Ricci Flows of Ricci flat Cones

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Abstract

In this thesis we study the evolution of smooth Riemannian cones under Hamilton’s Ricci flow. In the first part we define Ricci flows coming out of a Riemannian cone and prove using Perelman’s Pseudolocality Theorem that the quadratic curvature decay is preserved. In fact, in combination with Shi’s estimates all derivatives decay at the appropriate rate for \( t > 0 \) as \( x \to \infty \). We go on to consider Ricci flat cones as initial conditions. By construction of a suitable supersolution we show that the Ricci curvature decays exponentially for any smooth Ricci flow coming out of a Ricci flat cone. This result corresponds to the exponential curvature decay for the 2-dimensional expanding gradient Ricci solitons coming out of Ricci flat cones described in [CLN06] and the Kähler gradient Ricci solitons in [FIK03]. Inspiration comes from similar results for mean curvature flows from stationary cones in [Ilm95].

In the second part we prove that gradient expanding Ricci solitons with quadratic curvature decay are solutions of the Ricci flow coming out of a cone as defined in the first part. To this end we show that the gradient potential \( f \) converges as \( t \downarrow 0 \) to a continuous function on \( M \) and locally smoothly outside of a compact subset. The deRham splitting theorem then gives the initial cone.

The third part also contains the main result of this thesis. We construct expanding Kähler Ricci solitons coming out of Ricci flat Kähler cones. We show that if a Ricci flat Kähler cone (a Riemannian cone over a Sasaki Einstein manifold) admits an equivariant resolution with positive canonical line bundle (negative first Chern class) then there exists an expanding self-similar solution coming out of that cone as defined in the first part. By the work of van Coevering Kähler cones can always be understood as complex varieties and we follow the arguments there to set up the corresponding complex Monge Ampere equation on such a resolution. This complex Monge Ampere equation is elliptic and has unbounded coefficients in the first order term. To show existence of a solution we combine Aubin’s and Yau’s estimates for the proof the the Calabi conjecture with technique’s developed by Lunardi to deal with elliptic operators with unbounded coefficients acting on certain weighted spaces. Finally we present some old examples of expanding Kähler Ricci solitons in the light of this of this theorem and also construct new examples using toric geometry.

We conclude this thesis with a list of open problems and possible directions for further research.
Zusammenfassung


Wir schliessen mit einer Reihe offener Fragen sowie möglichen zukünftigen Richtungen unserer Forschung.
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CHAPTER 1

Introduction

Geometric heat flows have a long and successful history. Presumably starting with the harmonic map heat flow introduced by Eells and Sampson [ES64], these geometric flows have helped to solve various problems in geometry and topology. One of the more famous exponents is the Ricci flow introduced by Hamilton in his seminal work [Ham82] from 1982 which deforms a Riemannian metric \( g \) by the evolution equation

\[
\frac{\partial g_{ij}}{\partial t} = -2R_{ij}(g(t))
\]

towards a canonical metric. In this paper and subsequent work Hamilton laid out a program using Ricci flow to approach Thurston’s geometrization conjecture which entails the famous Poincaré conjecture as a corollary. In the following decades Hamilton and numerous others contributed to the execution of this program culminating in the celebrated work of Perelman ([Per02], [Per03b], and [Per03a]) whose results completed the proof of Thurston’s conjecture (see [MT07], [KL08]). While the Ricci flow was originally developed only for 3-dimensional manifolds it has also been successfully applied in higher dimensions for example in the work of Böhm and Wilking [BW08] who proved, using Ricci flow, that manifolds with positive curvature are space forms and more recently in Brendle and Schoen’s proof of the differentiable sphere conjecture [BS11].

Parallel to this development Ricci flow opened up a new field of research in Kähler geometry (with its own geometrization program) that has grown rapidly in recent years. The Ricci flow equation on Kähler manifolds reduces to a scalar Monge Ampère equation and even in higher dimensions the formation of singularities is much better understood, at least in the case of algebraic varieties.

Here we are interested in the “un-formation” of singularities under Ricci flow in general and Kähler Ricci flow in particular. As a heat flow the Ricci flow has the tendency to improve regularity and in this thesis we investigate whether the Ricci flow can smoothen out certain initial singularities. Apart from that we also study possible non-uniqueness issues after the formation of singularities.

Our driving motivation is a question raised in [FIK03] which contains both of these aspects: When does a Ricci flat Kähler cone (a stationary cone for the Ricci flow) possess a non-singular forward evolution by Ricci flow? This is illustrated by the following example in [FIK03]: the orbifold \( \mathbb{C}^n/\mathbb{Z}_k \) can be equipped with the flat metric \( g_0 \) to become a Ricci flat cone. At the same time, for \( k > n \), it is shown that there exists a smooth Ricci flow \( (M,g(t))_{t>0} \) which converges to the Ricci flat cone as \( t \searrow 0 \) in the Gromov Hausdorff sense and locally smoothly away from the vertex. These Ricci flows generally do not have curvature bounded from below. The goal of this thesis is to further generalize this example and provide a general necessary criterion for the existence of such flows.

In the following sections we present some related topics and results that guide and motivate our research. In the last section we give an overview of the chapters that follow and the main results of this thesis.

