MEASUREMENTS OF VELOCITY, VELOCITY FLUCTUATIONS AND DENSITY IN DRY GRANULAR FLOWS: SIGNIFICANCE FOR THE CONSTITUTIVE MODELING IN THE FRICTIONAL-COLLISIONAL REGIME

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Summary

A common feature of snow avalanches, rockslides and debris flows is that they are made of a large number of grains moving as a continuum. The grains are simultaneously accelerated by gravity and slowed down by the multiple contacts that they experience with each other or with the bed surface. Both processes result in variations of density, velocity and internal stresses within the flow. The rheology of granular flows, i.e. the dependency of the internal stresses on the relative motion of the individual grains, is poorly understood, in particular for flows with multiparticle, enduring contacts (frictional-collisional regime). The subject of this thesis is the formulation of mathematical relations for granular flows which reproduce the experimentally observed flow variables.

Laboratory chute experiments with finite volumes of glass beads in a dam break set-up are performed. The unsteady flows are filmed through a transparent sidewall with a high-speed camera at a fixed location of the chute. Pattern matching and particle tracking methods are developed to extract density, velocity and velocity fluctuations from the recordings. Thanks to the accuracy of the data, time derivatives and spatial gradients of the flow variables can be computed, allowing the calculation of the internal stresses from the mass and momentum conservation equations for two-dimensional, shallow flows. Constitutive laws needed to close the system of governing equations are derived by examination of the data over a continuous region of the flow variable space which identifies with the frictional-collisional regime.

A first constitutive law relates density to shear rate (dilatancy) and normal stress (compaction), and includes a rate-dependent limit density at low normal stress. A second constitutive law quantifies the shear stress as an increasing function of the shear rate, the normal stress and the velocity fluctuations. Velocity fluctuations are found to be isotropic and in the same order of magnitude as the vertical component of the mean velocity, leading to a simplified form of the energy conservation equation. The results argue
in favor of dry granular flow models accounting for non-constant density and non-zero velocity fluctuations also in the frictional-collisional regime. The new constitutive laws may find application in the study of more complex granular flows, such as two-phase flows, which exhibit non-constant solid volume fraction.
Résumé

Les avalanches, les glissements de terrain et les laves torrentielles ont ceci en commun qu’ils sont constitués d’un grand nombre de grains se mouvant de façon cohérente. Les grains sont accélérés par la force de gravité en même temps qu’ils sont freinés par les multiples contacts qu’ils subissent entre eux ou avec le milieu environnant. Les deux effets ont pour conséquence des variations de densité, de vitesse et de l’état des contraintes internes de l’écoulement. La rhéologie des écoulements granulaires, c’est-à-dire la dépendance de l’état des contraintes internes vis-à-vis du mouvement relatif des grains, est mal comprise, en particulier pour des écoulements qui se caractérisent par des contacts prolongés et/ou impliquant plusieurs grains (régime frictionnel-collisionel). Le sujet de cette thèse est la formulation de relations mathématiques qui sont à même de reproduire les variables d’écoulement observées expérimentalement pour des écoulements granulaires secs.

Des expériences sur un plan incliné dans une configuration de rupture de barrage ont été réalisées en laboratoire avec des volumes finis de billes de verre. Les écoulements non-stationnaires sont filmés à travers une paroi transparente à l’aide d’une caméra ultra rapide fixée le long du plan incliné. Des méthodes de reconnaissance de motifs et de suivi des particules ont été développées pour extraire des valeurs de densité, de vitesse et de fluctuation de la vitesse à partir des enregistrements. Grâce à la précision des données récoltées, les dérivées temporelles et les gradients dans l’espace des variables d’écoulement peuvent être calculés, permettant ainsi de déterminer l’état des contraintes internes à partir des équations de conservation de la masse et de l’impulsion pour des écoulements bidimensionnels et peu profonds. Les lois constitutives nécessaires à la fermeture du système d’équations sont dérivées en examinant les données sur une région continue de l’espace des variables d’écoulement qui correspond au régime frictionnel-collisionnel.

Une première loi constitutive établit une relation entre la densité et le taux
de cisaillement (dilatance) d’une part, et entre la densité et la contrainte normale (compaction) d’autre part. De plus, elle introduit une densité limite à faible contrainte normale qui dépend du taux de cisaillement. Une deuxième loi constitutive définit la contrainte de cisaillement en tant que fonction croissante du taux de cisaillement, de la contrainte normale et de la fluctuation de la vitesse. Il ressort que la fluctuation de la vitesse est isotrope et, du même ordre de grandeur que la composante verticale de la vitesse, ce qui se traduit par une forme simplifiée de l’équation de conservation de l’énergie. Ces résultats plaident en faveur de modèles prenant en compte une densité non-constante et une fluctuation de la vitesse non-nulle pour les écoulements granulaires dans le régime frictionnel-collisionnel. Les nouvelles lois constitutives peuvent trouver des applications dans l’étude d’écoulements granulaires plus complexes, tels que des écoulements biphasiques présentant des fractions volumiques non-constantes.
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Chapter 1

Introduction

Geophysical flows are frequent in many regions in the world. Snow avalanches, landslides, debris flows and rockslides are some examples. Recently, during the writing of this dissertation, a large rock avalanche released in the Bergell range (Switzerland). The deposit is depicted in Fig. 1.1. Between 2 and 3 millions m$^3$ of granite broke away from the side of Pizzo Cengalo (3369 m) and flowed over the glacier down to the valley. The winter 2011/2012 saw some heavy snow fall producing large avalanches that threatened infrastructure. Switzerland is especially affected by these hazardous flows.

Predicting the runout and velocity of these flows in real terrain remains one of the fundamental problems in natural hazards research. Although the conditions for their occurrence are wide-ranging, many geophysical flows have common physical features: (1) They consist of a finite amount of solid granular material. (2) They are driven by gravity. (3) They have a free surface. (4) They are shallow flows i.e. they have small flow heights compared to the flow lengths (snow avalanches have typically flow heights of several meters and are hundreds of meters long). (5) They are dense (large solid or liquid volume fractions). The solid granular material typically consists of rocks, snow clods, earthen clumps or woody debris whereas the interstitial fluids may be air and/or water. Therefore, many geophysical flows can be classified as two- or three-phase flows, consisting of air and/or water in addition to a solid granular phase whose size, shape and mass distributions may be very diverse.

Geophysical flows have been the object of intense research over the last decades, combining several different scientific fields: fluid dynamics, geophysics, computer science, applied mathematics, soil and snow mechanics and geography. Their study unites practitioners concerned with the mitigation of natural hazards and engineers and physicists who focus on developing
Figure 1.1: Deposit of a rock avalanche that released on the 27th December 2011 in the Bergell range (Switzerland). Between 2 and 3 million m$^3$ of granit rock broke away from the side of Pizzo Cengalo (3369 m) and flowed over the glacier down to the valley. Source: Kanton Graubünden
the system of equations allowing an accurate description of the flow processes in realistic terrain. Practitioners require reliable and pragmatic models to design protective measures and to delimit hazard zones. For researchers, the interest goes beyond the scope of natural hazards and the knowledge acquired finds application in many industrial processes involving primarily powder and grains (Duran 1999).

The study of geophysical flows has involved detailed observation of natural real-scale flows (sometimes artificially triggered) as well as the monitoring of well-instrumented laboratory experiments. Natural real-scale flows or full-scale tests are much more complex to investigate: they are not repeatable, many flow parameters are not controlled, the instrumentation is expensive. In laboratory, researchers resort to idealized materials and simplified flow geometries. The flows are reproducible i.e. flow parameters are controlled and can be varied systematically to determine dependencies between flow variables. Typically, rocks and cobbles are replaced by spherical and monodisperse grains and the density of the fluid phase is chosen so that the solid phase is neutrally buoyant (Bagnold 1954). If natural materials are used, the size, shape and mass distributions of the solid phase and the water content are chosen carefully (Coussot 1997, Iverson 1997). Flow paths without curvature and delineated by solid boundaries (e.g. sidewalls) are preferred to more complex geometries. To remove the unsteadiness inherent to finite-mass flows, steady-state flows are achieved using constant supply of material or circular geometries (e.g. rheometer, rotating drum, chute with conveyor belt) (Coussot 1995, Davies 1990, Kaitna 2006). In some rheometer and triaxial tests investigating the material rheology, no free surface is present and the effect of gravity is eliminated by using very thin layers of material (Phillips and Davies 1991).

The dynamic modeling of geophysical flows started by importing concepts from open channel flow, shallow water hydraulics and soil mechanics (Saint-Venant 1886, Terzaghi 1925, Voellmy 1955, Savage and Hutter 1989, Coussot 1997, Iverson 1997) and suspensions rheology (Einstein 1906, Bingham 1922, Bagnold 1954, Boyer et al. 2011). Constitutive laws for granular flows were derived using the formalism of kinetic theory originally developed to model gases (Chapman and Cowling 1970, Jenkins and Richman 1985, Goldhirsch and Tan 1996). With the improvements in computer science, molecular dynamics methods, in which the trajectories and the interactions of the individual flow constituents are computed, have been applied to granular flows (Walton 1993, Silbert et al. 1997, da Cruz et al. 2005).
In this work, we restrict our investigations to two-dimensional, free surface, dense, gravitational, shallow, unsteady flows of monodisperse grains on a rough incline. In the following, we simply refer to this category of flows as “dry granular flows”. Dry granular flows are idealizations of natural flows with the objective that: (1) the governing equations remain relatively simple, (2) the flow parameters can be well controlled experimentally and (3) nonetheless they are complex enough so that we can learn about natural flows. We investigate dry granular flows by performing small-scale chute experiments with glass beads. High-speed video recordings allow us to measure, at high temporal and spatial resolution, flow variables including mean velocity, velocity fluctuations and density over the flow height during the passing of the flow at one location of the chute. The normal and shear stresses are computed using the governing equations. Thanks to the unsteady character of the flows, we are able to collect data over continuous ranges of the flow variables. The constitutive laws needed to close the system of governing equations are directly determined from the experimental data. The isotropy and the scaling of the velocity fluctuations are examined leading to a simplified form of the energy equation and the relevance of an energy equation in a dry granular flow model is discussed.

1.1 Dry granular flows

In this section, we introduce the theoretical framework and the system of governing equations allowing the physical description of dry granular flows. We derive a system of partial differential equations starting from the mass and momentum conservation equations. We discuss each of the underlying assumptions: continuity, single-phase, compressibility, grain interaction, two-dimensional character, shallowness, gravity, boundaries, unsteady flow. The closure of the system of equations is addressed since there are more unknowns than equations. Special cases (steady flow, depth-averaging) as well as generalizations of the flow model are eventually discussed.

1.1.1 Continuum assumption

The continuum assumption is central since it allows a granular flow to be regarded as a continuum with well-defined flow variables instead as a collection of individual grains, each one with its own variables. The continuum assumption requires, first, that the ensembles over which the grain variables are averaged contain a large number of grains to ensure continuity and differentiability of the flow variables and, second, that the dimensions of the
ensembles are much smaller than the dimensions of the flow. In fluid flows, the molecules are twenty orders of magnitude smaller than the flow dimensions and thus the continuum assumption is valid for most measurements. In granular flows, the grain size is only a few orders of magnitude smaller than the flow dimensions so that not all choices of the average ensemble dimensions satisfy the continuum assumption. The lack of scale separation between the grain size and the flow dimensions is a characteristic feature of granular flows. Not only the grain size is important to ensure that average ensembles contain a large number of grains but also the concentration of the grains i.e. the density in the average ensembles. Thus, for a given grain size, dense flows are more likely to satisfy the continuum assumption than dilute flows.

1.1.2 Single-phase compressible flow

The flow density plays an important role with respect to the single-phase compressible flow assumption. Dry granular flows are two-phase flows composed of a solid incompressible phase (grains) and a gaseous compressible phase (interstitial air). In the following, we neglect the interaction between the grains and the interstitial air as well as the exchange of air between the flow and the atmosphere. Moreover, we assume that the grains are homogeneously distributed over the average ensembles. Thus, the interstitial air has no influence on the flow dynamics and the two-phase flow problem reduces to a single-phase, compressible flow problem with a bulk density $\rho$ that is smaller than the solid density of the grains $\rho_{\text{solid}}$. However, the single-phase compressible flow assumption is more likely to be valid if the granular flow is dense (for realistic values of grain inelasticity/friction). In a dilute flow, the velocity difference between the phases, the velocity fluctuations of the grains and the exchange of air between the flow and the atmosphere are larger, all contributing to enhance the momentum exchange between the phases.

1.1.3 Grain interaction

Since the interaction between grains and the interstitial air is neglected, the important interaction is the one between grains. Grains interact through contacts during which momentum is exchanged and energy is dissipated (kinetic energy is dissipated into heat). The momentum exchange and the energy dissipation are determined by the mechanical properties of the grains (inelasticity, friction, plasticity, fracture load) as well as by the velocities of the grains before the contact. Grain contacts play an important role in the perspective of a continuum description of the flows since they “materialize”
internal stresses. In dry granular flows, we distinguish between collisional and frictional flow regimes. The definitions of the different flow regimes are based on the predominant type of contact between grains. The collisional regime (or rapid flow regime) is characterized by binary contacts of short duration whereas the frictional regime (or quasi-static regime) refers to multiparticle, enduring contacts. The distinction between the two flow regimes is made by evaluating the dimensionless inertial number $N_I$. For a simple shear flow, it is written as:

$$N_I \equiv \frac{\partial u}{\partial y} d \sqrt{\frac{\rho_{\text{solid}}}{\sigma_{yy}}} \equiv \frac{t_r}{t_c} \tag{1.1}$$

where $\frac{\partial u}{\partial y}$ is the shear rate, $d$ is the grain diameter and $\sigma_{yy}$ is the normal stress in the $y$-direction (see frame of reference in Fig. 1.2). The inertial number is defined as the ratio between two typical times at the grain scale. The contact time $t_c \equiv (\partial u/\partial y)^{-1}$ is the time needed by two grains initially aligned along the cross-flow direction ($y$-direction) to be separated by the same distance in both flow and cross-flow directions. For a grain, it can be interpreted as the time between two successive contacts given a periodic arrangement of grains under shear (assuming the same periodicity in the flow and cross-flow directions and identical grains with the same orientation). The rearrangement time $t_r \equiv d \sqrt{\rho_{\text{solid}}/\sigma_{yy}}$ represents the time needed by a grain of solid density $\rho_{\text{solid}}$ subjected to a stress gradient $\sigma_{yy}/d$ to travel across a distance $2d$ if its initial velocity is zero (accelerating grain). Thus, it can be interpreted as the typical time for a grain to rearrange in the cross-flow direction. If $t_c$ is much larger than $t_r$ ($N_I \ll 1$), the flow is frictional whereas in the opposite case ($N_I \gg 1$), the flow is collisional. A shortcoming of the inertial number definition is that the interpretation of the contact time is independent of the density whereas the definition of the rearrangement time makes sense only if the distance between neighbouring grains is of the order of $2d$. Nevertheless, the inertial number is of practical use since it does not require knowledge on the grain mechanical properties or on the flow constitutive laws. In this work, we do not make any assumptions on the type of contact. For dry granular flows though, it is generally accepted that grain contacts are dissipative and non-adhesive (Pöschel and Schwager 2005). Most “real” dry granular flows, including the flows investigated in this work, belong to the intermediate collisional-frictional regime in which both types of contact coexist. The combined effect of gravitational acceleration and flow confinement (bed surface, sidewalls) explains why the collisional-frictional regime is so common among granular flows.
1.1.4 Two-dimensional flows

