $\mathscr{S}(S, X, \beta^+)$ be the dynamical cone as in Definition 2.24. Moreover let $\partial : HC_{*+1}(\{0\} \times S_0) \to HC_*(\{1\} \times S_1)$ is the boundary operator in the long exact sequence (1.5) of the attractor-repeller pair $(\{0\} \times S_0, \{1\} \times S_1)$ of $\mathscr{S}(S, X, \beta^+)$. Then the boundary operator of (1.5) of the attractor-repeller pair $(\{0\} \times S_0, \{1\} \times S_1)$ of $\mathscr{S}(S, X, \beta^+)$ is computed via the Conley continuation map, i.e. the diagramm*

$$\begin{tikzcd}
HC_{k+1}(\{0\} \times S_0) \arrow[r, "\partial"] & HC_k(\{1\} \times S_1) \\
HC_k(S_0) \arrow[u, "\sigma_0"] \arrow[r, "\Phi_{\mathbf{con}}(S,X)"] & HC_k(S_1) \arrow[u, "\sigma_1"']
\end{tikzcd}$$

commutes.

Proof. Step 1. First assume that $(S,X) = (S_0, X_0)$ is a constant continuation. By assumption S_0 is an isolated invariant set. It follows from Remark 2.20 (i) that $\mathscr{S}(S_0, X_0, \beta^+) = [0,1] \times S_0$ for any $\beta^+ \in \mathscr{B}^+(\delta)$ with $\|\beta^+\| < \varepsilon_0$. Pick a constant index pair (N, L) for S_0 the following triple

$$\tilde{N}_2 := [-\delta, 1+\delta] \times N$$
$$\tilde{N}_1 := \big([1/2, 1+\delta] \cup \{-\delta\}\big) \times N \cup [-\delta, 1+\delta] \times L$$
$$\tilde{N}_0 := \{-\delta\} \times N \cup [-\delta, 1+\delta] \times L$$

is an index triple of the attractor-repeller pair $(\{0\} \times S_0, \{1\} \times S_1)$ for $\mathscr{S}(S, X, \beta^+)$. The index triple $(\tilde{N}_2, \tilde{N}_1, \tilde{N}_0)$ is homotopic to

$$\big([0,1] \times N, (\{0\} \cup [1/2, 1]) \times N \cup [0,1] \times L, \{0\} \times N \cup [0,1] \times L\big).$$

Using the definitions of boundary map and the definition of the index triple one sees that the boundary map is just projection to the second factor i.e. the diagramm

$$\begin{tikzcd}
H_{k+1}(\tilde{N}_2, \tilde{N}_1) \arrow[r, "\partial^{X_0}"] & H_k(\tilde{N}_1, \tilde{N}_0) \\
H_1([0,1], \{0,1\}) \otimes H_k(N, L) \arrow[u, "\cong"] \arrow[r, "pr_2"] & H_k(N, L) \arrow[u, "\sigma(X_0)"'] \arrow[d, "\mathbb{1}"] \\
H_k(N, L) \arrow[u, "\sigma(X_0)"] \arrow[r, "\Phi_{\mathbf{flo}} = \mathbb{1}"] & H_k(N, L)
\end{tikzcd}$$

commutes. This proves step one.

Step 2. Pick a homotopy of families

$$\mathscr{X} = \{X_s\}_{s \in [0,1]} := \{X_{s \cdot \lambda}\}_{s, \lambda \in [0,1]}$$

connecting the constant family X_0 to $X_1 = \{X_\lambda\}_{\lambda \in [0,1]}$. Let the isolated invariant set in the product be defined by

$$\mathscr{S} := \{(s, \lambda, x) \mid x \in S_{s \cdot \lambda}\} \subset [0,1] \times ([-\delta, 1+\delta] \times M)\,.$$

This determines a continuation $\big(\mathscr{S}, \mathscr{X}\big)$ relating $\mathscr{S}_0 := \mathscr{S}(S_0, X_0, \beta^+) = [0,1] \times S_0$ to $\mathscr{S}_1 := \mathscr{S}(S, X, \beta^+)$. For $s = 1$ we get the following commuting diagramm of attractor-repeller pairs, see McCord-Mishaikov [7]:

$$\begin{tikzcd}
\cdots \arrow[r] & HC_{k+1}(\mathscr{S}(S, X, \beta^+)) \arrow[r] & HC_{k+1}(\{0\} \times S_0) \arrow[r, "\partial^X"] & HC_k(\{1\} \times S_1) \arrow[r] & \cdots \\
\cdots \arrow[r] & HC_{k+1}(\mathscr{S}(S_0, X_0, \beta^+)) \arrow[r] \arrow[u, "\Phi_{\mathbf{con}}(\mathscr{S}, \mathscr{X})"] & HC_{k+1}(\{0\} \times S_0)) \arrow[r, "\partial^{X_0}"] \arrow[u, "\Phi_{\mathbf{con}}(0, \mathscr{S}, \mathscr{X})"] & HC_k(\{1\} \times S_0) \arrow[r] \arrow[u, "\Phi_{\mathbf{con}}(1, \mathscr{S}, \mathscr{X})"'] & \cdots
\end{tikzcd}$$

Note that $\Phi_{\mathbf{con}}(0, \mathscr{S}, \mathscr{X})$ and $\Phi_{\mathbf{con}}(1, \mathscr{S}, \mathscr{X})$ are the continuation maps induced by continuations of the repeller and attractor respectively. The map $\Phi_{\mathbf{con}}(1, \mathscr{S}, \mathscr{X})$ factors as

$$\begin{tikzcd} HC_k(\{1\} \times S_0) \arrow[r, "\Phi_{\mathbf{con}}(1,\mathscr{S},\mathscr{X})"] & HC_k(\{1\} \times S_1) \\ HC_k(S_0) \arrow[u, "\sigma_1(X_0)"] \arrow[r, "\Phi_{\mathbf{con}}(S,X)"] & H_k(S_1) \arrow[u, "\sigma_1(X)"'] \end{tikzcd}$$

hence

$$\Phi_{\mathbf{con}}(1, \mathscr{S}, \mathscr{X}) = \sigma_1(X) \circ \Phi_{\mathbf{con}}(S, X) \circ \sigma_1^{-1}(X_0).$$

On the other hand the middle arrow is the identity map. Putting these equalities together we get

$$\begin{aligned} \partial^X \circ \sigma_0(X) &= \Phi_{\mathbf{con}}(1, \mathscr{S}, \mathscr{X}) \circ \partial^{X_0} \circ \Phi_{\mathbf{con}}^{-1}(\{0\} \times S, X) \circ \sigma_0(X) \\ &= \sigma_1(X) \circ \Phi_{\mathbf{con}}(S, X) \circ \sigma_1^{-1}(X_0) \circ \partial^{X_0} \circ \underbrace{\Phi_{\mathbf{con}}(1, \mathscr{S}, \mathscr{X})^{-1} \circ \sigma_0(X)}_{=\sigma_0(X_0)} \\ &= \sigma_1(X) \circ \Phi_{\mathbf{con}}(S, X) \circ \underbrace{\sigma_1^{-1}(X_0) \circ \partial^{X_0} \circ \sigma_0(X_0)}_{=\mathbb{1}} \\ &= \sigma_1(X) \circ \Phi_{\mathbf{con}}(S, X) \end{aligned}$$

This proves the theorem. ☐

The dynamical cone $\mathscr{S}(S, X, \beta^+)$ may be related by continuation to the empty set by adding a vector field to β^+. In that sence the dynamical cone is trivial. Hence its Conley index is trivial.

Lemma 2.28. *Let $\mathscr{S}(S, X, \beta^+)$ be the dynamical cone of (S, X). Then*

$$HC_k(\mathscr{S}(S, X, \beta^+)) = 0 \quad \textit{for all } k \in \mathbb{Z}.$$

Proof. Note that $\mathscr{S}(S, X, \beta)$ is related by continuation to $[0, 1] \times S_0$ as in Step 2 in the proof of Proposition 2.27. Hence by invariance of the Conley index (Theorem 2.14) and Example 2.11 we get

$$HC_k\big(\mathscr{S}(S, X, \beta)\big) \cong HC_k\big([0, 1] \times S_0\big) = \underbrace{HC_k([0, 1])}_{=0} \otimes HC_k(S_0) = 0$$

for all integers k. The last equality holds by the smash product formula of Lemma 2.10 part **(ii)**. ☐

Chapter 3

Morse Theory

In this chapter we give some background on local Morse theory. Any isolated invariant set S admits a Conley index $HC_*(S)$. On the other hand a *Morse-Smale triple* (see Definition 3.1) admits a *local Morse homology* $HM_*(S, f, g)$. Theorem A asserts that for a Morse-Smale triple (S, f, g) its Conley index $HC_*(S) = HC_*(S, f, g)$ and its local Morse homology $HM_*(S, f) = HM_*(S, f, g)$ are isomorphic. A continuation (S, f, g) relating two Morse-Smale triples (S_0, f_0, g_0) and (S_1, f_1, g_1) induces a *dynamical cone* $\mathscr{S}_\varepsilon(S, f, g)$. For generic (f, g) the dynamical cone $\mathscr{S}_\varepsilon(S, f, g)$ determines a *local Morse homology of the dynamical cone* $HM_*(\mathscr{S}_\varepsilon(S, f, g))$. The local Morse homology of the cone $HM_*(\mathscr{S}_\varepsilon(S, f, g))$ fits into a long exact sequence where also the Morse homologies $HM_*(S_0, f_0, g_0)$ and $HM_*(S_1, f_1, g_1)$ appear.

3.1 The local Morse complex

Let M be a smooth manifold without boundary, possibly non-compact and let $f : M \to \mathbb{R}$ be a smooth function. Moreover pick a smooth metric g on M. We call the pair (f, g) *gradient pair* and denote its negative gradient by $-\nabla f$. Consider the flow φ^t generated by the gradient pair (f, g) via equation (1.3) being

$$\dot{x} = -\nabla f(x).$$

If f is Morse the hyperbolic rest points of φ^t correspond to the critical points $\mathrm{Crit}(f) := \{x \in M \mid df(x) = 0\}$. For a critical point x the unstable set

$$W^u(x) := \left\{p \in M \mid \lim_{t\to-\infty} \varphi^t(p) = x\right\}$$

is a submanifold of M called the *unstable manifold* of x. Likewise we define the *stable manifold* by

$$W^s(x) := \{p \in M \mid \lim_{t\to\infty} \varphi^t(p) = x\}.$$

The dimension of the unstable manifold is the *Morse index* of the critical point x and is denoted by $|x| := \dim W^u(x)$.

Definition 3.1. Let $S \subset M$ be an isolated invariant set of the gradient flow φ^t generated by (f, g). The triple (S, f, g) is called **Morse-Smale triple** if the following holds:

(i) $\mathrm{Crit}(S, f) := \{x \in S \mid df(x) = 0\}$ are non-degenerate.

(ii) If $x, y \in \mathrm{Crit}(S, f)$ and $p \in W^u(y) \cap W^s(x) \cap S$ then $W^u(y)$ and $W^s(x)$ intersect *transversally* at p.

It follows that for all $p \in S \setminus \mathrm{Crit}(S, f)$ there exists a pair $y, x \in \mathrm{Crit}(S, f)$ such that $p \in W^u(y) \cap W^s(x)$. If (S, f, g) is Morse-Smale the critical points $\mathrm{Crit}(S, f)$ together with the connecting trajectories of the negative gradient flow define a chain complex $(CM_*(S, f), \partial^S)$ in the following way: First choose orientations $\{o_x\}$ of the tangent spaces $T_xW^u(x)$ for every critical point $x \in \mathrm{Crit}(S, f)$. Since the unstable manifold $W^u(x)$ is contractible o_x induces an orientation of $W^u(x)$ for all critical points x. Denote by $\langle x\rangle := (x, o_x)$ the pair consisting of the critical point $x \in \mathrm{Crit}(S, f)$ and the orientation of its unstable tangent spaces. The free abelian groups generated by $\langle x\rangle$ are denoted by

$$CM_k(S, f) := \bigoplus_{|x|=k} \mathbb{Z}\langle x\rangle.$$

Here the sum runs over critical points in S of Morse index k. Since (S, f, g) is Morse-Smale the intersection $W^u(y) \cap W^s(x) \cap S$ consists of finitely many trajectories in case $|y| - |x| = 1$. A solution curve $\gamma : \mathbb{R} \to S \subset M$ of (1.3) with boundary condition $\lim_{t\to-\infty} \gamma(t) = y$ and $\lim_{t\to\infty} \gamma(t) = x$ is called *connecting trajectory from y to x*. We attach a sign $n_S(\gamma)$ to any connecting trajectory from y to x in the following way. The pair $\langle y\rangle$ induces an orientation on the orthogonal complement $E^u_\gamma(y)$ of the vector

$$v := \lim_{t\to-\infty} \frac{\dot\gamma(t)}{|\dot\gamma(t)|} \in T_yW^u(y).$$

Since $|y| - |x| = 1$ the flow $d\varphi^t : T_\gamma M \to T_\gamma M$ induces an isomorphism $E^u_\gamma(y) \xrightarrow{i} T_xW^u(x)$. We define the sign $n_S(\gamma)$ of the connecting trajectory γ

to be $+1$ if the isomorphism is orientation preserving and -1 otherwise. The *algebraic number of connecting trajectories in S* from y to x is then defined to be

$$n_S(y,x) := \sum_{\gamma} n_S(\gamma).$$

Here the sum runs over (finitely many) connecting trajectories $\gamma : \mathbb{R} \to S$ from $y \in S$ to $x \in S$. If (S, f, g) is a Morse-Smale triple then the connecting trajectories determine a boundary operator $\partial^S : CM_*(S, f) \to CM_*(S, f)$ by

$$\partial^S \langle y \rangle := \sum_{|x|=|y|-1} n_S(y,x)\langle x \rangle$$

The sum runs over all critical points x with Morse index one lower that y. The chain complex $\big(CM_*(S, f), \partial^S\big)$ is called the **local Morse complex** of the Morse triple (S, f, g) and its homology $HM_*(CM_*(S, f))$ is the **local Morse homology** of the Morse-Smale triple (S, f, g).

3.2 The local Morse complex of the dynamical cone

As in Section 2.4 we discuss the dynamical cone. However we specialize to continuations of gradient triples. Assume that the Morse-Smale triples (S_0, f_0, g_0) and (S_1, f_1, g_1) are related by continuation via (S, f, g). We extend this family to $[-\delta, 1+\delta]$ by requiring that it is constant outside $(\delta, 1-\delta)$. Let

$$N := \bigcup_{\lambda \in [-\delta, 1+\delta]} \{\lambda\} \times N_\lambda$$

satisfying

$$N_\lambda = \begin{cases} N_0 & \text{for } -\delta \leqslant \lambda \leqslant \delta \\ N_1 & \text{for } 1-\delta \leqslant \lambda \leqslant 1+\delta \end{cases}$$

be an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, p) := (\lambda, \varphi_\lambda^t(p))$. Here φ_λ^t is the gradient flow of (f_λ, g_λ). For $\varepsilon > 0$ pick $\beta_\varepsilon \in \mathscr{B}^+(\delta)$ such that $\beta_\varepsilon^{-1}(0) = \{0, 1\}, \beta_\varepsilon'(0) > 0, \beta_\varepsilon'(1) < 0$ and $\beta_\varepsilon(\lambda) = \varepsilon \quad$ for $\delta \leqslant \lambda \leqslant 1-\delta$.

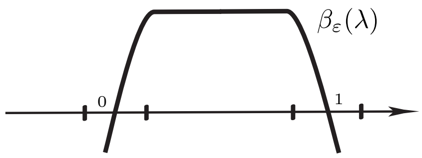


Figure 11. The smooth function $\beta_\varepsilon(\lambda)$ is constant $=\varepsilon$ on $[\delta, 1-\delta]$, decreasing outside and has its only zeroes at $\{0,1\}$.

The triple $(f, g, \beta_\varepsilon)$ determines a vector field on $[-\delta, 1+\delta]\times M$ by

$$(\lambda, p) \longmapsto \xi := (\beta_\varepsilon(\lambda), -\nabla f_\lambda(p))$$

with associated flow $\Psi^t_{(f,g,\beta_\varepsilon)}$ generated via

$$\left\{ \begin{array}{rcl} \dot\lambda(t) & = & \beta_\varepsilon(\lambda(t)) \\ \dot x(t) & = & -\nabla f_{\lambda(t)}(x(t)). \end{array} \right. \tag{3.2}$$

For small $\varepsilon_0 > 0$ the *dynamical cone*

$$\mathscr{S}_\varepsilon(S, f, g) := I\left(N, \Psi^t_{(f,g,\beta_\varepsilon)}\right)$$

is well-defined for all $\varepsilon \in (0, \varepsilon_0)$. For generic (f, g) the dynamical cone $\mathscr{S}_\varepsilon(S, f, g)$ is Morse-Smale in the sense of Definition 3.1. Slightly abusing notation we define

$$\mathrm{Crit}\left(\mathscr{S}_\varepsilon(S,f,g), \xi\right) := \{(\lambda, x) \in N \mid \xi(\lambda, x) = (0,0)\} \subset \{0,1\}\times M$$

for the zeroes of the vector field ξ. It follows from the definition that $\mathrm{Crit}(\mathscr{S}_\varepsilon(S,f,g),\xi)$ are non-degenerate. Denote by $W^u(\tilde x, \xi)$ the unstable manifold of $\tilde x$ with respect to the flow $\Psi^t_{(f,g,\beta_\varepsilon)}$ and by $|\tilde x| = \dim W^u(\tilde x, \xi)$ its Morse index. Choose orientations $\{o^0_y\}$ and $\{o^1_x\}$ for $y \in \mathrm{Crit}(S_0, f_0)$ and $x \in \mathrm{Crit}(S_1, f_1)$ of the unstable manifolds $W^u(y, f_0)$ and $W^u(x, f_1)$. Adding the standard orientation e_λ of $[-\delta, 1+\delta]$ to the orientations of $W^u(y, f_0)$ and $W^u(x, f_1)$ we orient the unstable manifolds $W^u(\tilde x, \xi)$ by

$$o_{\tilde x} := \left\{ \begin{array}{ll} e_\lambda \oplus o^0_x & \text{if } \tilde x = (0, x) \\ o^1_x & \text{if } \tilde x = (1, x) \end{array} \right.$$

Denote by $\langle \tilde x\rangle := (\tilde x, o_{\tilde x})$ the pair consisting of a critical point $\tilde x$ together with the orientation $o_{\tilde x}$ of its unstable manifold $W^u(\tilde x, \xi)$.

Definition 3.3. The local Morse complex of the dynamical cone $\mathscr{S}_\varepsilon(S, f, g)$ is defined by

$$CM_k(\mathscr{S}_\varepsilon(S,f,g),\xi) := \bigoplus_{|\tilde x| = k} \mathbb{Z}\langle \tilde x\rangle$$

where $\tilde x \in \mathrm{Crit}(\xi)$ and $|\tilde x| = \dim W^u(\tilde x, \xi)$. The boundary operator $\partial^{\mathscr{S}_\varepsilon(S,f,g)}$: $CM_k(\mathscr{S}_\varepsilon(S,f,g),\xi) \to CM_{k-1}(\mathscr{S}_\varepsilon(S,f,g),\xi)$ is defined by counting trajectories in $\mathscr{S}_\varepsilon(S,f,g)$:

$$\partial^{\mathscr{S}_\varepsilon(S,f,g)}\langle \tilde y\rangle = \sum_{|\tilde x| = k-1} n_{\mathscr{S}_\varepsilon(S,f,g)}(\tilde y, \tilde x)\langle \tilde x\rangle.$$

Here $n_{\mathscr{S}_\varepsilon(S,f,g)}(\tilde{y},\tilde{x})$ is the algebraic number of trajectories $\tilde{\gamma}:\mathbb{R}\to\mathscr{S}_\varepsilon(S,f,g)$ going from $\tilde{y}$ to $\tilde{x}$.

Lemma 3.4 (Hutchings)**.** *Let (S,f,g) be a continuation relating the Morse-Smale triples (S_0,f_0,g_0) and (S_1,f_1,g_1) and let $\mathscr{S}_\varepsilon(S,f,g)$ be the dynamical cone of (S,f,g). Then*

(i) *the local Morse complex $\big(CM_*(\mathscr{S}_\varepsilon(S,f,g),\xi),\partial^{\mathscr{S}_\varepsilon(S,f,g)}\big)$ decomposes canonically as*

$$CM_k(\mathscr{S}_\varepsilon(S,f,g),\xi)=CM_{k-1}(S_0,f_0)\oplus CM_k(S_1,f_1).$$

(ii) *the boundary operator takes the form*

$$\partial^{\mathscr{S}_\varepsilon(S,f,g)}=\begin{bmatrix}-\partial^{S_0} & 0\\ \Phi & \partial^{S_1}\end{bmatrix}$$

with respect to the decomposition in **(i)**.

Proof. (i). Let $\langle\tilde{x}\rangle=(\tilde{x},o_{\tilde{x}})$ be a generator of $CM_k(\mathscr{S}_\varepsilon(S,f,g),\xi)$. The decomposition $CM_k(\mathscr{S}_\varepsilon(S,f,g),\xi)=CM_{k-1}(S_0,f_0)\oplus CM_k(S_1,f_1)$ is given by the isomorphism

$$(\tilde{x},o_{\tilde{x}})\longmapsto\begin{cases}\big((x,o_x^1),0\big) & \text{if } \tilde{x}=(0,x)\\ \big(0,(x,o_x^0)\big) & \text{if } \tilde{x}=(1,x).\end{cases}$$

(ii). *Claim.* If $\tilde{y}=(0,y)$ and $\tilde{x}=(0,x)$ are critical points of ξ of Morse index difference $|y|-|x|=1$, then

$$n_{\mathscr{S}_\varepsilon(S,f,g)}(\tilde{y},\tilde{x})=-n_{S_0}(y,x).$$

Proof of the Claim. Pick a flow line $\gamma:\mathbb{R}\to M$ solving $\dot{x}=-\nabla f_0(x)$ connecting y to x and define $\tilde{\gamma}:\mathbb{R}\to\{0\}\times M$ being a solution of (3.2) connecting $\tilde{y}$ to $\tilde{x}$. The tangent flow induces an isomorphism

$$\tilde{E}_{\tilde{\gamma}}\to T_{\tilde{x}}W^u(\tilde{x},\xi).$$

Where $\tilde{E}_{\tilde{\gamma}}$ is the orthogonal complement of $\tilde{v}:=\lim_{t\to-\infty}\frac{\dot{\tilde{\gamma}}}{|\dot{\tilde{\gamma}}|}$ in $T_{\tilde{y}}W^u(\tilde{y},\xi)$. Pick a positive basis

$$e_\lambda,\tilde{v},\xi_1,\dots,\xi_k \quad\text{of}\quad T_{\tilde{y}}W^u(\tilde{y},\xi).$$

Then $-\{\tilde{v},e_\lambda,\xi_1,\dots,\xi_k\}$ is a positive basis of $W^u(\tilde{y},\xi)$ and $-\{e_\lambda,\xi_1,\dots,\xi_k\}$ is a positive basis of $\tilde{E}_{\tilde{\gamma}}$. The tangent flow sends the latter basis to the positive basis

$$-n_{S_0}(y,x)\{e_\lambda,\eta_1,\dots,\eta_k\}\quad\text{of}\quad T_{\tilde{x}}W^u(\tilde{x},\xi).$$

Hence the tangent flow sends $\{e_\lambda, \xi_1, \ldots, \xi_k\}$ to

$$-n_{S_0}(y,x)\{e_\lambda, \eta_1, \ldots, \eta_k\}.$$

This proves the claim.
If $\tilde{y} = (1,y)$ and $\tilde{x} = (1,x)$ the sign does not change, hence

$$n_{\mathscr{S}_\varepsilon(S,f,g)}(\tilde{y},\tilde{x}) = n_{S_1}(y,x).$$

If $\tilde{y} = (0,y)$ and $\tilde{x} = (1,y)$ with $|\tilde{y}| - |\tilde{x}| = 1$ then $n_{\mathscr{S}_\varepsilon(S,f,g)}(\tilde{y},\tilde{x})$ is the algebraic number of trajectories going from $\tilde{y}$ to $\tilde{x}$. This proves the Lemma.

□

The long exact sequence of the dynamical cone

The local Morse complex of the dynamical cone $\left(CM_*(\mathscr{S}_\varepsilon(S,f,g),\xi), \partial^{\mathscr{S}_\varepsilon(S,f,g)}\right)$ determines the short exact sequence

$$0 \longrightarrow CM_*(S_1,f_1) \longrightarrow CM_*\left(\mathscr{S}_\varepsilon(S,f,g),\xi\right) \longrightarrow CM_{*-1}(S_0,f_0) \longrightarrow 0. \tag{3.5}$$

The short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & CM_i(S_1,f_1) \arrow[r] \arrow[d, "\partial"'] & CM_{i-1}(S_0,f_0) \oplus CM_i(S_1,f_1) \arrow[r] \arrow[d, "\begin{bmatrix} -\partial^{S_0} & 0 \\ \Phi & \partial^{S_1} \end{bmatrix}"'] & CM_{i-1}(S_0,f_0) \arrow[r] \arrow[d, "\partial"'] \arrow[l, bend right, "s"'] & 0 \\
0 \arrow[r] & CM_{i-1}(S_1,f_1) \arrow[r] & CM_{i-2}(S_0,f_0) \oplus CM_{i-1}(S_1,f_1) \arrow[r] \arrow[l, bend left, "r"] & CM_{i-2}(S_1,f_1) \arrow[r] & 0
\end{tikzcd}$$

splits with maps $s(c) = (c,0)$ and $r(c,d) = d$. Hence the long exact sequence of (3.5) takes the form

$$HM_k(S_0,f_0) \longrightarrow HM_k(\mathscr{S}_\varepsilon(S,f,g),\xi) \longrightarrow HM_{k-1}(S_0,f_0) \xrightarrow{\Phi} HM_{k-1}(S_1,f_1)\ . \tag{3.6}$$

Proposition 3.7. *Let (S,f,g) be a continuation relating the Morse-Smale triple (S_0,f_0,g_0) to the Morse-Smale triple (S_1,f_1,g_1). Then*

$$\Phi(S,f,g) : HM_*(S_0,f_0,g_0) \xrightarrow{\cong} HM_*(S_1,f_1,g_1)$$

is an isomorphism.

Proof. Theorem A and Lemma 2.28 imply $HM_k(\mathscr{S}_\varepsilon(S,f,g)) \cong HC_k(\mathscr{S}_\varepsilon(S,f,g)) = 0$ for all $k \in \mathbb{Z}$. Inserting this into the long exact sequence (3.6) and get for any integer

$$0 \longrightarrow HM_{k-1}(S_0,f_0,g_0) \xrightarrow{\Phi} HM_{k-1}(S_1,f_1,g_1) \longrightarrow 0$$

which proves the assertion. □

3.3 Proof of Theorem A and Theorem B

First we show that the Conley connection matrix commutes with the Morse boundary operator; using this in a second step we prove Theorem A. For $y, x \in \mathrm{Crit}(S, f)$ with $|y| = k$ and $|x| = k-1$ it follows by the Morse-Smale condition that $C(y, x, S) := W^u(y) \pitchfork W^s(x) \cap S$ is a submanifold of dimension 1. Since $|y| - |x| = 1$ the trajectories in $C(y, x, S)$ can not break, so

$$S(y, x) := C(y, x) \cup \{y, x\}$$

is an *isolated invariant set.* Hence there exists an index pair (N_2, N_0) of $S(y, x)$. Defining $N_1 := \big(N_2 \cap \{f \leqslant a\}\big) \cup N_0$ for a regular value $f(y) < a < f(x)$, we get an index filtration (N_2, N_1, N_0) such that (N_2, N_1) is an index pair of y and (N_1, N_0) is an index pair for x. Define the map

$$\Delta_k^S(y, x) : H_k(N_y, L_y) \cong H_k(N_2, N_1) \xrightarrow{\partial} H_{k-1}(N_1, N_0) \cong H_{k-1}(N_x, L_x).$$

Here ∂ is the connecting homomorphism of triples and the isomorphisms are induced by corresponding flow maps (2.5).

Definition 3.8. The **cellular Conley complex** $C\Delta_*(S)$ is defined by

$$C\Delta_k(S) := \bigoplus_{|x|=k} H_k(N_x, L_x) = \bigoplus_{|x|=k} HC_k(x) \tag{3.9}$$

with boundary map $\Delta^S : C\Delta_k(S) \to C\Delta_{k-1}(S)$ defined by

$$\Delta^S\langle y\rangle = \sum_{|x|=k-1} \Delta^S(y, x)\langle x\rangle \tag{3.10}$$

being called **Conley connection matrix**.

Note that there exists an isomorphism $\iota : C\Delta_k(S) \to CM_k(S, f)$ after one choses orientations of unstable manifolds $W^u(x, f)$.