1.1. Ricci flow through singularities

In general the Ricci flow develops singularities. To continue the flow past such singularities Perelman introduced a surgery algorithm. The basic idea is to replace the parts of the manifold where the curvature blows up just before the singular time by a smooth piece in such a way that it satisfies certain a priori estimates and restart the flow there. However, this procedure depends on a number of choices. So Perelman suggests that there should be a way of flowing “into” the singularity and out of the resulting singular space in an unambiguous way. Such a procedure is described in [ACK09]. This is one motivation to study Ricci flows with singular initial conditions.
1.2. Comparison with other flows

It is instructive to compare the behavior of the Ricci flow near and at singularities with similar situations for other geometric flows. For the three flows described below non-uniqueness phenomena for stationary cones have been described in the literature (see \cite{Ilm95} and references therein).

1.2.1. Harmonic map heat flow. For the harmonic map heat flow (in supercritical dimension \( n \geq 3 \)) an example of a stationary cone is the map \( u_0 : \mathbb{R}^n \to S^n, x \mapsto (\pi/2, x/|x|) \) where \((s, \omega)\) are corotational coordinates on \(S^n\). The map \(u_0\) is weakly harmonic and hence in particular a self-similar (weak) solution of the harmonic map heat flow. In \cite{Ilm95} (see also \cite{Gr11} for a more general result) it is shown that there exists a non-constant solution of the harmonic map heat flow with initial condition \( u_0 \) if \( u_0 \) is not energy minimizing. In particular, Angenent, Ilmanen, and Velazquez \cite{Ilm95} (see \cite{Gr11}) show that there exists a self similar solution not identical with the cone if \( 3 \leq n < 7 \). Other examples of non-uniqueness were found by Coron \cite{Cor90}. In critical dimension 2 the weak flow is unique under the additional requirement that the energy is monotone \((\text{Str85}, \text{Fre95})\).

1.2.2. Mean curvature flow. For the mean curvature flow the stationary cones are submanifolds which are invariant under homotheties and have vanishing mean curvature (on the smooth part, i.e. away from the origin). It is shown in \cite{Ilm95} that a codimension one stationary cone has a non-unique evolution if it is not area minimizing. As an example consider the Simons cones which are defined, for \( p + q \geq 2 \), as
\[
C_{p,q} := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : |y|^2/(p-1) = |x|^2/(q-1)\}.
\]
Simons showed in \cite{Sim68} that for \( p + q > 7 \) these cones are area minimizing and Bombieri, de Giorgi, and Giusti showed in \cite{BDGG69} that these cones are unstable (i.e. not area minimizing) if \( p + q \leq 7 \). This reflects a general result that the singular set of an area minimizing hypersurface has codimension 7. In the cases \( p + q \leq 7 \) there exists a self similarly expanding solution different from \( C_{p,q} \) (see \cite{Ilm95} and example \( 3.3.3 \) in this thesis).

1.2.3. Lagrangian mean curvature flow. Lagrangian mean curvature flow is a special case of mean curvature flow. If \( M \) is a Lagrangian submanifold of a Calabi Yau manifold then the Lagrangian property is preserved under the mean curvature flow. A particularly simple example of a stationary Lagrangian cone is the union of two Lagrangian \( n \)-planes in \( \mathbb{C}^n \) which intersect only in the origin. Lawlor and Nance (\cite{Law89} and \cite{Mac87}) showed that such a cone is area minimizing if and only if it satisfies an angle criterion. Joyce, Lee and Tsui \cite{JLT10} constructed self-similar solutions coming out of such cones that do not satisfy the angle criterion. Moreover, it is shown in \cite{LN12} that these are the only Lagrangian expanders with zero Maslov class for each pair of \( 2 \)-planes in \( \mathbb{C}^2 \).

1.2.4. Mean curvature flow and Ricci flow. Mean curvature flow and Ricci flow bear several interesting similarities. In particular, mean curvature flow in dimension 2 and Ricci flow in dimension 4 seem to be closely related. However, this analogy does not work if one only considers mean curvature flows in codimension 1 (see \cite{HHS11}). There is much less known about mean curvature flow and minimal submanifolds in higher codimension (e.g. minimal surfaces in higher codimension have only isolated singular points). It is expected that some results for mean curvature flow of Lagrangian surfaces in \( \mathbb{C}^2 \) (which is much better understood) will have their analogue in the Kähler Ricci flow on complex surfaces and vice versa. But even this analogy breaks down in general (see \cite{Nev07}, \cite{FIK03}).

In this spirit the singular set of a Ricci flow should have (parabolic) codimension 4 which corresponds to Almgren’s regularity result \cite{Alm00} for area minimizing currents in general codimension (see also \cite{Wli00}). In particular, this would make Ricci flat cones valid solutions of the Ricci flow (see remarks in \cite{FIK03}).

1.3. Non-uniqueness for Ricci flow and Ricci flow of cones

On a smooth manifold a solution of the Ricci flow is unique if the curvature is bounded along the flow and the initial metric is complete \((\text{CZ06})\). Topping showed that uniqueness fails in general if either of
these conditions is dropped (see Top11 and GT01). Non-uniqueness also arises for singular initial conditions. In the following subsections we describe some partial results related to non-uniqueness for the Ricci flow of Ricci flat cones.

1.3.1. Ricci flat cones. Ricci flat cones are the stationary cones for the Ricci flow. As such they should be considered solutions of the Ricci flow equation (see remarks above). A special class of Ricci flat cones are Ricci flat Kähler cones, i.e. cones over Sasaki Einstein manifolds, which are an interesting topic of research in its own right. These cones have come up in the work of Cheeger, Colding and Tian (CT11) (note, however, that not every Ricci flat Kähler cone is the limit of Calabi Yau manifolds) and numerous examples of Sasaki Einstein manifolds have been constructed more recently (see section 2.5).