A two-dimensional flow (in the $x$-$y$ plane) is an idealized representation of a real three-dimensional flow which can be described with good precision by two-dimensional equations and flow variables. It is equivalent to the condition that the velocity component in the $z$-direction and the stress gradients in the $z$-direction are equal to zero. Experimentally, a way to approach a two-dimensional flow is to confine the flow in the $z$-direction with smooth sidewalls. The gradient of the shear stress in the $z$-direction is small due to low sidewall friction and the gradient of the normal stress in the $z$-direction is equal to zero since the sidewalls are fixed. An alternative way consists in considering only the flow regions around the centerline of a sufficiently wide flow. The shear stress behavior at the sidewalls is complex (Jop et al. 2005). It depends not only on the sidewall and grain properties but also on the flow variables (flow height, slip velocity). Other categories of flows to which the two-dimensional representation applies are infinite flows and flows which are only one constituent wide (one grain or one molecule in the case of granular or fluid flow, respectively). Whereas infinite flows and one molecule wide flows are utopian, one grain wide flows have been studied experimentally (Drake 1991, Azanza et al. 1999). In computer simulations, two-dimensional flows are achieved by imposing periodic boundary conditions at the flow lateral boundaries and a zero component of the external force in the $z$-direction.
1.1.5 Shallow flows

Shallowness is a common feature of many geophysical flows which, when formulated in terms of scaling relations between flow variables, leads to important simplifications in the governing equations. Mathematically, shallowness refers to the geometry of the flows and is expressed by a parameter $\varepsilon$ much smaller than 1: $\varepsilon \equiv H/L \ll 1$ where $H$ is the typical flow height and $L$ is the typical flow length (the typical flow width $W$ is irrelevant for two-dimensional flows). Importantly, the shallowness introduces a scaling for the mean velocity components $u$ and $v$, the stress tensor components $\tau_{xy}$, $\tau_{yx}$, $\sigma_{xx}$ and $\sigma_{yy}$, the time derivative $\partial / \partial t$ and the spatial gradients $\partial / \partial x$ and $\partial / \partial y$. The scaling helpfully allows an order of magnitude estimation of the flow variables in terms of constant typical flow quantities:

$$
\begin{align*}
    u &= O \left( \sqrt{gL} \right), \quad v = O \left( \sqrt{gH^2 / L} \right) \\
    \tau_{xy} &= O \left( \frac{PgH}{L} \right), \quad \tau_{yx} = O \left( \frac{PgH}{L} \right), \quad \sigma_{xx} = O \left( \frac{PgH}{L} \right), \quad \sigma_{yy} = O \left( \frac{PgH}{L} \right) \\
    \frac{\partial}{\partial t} &= O \left( \sqrt{g / L} \right), \quad \frac{\partial}{\partial x} = O \left( \frac{1}{L} \right), \quad \frac{\partial}{\partial y} = O \left( \frac{1}{H} \right)
\end{align*}
$$

where $P$ is a typical flow density. If the scaling is verified, the initial ordering $\varepsilon \ll 1$ (or $H \ll L$) results in many other orderings between flow variables. The orderings between flow variables can be combined into orderings between terms of the governing equations. The neglection of the smaller terms simplifies the analysis of the governing equations and the inspection of solutions.

![Flow scheme with frame of reference, gravitational acceleration vector $g$, flow height $h(x,t)$ and an arbitrary bed surface $b(x) \neq 0$.](image)

Figure 1.3: Flow scheme with frame of reference, gravitational acceleration vector $g$, flow height $h(x,t)$ and an arbitrary bed surface $b(x) \neq 0$. 

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1.1.6 Gravitational flows on a rough incline

Dry granular flows are subjected to two different external stresses; (1) the stress exerted by the bed surface and (2) the gravitational acceleration acting on the whole flow volume. The gravitational acceleration is the driving force whereas the shear stress at the bed surface is a resistive force opposing grain displacement at the bed surface (the shear stress acting on the flow at the sidewalls is neglected). The adjective “rough” is important (and loosely defined) since it means that the shear stress at the bed surface is not equal to zero. In case of zero (or very small) shear stress at the bed surface in comparison to the internal shear stresses, the scaling of the spatial gradient $\partial/\partial y$ in Eqs. (1.2) would not be valid. Generally, rough applies to a bed surface with a typical roughness size $d_r$ of the same order of magnitude as the grain diameter $d \approx d_r$. The combination of gravity and of the bed surface reaction are essential since the flow weight contributes largely to the flow normal stress. In the special case of shallow flows, the normal stress is totally determined by the flow weight (see below).

The incline geometry has two consequences: (1) it bounds the inclination angle $0^\circ < \theta < 90^\circ$ and ensures that the flow weight has non-zero components in the $x$- and $y$-direction. Second, the incline geometry results in the simple definition of the bed surface $y = 0$ (in the frame of reference of Fig. 1.2). If the bed surface would not be an incline, it would be parametrized by the function $b = b(x)$ with the definition $b - y = 0$ (see Fig. 1.3).

1.1.7 Finite-mass flows

If the mass flow rate is not controlled, the release of a finite mass of granular material will result in an unsteady flow. Unsteadiness is not only characterized by non-zero time derivatives of the flow variables but also by zero flow height values on the perimeter of the contact surface between the bed surface and the flow. In two-dimensional chute flows, the flow height goes to zero at the leading edge and at the rear of the flow represented by two coordinates along the $x$-axis. Thus, we can divide the flow in three regions: the flow “front” where the flow height increases (opposite to the flow direction), the flow “body” around the flow height maximum and the flow “tail” where the flow height decreases (opposite to the flow direction).
1.1.8 Mass and momentum conservation for continuum, two-dimensional flows

If the continuum assumption is satisfied, the flow obeys the principles of mass and momentum conservation. In differential form, for a two-dimensional flow and with the flow weight as the only body force, the mass and linear momentum conservation equations are:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0
\]

\[
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2) + \frac{\partial}{\partial y} (\rho vu) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho g \sin (\theta)
\]

\[
\frac{\partial}{\partial t} (\rho v) + \frac{\partial}{\partial x} (\rho uv) + \frac{\partial}{\partial y} (\rho v^2) = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \rho g \cos (\theta)
\]

where \( \rho \) is the flow density. The stress tensor \( \sigma \) is a two-dimensional tensor of order two, written as:

\[
\sigma = \begin{pmatrix}
\sigma_{xx} & \tau_{xy} \\
\tau_{yx} & \sigma_{yy}
\end{pmatrix}.
\]

(1.4)

Conservation of angular momentum ensures that the shear stresses (off-diagonal components of the stress tensor) are equal, and thus \( \tau_{xy} = \tau_{yx} \).

We also make the assumption that the material is isotropic by equating the normal stresses (diagonal components of the stress tensor) \( \sigma_{xx} = \sigma_{yy} \). Using the shallow flow assumption and the related scaling, several terms in Eqs. (1.3) can be neglected. Eqs. (1.3) are adimensionalized according to the shallow flow scaling and only terms of order \( O(\varepsilon^0) \) or lower are kept. Eqs. (1.3) thus reduce to:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0
\]

\[
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2) + \frac{\partial}{\partial y} (\rho vu) = \frac{\partial \tau_{xy}}{\partial x} + \rho g \sin (\theta)
\]

\[
0 = \frac{\partial \sigma_{yy}}{\partial y} - \rho g \cos (\theta)
\]

(1.5)

Some authors make the choice to keep the term \( \partial \sigma_{xx}/\partial x \) in the \( x \)-component of the momentum equation although it is of order \( O(\varepsilon^1) \) and they introduce anisotropy by assuming active/passive states of stress (Savage and Hutter 1989, Gray et al. 1999). We note that if the shallow flow scaling is applied consistently, the isotropy assumption has no importance since the term
comprising $\sigma_{xx}$ is neglected. The $y$-component of the momentum equation implies that the normal stress is hydrostatic i.e. the normal stress is only determined by the flow weight. We stress that hydrostatic normal stress is a consequence of the shallow flow assumption (inertial terms on the left of the momentum equation in the $y$-direction are negligible). The system of Eqs. (1.5) is made of three equations and includes five unknowns, $u$, $v$, $\rho$, $\tau_{xy}$ and $\sigma_{yy}$. It must be completed by boundary and initial conditions that are consistent with the dry granular flow assumptions, in particular with the shallow flow assumption. The problem is underdetermined and two additional equations are required to close the system of equations. The additional equations are termed constitutive laws and are supposed to account for the material properties. One solution is to find two equations relating the density $\rho$ and the shear stress $\tau_{xy}$ to the other unknowns.

One special case of Eqs. (1.5) has been given much attention. In the steady, incompressible case, Eqs. (1.5) are rewritten as:

$$0 = \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin (\theta)$$

$$0 = \frac{\partial \sigma_{yy}}{\partial y} - \rho g \cos (\theta).$$

(1.6)

The system of Eqs. (1.6) is made of two equations and includes three unknowns $\rho$, $\tau_{xy}$ and $\sigma_{yy}$. The one constitutive law needed to close the system comes down to assigning a constant value to the density. If the value of the constant density is known, Eqs. (1.6) can be used to determine the shear and normal stress profiles: they are linear and proportional to each other by a factor $\tan (\theta)$. On the contrary, Eqs. (1.6) do not allow the flow velocities to be determined. Whereas the velocity component $v$ is equal to zero in agreement with the steady flow assumption, the profile of the velocity component $u$ is undetermined, or, in other words, any profile of the velocity component $u$ satisfies Eqs. (1.6). Only experimental data can be used to relate the velocity component $u$ with the flow unknowns, in the form of an additional (artificial) constitutive law (Pouliquen 1999, Aranson and Tsimring 2002).

1.1.9 Free surface flows

The flow free surface is the fraction of the flow total surface which is in contact with the atmosphere, whereas the rest of the flow total surface is in contact with solid boundaries. In the special case of two-dimensional flows,
the free surface can be reduced to the curve \( h - y = 0 \) where \( h = h(x, t) \) is the flow height. In the special case of two-dimensional chute flows, the flow total surface comprises the free surface, the flow surface in contact with the sidewalls and the flow surface in contact with the bed surface. The interaction of the flow with the sidewalls at \( z = 0 \) is neglected, i.e., the mean velocity component \( w \) and the spatial gradients in the \( z \)-direction of the normal and shear stresses are equal to zero. At the free and bed surfaces, mass conservation is accounted for by so-called kinetic boundary conditions:

\[
\left. \left( \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = v \right) \right|_{y=h} \quad (0 = v)|_{y=0}.
\]

(1.7)

Momentum conservation at the free and bed surfaces is accounted for by specifying boundary conditions for the stresses. At the free surface, a traction-free condition is generally assumed (no interaction with the atmosphere):

\[
(\sigma n = 0)|_{y=h}
\]

(1.8)

where \( n|_{y=h} = 1/\sqrt{(\partial h/\partial x)^2 + 1(-\partial h/\partial x)} \) is the unit vector perpendicular to the free surface (outgoing) for a two-dimensional flow. For shallow flows, \( n|_{y=h} = (0 \ 1) \) and the traction-free condition of Eq. (1.8) reduces to \( \tau_{xy}|_{y=h} = \sigma_{yy}|_{y=h} = 0 \).

At the bed surface, defined by the line \( y = 0 \), the normal stress is hydrostatic and a Coulomb friction law is generally used to relate the shear stress to the normal stress:

\[
\left. \left( \sigma n - n \cdot (\sigma n) \right) = \tan(\phi) \frac{u}{||u||} \left( n \cdot (\sigma n) \right) \right|_{y=0}
\]

(1.9)

where \( n|_{y=0} = (0 \ 1) \) is the unit vector perpendicular to the bed surface (outgoing) and \( \tan(\phi) \) is the basal coefficient of friction. The Coulomb friction law introduces a singularity for \( u|_{y=0} = (0 \ 0) \). However, for two-dimensional flows \( u|_{y=0} = (n|_{y=0}) \) (see Eq. (1.7)) and thus \( (u/||u||)|_{y=0} \) in Eq. (1.9) can be replaced by \( (1 \ 0) \).
1.1.10 Depth-averaging

A simplification of the system of Eqs. (1.5) consists in reducing the two-dimensional equations to one-dimensional equations. The density is assumed constant in the y-direction and Eqs. (1.5) are integrated with respect to y from the bed surface to the free surface and divided by the flow height \( h \). The (two-dimensional) flow variables are replaced by their corresponding depth-averaged (one-dimensional) flow variables. Let us consider the scalar \( a \). Its depth-averaged value \( \bar{a} \) is defined as:

\[
\bar{a} \equiv \frac{1}{h} \int_{0}^{h} a \, dy.
\] (1.10)

Eqs. (1.5) are depth-averaged using the Leibniz rule of integration, the kinetic boundary conditions from Eqs. (1.7), and the traction-free condition at the bed surface for shallow two-dimensional flows from Eq. (1.8) (for details of the calculation, see for example Savage and Hutter 1989):

\[
\begin{align*}
\frac{\partial}{\partial t} (h\rho) + \frac{\partial}{\partial x} (h\rho \bar{u}) &= 0 \\
\frac{\partial}{\partial t} (h\rho \bar{u}) + \frac{\partial}{\partial x} (h \rho \bar{u}^2) &= -\tau_{xy}|_{y=0} + h \rho g \sin (\theta) \\
0 &= -\sigma_{yy}|_{y=0} - h \rho g \cos (\theta).
\end{align*}
\] (1.11)

The number of equations is the same as in Eqs. (1.5) but there are six unknowns instead of five \( \bar{u}, \bar{u}^2, \rho, \tau_{xy}|_{y=0}, \sigma_{yy}|_{y=0} \) and \( h \) so that three constitutive laws are required to close the system. In practice, a constant value is assigned to the density and the equality \( \bar{u}^2 = \bar{u}^2 \) (shape factor \( \alpha \equiv \bar{u}^2/\bar{u}^2 \) equal to 1) make up two of the three constitutive laws (Voellmy 1955, Savage and Hutter 1989, Christen et al. 2010). The last constitutive law typically specifies the shear stress at the bed surface and thus coincides with the boundary condition at the bed surface. The system of Eqs. (1.11) is widely used to simulate geophysical flows. The reasons for it are that it can be easily implemented and efficiently solved numerically (Kowalski 2008). The increase of the number of unknowns is explained by the fact that some flow variables evaluated at the flow boundaries in the y-direction become unknowns in the depth-averaged system of equations (1.11).