Remark 3.11. Let (S, f, g) be Morse-Smale. By Remark 2.8 the identitification

$$\iota : [W^u(x, f, N_x)] \mapsto (x, o_x) = \langle x\rangle \tag{3.12}$$

yields the canonical isomorphism $\iota : C\Delta_k(S) = \bigoplus_{|x|=k} H_k(N_x, L_x) \to CM_k(S, f)$.

The notion of an index triple can be generalized as follows: define subsets $S_{kj} \subset S$ by

$$S_{kj} := \bigcup_{j \leqslant |x| \leqslant |y| \leqslant k} C(y, x, S).$$

Since (S, f, g) is Morse-Smale the sets S_{kj} are compact for any pair $k \leqslant j$, or equivalently, S_{kj} is an isolated invariant set. Compactness follows by the fact that any sequence of connecting trajectories $\gamma_\nu(t)$ from y to x converges (uniformely on compact sets) to a finite collection of trajectories $\gamma^j(t)$ connecting x^j to x^{j-1} with $|x| \leqslant |x^{j-1}| < |x^j| \leqslant |y|$. The set S_{kj} is an isolated invariant set with $S_{kk} = \mathrm{Crit}_k(S, f)$. One crucial fact in the proof of Theorem A is the existence an *index filtration*

$$L = N_{-1} \subset N_0 \subset \cdots \subset N_n = N \tag{3.13}$$

such that (N_k, N_{j-1}) is an index pair for S_{kj}, for a proof see [2]. In particular (N_k, N_{k-1}) and (N, L) are index pairs for $\mathrm{Crit}_k(S, f)$ and S respectively. The essential property of (3.13) is that it is *cellular* meaning that the homology groups $H_*(N_k, N_{k-1})$ are zero in all degrees except of degree k, as proven in part (i) of Lemma 3.17. The other main observation to prove Theorem A is that the Conley complex $(C\Delta_*(S), \Delta^S)$ is chain isomorphic to the Morse complex $(CM_*(S, f), \partial^S)$.

Lemma 3.14 (Salamon [11]). *For all $k \in \mathbb{N}$ the following diagram commutes*

$$\begin{tikzcd}
CM_{k+1}(S,f) \arrow[r, "\partial^S"] & CM_k(S,f) \\
C\Delta_{k+1}(S) \arrow[u, "\iota"] \arrow[r, "\Delta^S"] & C\Delta_k(S) \arrow[u, "\iota"'] \\
H_{k+1}(N_{k+1}, N_k) \arrow[u, "\phi"] \arrow[r, "\partial"] & H_k(N_k, N_{k-1}) \arrow[u, "\phi"]
\end{tikzcd} .$$

The vertical maps ι are the canonical isomorphisms of Remark 3.11 and the vertical maps ϕ are the flow maps (2.5).

Proof of Lemma 3.14. Let N be an isolating neighborhood of S. By altering the Morse function $f : M \to \mathbb{R}$ outside $S(y, x)$ we may assume that y and x are the only critical points in $f^{-1}([a, b]) \cap N$ with $f(y) = b$ and $f(x) = a$. This alteration does *not* affect the homomorphisms ∂^S and Δ^S. For sufficiently small $\varepsilon > 0$ and sufficiently large $T > 0$ we define index pairs

$$N_y := \left\{p \in N \mid f(\varphi^{-T}(p)) \leqslant b+\varepsilon,\ f(p) \geqslant c\right\} \text{ and } L_y := \left\{p \in N_y \mid f(p) = c\right\}$$

for y and

$$N_x := \left\{p \in N \mid f(\varphi^{T}(p)) \geqslant a-\varepsilon, f(p) \leqslant c\right\},\ L_x := \left\{p \in N_x \mid f(\varphi^{T}(p)) = a-\varepsilon\right\}$$

for x. Note: N_y is a tubular neighborhood of $W^u(y,N) \cap \{f \geqslant c\}$ and N_x is a tubular neighborhood of $W^s(x,N) \cap \{f \leqslant c\}$. Their widths converge to zero as $T \to \infty$. Now define the index triple filtration (N_2, N_1, N_0) of $S(y,x)$ in N by

$$N_2 := N_y \cup N_x, \ N_1 := N_x \cup L_y \text{ and } N_0 = L_x \cup \mathrm{cl}(L_y \backslash N_x).$$

We will use the triple (N_2, N_1, N_0) in order to prove the lemma. Since $W^u(y)$ and $W^s(x)$ intersect transversally, it follows that $N_x \cap W^u(y) \cap \{f = c\}$ consists of finitely many $k-$dimensional components $V_1, \ldots, V_n$ each containing a unique intersection point $z_j \in C(y,x,S) \cap V_j$.

Claim. Denote by $B^\ell \subset \mathbb{R}^\ell$ the closed ball in the euclidean space $\mathbb{R}^\ell$. There exists a diffeomorphism

$$\Psi_x : N_x \to B^k \times B^{m-k}$$

satisfying the following

(i) $\Psi_x(N_x \cap W^s(x)) = \{0\} \times B^{m-k}$

(ii) $\Psi_x(L_x) = \partial B^k \times B^{m-k}$

(iii) $\Psi_x(V_j) = B^k \times \{\tilde{z}_j\}, \ \Psi_x(z_j) = \tilde{z}_j \in \partial B^{m-k}$ for all $j = 1, \ldots, n$.

(iv) $\partial V_j = V_j \cap L_x$ for all $j = 1, \ldots, n$.

In particular V_j is diffeomorphic to B^k with $\partial V_j = V_j \cap L_x$

Proof of the Claim. Since N_x is a tubular neighborhood of $W^s(x) \cap N_x$ and $W^s(x) \cap N_x$ is contractible, we can pick a (global) trivialization Ψ_x satisfying (i)-(iii). Property (iv) follows from (i) and (ii). This shows the Claim.

Note that the inclusion of pairs $i : (V_j, \partial V_j) \to (N_x, L_x)$ induces an isomorphism

$$H_k(V_j, \partial V_j) \cong H_k(N_x, L_x).$$

The generator $\langle x \rangle = (x, o_x)$ determines a generator $[W^u(x)] \in H_k(N_x, L_x)$, which under the above isomorphism determines a generator

$$\alpha_j\big([W^u(x)]\big) \in H_k(V_j, \partial V_j).$$

The generator $\alpha_j([W^u(x)])$ is determined by the orientation of V_j induced from $T_x W^u(x)$ via the flow $d\varphi^t : T_{z_j} V_j \to T_x W^u(x)$. There is another generator $\beta_j([W^u(y)]) \in H_k(V_j, \partial V_j)$ induced by the generator $\langle y \rangle = (y, o_y)$ by completing $-\nabla f(z_j)$ to a positive basis

$$-\nabla f(z_j), \xi_2, \ldots, \xi_{k+1}.$$

Then the orientation of $T_{z_j}V_j$ induced by $\xi_2, \dots, \xi_{k+1}$ determines a generator

$$\beta_j([W^u(y)]) \in H_k(V_j, \partial V_j).$$

The generators $\alpha_j([W^u(x)]), \beta_j([W^u(y)])$ agree if and only if $n(\gamma) = +1$, i.e. the following equation holds

$$\beta_j\big([W^u(y)]\big) = n(\gamma_j)\alpha_j\big([W^u(x)]\big) \quad \text{for } j = 1, \dots, n. \tag{3.15}$$

where $\gamma_j(t) := \varphi^t(z_j)$ is the connecting trajectory going through z_j. So far we have a space $H_k(V_j, \partial V_j)$ where we can compare the orientations coming from $H_{k+1}(N_y, L_y)$ and $H_k(N_x, L_x)$. Next we relate this to the connecting homomorphism ∂ of the triple (N_2, N_1, N_0). Since $(W^u(y) \cap N_y, W^u(y) \cap L_y)$ is a neighborhood deformation retract of (N_y, L_y) there is a canonical isomorphism

$$H_{k+1}\big(W^u(y) \cap N_y, W^u(y) \cap L_y\big) \cong H_{k+1}(N_y, L_y).$$

Consider the triple

$$\mathrm{cl}\Big(W^u(y) \cap (L_y \backslash V_j)\Big) \subset \underbrace{W^u(y) \cap L_y}_{=:S^u(y)} \subset W^u(y) \cap N_y$$

with connecting homomorphism ∂_j. From the exision theorem it follows

$$H_k\big(S^u(y) - L_y\backslash V_j, \mathrm{cl}(W^u(y) \cap (L_y\backslash V_j)) - L_y\backslash V_j\big) \cong H_k(V_j, \partial V_j).$$

In the case of one connecting trajectory γ_j Conleys connection matrix Δ_j factors as:

$$\begin{tikzcd}
H_{k+1}(W^u(y) \cap N_y, W^u(y) \cap L_y) \arrow[r, "\partial_j"] & H_k(S^u(y), \mathrm{cl}(W^u(y) \cap L_y \backslash V_j)) \arrow[d] \\
 & H_k(V_j, \partial V_j) \arrow[d] \\
H_{k+1}(N_y, L_y) \arrow[uu] \arrow[r, "\Delta_j"] & H_k(N_x, L_x).
\end{tikzcd}$$

For a generator $[W^u(y)] \in H_{k+1}(N_y, L_y)$ given by $\langle y \rangle = (y, o_y)$ we have that

$$\partial_j\big([W^u(y)]\big) = \beta_j\big([W^u(y)]\big). \tag{3.16}$$

We used the canonical vertical isomorphisms without mentioning it. In the general case of many connecting orbits the diagram becomes

$$\begin{tikzcd}
H_{k+1}(W^u(y)\cap N_y, S^u(y)) \arrow[r, "\partial"] & H_k(S^u(y), \mathrm{cl}(W^u(y)-L_y\backslash \sqcup V_j)) \arrow[d] \\
& \bigoplus_j H_k(V_j,\partial V_j) \arrow[d] \\
H_{k+1}(N_y, L_y) \arrow[uu] \arrow[r, "\Delta"] & H_k(N_x, L_x).
\end{tikzcd}$$

Pick a generator $[W^u(y)] \in H_{k+1}(N_y, L_y)$ correspondig to $\langle y\rangle = (y, o_y)$. Following the diagram we get

$$\begin{aligned}
\Delta([W^u(y)]) &= \partial([W^u(y)]) \\
&= \sum_{j=1}^{n} \beta_j([W^u(y)]) \in \bigoplus H_k(V_j,\partial V_j) \\
&= \sum_{j=1}^{n} n(\gamma_j)\alpha_j([W^u(x)]) \\
&= n_S(y,x)[W^u(x)] \in H_x(N_x, L_x)
\end{aligned}$$

We have used the vertical homomorphisms without denoting it. Using this equation we get

$$\Delta\circ\iota^{-1}\langle y\rangle = \Delta([W^u(y)]) = n_S(y,x)[W^u(x)] = \iota\circ\partial^S\langle x\rangle$$

where $\iota([W^u(x)]) = \langle x\rangle$. This proves Lemma 3.14. □

The properties of the index filtration (3.13) are crucial for the proof of Theorem A and are summarized in the following lemma.

Lemma 3.17. *Any index filtration as in (3.13) is celluar meanig:*

(i) $H_n(N_k, N_{k-1}) = 0$ *for all* $n \neq k$.

(ii) $H_n(N_k, N_\ell) = 0$ *for* $k \geqslant \ell \geqslant n$ *or* $n > k \geqslant \ell$.

(iii) $H_n(N, N_j) \cong H_n(N_k, N_j)$ *for* $k > n$ *and* $k \geqslant j$.

Proof of Lemma 3.17. Since the gradient flow is Morse-Smale it follows by Remark 2.8 and Lemma 2.10 that $N_k/N_{k-1} \simeq \vee S^k$. Hence (i) follows by

$$H_j(N_k, N_{k-1}) = \begin{cases} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \text{for } j = k \\ 0 & \text{for } j \neq k. \end{cases}$$

Property (ii) is shown by induction over the difference $k - \ell$. For $k - \ell = 0$ (ii) is trivial. Consider the exact sequence of the triple $(N_k, N_{\ell+1}, N_\ell)$

$$\cdots \to \underbrace{H_n(N_{\ell+1}, N_\ell)}_{=0} \to H_n(N_k, N_\ell) \to \underbrace{H_n(N_k, N_{\ell+1})}_{=0} \to \ldots$$

The left group vanishes because of (i) the right one vanishes because of the induction assumption. Hence the middle group vanishes and so (ii) is proved. In order to show (iii) consider the exact sequence of the triple (N, N_k, N_j)

$$\cdots \to \underbrace{H_n(N_k, N_j)}_{=0} \to H_n(N, N_j) \to H_n(N, N_k) \to \underbrace{H_{n-1}(N_k, N_j)}_{=0} \to \ldots$$

The terms on the left and right vanish by (ii). This proves (iii) and the Lemma 3.17. $\square$

Proof of Theorem A. Consider the commutative diagram

$$\begin{tikzcd}
 & H_{k+1}(N_{k+1}, N_k) \arrow[d, "\delta_*"'] \arrow[dr, "\partial_{k+1}"] & & 0 \arrow[d] \\
0 \arrow[r] & H_k(N_k, L) \arrow[r, hook, "\beta"'] \arrow[d, two heads] & H_k(N_k, N_{k-1}) \arrow[r, "\delta_k"] \arrow[dr, "\partial_k"] & H_{k-1}(N_{k-1}, L) \arrow[d, hook, "j_*"] \\
 & H_k(N_{k+1}, L) \arrow[d] & & H_{k-1}(N_{k-1}, N_{k-2}) \\
 & 0 & &
\end{tikzcd} \tag{3.18}$$

Any sequence in diagram (3.18) is a part of an exact sequence of the appropriate triple (N_{k+1}, N_k, L), (N_k, N_{k-1}, L) and (N_{k-1}, N_{k-2}, L). The zeroes in the diagram above appear because of Lemma 3.17 part (i) and (ii). We have the following canonical isomorphisms

$$\begin{array}{rcll} H_k(N, L) & \cong & H_k(N_{k+1}, L) & \text{Lemma 3.17 (iii)} \\ & \cong & H_k(N_k, L)/\mathrm{im}(\delta_*) & \text{exactness of the left column} \end{array}$$

and

$$\begin{array}{rcll} \mathrm{im}(\beta)/\mathrm{im}(\beta\delta_*) & \cong & \ker(\delta_k)/\mathrm{im}(\Delta_{k+1}) & \text{exactness of the row} \\ & \cong & \ker(j_*\delta_k)/\mathrm{im}(\Delta_{k+1}) & \text{injectivity of } j_* \\ & \cong & \ker(\partial_k)/\mathrm{im}(\partial_{k+1}) & \text{commutativity} \\ & \cong & \ker(\Delta_k)/\mathrm{im}(\Delta_{k+1}) & \text{Lemma 3.14} \end{array}$$

The injectivity of β together with these isomorphisms gives an isomorphism

$$\begin{array}{rcl} \beta : H_k(N,L) & \longrightarrow & \ker(\Delta_k)/\mathrm{im}(\Delta_{k+1}) =: H_k(C\Delta(S)) \\ z & \longmapsto & [\beta(z)] \end{array}$$

It follows that $H_*(N,L)$ is isomorphic to the homology of the Conley complex $(C\Delta_*S), \Delta(S))$. Lemma 3.14 implies that $H_*(C\Delta(S))$ is isomorphic to the Morse homology $HM_*(S,f,g)$. Hence we obtain the following isomorphism

$$\alpha := \iota\circ\beta : H_*(N,L) \xrightarrow{\beta} H_*\big(C\Delta(S)\big) \xrightarrow{\iota} HM_*(S,f,g).$$

This proves Theorem A. □

Proof of Theorem B. Let (S,f,g) be a Morse-Smale triple with the attractor-repeller pair (A^+,A^-) of S. We prove the commutativity of the following diagram:

$$\begin{tikzcd} \arrow[r] & HC_*(A^+) \arrow[r] \arrow[d, "\cong"', "\alpha"] & HC_*(S) \arrow[r] \arrow[d, "\cong"', "\alpha"] & HC_*(A^-) \arrow[r, "\partial"] \arrow[d, "\cong"', "\alpha"] & HC_*(A^+) \arrow[r] \arrow[d, "\cong"', "\alpha"] & {} \\ \arrow[r] & HM_*(A^+,f) \arrow[r] & HM_*(S,f) \arrow[r] & HM_*(A^-,f) \arrow[r, "\partial"] & HM_*(A^+,f) \arrow[r] & {} \end{tikzcd}$$

We prove Theorem B in several steps.

Step 1. It follows from Lemma 3.14 that we have a commutative diagram of the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & C\Delta_*(A^+) \arrow[r] \arrow[d, "\iota"'] & C\Delta_*(S) \arrow[r] \arrow[d, "\iota"'] & C\Delta_*(A^-) \arrow[r] \arrow[d, "\iota"'] & 0 \\ 0 \arrow[r] & CM_*(A^+,f) \arrow[r] & CM_*(S,f) \arrow[r] & CM_*(A^-,f) \arrow[r] & 0. \end{tikzcd}$$

The naturality of the boundary operator induces a commutative diagram of long exact sequences

$$\begin{tikzcd} \arrow[r] & H_*(C\Delta(A^+)) \arrow[r] \arrow[d, "\iota_*"'] & H_*(C\Delta(S)) \arrow[r] \arrow[d, "\iota_*"'] & H_*(C\Delta(A^-)) \arrow[r, "\partial"] \arrow[d, "\iota_*"'] & H_{*-1}(C\Delta(A^+)) \arrow[d, "\iota_*"'] \\ \arrow[r] & HM_*(A^+,f) \arrow[r] & HM_*(S,f) \arrow[r] & HM_*(A^-,f) \arrow[r, "\partial"] & HM_{*-1}(A^+,f) \end{tikzcd}$$

The top row is the homolgy of the Conley complex defined by (3.9) and (3.10). The vertical arrows are the isomorphisms induced by ι defined in (3.12).

Step 2 (Filtrations)**.** Denote by

$$\mathrm{Crit}_{\leqslant k}(A^{\pm}) := \left\{x \in A^{\pm} \mid df(x) = 0,\ |x| \leqslant k\right\}$$

the set of critical points in $A^{\pm}$ with Morse index lower or equal than k. For $k = -1, 0, \ldots, n$ define a set of attractors by

$$\begin{array}{lcl} A_{-1} & = & \emptyset \\ A_0 & = & \mathrm{Crit}_0(S,f) \\ A_1 & = & \mathrm{Crit}_{\leqslant 1}(A^-) \cup \mathrm{Crit}_0(A^+) \cup \bigcup_{y \in \mathrm{Crit}_{\leqslant 1}(A^-), x \in \mathrm{Crit}_0(A^+)} C(y,x) \\ \vdots & = & \vdots \\ A_k & = & \mathrm{Crit}_{\leqslant k}(A^-) \cup \mathrm{Crit}_{\leqslant k-1}(A^+) \cup \bigcup_{y \in \mathrm{Crit}_{\leqslant k}, x \in \mathrm{Crit}_{\leqslant k-1}} C(y,x) \end{array}$$

We define the set of all attractors by

$$S_{k,\ell} := A_k \cup \underbrace{\left(\mathrm{Crit}_{\leqslant \ell}(A^+) \cup \bigcup_{x,y \in \mathrm{Crit}_{\leqslant \ell}(A^+)} C(y,x)\right)}_{=:A_\ell^+}.$$

By definition it follows that $S_{-1,\ell} = A_\ell^+,\ S_{k,k-1} = A_k$ and $S_{-1,-1} = \emptyset$. To the set of attractors $\{S_{k,\ell}\}_{-1 \leqslant k,l \leqslant n}$ one can attach a collection of compact sets called *attractor network*

$$\{N_{k,\ell}\}_{-1 \leqslant k,l \leqslant n}$$

such that the follwoing holds

(i) $(N,L) := (N_{n,n}, N_{-1,-1})$ is an index pair for S.

(ii) $N_{k,\ell}$ is positively invariant in N for all k, ℓ.

(iii) $N_{k,\ell} \cap \mathrm{Crit}(S,f) = \mathrm{Crit}_{\leqslant k}(A^-) \cup \mathrm{Crit}_{\leqslant \ell}(A^+)$.

(iv) For $k \leqslant k', \ell \leqslant \ell'$ we have $N_{k,\ell} \subset N_{k',\ell'}$.

Out of this attrator network we can define three index filtrations of S, A^+ and A^- by

$$N_k := N_{k,k} \quad N_\ell^+ := N_{-1,\ell} \quad N_j^- := N_{j,n}$$

such that

$$\begin{array}{lcll} (N_k, N_{k-1}) & := & (N_{k,k}, N_{k-1,k-1}) & \text{is an index pair for } \mathrm{Crit}_k(S). \\ (N_\ell^+, N_{\ell-1}^+) & := & (N_{-1,\ell}, N_{-1,\ell-1}) & \text{is an index pair for } \mathrm{Crit}_\ell(A^+) \\ (N_j^-, N_{j-1}^-) & := & (N_{j,n}, N_{j-1,n}) & \text{is an index pair for } \mathrm{Crit}_k(A^-) \end{array}$$

Step 3. For all $k \geqslant 0$ we show the commutativity of the diagram

$$
\begin{tikzcd}
H_k(A^+) \arrow[r, "i"] \arrow[d, "\beta_+"'] & H_k(S) \arrow[r, "p"] \arrow[d, "\beta"'] & H_k(A^-) \arrow[r, "\partial"] \arrow[d, "\beta_-"'] & H_{k-1}(A^+) \arrow[d, "\beta_+"'] \\
H_k(C\Delta(A^+)) \arrow[r, "i"] & H_k(C\Delta(S)) \arrow[r, "p"] & H_k(C\Delta(A^-)) \arrow[r, "\partial"] & H_{k-1}(C\Delta(A^+))
\end{tikzcd}
\tag{3.19}
$$

We show that each square commutes separately. Canonical inclusions of pairs give rise to the following commutative diagram:

$$
\begin{tikzcd}
H_{k+1}(N^+_{k+1}, N^+_k) \arrow[dd, dashed, "\delta_+"'] \arrow[dr, "i"] \arrow[ddr, dashed] & & & \\
& H_{k+1}(N_{k+1}, N_k) \arrow[dd] \arrow[ddr] & & \\
H_k(N^+_k, L) \arrow[r, hook, dashed, "\beta_+"] \arrow[dr, "i"] \arrow[dd, dashed, two heads, "j_+"'] & H_k(N^+_k, N^+_{k-1}) \arrow[r, dashed] \arrow[dr, "i"] \arrow[ddr, dashed] & H_{k-1}(N^+_{k-1}, L) \arrow[dr, "i"] \arrow[dd, hook, dashed] & \\
& H_k(N_k, L) \arrow[r, hook, "\beta"'] \arrow[dd, "j"] & H_k(N_k, N_{k-1}) \arrow[r] \arrow[ddr] & H_{k-1}(N_{k-1}, L) \arrow[dd, hook] \\
H_k(N^+_{k+1}, L) \arrow[dr, "i"] & & H_{k-1}(N^+_{k-1}, N^+_{k-2}) \arrow[dr, "i"] & \\
& H_k(N_{k+1}, L) & & H_{k-1}(N_{k-1}, N_{k-2})
\end{tikzcd}
$$

The diagrams on the left hand side and on the right hand side commute by the naturality of the long exact sequence. They are induced by the commutative diagrams

$$
\begin{tikzcd}
(N^+_k, L) \arrow[r] \arrow[d, dashed] & (N_k, L) \arrow[d] \\
(N^+_{k+1}, L) \arrow[r] \arrow[d, dashed] & (N_{k+1}, L) \arrow[d] \\
(N^+_{k+1}, N^+_k) \arrow[r] & (N_{k+1}, N_k)
\end{tikzcd}
\qquad
\begin{tikzcd}
(N^+_{k-1}, L) \arrow[r] \arrow[d, dashed] & (N_{k-1}, L) \arrow[d] \\
(N^+_k, L) \arrow[r] \arrow[d, dashed] & (N_k, L) \arrow[d] \\
(N^+_k, N^+_{k-1}) \arrow[r] & (N_k, N_{k-1})
\end{tikzcd}
$$

where all arrows represent canonical inclusions of pairs. The diagonal diagrams are induced by corresponding commutative inclusions of pairs. We prove

$$\beta \circ i = i \circ \beta_+ \tag{3.20}$$

which is the commutativity of the left square in (3.19). Pick an element $x \in H_k(A^+) = H_k(N^+, L) \cong H_k(N^+_{k+1}, L)$. By surjectivity of j_+ we may choose a preimage $z_+ \in H_k(N_k^+, L)$ with $j_+(z_+) = x$. We apply β_+ and i to get the map

$$x \mapsto i \circ \beta_+(z_+) \in \ker \partial_k.$$

For $z'_+ \in j_+^{-1}(x)$ is an other preimage, commutativity and exactness imply that $i \circ \beta(z_+)$ and $i \circ \beta_+(z'_+)$ differ by an image of ∂_{k+1}. So the map $x \mapsto i \circ \beta_+(z_+)$ induces $i \circ \beta_+ : H_k(A^+) \to H_k(C\Delta(S))$ for some preimage z_+. Now we apply to $x \in H_k(N_{k+1}^+, L)$ the inclusion i and by surjectivity of j we may take a lift $z \in H_k(N_k, L)$ such that $j(z) = i(x)$. If we pick z_+ such that $i \circ j_+(z_+) = j \circ i(z_+)$ then it follows from

$$i \circ \beta_+(z_+) = \beta \circ i(z_+)$$

that $i \circ \beta_+ = \beta \circ i$. This proves the commutativity of the left square in (3.19). Likewise the commutative diagrams

$$\begin{tikzcd}
(N_k, L) \arrow[r] \arrow[d, dashed] & (N_k^-, N^+) \arrow[d] \\
(N_{k+1}, L) \arrow[r] \arrow[d, dashed] & (N_{k+1}^-, N^+) \arrow[d] \\
(N_{k+1}, N_k) \arrow[r] & (N_{k+1}^-, N_k^-)
\end{tikzcd}
\qquad
\begin{tikzcd}
(N_{k-1}, L) \arrow[r] \arrow[d, dashed] & (N_{k-1}^-, N^+) \arrow[d] \\
(N_k, L) \arrow[r] \arrow[d, dashed] & (N_k^-, N^+) \arrow[d] \\
(N_k, N_{k-1}^+) \arrow[r] & (N_k^-, N_{k-1}^-)
\end{tikzcd}$$

toghether with Lemma 3.14 imply the commutativity of

$$\begin{tikzcd}
H_{k+1}(N_{k+1}, N_k) \arrow[dr, "p"] \arrow[dd, dashed] \arrow[ddr, dashed] & & & \\
& H_{k+1}(N_{k+1}^-, N_k^-) \arrow[dd] \arrow[ddr] & & \\
H_k(N_k, L) \arrow[r, dashed, "\beta"] \arrow[dr, "p"] \arrow[dd, dashed, "j"'] & H_k(N_k, N_{k-1}) \arrow[r, dashed] \arrow[dr, "p"] \arrow[ddr, dashed] & H_{k-1}(N_{k-1}, L) \arrow[dr, "p"] \arrow[dd, dashed] & \\
& H_k(N_k^-, N^+) \arrow[r, "\beta_-"'] \arrow[dd, "j_-"'] & H_k(N_k^-, N_{k-1}^-) \arrow[r] \arrow[ddr] & H_{k-1}(N_{k-1}^-, N^+) \arrow[dd] \\
H_k(N_{k+1}, L) \arrow[dr, "p"] & & H_{k-1}(N_{k-1}, N_{k-2}) \arrow[dr, "p"] & \\
& H_k(N_{k+1}^-, N^+) & & H_{k-1}(N_{k-1}^-, N_{k-1}^-)
\end{tikzcd} \tag{3.21}$$

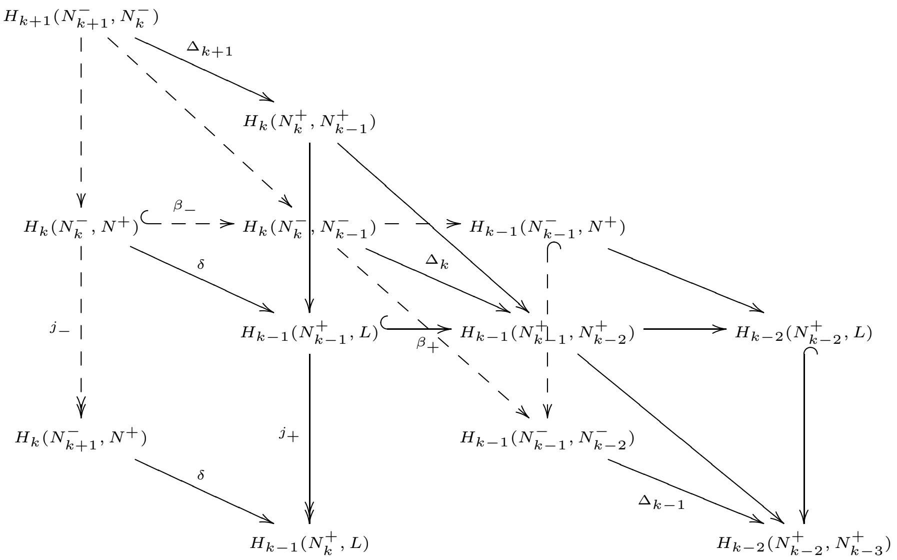
We prove the commutativity of the middle square in (3.19) which is

$$\beta_- \circ p = p \circ \beta.$$

This follows from the same argument, i.e. from the commutativity

$$\beta_- \circ p(z) = p \circ \beta(z)$$

in (3.21) for some preimage $z \in j^{-1}(x)$ of $x \in H_k(N_{k+1}, L)$. The last square involving the boundary operators commutes by the commutativity of the following diagram:



(3.22)

The cofmmutativity of the lower square of the left diagram in (3.22): Consider the following inclusions

$$\begin{array}{ccccc} L & \subset & N^+_{k-1} & \subset & N_{k,k-1} \\ & & \cap & & \cap \\ L & \subset & N^+_k & \subset & N_k. \end{array}$$

The long exact sequences of the horizontal triples give rise to the following commutative diagram

$$\begin{tikzcd} H_k(N_{k,k-1}, N^+_{k-1}) \arrow[r, "\partial"] \arrow[d, dashed, "j_-"'] & H_{k-1}(N^+_{k-1}, L) \arrow[d, "j_+"] \\ H_k(N_k, N^+_k) \arrow[r, "\partial"] & H_{k-1}(N^+_k, L) \end{tikzcd}$$

Now (N_k^-, N^+) and $(N_{k,k-1}, N_{k-1}^+)$ are index pairs of the same isolated invariant set $S_{k,-1}$. Likewise (N_{k+1}^-, N^+) and (N_k, N_k^+) are index pairs of the same isolated invariant set $S_{k,-1}$. Hence using the flow induced isomorphisms ϕ defined in (2.5) we get the commutative diagram

$$\begin{tikzcd}
\delta : H_k(N_k^-, N^+) \arrow[r, "\phi"] \arrow[d, "j_-"'] & H_k(N_{k,k-1}, N_{k-1}^+) \arrow[r, "\partial"] \arrow[d, dashed, "j_-"'] & H_{k-1}(N_{k-1}^+, L) \arrow[d, "j_+"] \\
\delta : H_k(N_{k+1}^-, N^+) \arrow[r, "\phi"] & H_k(N_k, N_k^+) \arrow[r, "\partial"] & H_{k-1}(N_k^+, L)
\end{tikzcd}$$

The commutativity of the left upper square in (3.22):
This follows from trivial reasons, namely because the compositions are zero. Using flow maps Conley's connection matrix Δ_{k+1} factors through the boundary operator of the triple $(N_{k+1,k}, N_k, N_{k,k-1})$ and using that (N_k^-, N^+) and $(N_{k,k-1}, N_{k-1}^+)$ are index pairs of the same isolated invariant set.

$$\begin{tikzcd}
H_{k+1}(N_{k+1,k}, N_k) \arrow[rr, "\partial"] & & H_k(N_k, N_{k,k-1}) \arrow[d, "\phi"] \\
H_{k+1}(N_{k+1}^-, N_k^-) \arrow[u, "\phi"'] \arrow[rr, "\Delta_{k+1}"] \arrow[d, dashed, "\partial"'] & & H_k(N_k^+, N_{k-1}^+) \arrow[d, "\partial"] \\
H_k(N_k^-, N^+) \arrow[rr, "\delta"] \arrow[dr, "\phi"'] & & H_{k-1}(N_{k-1}^+, L) \\
& H_k(N_{k,k-1}, N_{k-1}^+) \arrow[ur, "\partial"'] &
\end{tikzcd}$$

Now we see that the compositions are zero:

$$\partial \circ \Delta_{k+1} = \partial \circ \phi \circ \partial \circ \phi = 0 = \partial \circ \phi \circ \partial = \delta \circ \partial.$$

The commutativity of the right horizontal square in (3.22):
Again the compositions of any two maps in this square is zero.

$$\begin{tikzcd}
H_k(N_{k,k-1}, N_{k-1}) \arrow[r, "\phi"] \arrow[dd, "\partial"'] & H_k(N_k^-, N_{k-1}^-) \arrow[r, "\partial"] \arrow[dd, "\Delta"'] & H_{k-1}(N_{k-1}^-, N^+) \arrow[dd, "\delta"] \arrow[dr, "\phi"] & \\
& & & H_{k-1}(N_{k-1,k-2}, N_{k-2}^+) \arrow[dl, "\partial"] \\
H_k(N_{k-1}, N_{k-1,k-2}) \arrow[r, "\phi"] & H_{k-1}(N_{k-1}^+, N_{k-2}^+) \arrow[r, "\partial"] & H_{k-2}(N_{k-2}^+, L) &
\end{tikzcd}$$

The commutativity of the left horizontal square in (3.22):
Consider the commutative diagram of inclusions

$$\begin{tikzcd}
(N^+_{k-1}, L) \arrow[r] \arrow[d, dashed] & (N_{k-1,k}, N_{k-1,k-2}) \arrow[d] \\
(N_{k,k-1}, L) \arrow[r] \arrow[d, dashed] & (N_{k,k-1}, N_{k-1,k-2}) \arrow[d] \\
(N_{k,k-1}, N^+_{k-1}) \arrow[r] & (N_{k,k-1}, N_{k-1,k}).
\end{tikzcd}$$

The long exact sequences of the vertical triples give the following commutative diagram

$$\begin{tikzcd}
H_k(N_{k,k-1}, N^+_{k-1}) \arrow[r, "\beta_-"] \arrow[d, "\partial"'] & H_k(N_{k,k-1}, N_{k-1,k}) \arrow[d, "\partial"] \\
H_{k-1}(N^+_{k-1}, L) \arrow[r, "\beta_+"] & H_{k-1}(N_{k-1,k}, N_{k-1,k-2})
\end{tikzcd}$$

Using the flow maps (2.5) this diagram is canonically isomorphic to

$$\begin{tikzcd}
H_k(N^-_k, L) \arrow[d, "\delta"'] & H_k(N_{k,k-1}, N^+_{k-1}) \arrow[l, "\phi"'] \arrow[r, "\beta_-"] \arrow[d, "\partial"'] & H_k(N_{k,k-1}, N_{k-1,k}) \arrow[r, "\phi"] \arrow[d, "\partial"] & H_k(N^-_k, N^-_{k-1}) \arrow[d, "\Delta_{k+1}"] \\
H_{k-1}(N^+_{k-1}, L) \arrow[r, "\phi"] & H_{k-1}(N^+_{k-1}, L) \arrow[r, "\beta_+"] & H_{k-1}(N_{k-1,k}, N_{k-1,k-2}) \arrow[r, "\phi"] & H_{k-1}(N^+_{k-1}, N^+_k)
\end{tikzcd}$$

which implies the commutativity.
The commutativity of the upper square in the diagonal of (3.22):
This diagram again commutes since the compositions are zero. Δ_{k+1} and Δ_k factor through flow maps as

$$\begin{tikzcd}
H_{k+1}(N_{k+1,k}, N_k) \arrow[r, "\partial"] & H_k(N_k, N_{k,k-1}) \arrow[d, "\phi"] \\
H_{k+1}(N^-_{k+1}, N^-_k) \arrow[u, "\phi"] \arrow[r, "\Delta_{k+1}"] \arrow[d, "\partial"'] & H_k(N^+_k, N^+_{k-1}) \arrow[d, "\partial"] \\
H_k(N^-_k, N^-_{k-1}) \arrow[r, "\Delta_k"] \arrow[d, "\phi"'] & H_{k-1}(N^+_{k-1}, N^+_{k-2}) \\
H_k(N_{k,k-1}, N_{k-1}) \arrow[r, "\partial"] & H_{k-1}(N_{k-1}, N_{k-1,k-2}) \arrow[u, "\phi"']
\end{tikzcd}$$

We see that $\Delta_{k+1} \circ \partial = \partial \circ \Delta_k = 0$. Note that the lower square of the diagonal also commutes by writing $k-1$ instead of k.

The commutativity of the right square in (3.22):
Consider the following diagram of inclusions:

$$\begin{tikzcd}
(N^+_{k-2}, L) \arrow[r] \arrow[d] & (N_{k-1,k-2}, L) \arrow[r] \arrow[d] & (N_{k-1,k-2}, N^+_{k-2}) \arrow[d] \\
(N_{k-2}, N_{k-2,k-3}) \arrow[r] & (N_{k-1,k-2}, N_{k-2,k-3}) \arrow[r] & (N_{k-1,k-2}, N_{k-2}).
\end{tikzcd}$$

The long exact sequences of the horizontal triples give the commutative diagram

$$\begin{tikzcd}
H_{k-1}(N_{k-1,k-2}, N^+_{k-2}) \arrow[r, "\partial"] \arrow[d, dashed] & H_{k-2}(N^+_{k-2}, L) \arrow[d] \\
H_{k-1}(N_{k-1,k-2}, N_{k-2}) \arrow[r, "\partial"] & H_{k-2}(N_{k-2}, N_{k-2,k-3}).
\end{tikzcd}$$

Using different index pairs for the same isolated invariant set and follow induced isomorphisms we get the commutative diagram

$$\begin{tikzcd}
H_{k-1}(N^-_{k-1}, N^+) \arrow[r, "\delta"] & H_{k-2}(N^+_{k-2}, L) \arrow[d, "\phi"] \\
H_{k-1}(N_{k-1,k-2}, N^+_{k-2}) \arrow[r, "\partial"] \arrow[u, "\phi"'] \arrow[d, dashed] & H_{k-2}(N^+_{k-2}, L) \arrow[d] \\
H_{k-1}(N_{k-1,k-2}, N_{k-2}) \arrow[r, "\partial"] \arrow[d, "\phi"'] & H_{k-2}(N_{k-2}, N_{k-2,k-3}) \\
H_{k-1}(N^-_{k-1}, N^-_{k-2}) \arrow[r, "\Delta"] & H_{k-2}(N^+_{k-2}, N^+_{k-3}). \arrow[u, "\phi"]
\end{tikzcd}$$

This shows the commutativity of the right square in (3.22). We have proved that the diagram (3.22) commutes. The commutativity of the right square in (3.19), i.e.

$$\beta_+ \circ \partial = \partial \circ \beta_-$$

follows by using the commutativity of (3.22) and applying the same diagram chase as in the proof of (3.20).

Step 4. By Step 1 and Step 3 the following diagram commutes

$$\begin{tikzcd}
H_k(A^+) \arrow[r] \arrow[d, "\beta_+"'] & H_k(S) \arrow[r] \arrow[d, "\beta"'] & H_k(A^-) \arrow[r, "\partial"] \arrow[d, "\beta_-"'] & H_{k-1}(A^+) \arrow[d, "\beta_+"'] \\
H_k(C\Delta(A^+)) \arrow[r] \arrow[d, "\iota"'] & H_k(C\Delta(S)) \arrow[r] \arrow[d, "\iota"'] & H_k(C\Delta(A^-)) \arrow[r, "\partial"] \arrow[d, "\iota"'] & H_{k-1}(C\Delta(A^+)) \arrow[d, "\iota"'] \\
H_*(A^+, f) \arrow[r] & H_*(S, f) \arrow[r] & H_*(A^-, f) \arrow[r, "\partial"] & H_{k-1}(A^+, f)
\end{tikzcd}$$

This proves Theorem B.

$\square$

Chapter 4

Continuation in Morse Thoery

Motivated by Conley continuation thogether with Theorem A Floer defined in [4] continuation maps for families of Hamiltonians together with almost complex structures. By replacing the parameter by the time Floer got a perturbed Cauchy-Riemann equation whose solution space led to continuation maps between Floer homology groups. Floers method in [4] using Fredholm theory on Banach manifolds was adapted to Morse theory by Schwarz [12] in the case where M is a compact manifold. Schwarz proved in [12] Theorem A part (i) by veryfying the Eilenberg-Steenrod axioms. There is a beautiful exposition of Weber [15], where he proves compactness and gluing using theorems from dynamical systems such as the Hartman-Grobman to show compactness and the inclination lemma to show gluing. In this thesis we adapt Floers method for Morse-Smale triples (S, f, g) in a possibly non-compact manifold M without boundary. We will see how a continuation induces a isolating quadruple which determines a homomorphism between two Morse complexes. We then prove compactness, transversality and the chain map property of the map. We start with a more general concept: We define a continuation map $\Phi_{\mathbf{flo}}(K, f, g) : HM_*(S_0, f_0) \to HM_*(S_1, f_1)$ for general families of gradient pairs connecting two Morse-Smale triples (S_0, f_0, g_0) and (S_1, f_1, g_1).

4.1 Isolated connecting sets

Let M be a possibly non-compact manifold without boundary and let (S_0, f_0, g_0) and (S_1, f_1, g_1) be two Morse-Smale triples on M. Pick family of gradient pairs $(f, g)_{\mathbb{R}} = \{(f_\lambda, g_\lambda) \mid \lambda \in \mathbb{R}\}$ being constant outside an interval $(\delta, 1-\delta)$ meaning

$$(f_\lambda, g_\lambda) = \begin{cases} (f_0, g_0) & \text{for } \lambda \leqslant \delta \\ (f_1, g_1) & \text{for } \lambda \geqslant 1-\delta \end{cases} \tag{4.1}$$

Replacing the parameter λ by t we get the **time-dependent gradient equation**

$$\dot{x}(t) = -\nabla f_t(x(t)) \tag{4.2}$$

with local flow φ_f^{t,t_0}. The set of initial points $p \in M$ for which the zero flow $t \mapsto \varphi_f^{t,0}(p)$ is globally defined with asymptotics in S_0, S_1 is denoted by

$$C := \left\{ p \in M \;\middle|\; \lim_{t\to-\infty} \varphi_f^{t,0}(p) \in S_0 \text{ and } \lim_{t\to\infty} \varphi_f^{t,0}(p) \in S_1 \right\}$$

and called the **connecting set** of $(f,g)_{\mathbb{R}}$. It follows that for all $p \in C_0$ there exists $x_0 \in \mathrm{Crit}(S_0)$ and $x_1 \in \mathrm{Crit}(S_1)$ such that $p \in W^u(x_0, f_0) \cap \varphi^{0,1}(W^s(x_1, f_1))$. Let S_0, S_1 be the isolated invariant sets with respect to the flows $\varphi_{f_0}^t$ and $\varphi_{f_1}^t$ respectively. For isolating neighborhoods N_0, N_1 of S_0, S_1 we define the unstable and stable sets of S_0 and S_1 by

$$W^u(S_0, f_0, N_0) := \left\{ p \in M \mid \varphi_{f_0}^{(-\infty,0]}(p) \subset N_0 \right\}$$

and

$$W^s(S_1, f_1, N_1) := \left\{ p \in M \mid \varphi_{f_1}^{[0,\infty)}(p) \subset N_1 \right\}.$$

Definition 4.3. Let (S_0, f_0, g_0) and (S_1, f_1, g_1) be two Morse-Smale triples and (f,g) be a family of gradient pairs satisfying (4.1). A subset $K \subset C$ is called and **isolated connecting set** from S_0 to S_1 if there exists

(i) a compact isolating neighborhood N_0 of S_0

(ii) a compact isolating neighborhood N_1 of S_1

(iii) a *compact* neighborhood W of K

(iv) a time $T > 0$ such that $t \mapsto \varphi_f^{t,0}(p)$ is defined $\forall t \in [-T, T]$ and every $p \in W$

and

$$K_{\Gamma,f} = \left(W \cap \varphi_f^{0,-T}\big(W^u(S_0, f_0, N_0)\big) \right) \cap \varphi_f^{0,T}\big(W^s(S_1, f_1, N_1)\big) \tag{4.4}$$

In this case $\Gamma = (N_0, N_1, W, T)$ is called an **isolating quadruple.** The isolated connecting set is also denoted by $K = K_{\Gamma,f}$.

Definition 4.5. An isolated connecting set $K \subset C$ from S_0 to S_1 is called **transverse** if for every $x_0 \in \mathrm{Crit}(S_0, f_0)$, every $x_1 \in \mathrm{Crit}(S_1, f_1)$ and every

$$p \in \Big(W \cap \varphi_f^{0,-T}((W^u(x_0, f_0, N_0))\Big) \cap \varphi_f^{0,T}(W^s(x_1, f_1, N_1)),$$

the submanifolds $\varphi_f^{0,-T}((W^u(x_0, f_0, N_0))$ and $\varphi_f^{0,T}(W^s(x_1, f_1, N_1))$ intersect transversally at p.

Remark 4.6. Note that isolated connecting sets are compact and are stable with respect to nearby flows:

(i) $K \subset W$ is closed and hence compact.

(ii) Being an isolated connecting set is an open condition meaning: An isolated connecting set $K \subset W$ remains isolated with respect to near by flows.

Let $\Gamma = (N_0, N_1, W, T)$ be an isolating quadruple. We denote the space of solutions $u : \mathbb{R} \to M$ of (1.8) such that $u(0) \in W$ with asymptotics $y_0 \in \mathrm{Crit}_k(S_0, f_0)$ and $x_1 \in \mathrm{Crit}_k(S_1, f_1)$ by

$$\mathscr{M}_f(x_0, x_1, \Gamma) := \left\{ u : \mathbb{R} \to M \;\middle|\; \begin{array}{l} \dot{u} = -\nabla f_t(u), u(0) \in W \\ u(t) \in N_0 \ \forall t \leqslant -T, u(t) \in N_1 \ \forall t \geqslant T \\ \lim_{t\to\infty} u(t) = x_0, \ \lim_{t\to\infty} u(t) = x_1 \end{array} \right\}.$$

For critical points $y_\mu, x_\mu \in \mathrm{Crit}(S_\mu, f_\mu)$ with $\mu = 0, 1$ we consider solutions $\gamma_\mu : \mathbb{R} \to S_\mu$ of the equation

$$\dot{x} = -\nabla f_\mu(x) \tag{4.7}$$

with boundary conditions

$$\lim_{t\to-\infty} \gamma_\mu(t) = y_\mu \quad \text{and} \quad \lim_{t\to\infty} \gamma_\mu(t) = x_\mu \quad \mu = 0, 1 \tag{4.8}$$

Define the set of solutions

$$\mathscr{M}_{f_\mu}(y_\mu, x_\mu, S_\mu) := \{\gamma_\mu : \mathbb{R} \to S_\mu \mid \gamma_\mu \text{ solves (4.7) with (4.8)}\}.$$

The following splitting result holds

Theorem 4.9 (Compactness)**.** *Let $\Gamma = (N_0, N_1, W, T)$ be an isolating quadruple and u_ν be a sequence in $\mathscr{M}_f(x_0, x_1, \Gamma)$. Then there exists*

(i) *a subsequence still denoted by u_ν*

(ii) *sequences* $s_\nu^{-N^-} < \cdots < s_\nu^{-1} < s_\nu^0 = 0 < s_\nu^1 < \cdots < s_\nu^{N^+}$ *in* $\mathbb{R}$

(iii) *a finite number of critical points*

$$\begin{aligned} x^{-1}, x^{-2}, \ldots, x^{-N^-} &= x_0 \in \mathrm{Crit}(S_0, f_0) \\ \text{and } x^0, x^1, \ldots, x^{N^+} &= x_1 \in \mathrm{Crit}(S_1, f_1) \end{aligned}$$

(iv) *solutions* $\gamma^i : \mathbb{R} \to M$ *with*

$$\gamma^i \in \begin{cases} \mathscr{M}_{f_0}(x^{i-1}, x^i, S_0) & \text{for } i = -N^- + 1, \ldots, -1 \\ \mathscr{M}_f(x^{-1}, x^0, \Gamma) & \text{for } i = 0 \\ \mathscr{M}_{f_1}(x^{i-1}, x^i, S_1) & \text{for } i = 1, \ldots, N^+ \end{cases}$$

such that the subsequence $u_\nu(t + s_\nu^i)$ *converges uniformly with all its derivatives on compact sets to solution* $\gamma^i(t)$ *as* $\nu \to \infty$.

Proof of Theorem 4.9. We reduce the compactness to the time-independent case by redefining time as a new variable. Pick a smooth function $\beta : [-\delta, 1+\delta] \to [0,1]$ with $\beta_1^{-1}(0) = \{0,1\}, \beta'(0) > 0, \beta'(1) < 0$ such that $\beta_1(\lambda) = 1$ for $\delta \leqslant \lambda \leqslant 1 - \delta$.

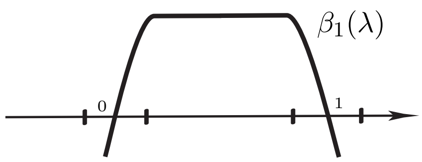


Figure 12. The smooth function $\beta(\lambda)$ is constant $= 1$ on $[\delta, 1-\delta]$, decreasing outside and has its only zeroes at $\{0,1\}$.

We define the vector field on $[-\delta, 1+\delta] \times M$ by

$$\xi(x, \lambda) := \big(\beta_1(\lambda), -\nabla f_\lambda(x)\big).$$

Let (N_0, L_0) and (N_1, L_1) be index pairs for S_0 and S_1 respectively. Define the neighborhood

$$N := \big([-\delta, \delta] \times N_0\big) \cup \big([1-\delta, 1+\delta] \times N_1\big) \cup \left(\bigcup_{t \in [-T,T]} \lambda(t) \times \varphi_f^{t,0}(W) \right)$$

and denote its isolated invariant set by

$$\mathscr{S}(S_0, \Gamma, S_1) := I\left(N, \Phi^t_{(f,g,\beta_1)}\right)$$

Any sequence $u_\nu \in \mathscr{M}_f(y_0, x_1, \Gamma)$ can be viewed as a solution of (2.19) by extending it

$$\tilde{u}_\nu(t) = (\lambda(t), u_\nu(t)) \in \mathscr{S}(S_0, \Gamma, S_1)$$

where $\lambda(t)$ solves $\dot\lambda(t) = \beta_1(\lambda(t))$ with initial condition $\lambda(\delta) = \delta$. For the time-independent gradient of (2.19). Applying the compactness theorem to u_ν and then project back to M the assertion of the theorem follows. $\square$

4.2 The Floer cobordism construction

Every isolating quadruple $\Gamma = (N_0, N_1, W, T)$ from S_0 to S_1 determines a continuation map

$$\Phi_{\mathbf{flo}}(\Gamma, f, g) : HM_*(S_0, f_0) \to HM_*(S_1, f_1)$$

on local Morse homology defined via the cobordism construction of Floer [4]. On chain level the map

$$\Phi_{\mathbf{flo}}(\Gamma, f, g) : CM_k(S_0, f_0) \longrightarrow CM_k(S_1, f_1)$$

is defined by counting the points in the isolated connecting set $K \subset W$ that connect critical points of the same Morse index. More precisely for $x_0 \in \mathrm{Crit}_k(S_0, f_0)$ and $x_1 \in \mathrm{Crit}_k(S_1, f_1)$ define

$$K_{\Gamma,f}(x_0, x_1) := \Big(\varphi_f^{0,-T}(W^u(x_0, f_0, N_0)) \cap W\Big) \cap \varphi_f^{0,T}(W^s(x_1, f_1, N_1)).$$

This is a finite set and we denote by

$$n_{\Gamma,f}(x_0, x_1) := \#\left\{\big(\varphi_f^{0,-T}(W^u(x_0, f_0, N_0)) \cap W\big) \cap \varphi_f^{0,T}(W^s(x_1, f_1, N_1))\right\}$$

the algebraic number of intersection points of $\varphi^{0,-T}(W^u(x_0, f_0, N_0)) \cap W$ and $\varphi^{0,T}(W^s(x_1, f_1, N_1))$. Then define

$$\Phi_{\mathbf{flo}}(\Gamma, f, g)\langle x_0\rangle := \sum_{|x_1|=k} n_{\Gamma,f}(x_0, x_1)\langle x_1\rangle$$

where the sum runs over $x_1 \in \mathrm{Crit}_k(S_1, f_1)$. This is a chain map, see Theorem 4.12. This definition of $\Phi_{\mathbf{flo}}(\Gamma, f, g)$ extends to the non-transverse with $n_{\Gamma,f}(x_0, x_1)$ understood as the intersection number of the submanifolds $\varphi^{0,-T}(W^u(x_0, f_0, N_0)) \cap W$ and $\varphi^{0,T}(W^s(x_1, f_1, N_1))$ in terms of standard intersection theory. So $\Phi_{\mathbf{flo}}(\Gamma, f, g)$ by definition counts intersection points of $\varphi^{0,-T}(W^u(x_0, f_0, N_0))$ and $\varphi^{0,T}(W^s(x_1, f_1, N_1))$ in W. Next we prove that transversality is a generic property. Fix the smooth family $g = \{g_\lambda\}_{\lambda\in\mathbb{R}}$ of Riemannian metrics on M satisfying (4.1), i.e. $g_\lambda = g_0$ for $\lambda \leqslant \delta$ and $g_\lambda \geqslant g_1$ for $\lambda \geqslant 1 - \delta$. Define the set of smooth families

$$\mathscr{F}(\Gamma) := \left\{ h : \mathbb{R} \times M \to \mathbb{R} \;\middle|\; \begin{array}{l} h_\lambda = f_0 \text{ for } \lambda \leqslant \delta, h_\lambda = f_1 \text{ for } \lambda \geqslant 1 - \delta \\ \forall x_0 \in \mathrm{Crit}(S_0, f_0), \forall x_1 \in \mathrm{Crit}(S_1, f_1) \\ K_{\Gamma,h}(x_0, x_1) \subset \mathrm{int}(W) \end{array} \right\}$$

with the $C^\infty-$topology. $\mathscr{F}(\Gamma)$ is a Frechét space and can be quipped with a complete metric. Hence it has the Baire property. The set of familes for which the isolated connecting set K is transverse is denoted by

$$\mathscr{F}_{reg}(\Gamma) := \left\{ f \in \mathscr{F}(\Gamma) \;\middle|\; \begin{array}{c} \forall x_0 \in \mathrm{Crit}(S_0, f_0), \forall x_1 \in \mathrm{Crit}(S_1, f_1) \\ K_{\Gamma,f}(x_0, x_1) \text{ is transverse} \end{array} \right\}.$$

The spaces $\mathscr{F}(\Gamma)$ and $\mathscr{F}_{reg}(\Gamma)$ are endowed with the $C^\infty-$topology.

Theorem 4.10 (Transversality). *Let (S_0, f_0, g_0) and (S_1, f_1, g_1) be two Morse-Smale pairs in M and $\Gamma = (N_0, N_1, W, T)$ be an isolating quadruple, see Definition 4.3. Then $\mathscr{F}_{reg}(\Gamma)$ is an open and dense subset of $\mathscr{F}(\Gamma)$.*

Proof of Theorem 4.10. Now fix two critical points $x_0 \in \mathrm{Crit}(S_0, f_0)$ and $x_1 \in \mathrm{Crit}(S_1, f_1)$ and denote by $\Psi_f := \varphi_f^{0,-T} : M \to M$ the one-zero flow of (4.2). Note that a priori the $-T$ to 0 flow might be not defined for all points $p \in M$. However for the isolating quadruple $\Gamma = (N_0, N_1, W, T)$ the map

$$\begin{array}{rcl} \Psi_{\Gamma,f} : W^u(x_0, f_0, N_0) \times \mathscr{F}(\Gamma) & \longrightarrow & \varphi^{0,T}(W^s(x_1, f_1, N_1)) \cap W \\ (x, f) & \longmapsto & \Psi_f(x) \end{array}$$

is defined. Let $p \in W^u(x_0, f_0, N_0)$, pick a path $s \mapsto (p, f_s)$ with $(p, f_0) = (p, f)$ and $\partial_s|_{s=0} f_{t,s} = \hat{f}_t$. Define the variations

$$u_s(t) := \varphi_{f_s}^{t,-T}(p) \text{ and } \xi_s(t) = \partial_s u_s(t) \quad t \in [-T, 0].$$

The family $u_s(t)$ of solutions satisfies the eqaution

$$\begin{array}{rcll} \partial_t u_s(t) & = & -\nabla f_{t,s}(u_s(t)) & \mid \nabla_s \\ \nabla_s \partial_t (u_s(t)) & = & -\nabla_s(\nabla f_{t,s}(u_s(t))) & \\ \partial_t \nabla_s u_s(t) & = & -\nabla_{\xi_s} f_{t,s}(u_s(t)) - \nabla \partial_s f_{t,s}(u_s(t)) & \mid s = 0 \\ \partial_t \xi(t) & = & -\nabla_\xi f_t(u(t)) - \nabla \hat{f}_t(u(t)) & \end{array}$$

The covariant derivative is taken with respect to the Levi-Civita connection. In a orthonromal frame along $u(t) = u_0(t)$ the equation has the follwing form

$$\dot{\xi}(t) = A(t)\xi(t) + b(t)$$

where $A : [-T, 0] \to \mathbb{R}^{n \times n}$ and $b : [-T, 0] \to \mathbb{R}^n$ are smooth maps. In that frame the differential $d\Psi_f(x, f)(0, \hat{f})$ equals the solution operator $U(0, -T, b) \cdot 0$ defended by

$$U(t, -T, b) \cdot 0 = \int_0^t \Phi(-T, \tau) b(\tau) d\tau$$

where $\Phi(t, -T)$ is the solution operator of $\dot{\xi} = A(t)\xi$. Now pick a cutt-of function $\beta : [-T, -T + \varepsilon] \to \mathbb{R}$ with $\mathrm{supp}\beta \subset (-T, -T + \varepsilon)$ and $\int_{\mathbb{R}} \beta = 1$.