1.3.2. Stability of Ricci flat cones. As described above, unstable (with respect to some energy functional) stationary cones cause non-uniqueness for various geometric flows. In HHS11 we discuss the relation between unstable cones and non-uniqueness of the Ricci flow. There, a Ricci flat cone is considered to be stable if the renormalized Perelman functional $\lambda_{nc}$ introduced in Has11 has a local maximum in $(C, g_C)$. This means that for any nearby manifold $(M, g)$ asymptotic to the original cone $(C, g_C)$ we have $\lambda_{nc}(g) \leq \lambda_{nc}(g_C)$. It is conjectured that in the case that $(C, g_C)$ is not stable a Ricci flow not identical with the cone exists. However, this flow might very well be singular. Conversely, if a smooth flow coming out of the Ricci flat cone exists then there exists a competitor $(M, g)$ asymptotic to the cone with $\lambda_{nc}(g) > \lambda_{nc}(g_C)$ (see HHS11).

1.3.3. Ricci flow and the positive mass theorem. It was conjectured by Ilmanen [Ilm] that there exists a relation between non-uniqueness of Ricci flows of Ricci flat cones and the failure of the positive mass theorem. Recall that the positive mass theorem is not true in general for ALE manifolds (see section 2.5). Theorem 1.3.1. Let $M^n (n \geq 3)$ be a complete manifold with one end, and assume the end is diffeomorphic to $S^{n-1}/\Gamma \times \mathbb{R}$, where $\Gamma$ is a finite subgroup of $SO(n)$. Then the following are equivalent:

1. (lambda not a local maximum) There exists a metric $g$ on $M$ that agrees with the flat conical metric outside a compact set such that $\lambda_{nc}(g) > 0$.

2. (failure of positive mass) There exists an asymptotically locally euclidean metric $g$ on $M$ such that $R_g \geq 0$ and $m(g) < 0$.

Examples where the positive mass theorem fails for ALE manifolds were constructed by LeBrun [LeB88]. He shows that there exists a complete scalar flat metric on $L^{-k} \to \mathbb{CP}^1$ for $k > 2$ such that $m(g) < 0$. This is also true for the scalar flat metric on toric resolutions of cyclic singularities of $\mathbb{C}^2$ in [CS04], [Joy95] and [ASD12]. This theorem does not give a sufficient criterion for the existence of non-stationary Ricci flow coming out of the cone. However, if such a flow exists then it is shown in [HHS11] that (1) and (2) hold. The correct converse statement would have to take into account possible singular non-stationary solutions of the Ricci flow coming out of the cone. Our results for smooth (Kähler) Ricci flows on toric resolutions of cyclic singularities further support this conjecture.

1.3.4. Expanding Ricci solitons. Ricci solitons model the behavior of the Ricci flow near a singularity and they appear as blow up limits of a Ricci flow at a singular time. Ricci solitons are self-similar solutions of the Ricci flow (in other words solutions which move only under scaling and diffeomorphisms) and at any fixed time they satisfy the equation

$$2Rc - L_X g + \lambda g = 0,$$

where $X$ generates the diffeomorphisms and the sign of $\lambda$ determines the scaling is shrinking ($\lambda < 0$) or expanding ($\lambda > 0$). Understanding Ricci solitons was essential to carry out Hamilton's program. While shrinking Ricci solitons describe Ricci flows flowing into singularities expanding Ricci solitons are Ricci flows coming out of singularities. This means that expanding Ricci solitons are the first thing to study if one wants to understand Ricci flows coming out of cones. The behavior of expanding Ricci solitons as $t \searrow 0$ can be compared with shrinking Ricci solitons as $t \nearrow 0$. In both cases the flows converge to a cone under the additional assumption of quadratic curvature decay. Given a cone one expects that a Ricci...
flow and probably an expanding Ricci soliton asymptotic to the cone exist (possibly with a singularity). On the other hand shrinking solitons seem to be quite rare and hence it is unlikely that a given cone is the limit of a shrinking Ricci soliton. This means that it is much easier to construct expanding Ricci solitons (and much harder if not impossible to prove their uniqueness) than their shrinking counterparts. Note that Ricci flat cones are expanding and shrinking Ricci solitons at the same time (with a singularity though). Examples of such self-similar expanding solutions (coming out of Riemannian cones) were constructed in [GK04, FIK03, DW11, SS10]. The examples in [FIK03] are rotationally symmetric expanding Kähler Ricci solitons on resolutions of quotients of $\mathbb{C}^n$ by diagonal cyclic subgroups $U(n)$. The non-uniqueness of the Ricci flow occurs as follows: Cao showed in [Cao96] that for any $p \geq 1$ there exists a smooth expanding Kähler Ricci soliton metric $g_{\text{Cao},p}$ on $\mathbb{C}^n$ giving a Kähler Ricci flow with initial condition $\omega_{g_p} := \sqrt{-1} \partial \bar{\partial} |z|^2/p$. The metric also gives a well-defined Kähler Ricci soliton metric $g_{\text{Cao},p}$ on $\mathbb{C}^2/\mathbb{Z}_k$ giving a singular Kähler Ricci flow ($\mathbb{C}^n/\mathbb{Z}_k, g_{\text{Cao},p}(t)$). At the same time it is shown in [FIK03] that for $k > n$ there exists a smooth expanding Kähler Ricci soliton metric $g_{\text{FIK},p}$ on the $k$-th power of the tautological line bundle $L^{-1}$ over $\mathbb{C}P^{n-1}$ with associated Kähler Ricci flows $g_{\text{FIK},p}(t)$ coming out of the cone ($\mathbb{C}^n, g_p$). Hence there exist two different Ricci flows starting from the same cone. In particular, if $p = 1$ the initial cone is flat and $g_{\text{Cao},1}$ is the euclidean expander on the quotient while ($L^{-k}, g_{\text{FIK},1}$) is a non-stationary expanding Kähler Ricci soliton coming out of a Ricci flat cone. It is worth noting that the expanding Kähler Ricci solitons constructed in [FIK03] are ALE and coincide (in complex dimension 2) with the manifolds which were used to construct the counterexamples to the positive mass theorem in [LeB88].