1.1.11 Energy conservation

The system of Eqs. (1.5) can be generalized by considering an equation for conservation of energy. In fluid flows, the flow variable related to internal
energy is the molecular temperature. In granular flows, the molecular temperature does not play an important role (except for solid grains which are close to their melting point). Kinetic energy is dissipated into heat during grain contacts but the rise in molecular temperature does not affect the mechanical properties of the grains. In granular flows, the fluctuation energy that accounts for the fluctuations of the individual grain velocities is of interest. As a measure of the fluctuations of the individual grain velocities, the granular temperature $T$ is defined per analogy to the molecular temperature. The velocity of an individual grain is decomposed into a mean part and a fluctuating part which cancels over the average ensembles. The mean velocity components $u$ and $v$ identify with the mean of the individual grain velocities over the average ensembles, whereas the velocity fluctuation components $u'$ and $v'$ are equal to the standard deviation of the individual grain velocities over the average ensembles:

$$
\begin{align*}
  u_i &= u + u'_i \\
  v_i &= v + v'_i \\
  u &\equiv \langle u_i \rangle \\
  v &\equiv \langle v_i \rangle \\
  u' &\equiv \langle (u - u_i)^2 \rangle^{\frac{1}{2}} \\
  v' &\equiv \langle (v - v_i)^2 \rangle^{\frac{1}{2}}.
\end{align*}
$$

The granular temperature $T$ is defined as one half of the norm of the velocity fluctuation vector (for two-dimensional flows) and coincides with the fluctuation kinetic energy per unit mass (difference between total kinetic energy per unit mass and mean kinetic energy per unit mass):

$$
T \equiv \frac{1}{2} \left( u'^2 + v'^2 \right).
$$

The derivation of the fluctuation energy conservation equation is based on the splitting of the thermal and kinetic terms in the total energy equation. It can also be obtained from the Boltzmann equation for the single-particle distribution function (Jenkins and Richman 1985, Goldhirsch 2003). For two-dimensional, shallow flows, it can be written as:

$$
\begin{align*}
  \rho \frac{\partial T}{\partial t} + \rho \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \\
  \sigma_{yy} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \tau_{xy} \frac{\partial u}{\partial y} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho \Gamma
\end{align*}
$$

where $Q$ and $\Gamma$ are respectively the fluctuation flux vector and the decay coefficient. Eq. (1.14) states that the material derivative of fluctuation energy (time derivative plus convection) is equal to the sum of five terms. The
first and second terms (from left to right on the second line of Eq. (1.14)) account for the change in fluctuation energy associated with the work rate of the normal stress in case of volume change and the work rate of the shear stress in case of shear deformation. The third and fourth terms quantify the transport of fluctuation energy across the flow. The fifth term represents the dissipation of fluctuation energy related to grain contacts. The decay coefficient $\Gamma$ is not a constant since it must cancel for $T = 0$. The system of equations formed by Eqs. (1.5) and Eq. (1.14) is made of four equations and includes nine unknowns $u$, $v$, $\rho$, $\tau_{xy}$, $\sigma_{yy}$, $T$, $Q_x$, $Q_y$ and $\Gamma$. Five constitutive laws are needed to close the system in addition to initial and boundary conditions. Constitutive laws for the shear and normal stresses, the fluctuation flux vector and the decay coefficient have been derived for the collisional regime using the kinetic theory (Chapman and Cowling 1970, Jenkins and Richman 1985, Goldhirsch and Tan 1996). Attempts have also been made to derive constitutive laws for the density, the shear stress, the fluctuation flux vector and the decay coefficient for the collisional regime from heuristic arguments (Haff 1983, Jenkins and McTigue 1990, Bartelt et al. 2006). Experiments investigating the motion of vibrated grains at steady state (D’Anna et al. 2003) and during collapse (transition between dynamic and static states) (Son et al. 2008) have improved the understanding of fluctuation energy in granular systems. Given that the complexity of the problem is sensibly enhanced by including an energy equation and given that the knowledge of the granular temperature is generally not an end in itself, the use of a flow model that is based on an energy equation is relevant only if the coupling between the system of Eqs. (1.5) and the velocity fluctuations is demonstrated.

### 1.1.12 Other flow models

Until now, we have restricted ourselves to the discussion of dry granular flows, of their related assumptions and of their physical description. The energy equation has been mentioned as well as the simplifications in the cases of steady, incompressible flows or depth-averaged flows. In this section,
we give an overview of more complex flow problems and related theoretical approaches or experimental work.

- The generalization from two-dimensional to three-dimensional flows is straightforward, at least concerning the derivation of the system of equations. The implementation and the research of numerical solutions are more complicated, however.

- Another generalization concerns two- or three-dimensional flows over complex topography for which curvature effects come into play (Gray et al. 1999, Pudasaini and Hutter 2003, Fischer et al. 2012). The difficulty lies in the choice of the coordinate system and in the derivation of the governing equations in that coordinate system. One solution consists in superposing a three-dimensional, shallow topography over a two-dimensional plane surface. The inclination of the two-dimensional plane surface is chosen so that the three-dimensional topography satisfies the shallowness assumption (typical topography height $B \approx H \ll L$) and the related scaling.

- Granular flows subjected to heat transfer have been studied, mainly from a theoretical point of view (Massoudi 2006). Constitutive laws for heat conduction and radiant heating are derived and the system of equations including an energy equation (parameterized with the molecular temperature) are solved for different boundary conditions.

- Two-phase, gravitational, free surface flows including a solid granular phase and a liquid phase have been investigated (Iverson 1997, Kowalski 2008). The conservation equations are written for each phase assuming a homogenous mixture (constant phase volume fractions) and stress separation (no momentum exchange between phases). An additional equation is derived to account for non-hydrostatic liquid pressure. Non-homogenous mixtures over the flow height are modeled taking into account sedimentation and resuspension processes.

- Polydisperse granular flows have been investigated experimentally (Félix and Thomas 2004, Phillips et al. 2006, Moro et al. 2010). Segregation processes have been quantified as functions of the grain size and grain solid density ratios. The effect of polydispersity on the mobility of flows has also been studied. However a theoretical framework is lacking.

- Entrainment on a erodible bed surface and the effect on flow mobility has been simulated numerically (Mangeney et al. 2007).
Chapter 2

Methods

The primary objective of the experimental work presented in this thesis is to produce dry granular flows at laboratory scale and to instrument them in a non-invasive way in order to determine the mean velocity components $u$ and $v$, the density $\rho$ and the shear and normal stresses $\tau_{xy}$ and $\sigma_{yy}$, which are the model unknowns in Eqs. (1.5). Mean velocity components and density are directly measured, whereas shear and normal stresses are computed using the model equations. Flow height $h$ and velocity fluctuation components $u'$ and $v'$ are directly measured too. New methods or refinements of existing techniques for the measurement of mean velocity, velocity fluctuations and density are presented. The goal of our experimental setup is to provide a complete picture of unsteady, compressible, dry granular flows in the frictional-collisional regime.

2.1 Experimental setup

We perform chute flow experiments on a 2 m long, 0.2 m wide chute with variable inclination parameterized by the angle $\theta$ (see Fig. 1.2 and Fig. 2.1). Perpendicular to the bed surface, 50 cm high sidewalls made of Plexiglas bound the flow laterally. The bed surface is covered with sand paper with 0.1 mm (P150) roughness. At a 30 cm distance from the upper end of the chute, the grains are retained by a trap door. The granular material is released by rotating the door around an axis fixed at the top of the sidewalls. Small brushes are fixed along the perimeter of the door to prevent grains from flowing through.

We choose the reference frame from Fig. 1.2, with the $x$-axis parallel to the chute length and directed towards the lower end of the chute and the
Table 2.1: Properties of the glass beads. The diameter and roundness are specified by the manufacturer. The densities and angle of repose are measured independently.

<table>
<thead>
<tr>
<th>Property</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>1.4 mm (92.5% between 1.18 and 2 mm)</td>
</tr>
<tr>
<td>Roundness</td>
<td>92% perfectly spherical</td>
</tr>
<tr>
<td>Solid density</td>
<td>2400 kg/m³</td>
</tr>
<tr>
<td>Poured density</td>
<td>1590 kg/m³</td>
</tr>
<tr>
<td>Tapped density</td>
<td>1660 kg/m³</td>
</tr>
<tr>
<td>Angle of repose</td>
<td>25°</td>
</tr>
</tbody>
</table>

The granular material consists of glass beads with a mean diameter $d = 1.4$ mm. According to the manufacturer’s specifications, 92.5% of the glass beads have a diameter between 1.18 and 2 mm and at least 92% of the glass beads are spherical (see Tab. 2.1). The solid density of the glass beads is equal to $\rho_{\text{solid}} = 2400$ kg/m³. We distinguish between the poured density measured after pouring the material in a container (1590 kg/m³) and the tapped density measured after vibrating the container (1660 kg/m³).

Volumes of 30 l and 15 l of glass beads are released at four different chute inclinations $\theta$: 22°, 24.5°, 27° and 29.5°. The resulting flows are denoted by the letters A to H (see Tab. 2.2). The range of the inclination angle is centered around the angle of repose of the glass beads which is equal to 25°. At the location $x = 1.5$ m, a digital high-speed camera is set up perpendicularly to the chute. The high-speed camera is a HCC-1000 Vosskühler camera with a 25 mm focus lens and a 1 GB internal memory. For a 512 × 1024 pixels image resolution with a 8 bit grey scale (0 = black, 255 = white), the maximum frame rate is equal to 922 fps enabling a maximum 2.2 s recording time. In the present work, the frame rate is adapted to the flow velocity/duration and ranges between 184.6 fps for 22° flows and 461.5 fps for 29.5° flows. In order to provide sufficient lighting, two strobes are located on each side of the camera. The strobes are synchronised with the camera and deliver 0.1 ms long lighting pulses (Schaefer et al. 2010).
Table 2.2: Summary table of the flows denoted by the letters A to H. They differ in the starting volume of grains in liters and in the chute inclination in degrees.
2.1.1 Ensemble averaging

In accordance with the continuum assumption, the flow variables are averages of the properties of the individual grains. The average ensemble dimensions are defined so that 1) an average ensemble contains a large number of grains and 2) they are much smaller than the flow dimensions. Moreover, we choose the average ensemble dimensions so that the variation of the flow variables over the average ensembles has the same order of magnitude in the x-, y- and t-directions. This assumption is equivalent to stipulating that the average ensemble dimensions $\Delta x$, $\Delta y$ and $\Delta t$ are respectively of order $O(cL)$, $O(cH)$ and $O\left(c\sqrt{L/g}\right)$ where $c$ is an arbitrary constant, and thus that:

$$\Delta x \frac{\partial}{\partial x} \simeq \Delta y \frac{\partial}{\partial y} \simeq \Delta t \frac{\partial}{\partial t} = O(c). \quad (2.1)$$

In practice, the dimensions of the average ensembles are set to: $\Delta x = 1024$ pixels = 76 d, $\Delta y = 16$ pixels = 1.2 d and $\Delta t = 40$ frames = 0.09 to 0.22 s (depending on the frame rate). In comparison with the flow dimensions, $\Delta x$ and $\Delta y$ represent respectively 1/20 of the flow typical length $L$ and of the flow typical height $H$, whereas $\Delta t$ represents 1/30 of the typical flow duration (2 to 6 s). In terms of number of grains, approximately 100 grains are visible in a $\Delta x$ long and $\Delta y$ high flow region. Since one grain needs about $\Delta t$ to travel across $\Delta x$ and $\Delta y$ (according to the shallow flow scaling and the definition of the average ensembles), around 400 different grains participate in one average ensemble (grain fluxes across the flow region $\Delta x \Delta y$ in both x- and y-directions). We can also interpret the average ensemble dimensions in terms of grain “realizations” instead of a number of grains. If we assume that the velocities of a grain before and after a contact are not correlated (molecular chaos assumption), we can speak of different realizations of the same grain before and after a contact. Considering that the typical contact time $t_c \approx 1/20$ to $1/50$ s, an ensemble average is calculated over approximately 500 grain realizations ($76 \times 1.2 \times \Delta t/t_c$).

2.2 Flow height measurement

The flow height $h = h(t)$ at the location $x = 1.5$ m is determined from the high-speed camera recordings. We take advantage of the brightness difference between the flow and the background. In each frame, the first line of pixels (starting from the bed surface) with an average brightness less than a limit value of 40 determines the flow height. In Fig. 2.2, a frame from flow A at the instant $t = 5.4$ s is represented together with the green line indicating
the flow height for that frame. The flow height $h(t)$ is obtained by ensemble averaging over all frames belonging to $\Delta t$.

Figure 2.2: Frame from flow A together with the green line indicating the flow height for that frame.

### 2.3 Density measurement

In this section, we present the method used to determine the flow density from the high-speed camera recordings. The principle of the measurement is the following: in each frame, the positions of the visible grains are identified and the grain surface concentration in $1/m^2$ is computed over the average ensembles. The grain surface concentration is then transformed into a flow density in $kg/m^3$.

The glass beads are transparent and their image is the result of the reflection/refraction of the strobe light on/through the grains, the neighbouring grains and the Plexiglas sidewall. The image of one grain shows a complex texture made of patches of different brightnesses. The texture varies from one grain to another and depends on the grain location as well as on the arrangement of its neighbouring grains (see Fig. 2.3a). Whereas a human eye is able to put the brightness patches together to identify the individual grains, the task turns out to be much more complicated for a computer. We take advantage of a characteristic pattern formed by indirect reflections of the strobes light that is visible on each grain and which is easy to recognize in an automated way. The pattern is made of two reflections. They are about one pixel large and are separated by a distance of two to four pixels on a
horizontal line passing in the upper half of the grain (see Fig. 2.3b).

In a first step, pixels which are local brightness maxima are identified. To be categorized as a local brightness maximum, a pixel \( i \) with brightness \( b(i) \) (ranging between 0 and 255) and surrounding pixel brightnesses \( b(i,l) \) with \( l = 1, \ldots, 8 \) (see Fig. 2.4) has to fulfil the following conditions:

- The pixel \( i \) must satisfy \( b(i) < b_{\text{max}} \) with \( b_{\text{max}} = 130 \). Large size reflections from neighbouring grains (see Fig. 2.3b) can be brighter than the reflections of the characteristic pattern which we are looking for. The brightness of a pixel \( i \) with \( b(i) \geq b_{\text{max}} \) is set to 0.

- The pixel \( i \) must have a brightness \( b(i) \) larger than the brightnesses \( b(i,l) \) of either: its eight surrounding pixels, or 7 out of its 8 surrounding pixels with \( \text{nbd}(i) \geq -5 \), or 6 out of its 8 surrounding pixels with \( \text{nbd}(i) \geq -1 \). The parameter \( \text{nbd}(i) \) is defined as the sum of the negative brightness differences of pixel \( i \) with its surrounding pixels \( i, l \):

\[
\text{nbd}(i) = \sum_{b(i) - b(i,l) < 0} b(i) - b(i,l) \quad (2.2)
\]

In summary, the pixel \( i \) does not need to be brighter than all its surrounding pixels to be categorized as a local brightness maximum. It may have one or two equally bright or brighter surrounding pixels under the condition that they are not much brighter. Indeed, it often occurs that the approximately one pixel large reflections of the characteristic pattern are positioned over more than one pixel.

- The relative mean brightness difference \( \text{bd}(i) / b(i) \) must be larger than the minimum value \( (\text{bd}/b)_{\text{min}} = 0.06 \). The parameter \( \text{bd}(i) \) is defined as the mean brightness difference of pixel \( i \) with its surrounding pixels \( i, l \):

\[
\text{bd}(i) = \frac{1}{8} \sum_{l} b(i) - b(i,l) \quad (2.3)
\]

It ensures that pixel \( i \) is, on average, sufficiently brighter than its surrounding pixels.