Define $\tilde{b}(t) := \mathbb{1} \cdot \beta(t)$. Then

$$\begin{aligned} \| \textstyle\int_{-T}^{-T+\varepsilon} \Phi(-T,\tau)\tilde{b}(\tau)d\tau - \mathbb{1}\| &= \textstyle\int_{-T}^{-T+\varepsilon} \big(\Phi(-T,\tau) - \mathbb{1}\big)\beta(\tau)d\tau \\ &\leqslant \sup_{\tau\in[-T,-T+\varepsilon]} \|\Phi(-T,\tau) - \mathbb{1}\| \end{aligned}$$

By making $\varepsilon > 0$ small we see that $\int_{-T}^{-T+\varepsilon} \Phi(-T,\tau)\beta(\tau)d\tau$ is an isomorphism. For any $\xi_0 \in \mathbb{R}^n$ there is a $v \in \mathbb{R}^n$ such that

$$\int_{-T}^{-T+\varepsilon} \Phi(-T,\tau)\beta(\tau)d\tau \cdot v = \xi_0.$$

So pick $\tilde{b}(t) = \beta(t)v$ it follows that $U(0,-T,\tilde{b})\xi_0 = \xi_1$. This shows that $\Psi_{\Gamma,f}(x)$ is a submersion. Define the preimage

$$\mathscr{W} := \big\{(p,f) \mid p \in W^u(x_0,f_0,N_0),\ \Psi_{\Gamma,f}(p) \in \varphi^{0,T}(W^s(x_0,f_0,N_0)) \cap W\big\}.$$

The isolated connecting set

$$K_{\Gamma,f}(x_0,x_1) = \big(\varphi_f^{0,-T}(W^u(x_0,f_0)) \cap W\big) \cap \varphi_f^{0,-T}\big(W^s(x_1,f_1)\big)$$

is transverse if and only if the family $f \in \mathscr{F}(\Gamma)$ is a regular value of the smooth projection $pr_2 : \mathscr{W} \to \mathscr{F}(\Gamma)$. The theorem of Sard-Smale says that the set of regular values $\mathscr{F}_{reg}(\Gamma)$ of pr_2 is an open and dense subset of $\mathscr{F}(\Gamma)$. □

In the remainder of this section we show that

$$\Phi_{\mathbf{flo}}(\Gamma,f,g) : CM_k(S_0,f_0) \to CM_k(S_1,f_1)$$

is a chain homomorphism. To do so we again will consider the the product flow (3.2) for $\varepsilon = 1$: Pick a smooth function $\beta_1 : [-\delta, 1+\delta] \to [0,1]$ with $\beta_1^{-1}(0) = \{0,1\}, \beta_1'(0) > 0, \beta_1'(1) < 0$ such that $\beta_1(\lambda) = 1 \quad$ for $\delta \leqslant \lambda \leqslant 1-\delta$.

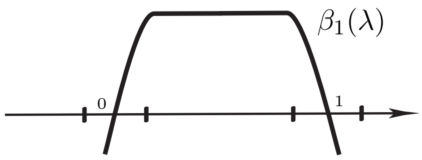


Figure 12. The smooth function $\beta(\lambda)$ is constant $= 1$ on $[\delta, 1-\delta]$, decreasing outside and has its only zeroes at $\{0,1\}$.

We define the vector field on $[-\delta, 1+\delta] \times M$ by

$$\xi(x,\lambda) := \big(\beta_1(\lambda) - \nabla f_\lambda(x)\big)$$

with associated flow $\Psi^t_{(f,g,\beta_1)}(x,\lambda)$ generated via equation (3.2) being

$$\begin{cases} \dot\lambda(t) &= \beta_1(\lambda(t)) \\ \dot x(t) &= -\nabla f_{\lambda(t)}(x(t)) \end{cases}$$

Let $\Gamma = (N_0, N_1, W, T)$ is an isolating quadruple. We define the associated isolating neighborhood

$$N := \big([-\delta,\delta]\times N_0\big) \cup \big([1-\delta, 1+\delta]\times N_1\big) \cup \left(\bigcup_{t\in[-T,T]} \lambda(t)\times \varphi_f^{t,0}(W)\right)$$

The dynamical cone of the isolated quadruple is defined by

$$\mathscr{S}(\Gamma, f, g) := I\left(N, \Psi^t_{(f,g,\beta_1)}\right).$$

For generic (f,g) the dynamical cone $(\mathscr{S}(\Gamma,f,g),\xi)$ is Morse-Smale in the sense of Definition 3.1. Slightly abusing notation we write

$$\mathrm{Crit}(\mathscr{S}(\Gamma,f,g),\xi) = \{(\lambda,\xi)\mid \xi(\lambda,x)=0\} \subset \big(S_0\cup S_1\big).$$

It follows from the definition of ξ that its zeroes $\mathrm{Crit}(\mathscr{S}(\Gamma,f,g),\xi)$ are non-degenerate. Let $\{o_{x_0}\}_{x_0\in \mathrm{Crit}(S_0,f_0)}$ and $\{o_{x_1}\}_{x_1\in \mathrm{Crit}(S_1,f_1)}$ be choices of orientations of the unstable manifolds. Adding the standard orientation e_λ of $[-\delta, 1+\delta]$ to the orientations $\{o_{x_0}\}_{x_0\in\mathrm{Crit}(S_0,f_0)}$ we orient the unstable manifolds $W^u(\tilde x,\xi,N)$ by

$$\{o_{\tilde x}\}_{\tilde x\in \mathrm{Crit}(N,\Gamma,\xi)} = e_\lambda \oplus \{o_{x_0}\}_{x_0\in\mathrm{Crit}(S_0,f_0)} \cup \{o_{x_1}\}_{x_1\in\mathrm{Crit}(S_1,f_1)}.$$

As in Definition 3.3 we define the local Morse complex of the cone $\mathscr{S}(\Gamma,f,g)$ by

$$CM_k(\mathscr{S}(\Gamma,f,g),\xi) := \bigoplus_{|\tilde x|=k} \mathbb{Z}\langle \tilde x\rangle$$

where $\tilde x = (\lambda, x)\in \mathrm{Crit}(\mathscr{S}(\Gamma,f,g),\xi)$ and $|\tilde x| = \dim W^u(\tilde x,\xi)$. The boundary operator $\partial^{\mathscr{S}(\Gamma,f,g)} : CM_k(\mathscr{S}(\Gamma,f,g),\xi)\to CM_{k-1}(\mathscr{S}(\Gamma,f,g),\xi)$ is defined by counting trajectories in $\{0\}\times S_0$, $\{1\}\times S_1$ and trajectories of (1.8) in N going from $\{0\}\times S_0$ to $\{1\}\times S_1$:

$$\partial^{\mathscr{S}(\Gamma,f,g)}\langle\tilde y\rangle := \sum_{|\tilde x|=k-1} n_{\mathscr{S}(\Gamma,f,g)}(\tilde y,\tilde x)\langle\tilde x\rangle.$$

The local Morse complex $\big(CM_*(\mathscr{S}(\Gamma,f,g),\xi),\partial^{\mathscr{S}(\Gamma,f,g)}\big)$ is the *algebraic mapping cone* of $\Phi_{\mathbf{flo}}(\Gamma,f,g) : CM_k(S_0,f_0)\to CM_k(S_1,f_1)$; this is the assertion of the next lemma.

Lemma 4.11. *Let (S_0, f_0, g_0) and (S_1, f_1, g_1) be two Morse-Smale triples and let (f, g) be a continuation satisfying (4.1). Let moreover $\Gamma = (N_0, N_1, W, T)$ be an isolating quadruple. Then*

(i) *the local Morse complex $\big(CM_*(\mathscr{S}(\Gamma, f, g), \xi), \partial^{\mathscr{S}(\Gamma,f,g)}\big)$ decomposes as*

$$CM_k(\mathscr{S}(\Gamma, f, g), \xi) = CM_{k-1}(S_0, f_0) \oplus CM_k(S_1, f_1).$$

(ii) *the boundary operator $\partial^{\mathscr{S}(\Gamma,f,g)}$ takes the form*

$$\partial^{\mathscr{S}(\Gamma,f,g)} = \begin{bmatrix} -\partial^{S_0} & 0 \\ \Phi_{\mathbf{flo}} & \partial^{S_1} \end{bmatrix}$$

with respect to the composition in **(i)**. *Here $\Phi_{\mathbf{flo}} = \Phi_{\mathbf{flo}}(\Gamma, f, g)$.*

Proof. It follows as in Lemma 3.4 that the boundary operator has the form

$$\partial^{\mathscr{S}(\Gamma,f,g)} = \begin{bmatrix} -\partial^{S_0} & 0 \\ \Phi & \partial^{S_1} \end{bmatrix}.$$

It remains to show that $\Phi = \Phi_{\mathbf{flo}}(\Gamma, f, g)$. The solution of the initial value problem

$$\dot{\lambda}(t) = \beta(\lambda(t)) \quad \lambda(0) = \delta$$

satisfies

$$\lambda(t) = t \quad \text{for } \delta \leqslant t \leqslant 1 - \delta.$$

We see that

$$(f_{\lambda(t)}, g_{\lambda(t)}) = \begin{cases} (f_0, g_0) & \text{for } t \leqslant \delta \\ (f_t, g_t) & \text{for } \delta \leqslant t \leqslant 1 - \delta \\ (f_1, g_1) & \text{for } t \geqslant 1 - \delta \end{cases}$$

On $[-\delta, 1 + \delta] \times M$ be observe the following time independent gradient-equation

$$\begin{cases} \dot{\lambda}(t) &= \beta_1(\lambda(t)) \\ \dot{x}(t) &= -\nabla f_{\lambda(t)}(x(t)) \end{cases}$$

Assume that $\tilde{y} = (0, y_0)$ and $\tilde{x} = (1, x_1)$ with $y_0 \in \mathrm{Crit}_k(S_0, f_0)$ and $x_1 \in \mathrm{Crit}_k(S_1, f_1)$. There exists an orientation preserving bijection

$$W^u(\tilde{y}, \xi, N) \pitchfork W^s(\tilde{x}, \xi, N) \cap \{\delta\} \times W \cong K_{\Gamma,f}(y_0, x_1)$$

where

$$K_{\Gamma,f}(y_0, x_1) = \Big(\varphi_f^{0,-T}(W^u(x_0, f_0, N_0)) \cap W\Big) \cap \varphi_f^{0,T}(W^s(x_1, f_1, N_1)).$$

This bijection is achieved by the projection $(\delta, p) \mapsto p$. Let $(\delta, p) \in W^u(\tilde{y}, \xi, N) \pitchfork W^s(\tilde{x}, \xi, N) \cap \{\delta\} \times W$. Then $u(t) = (\lambda(t+\delta), \varphi_f^{t,0}(p))$ solves (3.2) with $u(0) = (\delta, p)$. It follows that $t \mapsto \varphi_f^{t,0}(p)$ solves (1.8). Hence

$$n_{\mathscr{S}(\Gamma,f,g)}(\tilde{y}, \tilde{x}) = n_{\Gamma,f}(y_0, x_1)$$

which is equivalent to $\Phi = \Phi_{\mathbf{flo}}(\Gamma, f, g)$ It follows immediately that

$$CM_k(S_0, f_0) \oplus CM_{k+1}(S_1, f_1) \xrightarrow{\begin{bmatrix} -\partial^{S_0} & 0 \\ \Phi_{\mathbf{flo}} & \partial^{S_1} \end{bmatrix}} CM_{k-1}(S_0, f_0) \oplus CM_k(S_1, f_1)\ .$$

□

Next we prove that $\Phi_{\mathbf{flo}}(\Gamma, f, g)$ is a chain homomorphism.

Theorem 4.12. *The map $\Phi_{\mathbf{flo}} = \Phi_{\mathbf{flo}}(\Gamma, f, g)$ is a chain map, i.e.*

$$\partial^{S_1} \circ \Phi_{\mathbf{flo}} = \Phi_{\mathbf{flo}} \circ \partial^{S_0}.$$

Proof. This is an immediate consequence of $\partial^{\mathscr{S}(\Gamma,f,g)} \circ \partial^{\mathscr{S}(\Gamma,f,g)} = 0$ and Lemma 4.11 part **(ii)**:

$$\begin{bmatrix} -\partial^{S_0} & 0 \\ \Phi_{\mathbf{flo}} & \partial^{S_1} \end{bmatrix}^2 = \begin{bmatrix} (\partial^{S_0})^2 & 0 \\ -\Phi_{\mathbf{flo}}\partial^{S_0} + \partial^{S_1}\Phi_{\mathbf{flo}} & (\partial^{S_1})^2 \end{bmatrix} = 0.$$

This implies $-\Phi_{\mathbf{flo}}\partial^{S_0} + \partial^{S_1}\Phi_{\mathbf{flo}} = 0$ and hence the assertion. □

4.3 A special case

Let (S, f, g) be a continuation relating the Morse-Smale triple (S_0, f_0, g_0) to the Morse-Smale triple (S_1, f_1, g_1). Denote by $(f, g)_{\mathbb{R}} := \{(f_\lambda, g_\lambda) \mid \lambda \in \mathbb{R}\}$ the on $\mathbb{R}$ extended family such that (f_λ, g_λ) is constant outside $(\delta, 1-\delta)$. For $\varepsilon > 0$ we consider the slow time dependent gradient equation

$$\dot{x}(t) = -\nabla f_{\varepsilon t}(x(t)) \tag{4.13}$$

with local flow $\varphi_\varepsilon^{t,t_0}$. We pick an isolating neighborhood

$$N := \bigcup_{\lambda \in \mathbb{R}} \{\lambda\} \times N_\lambda$$

of S with respect to the product flow Φ^t such that

$$N_\lambda = \begin{cases} N_0 & \text{for } \lambda \leqslant \delta \\ N_1 & \text{for } 1-\delta \leqslant \lambda \end{cases}$$

Example 4.14. Pick a smooth function $\beta_\varepsilon : [-\delta, 1+\delta] \to [0,1]$ in $\mathscr{B}^+(\delta)$ with $\operatorname{supp}(\beta_\varepsilon) = (0,1), \beta^{-1}(0) = \{0,1\}, \beta_\varepsilon'(0) > 0, \beta_\varepsilon'(1) < 0$ such that

$$\beta_\varepsilon(\lambda) = 1 \quad \text{for } \delta \leqslant \lambda \leqslant 1-\delta.$$

The solution of the initial value problem

$$\dot\lambda(t) = \beta_\varepsilon(\lambda(t)) \quad \lambda\left(\frac{\delta}{\varepsilon}\right) = \delta$$

satisfies

$$\lambda(t) = \varepsilon t \quad \text{for } \frac{\delta}{\varepsilon} \leqslant t \leqslant \frac{1-\delta}{\varepsilon}.$$

It follows that

$$(f_{\lambda(t)}, g_{\lambda(t)}) = \begin{cases} (f_0, g_0) & \text{for } t \leqslant \frac{\delta}{\varepsilon} \\ (f_{\varepsilon t}, g_{\varepsilon t}) & \text{for } \frac{\delta}{\varepsilon} \leqslant t \leqslant \frac{1-\delta}{\varepsilon} \\ (f_1, g_1) & \text{for } t \geqslant \frac{1-\delta}{\varepsilon} \end{cases}$$

For the solution $t \mapsto \lambda_\varepsilon(t)$ we get

$$\dot x(t) = -\nabla f_{\varepsilon t}(x(t)) \quad \text{for } \frac{\delta}{\varepsilon} \leqslant t \leqslant \frac{1-\delta}{\varepsilon}.$$

Now we are ready to prove that a continuation (S, f, g) induces a triple $(K_\varepsilon, f_\varepsilon, g_\varepsilon)$.

Theorem 4.15. *Let (S, f, g) be a continuation relating two Morse-Smale triples (S_0, f_0, g_0) and (S_1, f_1, g_1). Then for small enough $\varepsilon_0 > 0$ and any $\varepsilon \in (0, \varepsilon_0)$ the set*

$$K_\varepsilon := \left\{ x \in M \mid \lim_{t\to-\infty} \varphi_\varepsilon^{t,0}(x) \in S_0, \lim_{t\to\infty} \varphi_\varepsilon^{t,0}(x) \in S_1,\ \varphi_\varepsilon^{t,0}(x) \in N_{\lambda(t)} \right\}$$

is an isolated connecting set.

Proof. Having Example 4.14 in mind, we define an isolating quadruple induced by any continuation $S, f, g)$. Let $t \mapsto \lambda(t)$ be a solution with $\lambda(\delta/\varepsilon) = \delta$ as in Example 4.14 and define $T_\varepsilon := (1-\delta)/\varepsilon$. Pick a neighborhood $\{\lambda(0)\} \times W_\varepsilon \subset N_{\lambda(0)}$ of

$$\mathscr{S}_\varepsilon(S, f, g) \cap \{\lambda(0)\} \times M$$

such that

$$t \mapsto \Phi^t_{(f,g,\beta_\varepsilon)}(\lambda(0), p) = (\lambda(t), \varphi_f^{t,0}(p))$$

is defined for all $t \in [-T_\varepsilon, T_\varepsilon]$ and every $p \in W_\varepsilon$. This can be achieved by making W_ε small enough. It follows that

$$t \longmapsto \varphi_f^{t,0}(p) \quad \text{for } t \in [-T_\varepsilon, T_\varepsilon] \text{ and every } p \in W.$$

Denote by $\Gamma_\varepsilon = (N_0, N_1, W_\varepsilon, T_\varepsilon)$ the induced isolating quadruple of (S, f, g). Now define an isolating neigbhorhood

$$W := [-\delta, \delta] \times N_0 \cup \left(\bigcup_{t \in [-T_\varepsilon, T_\varepsilon]} \{\lambda(t)\} \times \varphi_\varepsilon^{t,0}(W_\varepsilon) \right) \cup [1-\delta, 1+\delta] \times N_1.$$

of the dynamical cone $\mathscr{S}_\varepsilon(S, f, g) \subset [0,1] \times M$. Using the product flow and projecting to the second variable, we check (i)-(iv) of Definition 4.3. Again by definition it follows

$$K_\varepsilon = \big(W_\varepsilon \cap \varphi_\varepsilon^{0,-T_\varepsilon}\big(W^u(S_0, f_0, N_0)\big)\big) \cap \varphi_\varepsilon^{0,T_\varepsilon}\big(W^s(S_1, f_1, N_1)\big)$$

Let β_ε and $t \mapsto \lambda(t)$ be as in Example 4.14. By Lemma 2.22 there exists an $\varepsilon_0 > 0$ such that the dynamical cone $\mathscr{S}_\varepsilon(S, f, g)$ is an isolated invariant set hence compact for $\varepsilon \in (0, \varepsilon_0)$. Fix $\varepsilon \in (0, \varepsilon_0)$. We show that K_ε is compact. Pick a sequence $x_k \subset K_\varepsilon$. Then the curve

$$u_k(t) := \left(\lambda \left(t + \frac{\delta}{\varepsilon} \right), \varphi_\varepsilon^{t,0}(x_k) \right)$$

is a solution of (2.19) whose image is in $\mathscr{S}_\varepsilon(S, f, g)$. Since $\mathscr{S}_\varepsilon(S, f, g) \subset [0,1] \times M$ is compact it follows that the $(\delta, x_k) = \tilde{u}_k(0) \in \mathscr{S}_\varepsilon(S, f, g)$ has an convergent subsequence $(\delta, x_0) \in \mathscr{S}_\varepsilon(S, f, g)$ and $x_0 \in K_\varepsilon$. This proves the compactness of K_ε. □

It follows from Lemma 4.11 and Proposition 3.7 that for small $\varepsilon > 0$ Floer continuation map $\Phi_{\mathbf{flo}}(\Gamma_\varepsilon, f_\varepsilon, g_\varepsilon)$ is an *isomorphism.*

Definition 4.16. Let (S_0, f_0, g_0) and (S_1, f_1, g_1) be two Morse-Smale pairs related by continuation via (S, f, g). Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the set K_ε is an isolated connecting set of $(f_\varepsilon, g_\varepsilon) = \{(f_{\varepsilon t}, g_{\varepsilon t}) \mid \lambda \in [0, 1/\varepsilon]\}$. The triple $(K_\varepsilon, f_\varepsilon, g_\varepsilon)$ determines a Floer type homomorphism

$$\Phi_{\mathbf{flo}}(K_\varepsilon, f_\varepsilon, g_\varepsilon) : HM_*(S_0, f_0, g_0) \to HM_*(S_1, f_1, g_1)$$

is called the **Floer continuation map** of (S, f, g).

4.4 Proof of Theorem C

Let (S, f, g) be a continuation relating the Morse-Smale triple (S_0, f_0, g_0) to the Morse-Smale triple (S_1, f_1, g_1). Next pick a smooth function β_ε : $[-\delta, 1+\delta] \to [0,1]$ with $\beta_\varepsilon^{-1}(0) = \{0,1\}, \beta_\varepsilon'(0) > 0, \beta_\varepsilon'(1) < 0$ and $\beta_\varepsilon(\lambda) = \varepsilon$ for $\lambda \in [\delta, 1-\delta]$. We define vector field $\xi(\lambda, x) = (\beta_\varepsilon(\lambda), -\nabla f_\lambda(x))$ on the product $[-\delta, 1+\delta] \times M$ by

$$\begin{cases} \dot\lambda(t) &= \beta_\varepsilon(\lambda(t)) \\ \dot x(t) &= -\nabla f_{\lambda(t)}(x(t)) \end{cases}$$

with local flow $\Psi^t_{(f,g,\beta_\varepsilon)}$. Denote by

$$N := \bigcup_{\lambda \in [-\delta, 1+\delta]} \{\lambda\} \times N_\lambda \tag{4.17}$$

where $N_\lambda = N_0$ for $\lambda \leqslant \delta$ and $N_\lambda = N_1$ for $\lambda \geqslant 1-\delta$ an isolating neighborhood of S with repsect to the product flow $\Phi^t(\lambda, x) = (\lambda, \varphi^t_\lambda)$ where φ^t_λ is the gradient flow of (f_λ, g_λ). Denote by

$$\mathscr{S}_\varepsilon(S, f, g) := I\left(N, \Psi^t_{(f,g,\beta_\varepsilon)}\right) \subset [0,1] \times M$$

the *dynamical cone* of (S, f, g). We prove the main ingredient for the proof of Theorem C.

Lemma 4.18. *Let (S, f, g) be a continuation relating the Morse-Smale triple (S_0, f_0, g_0) to the Morse-Smale triple (S_1, f_1, g_1) and let $\mathscr{S}_\varepsilon(S, f, g)$ be the dynamical cone. Then $\big(\{0\} \times S_0, \{1\} \times S_1\big)$ is an attractor-repeller decomposition of $\mathscr{S}_\varepsilon(S, f, g)$ and the following diagram commutes:*

$$\begin{tikzcd} HM_{k+1}(\{0\} \times S_0, \xi) \arrow[r, "\partial"] & HM_k(\{1\} \times S_1, \xi) \\ HM_k(S_0, f_0, g_0) \arrow[u, "\sigma"] \arrow[r, "\Phi_{\mathbf{flo}}(\Gamma_\varepsilon, f_\varepsilon, g_\varepsilon)"'] & HM_k(S_1, f_1, g_1) \arrow[u, "\sigma"'] \end{tikzcd} \tag{4.19}$$

Proof. The two short exact sequences are isomorphic

$$\begin{tikzcd} 0 \arrow[r] & CM_*(\{1\} \times S_1) \arrow[r] \arrow[d, "\sigma^{-1}"'] & CM_*\left(\mathscr{S}_\varepsilon(S, f, g), \xi\right) \arrow[r] \arrow[d, "\mathrm{id}"'] & \frac{CM_*\left(\mathscr{S}_\varepsilon(S,f,g),\xi\right)}{CM_*(\{1\}\times S_1)} \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & CM_*(S_1, f_1) \arrow[r] & CM_*\left(\mathscr{S}_\varepsilon(S, f, g), \xi\right) \arrow[r] & CM_{*-1}(S_0, f_0) \arrow[r] & 0 \end{tikzcd}$$

where c is the following composition

$$c : \frac{CM_*\big(\mathscr{S}_\varepsilon(\tilde{S},\tilde{f},\tilde{g}),\xi\big)}{CM_*(\{1\}\times S_1)} \xrightarrow{\cong} \frac{CM_{*-1}(S_0,f_0)\oplus CM_*(S_1,f_1)}{CM_*(\{1\}\times S_1)} \xrightarrow{\cong} CM_{*-1}(S_0,f_0).$$

The diagram of short exact sequences above induces isomorphic long exact sequences

$$\begin{tikzcd}
HM_{k+1}(\{0\}\times S_0,\xi) \arrow[r, "\partial"] & HM_k(\{1\}\times S_1,\xi) \\
HM_k(S_0,f_0,g_0) \arrow[u, "\sigma"] \arrow[r, "\Phi_{\mathbf{flo}}(\Gamma_\varepsilon,f_\varepsilon,g_\varepsilon)"'] & HM_k(S_1,f_1,g_1). \arrow[u, "\sigma"']
\end{tikzcd}$$

$\square$

Now we are ready to prove Theorem C.

Proof of Theorem C. Observe the following diagram

$$\begin{tikzcd}
HC_k(S_0) \arrow[r, "\Phi_{\mathbf{con}}(S,f,g)"] \arrow[d, "\sigma"'] \arrow[ddd, bend right=60, "\alpha_0"'] & HC_k(S_1) \arrow[d, "\sigma"] \arrow[ddd, bend left=60, "\alpha_1"] \\
HC_{k+1}(\{0\}\times S_0) \arrow[r, "\partial"] \arrow[d, "\{0\}\times\alpha_0"'] & HC_k(\{1\}\times S_1) \arrow[d, "\{1\}\times\alpha_1"] \\
HM_{k+1}(\{0\}\times S_0,\xi) \arrow[r, "\partial"] & HM_k(\{1\}\times S_1,\xi) \\
HM_k(S_0,f_0) \arrow[u, "\sigma"] \arrow[r, "\Phi_{\mathbf{flo}}(\Gamma_\varepsilon,f_\varepsilon,g_\varepsilon)"] & HM_k(S_1,f_1) \arrow[u, "\sigma"']
\end{tikzcd}$$

The top square of this diagram commutes by Propostion 2.27. The middle square commutes by Theorem B. And the bottom square commutes by Lemma 4.18. The only thing left to show is that $\alpha_0 = \sigma^{-1}\circ\{0\}\times\alpha_0\circ\sigma$ and $\alpha_1 = \sigma^{-1}\circ\{1\}\times\alpha_1\circ\sigma$. But this is clear since σ is just a $+1$ shift in degree and σ^{-1} is a -1 shift in degree. This proves Theorem B. $\square$

4.5 The Floer bifurcation construction

In this section we define the Floer bifurcational map using bifurcation analysis as in [3]. Consider a continuation (S,f,g) relating the Morse-Smale triple (S_0,f_0,g_0) to the Morse-Smale triple (S_1,f_1,g_1). We assume throughout that

$$\boxed{f_\lambda \quad \text{is Morse} \quad \forall\lambda\in[0,1].}$$

We observe the orbit structure of the local flow φ^t_λ generated by equation (1.12) being

$$\dot{x} = -\nabla f_\lambda(x).$$

Pick an isolating neighborhood

$$N := \bigcup_{\lambda\in[0,1]} \{\lambda\} \times N_\lambda$$

of S with respect to the product flow $\Phi^t(\lambda, p) = (\lambda, \varphi^t_\lambda(p))$. Let $y, x : [0,1] \to M$ such that $y_\lambda := y(\lambda) \in \mathrm{Crit}(S_\lambda, f_\lambda)$ and $x_\lambda := x(\lambda) \in \mathrm{Crit}(S_\lambda, f_\lambda)$ for all $\lambda \in [0,1]$. The unstable manifold of $x_\lambda \in \mathrm{Crit}(S_\lambda, f_\lambda)$ relative to N_λ is denoted by

$$W^u(x_\lambda, f_\lambda, N_\lambda) := \left\{ p \in M \;\middle|\; \lim_{t\to-\infty} \varphi^t_\lambda(p) = x_\lambda,\ \varphi_\lambda^{(-\infty,0]}(p) \subset N_\lambda \right\}.$$

The parametrized unstable manifold of $x : [0,1] \to M, x_\lambda \in \mathrm{Crit}(S_\lambda, f_\lambda)$ is defined by

$$\mathcal{W}^u(x, f, N) := \{(\lambda, p) \mid p \in W^u(x_\lambda, f_\lambda, N_\lambda)\} \subset [0,1] \times M.$$

Define the space of solutions

$$\mathscr{M}(y_\lambda, x_\lambda, N_\lambda) := \left\{ u : \mathbb{R} \to N_\lambda \;\middle|\; \begin{array}{c} u \text{ solves } \dot{u} = -\nabla f_\lambda(u) \\ \lim_{t\to-\infty} u(t) = y_\lambda, \lim_{t\to\infty} u(t) = x_\lambda \end{array} \right\}.$$

The **index of a solution** $u \in \mathscr{M}_f(y_\lambda, x_\lambda, N_\lambda)$ to be the difference of the Morse indices of its asymptotics, i.e.

$$\mathrm{ind}(u) := |y| - |x|.$$

The intersection $W^u(y, f, N) \cap W^s(x, f, N)$ is diffeomorphic to the parametrized space of solutions given by

$$\mathscr{M}_f(y, x, N) := \{(\lambda, u) \mid u \in \mathscr{M}(y_\lambda, x_\lambda, N_\lambda)\}.$$

Denote by

$$\mathscr{F}^{mor}(N) := \{h : [0,1] \times M \to \mathbb{R} \mid h_\lambda : N_\lambda \to \mathbb{R} \text{ is Morse } \forall\lambda\}$$

the space of Morse families endowed with the C^∞−topology on compact sets. Using the C^k−topologies we can define a complete metric on $\mathscr{F}^{mor}(N)$.