Another example of an expanding Ricci soliton coming out of a Ricci flat cone is the 2-dimensional expanding Ricci soliton from [GHS03] described in [CLN06] (see also chapter 3). As gradient Ricci solitons on surfaces they are rotationally symmetric. Note that also higher dimensional gradient Kähler Ricci solitons have at least a one-dimensional symmetry. The expanders in [SS10] are obtained as smooth limits of Ricci flows on non-compact manifolds with positive sectional curvature. The evolving manifolds are homeomorphic to the initial cone (the asymptotic cone). However, none of these flows (with exception of the Gaussian expander) are coming out of Ricci flat cones.

Finally, we should mention the examples constructed by Futaki and Wang in [FW11] which extend the shrinking and expanding soliton construction of [Cao85] and [FIK03] to Ricci flat Kähler cones (compared to $\mathbb{C}^n$ or quotients of $\mathbb{C}^n$) to obtain global solutions of the Ricci flow on complex line bundles over Kähler Einstein manifolds with positive scalar curvature. Note that at time $t = 0$ their metric $g = g(0)$ is a cone metric but not Ricci flat. This generalizes the global solutions in [FIK03] which are shrinking Kähler Ricci solitons on $L^{-1} \rightarrow \mathbb{C}P^{n-1}$, blow down the zero section at $t = 0$, and then continue as the corresponding expanding Ricci solitons on $\mathbb{C}^n$ from [Cao96]. The examples of expanders constructed in [FW11] are not complete in general. While the shrinking Ricci solitons extend smoothly to the zero section the expanding Ricci solitons remain singular along the zero section (i.e. they are incomplete in the smooth part).

The Kähler Ricci solitons constructed in [DWT1] are of cohomogeneity one and live on complex line bundles over products of Kähler Einstein manifolds. They are generalizations of the solitons constructed in [FIK03] and they extend smoothly across the zero section. However, only few of them come out of Ricci flat cones.

As described above Ricci flat cones are critical points of the renormalized Perelman functional and $\lambda_{ac}(C, gc) = 0$. At the same time Perelman’s shrinker entropy of the cone is $\nu_-(C) = \log(\text{Vol}(S)) - \log(\text{Vol}(S^{n-1}))$ where $S$ is the link of he cone and $\text{Vol}(S^{n-1})$ is the volume of the round sphere (see [CH04]). The expander entropy $\nu_+$ as defined in [FIN05] on the other hand is not finite on Ricci flat cones in general. In chapter 6 we speculate about the existence of a relative expander entropy.

1.4. Reverse bubbling

The term “reverse bubbling” was originally used by Topping to describe a phenomenon occurring in harmonic map heat flow. In general this flow develops singularities and does so (in critical dimension
2) by blowing up the gradient of the evolving map while concentrating energy in one point (or isolated points). One says that the flow generates a bubble which causes the topology of the map (the image) to change. To continue the flow one removes all the bubbles and restarts the flow smoothly. During the process the energy drops by a fixed amount. Struwe [Str85] and Freire [Fre95] showed that there exists a unique global solution with decreasing energy. Topping subsequently investigated possible non-uniqueness by inserting bubbles at the singular times (therefore increasing the energy) whose energy deconcentrates for positive times. Rupflin showed in [Rup08] that this is the only way non-uniqueness can occur. For the Ricci flow on surfaces a similar phenomenon can be observed. Recall that the rescaled Ricci flow on a compact Riemannian surface exists globally and converges to a constant curvature metric so no forward singularities are expected. Reverse bubbles, on the other hand, can be created by either starting with an incomplete metric that becomes complete instantaneously or by allowing instant topological change to happen (see [Top11] for a nice survey). In higher dimensions the Kähler Ricci flow has a similar behavior. In [SW11] Song and Weinkove describe carefully how the Kähler Ricci flow contracts an exceptional divisor (creating a “forward bubble”) and converges to the original manifold. This is part of a more general program outlined by Song and Tian [ST09] to show that the Ricci flow on algebraic varieties performs algebraic surgeries and eventually carries out an analytical version of Mori’s (conjectural) minimal model program (the geometrization program for Kähler manifolds mentioned before). A higher dimensional analogue of reverse bubbling is the “extraction” of an exceptional set.

This happens for example with the expanders constructed in [FIK03] or in higher codimension when the Kähler Ricci flow performs a flip as described in [SY12] (these are modelled on higher rank vector bundle versions of the solitons in [FIK03] also described in [DW11]. At the singular time these are projective cones).