In a second step, pairs of local brightness maxima which are likely to correspond to the characteristic patterns are identified. (1) Since the horizontal and vertical distances between the two reflections may vary by one or
Figure 2.3: Two enlargements of the frame from flow A at the instant $t = 5.4$ s are displayed twice. On the 1 grain enlargement, the reflections of the characteristic pattern are highlighted with green filled squares and a large size, bright reflection from a neighbouring grain visible in the left lower part of the grain is encircled by a green oval. On the 100 grains enlargement, the grain positions determined by the particle tracking method are indicated by green circles.
two pixels, we define seven possible pair types \( j_m \) with \( m = 1, \ldots, 7 \) (see Fig. 2.5). (2) We do not want that the algorithm identifies characteristic patterns within the large size, bright reflexions caused by neighbouring grains (see Fig. 2.3). We thus verify that the intermediate pixels \( k_n \) with \( n = 1, \ldots, N_m \) (see Fig. 2.5) of a potential pair of local brightness maxima are not part of a reflection from neighbouring grains by imposing that \( b(k_n) \neq 0 \). Once pairs of local brightness maxima \( \{i, j\} \) satisfying the two above conditions have been identified, we eliminate possible false pairs by keeping only one pair per circular surface \( \pi r_b^2 \). The radius \( r_b \) is set to 8.5 pixels = 0.63 \( d \) and therefore defines both the maximum possibly measurable grain surface concentration (hexagonal arrangement with distance 0.63 \( d \)) and the minimum possibly measurable grain surface concentration (hexagonal arrangement with distance 1.25 \( d \)). Indeed, a correct pair of local brightness maxima will not eliminate another correct pair located at a distance down to 0.63 \( d \), whereas a false pair located in between two correct pairs separated by a distance up to 1.25 \( d \) will be eliminated. In effect, the algorithm is able to measure lower grain surface concentrations since the surface concentration of false pairs of local brightness maxima is low in comparison to the surface concentration of correct pairs and since most false pairs are located within a grain i.e. close to a correct pair. The comparison between two pairs of local brightness maxima \( \{i_1, j_1\} \) and \( \{i_2, j_2\} \) located in the same circular surface \( \pi r_b^2 \) is made on the basis of the following criteria:

- The relative mean brightness differences between the local brightness maxima and their surrounding pixels \( b_d(i)/b(i) \) are examined. The pair of local brightness maxima with the minimum relative mean brightness difference is less likely to be the correct pair:

\[
\min \left( \frac{b_d(i_1)}{b(i_1)}, \frac{b_d(j_1)}{b(j_1)}, \frac{b_d(i_2)}{b(i_2)}, \frac{b_d(j_2)}{b(j_2)} \right). \tag{2.4}
\]
The relative mean brightness differences between the local brightness maxima and their intermediate pixels \( pbd(i, j) / (b(i) + b(j)) \) are compared where

\[
pbd(i, j) = \frac{1}{N_m} \sum_n b(i) + b(j) - 2b(k_n). \tag{2.5}
\]

The pair of local brightness maxima with the maximum relative mean brightness difference is more likely to be the correct pair:

\[
\max \left( \frac{pbd(i_1, j_1)}{b(i_1) + b(j_1)}, \frac{pbd(i_2, j_2)}{b(i_2) + b(j_2)} \right). \tag{2.6}
\]

Some pair types are more represented than others among correct pairs of local brightness maxima. Thus, the pair types are classified from one to seven according to their representation among the population of correct pairs. The pair of local brightness maxima with the lowest pair type is more likely to be the correct pair:

\[
\min (m_1, m_2). \tag{2.7}
\]

The brightness of the surrounding pixels of the local brightness maxima is examined to examine that they are not part of a reflection from neighbouring grains. More precisely, the number \( N \) per pair of local brightness maxima of surrounding pixels \( i, l \) and \( j, l \) which are reflections from neighbouring grains i.e. with \( b(i, l) = 0 \) or \( b(j, l) = 0 \) is compared. The pair of local brightness maxima with the minimum \( N \) is more likely to be the correct pair:

\[
\min (N_1, N_2). \tag{2.8}
\]

Each significant comparison between pairs of local brightness maxima (i.e. for which the difference between the tested parameters is significant, minimum relative difference values of 5% are used) returns one point to the pair which is more likely to be a correct pair. Eventually, the pair with the largest sum of points is kept. In case of equality between the sums of points, the first significant comparison in the above order is decisive.

The position of the grain is defined at the mid-distance between the pixels forming the pair of local brightness maxima. The grain surface concentration
Figure 2.5: Types of pair of local brightness maxima $j_m$ with $m = 1, ..., 7$ and intermediate pixels $k_n$ with $n = 1, 2$ and $n = 1, ..., 4$ for pair types $j_1$ and $j_3$ respectively.

is computed by ensemble averaging the number of grains per unit surface. For the density measurements, the size of the average ensembles in the $x$-direction $\Delta x$ is reduced from 1024 pixels to 512 pixels. Only the central half of the frames is used since the right/left strobe reflections in the left/right quarter of the frames do not always have a sufficient brightness. The surface density $\rho_s$ is obtained from the grain surface concentration by multiplication with the mean grain mass $\rho_{\text{solid}} \frac{4}{3} \pi (d/2)^3$. Finally, we compute the density $\rho$ from the surface density using the relation:

$$\rho = \rho_{\text{max}} \left( \frac{\rho_s}{\rho_{s,\text{max}}} \right)^{\frac{2}{3}}$$

(2.9)

where $\rho_{\text{max}} = \rho_{\text{solid}} \nu_{\text{max}}$ and $\rho_{s,\text{max}} = \rho_{\text{solid}} \nu_{s,\text{max}}$ are the densities corresponding the maximum packing concentrations in two- and three-dimensions. The maximum volume and surface fractions $\nu_{\text{max}}$ and $\nu_{s,\text{max}}$ are equal to 0.74 and 0.91 for a hexagonal arrangement of spheres. Eq. (2.9) is valid for any periodic arrangement of the grains under the condition that only one layer of grains is taken into account in the three-dimensional arrangement. In dilute flows, grains from several layers would be visible when filming through the sidewall (although the focus would not be good enough to distinguish
the characteristic pattern on grains which would be more distant from the sidewall). In the present flows, the grains are densely packed and only grains from the first layer are completely visible whereas grains from the second layer are only partly visible, reducing the probability that both reflections of the characteristic pattern are visible. Moreover, the comparison between pairs of local brightness maxima favours pairs on grains from the first layer whose reflections have sharper contrasts. The method is developed and validated over flow regions of approximately 100 grains by verifying that grains recognizable to the naked eye are identified by the algorithm.

2.4 Velocity measurement

In this section, we present the method used to measure mean velocity and velocity fluctuations over the flow height during the passing of the flow at the location $x = 1.5$ m using the high-speed camera recordings. The method is based on the pattern matching algorithm developed in Schaefer et al. (2010) for snow and glass beads in chute flows. Pattern matching algorithms distinguish themselves from particle tracking methods (Capart et al. 2002) in that they do not require the identification of the individual grains. For granular flows whose grains have a complex shape/image (for example snow or glass beads) or are of a size smaller than the camera resolution, pattern matching algorithms are of particular interest. The method takes advantage of the fact that the pattern of a flow region or equivalently the image of the grains in a flow region does not change significantly during a short time interval whereas its displacement during the same time interval is large enough to be measured with good accuracy. This condition is straightforward in a flow without material deformation. However, in a flow with material deformation, the time interval must be chosen carefully.

We determine the mean velocity of a flow region (probe window) in the time interval between two successive frames representing the flow at the instants $t_i$ and $t_{i+1}$ by computing the brightness two-dimensional cross-correlation function of the probe window in the frame at instant $t_i$ over a larger flow region (search window) in the frame at instant $t_{i+1}$. The position of the maximum of the cross-correlation function relative to the position of the probe window is equivalent to the displacement of the probe window in the time interval $t_{i+1} - t_i$ and can be converted into a two-dimensional velocity vector. The maximum of the two-dimensional cross-correlation function (represented by a surface in a three-dimensional space) is determined with sub-pixel accuracy. By repeating the operation for probe windows over the
whole flow region visible in the frames and for all pairs of successive frames of the flow recording, we obtain a discrete field of two-dimensional velocity vectors with components $u_i(x_i, y_i, t_i)$ and $v_i(x_i, y_i, t_i)$ which can be used to compute mean velocity and velocity fluctuations by ensemble averaging. The brightness two-dimensional cross-correlation function $f$ is written:

$$f(x_i, y_i, \Delta x_i, \Delta y_i, t_i) = \frac{1}{L_{pw}H_{pw}} \int_{-\frac{L_{pw}}{2}}^{\frac{L_{pw}}{2}} \int_{-\frac{H_{pw}}{2}}^{\frac{H_{pw}}{2}} b(x_i + x', y_i + y', t_i) b(x_i + \Delta x_i + x', y_i + \Delta y_i + y', t_{i+1}) \, dx' \, dy'$$  

(2.10)

where $x_i$ and $y_i$ are the coordinates of the probe window in the frame at instant $t_i$, $\Delta x_i$ and $\Delta y_i$ are the distances in the $x$- and $y$-directions between the probe window in the frame at instant $t_i$ and the probe window in the search window in the frame at instant $t_{i+1}$, $x'$ and $y'$ are the coordinates used to skim through the probe window and $L_{pw}$ and $H_{pw}$ are respectively the length and the height of the probe window. The $\Delta x_i$ and $\Delta y_i$ values which maximize the cross-correlation function $f$ are interpreted as the displacement of the probe window during the time interval $t_{i+1} - t_i$.

![Figure 2.6: Probe window with fixed search window.](image)

### 2.4.1 Fixed search window

In a first step, we compute velocities using a probe window size of $16 \times 64(128)$ pixels equivalent to $1.2 \times 4.7(9.4)$ $d$. The search window has a size
48 × 128(256) pixels and is positioned with its left side aligned with the left side of the probe window and is centered around the probe window in the \( y \)-direction (see Fig. 2.6). The size and the position of the search window determine the minimum and maximum velocities possibly measurable by the pattern matching algorithm. If the position of the search window (relative to the probe window) is fixed for all flow regions and for the whole duration of the flow, the size of the search window must be large enough so that the minimums and maxima of both \( u_i \) and \( v_i \) velocity components can be measured. However, small probe windows combined with large search windows increase the probability of false velocities (maxima of the cross-correlation function which do not correspond to a physical displacement of the flow region visible in the probe window). If more than one grain is visible in the probe window, the velocity obtained using the pattern matching algorithm is a measure of the mean of the velocity of the individual grains visible in the probe window. The mean velocity components \( u \) and \( v \) computed over an average ensemble are mean values of the velocity components \( u_i \) and \( v_i \) and coincide with the mean values of the individual grains velocities. However, the same is not true for the velocity fluctuation components \( u' \) and \( v' \). If more than one grain is visible in the probe window, the standard deviation of the velocity components \( u_i \) and \( v_i \) over an average ensemble does not coincide with the velocity fluctuation components \( u' \) and \( v' \) defined in Eqs. (1.12).

Figure 2.7: Probe window with moving search window.

### 2.4.2 Moving search window

In order to measure velocity fluctuations correctly, we compute velocities a second time using a smaller probe window size of 16 × 16 pixels equivalent
to $1.2 \times 1.2 d$ and a moving search window with size ranging between $68 \times 68$ pixels for flows A and E and $38 \times 38$ pixels for flows D and H (see Fig. 2.7). The position of the search window is centered around the displacement determined in the first step with the $16 \times 64(128)$ pixels probe window and the fixed search window. The size of the search window corresponds to a velocity range of 0.3 m/s for flows A-C and E-G and of 0.5 m/s for flows D and H and is chosen on the basis of the order of magnitude of the velocity fluctuations. The use of the pattern matching algorithm with moving search windows allows the size of the search window to be reduced (saves computational time), and limits the occurrence of false velocities.

In order to improve the accuracy of the pattern matching algorithm, we apply image processing methods:

- Reflections from neighbouring grains characterized by large brightness values appear and disappear very rapidly due to slight changes in the grain orientations. We thus replace brightness values larger than 200 by the mean brightness value of the frame (neutral weight in the computation of the cross-correlation function).

- Contrasts are enhanced using a Contrast-Limited Adaptive Histogram Equalization method (MATLAB routine, Zuiderveld 1994) which divides the frames in small domains and flattens their brightness histograms. The small regions are put back together and the modifications of the histograms are smoothed between neighbouring small regions to avoid large discontinuities.

- The mean brightness computed over all frames belonging to the same average ensemble (a $512 \times 1024$ matrix of brightness values) is subtracted from all frames of the average ensemble.

- Normalized brightnesses over the probe window are used in the computation of the cross-correlation function. The product of brightnesses $b(x_i + x', y_i + y', t_i)$ $b(x_i + \Delta x_i + x', y_i + \Delta y_i + y', t_{i+1})$ in Eq. (2.10) is replaced by:

$$\frac{[b(x_i + x', y_i + y', t_i) - \bar{b}(x_i, y_i, t_i)]}{\sigma_b(x_i, y_i, t_i)}$$

$$\frac{[b(x_i + \Delta x_i + x', y_i + \Delta y_i + y', t_{i+1}) - \bar{b}(x_i + \Delta x_i, y_i + \Delta y_i, t_{i+1})]}{\sigma_b(x_i + \Delta x_i, y_i + \Delta y_i, t_{i+1})}$$

(2.11)
where $\bar{b}$ is a mean brightness computed over the probe window and $\sigma_b$ is the standard deviation of the brightness over the probe window.

In Figs. 2.8 and 2.9 histograms of the velocities $u_i$ and $v_i$ from different average ensembles over the flow height for flows A and D are plotted. The limits of the histograms are explained by the finite size of the search window which determine minimum and maximum possibly measurable velocities. The monotonous decrease of the histograms at their limits results from the cross-correlation of probe windows which only partly overlap with the search window. In that case, the cross-correlation function value is computed over a surface smaller than the probe window surface and is less likely to be a maximum of the cross-correlation function. The histograms show a main peak which corresponds to the velocity spectrum centered around the mean velocity and two secondary peaks surrounding the main peak at a velocity interval between 0.2 and 0.5 m/s. The secondary peaks are explained by false velocities resulting from the cross-correlation of a grain visible in the probe window in the frame at instant $t_i$ with a neighbouring grain in the frame at instant $t_{i+1}$, i.e. a neighbouring grain behind or in front for the $u_i$ velocity component and below or above for the $v_i$ component. The velocity intervals between the main peak and the secondary peaks are determined by the mean distance between neighbouring grains and by the frame rate. Following the above explanation, one would expect that the histograms cancel between the main and the secondary peaks. In effect, neighbouring grains below or above (for the $u_i$ velocity component) and behind or in front (for the $v_i$ component) are also responsible for false velocities. These false velocities do not form secondary peaks, but are homogeneously distributed over the histogram velocity range.

In summary, not all velocities $u_i$ and $v_i$ represented in the histograms in Figs. 2.8 and 2.9 correspond to the physical displacement of a grain during the time interval between two frames. We assume that the velocities of the main peak are correct and exclude non-physical velocities by truncating the histograms around the main peak. We identify the main peak by looking for the histogram maximum and we truncate the histogram at the limit velocities $u_{inf}$, $u_{sup}$ and $v_{inf}$, $v_{sup}$ respectively (green dashed lines in Figs. 2.8 and 2.9). The limit velocities are determined as the first velocity values left and right of the histogram maximum that are lower than the histogram median value (red line in Figs. 2.8 and 2.9). We assume that the truncated histogram identifies with the truncated velocity spectrum, allowing the computation of mean velocities and velocity fluctuations, see Eq. (1.12). To recover the truncated parts of a spectrum, we fit it with a normal distribution (pink dashed
Figure 2.8: Histograms of the velocities $u_i$ and $v_i$ from different average ensembles taken over the flow height for the flow A at instant $t = 5.4$ s. The indices 1 to 4 refer to the different heights of the average ensembles regularly spaced from the free surface to the bed surface.
Figure 2.9: Histograms of the velocities $u_i$ and $v_i$ from different average ensembles taken over the flow height for the flow D at instant $t = 1.3$ s. The indices 1 to 4 refer to the different heights of the average ensembles regularly spaced from the free surface to the bed surface.
line in Figs. 2.8 and 2.9), directly yielding the mean velocity (distribution mean) and velocity fluctuations (distribution standard deviation).