Hence the space $\mathscr{F}^{mor}(N)$ has the Baire property. Define the *bifurcation values* of (S, f, g) by

$$\Lambda_{\text{bif}}(N, f) := [0,1] \backslash \{\lambda \mid (N_\lambda, f_\lambda, g_\lambda) \text{ Morse-Smale}\}$$

The set of bifurcation values where an *index zero* solution occurs is denoted by

$$\Lambda_0(N, f) : \{\lambda \in \Lambda_{\text{bif}}(N, h) \mid \exists u \in \mathscr{M}(y_\lambda, x_\lambda, f_\lambda, N_\lambda) \text{ with } \operatorname{ind}(u) = 0\}.$$

Since f_λ is assumed to be Morse the only bifurcation that occurs for $\lambda_{\text{bif}} \in \Lambda_{\text{bif}}(N, f)$ is a solution $\gamma : \mathbb{R} \to N_\lambda$ whose trajectory $\gamma(\mathbb{R})$ is the non-transversal intersection of some manifolds $W^u(y_\lambda, f_\lambda, N_\lambda)$ and $W^s(x_\lambda, f_\lambda, N_\lambda)$. Such a solution or trajectory is called **non-transversal**.

$$\mathscr{F}^{mor}_{reg}(N) := \left\{ h \in \mathscr{F}^{mor}(N) \;\middle|\; \begin{array}{ll} (i) & \forall x, y : [0,1] \to M, y_\lambda, x_\lambda \in \operatorname{Crit}(S_\lambda, f_\lambda) \\ & \qquad \mathcal{W}^u(y, h, N) \pitchfork \mathcal{W}^u(x, h, N) \\ (ii) & \forall \lambda_0 \in \Lambda_0(N, h) \text{ there is exactly one} \\ & \qquad \text{non-transversal orbit of } \varphi^t_{\lambda_0} \end{array} \right\}.$$

It follows from the transversality assumption (i) that $\Lambda_{\text{bif}}(N, h)$ is finite. Now we are ready to state the *transverality* theorem.

Theorem 4.20 (Floer [3]). *Let (S_0, f_0, g_0) and (S_1, f_1, g_1) be two Morse-Smale triples that are related by continuation via (S, f, g) such that f_λ is Morse for all $\lambda \in [0,1]$. Moreover let N be an isolating neighborhood of S with respect to the product flow. Then $\mathscr{F}^{mor}_{reg}(N)$ is open and dense in $\mathscr{F}^{mor}(N)$.*

Proof. In order to interpret the solution spaces as preimages of regular value of some Fredholm sections, we introduce convenient Banach manifolds. Let $u : \mathbb{R} \to M$ be a continuous map such that

$$(*) = \begin{cases} u \in W^{1,2}_{\text{loc}}(\mathbb{R}, M) \\ \exists T > 0 \text{ and} \\ \xi \in W^{1,2}([T, \infty), T_{x_\lambda} M) \\ \eta \in W^{1,2}((-\infty, -T], T_{y_\lambda} M) \\ \text{such that} \\ u(t) = \exp_{y_\lambda}(\eta(t)) \text{ for } t \leqslant -T \\ u(t) = \exp_{x_\lambda}(\xi(t)) \text{ for } t \geqslant T \end{cases}$$

Define the Banach manifold consisting of paths connecting y_λ to x_λ by

$$\mathscr{P}(y_\lambda, x_\lambda, N_\lambda) := \{u : \mathbb{R} \to N_\lambda \mid u \text{ is continuous and satisfies } (*)\}.$$

Next we define the parametrized space of paths by

$$\mathscr{B}(y,x,N) := \{(\lambda,u) \mid \lambda \in [0,1], u \in \mathscr{P}(y_\lambda, x_\lambda, N_\lambda)\}.$$

The tangent space of $\mathscr{P}(y_\lambda, x_\lambda, N_\lambda)$ at u is given by

$$T_u\mathscr{P}(y_\lambda, x_\lambda, N_\lambda) = W^{1,2}(\mathbb{R}, u^*TM).$$

To explore the tangent space of $\mathscr{B}(y,x,N)$ we pick a path

$$(-\delta,\delta) \to \mathscr{B}(y,x,N), s \mapsto (\lambda(s), u(s))$$

with $(\lambda(0), u(0)) = (\lambda, u)$. Moreover note that $E_1(x,0) = E_2(x,0) = \mathbb{1}$. It follows that

$$\hat{u}(t) = E_1(x,\xi)\hat{x} + E_2(x,\xi)\hat{\xi}.$$

We differentiate this path at $s=0$ and get

$$\left.\frac{d}{ds}\right|_{s=0} u(s,t) = \left.\frac{d}{ds}\right|_{s=0} \exp_{x(\lambda(s))}(\xi(t)) \quad \text{for } t \geqslant T.$$

For $t \geqslant T$ it follows that

$$\hat{u}(t) := \left.\frac{d}{ds}\right|_{s=0} u(s,t) = \underbrace{\sum_i \frac{\partial E}{\partial x^i}\hat{x}^i}_{=:E_1(x,\xi)\hat{x}} + \underbrace{\sum_k \frac{\partial E}{\partial \xi^k}\hat{\xi}^k + \sum_{i,j,k}\Gamma_{ij}^k(x)\hat{x}^i\xi^j}_{=:E_2(x,\xi)\hat{\xi}}$$

where $E_1, E_2 : T_{x_\lambda}M \to T_{\exp_{x_\lambda}(\xi)}M$ and $\frac{d}{ds}|_{s=0}x(\lambda(s)) = \hat{x} \in T_xM, \frac{d}{ds}|_{s=0}\lambda(s) = \hat{\lambda}, \nabla_s|_{s=0}\xi = \hat{\xi}, x_{\lambda(0)} = x$. Fix the conditions

$$(**) = \begin{cases} \hat{\lambda} \in \mathbb{R},\ \hat{u} : \mathbb{R} \to TM \\ \hat{u}(t) \in T_{u(t)}M \\ u(t) = \exp_{x_\lambda}(\xi(t))\ \forall t \geqslant T, \hat{x} := dx(\lambda)\hat{\lambda} \\ \hat{u} - E_1(x,\xi)\hat{x} \in W^{1,2}([T,\infty)), u^*TM) \\ u(t) = \exp_{y_\lambda}(\eta(t))\ \forall t \leqslant -T, \hat{y} := dy(\lambda)\hat{\lambda} \\ \hat{u} - E_1(y,\eta)\hat{y} \in W^{1,2}((-\infty,-T]), u^*TM) \end{cases}$$

Then the tangent space of $\mathscr{B}(y,x,N)$ at (λ,u) is given by

$$T_{(\lambda,u)}\mathscr{B}(y,x,N) := \left\{(\hat{\lambda},\hat{u}) \mid (\hat{\lambda},\hat{u}) \text{ satisfies } (**)\right\}.$$

Now denote by $\mathscr{E}(y,x,N)$ the Banach space bundle with fibre $\mathscr{E}_{(\lambda,u)} := L^2(\mathbb{R}, u^*TM)$. Define the section

$$\mathcal{S} : \mathscr{P}(x,y,N) \to \mathscr{E}(y,x,N),\ (\lambda,u) \mapsto \dot{u} + \nabla f_\lambda(u)$$

whose differential

$$D\mathcal{S}(\lambda,u)(\hat{\lambda},\hat{u}) = \underbrace{\partial_t \hat{u} + \nabla_{\hat{u}}\nabla f_\lambda(u)}_{=:D_u(\hat{u})} + \hat{\lambda}\partial_\lambda \nabla f_\lambda(u).$$

is Fredholm of *Fredholm index* $\operatorname{ind} D\mathcal{S}(\lambda,u) = |y| - |x| + 1$. Define

$$\mathscr{W} := \left\{ (\lambda,u,f) \;\middle|\; \begin{array}{c} \lim_{t\to-\infty} u(t) = y_\lambda(f), \lim_{t\to\infty} u(t) = x_\lambda(f) \\ u \in \mathscr{P}(x_\lambda(f), y_\lambda(f), N) \end{array} \right\}.$$

$$\tilde{\mathcal{S}} : \mathscr{W} \longrightarrow \mathscr{E}(y,x,N),\ (\lambda,u,f) \longmapsto \dot{u} + \nabla f_\lambda(u).$$

Denote by $pr_2 : \mathscr{M}_f(y,x,N) := \tilde{\mathcal{S}}^{-1}(0) \to \mathscr{F}^{mor}(N)$ the smooth projection on the space of Morse families. From the Sard-Smale theorem for Banach manifolds it follows that

$$\mathscr{M}_f(y,x,N) = pr_2^{-1}(f)$$

is a smooth manifold of dimension $|y| - |x| + 1$.

$$X := \left\{ (\lambda,u,v,f) \mid u \in \hat{\mathscr{M}}(y_\lambda, y'_\lambda, N_\lambda), v \in \hat{\mathscr{M}}(x_\lambda, x'_\lambda, N_\lambda), f \in \mathscr{F}^{mor}(N) \right\}$$

is a Banach manifold. The projection on the families of Morse functions $X \to \mathscr{F}^{mor}(N)$ has Fredholm index -1. Hence for generic f the space X is empty. $\square$

The critical points of $f_\lambda : N_\lambda \to \mathbb{R}$ depend smoothly on λ and the number of critical points and their Morse indices do not change. In other words there exists smooth maps $x_1, \ldots, x_\ell : [0,1] \to M$ called **critical families** such that

(i) $\operatorname{Crit}(N_\lambda, f_\lambda) = \{x_1(\lambda), \ldots, x_\ell(\lambda)\}$ for all $\lambda \in [0,1]$

(ii) $|x_i| := |x_i(\lambda)|$ for $i = 1, \ldots, \ell$ is welldefined.

Chose orientations $\{\mathbf{o}_{x_i}\}_{i=1,\ldots,\ell}$ of $\mathcal{W}^u(x_i, f, N)$. These orientations induce orientations $\{o_{x_\lambda}\}_{x_\lambda \in \operatorname{Crit}(S_\lambda, f_\lambda)}$ on $W^u(x_\lambda, f_\lambda, N_\lambda)$. Using these orientations we get the local Morse complex

$$\big(CM_*(S_\lambda, f_\lambda), \partial^{S_\lambda}\big) \quad \text{for } \lambda \in [0,1] \backslash \Lambda_{\text{bif}}.$$

If there are no index zero trajectories meaning $\Lambda_0(N,f) = \emptyset$ then the local Morse complex $(CM_*(S_\lambda, f_\lambda), \partial^{S_\lambda})$ does not change through the deforamtion. This is the content of the next lemma.

Lemma 4.21 (Floer)**.** *Let (S, f, g) be a continuation relating the Morse-Smale pairs (S_0, f_0, g_0) and (S_1, f_1, g_1) such that $f \in \mathscr{F}^{mor}_{reg}(N)$ for some isolating neighborhood N of S. If $\Lambda_0(N, f) = \emptyset$ then the isomorphism*

$$\Phi_{\mathbf{bif}}(S, f, g) : CM_*(S_0, f_0) \xrightarrow{\cong} CM_*(S_1, f_1)$$

defined by $\langle x_0 \rangle \mapsto \langle x_1 \rangle$ is a chain isomorphism.

Proof. Pick two critical families $y, x : [0, 1] \to N$ of Morse index difference one, $|y| - |x| = 1$. We have to prove that

$$n_{S_0}(y_0, x_0) = n_{S_1}(y_1, x_1).$$

Note that $\mathscr{M}_f(y, x, N)$ is compact. Its boundary points counted with sign is $n_{S_0}(y_0, x_0) - n_{S_1}(y_1, x_1) = 0$ since $\mathscr{M}_f(y, x, N)$ is one dimensional. Hence it is a chain homomorphism since

$$\partial^{S_1} \circ \Phi_{\mathbf{bif}}(y_0) = \partial^{S_1}(y_1) = \sum_{x_1} n_{S_1}(y_1, x_1)x_1$$

and

$$\Phi_{\mathbf{bif}} \circ \partial^{S_0}(y_0) = \Phi_{\mathbf{bif}}\big(\sum_{x_0} n_{S_0}(y_0, x_0)x_0\big) = \sum_{x_0} n_{S_0}(y_0, x_0)\Phi_{\mathbf{bif}}(x_0)$$

are equal using $n_{S_0}(y_0, x_0) = n_{S_1}(y_1, x_1)$ and the definition of $\Phi_{\mathbf{bif}}$. □

Now we want to explore how the Morse complex $CM_*(S_\lambda, f_\lambda)$ changes as we cross a bifurcation value $\lambda_0 \in \lambda_{\text{bif}}(N, f)$. Pick a small $\varepsilon > 0$ such that for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \backslash \{\lambda_0\}$ such that $(N_\lambda, f_\lambda, g_\lambda)$ is Morse-Smale. To simplify notation we may assume that $\lambda_0 = 0$. Then the interval becomes $[-\varepsilon, \varepsilon]$ and the unique index zero trajectory occurs at $\lambda = 0$. A bifurcation at a time. Denote by

$$\text{Crit}_k(S_\lambda, f_\lambda) = \{x_1(\lambda), \ldots, x_\ell(\lambda)\} \quad \text{for } \lambda \in [-\varepsilon, \varepsilon]$$

and $\gamma_{ij} \in \mathscr{M}_{f_0}(x_i(0), x_j(0), N_0)$ the unique index zero solution going from $x_i(0)$ to $x_j(0)$.

Lemma 4.22 (Floer [3])**.** *Let (S, f, g) be a continuation relating the Morse-Smale pairs (S_0, f_0, g_0) and (S_1, f_1, g_1) such that $f \in \mathscr{F}^{mor}_{reg}(N)$ for some isolating neighborhood N of S. Assume that $\Lambda_0(N, f) = \{0\}$. Then the following holds:*

(i) *There exists a chain homomorphism* $\Phi^{\varepsilon,-\varepsilon}_{\mathbf{bif}}(S,f,g) : CM_*(S_{-\varepsilon}, f_{-\varepsilon}) \xrightarrow{\cong} CM_*(S_\varepsilon, f_\varepsilon)$ *defined by the elementary homomorphism*

$$\Phi^{\varepsilon,-\varepsilon}_{\mathbf{bif}}(\langle x_k(-\varepsilon)\rangle) := \begin{cases} \langle x_i(\varepsilon)\rangle - n(\gamma_{ij})\langle x_j(\varepsilon)\rangle & \text{for } k = i \\ \langle x_k(\varepsilon)\rangle & \text{for } k \neq i \end{cases}$$

Here $n(\gamma)$ *is the sign of* γ *defined in Section 5.2.*

(ii) *The homomorphism* $\Phi^{\varepsilon,-\varepsilon}_{\mathbf{bif}}(S,f,g)$ *has the matrix representation*

$$E(\gamma_{ij}) := (\delta_{mn} - n(\gamma_{ij})\delta_{mj}\delta_{in})^{\ell}_{m,n=1}$$

with respect to the bases $\{\langle x_k(-\varepsilon)\rangle\}^{\ell}_{k=1}$ *and* $\{\langle x_k(\varepsilon)\rangle\}^{\ell}_{k=1}$. *The elementary matrix* $E(\gamma_{ij}) \in \mathbb{R}^{\ell\times\ell}$ *has units on the diagonal and exactly one non-zero entry* $n(\gamma_{ij})$ *in the* $(j,i)-$*slot.*

Proof. The chain map property follows by the two orientation preserving bijection. For $z : [-\varepsilon, \epsilon] \to N,\ y_\lambda \in \mathrm{Crit}_{k-1}(S_\lambda, f)$ we get

$$\mathscr{M}_{-\varepsilon}(x_i, z, N) = \mathscr{M}_\varepsilon(x_i, z, N) - \mathscr{M}_0(x_i, x_j, N) \times \mathscr{M}_\varepsilon(x_j, z, N).$$

We shortened the notation by $\mathscr{M}_\varepsilon(x_i, z, N) = \mathscr{M}_{f_\varepsilon}(x_i(\varepsilon), z(\varepsilon), N_\varepsilon)$. Using this equation we get

$$n_{S_{-\varepsilon}}(x_i, z) = n_{S_\varepsilon}(x_i, z) - n(\gamma_{ij})n_{S_\varepsilon}(x_j, z).$$

It follows that

$$\begin{aligned} \partial^{S_\varepsilon} \circ \Phi_{\mathbf{bif}}(\langle x_i\rangle) &= \partial^{S_\varepsilon}\langle x_i\rangle - n(\gamma_{ij})\partial^{S_\varepsilon}\langle x_j\rangle \\ &= \textstyle\sum_z n_{S_\varepsilon}(x_i, z)\langle z\rangle - \sum_z n(\gamma_{ij})n_{S_\varepsilon}(x_j, z)z \\ &= \textstyle\sum_z \big(n_{S_\varepsilon}(x_i, z) - n(\gamma_{ij})n_{S_\varepsilon}(x_j, z)\big)\langle z\rangle \\ &= \textstyle\sum_z n_{S_{-\varepsilon}}(x_i, z)\langle z\rangle \\ &= \Phi_{\mathbf{bif}} \circ \partial^{S_{-\varepsilon}}(\langle x_i\rangle) \end{aligned}$$

where $\Phi_{\mathbf{bif}} = \Phi^{\varepsilon,-\varepsilon}_{\mathbf{bif}}(S,f,g)$. If $z : [-\varepsilon, \varepsilon] \to N$ is a critical family with $y(\lambda) \notin \mathrm{Crit}_k(S_\lambda, f_\lambda)$ then it follows by 4.21 that

$$n_{S_{-\varepsilon}}(y, z) = n_{S_\varepsilon}(y, z).$$

This immediatley implies

$$\begin{aligned}\partial^{S_\varepsilon} \circ \Phi_{\mathbf{bif}}(\langle y\rangle) &= \textstyle\sum_z n_{S_\varepsilon}(y,x)\langle x\rangle \\ &= \textstyle\sum_z n_{S_{-\varepsilon}}(y,x)\langle x\rangle \\ &= \Phi_{\mathbf{bif}} \circ \partial^{S_{-\varepsilon}}(\langle y\rangle).\end{aligned}$$

It follows that $\Phi_{\mathbf{bif}}^{\varepsilon,-\varepsilon}(S,f,g)$ is a chain map. □

Now we are ready to define Floers bifurcational map.

Definition 4.23. Let (S,f,g) be a continuation relating the two Morse-Smale triples (S_0,f_0,g_0) and (S_1,f_1,g_1) such that $f \in \mathscr{F}_{reg}^{mor}(N)$. Moreover denote by

$$\{\lambda_1,\dots,\lambda_N\} = \Lambda_0(N,f)$$

the bifurcation values $0 < \lambda_1 < \cdots < \lambda_N < 1$ where an index zero connecting orbit occurs. Pick a small enough $\varepsilon > 0$ such that for all $k \in \{1,\dots,N\}$ the interval $[\lambda_k - \varepsilon, \lambda_k - \varepsilon] \backslash \{\lambda_k\}$ has no bifurcation values. Then the **Floer bifurcational continuation map**

$$\Phi_{\mathbf{flo}}(S,f,g) : HM_*(S_0,f_0,g_0) \longrightarrow HM_*(S_1,f_1,g_1)$$

is induced by the chain isomorphism $\Phi_{\mathbf{flo}}^{1,0}(S,f,g) : CM_*(S_0,f_0,g_0) \to CM_*(S_1,f_1,g_1)$ defined via the composition

$$\boxed{\Phi_{\mathbf{bif}}^{1,0}(S,f,g) := \Phi_{\mathbf{bif}}^{\lambda_N+\varepsilon,\lambda_N-\varepsilon}(S,f,g) \circ \cdots \circ \Phi_{\mathbf{bif}}^{\lambda_1+\varepsilon,\lambda_1-\varepsilon}(S,f,g)}$$

of elementary isomorphisms $\Phi_{\mathbf{bif}}^{\lambda_k+\varepsilon,\lambda_k-\varepsilon}(S,f,g)$ of Lemma 4.22 part **(i)**.

Floers bifurcation map $\Phi_{\mathbf{bif}}(S,f,g) : CM_k(S_0,f_0) \to CM_k(S_1,f_1)$ is a chain isomorphism since it is a composition of chain isomorphisms. Hence the Floer bifurcation map

$$\Phi_{\mathbf{bif}} : HM_*(S_0,f_0,g_0) \to HM_*(S_1,f_1,g_1)$$

is well defined and an isomorphism.

Remark 4.24. Let $\{\lambda_1,\dots,\lambda_N\} = \Lambda_0(N,f)$ be the set of bifurcation values where an index zero trajectory occurs. Denote for any $\lambda_\nu \in \Lambda_0(N,f)$ the *unique* index zero trajectory

$$\gamma_{i_\nu j_\nu} : \mathbb{R} \to N_{\lambda_\nu}$$

connecting the critical points x_{i_ν} and x_{j_ν} of the same Morse index, i.e. $\gamma_{i_\nu j_\nu} \in \mathscr{M}_{f_{\lambda_\nu}}(x_{i_\nu}, x_{j_\nu}, N_{\lambda_\nu})$. The Floer bifurcation map $\Phi_{\mathbf{bif}}(S, f, g) : CM_k(S_0, f_0, g_0) \to CM_k(S_1, f_1, g_1)$ acts on generators by

$$\Phi_{\mathbf{bif}}(S, f, g)(\langle x_j \rangle) = \sum_i a_{ji} \langle x_i \rangle$$

where the representation matrix $A = (a_{ij})$ is the product

$$A = E(\gamma_{i_N j_N}) \cdot \dots \cdot E(\gamma_{i_1 j_1})$$

of elementary matrices $E(\gamma_{i_\nu j_\nu}) = \big(\delta_{ij} - n_{\gamma_{i_\nu j_\nu}} \delta_{i i_\nu} \delta_{j_\nu j}\big)$ having units on the diagonal and exactly one non-zero entry $-n(\gamma_{i_\nu j_\nu})$ in the (i_ν, j_ν) slot.

Remark 4.25. Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda[-\varepsilon, \varepsilon]\}$ be a continuation satisfying the parametrized Morse-Smale condition such that $\Lambda_{\mathrm{bif}}(N, f) = \Lambda_0(N, f) = \{0\}$. Let $x_1, \dots, x_\ell : [0, 1] \to M$ be the critical families and

$$\gamma_1, \dots, \gamma_k : \mathbb{R} \to \mathbb{N}_0$$

be the connecting trajectories from $x_i(0)$ to $x_j(0)$. Denote by

$$n_{ij} = \sum_{m=1}^{k} n(\gamma_m).$$

The Floer bifurcation map $\Phi_{\mathbf{bif}}^{-\varepsilon,\varepsilon}(S, f, g)$ has the matrix representation

$$E(\gamma_{ij}) := (\delta_{mn} - n_{ij} \delta_{mj} \delta_{in})_{m,n=1}^{\ell}.$$

having ones on the diagonal and exactly one off diagonal non-zero entry n_{ij} in the (j, i) slot. Perturbing f to an element in $\mathscr{F}_{reg}^{mor}(N, f)$ we get the product of elementary matrices whose (j, i) slots sum up to n_{ij}.

Chapter 5

Proof of Theorem D

Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let

$$N := \bigcup_{\lambda \in [-\varepsilon,\varepsilon]} \{\lambda\} \times N_\lambda$$

an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_{\mathrm{bif}}(N, f) = \Lambda_0(N, f) = \{0\}$.
Hence there exists critical families $y, x : [-\varepsilon, \varepsilon] \to N$ with $y_\lambda, x_\lambda \in \mathrm{Crit}_k(S_\lambda, f_\lambda)$ and a unique index zero trajectory $\gamma : \mathbb{R} \to M$ connecting y_0 to x_0. Since $f \in \mathscr{F}^{mor}_{reg}(N)$ it follows

$$\mathcal{W}^u(y, f, N) \pitchfork \mathcal{W}^s(x, f, N) = \{0\} \times \gamma(\mathbb{R}).$$

We will deform the family of negative gradients $-\nabla f_\lambda$ parametrized by $\lambda \in [-\varepsilon, \varepsilon]$ to a preferred family of vector fields Y_λ parametrized by $\lambda \in [-\varepsilon, \varepsilon]$ fixing the endpoints $-\nabla f_{-\varepsilon} = X_{-\varepsilon}$ and $-\nabla f_\varepsilon = X_\varepsilon$. We will do that such that the transversality property is not violated. This means that during the deformation the parametrized unstable $\mathcal{W}^u(y, f, N)$ and the pramaetrized stable $\mathcal{W}^s(x, f, N)$ manifolds keep intersecting transversally in the connecting orbit γ and that no new connecting orbit is created. That's the assertion of the next

Lemma 5.1. *Let (S, f, g) be a continuation relating the Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ to $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let N be an isolating neighborhood of S such that $f \in \mathscr{F}^{mor}_{ref}(N)$ with $\Lambda_{bif}(N, f) = \Lambda_0(N, f) = \{0\}$. Then*

$$W^u(y, f, N) \pitchfork W^s(x, f, N)$$

holds if and only if the following assertions are true:

(i) *The space of bounded solutions of* $\nabla_t \zeta = -\nabla_\zeta f_0(\gamma)$ *is one-dimensional.*

(ii) *The variational equation*

$$\nabla_t \zeta = -\nabla_\zeta \nabla f_0(\gamma) - \partial_\lambda f_0(\gamma) \tag{5.2}$$

has no bounded solutions.