1.5. Kähler Ricci flow

It was shown in [Cao85] that the (normalized) Ricci flow deforms a Kähler metric on a compact Kähler manifold with first Chern class $c_1(M) < 0$ or $c_1(M) = 0$ to a Kähler Einstein metric therefore reproving the results of Aubin and Yau. In a sense Kähler Ricci flow is the generalization of Ricci flow on surfaces in higher dimensions. Much like the Ricci flow on the sphere the case $c_1(M) > 0$ has attracted the most attention. If $c_1(M)$ is not trivial or definite the Kähler Ricci flow develops singularities. Its behavior before and after these singularities has been studied intensively in recent years. While the Kähler Ricci flow can be studied independently from general Ricci flow as questions that arise in Kähler geometry have no analogy in Riemannian geometry, Kähler Ricci flow serves also as a good testing ground for general Ricci flow. As mentioned before there is a promising theory under development that would allow the Kähler Ricci flow to flow through singularities supporting Perelman’s conjecture mentioned before. In chapter 5 of this thesis we will construct self-similar solutions of the Kähler Ricci flow coming out of Ricci flat Kähler cones.

1.6. Asymptotically conical Ricci flat Kähler manifolds

The construction of nearby Ricci flat manifolds or minimal surfaces asymptotic to a given Ricci flat or stationary cone complements the above mentioned results on the existence of self-similar solutions coming out of such cones. In [Im05] it is shown that the Simons cones $C_{p,q}$ with $p+q \leq 7$ admit asymptotic smooth minimal surfaces. Kronheimer [Kro89a] showed the existence of so called gravitational instantons, i.e. ALE hyperkähler manifolds. These manifolds are deformations of crepant resolutions of quotients of $\mathbb{C}^2$ by finite subgroups of $SU(2)$ which are completely classified by the McKay correspondence. Joyce [Joy00] proved that ALE Ricci flat Kähler metrics exist on any crepant resolution (vanishing first Chern class) of a quotient $\mathbb{C}^n/\Gamma$ by a finite subgroup $\Gamma \subset SU(n)$. More recently these results were generalized for crepant resolutions of general Calabi Yau cones by van Coevering [VC10], Goto [Got12], and their deformations by Conlon and Hein [CH12].

These results should be compared to our theorem 6 which roughly says that a resolution of a Ricci flat Kähler cone with negative first Chern class admits an expanding Kähler Ricci solitons metric. In particular there exist ALE hyper-Kähler structures on crepant resolutions of $\mathbb{C}^2/\Gamma_k$, where $\Gamma_k$ is a
cyclic subgroup of $SU(2)$ of order $k$. Such a cone is diffeomorphic to $\mathbb{C}^2/\mathbb{Z}_k$ where $\mathbb{Z}_k$ is a cyclic subgroup of $U(2)$ that acts diagonally on $\mathbb{C}^2$ and has a non-stationary solution of the Ricci flow as shown in [FTK03].

1.7. Overview

This thesis is concerned with Ricci flows which are initially Riemannian cones and become smooth instantaneously. In particular we study Ricci flows starting from Ricci flat cones and Kähler Ricci flows starting Ricci flat Kähler cones. The next chapters are organized as follows:

In chapter 2 we recall some notation and definitions in Kähler geometry and present some classical results such as the famous Calabi Yau theorem. We also prove the following long time existence result for Kähler Ricci flow on non-compact Kähler manifolds (which follows almost directly from the arguments in [Cha04] and [TZ06]).

**Theorem 1.** Let $(M, J, g)$ be a complete Kähler manifold with bounded geometry. Assume that $K_M$ is non negative in the sense that there exists a hermitian metric $h$ on $K_M$ such that the associated curvature form $\Theta_h$ is bounded and non-negative with $\rho(\omega) + 2\pi \Theta_h = -\partial \bar{\partial} F$; for some bounded smooth $F$. Then the Kähler Ricci flow on $M$ exists for all times.

We go on to study Ricci flat Kähler cones and closely related Sasaki Einstein manifolds. Apart from the basic definitions some of these results are more recent or less classical so we include some proofs in the appendix. These cones have been studied by various authors mostly to construct asymptotic Ricci flows on $\mathbb{C}^2$, with the fact that any Ricci flat Kähler cone (including the vertex) can be understood as a normal affine algebraic variety with an isolated rational singularity (more precisely a $Q$-Gorenstein singularity) and smoothly away from some compact subset. For such flows we show that the quadratic curvature decay is inherited for positive times from the initial cone metric. In particular, we use their results for our purposes. In particular, we build on the proof of this result uses Perelman’s pseudolocality theorem for non-compact manifolds in combination with Shi’s curvature estimates for the Ricci flow. This gives a rough estimate for the spatial asymptotics of the Ricci flow which can improved if the initial cone is Ricci flat:

**Theorem 2.** Let $(M, g(t))_{t \in (0, T]}$ be a complete Ricci flow with bounded curvature starting from the Riemannian cone $(C, g_C)$. Then $|\text{Rm}(p, t)|$ decays quadratically. In other words, for each $t \in (0, T]$ we have

$$\sup_{x \in M} |\text{Rm}(x, t)| d_{g(t)}(p, x)^2 < \infty.$$  

Moreover, for any $k \in \mathbb{N}$

$$\sup_{x \in M} |\nabla^k \text{Rm}(x, t)| d_{g(t)}(p, x)^{2+k} < \infty.$$  

The proof of this result uses Perelman’s pseudolocality theorem for non-compact manifolds in combination with Shi’s curvature estimates for the Ricci flow. This gives a rough estimate for the spatial asymptotics of the Ricci flow which can improved if the initial cone is Ricci flat:

**Theorem 3.** Let $(M, g(t))_{t \in (0, T]}$ be a complete Ricci flow with bounded curvature starting from the Ricci flat cone $(C, g_C)$. Then for each $\kappa > 0$ and $t \in (0, T]$

$$\sup_{x \in M} t |\text{Rc}(x, t)| \sigma(x, t)^{-\kappa-1} e^{\sigma(x, t)} < \infty,$$

where $\sigma(x, t) = d_{g(t)}(p, x)^2/4t$.  