Figure 2.10: Illustration of a cross-correlation function $f$ as the sum of two local maxima: in the case of close and steep maxima (left) and in the case of close but not steep maxima (right) and the consequences for the maximum(s) of the sum. The cross-correlation functions that we compute are not two-dimensional curves but three-dimensional surfaces.

If false velocities are homogeneously distributed over the velocity range, they are not a source of error for the mean velocity values but only for the velocity fluctuations. The main peaks of the histograms are less continuous in flow regions with large time derivatives and spatial gradients of the mean velocity components, resulting in less accurate truncation of the histograms. This effect is not due to the failure of the cross-correlation method but instead due to grains which experience a contact in the time interval between two frames. The use of the pattern matching algorithm supposes that the time interval between two frames is smaller than the typical contact time $t_c$, so that a grain visible in a frame at instant $t_i$ is only rarely involved in a contact in the time interval $t_{i+1} - t_i$. In the present work, $t_c$ is about ten times larger than the inverse of the frame rate. However, in flow regions with large time derivatives and spatial gradients of the mean velocity components, the $t_c$ values are smaller and a larger proportion of grains experience a contact in the time interval between two successive frames. This explains why the histograms are less continuous at large deformation rates. A solution consists in enlarging the dimensions of the average ensembles in these particular flow regions. The image of a grain (reflection/refraction pattern) is principally determined by its distance from the sidewall and by the arrangement of its neighbouring grains. For the pattern matching algorithm to be accurate, it is important that the image of a grain does not change too much in the
time interval between two successive frames. Although the expression for the contact time $t_c$ is only an approximation (valid in the collisional regime), we use the $t_c$ values as a measure of the typical time during which changes of the grain image become significant.

Another source of error is related to the size of the probe window. It is likely that more than one grain is visible in a $1.2 \times 1.2 \ d$ probe window. If two grains with different velocities are partially visible in the same probe window, we expect the cross-correlation function to exhibit two local maxima. If the local maxima are steep or far from each other, their sum will yield the same two local maxima. In that case, the algorithm will select the largest of the two local maxima (global maximum) and no error will be introduced. On the other hand, if the local maxima are not steep or close to each other, the sum will exhibit only one maximum which lies in between the two initial local maxima. Again, the mean velocities are not affected by such errors whereas the velocity fluctuations are underestimated. In Fig. 2.10, we illustrate a case in which two cross-correlation function maxima have a similar magnitude and shape. In practice, one grain is very often more visible than the other one resulting in local maxima with different amplitude, thus reducing the probability of error.

A last source of error concerns non-zero time derivatives and spatial gradients of the mean velocities. The velocity fluctuations are defined as the standard deviation of the individual grain velocities from the mean velocity, the latter being defined as the mean of the individual grain velocities, Eq. (1.12). A consequence of these definitions is that errors in the measurement of the individual grain velocities which have opposite sign add up in the calculation of the velocity fluctuations whereas they cancel out in the calculation of the mean velocity. In particular, non-zero time derivatives and spatial gradients of the mean velocity components $u$ and $v$ over the average ensembles result in spurious contributions to the velocity fluctuations. These unwanted contributions can be computed under the condition that the time derivatives and spatial gradients are known; see Drake (1991) for the one-dimensional case. In the following, we compute the unwanted contributions for the $u'$ component of the velocity fluctuations assuming that the time derivatives and spatial gradients are constant over the average ensembles. In the presence of non-zero time derivatives and spatial gradients of the mean velocities, the individual grain velocity component $u_i$ in the average ensemble centered around $x_i, y_i$ and $t_i$ is:
\[ u_i = u + u'_i + x_i \frac{\partial u}{\partial x} + y_i \frac{\partial u}{\partial y} + t_i \frac{\partial u}{\partial t} \quad (2.12) \]

with \(-\Delta x/2 < x_i < \Delta x/2, -\Delta y/2 < y_i < \Delta y/2\) and \(-\Delta t/2 < t_i < \Delta t/2\).

We also assume that the fluctuating parts \(u'_i\) and \(v'_i\) do not depend on the coordinates \(x_i, y_i\) and \(t_i\) i.e. that the averages \(\langle u'_i x_i \partial u/\partial x \rangle\), \(\langle u'_i y_i \partial u/\partial y \rangle\) and \(\langle u'_i t_i \partial u/\partial t \rangle\) are equal to zero. Using Eq. (2.12), the velocity fluctuation component \(u'\) is equal to:

\[
\begin{align*}
\langle u'_i \rangle &= \left( \frac{1}{\Delta x \Delta y \Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} \int_{-\Delta t/2}^{\Delta t/2} \frac{(u - u_i)^2 \, dx \, dy \, dt}{(u'_i + x_i \partial u/\partial x + y_i \partial u/\partial y + t_i \partial u/\partial t)^2} \right)^{1/2} = \\
&= \left( \frac{1}{\Delta x \Delta y \Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_{-\Delta y/2}^{\Delta y/2} \int_{-\Delta t/2}^{\Delta t/2} \left( u'_i + x_i \partial u/\partial x + y_i \partial u/\partial y + t_i \partial u/\partial t \right)^2 \, dx \, dy \, dt \right)^{1/2} = \\
&= \left( \langle u'^2_i \rangle + \frac{\Delta x^2}{12} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\Delta y^2}{12} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\Delta t^2}{12} \left( \frac{\partial u}{\partial t} \right)^2 \right)^{1/2}. \quad (2.13)\end{align*}
\]

The unwanted contribution to the velocity fluctuations is made up by the three last terms in Eq. (2.13). We subtract it from our velocity fluctuations measurements.

2.5 Validation of model assumptions

The objective of the present section is to evaluate to what extent the flows investigated meet the definition of dry granular flows. We discuss the validity of the two-dimensional, single-phase, compressible and shallow flow assumptions. We also characterize the flow regime on the base of the inertial number.

2.5.1 Two-dimensional flow

The two-dimensional character of the flow (i.e. the variation of the flow variables in the z-direction) is difficult to assess since we only measure the mean velocity at the sidewall. Camera recordings of the free surface cannot be used to measure the surface mean velocity since the flow height is not constant and the images are not focused. An alternative solution consists in filming the flow through the bed surface requiring a transparent (i.e. smooth) bed surface. A smooth bed surface though reduces the possibility to achieve a
two-dimensional, shallow flow since it results in much smaller variations of the flow variables in the x- and y-direction (small shear stress at the bed surface) whereas the variations of the flow variables in the z-direction are the same. To estimate the interaction between the flow and smooth sidewalls, we refer to experiments investigating sidewall effects in steady granular flows with constant flow height. In Savage (1979), glass beads with a mean diameter of 0.5 mm are used. The chute is 10.8 cm wide and has smooth sidewalls and bed surface (aluminium). They measure surface mean velocity over the flow width using a camera and tracers (coloured glass beads). Surface mean velocity profiles over the flow width for two different flows are reported, the first one with 18.8° chute inclination and 1 cm (20 d) flow height and the second one with 20.2° chute inclination and 0.65 cm (13 d) flow height. In Jop et al. (2005), glass beads with a mean diameter of 0.53 mm are used. The chute has glass sidewalls and an erodible bed surface made of the same granular material. From camera recordings of the free surface, they measure surface mean velocity profiles over the flow width for various flow widths and flow rates. We want to know if the variations of the surface mean velocity in the z-direction scale with the inverse of the typical flow width $\partial/\partial z = O(1/W)$ or if they are smaller, in which case the two-dimensional flow assumption is verified. We compute typical values of the gradient in the z-direction from the variation of the surface mean velocity over the flow width (in a chute flow, the typical flow width equals the constant flow width):

$$\frac{\partial}{\partial z} \approx \frac{1}{W} \frac{u(z = W/2) - u(z = 0)}{u(z = W/2)}.$$  \hspace{1cm} (2.14)

We obtain values of the gradient in the z-direction which are always smaller than 1/W, by a factor 1.15 for slow flows (0.07 m/s maximal surface velocity) to a factor 10 for fast flows (0.77 m/s maximal surface velocity) in Jop et al. (2005) and by factors 26 and 18, respectively, for the slower and faster flows in Savage (1979) (see Fig. 2.11). In Savage (1979), the factor values correlate differently with the maximum surface velocity indicating that other flow variables may influence the flow/sidewall interaction. However, only two data points are available in a velocity range which is different than in Jop et al. (2005). Since we have no information about the variation of the velocity over the flow width below the free surface, we cannot ensure that the two-dimensional flow assumption is verified in all flow regions. We can only say that the two-dimensional assumption is valid at the free surface for fast flows.
2.5.2 Single-phase flow

The flow density is not constant and ranges between 450 and 1350 kg/m$^3$ or equivalently between 0.19 and 0.56 in terms of volume fraction. Other experimental work investigating dry granular flows in the frictional-collisional regime report volume fractions between 0.02 and 0.66 for chute inclinations between 27$^\circ$ and 37$^\circ$ (steady flows) (Ancey 2001). The density is undoubtedly not constant and density time derivatives and spatial gradients must be accounted for. In Drake (1991), rates of energy loss per grain due to air drag and to grain contacts are estimated for one grain wide flows with surface fractions ranging between 0.02 and 0.65. It is shown that the rate of energy loss due to air drag is negligible, i.e. smaller than the rate of energy loss due to grain contacts by a factor 10 or more, if the surface fraction is more than or equal to 0.25-0.46. It is also shown that the rate of energy loss due to air drag is not negligible, i.e. of the same order of magnitude than the rate of energy loss due to grain contacts, if the surface fraction is less than or equal to 0.1-0.08. Thus we conclude that the single-phase assumption is valid for most of the density range investigated in this work.

2.5.3 Shallow flow

The typical flow length $L$ is similar at all chute inclinations for a given starting volume and is equal to approximately 2 m for flows A-D. We assign to the typical flow height $H$ the maximum flow height value. It ranges between
3.5 cm for flow A and 4 cm for flow D. The shallowness parameter \( \varepsilon \) is equal to 2\%, fulfilling the shallow flow criterion for the flow geometry. Next, we verify the scaling of the flow variables from Eqs. (1.2). In Fig. 2.12, we plot the ratio of the velocity components \( \nu/u \) and the ratio \( \partial u/\partial x (\partial u/\partial y)^{-1} \) of the spatial gradients of \( u \) at the location \( x = 1.5 \) m during the flows A and D. According to the scaling, both ratios shall be of order \( O(H/L) \) i.e. we expect them to be close to 2\% (or generally much smaller than 1).

Figure 2.12: Shallow flow scaling for the ratio of the velocity components and the ratio of the spatial gradients of \( u \) for flows A and D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that \( \nu/u \) is respectively less than or equal to \(-0.1\) and more than or equal to \(0\) and that \( \partial u/\partial x (\partial u/\partial y)^{-1} \) is respectively less than or equal to \(0\) and more than or equal to \(0.1\).

We observe that for both flows A and D, the absolute values of the ratios
are close to 2% in the front, in the tail and close to the bed surface. In the flow body, close to the free surface, the absolute values of the ratios are somewhat larger. Generally, the absolute values of the ratios are always smaller than 10%, verifying the shallow flow scaling. We have not observed any significative deviation from shallow flow scaling in the experiments.

2.5.4 Frictional-collisional regime

A flow regime is related to a particular region of the flow variable space and a predominant type of contact between grains. Since we do not have access to the details of the individual grain contacts, we concentrate on delineating the regions of the flow variable space associated with the different flow regimes. If the interstitial fluid is air, the transition between the collisional and frictional flow regimes is similar to the one between significant and negligible air drag in the volume fraction range between 0.08 and 0.46. In the following, we compute the inertial number $N_I$ at the chute location $x = 1.5$ m during the passing of flows A and D (Fig. 2.13). The inertial number ranges between 0.02 and 0.5 and is larger for faster flows. It is highest in the flow front and close to the free surface and lowest in the flow body, close to the bed surface. In agreement with the volume fraction range, the flows investigated in this work are not collisional but range over frictional ($N_I \ll 1$) and frictional-collisional regimes ($N_I \approx 1$). GDR Midi (2004) reports on chute experiments with glass beads in the dense inertial regime with depth-averaged inertial number $\bar{N}_I$ values ranging between 0.05 and 0.35. They assume the transition to the frictional regime, characterized by shear-rate independent shear stress, around an $\bar{N}_I$ value of 0.01.
Figure 2.13: Inertial number $N_I$ for flows A and D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that the inertial number is less than or equal to 0.025 and more than or equal to 0.5, respectively.
Chapter 3

Results

Results for flows A-H are shown. The flow variables are either directly measured (mean velocity components $u$, $v$ and density $\rho$) or calculated using the model Eqs. (1.5) (internal stresses $\tau_{xy}$ and $\sigma_{yy}$). The computation of the internal stresses is possible since there are three equations and only two unknowns which are not directly measured. Flow height $h$, shear rate $\partial u/\partial y$ and velocity fluctuation components $u'$ and $v'$ are also presented. The flow variables are measured over the flow height at the chute location $x = 1.5$ m during the passing of the flows.

3.1 Flow height

The finite size of the granular flows is depicted in the flow height measurements (Fig. 3.1). At the very tail of flows (principally at large chute inclination), the free surface is very dilute (bouncing grains with low density) and the flow height is not well defined. Thus, the flow height time series is truncated where the brightness limit method is not able to determine the flow height any more. The maximum flow heights are larger for faster flows and

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$, $v$</td>
<td>Measured</td>
</tr>
<tr>
<td>$\tau_{xy}$, $\sigma_{yy}$</td>
<td>Calculated</td>
</tr>
<tr>
<td>$h$</td>
<td>Measured</td>
</tr>
<tr>
<td>$u'$, $v'$</td>
<td>Measured</td>
</tr>
<tr>
<td>$\partial u/\partial y$, $\partial u/\partial x$, $\partial v/\partial y$</td>
<td>Calculated</td>
</tr>
</tbody>
</table>

Table 3.1: Summary table of flow variables.
range between 3.3 and 4 cm for flows A-D and between 2.5 and 3 cm for flows E-H. The differences in mean velocity and density among the flows explain that for the same starting volume, the surfaces below the flow height curves are not the same (the two-dimensional mass flow rate in the \(x\)-direction in kg/(m·s) is given by \(\int_0^h \rho u \, dy\)).

![Figure 3.1: Flow height versus time at the chute location \(x = 1.5\) m for flows A-D and E-H.](image)

Figure 3.1: Flow height versus time at the chute location \(x = 1.5\) m for flows A-D and E-H.

### 3.2 Density

The density is represented for flows A-D in intensity maps (Fig. 3.2). The mean density over the flow height is plotted for flows A-D and E-H to facilitate comparison between flows and flow regions (Fig. 3.3). We make the following observations:

- The density is always smaller than the static poured density which is equal to 1590 kg/m\(^3\).
- The density is larger for slower flows.
- The density is smaller in the flow fronts and larger in the flow body and in the flow tail. In the very tail of the flows, the density is very small.
- The density is large close to the bed surface and decreases towards the free surface.
• Close to the free surface, the density increases again, in particular in the flow tail of slower flows. This is counterintuitive and is discussed further in chapter 4, Discussion.

• The density is not influenced much by the starting volume.