Proof. Pick a path $\lambda \mapsto (\lambda, p(\lambda)) \in \mathcal{W}^u(y,f,N)$ with $(0,p(0)) \in \mathcal{W}^u(y,f,N) \pitchfork \mathcal{W}^s(x,f,N)$. By definition $\varphi^t_\lambda\big(p(\lambda)\big) \longrightarrow y(\lambda)$ for $t \to -\infty$. A variation of the connectiong orbit γ in $\mathcal{W}^u(y,f,N)$ is given by

$$\gamma_\lambda(t) := \varphi^t_\lambda(p(\lambda)) \quad \text{and} \quad \xi_\lambda(t) := \partial_\lambda \gamma_\lambda(t).$$

It follows that $\lim_{t\to-\infty} \xi_\lambda(t) = \partial_\lambda y(\lambda)$. Now we differentiate $\xi_\lambda(t)$ and see what happens.

$$\begin{aligned} \nabla_t \xi_\lambda &= \nabla_t(\partial_\lambda \gamma_\lambda) \\ &= \nabla_{\xi_\lambda}(\partial_t \gamma_\lambda) \\ &= \nabla_{\xi_\lambda}(-f_\lambda(\gamma_\lambda)) \\ &= -\nabla_{\xi_\lambda} f_\lambda(\gamma_\lambda) - \partial_\lambda f_\lambda(\gamma_\lambda) \end{aligned}$$

After setting $\lambda = 0$ we have proved that $\xi(t) := \xi_0(t)$ satisfies (5.2) with $\lim_{t\to-\infty} \xi(t) = \partial_\lambda y_0$. By a similar procedure we may find a $\eta(t) := \eta_0(t)$ of γ satisfying (5.2) with $\lim_{t\to\infty} \eta(t) = \partial_\lambda x(0)$. By definition of transversality, at any intersection point $(0,p_0) = (0,p(0)) \in \mathcal{W}^u(y,f,N) \pitchfork \mathcal{W}^s(x,f,N)$ we have that

$$T_{(0,p_0)}\mathcal{W}^u(y,f,N) + T_{(0,p_0)}\mathcal{W}^s(x,f,N) \cong \mathbb{R} \oplus T_{p_0}M$$

The subspace $T_{p_0}W^u(y_0,f_0,N_0) \subset T_{(0,p_0)}\mathcal{W}^u(y,f,N)$ has codimension one. Hence $T_{(0,p_0)}\mathcal{W}^u(y,f,N)$ splits into the direct sum

$$T_{(0,p_0)}\mathcal{W}^u(y,f,N) \cong \mathbb{R}\cdot(\lambda,\xi_0) \oplus T_{p_0}W^u(y_0,f_0,N_0)$$

for any $(\lambda,\xi_0) \in T_{(0,p_0)}\mathcal{W}^u(y,f,N)\backslash T_{p_0}W^u(y_0,f_0,N_0)$. Analogously for any $(\lambda,\eta_0) \in T_{(0,p_0)}\mathcal{W}^s(x,f,N)\backslash T_{p_0}\mathcal{W}^s(x_0,f_0,N_0)$ we get a splitting of $T_{(0,p_0)}\mathcal{W}^s(x,f,N)$. With these splittings, transverality means that there exists an isomorphism

$$\big[\mathbb{R}\cdot(\lambda,\xi_0) \oplus T_{p_0}W^u(y_0,f_0,N_0)\big] + \big[\mathbb{R}\cdot(\lambda,\eta_0) \oplus T_{p_0}W^s(x_0,f_0,N_0)\big] \cong \mathbb{R} \oplus T_{p_0}M.$$

Transversality reduces to the assertion that for any element $(\lambda, \xi_0) \in T_{(0,p_0)}\mathcal{W}^u(y, f, N)\backslash T_{p_0}W^u(y_0, f_0, N_0)$ and any element $(\lambda, \eta_0) \in T_{(0,p_0)}\mathcal{W}^s(x, f, N)\backslash T_{p_0}W^s(x_0, f_0, N_0)$ there exists an isomorphism

$$\mathbb{R} \cdot (\lambda, \xi_0) + \mathbb{R} \cdot (\lambda, \eta_0) \cong \frac{\mathbb{R} \oplus T_{p_0}M}{T_{p_0}W^u(y_0, f_0, N_0) + T_{p_0}W^s(x_0, f_0, N_0)}.$$

$(\Rightarrow)$. Assume that $\mathcal{W}^u(y, f, N) \pitchfork \mathcal{W}^s(x, f, N)$. Assertion **(i)** follows from the fact that there exists an isomorphism between the space of bounded solutions of the equation $\nabla_t\zeta + \nabla_\zeta f_0(\gamma)\zeta = 0$ and the space

$$T_{\gamma(t)}W^u(y_0, f_0, N_0) \cap T_{\gamma(t)}W^s(x_0, f_0, N_0)$$

which by assumption one dimensional, hence **(i)** follows. We prove **(ii)** by contradiction: Assume that there is a $\zeta(t)$ solving (5.2) with boundary conditions $\lim_{t\to\infty} \zeta(t) = \partial_\lambda y(0)$ and $\lim_{t\to-\infty} \zeta(t) = \partial_\lambda x(0)$. Then there exists $(1, \zeta_0) = (1, \xi_0) = (1, \eta_0)$ with

$$T_{(0,p_0)}\mathcal{W}^u(y, f, N) \cong \mathbb{R} \cdot (1, \zeta_0) \oplus T_{p_0}W^u(y_0, f_0, N_0)$$

and

$$T_{(0,p_0)}\mathcal{W}^s(x, f, N) \cong \mathbb{R} \cdot (1, \xi_0) \oplus T_{p_0}W^s(x_0, f_0, N_0).$$

Since $\xi_0 = \eta_0$ it would follow that

$$\mathbb{R} \cdot (1, \xi_0) \cong \frac{\mathbb{R} \oplus T_{p_0}M}{T_{p_0}W^u(y_0, f_0, N_0) + T_{p_0}W^s(x_0, f_0, N_0)}$$

which is not possible by dimension reasons. This is a contradiction. Hence **(ii)** is proved.

$(\Leftarrow)$. By contradiction. Assume **(i)** and **(ii)** hold true and that the parametrized manifolds intersect non-transversally. Then it follows that

$$\mathbb{R} \cdot (\xi_0, 1) + \mathbb{R} \cdot (\eta_0, 1) \ncong \frac{T_pM \oplus \mathbb{R}}{(T_pW^u + T_pW^s)}.$$

This can only happen if $\xi_0 = \eta_0$. We extend it to $\zeta(t) = \xi(t) = \eta(t)$ that solves (5.2) with boundary conditions. This is a contradiction to **(ii)**. This proves the lemma. $\square$

5.1 The Model Case

Next we give a family of vectorfields in $\mathbb{R}^2$ having an index zero trajectory. This example is the motivation for this work. It appeared already in Conley's monograph [1] (p.51). Conley observed that the Conley index is sensitive with respect to the occurance of index zero trajectories. For giving the example in a general way specify tree functions.

Definition 5.3. We define a triple of smooth functions $v, w, h : \mathbb{R} \to \mathbb{R}$ such that

(i) the function v is constant outside an open interval $\subset [-\delta, 1-\delta] \subset [0,1]$ with $v(0) = 1$ and $v(1) = -1$. h is compactly supported, i.e. $\mathrm{supp}(h) \subset (0,1)$.

(ii) the function w has only two zeros at 0 and 1 with derivatives $w'(0) = -1$ and $w'(1) = 1$.

(iii) the function h is compactly supported, i.e. $\mathrm{supp}(h) \subset (0,1)$.

A triple of functions satisfying (i)-(iii) is called **function triple** and is denoted by (v, w, h).

We will see in Proposition 5.5 that any index zero trajectory admits a function triple (v, w, h). To construct the model case we consider the following function triple:

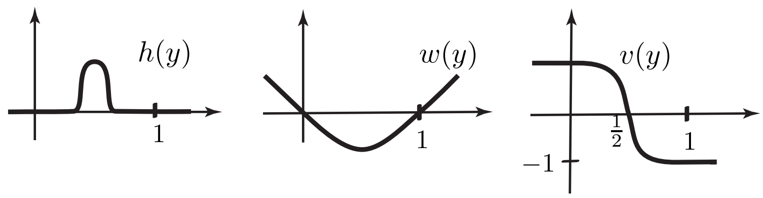


Figure 13. $h(y) \geqslant 0,\ w(y) < 0$ for $y \in (0,1)$ and $v^{-1}(0) = 1/2$.

A function triple (v, w, h) as in the figure is called **model triple.** Using the model triple (v, w, h) of the Figure 8 we will construct a family of vectorfields $\{X_\lambda\}_{\lambda \in [-\varepsilon, \varepsilon]}$ on $\mathbb{R}^2$ by

$$X_\lambda(x, y) := \begin{pmatrix} v(y) \cdot x + \lambda \cdot h(y) \\ w(y) \end{pmatrix} \quad \text{for } \lambda \in [-\varepsilon, \varepsilon].$$

Denote by $Q := [-1,1] \times [-1,2]$ the box which is an isolating neighborhood for all $\lambda \in [-\varepsilon, \varepsilon]$. Morover

$$N := [-\varepsilon, \varepsilon] \times Q$$

is an isolating neighborhood of the product flow $\Phi^t(\lambda, x) = (\lambda, \varphi_\lambda^t(x))$ where φ_λ^t is the local flow generated by X_λ. The phase portraits of this family of dynamical systems look like

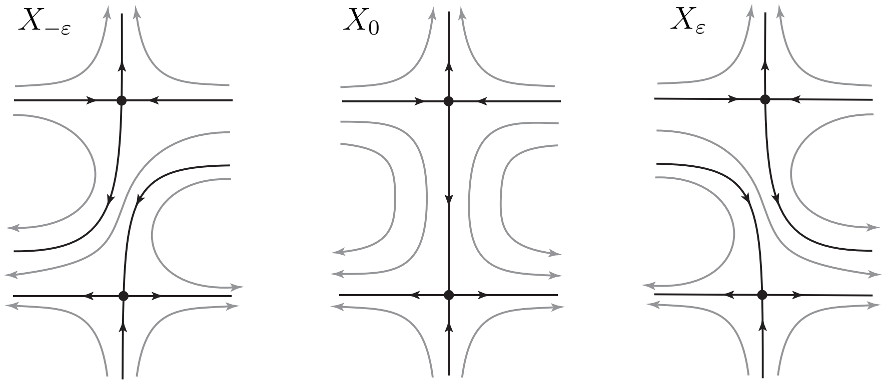


Figure 14. This is the phase portrait of the model case in $\mathbb{R}^2$. Near the critical points the family $\{X_\lambda\}$ does not depend on λ. The unstable manifold $W^u(y_\lambda, X_\lambda, Q)$ and the stable manifold $W^s(x_\lambda, X_\lambda, Q)$ cross from one side to the other. The way they cross is determined by the function h.

First of all we have for every parameter λ two critical points

$$y_\lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad x_\lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

These critical points are hyperbolic of course

$$dX_\lambda(y_\lambda) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad dX_\lambda(x_\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By starting with the vector field

$$X_0(x, y) = \begin{pmatrix} v(y)x \\ w(y) \end{pmatrix}$$

we see that it gives the phase portrait with a connecting orbit. Here the function w generates the connecting orbit. The function v forces the indices of the critical points to be the same. Last the function $\lambda \cdot h$ changes the vectorfield X_0 in the x-direction. This change happens only where h is supported hence in a horizontal strip fbetween the critical points in $\mathbb{R}^2$. Since for $\lambda = 0$ there is no perturbation for $0 < y < 1$ there exists a solution $\gamma : \mathbb{R} \to \mathbb{R}^2$ of $\dot{\gamma}(t) = X_0(\gamma(t))$ with $\gamma(0) = (0, y)$ connecting y_0 and x_0, i.e. $\lim_{t \to -\infty} \gamma(t) = y_0$ and $\lim_{t \to \infty} \gamma(t) = x_0$.

5.2 Orientations and Signs

Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let

$$N := \bigcup_{\lambda \in [-\varepsilon,\varepsilon]} \{\lambda\} \times N_\lambda$$

an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_{\text{bif}}(N, f) = \Lambda_0(N, f) = \{0\}$.
Denote by $y, x : [-\varepsilon, \varepsilon] \to N$ the critical points $y_\lambda, x_\lambda \in \text{Crit}_k(S_\lambda, f_\lambda)$ and by $\gamma : \mathbb{R} \to N_0$ the index zero trajectory connecting y_0 to x_0. We want to attach to the connecting orbit γ a sign $n(\gamma) \in \{-1, 1\}$. The unstable manifold of $x_\lambda \in \text{Crit}(S_\lambda, f_\lambda)$ relative to N_λ is denoted by

$$W^u_\lambda(x) = W^u(x_\lambda, f_\lambda, N_\lambda) := \left\{ p \in M \;\middle|\; \lim_{t \to -\infty} \varphi^t_\lambda(p) = x_\lambda,\ \varphi^{(-\infty,0]}_\lambda(p) \subset N_\lambda \right\}.$$

The parametrized unstable manifold of $x : [0, 1] \to M, x_\lambda \in \text{Crit}(S_\lambda, f_\lambda)$ is defined by

$$\mathcal{W}^u(x) = \mathcal{W}^u(x, f, N) := \{(\lambda, p) \mid p \in W^u_\lambda(x)\} \subset [0, 1] \times M.$$

Now fix orientations of $\mathcal{W}^u(y, f, N)$ and $\mathcal{W}^u(x, f, N)$. These induce orientations of $W^u_\lambda(y)$ and $W^u_\lambda(x)$. The (induced) orientations of $W^u_0(y)$ and $W^u_0(x)$ induce an orientation of the direct sum

$$T\gamma \oplus \frac{T_\gamma M}{T_\gamma W^u_0(y) + T_\gamma W^s_0(x)}$$

in the following way: First pick a vector $\xi \in T\gamma$ and extend it to a *positive* basis $\xi, v_2, \ldots, v_k$ of $T_\gamma W^u_0(y)$. By the canonical isomorphism

$$\frac{T_\gamma W^u_0(y)}{T_\gamma W^u_0(y) \cap T_\gamma W^s_0(x)} \oplus \frac{T_\gamma M}{T_\gamma W^u_0(x) + T_\gamma W^s_0(x)} \cong \underbrace{\frac{T\gamma M}{T_\gamma W^s_0(x)}}_{\text{oriented}}$$

and the fact that $v_2, \ldots, v_k$ descends to a basis of $T_\gamma W^u_0/(T_\gamma W^u_0(y) \cap T_\gamma W^s_0(x))$ we pick $\eta \in T_\gamma M/(T_\gamma W^u_0(y) + T_\gamma W^s_0(x))$ such that $v_2, \ldots, v_k, \eta$ gives a *positive* basis of $T\gamma M/T_\gamma W^s_0(x)$. Since we have the pair ξ, η we are ready for the next

Definition. The pair of vectors

$$\{\xi, \eta\} \in T\gamma \oplus \frac{T_\gamma M}{T_\gamma W^u_0(x) + T_\gamma W^s_0(x)}$$

defines an orientation of the direct sum and is called the **induced frame.**

There is an other way of orienting the direct sum $T\gamma \oplus T_\gamma M/(T_\gamma W_0^u(y) + T_\gamma W_0^s(x))$ coming from the dynamics of the negative gradients $\dot{x} = -\nabla f_\lambda(x)$. Namely

$$-\nabla f_0(\gamma) \in T\gamma$$

orients $T\gamma$. Observe the *variational equation*

$$\nabla_t \zeta = -\nabla_\zeta f_0(\gamma) - \partial_\lambda f_0(\gamma).$$

Now pick a solution $\xi(t)$ of the variational equation satisfying $\lim_{t\to-\infty} \xi(t) = \partial_\lambda y_0$. Pick another solution $\eta(t)$ of the variational equation satisfying $\lim_{t\to\infty} \eta(t) = \partial_\lambda x_0$. Then we have that $\xi(t) - \eta(t) \neq 0$ for all $t \in \mathbb{R}$ otherwise by the previous lemma the transversality condition would be violated. So we define

$$v(\gamma)(t) := \xi(t) - \eta(t)$$

which is a non-vanishing section of $T_\gamma M/(T_\gamma W_0^u(y) + T_\gamma W_0^s(x))$. This leads us to the next

Definition. The pair of vectors

$$\{-\nabla f_0(\gamma), v(\gamma)\} \in T\gamma \oplus \frac{T_\gamma M}{T_\gamma W_0^u(y) + T_\gamma W_0^s(x)}$$

defines an orientation of the direct sum and is called the *dynamical frame.*

Definition 5.4. Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let N an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}_{reg}^{mor}(N)$ and $\Lambda_{\text{bif}}(N, f) = \Lambda_0(N, f) = \{0\}$. We compare the dynamical frame to the induced frame and attach to the connecting orbit the *sign* $n(\gamma) = +1$ if these two orientations agree, $n(\gamma) = -1$ if they do not, i.e.

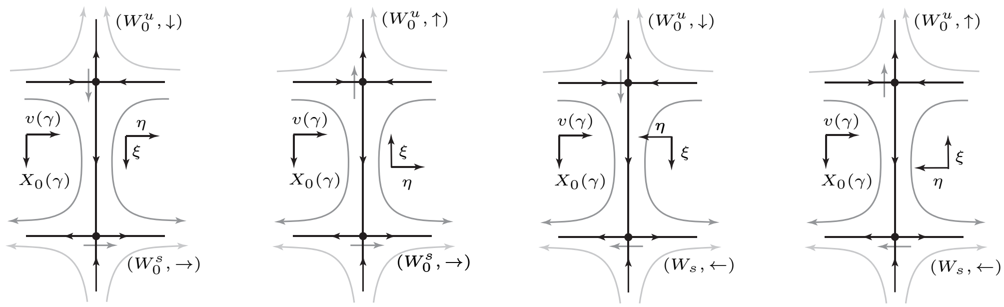
$$n(\gamma) := \begin{cases} +1 & \text{if } \{-\nabla f_0(\gamma), v(\gamma)\} \text{ is oriented as } \{\xi, \eta\} \\ -1 & \text{if } \{-\nabla f_0(\gamma), v(\gamma)\} \text{ is } \textit{not} \text{ oriented as } \{\xi, \eta\}. \end{cases}$$

Signs in the model case

The sign $n(\gamma)$ depends on the choice of the orientations of $\mathcal{W}^u(y)$ and $\mathcal{W}^s(x)$. As in the example above we will study the sign $n(\gamma)$ of the connecting orbit γ arising in the family of vectorfields X_λ on the plane $\mathbb{R}^2$ given by

$$X_\lambda(x, y) := \begin{pmatrix} v(y) \cdot x + \lambda \cdot h(y) \\ w(y) \end{pmatrix} \quad \text{for } \lambda \in [-\varepsilon, \varepsilon].$$

where (v, w, h) is a model triple as in Figure 8. In the next picture one sees which cases that may occur. The sing $n(\gamma)$ depends fully on how the orientations of $W^u_\lambda(y)$ and $W^u_\lambda(x)$ are induced. There are four different ways of orienting these manifolds as you can see in this picture.



$n(\gamma) = +1$ $\quad$ $n(\gamma) = -1$ $\quad$ $n(\gamma) = -1$ $\quad$ $n(\gamma) = +1$

Figure 15. The dynamical frame $\{X_0(\gamma), v(\gamma)\}$ does not change. The reason that $v(\gamma)$ points to the right is that the unstable manifold $W^u_\lambda(y)$ goes from left to right while the stable manifold $W^s_\lambda(x)$ goes from right to left. The vector $v(\gamma)$ tells you in which way these manifolds cross. The induced frame $\{\xi, \eta\}$ depends on the orientations of $\mathcal{W}^u(y)$ and $\mathcal{W}^s(x)$ as one sees.

5.3 Perturbation to a normal family

Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let N an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_{\mathrm{bif}}(N, f) = \Lambda_0(N, f) = \{0\}$.
Hence there exists critical families $y, x : [-\varepsilon, \varepsilon] \to N$ with $y_\lambda, x_\lambda \in \mathrm{Crit}_k(S_\lambda, f_\lambda)$ and a unique index zero trajectory $\gamma : \mathbb{R} \to M$ connecting y_0 to x_0. For any parameter $\lambda \in [-\varepsilon, \varepsilon]$ we denote the unstable manifold of φ^t_λ in N_λ by

$$W^u_\lambda(y) = W^u(y_\lambda, f_\lambda, N_\lambda) := \left\{ p \in N_\lambda \;\middle|\; \lim_{t \to -\infty} \varphi^t_\lambda(p) = y_\lambda,\ \varphi^{(-\infty,0]}_\lambda(p) \subset N_\lambda \right\}.$$

And the parametrized unstable manifold of $y : [-\varepsilon, \varepsilon] \to N, y_\lambda = y(\lambda) \in \mathrm{Crit}_k(S_\lambda, f_\lambda)$ for all $\lambda \in [-\varepsilon, \varepsilon]$ by

$$\mathcal{W}^u(y) = \mathcal{W}^u(y, f, N) := \left\{ (\lambda, p) \;\middle|\; p \in W^u_\lambda(y) = W^u(y_\lambda, f_\lambda, N_\lambda) \right\}.$$

Now we perturb any family $\{-\nabla f_\lambda\}_{\lambda \in [-\varepsilon, \varepsilon]}$ to a nearby family of vectorfields $\{X_\lambda\}_{\lambda \in [-\varepsilon, \varepsilon]}$ fixing the endpoints, i.e. $-\nabla f_{-\varepsilon} = X_{-\varepsilon}$ and $-\nabla f_\varepsilon = X_\varepsilon$. We want that in certain coordinates the family $\{X_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$ can be presented in a normal form. The sign of the connecting orbit $n(\gamma)$ is the same for the family $\{-\nabla f_\lambda\}$ as for $\{X_\lambda\}$. This is the assertion of the next propostition. Let (v, w, h) be a function triple as in Definition 5.3. We denote by D^m the the closed unit ball in $\mathbb{R}^m$ and by $Q := [-1, 1] \times [-1, 2]$ a two dimensional box.

Proposition 5.5 (Normal Form). *Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let N an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_{bif}(N, f) = \Lambda_0(N, f) = \{0\}$.*
Let $\Gamma := \gamma(\mathbb{R}) \cup \{y_0, x_0\}$ be the closure of the connecting orbit γ. Then for every neighborhood U of Γ there exists a family of compact neighborhoods $K_\lambda \subset U$ of Γ, and a family of diffeomorphisms

$$\phi_\lambda : Q \times D^{k-1} \times D^{n-k-1} \longrightarrow K_\lambda$$

and a family of vectorfields $\{X_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon}$ on M satisfying the following conditions

(i) *For all parameters $\lambda \in [-\varepsilon, \varepsilon]$ we have*

$$\begin{aligned} \phi_\lambda(0,1,0,0) &= y_\lambda \\ \phi_\lambda(0,0,0,0) &= x_\lambda \\ \phi_0(0,[0,1],0,0) &= \Gamma. \end{aligned}$$

(ii) *For all $\lambda \in [-\varepsilon, \varepsilon], x \in [-1, 1], y \in [-1, 2], \xi \in D^{k-1}$ and $\eta \in D^{n-k-1}$ we have the normal form*

$$\phi_\lambda^* X_\lambda(x, y, \xi, \eta) = \begin{pmatrix} v(y)x + \lambda h(y) \\ w(y) \\ \xi \\ -\eta \end{pmatrix}.$$

(iii) *The family $\{-\nabla f_\lambda\}$ agrees with $\{X_\lambda\}$ outside U, and the vectorfield $-\nabla f_\lambda$ is Morse-Smale equivalent to $\{X_\lambda\}$ for every parameter $\lambda \in [-\varepsilon, \varepsilon]\backslash\{0\}$. Moreover the end points are fixed, i.e. $-\nabla f_{-\varepsilon} = X_{-\varepsilon}$ and $-\nabla f_\varepsilon = X_\varepsilon$.*

(iv) *The family $\{X_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$ also has a unique index zero connecting orbit γ at $\lambda = 0$. And this orbit is the transversal intersection of the parametrized unstable and stable manifolds. The sign $n(\gamma)$ of the connecting orbit in the deformed family $\{X_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$ is the same as the sign of the connecting orbit in the initial family $\{-\nabla f_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$.*

Proof. We will perform this perturbation around the connecting orbit in seven steps. We will perform three different types of deformations. The first type is performed in Step 1 and Step 2. The second type of perturbation is performed in Step 6. The third and last is performed in the last Step 7.

Step 1. There exists a family of charts $\Phi_{\beta,\lambda} : U_\beta \to \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k}$ around y_λ with

(i) $\Phi_{\beta,\lambda}(y_\lambda) = (1, 0, 0)$

(ii) $\Phi_{\beta,\lambda}(W^u_\lambda(y) \cap U_\beta) = \mathbb{R} \times \mathbb{R}^{k-1} \times \{0\} \cap \Phi_{\beta,\lambda}(U_\beta)$

(iii) $\Phi_{\beta,\lambda}(W^s_\lambda(y) \cap U_\beta) = \{1\} \times \mathbb{R}^{n-k} \cap \Phi_{\beta,\lambda}(U_\beta)$

for every $\lambda \in [-\varepsilon, \varepsilon]$. Moreover there exists an admissible perturbation X'_λ of X_λ on M such that

$$\tilde{X}'_\lambda := (\Psi_{\beta,\lambda})_* X'_\lambda = \begin{pmatrix} y-1 \\ \xi \\ -\eta \end{pmatrix}$$

near $(1, 0, 0) \in \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1}$. Here $y \in \mathbb{R}, \xi \in \mathbb{R}^{k-1}$ and $\eta \in \mathbb{R}^{n-k}$.

Proof of Step 1. The idea is to put the unstable and stable manifolds into coordinate spaces and then to perturb only the flow on these unstable and stable manifolds and *not* the manifolds themselves. So first of all pick a chart domain (U_β, Φ_β) with $\Phi_\beta(y_0) = 0$ and such that y_λ stays for all parameters λ in U_β. This can be achieved by picking $\varepsilon > 0$ small enough. By compactness of the interval $[-\varepsilon, \varepsilon]$ there exists a compact set $N_\beta \subset U_\beta$ with $y_\lambda \in N_\beta$ for all $\lambda \in [-\varepsilon, \varepsilon]$. We define a smooth family of vectors $v(\lambda) \in \mathbb{R}^n$ by $\Phi_\beta(y_\lambda) + v(\lambda) = 0$. We multiply with a smooth cut-off function $\beta : M \to [0, 1]$ defined by $\beta_{|N_\beta} \equiv 1$ and $\beta_{|U^c_\beta} \equiv 0$ and solve the equation

$$\frac{d}{dt} x(t) = \tilde{\beta}(x(t)) \cdot v(\lambda)$$

where $\tilde{\beta} := \beta \circ \Phi_\beta^{-1}$. This gives a flow $\varphi^t_\lambda(x)$ such that the time one map satisfies $\varphi^1_\lambda(y_\lambda) = y_\lambda + 1 \cdot v_\lambda = 0$. We put $\Phi_{\beta,\lambda} := \varphi^1_\lambda \circ \Phi_\beta : U_\beta \to \mathbb{R}^n$. Since $\Phi_{\beta,\lambda}(y_\lambda)$ is hyperbolic in $\Phi_{\beta,\lambda}(U_\beta) \subset \mathbb{R}^n$, there exists for every parameter λ there exists a splitting $E^u_\lambda \oplus E^s_\lambda \cong \mathbb{R}^n$. By the theorem of Hadamard-Perron there exsists for any parameter λ smooth maps h^+_λ and h^-_λ such that

$$\begin{array}{rcl} h^+_\lambda : E^s_\lambda & \to & E^u_\lambda \\ h^-_\lambda : E^u_\lambda & \to & E^s_\lambda \end{array}$$

whose graphs $(\mu, h^+_\lambda(\mu)) \in W^s(y_\lambda)$ and $(h^-_\lambda(\zeta), \zeta) \in W^u(y_\lambda)$ locally parametrize the unstable and stable manifolds. Now we define the following family of maps

$$\begin{array}{rcl}\Phi_{\beta,\lambda} : E^u_\lambda \oplus E^s_\lambda \cong \mathbb{R}^n & \longrightarrow & \mathbb{R}^k \times \mathbb{R}^{n-k} \\ (\mu, \zeta) & \longmapsto & \begin{pmatrix} \mu - h^-_\lambda(\zeta) \\ \zeta - h^+_\lambda(\mu) \end{pmatrix}.\end{array}$$

So he have got a family of coordinate charts $\Phi_{\beta,\lambda} : U_\beta \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ that for every parameter $\lambda \in [-\varepsilon, \varepsilon]$ satisfies

(i) $\Phi_{\beta,\lambda}(y_\lambda) = 0$

(ii) $\Phi_{\beta,\lambda}(W^u_\lambda(y) \cap U_\beta) = \mathbb{R}^k \times \{0\} \cap \Phi_{\beta,\lambda}(U_\beta)$

(iii) $\Phi_{\beta,\lambda}(W^s_\lambda(y) \cap U_\beta) = \{0\} \times \mathbb{R}^{n-k} \cap \Phi_{\beta,\lambda}(U_\beta)$.

Next we will perturb the vector field $-\nabla f_\lambda$ in U_β without perturbing the stable and unstable manifolds. To do so we pick a smooth cut-off function $\beta : M \to [0,1]$ defined by $\beta_{|N_\beta} \equiv 1$ and $\beta_{|U^c_\beta} \equiv 0$. In order to keep the end points fixed we pick an other cut-off function $a : [-\varepsilon, \varepsilon] \to [0,1]$ with $a(\pm\varepsilon) = 0$ and $a \equiv 1$ on $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. Multiply the cut-off functions to get $\beta_\lambda := a(\lambda) \cdot \beta : M \to [0,1]$. Define the one parameter family

$$X'_\lambda := \beta_\lambda \cdot (\Phi_{\beta,\lambda})^* \begin{pmatrix} \xi \\ -\eta \end{pmatrix} + (1 - \beta_\lambda) \cdot (-\nabla f_\lambda).$$

It follows that $-\nabla f_{\pm\varepsilon} = X'_{\pm\varepsilon}$. The perturbed vectorfield X'_λ represented in the charts $\Phi_{\beta,\lambda}$ is denoted by $\tilde{X}'_\lambda := (\Phi_{\beta,\lambda})_* X'_\lambda$ and on $\tilde{N}_{\beta,\lambda} := \Phi_{\beta,\lambda}(N_\beta)$ it a linear vector field. We pick a point $p \in \Gamma \backslash U_\beta$ and flow back with $\varphi^{-t}_{X'_0}(p)$ till we hit the boundary ∂N_β in $y' = \varphi^{-t}_{X'_0}(p) \cap \partial N_\beta$. In the coordinate chart $(U_\beta, \Phi_{\beta,0})$ this point is denoted by $\Phi_{\beta,0}(y') = (\xi', 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Now since in $\tilde{N}_{\beta,0}$ the vector field is linear we se that the negative flow in the chart is given by

$$(e^{-t} \cdot \xi', 0).$$

We want to split the one dimensional vectorspace spanned by ξ' off $\mathbb{R}^k$. To do so we choose the linear transformation $A : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k}$ that is defined by sending $-\xi'$ to $(1,0) \in \mathbb{R} \times \mathbb{R}^{k-1}$. Composing the family $\Phi_{\beta,\lambda}$ with A we get that

$$\Phi_{\beta,\lambda} : U_\beta \to \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k}$$

where we wrote again $\Phi_{\beta,\lambda}$ for $A \circ \Phi_{\beta,\lambda}$. After translating with the vector $(1,0,0) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k}$ we get a family $\Phi_{\beta,\lambda}$ and a small enough perturbation X'_λ such that the assertions of Step 1 are fullfilled. This proves Step 1.