This has also been observed by Lott and Zhang, see theorem 4.1 and theorem 5.1 in [LZ11].
In other words, \((M, g(t))\) approaches the asymptotic cone at infinity exponentially fast. In contrast, the asymptotically conical Ricci flat manifolds mentioned in the introduction approach the asymptotic Ricci flat cone only at polynomial rate.

In chapter 4 we study expanding gradient Ricci solitons. We recall some well known facts about Ricci solitons and prove a special converse of the previous theorem: If the Ricci flow is self similar expanding and has quadratic curvature decay then its initial condition is a cone. More precisely, the self expanding Ricci flow \((M, g(t))\) converges to a Riemannian cone as \(t \searrow 0\), i.e. it is a Ricci flow starting from a cone in the sense above.

**Theorem 4.** Let \((M, f, g)\) be an expanding gradient soliton such that for some point \(p \in M\) and for each \(k \in \mathbb{N}_0\)

\[
\sup_{x \in M} |\nabla^k \text{Rm}|(x) d_g(p, x)^{2+k} < \infty
\]

then the associated Ricci flow \((M, g(t))\) converges to a Riemannian cone \(C\) as \(t \searrow 0\).

This means that expanding gradient Ricci solitons are (under the assumption of quadratic curvature decay) always examples of smooth Ricci flows coming out of cones.

Chapter 5 contains the main result of this thesis. We return to the Kähler setting and prove that if a Ricci flat cone admits a resolution which satisfies a certain negativity condition then there exist a self-expanding Kähler Ricci flow coming out of the cone:

**Theorem 5.** Let \((M, J)\) be a complex manifold biholomorphic via \(\Phi\) to a Ricci flat Kähler cone \((C, J_C, g_C)\) outside a compact subset. Suppose in addition that

1. \(g\) is a complete Kähler metric on \(M\) such that \(\Phi_* g - g_C = O(r^{-\tau})\), \(\tau > 2\), as \(r \to \infty\) and
2. there exists a complete holomorphic vector field \(X\) on \(M\) such that
   a. \(L_{\Phi(X)} \omega_g = 0\), where \(\Phi(X) = -X + X\)
   b. \(\Phi_*(X) = \frac{1}{2} (r \partial_r - \sqrt{-1} J r \partial_r)\) outside of a compact set
   c. \(\Re(X) = \nabla \theta_X\), for some real valued \(\theta_X\).

Then, if there exists a smooth function \(F \in C^\infty(M)\) such that

\[
\rho(\omega_g) + \omega_g = \sqrt{-1} \partial \bar{\partial} F + L_X \omega_g,
\]

there exists a smooth and complete Kähler metric \(\omega'_g = \omega_g + \sqrt{-1} \partial \bar{\partial} \varphi\) such that

\[
\rho(\omega'_g) + \omega'_g = L_X \omega'_g
\]

which is asymptotic to the cone at exponential rate.

We will see that in the case above \(M\) has to be a resolution of \(C\) and the assumption \(K_M > 0\) is sufficient:

**Theorem 6.** Let \((C, J_C, g_C)\) be a Ricci flat Kähler cone. Suppose that \(\pi : M \to C\) is an equivariant resolution of \(C\) and \(X\) the holomorphic extension of the pull back the radial vector field across the exceptional locus \(E\). Then there exists an expanding (gradient) Kähler Ricci soliton metric on \(M\) with vector field \(X\) if there exists a hermitian metric \(h\) on \(K_M\) such that the associated curvature form \(\Theta_h\) is positive.

The theorem can be reduced to an elliptic Monge Ampère equation. We prove that a solution to this equation exists and decays exponentially fast towards the cone metric.

**Theorem 7.** Let \((M, J, g)\) be a Kähler manifold biholomorphic via \(\Phi\) outside a compact set to \(C \setminus B_R(o)\) such that \(\Phi_* g = g_C\) outside a possibly larger compact set. Moreover let \(X\) be a holomorphic vector field on \(M\) such that

1. \(\Phi_* X = \frac{1}{2} (r \frac{\partial_r}{r} - \sqrt{-1} J r \frac{\partial_r}{r})\) outside a compact set.

\(^2\)This is basically a corollary of similar result in [CD11] and we merely spell out some of the details of remark 1.5 in [CD11].
1. Introduction

(2) $\Re(X) = \nabla \theta_X$ for some real valued function $\theta_X$ on $M$.

(3) $L_{\Re(X)} \omega = 0$.

Then for any $F \in C^\infty_c(M)$ there exists a unique $\varphi \in C^\infty_w(M)$ such that

(1) $\omega \varphi > 0$.

(2) $\varphi$ satisfies the equation

$$\omega^n = e^{\varphi - X(\varphi)} + F \omega^n.$$ 

The proof of this theorem is very similar to the proof of the Calabi conjecture. The main difficulty are the unbounded coefficients in the first order term. Here we will use global Schauder estimates of Lunardi (Theorem 9 below) which we will describe in the appendix.