![Figure 3.2: Maps of the flow density versus time at the chute location $x = 1.5$ m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that the density is less than or equal to 1000 kg/m$^3$ and more than or equal to 1500 kg/m$^3$, respectively.](image)

### 3.3 Velocity

The mean velocity components $u$ and $v$ for flows A-D are represented in intensity maps (Figs. 3.4 and 3.5). The means of the mean velocity com-
Figure 3.3: Mean flow density over the flow height versus time at the chute location $x = 1.5$ m for flows A-D and E-H. The dilute flow tails are not represented.

The absolute mean velocity components over the flow height are plotted for flows A-D and E-H to facilitate comparison between flows and flow regions (Fig. 3.6). We observe the following:

- The absolute mean velocity components are higher for larger chute inclinations.
- The absolute mean velocity components decrease during the passing of the flow.
- The absolute mean velocity components $u$ and $v$ increase from the bed surface to the free surface. Close to the free surface, the absolute mean velocity component $v$ decreases slightly. It indicates that the gradient $\partial v/\partial y$ changes its sign and contributes to a positive velocity divergence (decrease in density) close to the free surface.
- The absolute mean velocity component $v$ is larger for flows A-D (factor 1.3) in agreement with the shallow flow scaling.

### 3.4 Velocity fluctuations

The velocity fluctuation components $u'$ and $v'$ for flows A-D are represented in intensity maps (Figs. 3.7 and 3.8). The mean velocity fluctuation components over the flow height are plotted for flows A-D and E-H to facilitate
Figure 3.4: Intensity maps of the mean velocity component $u$ versus time at the chute location $x = 1.5$ m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that $u$ is less than or equal to 0 m/s and more than or equal to 2.5 m/s, respectively.
Figure 3.5: Intensity maps of the mean velocity component $v$ versus time at the chute location $x = 1.5$ m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that $v$ is less than or equal to $-0.1$ m/s and more than or equal to 0 m/s, respectively.
Figure 3.6: Mean of the mean velocity components $u$ and $v$ over the flow height versus time at the chute location $x = 1.5$ m for flows A-D and E-H. The dilute flow tails are not represented.
comparison between flows and flow regions (Fig. 3.9). We make the following observations:

- The velocity fluctuations are larger for faster flows.
- Velocity fluctuations are large in the flow fronts. They are small in the flow body close to the free surface and in the flow tail close to the bed surface. They are large again in the very tail of the flows.
- The velocity fluctuation components \( u' \) and \( v' \) have the same order of magnitude (isotropic velocity fluctuations). Their order of magnitude is also similar to the one of the mean velocity component \( v \). Thus, the velocity fluctuations scale like \( \varepsilon u \) with \( \varepsilon = 2\% \).
- The velocity fluctuations are underestimated (in particular the component \( u' \)) in flow regions with large time derivative and spatial gradients of the mean velocity components. In these flow regions, the correction of the unwanted contributions due to non-zero time derivative and spatial gradients results in negative velocity fluctuations. Negative velocity fluctuations are not represented in Figs. 3.7, 3.8 and 3.9.
- The means of the velocity fluctuation components over the flow height do not vary significantly during the passing of the flows in comparison to the means of \( u \) and \( v \) over the flow height or to the mean of \( \partial u/\partial y \) over the flow height.
- The velocity fluctuations are not influenced much by the starting volume.

### 3.5 Shear rate

The shear rate \( \partial u/\partial y \) versus time derived from the mean velocity component \( u \) for flows A-D is represented in intensity maps (Fig. 3.10). The mean shear rate over the flow height is plotted for flows A-D and E-H to facilitate the comparison between flows and flow regions (Fig. 3.11). The shear rate is represented in detail since it is the largest material deformation. We observe the following:

- The shear rate is always positive i.e. the mean velocity component \( u \) increases monotonously from the bed surface to the free surface.
- The shear rate is lower for slower flows.
Figure 3.7: Intensity maps of the velocity fluctuation component $u'$ versus time at the chute location $x = 1.5$ m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that $u'$ is less than or equal to 0 m/s and more than or equal to 0.08 m/s, respectively.
Figure 3.8: Intensity maps of the velocity fluctuation component $v'$ versus time at the chute location $x = 1.5$ m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that $v'$ is less than or equal to 0 m/s and more than or equal to 0.08 m/s, respectively.
Figure 3.9: Mean velocity fluctuation components $u'$ and $v'$ over the flow height versus time at the chute location $x = 1.5$ m for flows A-D and E-H. If velocity fluctuations are negative, the mean velocity fluctuations over the flow height are not represented.
• The shear rate is higher in the flow fronts and lower in the flow body and in the flow tail. In the very tail of the flows, the shear rate is high again.

• High shear rates are measured close to the free surface in the flow tail, in particular for slower flows.

• The shear rate is higher for flows E-H (factor 1.3) in agreement with the shallow flow scaling.

Figure 3.10: Intensity maps of the shear rate versus time at the chute location \( x = 1.5 \) m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that the shear rate is less than or equal to 0 1/s and more than or equal to 80 1/s, respectively.
3.6 Normal stress

The normal stress $\sigma_{yy}$ is not directly measured but it is integrated using the momentum equation in the $y$-direction in Eqs. (1.5) and the measured density. We present the normal stress for flows A-D in intensity maps. The normal stress ranges between 0 and 400 Pa for flows A-D and between 0 and 300 Pa for flows E-H and monotonously increases from the free surface to the bed surface. The mean normal stress over the flow height is plotted for flows A-D and E-H versus time in Fig. 3.13. Lower density for faster flows explains that normal stress maxima are almost equal for all flows whereas flow height maxima are not.

3.7 Shear stress

The shear stress $\tau_{xy}$ is not directly measured (see Platzer et al. 2007) but instead is integrated from Eqs. (1.5). The mass equation is multiplied by $u$ and substituted in the left hand-side of the momentum equation in the $x$-direction:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau_{xy}}{\partial y} + \rho g \sin(\theta).$$  (3.1)

The shear stress is integrated from Eq. (3.1) knowing the flow density, the mean velocity components $u$ and $v$ and the time derivative and spatial

Figure 3.11: Mean shear rate over the flow height versus time at the chute location $x = 1.5$ m for flows A-D and E-H. The dilute flow tails are not represented.
Figure 3.12: Intensity maps of the normal stress versus time at the chute location $x = 1.5$ m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that the normal stress is less than or equal to 0 Pa and more than or equal to 400 Pa, respectively.
Figure 3.13: Mean normal stress over the flow height versus time at the chute location $x = 1.5$ m for flows A-D and E-H. The dilute flow tails are not depicted.

gradients of $u$. We present the shear stress for flows A-D in intensity maps (Fig. 3.14). The mean shear stress over the flow height is plotted for flows A-D and E-H versus time (Fig. 3.15). In contrast to the normal stress, the maxima of the mean shear stress over the flow height are not equal for all flows. This indicates that, at constant normal stress, the shear stress is larger (in absolute terms) for faster flows. It is also larger in the flow fronts than in the flow body and tail.
Figure 3.14: Intensity maps of the shear stress versus time at the chute location $x = 1.5$ m for flows A-D. The same scale is used for all flows to facilitate comparison. Dark blue and dark red flow regions indicate that the shear stress is less than or equal to $-250$ Pa and more than or equal to $0$ Pa, respectively.
Figure 3.15: Mean shear stress over the flow height versus time at the chute location $x = 1.5$ m for flows A-D and E-H. The dilute flow tails are not depicted.
Chapter 4
Discussion

The data sets presented in the previous chapter serve as a basis for the search of constitutive laws for dry granular flows. We investigate the dependences of the flow density and of the shear stress on both the shear rate and the normal stress. The correlations with the shear rate and with the normal stress are tested separately by considering groups of data points with one of the two variables kept constant. We also evaluate the dependence of the velocity fluctuations on the shear rate and on the normal stress and discuss the relevance of an energy equation. The choice of the shear rate and of the normal stress as variables of the constitutive laws is motivated by: (1) the observation of the data, (2) existing theories or models and (3) practical reasons detailed in the following. The number of variables of the constitutive laws is limited first by the number of data points and second by their accuracy. The dependences of density and shear stress on the velocity divergence $\partial u/\partial x + \partial v/\partial y$ and on the granular temperature $T$ would be of interest. However, the uncertainties related to the measurement of velocity fluctuations and to the differentiation of the mean velocity components make it difficult to perform the analysis systematically for all four variables. Despite this, mean velocity fluctuations over the flow are included in the constitutive law for the shear stress. Finally, experimental data and models from the literature are presented and the agreement with our results is discussed.

4.1 Density

We evaluate the correlation of density with the shear rate $\partial u/\partial y$ and normal stress $\sigma_{yy}$. We partition the density data $\rho(y,t)$ (Fig. 3.2) in groups of data points with equal shear rate (shear rate intervals of 5 1/s) and plot the groups of data points individually versus the normal stress (Fig. 4.1).
Figure 4.1: Groups of density data points with equal shear rate versus normal stress at the chute location $x = 1.5$ m for flows B and G. The dashed black line represents the separation between low and high normal stress mentioned in the analysis.

For normal stress $\sigma_{yy} > 150$ Pa for flows A-D (and 100 Pa for flows E-H), the density is proportional to the normal stress with a proportionality coefficient that is a monotonous function of the shear rate. We determine the proportionality coefficient using linear fits of the density data point groups in Fig. 4.1 with respect to the normal stress. We write the following relation for the density:

$$\rho = \rho_0 + m_c \sigma_{yy}$$

where $m_c$ and $\rho_0$ are functions of the shear rate. In Fig. 4.2, we plot
the compaction coefficient $m_c$ and the limit density $\rho_0$ and the density $\rho_{\text{max}}$ versus shear rate for the flows A-D and E-H. The limit density $\rho_0$ is the density at zero normal stress computed from Eq. (4.1) using the $m_c$ and $\rho_0$ values whereas the density $\rho_{\text{max}}$ is the density at 400 Pa for flows A-D and 300 Pa for flows E-H. The compaction coefficient $m_c$ ranges between 0.2 and 0.9 $s^2/m^2$ and increases with the shear rate. For a given normal stress, the density is smaller at higher shear rates. Moreover, the granular flow dilates more if it is highly sheared. The dependence of $\rho_0$ on the shear rate is consistent with the dependence of $m_c$. The limit density $\rho_0$ ranges between 1250 kg/m$^3$ at a shear rate of 15 $l/s$ and 1000 kg/m$^3$ at a shear rate of 70 $l/s$. Thus, the density at zero normal stress is lower at higher shear rates. The $m_c$ and $\rho_0$ values are obtained from density data points at normal stresses larger than 150 Pa. Therefore, the interpretation of the limit density $\rho_0$ results from the extrapolation of the observations made at large normal stresses. The $m_c$ and $\rho_0$ values collapse relatively well for the different flows i.e. we do not detect systematic variations with chute inclination or starting volume (Figs. 4.2). Thus, we fit them linearly with respect to the shear rate. The equations of the linear fits are:

$$m_c = m_{cd} \frac{\partial u}{\partial y}$$

$$\rho_0 = \rho_{00} + m_d \frac{\partial u}{\partial y}$$  \hspace{1cm} (4.2)

where $m_{cd} = 1.2 \cdot 10^{-2} \text{ s}^3/\text{m}^2$, $m_d = -5 \text{ kg} \cdot \text{s} / \text{m}^3$ and $\rho_{00} = 1320 \text{ kg} / \text{m}^3$. We call $m_d$ the dilatancy coefficient since it accounts for the relation between density and shear rate independently of the normal stress. The static density $\rho_{00}$ is the limit density $\rho_0$ at zero shear rate. The value of $\rho_{00}$ coincides with the density values $\rho_{\text{max}}$. Although we cannot assess a saturation of the density at large normal stress in Fig. 4.1, the linear fits of the density data points predict that at a normal stress of 400(300) Pa, the density is equal to the static density independently of the shear rate. Likewise, the density at zero shear rate $\rho_{00}$ is predicted to be equal to 1320 kg/m$^3$, independent of the normal stress. We interpret the static density $\rho_{00}$ as the density of static grains in a random packing state. The fact that $\rho_{00}$ is lower than the poured and tapped densities by 17-20% can be due to a different arrangement of the grains at the sidewall (constrained by the plane surface). This would indicate a systematic error in the density measurement method. It could also be explained by a non-linearity of $\rho_0$ in the shear rate range 0-15 $l/s$. 

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Figure 4.2: Compaction coefficient $m_c$, limit density $\rho_0$ (coloured symbols) and density $\rho_{\text{max}}$ (black symbols) versus shear rate for flows A-D and E-H.
For normal stress $\sigma_{yy} < 150$ Pa for flows A-D (and 100 Pa for flows E-H), the scatter of the density data points is large, in particular for slower flows. This can be explained by the fact that relatively large density values are measured close to the free surface in the flow body and in the flow tail (in Fig. 3.2 in particular for the slower flows A-B and E-F) whereas small density values are measured close to the free surface in the flow fronts. Larger density values close to the free surface i.e. at low normal stress are counterintuitive and disagree with most previous results. We propose two explanations:

- At small normal stress, grains rearrange more easily and minimize their potential energy by “falling” in free spaces between neighbouring grains resulting in larger densities (similar arguments are put forward to explain size segregation processes, see Félix and Thomas (2004)). This argument is however unlikely to be correct. Indeed, large density values are measured at low normal stress at any shear rate whereas we expect such an effect to play a significant role at low shear rates only.

- The two-dimensional assumption is not always correct for flows with smooth sidewalls, in particular at low velocity. In Jop et al. (2005), larger flow heights are observed at the chute centerline than at the sidewall, and the flow height difference over the flow width scales with the surface velocity difference over the flow width. A flow height difference over the flow width results in an additional normal stress (and thus in a larger density) at the sidewall since any movement of grains in the $z$-direction is impeded by the sidewall. This explanation is supported by the fact that large density values close to the free surface are observed particularly at low flow surface velocity (in the flow body and in the flow tail of the flows A, B, E and F), condition for which the two-dimensional assumption is less valid (Fig. 2.11). Besides, we note that the high density region close to the free surface is smaller for flows with 15 l starting volume than for flows with 30 l starting volume (100 Pa range versus 150 Pa range in Fig. 4.1). This observation agrees with our explanation since we expect the extension of free surface processes to scale with the typical flow height. As a result, no relation for the density at small normal stress can be obtained. In our opinion, the scatter of the density data points at small normal stress is due to velocity and flow height gradients in the $z$-direction, and the constitutive laws in Eqs. (4.1) and (4.2) are valid also at small normal stress. Indeed, at small normal stresses they fit the density data satisfactorily for faster flows, and for flow fronts of slower flows.
4.2 Shear stress

Figure 4.3: Groups of shear stress data points with equal shear rate versus normal stress at the chute location $x = 1.5$ m for flows B and G.