Step 2. There exists a family of charts $\Phi_{\alpha,\lambda} : U_\alpha \to \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k-1}$ around x_λ with

(i) $\Phi_{\alpha,\lambda}(x_\lambda) = 0$

(ii) $\Phi_{\alpha,\lambda}(W^s_\lambda(x) \cap U_\alpha) = \mathbb{R} \times \{0\} \times \mathbb{R}^{n-k-1} \cap \Phi_{\alpha,\lambda}(U_\alpha)$

(iii) $\Phi_{\alpha,\lambda}(W^u_\lambda(x) \cap U_\alpha) = \{0\} \times \mathbb{R}^k \cap \Phi_{\alpha,\lambda}(U_\alpha)$

for every $\lambda \in [-\varepsilon, \varepsilon]$. Moreover there exists an admissible perturbation X''_λ of X'_λ on M such that

$$\tilde{X}''_\lambda := (\Phi_{\alpha,\lambda})_* X''_\lambda = \begin{pmatrix} -y \\ \zeta \\ -\mu \end{pmatrix}$$

near $(0,0,0) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k-1}$.

Proof of Step 2. The proof is exactly the same as in in Step 1. Just replace the index β by α and the critical point y_λ by x_λ. This proves Step 2.

Step 3. There is a non-compact one dimensional manifold $L \subset M$ containing Γ in its interior such that $L \backslash \{y_0, x_0\}$ consisting of three orbits of the flow of X''_λ. Thus the manifold L is represented in the coordinates $\Psi_{\beta,0}$ and $\Psi_{\alpha,0}$ of Steps 1 and 2 in the y-axis. Moreover one may choose a diffeomorphism $l : \mathbb{R} \to L$ such that $l(1) = y_0$ and $l(0) = x_0$ and the pulled back vector field

$$w(y) := l^* X''_0(y)$$

satisfies

$$w(y) = -y \quad \text{near } 0 \quad \text{and} \quad w(y) = y - 1 \quad \text{near } 1.$$

Proof of Step 3. Denote by π_y the projection on the y-axis. Then we define $(\Phi_{\beta,0} \circ \pi_y)^{-1} : \mathbb{R} \to M$ and $(\Phi_{\alpha,0} \circ \pi_y)^{-1} : \mathbb{R} \to M$. These maps are embeddings around 1 and 0. Moreover we define another map $(\delta, 1-\delta) \to M$ defined by

$$t \mapsto \varphi_{X''_0}^{\tan(\pi \cdot t - \frac{\pi}{2})}(p).$$

Here $p \in \gamma(\mathbb{R})$ and $\delta > 0$ e so small that it intersects the neighborhoods around 1 and 0 where the other two maps are defined. Not that these three embeddings together give an emmbeding

$$l : (-\delta, 1+\delta) \to M$$

with the desired conditions of Step 1. This embedding can be reparametrized to

$$l : \mathbb{R} \to M.$$

Here $L := l(\mathbb{R})$. This proves Step 3.

Step 4. The restriction of $T_L M$ of the tangent bundle TM to the embedded curve L admits a smooth direct sum decompostition

$$T_L M \cong H \oplus TL \oplus E^u \oplus E^s$$

which is invariant under the flow φ_0^t, i.e.

$$d\varphi_0^t(y)E_y^u = E^u_{\varphi_0^t(y)}, \; d\varphi_0^t(y)E_y^s = E^s_{\varphi_0^t(y)} \quad \text{and} \quad d\varphi_0^t(y)H_y = H_{\varphi_0^t(y)}$$

for all $y \in L$. Note that the invariance of TL is by construction of L established. Moreover we have

$$T_y W_0^u(y) \cong T_y L \oplus E_y^u \quad \text{and} \quad T_y W_0^s(x) \cong T_y L \oplus E_y^s$$

Proof of Step 4. Perturb the family of Riemannian metrics $\{g_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$ such that in the charts $\Phi_{\beta,\lambda}$ and $\Phi_{\alpha,\lambda}$ it agrees with the standard euclidean metric. By assumption the parametrized unstable and the parametrized stable manifolds intersect transversally, i.e. $\mathcal{W}^u(y) \pitchfork \mathcal{W}^s(x)$. We will work with the family of charts of $\Phi_{\alpha,\lambda}$ defined in Step 2. Define the extended chart

$$\Phi_\alpha : U_\alpha \times [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k-1} \times [-\varepsilon, \varepsilon]$$

by $\Phi_\alpha(p,\lambda) := (\Phi_{\alpha,\lambda}(p), \lambda)$. We perturb the family of metrics $\{g_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$ such that in the chart $\Phi_{\alpha,\lambda}$ it is standard euclidean . Using orthogonal decompostition with respect to the metric g_0 define $\widetilde{E}_p^u := (T_pL)^\perp \cap T_{(p,0)}\mathcal{W}^u(y)$ and $\widetilde{E}_p^s := (T_pL)^\perp \cap T_{(p,0)}\mathcal{W}^s(x)$. The subspace $\widetilde{E}_p^s \subset T_pM \times \mathbb{R}$ represented in the chart Φ_α is

$$\widetilde{E}_y^s = \{y\} \times \{0\} \times \mathbb{R}^{n-k-1} \times \mathbb{R}.$$

Here $y \in (\Phi_{\alpha,0}(L \cap U_\alpha), 0)$. By definition we have that $W_0^s = \mathcal{W}^s(x) \cap (M \times \{0\})$. It follows that in the chart Φ_α the space $E_y^s := T_y W_0^s$ has the form

$$E_y^s = \{y\} \times \{0\} \times \mathbb{R}^{n-k-1} \times \{0\}.$$

Claim 1. There is a linear map

$$(A_0, l_0) : \mathbb{R}^k \to \mathbb{R}^{n-k-1} \times \mathbb{R}$$

such that

$$\widetilde{E}^u_{y_0} = \{y_0\} \times \operatorname{graph}(A_0, l_0).$$

Proof of Claim 1. Note that transversality implies $\tilde{E}^u_p \oplus \tilde{E}^s_p \cong (T_pL)^\perp \cap M \oplus \mathbb{R}$. In charts this is equivalent to the map

$$\widetilde{E}^u_y \to \{y\} \times \mathbb{R}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R} \xrightarrow{\pi} \mathbb{R}^k$$

being surjective (and hence isomorphic). We can use this isomorphism to construct the map

$$(A_0, l_0) : \mathbb{R}^k \to \widetilde{E}^u_y \xrightarrow{\pi} \widetilde{E}^s_y = \{y\} \times \{0\} \times \mathbb{R}^{n-k-1} \times \mathbb{R}.$$

This proves Claim 1.
We have shown that

$$\widetilde{E}^u_y = \big\{(y, \zeta, \mu, \lambda) \mid \mu = A_0(\zeta),\ \lambda = l_0(\zeta)\big\}.$$

It follows by definition

$$E^u_y = \big\{(y, \zeta, \mu, 0) \mid \mu = A_0(\zeta),\ 0 = l_0(\zeta)\big\}.$$

Here $(y, \zeta, \mu, 0) \in \{y\} \times \mathbb{R}^k \times \mathbb{R}^{n-k-1} \times \{0\}$.
Claim 2. For any $\tilde{y} \in \Phi_{\alpha,0}(L \cap U_\alpha)$ we have that

$$E^u_{\tilde{y}} = \big\{(\tilde{y}, \zeta, \mu, 0) \mid \mu = \tilde{y}^2 \Lambda_0(\zeta),\ \zeta \in \ker l_0\big\}.$$

Proof of Claim 2. Let $\tilde{y} \in \Phi_{\alpha,0}(L \cap U_\alpha)$ then there exists a time $t \in \mathbb{R}$ such that $\varphi^t_0(y, 0, 0) = (\tilde{y}, 0, 0)$. Since in our chart the vector field is linear we have that

$$\varphi^t(y, \zeta, \mu, 0) = (e^{-t}y, e^t\zeta, e^{-t}\eta, 0)$$

By invariance we have

$$\begin{aligned}
E^u_{\tilde{y}} &= \varphi^t(E^u_y) \\
&= \big\{(e^{-t}y, e^t\zeta, e^{-t}\mu, 0) \mid \mu = A_0(\zeta)\big\} \\
&= \big\{(u, v, w, 0) \mid w = e^{-2t} A_0(v)\big\} \\
&= \big\{(\tilde{y}, \zeta, \mu, 0) \mid \mu = e^{-2t} A_0(v)\big\} \\
&= \big\{(\tilde{y}, \zeta, \mu, 0) \mid \mu = \tilde{y}^2 \tfrac{1}{y_0^2} A_0(\zeta)\big\} \\
&= \big\{(\tilde{y}, \zeta, \mu, 0) \mid \mu = \tilde{y}^2 \Lambda_0(\zeta)\big\}.
\end{aligned}$$

Note that ζ and v are in $\ker l_0$. This proves Claim 2.
We have constructed a subbundle E^u of $T_L M$. By using transversality and invariance we can construct a subbundle E^s of $T_L M$ in the same way by working in the chart Φ_β. So far we have a direct sum $TL \oplus E^u \oplus E^s$, which is a subbundle of $T_L M$. Define $A_y : \mathbb{R} \times \mathbb{R}^{k-1} \to \mathbb{R}^k$ with $A_y(\mathbb{R}) = \ker l_0$ and $A_y(\mathbb{R}^{k-1}) = E^u_y$. By invariance we can extend this map to $A_{\tilde{y}} : \mathbb{R} \times \mathbb{R}^{k-1} \to \mathbb{R}^k$. So we get a chart $\Phi_{\alpha,0} : U_\alpha \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1}$ by defining a map

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1} \xrightarrow{A} \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{m-k-1}$$

through $(x, \tilde{y}, \zeta, \mu) \longmapsto (\tilde{y}, A_{\tilde{y}}(x, \zeta), \mu)$. Now define

$$\Phi_{\alpha,\lambda} := A^{-1} \circ \Phi_{\alpha,\lambda} : U_\alpha \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1}.$$

Similarly we can construct a family of charts

$$\Phi_{\beta,\lambda} : U_\beta \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1}.$$

Finally define $H := \big(T_L \oplus E^u \oplus E^s\big)^\perp$, which in the chart $(U_\alpha, \Phi_{\alpha,0})$ is given by

$$H_y := (d\Phi_{\beta,0})_y^{-1}(\mathbb{R} \times \{y\} \times \{0\} \times \{0\}).$$

After extending it with the flow φ_0^t we have an orthogonal splitting

$$T_L = H \oplus TL \oplus E^u \oplus E^s.$$

This proves Step 4.

Step 5. There exists a diffeomorphism

$$\Phi_0 : Q \times D^{k-1} \times D^{n-k-1} \longrightarrow N_0 \subset U \subset M$$

such that

$$\tilde{X}_0'' := \Phi_0^* X_0''(x, y, \xi, \eta) = \begin{pmatrix} v(y)x \\ w(y) \\ \hat{a}(y)\xi \\ \hat{b}(y)\eta \end{pmatrix} + \mathcal{O}_2(x, \xi, \eta)$$

Here $Q := [-1,1] \times [-1,2]$ is a two dimensional closed box. The compact subset N_0 is a neighborhood of Γ in M. Moreover $v(y) \in \mathbb{R}, \hat{a}(y) \in \mathbb{R}^{(k-1)\times(k-1)}$ and $\hat{b}(y) \in \mathbb{R}^{(n-k-1)\times(n-k-1)}$ with

$$\hat{a}(y) = \mathbb{1} \quad \text{and} \quad \hat{b}(y) = -\mathbb{1}$$

near $y = 0$ and $y = 1$. Moreover $v(y) = 1$ near $y = 0$ and $v(y) = -1$ near $y = 1$.

Proof of Step 5. We have choosen the Riemannian metrics $\{g_\lambda\}$ on M to agree with the standard metric in the charts $\Phi_{\beta,\lambda}$ and $\Phi_{\alpha,\lambda}$ near y_0 and x_0. The coordinate systems $(U_\beta, \Phi_{\beta,0})$ and $(U_\alpha, \Phi_{\alpha,0})$ determine trivializations of

$$H \oplus E^u \oplus E^s \xrightarrow{\pi} L$$

near y_0 and x_0. We extend to a vectorbundle trivialisation τ that makes the following diagram commute

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1} & \xrightarrow{\quad\tau\quad} & H \oplus E^u \oplus E^s \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow{\quad l \quad} & L \end{array}$$

In other words we have chosen the trivialisation τ to cover the diffeomorphism $l : \mathbb{R} \to L$. Note that π_2 is the projection to the second factor and when extending the trivialisations to τ it may be necessary reverse the sign of one component of ξ or η to match the orientations. Now we define the map

$$\Phi_0 : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1} \xrightarrow{\tau} H \oplus E^u \oplus E^s \xrightarrow{\exp} M$$

by $\Phi_0(x, y, \xi, \eta) := \exp_{l(y)} \big(\tau(x, 0, \xi, \eta)\big)$. Next we want to mention that the families of charts can be extended $\Phi_{\beta,\lambda}$ and $\Phi_{\alpha,\lambda}$ to the family of trivializations

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1} & \xrightarrow{\quad\tau_\lambda\quad} & H \oplus E^u \oplus E^s \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow{\quad l \quad} & L \end{array}$$

Here $\tau_0 = \tau$. These are all trivializations covering l. Hence we may define

$$\Phi_\lambda : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1} \xrightarrow{\tau_\lambda} H \oplus E^u \oplus E^s \xrightarrow{\exp} M$$

by $\Phi_\lambda(x, y, \xi, \eta) := \exp_{l(y)} \big(\tau_\lambda(x, \xi, \eta)\big)$. Hence there is a family of charts

$$\Phi_\lambda : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^{n-k-1} \longrightarrow M.$$

The image of Φ_0 is a tubular neighborhood of L giving coordinates (x, y, ξ, η) on a neighborhood of Γ. We use the letters $\tilde{\varphi}_0$ and $\tilde{X}_0''$ for the flow and vector field in these coordinates. Therefore in these coordinates we have $\tilde{y}_0 = (0, 1, 0, 0)$ and $\tilde{x}_0 = (0, 0, 0, 0)$. And $\tilde{\Gamma}$ is the set of points $(0, y, 0, 0)$ where $0 \leqslant y \leqslant 1$. Since $\tilde{L} = \{0\} \times \mathbb{R} \times \{0\} \times \{0\}$ is invariant under $\tilde{\varphi}_0$ the restriction has the form

$$\tilde{\varphi}_0^t(0, y, 0, 0) = (0, \tilde{\phi}_0^t(y), 0, 0).$$

By invariance of the splitting, the differential $d\tilde{\varphi}_0^t(0, y, 0, 0)$ has the form

$$d\tilde{\varphi}_0^t(0, y, 0, 0) = v^t(y) \oplus w^t(y) \oplus a^t(y) \oplus b^t(y).$$

Here $w^t(y) > 0, v^t(y) \in \mathbb{R}, a^t(y) \in GL_{k-1}(\mathbb{R})$ and $b^t(y) \in GL_{n-k-1}(\mathbb{R})$. By differentiating the two last identities we get

$$\tilde{X}_0''(0, y, 0, 0) = (0, w(y), 0, 0)$$

and

$$d\tilde{X}_0''(0, y, 0, 0) = v(y) \oplus w'(y) \oplus \hat{a}(y) \oplus \hat{b}(y).$$

Here

$$v(y) := \frac{d}{dt} v^t(y)\big|_{t=0} \quad \hat{a}(y) := \frac{d}{dt} a^t(y)\big|_{t=0} \quad \hat{b}(y) := \frac{d}{dt} b^t(y)\big|_{t=0}$$

and $w'(y)$ is just the derivative of w at y. By construction the vector field X_0'' represented in the trivializations $\Phi_{\beta,0}$ and $\Phi_{\alpha,0}$ implies that $v(y), \hat{a}(y)$ and $b(y)$ verify the conditions near $y = 0$ and $y = 1$ mentioned in the assertion of Step 5. Now use Taylor's formula in (x, ξ, η) to obtain

$$\begin{aligned}
\tilde{X}_0''(x, y, \xi, \eta) &= \tilde{X}_0''(0, y, 0, 0) + d\tilde{X}_0''(0, y, 0, 0) \cdot \begin{pmatrix} x \\ 0 \\ \xi \\ \eta \end{pmatrix} + \mathcal{O}_2(x, \xi, \eta) \\
&= \begin{pmatrix} 0 \\ w(y) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v(y)x \\ w'(y) \cdot 0 \\ \hat{a}(y)\xi \\ \hat{b}(y)\eta \end{pmatrix} + \mathcal{O}_2(x, \xi, \eta) \\
&= \begin{pmatrix} v(y)x \\ w(y) \\ \hat{a}(y)\xi \\ \hat{b}(y)\eta \end{pmatrix} + \mathcal{O}_2(x, \xi, \eta)
\end{aligned}$$

Rescale (x, ξ, η) so that the coordinates are defined for $x \in [-1, 1], y \in [-1, 2], \xi \in D^{k-1}$ and $\eta \in D^{n-k-1}$. This proves Step 5.

Step 6. There exists an admissible perturbation X_λ''' of the family X_λ'' and coordinates

$$\Phi_0 : Q \times D^{k-1} \times D^{n-k-1} \to N_0 \subset U \subset M$$

such that

$$(\Phi_0)^* X_0'''(x,y,\xi,\eta) = \begin{pmatrix} v(y)x \\ w(y) \\ \xi \\ -\eta \end{pmatrix}.$$

Proof of Step 6. First we perform the admissible perturbation. Define first the cut-off function $\rho : Q \times D^{k-1} \times D^{n-k-1} \to [0,1]$ which is identically one near $\tilde{\Gamma}$ and zero near the boundary of $Q \times D^{k-1} \times D^{n-k-1}$. Moreover we want $\rho^{-1}((0,1])$ to be contained in a sufficiently small neighborhood of the boundary of $Q \times D^{k-1} \times D^{n-k-1}$. We need these restrictions in order the perturbation to be admissible. In order to keep the end points fixed we pick an other cut-off function $a : [-\varepsilon, \varepsilon] \to [0,1]$ with $a(\pm\varepsilon) = 0$ and $a \equiv 1$ on $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. Define $\rho_\lambda := a(\lambda)\rho$. We perturb $\tilde{X}_\lambda''$ to

$$\tilde{X}_\lambda'''(x,y,\xi,\eta) := \tilde{X}_\lambda''(0,y,0,0) + d\tilde{X}_\lambda''(0,y,0,0) \cdot \begin{pmatrix} x \\ 0 \\ \xi \\ \eta \end{pmatrix} + (1-\rho_\lambda) \cdot \mathcal{O}_2(x,\xi,\eta,\lambda).$$

In a neighborhood of $\tilde{\Gamma}$ have that

$$\tilde{X}_0'''(x,y,\xi,\eta) = \begin{pmatrix} v(y)x \\ w(y) \\ \hat{a}(y)x \\ \hat{b}(y)\eta \end{pmatrix}.$$

One may extend $\tilde{X}'''$ to M by extending ρ_λ to M and cutting off X_λ''. We want to transform the term $\hat{a}(y)$ and $\hat{b}(y)$ away. We will make an ansatz

$$g : (x,y,\xi,\eta) \longmapsto \big(x, y, A(y)\xi, B(y)\eta\big).$$

We want the following equation to hold

$$dg_{(x,y,\xi,\eta)} \tilde{X}_0''(x,y,\xi,\eta) = \tilde{X}_0'''(g(x,y,\xi,\eta)).$$

Computed to the end this gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \partial_y A(y)\xi & A(y) & 0 \\ 0 & \partial_y B(y)\eta & 0 & B(y) \end{pmatrix} \cdot \begin{pmatrix} v(y)x \\ w(y) \\ \hat{a}(y)\xi \\ \hat{b}(y)\eta \end{pmatrix} = \begin{pmatrix} v(y)x \\ w(y) \\ A(y)\xi \\ -B(y)\eta \end{pmatrix}$$

For this equality to hold, we get two necessary conditions of $A(y)$ and $B(y)$ namely

$$w(y)\partial_y A(y) = A(y)\big(\mathbb{1} - \hat{a}(y)\big) \quad \text{with } A(0) = \mathbb{1}$$

and

$$w(y)\partial_y B(y) = -B(y)\big(\mathbb{1} - \hat{b}(y)\big) \quad \text{and } B(0) = \mathbb{1}.$$

Now put $\Phi_\lambda := \Phi_\lambda \circ g^{-1}$. The chart Φ_λ pulls the vectorfield X_0''' on M back to our desired vectorfield $\tilde{X}_0'''$ on $Q \times D^{k-1} \times D^{n-k-1}$. This proves Step 6.

Step 7. Define $\tilde{X}_\lambda''' := (\Phi_\lambda)^* X_\lambda'''$. There exists another admissible perturbation X_λ'''' of X_λ''' such that

$$\tilde{X}_\lambda''''(x,y,\xi,\eta) = \begin{pmatrix} v(y)x \\ w(y) \\ \xi \\ -\eta \end{pmatrix} + \begin{pmatrix} \lambda \cdot h_1(y) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This deformation is performed such that $\mathcal{W}^u(y)$ and $\mathcal{W}^s(y)$ keep intersecting transversally.

Proof of Step 7. We perform the Taylor expansion of the vector field $\tilde{X}_\lambda$ around $\lambda = 0$ and cut off the second order terms $\mathcal{O}_2(\lambda)$ without changing the transversality condition. We get

$$\tilde{X}_\lambda'''(x,y,\xi,\eta) = \begin{pmatrix} v(y)x \\ w(y) \\ \xi \\ -\eta \end{pmatrix} + \lambda \cdot \begin{pmatrix} h_1(x,y,\xi,\eta) \\ h_2(x,y,\xi,\eta) \\ h_3(x,y,\xi,\eta) \\ h_4(x,y,\xi,\eta) \end{pmatrix}.$$

Next we interpolate $\tilde{X}_\lambda'''$ to another vector field

$$\tilde{X}_\lambda'''(x,y,\xi,\eta) = \begin{pmatrix} v(y)x \\ w(y) \\ \xi \\ -\eta \end{pmatrix} + \lambda \cdot \begin{pmatrix} h_1(0,y,0,0) \\ h_2(0,y,0,0) \\ h_3(0,y,0,0) \\ h_4(0,y,0,0) \end{pmatrix}.$$

We define $h_i(y) := h_i(0,y,0,0)$ for $i = 1,\dots,4$. Note that since the variational equation (5.2) in the charts Φ_λ given by

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{y}} \\ \dot{\hat{\xi}} \\ \dot{\hat{\eta}} \end{pmatrix} = \begin{pmatrix} v(y)\hat{x} \\ w'(y)\hat{y} \\ \hat{\xi} \\ -\hat{\eta} \end{pmatrix} + \begin{pmatrix} h_1(y) \\ h_2(y) \\ h_3(y) \\ h_4(y) \end{pmatrix}$$

does not change under the interpolation, the transversality condition is not violated. The $h_i(y)$ are compactly supported on $[-1,2]$. Define a deformation of $\tilde{X}'''_\lambda$ depending on the parameter $s \in [0,1]$ in the following way

$$\tilde{X}'''_{\lambda,s}(x,y,\xi,\eta) := \begin{pmatrix} v(y)x \\ w(y) \\ \xi \\ -\eta \end{pmatrix} + \lambda \cdot \begin{pmatrix} h_1(y) \\ s \cdot h_2(y) \\ s \cdot h_3(y) \\ s \cdot h_4(y) \end{pmatrix}.$$

We want to show that while we move s from 1 to 0 *no* non-trivial and bounded solution of the variational equation (5.2) occurs. The solutions in the $(\hat{\xi},\hat{\eta})$ coordinates satisfy

$$\hat{\xi}(t) = e^{t-t_0}\hat{\xi}_0 + s \cdot \int_{t_0}^{t} h_3(\tilde{\gamma}(\tau))e^{-(\tau-t_0)}d\tau$$

Note that $h_3(\tilde{\gamma}(t))$ and $h_4(\tilde{\gamma}(t))$ are only supported outside an interval $[-T,T]$, hence vanish outside that interval. For $t > T$ we get that

$$\hat{\xi}(t) = e^{(t-t_0)}\hat{\xi}_0 + s \cdot \int_{t_0}^{T} h_3(\tilde{\gamma}(\tau))e^{-(\tau-t_0)}d\tau.$$

For any $s \in [0,1]$ we have $\hat{\xi}(t) \longrightarrow \infty$ for $t \to \infty$. In the same way one shows that $\hat{\eta}(t) \longrightarrow \infty$ for $t \to -\infty$ for any $s \in [0,1]$. Hence we may move s from 1 to 0 without creating bounded solution of the variational equation. For any $s \in [0,1]$ there exists a non-unique bounded solution of the variational equation restricted to the coordiantes

$$\begin{array}{rcccl} \hat{y}' &=& w(\hat{y})\hat{y} &+& s \cdot h_2(\hat{y}) \\ \hat{\xi}' &=& \hat{\xi} &+& 0 \cdot h_3(\hat{y}) \\ \hat{\eta}' &=& -\hat{\eta} &+& 0 \cdot h_4(\hat{y}) \end{array}.$$

Note that a solution of the top equation has the form

$$\hat{y}(t) = e^{\int_{t_0}^{t} w'(\hat{y}(\tau))d\tau}\hat{y}_0 + s \cdot \int_{t_0}^{t} h_2(\hat{y}(\tau))e^{-\int_{t_0}^{\tau} w'(\hat{y}(r))dr}d\tau$$

hence whenever $\hat{y}_0 \in (0,1)$ the solution $\hat{y}(t)$ is bounded for any $s \in [0,1]$. Hence if we move s from 1 to 0 we don't change the number of bounded solutions of the variational equation (5.2). By assumption the variational equation (5.2)

$$\begin{array}{rcccl} \hat{x}' &=& v(\hat{y})\hat{x} &+& h_1(\hat{y}) \\ \hat{y}' &=& w(\hat{y})\hat{y} &+& 0 \cdot h_2(\hat{y}) \\ \hat{\xi}' &=& \hat{\xi} &+& 0 \cdot h_3(\hat{y}) \\ \hat{\eta}' &=& -\hat{\eta} &+& 0 \cdot h_4(\hat{y}) \end{array}.$$

has no bounded solutions. From the formula

$$\hat{x}(t) = e^{\int_{t_0}^{t} \hat{v}(\hat{y}(\tau))d\tau}\hat{x}_0 + \int_{t_0}^{t} h_1(\hat{y}(\tau)e^{-\int_{t_0}^{\tau} v(\hat{y}(r))dr}dr.$$

Hence we have that $h_1 \not\equiv 0$ otherwise we would have plenty of bounded solutions by setting $\hat{x}_0 = 0$. Next we interpolate between $\tilde{X}'''_{\lambda,1}$ and the desired vector field $\tilde{X}'''_{\lambda,0}$. This can be done without obtaining a bounded solution of the variational equation. So in coordiantes Φ_λ we have a family of vector fields

$$\tilde{X}''''_\lambda(x, y, \xi, \eta) = \begin{pmatrix} v(y)x + \lambda \cdot h_1(y) \\ w(y) \\ \xi \\ -\eta \end{pmatrix}$$

which is the desired normal form. The interpolation can be done with a cut-off function ρ_λ. Set $X_\lambda := X''''_\lambda$, $\phi_\lambda := \Phi_\lambda : Q \times D^{k-1} \times D^{n-k-1} \to U$ and $K_\lambda := \Phi_\lambda(Q \times D^{k-1} \times D^{n-k-1})$. This proves Step 7 and hence the Proposition. □

5.4 Computation of the Conley continuation map

Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ such that $\mathscr{F}^{mor}_{reg}(N)$. For any $\lambda \in [-\varepsilon, \varepsilon]$ denote the critical points of Morse index k in S_λ by

$$\mathrm{Crit}_k(S_\lambda, f_\lambda) = \{x_1(\lambda), \ldots, x_{\ell_k}(\lambda)\}.$$

and the parametrized critical points of Morse index k in S are denoted by

$$\mathrm{Crit}_k(S, f) := \{(\lambda, x_1(\lambda)), \ldots, (\lambda, x_{\ell_k}(\lambda))\} =: \{x_1, \ldots, x_{\ell_k}\}.$$

The parametrized set of critical points of Morse index k together with connecting trajectories is between them is denoted by

$$S_k := \mathrm{Crit}_k(S, f, g) \cup \bigcup_{i,j=1}^{\ell_k} C(x_j, x_i)$$

where $C(x_j, x_i) := \{p \in S \mid \omega^-(p) \subset x_j,\ \omega^+(p) \subset x_i\}$. Pick a filtration

$$L = N_{-1} \subset N_0 \subset \cdots \subset N_n = N$$

such that (N_k, N_{k-1}) is an index pair of S_k for $k \in \{1, \dots, n\}$. Denote by $N_k(\lambda)$ the projection of $N_k \cap \{\lambda\} \times M$ to M. Then

$$L(\pm\varepsilon) \subset N_1(\pm\varepsilon) \subset \cdots \subset N_n(\pm\varepsilon)$$

is an index filtration with $(N_k(\pm\varepsilon), N_{k-1}(\pm\varepsilon))$ index pair for $\mathrm{Crit}_k(S_{\pm\varepsilon}, f_{\pm\varepsilon})$ and $(N_k(\pm\varepsilon), L(\pm\varepsilon))$ index pair for $S_{\pm\varepsilon}$. For any λ the *Conley complex* is defined by

$$C\Delta_k(S_\lambda) = \bigoplus_{|x_\lambda|=k} HC_k(x_\lambda)$$

where the sum runs over critical points x in $\mathrm{Crit}_k(S_\lambda, f_\lambda)$ with homological Conley index $HC_k(x)$. For $\lambda = \pm\varepsilon$ it follows from Lemma 2.10 that

$$\begin{aligned} C\Delta_*(S_{-\varepsilon}) &\xrightarrow{\cong} H_k\big(N_k(-\varepsilon), N_{k-1}(-\varepsilon)\big) \\ C\Delta_*(S_\varepsilon) &\xrightarrow{\cong} H_k\big(N_k(\varepsilon), N_{k-1}(\varepsilon)\big) \end{aligned}$$

On the other hand S_k gives the following continuation map

$$\Phi_{\mathbf{con}}(S_k, f, g) : H_k\big(N_k(-\varepsilon), N_{k-1}(-\varepsilon)\big) \to H_k\big(N_k(\varepsilon), N_{k-1}(\varepsilon)\big).$$

Putting all this together we come to the next definition.