We conclude this chapter by reviewing some known examples of asymptotically Ricci flat expanders and present new examples which do not reduce to ordinary differential equations. We show:

Theorem 8. Let $\Gamma$ be the subgroup of $SU(2)$ generated by $\text{diag}(e^{2\pi i/p}, e^{2\pi i/q})$ for $p, q \in \mathbb{N}$ with $p > q$ and $\gcd(p, q)$ acting freely on $\mathbb{C}^2 \setminus \{0\}$. Then there exists an ALE expanding Ricci soliton metric on the toric resolution of $\mathbb{C}^2/\Gamma$ if and only if for all integers ($\neq 1$) in the continued fraction expansion of $p/q$ are strictly larger than 2. Moreover, these metrics approach the euclidean metric exponentially fast.

These resolutions can be seen as analogues of the resolutions considered by Kronheimer mentioned in section 1.6. The corresponding Ricci flows converge to $(\mathbb{C}^2/\Gamma, g_{\text{eucl}})$ as $t \searrow 0$.

In chapter 6 we discuss some open problems and further research directions. In particular, we mention the expander version of Perelman’s entropy functional constructed in [FIN05] which might have applications for Ricci flows coming out of cones.

In the appendix we state and prove a version of a result of Lunardi [Lun98] for elliptic operators with unbounded coefficients needed in chapter 5 which is suited for asymptotically conical manifolds.

Theorem 9. Let $(M, g)$ be an asymptotically conical manifold and $X$ the radial vector field outside a compact set. Then for any $f \in C^{0, \alpha}(M)$, $\alpha \in (0, 1)$ and $\mu > 0$ there exists a unique $u \in C^{2, \alpha}(M)$ such that

$$\Delta u - X(u) - \mu u = f.$$

Moreover, there exists a uniform constant $C = C(\alpha, \mu)$ such that

$$\|u\|_{C^{2, \alpha}(M)} \leq C\|f\|_{C^{0, \alpha}(M)}.$$ 

We also remind the reader about some basic toric geometry in the second part of the appendix. In this part we quote some maybe less known results for non-compact toric varieties.
CHAPTER 2

Preliminaries

2.1. Kähler Geometry

This chapter is meant to be a summary of material needed in the following chapters, in particular chapter 5. For general background on Kähler geometry and complex geometry see [Huy05, Bal06, GH94]; the Calabi Yau conjecture and related problems see [Aub98, Bes08, Tia00, Joy00], and [Blo12]; Ricci flow see [CK04, CCG+07, CCG+08, CCG+10, CLN06]; Kähler Ricci flow [CCG+07, SW12, BCG+12]; elliptic equations in general see [GT01, Kry96], (Complex) Monge Ampere equations in particular [PSS12] and [Aub98]; algebraic and complex analytic geometry see [GH94, KN198, Law89, Dem12]; for the minimal model program see [Mat02] and resolutions [Hau03]; for Sasaki geometry see [BG08] and [Spa11].

2.1.1. Complex manifolds. Let $M$ be a smooth manifold of dimension $2n$. The manifold $M$ is called a complex manifold if it can be covered by charts $(U_i, \varphi_i) = \{(z_1^i, \ldots, z_n^i) : U_i \rightarrow \mathbb{C}^n\}$ such that the transition maps $\varphi_j \circ \varphi_i^{-1}$ are biholomorphisms (a complex structure). This induces an almost complex structure $J \in \Gamma(T^*M \otimes TM)$ with $J^2 = -id_{TM}$ on $TM$. For the local holomorphic coordinates $z_j := x_j + \sqrt{-1}y_j$ the endomorphism $J$ is defined by

$$J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \quad J \frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$$

for $j = 1, \ldots, n$. In particular $J^2 = -id_{TM}$. Conversely any even dimensional manifold $M$ with an almost complex structure $J$ on $M$ such that the Nijenhuis tensor

$$N(J)(X, Y) := [X, Y] + J([X, JY] + [JX, Y]) - [JX, JY]$$

vanishes is a complex manifold in the sense that it admits a holomorphic coordinate charts such that the induced almost complex structure is equal to $J$.

The almost complex structure $J$ induces a splitting of the complexified tangent bundle $TM \otimes \mathbb{C}$ into an eigenspace $TM^{1,0}$ with eigenvalue $\sqrt{-1}$ and $TM^{0,1}$ with eigenvalue $-\sqrt{-1}$, respectively. Similarly $TM^* \otimes \mathbb{C} = (TM^{1,0})^* \oplus (TM^{0,1})^*$. In local holomorphic coordinates we can give bases over $\mathbb{C}$ for $TM^{1,0}$, $TM^{0,1}$, $(TM^{1,0})^*$ and $(TM^{0,1})^*$ respectively by

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - \sqrt{-1} \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial y_j} \right)$$

and

$$dz_j = dx_j + \sqrt{-1}dy_j, \quad d\bar{z}_j = dx_j - \sqrt{-1}dy_j.$$
where $\Omega^{k,l}(M) := C^\infty(M, \wedge^{k,l}TM)$ if and only if $J$ is integrable ($N(J) = 0$). In holomorphic coordinates $\Omega^{p,q}(M)$ are the forms of the type

$$
\alpha := \sum_{|I| = p, |J| = q} \alpha_{IJ} dz_i \wedge ... \wedge dz_p \wedge d\bar{z}_j \wedge ... \wedge d\bar{z}_q.
$$