In a first step, we evaluate the dependence of the shear stress on the shear rate and the normal stress. We partition the shear stress data $\tau_{xy}(y, t)$ (Fig. 3.14) in groups of data points with equal shear rates (shear rate intervals of 5 1/s) and plot the groups of data points individually versus the normal stress (Figs. 4.3 and 4.4). We observe that the shear stress is proportional to the normal stress with a proportionality coefficient that is a function of the shear rate. We determine the proportionality coefficient using linear fits of the shear stress data points groups as functions of the normal stress. We write the following relation for the shear stress:

$$\tau_{xy} = \mu \sigma_{yy}$$

(4.3)
Figure 4.4: Groups of shear stress data points with equal shear rate are plotted separately versus normal stress at the chute location $x = 1.5$ m for flows B and G. The grey dots represent all shear stress data points without distinction of shear rate for the respective flows. The line $-\tan(\theta) \sigma_{yy}$ is shown as reference.
We plot the dimensionless friction coefficient $\mu$ and the $-\tan(\theta)$ values versus the shear rate for the flows A-D and E-H (Fig. 4.5). The friction coefficient $\mu$ is equal to $-\tan(\theta)$ if all inertial terms (terms on the left-hand side) in Eq. (3.1) cancel out. However, the condition $\mu = -\tan(\theta)$ is not necessarily equivalent to steady-state in which case all inertial terms are identically zero. The friction coefficient $\mu$ ranges between $-0.35$ and $-0.75$ and increases (in absolute terms) with the shear rate. This implies that for a given normal stress, the absolute value of the shear stress is larger at higher shear rates. The $\mu$ values for different chute inclinations have a similar (linear) dependence on shear rate but do not collapse on a single curve. We make the data collapse by plotting $\mu - \mu (\partial u/\partial y = 0)$ versus shear rate (Fig. 4.5). The $\mu (\partial u/\partial y = 0)$ values are larger (in absolute terms) for faster flows. Thus, the values of the friction coefficient not only depend on the shear rate and the normal stress, but are influenced by other flow variables as well. We observe that the differences between the $\mu (\partial u/\partial y = 0)$ values scale with the variation of the $-\tan(\theta)$ values. A chute inclination dependent shear stress is conceivable at low shear rates since the limit of shear stress at zero shear rate equals $-\tan(\theta) \sigma_{yy}$. At higher shear rates we expect the shear stress to be independent of the chute inclination. Indeed, away from the static case the shear stress is the macroscopic quantity associated with momentum exchange between contacting grains, which is not influenced by the slope inclination. We recall that the velocity fluctuations are larger for faster flows and are relatively constant over the individual flows. For the reasons mentioned previously, we are not able to analyze the shear stress dependences on shear rate, normal stress and velocity fluctuations simultaneously. Nevertheless, we test the relation between the mean velocity fluctuations over the flow and the difference between the $\mu$ series for different chute inclinations. We plot the $\mu (\partial u/\partial y = 0)$ values versus the mean velocity fluctuation component $v'$ over the flow for all flows (Fig. 4.6). We observe that the absolute friction coefficient values at zero shear rate increase with the mean velocity fluctuations values over the flow and can be reasonably well fitted by a linear relation. We thus write the friction coefficient as the sum of a fluctuation term and a viscous term:

$$\mu = \mu_{00} + \mu_f v' + \mu_v \frac{\partial u}{\partial y}. \quad (4.4)$$

The coefficients $\mu_{00}$, $\mu_f$ and $\mu_v$ are obtained by linear fits of the $\mu (\partial u/\partial y = 0)$ values to the mean velocity fluctuation component $v'$ over the flow and fits of the $\mu - \mu (\partial u/\partial y = 0)$ values to the shear rate, respectively. They are equal to $\mu_{00} = -0.27$, $\mu_f = -2.4 \text{ s/m}$ and $\mu_v = -3.2 \cdot 10^{-3} \text{ s}$ and characterize the
Figure 4.5: Coefficient $\mu$, $-\tan(\theta)$ values and $\mu - \mu (\partial u/\partial y = 0)$ values with linear fit versus shear rate for flows A-D and E-H.
Figure 4.6: \( \mu (\partial u/\partial y = 0) \) values versus the mean velocity fluctuation component \( v' \) over the flow for flows A-H.

\[
\begin{array}{ccc}
\rho_{00} \text{ in kg/m}^3 & m_d \text{ in kg} \cdot \text{s/m}^3 & m_{cd} \text{ in s}^3/\text{m}^2 \\
1320 & -5 & 1.2 \cdot 10^{-2}
\end{array}
\]

Table 4.1: Summary table of the density constitutive law coefficients

Granular material. The coefficient \( \mu_{00} \) is interpreted as the friction coefficient at zero velocity fluctuations whereas \( \mu_f \) describes the enhanced friction due to increasing velocity fluctuations. We note that the relation is linear and differs significantly from expressions in kinetic theory models (Jenkins and Richman 1985) which predict the internal stresses to be proportional to the square of the granular temperature \( T^2 \) for non-gravitational, collisional flows. The coefficient \( \mu_v \) combines Newtonian viscosity and Coulomb friction and goes to zero at zero shear rate. The shear stress is not consistent with static equilibrium in the limit of zero shear rate and zero velocity fluctuations, but we are interested in the frictional-collisional regime. At zero normal stress, the shear stress is equal to zero independent of the velocity fluctuations or the shear rate.

\[
\begin{array}{ccc}
\mu_{00} & \mu_f \text{ in s/m} & \mu_v \text{ in s} \\
-0.27 & -2.4 & -3.2 \cdot 10^{-3}
\end{array}
\]

Table 4.2: Summary table of the shear stress constitutive law coefficients
4.3 Velocity fluctuations

In Fig. 3.9, we observe that the velocity fluctuation components \( u' \) and \( v' \) are similar indicating that the velocity fluctuations are isotropic (in contrast to the mean velocity). The use of one energy equation (rather than two) accounting for the conservation of the fluctuation energy \( \rho T \) (with \( T \) the square of the velocity fluctuations norm) is thus a reasonable assumption. Besides, the components \( u' \) and \( v' \) have the same order of magnitude as the mean velocity component \( v \). Thus, the scaling in Eqs. (1.2) is completed by two new equalities:

\[
 u' = O \left( \sqrt{\frac{gH^2}{L}} \right), \quad v' = O \left( \sqrt{\frac{gH^2}{L}} \right).
\]  

(4.5)

Using Eq. (4.5), we perform a new dimensional analysis of the energy equation (1.14). We recall that Eq. (1.14) applies to two-dimensional flows obeying the shallow flow scaling and ordering. We define the dimensionless variables of order \( O(1) \):

\[
 \hat{\rho} = \frac{\rho}{P}, \quad \hat{T} = \frac{T}{gH^2}, \quad \hat{u} = \frac{u}{\sqrt{gL}}, \quad \hat{v} = v\sqrt{\frac{L}{gH^2}},
\]

\[
 \frac{\partial}{\partial \hat{t}} = \sqrt{\frac{L}{g}} \frac{\partial}{\partial \hat{t}}, \quad \frac{\partial}{\partial \hat{x}} = \frac{L}{\sqrt{g}} \frac{\partial}{\partial \hat{x}}, \quad \frac{\partial}{\partial \hat{y}} = H \frac{\partial}{\partial \hat{y}},
\]

\[
 \hat{\sigma}_{yy} = \frac{\sigma_{yy}}{PgH}, \quad \hat{\tau}_{xy} = \frac{\tau_{xy}}{PgH}, \quad \hat{Q}_x = \frac{Q_x}{Q_x^*}, \quad \hat{Q}_y = \frac{Q_y}{Q_y^*}, \quad \hat{\Gamma} = \frac{\Gamma}{\Gamma^*}
\]  

(4.6)

where \( P \) is a typical flow density (for example \( P = \rho_0 \)) and where \( Q_x^* \), \( Q_y^* \) and \( \Gamma^* \) are typical components of the fluctuation flux vector and decay coefficient, respectively. We rewrite Eq. (1.14) in a dimensionless form:

\[
 \hat{\rho} \frac{\partial \hat{T}}{\partial \hat{t}} + \hat{\rho} \left( \hat{u} \frac{\partial \hat{T}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{T}}{\partial \hat{y}} \right) = \\
 \frac{1}{\varepsilon} \hat{\sigma}_{yy} \left( \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} \right) + \frac{1}{\varepsilon^2} \hat{\tau}_{xy} \frac{\partial \hat{u}}{\partial \hat{y}} \\
 + \frac{\sqrt{L}}{Pg\sqrt{gH^2}} \left( Q_x^* \frac{\partial \hat{Q}_x}{\partial \hat{x}} + \frac{1}{\varepsilon} Q_y^* \frac{\partial \hat{Q}_y}{\partial \hat{y}} \right) - \frac{\Gamma^*}{\varepsilon^2 g \sqrt{gL}} \hat{\rho} \hat{\Gamma}.
\]  

(4.7)

The material derivative terms on the first line of Eq. (4.7) are smaller than the shear work term on the second line by a factor \( \varepsilon^2 \) and can be neglected.
Likewise, the normal stress work rate term is negligible in comparison to the shear stress work rate term (by a factor of $\varepsilon$). On the third line, the orders of magnitude of the typical quantities $Q^*_x$, $Q^*_y$ and $\Gamma^*$ are not known and no direct comparisons are possible. We rewrite Eq. (1.14) in dimensional form after simplification:

$$0 = \tau_{xy} \frac{\partial u}{\partial y} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \rho \Gamma$$

(4.8)

Even using the simplified energy equation Eq. (4.8), we are not able to determine either the fluctuation flux vector components $Q_x$ and $Q_y$ or the decay coefficient $\Gamma$ from our experimental data (three unknowns for one equation). In Figs. 3.7, 3.8 and 3.10, we note a positive correlation between shear rate and velocity fluctuation values. We plot the velocity fluctuation component $v'(y, t)$ at the chute location $x = 1.5$ m (Fig. 3.8) versus shear rate (Fig. 4.7) for flows A and D. We observe that the velocity fluctuation component $v'$ increases with the shear rate. Thus, we fit $v'$ linearly with respect to the shear rate with a relation of the type:

$$v' = v'_0 + \alpha \frac{\partial u}{\partial y}$$

(4.9)

with $\alpha$ a proportionality coefficient and $v'_0$ limit velocity fluctuations at zero shear rate (Figs. 4.7 and 4.8). We observe that the dependence of $v'$ on the shear rate is small and badly constrained. The $\alpha$ values determined for each of the flows A-D range over an order of magnitude, between $2 \cdot 10^{-5}$ and $2.5 \cdot 10^{-4}$ m whereas the $v'_0$ values scale with the mean of the velocity fluctuations over the flow height. It shows that the velocity fluctuation components cannot be related locally to the shear rate and must be solved using an energy equation. We recall that the energy equation specifies the change in granular temperature over time. Thus, the value of the granular temperature at the coordinates $(x, y, t)$ is determined not only by the energy equation but also by initial and boundary conditions (for example, by the granular temperature at the bed surface or by the granular temperature in the first instants of the flow when the shallow flow assumption is not necessarily valid). Likewise, rotational velocities and rotational velocity fluctuations of the grains (see Drake 1991) may also be variables of interest, which would require additional momentum and energy equations.
Figure 4.7: Velocity fluctuation component $v'$ versus shear rate for flows A and D. The scale of the $y$-axis is the same to facilitate comparison.

Figure 4.8: Coefficient $\alpha$ and limit velocity fluctuations $v'_0$ versus shear rate for flows A-D.
4.4 Boundary conditions

At the free surface $y = h$, the kinetic boundary condition in Eq. (1.7) and the traction free condition in Eq. (1.8) are sufficient for granular flows in the frictional-collisional regime. We note that the constitutive law for the shear stress in Eq. (4.3) is in agreement with the traction free condition since a zero normal stress implies a zero shear stress. At the bed surface $y = 0$, the mean velocity component $v$ is defined by the kinetic boundary condition in Eq. (1.7), the normal stress is defined by the momentum equation in the $y$-direction from Eq. (1.5) and the shear stress must be specified by a boundary condition which characterizes the momentum exchange between the first layer of grains and the rigid bed surface. Instead of the Coulomb friction law, we propose to use the constitutive law for the shear stress in Eq. (4.3) where the shear rate is replaced by the discrete velocity difference $u_{y=\Delta y/2} / (\Delta y/2)$. We are not able to verify the validity of this boundary condition from our experimental data for two reasons: (1) we do not measure the mean velocity at a distance $\Delta y/2$ from the bed surface. Indeed, the search window is centered around the probe window in the $y$-direction and the minimal distance from the bed surface at which velocities can be measured is equal to the half of the search window size. (2) The number of data points at the minimum value $y$ is not sufficient to test a boundary condition with three variables. Nevertheless, we expect that for a rough bed surface with $d_r \approx d$ and mechanical properties comparable to the ones of the grains, the shear stress at the bed surface is similar to that in the flow. With some adjustments in the pattern matching algorithm (measurement of the velocities $u_i$ close to the bed surface only) or with another velocity measurement method (particle tracking), it would be possible to measure the velocity of the first layer of grains above the bed surface. Still, the problem of the limited number of data points can be only solved by a large number of experiments. Generally, explicit relations between the boundary conditions and the bed surface properties are still lacking.

4.5 Other experimental data

There have been few experimental studies which present comprehensive data sets including direct measurements of mean velocity and density and even less in the frictional-collisional regime. Velocity fluctuations have also been measured experimentally, principally in the collisional regime and for one grain wide flows. Besides, most experimental work on dry granular flows investigates steady, simple shear flows and we do not know any example of
systematic direct measurements of internal stresses. Generally, one grain wide, dilute, steady flows are easier to perform and to analyse than many grains wide, dense, unsteady flows. In the following, we present the results of some experimental work of reference and discuss them in the perspective of our results.

**Drake (1991)**

They investigate one grain wide flows on a rough bed surface. The grains are plastic spheres with diameter \( d = 6 \, \text{mm} \). They present measurements of flows in the collisional regime i.e. most grain contacts are binary and short (compared to the typical time between two contacts). Surface fractions \( \nu_s \), mean velocity \( u \) and velocity fluctuations \( \sqrt{T} \) are measured from high-speed camera recordings. They report two experiments with different flow rates on a 43° inclined chute. The flows are steady with a height of 18 \( d \). Surface fractions \( \nu_s \) decrease monotonously from the bed surface (\( \nu_s = 0.28 \) and 0.65) to the free surface (\( \nu_s = 0.02 \) and 0.04). The profiles are concave \( (\partial^2 \nu_s / \partial y^2 < 0) \) over the flow height except at the free surface where they are convex (we remind that the comparison between surface fraction and volume fraction is not straightforward, see Eq. (2.9)). The mean velocity profiles are concave, with mean velocity values ranging between 0.5 and 0.8 m/s at the bed surface and 3.2 and 3.6 m/s at the free surface. The velocity fluctuations either decrease slightly or increase slightly from the bed surface \( (\sqrt{T} = 0.4 \) and 0.49 m/s) to the free surface \( (\sqrt{T} = 0.35 \) and 0.57 m/s). The mean velocity and velocity fluctuations are larger compared to our flows (factor 1 to 2 for \( u \) and factor 4 to 6 for \( \sqrt{T} \)) whereas the surface densities are lower (factor 1.5 to 3).

**Azanza et al. (1999)**

Steady, collisional, one grain wide flows on a rough bed surface are studied. The granular material is made of aluminium beads with diameter \( d = 3 \, \text{mm} \). Surface fractions, mean velocities and velocity fluctuations are measured from high-speed camera recordings. At a 21° chute inclination (flow height of 12 \( d \)), they measure a linear decrease in the surface fraction from the bed surface \( (\nu_s = 0.54) \) to the free surface \( (\nu_s = 0.09) \). The profile of the velocity component \( u \) is linear over the flow height and varies between 0.1 m/s at the bed surface and 1.4 m/s at the free surface. The granular temperature profile is linear except close to the free surface where it is concave \( (\partial^2 T / \partial y^2 < 0) \) i.e. the profile of the velocity fluctuations \( \sqrt{T} \) is also concave. The velocity fluctuations \( \sqrt{T} \) increase from 0.1 m/s at the bed surface to 0.3 m/s at the
free surface. At a chute inclination of 23° (flow height of 21 d), the surface fraction profile is linear close to the bed surface ($v_s = 0.34$) and convex close to the free surface ($v_s = 0.01$). The profile of the velocity component $u$ is similar to the one of the surface fraction i.e. linear close the bed surface ($u = 0.2$ m/s) and concave close to the free surface ($u = 1.7$ m/s). The granular temperature profile is concave close to the bed surface ($\sqrt{T} = 0.25$ m/s) and constant close to the free surface ($\sqrt{T} = 0.4$ m/s). The velocity fluctuations are 2 to 8 times larger than in our flows. However, no corrections due to non-zero shear rate over the average ensembles are made. The mean velocity is similar compared to our flows. In contrast, the flow height, the surface fraction and thus the maximum normal stress are significantly smaller. We note that the flow behavior changes significantly with a 2° variation of the chute inclination.