Definition 5.6. Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ such that $\mathscr{F}^{mor}_{reg}(N)$. Denote by $C\Delta_*(S_\lambda)$ the cellular Conley complex. Then for any $k = \{1, \dots, n\}$ there is a **cellular Conley continuation map**

$$\Phi^{\Delta_k}_{\mathbf{con}}(S, f, g) : C\Delta_k(S_{-\varepsilon}) \longrightarrow C\Delta_k(S_\varepsilon)$$

defined via the composition

$$\begin{tikzcd} C\Delta_k(S_{-\varepsilon}) \arrow[r] \arrow[d, dotted, "\Phi^{C\Delta_k}_{\mathbf{con}}(S,f,g)"'] & H_k\big(N_k(-\varepsilon), N_{k-1}(-\varepsilon)\big) \arrow[d, "\Phi_{\mathbf{con}}(S_k,f,g)"] \\ C\Delta_k(S_\varepsilon) & H_k\big(N_k(\varepsilon), N_k(\varepsilon)\big) \arrow[l] \end{tikzcd}$$

By Theorem A we know that the homology of the Conley complex is isomorphic to the Conley index. The cellular continuation maps respect this isomorphism:

Proposition 5.7. *Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ such that $\mathscr{F}^{mor}_{reg}(N)$. Then the following diagram*

$$\begin{tikzcd}
H_k(C\Delta(S_{-\varepsilon})) \arrow[r, "\Phi^{\Delta}_{\mathbf{con}}(S,f,g)"] & H_k(C\Delta(S_\varepsilon)) \\
HC_k(S_{-\varepsilon}) \arrow[u, "\beta"] \arrow[r, "\Phi_{\mathbf{con}}(S,f,g)"] & HC_k(S_\varepsilon) \arrow[u, "\beta"']
\end{tikzcd}$$

where β are the maps induced by the diagram 3.18

Proof. For any k the follwing diagram commutes

$$\begin{tikzcd}
H_k(N_k(-\varepsilon), L(-\varepsilon)) \arrow[r, "\beta_-"] \arrow[d, "\Phi_{\mathbf{con}}(S,f,g)"'] & H_k(N_k(-\varepsilon), N_{k-1}(-\varepsilon)) \arrow[d, "\Phi_{\mathbf{con}}(S_k,f,g)"'] & C\Delta_k(S_{-\varepsilon}) \arrow[l] \arrow[d, "\Phi^{\Delta}_{\mathbf{con}}"'] \\
H_k(N_k(\varepsilon), L(\varepsilon)) \arrow[r, "\beta_+"] & H_k(N_k(\varepsilon), N_{k-1}(\varepsilon)) \arrow[r] & C_k\Delta(S_\varepsilon)
\end{tikzcd}$$

Applying the homology functor to this diagram as in the proof of Theorem 2 we get the commutative diagam

$$\begin{tikzcd}
H_k(S_{-\varepsilon}) \arrow[r, "\beta_-"] \arrow[d, "\Phi_{\mathbf{con}}(S,f,g)"'] & H_k(C\Delta(S_{-\varepsilon}) \arrow[d, "\Phi^{\Delta}_{\mathbf{con}}"] \\
H_k(S_\varepsilon) \arrow[r, "\beta_+"] & H_k(C\Delta(S_\varepsilon)
\end{tikzcd}$$

which proves the proposition. ☐

Now we have reduced the computation of the $\Phi_{\mathbf{con}}(S, f, g)$ to the comptutation of $\Phi^{\Delta}_{\mathbf{con}}(S, f, g)$ which we will do next. If no index zero bifurcation occurs the cellular continuation map $\Phi^{\Delta}_{\mathbf{con}}(S, f, g)$ is trivial in the following sense.

Lemma 5.8. *Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let N an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_0(N, f) = \emptyset$. Chose orientations of the parametrized unstable manifolds $\mathcal{W}^u(x, f, N)$ for all $x \in \mathrm{Crit}(S, f)$. Then the cellular Conley continuation map $\Phi^{\Delta}_{\mathbf{con}}(S, f, g) : C\Delta_k(S_{-\varepsilon}) \to C\Delta_k(S_\varepsilon)$ is given by*

$$\Phi^{\Delta}_{\mathbf{con}}(S, f, g)[W^u_{-\varepsilon}(x_\ell)] = [W^u_\varepsilon(x_\ell)] \qquad \ell \in \{1, \ldots, \ell_k\}$$

The generators $[W^u_{\pm\varepsilon}(x_\ell)]$ are given by the induced orientations of $W^u_{\pm\varepsilon}(x_\ell)$.

Proof. For $k \in \{1, \dots, n\}$ we get that

$$S_k = \mathrm{Crit}_k(S, f) = \{x_1, \dots, x_{\ell_k}\}$$

For any $i \in \{1, \dots, \ell_k\}$ pick an index pair (N_{x_i}, L_{x_i}) for x_i such that for $N_i \cap N_j = \emptyset$ for $i \neq j$. The disjoint union

$$N := \bigsqcup_{i=1}^{\ell_k} N_{x_i}$$

is an isolating neighborhood for $S_k = \mathrm{Crit}_k(S, f)$ with respect to the product flow. Note that any (N_{x_i}, L_{x_i}) determines a continuation map

$$F_{\varepsilon,-\varepsilon}(x_i) : N_{x_i}(-\varepsilon)/L_{x_i}(-\varepsilon) \to N_{x_i}(\varepsilon)/L_{x_i}(\varepsilon)$$

such that $F_{\varepsilon,-\varepsilon}(x_i)[W^u_{-\varepsilon}(x_i)] = [W^u_\varepsilon](x_i)]$. By the choice of (N, L) we have that

$$\begin{tikzcd}
\bigvee_{i=1}^{\ell} N_i(-\varepsilon)/L_i(-\varepsilon) \arrow[r] \arrow[d, dotted, "F_{\varepsilon,-\varepsilon}(x_1)\vee\cdots\vee F_{\varepsilon,-\varepsilon}(x_{\ell_k})"'] & \bigvee_{i=1}^{\ell} N_i(-\varepsilon)/L_i(-\varepsilon) \arrow[d, "F_{\varepsilon,-\varepsilon}"] \\
\bigvee_{i=1}^{\ell} N_i(\varepsilon)/L_i(\varepsilon) & \bigvee_{i=1}^{\ell} N_i(\varepsilon)/L_i(\varepsilon). \arrow[l]
\end{tikzcd}$$

Applying the homology functor to this diagram we get

$$\Phi^{C\Delta_k}_{\mathbf{con}}(S, f, g) = \Phi_{\mathbf{con}}(x_1) \oplus \cdots \oplus \Phi_{\mathbf{con}}(x_{\ell_k})$$

which implies the Lemma. □

Next we prove that the cellular Conley continuation map counts index zero connecting trajectories.

Proposition 5.9. *Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let N an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_{bif}(N, f) = \Lambda_0(N, f) = \{0\}$. Moreover assume that $S_\lambda = \mathrm{Crit}_k(S_\lambda, f_\lambda) = \{y_\lambda, x_\lambda\}$ for all $\lambda \in [-\varepsilon, \varepsilon]\backslash\{0\}$ and $S_0 = \{y_0, x_0\} \cup \gamma(\mathbb{R})$ where γ is the index zero trajectory from y_0 to x_0. Then the cellular Conley continuation map $\Phi^\Delta_{\mathbf{con}} : C\Delta_k(S_{-\varepsilon}) \to C\Delta_k(S_\varepsilon)$ is given by*

$$\Phi^\Delta_{\mathbf{con}}(S, f, g)[W^u_{-\varepsilon}(y)] = [W^u_\varepsilon(y)] - n(\gamma)[W^u_\varepsilon(x)]$$

and

$$\Phi^{C\Delta_k}_{\mathbf{con}}(S, f, g)[W^u_{-\varepsilon}(x) = [W^u_\varepsilon(x)]$$

Proof. By Propostition 5.5 we may deform the family of negative gradient fields $\{-\nabla f_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$ to a family of vector fields $\{X_\lambda\}_{\lambda\in[-\varepsilon,\varepsilon]}$ on N and find a family of charts $\phi_\lambda : Q \times D^{k-1} \times D^{n-k-1} \longrightarrow K_\lambda \subset U$ such that

$$\tilde{X}_\lambda := \phi_\lambda^* X_\lambda = \begin{pmatrix} v(y)x + \lambda \cdot h(y) \\ w(y) \\ \xi \\ -\eta \end{pmatrix}.$$

The dynamical system $\dot{x} = \tilde{Y}_\lambda(x)$ represented in the charts ϕ_λ splits into the products

$$(1) \left\{ \begin{array}{rcl} \dot{x} &=& v(y)x + \lambda h(y) \\ \dot{y} &=& w(y) \end{array} \right. \qquad \text{and} \qquad (2) \left\{ \begin{array}{rcl} \dot{\xi} &=& \xi \\ \dot{\eta} &=& -\eta \end{array} \right.$$

The dynamical system (1) is the two dimensional system on Q of the archetypical example whereas the system (2) is a system independent of λ on $D^{k-1} \times D^{n-k-2}$. In Step 1 we will prove Proposition 5.9 for the system (1). In Step 2 we will see that it extends to the whole dynamical system on $Q \times D^{k-1} \times D^{n-k-1}$ by using a Künneth formula.

Step 1. Let $N = Q = [-1,1] \times [-1,2]$ be an isolating neighbourhood of the dynamical system (1) and $L = \{\pm 1\} \times [-1, \frac{1}{2}] \cup [-1,1] \times \{2\}$ its exit set, both independent of $\lambda \in [-\varepsilon, \varepsilon]$. Then in $N_1(N, L)$ the following equations hold:

$$\begin{array}{rcl} [W^u_{-\varepsilon}(y)] &=& [W^u_\varepsilon(y)] - n(\gamma)[W^s_\varepsilon(x)] \\ [W^u_{-\varepsilon}(x)] &=& 0 + [W^u_\varepsilon(x)]. \end{array}$$

Proof of Step 1. (i). Note that the decomposition follows from Lemma 2.10 via the flow maps there exist isomorphisms

$$HC_1(y_{-\varepsilon}) \oplus HC_1(x_{-\varepsilon}) \xrightarrow{\cong} H_1(N, L)$$

and

$$HC_1(y_{-\varepsilon}) \oplus HC_1(x_{-\varepsilon}) \xrightarrow{\cong} H_1(N, L).$$

First observe that that the Conley continuation map of the system (1) is defined via

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, "\begin{bmatrix} 1 & -n(\gamma) \\ 0 & 1 \end{bmatrix}"'] & HC_1(y_{-\varepsilon}) \oplus HC_1(x_{-\varepsilon}) \arrow[d, "\Phi_{\mathbf{con}}"'] \arrow[dr, "\cong"] & \\
 & & H_1(N, L) \arrow[dl, "\cong"] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r] & HC_1(y_\varepsilon) \oplus HC_1(x_\varepsilon) &
\end{tikzcd}$$

Now pick generators $[W^u_{\pm\varepsilon}(y)]$ and $[W^u_{\pm\varepsilon}(x)]$ of $HC_1(y_{\pm\varepsilon})$ and $HC_1(x_{\pm\varepsilon})$. Then the flow maps send these generators to generators in $H_1(N,L)$. Four possible cases may occur as you see in the following figure

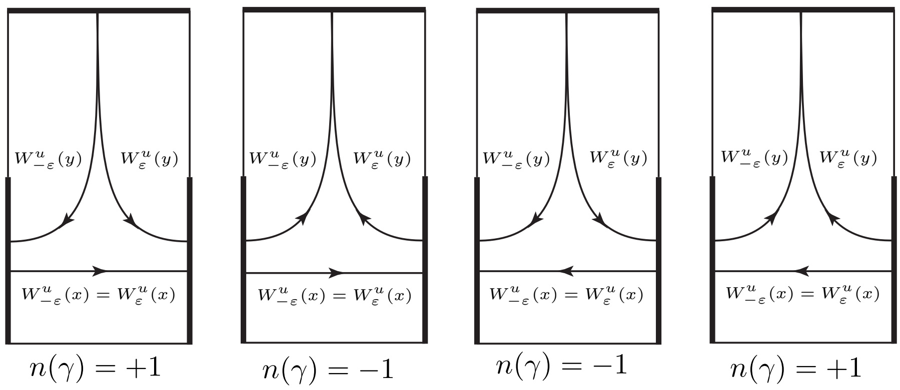


Figure 15. Four possible orientations may occur.

From the picture we see that the homological equations in $H_1(N,L)$ hold

$$\begin{aligned}[][W^u_{-\varepsilon}(y)] &= [W^u_{\varepsilon}(y)] - n(\gamma)[W^s_{\varepsilon}(x)] \\ [W^u_{-\varepsilon}(x)] &= 0 + [W^u_{\varepsilon}(x)].\end{aligned}$$

Since the $\Phi_{\mathbf{con}}$ is the base change matrix from $\{[W^u_{-\varepsilon}(y),[W^u_{-\varepsilon}(x)]\}$ to $\{[W^u_{\varepsilon}(y)],[W^u_{\varepsilon}(x)]\}$ we get

$$\Phi_{\mathbf{con}} = \begin{bmatrix} 1 & -n(\gamma) \\ 0 & 1 \end{bmatrix}.$$

This proves Step 1.

Step 2. Let $\tilde{N} = Q \times D^{k-1} \times D^{n-k-1}$ be an isolating neighborhood of the dynamical sytstem $\dot{x} = \tilde{Y}_\lambda(x)$ and L its leaving set. Then the following homological equations holds

$$\begin{aligned}[][W^u_{-\varepsilon}(y) \times D^{k-1}] &= [W^u_{\varepsilon}(y) \times D^{k-1}] - n(\gamma)[W^u_{\varepsilon}(x) \times D^{k-1}] \\ [W^u_{-\varepsilon}(x) \times D^{k-1}] &= 0 + [W^u_{\varepsilon}(x) \times D^{k-1}].\end{aligned}$$

Proof of Step 2. Let $(N_1,L_1) = (Q,L_1)$ be the index pair of the dynamical system (1) and $(N_2,L_2) = (D^{k-1} \times D^{n-k-1}, \partial D^{k-1} \times D^{n-k-1}) \sim (D^{k-1}, \partial D^{k-1})$ the index pair of the dynamical system (2). Denote by $e \in H_{k-1}(N_2,L_2) \cong H_{k-1}(D^{k-1},\partial D^{k-1})$ the canonical generator. Since all the homology groups involved are torsion-free we get by applying the Künneth formula for pairs the natural isomorphism

$$i : H^{\text{sing}}_1(N_1,L_1) \otimes H^{\text{sing}}_{k-1}(N_2,L_2) \longrightarrow H^{\text{sing}}_k(N_1 \times N_2, N_1 \times L_2 \sqcup N_2 \times L_2).$$

It follows that

$$\begin{array}{rcl} [W^u_{-\varepsilon}(y)\times D^k] & = & i([W^u_{-\varepsilon}(y)]\otimes e) \\ & = & i\big(([W^u_{\varepsilon}(y)]-n(\gamma)[W^u_{\varepsilon}(x)])\otimes e\big) \\ & = & i\big([W^u_{\varepsilon}(y)]\otimes e-n(\gamma)([W^u_{\varepsilon}(x)]\otimes e)\big) \\ & = & i([W^u_{\varepsilon}(y)]\otimes e)-n(\gamma)i([W^u_{\varepsilon}(y)]\otimes e) \\ & = & [W^u_{\varepsilon}(y)]\times D^{k-1}]-n(\gamma)[W^u_{\varepsilon}(y)]\times D^{k-1}] \end{array}$$

The second homological equation is proven similarly. This proves the homological equation and hence Step 2. The Conley continuation map $\Phi_{\mathbf{con}}(S,f,g)$ is the base change form $-\varepsilon$ to ε. This proves the Proposition. □

Corollary 5.10. *Let $(S,f,g)=\{(S_\lambda,f_\lambda,g_\lambda)\mid\lambda\in[-\varepsilon,\varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon},f_{-\varepsilon},g_{-\varepsilon})$ and $(S_\varepsilon,f_\varepsilon,g_\varepsilon)$ and let N an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda,x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_0(N,f)=\{0\}$ with index zero solution $\gamma\in\mathscr{M}_f(x_i(0),x_j(0),N(0))$. Then the cellular Conley continuation map*

$$\Phi^{\Delta}_{\mathbf{con}}(S,f,g):C\Delta_k(S_{-\varepsilon})\to C\Delta_k(S_\varepsilon)$$

is given by

$$\Phi^{\Delta}_{\mathbf{con}}(S,f,g)[W^u_{-\varepsilon}(x_\ell)]=\begin{cases}[W^u_{\varepsilon}(x_i)]-n(\gamma)[W^u_{\varepsilon}(x_j)] & \textit{for } \ell=i\\ [W^u_{\varepsilon}(x_\ell)] & \textit{for } \ell\neq i\end{cases}$$

Proof. Let $k\in\{1,\dots,n\}$. For every $x_\ell\in\mathrm{Crit}_k(S,f)\backslash\{x_i,x_j\}$ pick an isolating neighborhood of x_ℓ with respect to the product flows such that whenever $x_\ell\neq x_{\ell'}$ both in $\in\mathrm{Crit}_k(S,f)\backslash\{x_i,x_j\}$ it follows

$$N_\ell\cap N_{\ell'}=\emptyset.$$

Pick and isolating neighborhood N_{ij} of $\{x_i,x_j\}\cup\gamma(\mathbb{R})$ such that

$$N_{ij}\cap N_\ell=\emptyset\quad\text{for all } \ell\neq i,j.$$

and define an isolating neighborhood of S by the union

$$N:=\Big(\bigsqcup_{\ell\neq i,j}N_\ell\Big)\cup N_{ij}$$

Now apply Lemma 5.8 and Proposition 5.9 and get the result. □

Now we are ready to prove Theorem D.

Proof of Theorem D. Since any continuation (S, f, g) satisfying the parametrized Morse-Smale condition can be perturbed to a continuation with $f \in \mathscr{F}^{mor}_{ref}(N, f)$ leaving the continuation map $\Phi_{\mathbf{con}}(S, f, g)$ unchanged we may assume the latter. Let (S, f, g) be a continuation relating the two Morse-Smale triples (S_0, f_0, g_0) and (S_1, f_1, g_1) such that $f \in \mathscr{F}^{mor}_{reg}(N)$. Moreover denote by

$$\{\lambda_1, \ldots, \lambda_N\} = \Lambda_0(N, f)$$

the bifurcation values $0 < \lambda_1 < \cdots < \lambda_N < 1$ where an index zero connecting orbit occurs. Pick a small enough $\varepsilon > 0$ such that for all $k \in \{1, \ldots, N\}$ the interval $[\lambda_k - \varepsilon, \lambda_k - \varepsilon] \backslash \{\lambda_k\}$ has no bifurcation values. Then the Conley continuation map

$$\Phi_{\mathbf{con}}(S, f, g) : HC_*(S_0) \longrightarrow HC_*(S_1)$$

by definition is the composition

$$\boxed{\Phi_{\mathbf{con}}(S, f, g) := \Phi^{\lambda_N+\varepsilon, \lambda_N-\varepsilon}_{\mathbf{con}}(S, f, g) \circ \cdots \circ \Phi^{\lambda_1+\varepsilon, \lambda_1-\varepsilon}_{\mathbf{con}}(S, f, g)}$$

of isomorphisms $\Phi^{\lambda_k+\varepsilon, \lambda_k-\varepsilon}_{\mathbf{con}}(S, f, g)$ defined in (2.15). Since we may assume $[\lambda_k - \varepsilon, \lambda_k + \varepsilon] = [-\varepsilon, \varepsilon]$ it is enough to show the following

Claim. Let $(S, f, g) = \{(S_\lambda, f_\lambda, g_\lambda) \mid \lambda \in [-\varepsilon, \varepsilon]\}$ be a continuation connecting two Morse-Smale triples $(S_{-\varepsilon}, f_{-\varepsilon}, g_{-\varepsilon})$ and $(S_\varepsilon, f_\varepsilon, g_\varepsilon)$ and let N an isolating neighborhood of S with respect to the product flow $\Phi^t(\lambda, x)$. Assume moreover that $\mathscr{F}^{mor}_{reg}(N)$ and $\Lambda_{\mathrm{bif}}(N, f) = \Lambda_0(N, f) = \{0\}$. For $k \in \{0, \ldots, n\}$ and $\lambda \in [-\varepsilon, \varepsilon] \backslash$ we denote the set of critical points of Morse index k by

$$\mathrm{Crit}_k(S_\lambda, f_\lambda) = \{x_1(\lambda), \ldots, x_{\ell_k}(\lambda)\}.$$

By assumption for $\lambda = 0$ there exists a connecting trajectory $\gamma : \mathbb{R} \to N_0$ connecting $x_i(0)$ and $x_j(0)$. Then

$$\begin{tikzcd}
HM_k(S_{-\varepsilon}, f_{-\varepsilon}) \arrow[r, "\Phi_{\mathbf{bif}}(S,f,g)"] & HM_k(S_\varepsilon, f_\varepsilon) \\
HC_k(S_{-\varepsilon}) \arrow[u, "\alpha"] \arrow[r, "\Phi_{\mathbf{con}}(S,f,g)"] & HC_k(S_\varepsilon) \arrow[u, "\iota"']
\end{tikzcd}$$

Proof of the Claim. Observe the following diagram:

$$\begin{tikzcd}
 & CM_k(S_{-\varepsilon}, f_{-\varepsilon}) \arrow[rr, dotted, "\Phi_{\mathbf{bif}}(S,f,g)"] \arrow[dd, dotted] & & CM_k(S_\varepsilon, f_\varepsilon) \arrow[dd] \\
C\Delta_k(S_{-\varepsilon}) \arrow[ur, "\iota"] \arrow[rr, "\Phi^{\Delta}_{\mathbf{con}}(S,f,g)"] \arrow[dd, "\Delta_{-\varepsilon}"'] & & C\Delta_k(S_\varepsilon) \arrow[ur, "\iota"'] \arrow[dd] & \\
 & CM_{k-1}(S_{-\varepsilon}, f_{-\varepsilon}) \arrow[rr, dotted, "\Phi_{\mathbf{bif}}"'] & & CM_{k-1}(S_\varepsilon, f_\varepsilon) \\
C\Delta_{k-1}(S_{-\varepsilon}) \arrow[ur, "\iota"'] \arrow[rr, "\Phi^{\Delta}_{\mathbf{con}}(S,f,g)"] & & C\Delta_{k-1}(S_\varepsilon) \arrow[ur, "\iota"'] &
\end{tikzcd}$$

The squares on the left and right commute by Lemma 3.14. The square on the top and the bottom comutes by the formulas for $\Phi_{\mathbf{bif}}$ and $\Phi^{\Delta}_{\mathbf{con}}$. The square in the back commutes since $\Phi_{\mathbf{bif}}$ is a chain map. The commutativity of the square in the front follows by the commutativity of the other squares. Hence the the diagram commutes. Taking the homology we get the commuting diagram:

$$\begin{tikzcd}
HM_k(S_{-\varepsilon}, f_{-\varepsilon}) \arrow[r, "\Phi_{\mathbf{bif}}(S,f,g)"] & HM_k(S_\varepsilon, f_\varepsilon) \\
H_k(C\Delta(S_{-\varepsilon})) \arrow[u, "\iota"] \arrow[r, "\Phi^{\Delta}_{\mathbf{con}}(S,f,g)"] & H_k(C\Delta(S_\varepsilon)) \arrow[u, "\iota"']
\end{tikzcd}$$

Using the commutative diagram of Proposition 5.7 this proves the claim and hence Theorem D. $\square$

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Curriculum Vitae

Name:	Driton Komani
Date of Birth:	30.12.1982
Place of Birth:	Zhub, Gjakovë, Kosova
Date of CV:	February 22, 2013

Education

2007 – 2012	Ph.D. student and teaching assistant, ETH Zürich
2004-2007	Junior assistant, ETH Zürich
2002 – 2007	Undergraduate Student, ETH Zürich
1995 – 2002	High School, Luzern
1990 – 1995	Elementary School Maihof, Luzern
1988-1990	Elementary School "Zef Lush Marku", Gjakovë

I was born 30.12.1982 in a small village called Zhub near Gjakovë in the state of Kosova near the artificial border to Albania. This is not my only connection to Albania. The ancestors of the Komani family have their origin in the village Koman in the north of Albania. My grandfather *Rrustem Zenun Komani* was the first family member to enjoy a four year education in a austro-hungarian school. My grand father saw the value of education and motivated his children to undertake studies. Three out of six children of *Rrustem Zenun Komani* had university degrees. Among them was *Nik Rrustem Komani, Hil Rrustem Komani* and my father *Pal Rrustem Komani.* It should be mentioned that my uncle *Nik Rrustem Komani* (himself a judge) has eight children. Six of them have university degrees. At an early stage of my live I was influenced by my cousins *Dr.Tomë Nik Komani* and *Dr.Gjokë Nik Komani.* They were medicine students and the pictures on the wall showing them studying while holding a pen is still burned in my mind. My grandfather and these two cousins were my first heroes. At the time of my birth my father *Pal Rrustem Komani* was working in Switzerland. At the age of five I went to school for the first time. I remember those days very precisely. I liked school and finished the first grade. Due to political and financial circumstances my parents decided to emigrate to Lucern in

Switzerland in january 1990. At the age of seven I restarted elementary school in Maihof in Lucerne. When visiting us in 1992 my uncle *Hil Rrustem Komani* explained to me *Darvin's* evolution theory in complete detail. I liked it very much since everything was logically clear. Finishing elementary school in Maihof I continued for two more years in Sekundarschule before I was given permission to sign in for the Kantonsschule Alpenquai in Lucerne. I finished Kantonschule Alpenquai in spring 2002 without knowing for which faculty to sign in next. There were a number of accidents that led me to the decision to study mathematics at the ETH. My first year calculus teacher at ETH was *Prof.Eduard Zehnder*. His precise and demanding lectures influenced me. It was *Prof.Eduard Zehnder* who suggested to me *Prof.Dietmar Salamon* as an advisor. An excellent choice. During my studies I have been interested in various other things outside mathematics. I've read a lot *Goethe*, *Shakespeare* and *Fishta*. I've always considered myself to be 100% Albanian and nothing else.

Driton Komani