Ancey (2001)

Steady flows of glass beads with $d = 1$ mm on a 5 cm wide chute with rough bed surface are investigated. The chute inclination is varied between 27° and 37° resulting in flow height ranging between 26 d and 8 d. The velocity profile is linear at 27° with $u$ varying between 0.1 and 0.6 m/s (measured via high-speed camera recordings). At 37°, it is convex ($\partial^2 u/\partial y^2 > 0$) with $u$ varying between 0.2 and 1.9 m/s. Width-averaged volume fraction is measured using a $\gamma$-radiation technique. They obtain maximum volume fraction at the bed surface ranging between 0.66 and 0.56 and minimum volume fractions at the free surface ranging between 0.05 and 0.02. The volume fraction profiles are convex except at the free surface where they are concave. The mean velocity and volume fraction are similar compared to our flows whereas the flow height is similar (for slower flows) or smaller (for faster flows). The chute geometry, the bed surface roughness and the material properties make the flows in Ancey (2001), in particular the slower flows, very similar to our flows (frictional-collisional regime). However, we do not measure convex mean velocity profiles and our density profiles are concave (with an increase of density close to the free surface in the body and tail of the slower flows).

Steady chute flows are characterized by only one flow height, mean velocity profile, volume fraction profile and granular temperature profile. In the analysis of steady flows, the average ensemble dimension in the $t$-direction $\Delta t$ is not limited since the flow variables are constant in time. Thus, the time resolution of the measurement methods is not an issue (for mean flow variables, not for flow variable fluctuations) and the continuum assumption can be easily fulfilled by choosing average ensembles with large $\Delta t$. A drawback
is that the ranges of the measured flow variables are discrete. Consequently, measurements of many different flows are required in order to fill the flow variable space with data points. Besides, not all regions of the flow variable space are accessible (see Ancey 2001).

4.6 Other models

In steady chute flows (Drake 1991, Azanza et al. 1999, Ancey 2001, GDR Midi 2004), the shear stress is proportional to the normal stress and can be directly computed from the density profile (see Eqs. (1.6)). In Azanza et al. (1999), a model from Jenkins and Richman (1985) is used including some adjustments. The flow is described by a system of three equations (two momentum equations, one energy equation) with eight unknowns \( (u, \rho, \tau_{xy}, \sigma_{yy}, T, Q_x, Q_y \text{ and } \Gamma) \) and requires five constitutive laws for the shear and normal stresses, for the fluctuation flux vector and the decay coefficient. The constitutive laws are relatively simple functions of the velocity and of the granular temperature. However, they are complicated functions of the volume fraction, making their handling, interpretation and verification strenuous. The system of equations is solved for the velocity, the density and the granular temperature and compared with the experimental data. The model fails to reproduce the experimental data in the lower half of the flow, whereas qualitative agreement is obtained in the upper half of the flow for the 21° flow. For the 23° flow, qualitative agreement is obtained in the upper half of the flow by setting the restitution coefficient to 1 and by neglecting terms of order \( O(\nu^2) \) and higher.

In Ancey (2001), different models (Savage 1983, Mills et al. 1999 and Ancey and Evesque 2000) are tested. In the model of Savage (1983), the equations and unknowns are the same as in Jenkins and Richman (1985) but the constitutive laws combine Coulomb friction and kinetic theory. In Mills et al. (1999), the system of two equations (two momentum equations) counts four unknowns. The model assumes constant density and a shear stress combining solid friction and a power-law rheology of order two. The model of Ancey and Evesque (2000) is similar. The constitutive laws postulate constant density and a shear stress composed of a frictional and a collisional term, the relative importance of the terms being controlled by the inertial number \( N_I \). None of the models is able to reproduce the experimental data from Ancey (2001) satisfactorily. Furthermore, none of the models can explain the different behaviors (different profile shapes) observed when varying the chute inclination. Models using kinetic theory (Jenkins and Richman
1985) are valid in the collisional regime (binary, short duration contacts) and thus reproduce experimental data satisfactorily at low volume fraction. Extensions of models based on kinetic theory to the frictional-collisional regime (Savage 1983) which consist in adding a frictional term in the constitutive law for the shear stress are inconsistent from a physical point of view. Indeed, they imply that only the shear stress is modified and that the normal stress, the fluctuation flux vector and the decay coefficient are unchanged.

In GDR Midi (2004), an empirical model is proposed that is not based either on kinetic theory or on Coulomb friction theory. It is inspired by earlier work on steady granular flows relating flow height to chute inclination and mass flow rate (Pouliquen 1999). The model applies to unsteady, incompressible flows with no energy equation. The flows are described by a system of three equations (one mass equation, two momentum equations) with five unknowns \((u, v, \rho, \tau_{xy}, \sigma_{yy})\). A first constitutive law assigns a constant value to the density. A second constitutive law is based on considerations about typical times at the scale of the grain and relates the normal stress to the square of the shear rate \(\sigma_{yy} \propto (\partial u/\partial y)^2\). A (redundant) third constitutive law specifies the shear stress. However, the model is only applied to steady flows (two equations, four unknowns) using the first two constitutive laws (shear stress proportional to normal stress proportional to the square of the shear rate). It is equivalent to power-law rheology of order two or to the Bagnold law for solid suspensions (two-phase flows, Bagnold 1954). The model predicts concave velocity profiles and constant density profiles.

In the present work, we adopt a different approach. Instead of comparing experimental data with model predictions, we derive constitutive laws from the experimental data by determining the dependences of the density and of the shear stress on the other model unknowns. We take advantage of the fact that: (1) the flow variable ranges are continuous and (2) the data points are homogeneously distributed over the flow variable space. The constitutive laws proposed are intuitive and bring together accepted concepts and observed behaviors in granular flow rheology (dilatancy, Newtonian viscosity, Coulomb friction). They do not apply to all granular flows, though. Outside the ranges \(0.2 < \nu < 0.6, 10 < \partial u/\partial y < 100 \) \(1/s\) or \(100 < \sigma_{yy} < 500 \) Pa, or, equivalently, outside the range \(0.02 < N_I < 0.5\), the constitutive laws in Eqs. (4.1), (4.2), (4.3) and (4.4) may not be valid. Besides, the parameters \(m_{cd}, m_d, \rho_{00}, \mu_{00}, \mu_f\) and \(\mu_v\) are obtained for monodisperse glass beads and are expected to vary for different grain sizes, shapes or materials.
Chapter 5
Conclusions

In this work, we study free surface, gravitational, shallow, two-dimensional, dry granular flows on a rough incline in the frictional-collisional regime. We perform small-scale experiments consisting in the release of finite volumes of glass beads down a chute with smooth sidewalls and a rough bed surface. The chute inclination and the starting volume are varied and the flows are filmed at one location of the chute through the sidewall using a high-speed camera. Using a particle tracking method and a pattern matching algorithm and defining average ensembles, the density, mean velocity and velocity fluctuations are measured during the passing of the flow at the location of the camera.

The flows can be described by a system of three partial differential equations (mass and momentum) and five unknowns (mean velocity components $u$ and $v$, density $\rho$ and shear and normal stresses $\tau_{xy}$ and $\sigma_{yy}$). The problem is underdetermined and two constitutive laws associated with the material properties must be specified to close the system of equations. From the measurements of $u$, $v$ and $\rho$, their respective time derivatives and spatial gradients and the mass and momentum equations, the shear and normal stresses $\tau_{xy}$ and $\sigma_{yy}$ are computed. Thus, we know all five unknowns over time and over the flow height. The data points cover a large region of the flow variable space corresponding to the frictional-collisional flow regime. An analysis of correlations between selected flow variables is performed yielding the following qualitative statements:

- The density increases with increasing normal stress, and decreases with increasing shear rate. The dependency of density on shear rate is weaker at higher normal stresses. Conversely, the dependence of density on normal stress is stronger at higher shear rates. The asymptotic
behavior of the density is extrapolated: at large normal stress, the density tends to a maximum density value which is independent of the shear rate, and at zero normal stress, the density tends to a minimum density value which decreases with increasing shear rate.

- The shear stress increases with increasing normal stress, shear rate and velocity fluctuations. At zero normal stress, the shear stress tends to zero independent of the shear rate and the velocity fluctuations.

We propose two constitutive laws in Eqs. (4.1) and (4.3) locally relating density and shear stress to the shear rate, normal stress and velocity fluctuations. These constitutive laws a priori do not apply only to dry granular flows in contrast to the system of Eqs. (1.5). They are well-suited to model flows that are not shallow or that do not have a free surface. Generally, they characterize grains in the frictional-collisional regime i.e. in the region of the flow variable space (shear rate, volume fraction, normal stress) investigated experimentally in this work.

These results may also have implications for the modeling of two-phase, uncompressible flows like debris-flow suspensions. In suspensions, the shear stress is usually accounted for by a sum of three terms: two terms for the stress within the individual phases and one term for the coupling between the phases (Iverson 1997, Boyer et al. 2011). The solid phase contribution is obtained from constitutive laws for dry granular flows. The volume fraction of the solid phase in suspensions does not behave like the density in dry granular flows. However, constitutive laws for the volume fraction of the solid phase must include dilatancy and compaction effects (Coussot 1997, Kowalski 2008) which are similar in suspensions and in dry granular flows.

Measurements of velocity fluctuations show that they are smaller in the frictional-collisional regime than in the collisional regime (one grain wide flows). They are isotropic with both components scaling with the mean velocity component $v$. A simplified energy equation taking into account these findings is derived. However, the fluctuation flux vector and decay coefficients are unknown and cannot be approached with our experimental data. Experimental work on dynamic granular systems are needed to quantify the different processes (flux, decay, interaction with boundaries) associated with velocity fluctuations. Investigations of vibrated granular systems at zero shear rate using high-speed video recording (Son et al. 2008), mechanical probe (D’Anna et al. 2003), positron emission particle tracking (Wildman et al. 2000) or nuclear magnetic resonance (Yang et al. 2001)) have been first steps towards this goal.
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Appendix A

List of symbols

Flow variables

$x$: position coordinate parallel to the chute length (directed downwards)
$y$: position coordinate normal to the chute plane (directed upwards)
$z$: position coordinate parallel to the chute width (directed according to the right-hand rule)
$t$: time coordinate
$u$: velocity component in the $x$-direction
$v$: velocity component in the $y$-direction
$w$: velocity component in the $z$-direction
$u'$: velocity fluctuations component in the $x$-direction
$v'$: velocity fluctuations component in the $y$-direction
$w'$: velocity fluctuations component in the $z$-direction
$T$: granular temperature
$Q$: fluctuation flux vector
$\Gamma$: decay coefficient
$\partial u/\partial y$: shear rate
$\rho$: flow density
$\nu = \rho/\rho_{\text{solid}}$: flow volume fraction
$\sigma$: stress tensor
$\tau_{xy}, \tau_{yz}$: shear stresses
$\sigma_{xx}, \sigma_{yy}$: normal stresses
$g$: gravitational acceleration vector
$\theta$: slope/chute inclination
$h$: flow height
$b$: bed surface height
$n$: unit vector normal to the flow boundaries
$\bar{u}$: depth-averaged velocity component in the $x$-direction
\( \alpha \equiv \bar{u}^2/\bar{u}^2 \): shape factor  
\( L \): typical flow length  
\( H \): typical flow height  
\( W \): (typical) flow width  
\( \varepsilon \equiv H/L \): shallowness parameter  
\( P \): typical flow density  
\( B \): typical bed surface height  
\( d_r \): bed surface roughness size

**Grain variables**

\( d \): grain diameter  
\( \rho_{\text{solid}} \): grain density  
\( u_i \): velocity component in the \( x \)-direction of grain \( i \)  
\( u_i' \): fluctuating part of the velocity component in the \( x \)-direction of grain \( i \)  
\( t_r \equiv d\sqrt{\rho_{\text{solid}}/\sigma_{yy}} \): rearrangement time  
\( t_c \equiv (\partial u/\partial y)^{-1} \): contact time  
\( N_I \equiv t_r/t_c = d\sqrt{\rho_{\text{solid}}/\sigma_{yy}}\partial u/\partial y \): inertial number

**Model parameters**

\( \rho_0 \): limit density (at zero normal stress)  
\( \rho_{00} \): static density (at zero normal stress)  
\( m_c \): compaction coefficient  
\( m_d \): 1st dilatancy coefficient  
\( m_{cd} \): 2nd dilatancy coefficient  
\( \mu \): friction coefficient  
\( \mu_{00} \): static friction coefficient  
\( \mu_f \): fluctuation friction coefficient  
\( \mu_v \): viscous friction coefficient

**Measurement methods parameters**

\( \Delta x \): average ensemble dimension in \( x \)-direction  
\( \Delta y \): average ensemble dimension in \( y \)-direction  
\( \Delta t \): average ensemble temporal dimension  

\( b(i) \): pixel \( i \) brightness  
\( b(i, l) \) with \( l = 1, ..., 8 \): pixel \( i \) surrounding pixels \( i, l \) brightnesses  
\( nbd(i) \): negative brightness difference of pixel \( i \) with its surrounding pixels  
\( bd(i) \): mean brightness difference of pixel \( i \) with its surrounding pixels
$j_m$ with $m = 1, ..., 7$: pair type
$N_m$: number of intermediate pixels for pair type $m$
$k_n$ with $n = 1, ..., N_m$: intermediate pixels 1 to $N_m$ for pair type $m$
$pbd(i, j)$: mean brightness difference of pixel pair $\{i, j\}$ with its intermediate pixels $k_n$
$b_{\text{max}} = 130$: maximum brightness value for local brightness maxima
$(bd/b)_{\text{min}} = 0.06$: minimum relative mean brightness difference with surrounding pixels value for local brightness maxima
$r_b$: exclusion radius
$\nu_s$: surface fraction
$\nu_{s, \text{max}}$: maximum surface fraction
$\nu_{\text{max}}$: maximum volume fraction

t_i$: time of frame $i$
$x_i$: discrete position coordinate of probe window $i$ in $x$-direction
$y_i$: discrete position coordinate of probe window $i$ in $y$-direction
$u_i$: velocity component of probe window $i$ in $x$-direction
$v_i$: velocity component of probe window $i$ in $y$-direction
$f$: brightness two-dimensional cross-correlation function
$\Delta x_i$: distance in $x$-direction between the position of probe window $i$ in the frame $j$ and in the frame $j + 1$
$\Delta y_i$: distance in $y$-direction between the position of probe window $i$ in the frame $j$ and in the frame $j + 1$
$L_{\text{pw}}$: probe window length
$H_{\text{pw}}$: probe window height
$b_b$: mean brightness over probe window
$\sigma_b$: standard deviation of brightness over probe window
$u_{\text{inf}}, u_{\text{sup}}$: histogram limit velocity components in $x$-direction
$v_{\text{inf}}, v_{\text{sup}}$: histogram limit velocity components in $y$-direction
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Education
1990-1999 Primary and secondary school in Pully
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