And so in this case $d$ splits as

$$
d = \partial + \bar{\partial},
$$

where

$$
\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)
$$

and

$$
\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).
$$

Since $d^2 = 0$ it follows that $\partial^2 = 0$, $\bar{\partial}^2 = 0$, and $\partial \bar{\partial} = -\bar{\partial} \partial$. This observation allows for the definition of the Dolbeaut cohomology group

$$
H^{p,q}(M, \mathbb{C}) = \ker \left( \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M) \right) / \partial \Omega^{p,q-1}(M).
$$

And similarly the Bott-Chern cohomology

$$
H^{p,q}_{BC}(M, \mathbb{C}) := \{ \alpha \in \Omega^{p,q}(M) : d\alpha = 0 \} / \bar{\partial} \partial \Omega^{p,q-1}(M).
$$

2.1.2. Holomorphic line bundles and the canonical bundle. Holomorphic vector bundles are complex vector bundles (i.e. vector bundles with complex vector spaces as fibers) over complex manifolds whose total space carries a complex structure compatible with that of the base. The complex vector bundle $TM^{1,0}$ of a complex manifold $(M, J)$ is an example of a holomorphic vector bundle. More precisely

**Definition 2.1.1.** A holomorphic vector bundle of rank $r$ over a complex manifold $(M, J)$ is a complex manifold $E$ together with a holomorphic map $\pi : E \to M$ with each fibre $\pi^{-1}(x)$ having the structure of an $r$-dimensional complex vector space such that $M$ admits a covering $\{ U_i, i \in I \}$ of open sets and biholomorphic maps $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ which satisfy $(\pi(x), \cdot) = \bar{\pi}(\psi_i(x))$ (where $\bar{\pi}$ projects to the first factor) and $\psi_i$ restricted to each fibre $\pi^{-1}(x)$ is $\mathbb{C}$-linear. A holomorphic line bundle is a holomorphic vector bundle of rank 1.

**Remark 2.1.2.** The tensor product of 2 holomorphic vector bundles is a holomorphic vector bundle.

Every holomorphic vector bundle $E$ over a complex manifold $(M, J)$ admits a hermitian inner product $h(x)$ on each fibre that depends differentiably on $x$. Moreover, there exists a unique connection $D$ (called Chern connection) on a hermitian holomorphic vector bundle $(E, h)$ such that for any pair of local sections $\sigma_1$ and $\sigma_2$ of $E$ we have $d(h(\sigma_1, \sigma_2)) = h(D\sigma_1, \sigma_2) + h(\sigma_1, D\sigma_2)$ and $D^{0,1} = \bar{\partial}$, i.e. for a local section $\sigma$ and a vector field $X$ on $M$ we have $D_X \sigma = D_X^{1,0} \sigma + \bar{\partial}\sigma(X^{0,1})$ where $X = X^{1,0} + X^{0,1}$ are the parts of $X$ in $TM^{1,0}$ and $TM^{0,1}$ respectively. This connection gives rise to a curvature operator $D^2$ on $E$ and if $r = 1$ (a holomorphic line bundle) then the curvature operator can be identified with a real closed 2-form which will be denoted by $\Theta_h$ on $M$ (the curvature form of $(E, h)$). This curvature form can be locally expressed as $\Theta_h(x) = -\frac{i}{2\pi} \partial \bar{\partial} \log(h(x))$. For a holomorphic line bundle the first Chern class $c_1(L)$ is defined as the cohomology class in $H^2(M, \mathbb{R})$ represented by $\Theta_h$. One can check that this definition does not depend on the choice of $h$ (See [GH94] for details). For a complex manifold $(M, J)$ the canonical line bundle is the holomorphic line bundle

$$
K_M := \wedge^{n,0} TM^*
$$

and the first Chern class of $M$ is defined as

$$
c_1(M) := -c_1(K_M).
$$
2.1.3. Kähler geometry. A hermitian structure on a complex manifold \((M, J)\) is a Riemannian metric \(g\) on \(M\) such that for any \(x \in M\) and \(X, Y \in T_xM\)
\[
g(JX, JY) = g(X, Y).
\]
The hermitian metric \(h\) on \(M\) is then the complex sesquilinear extension of \(g\) to \(TM \otimes \mathbb{C}\) which splits as
\[
h(X, Y) = g(X, Y) - \sqrt{-1} \omega(X, Y),
\]
where \(\omega(\cdot, \cdot) = g(J\cdot, \cdot)\) is a real \((1, 1)\)-form. In local coordinates \(\omega\) is of the form
\[
\sqrt{-1} \sum_{i,j} h_{ij} dz_i \wedge dar{z}_j.
\]
In the following we will write \(g_{ij}\) instead of \(h_{ij}\).

**Definition 2.1.3.** A Kähler structure on a complex manifold \((M, J)\) is a hermitian structure \(h\) on \((M, J)\) such that \(\omega_g = 0\). The metric \(g\) induced by the hermitian structure is then called a Kähler metric and \(\omega_g\) a Kähler form. A complex manifold with a Kähler structure is a Kähler manifold.

The 2-form \(\omega_g\) is a representative of a cohomology class in \(H^{1,1}(M, \mathbb{R})\) called Kähler class. Note that the condition \(\omega_g = 0\) together with the integrability of \(J\) implies that \(\nabla J = 0\) where \(\nabla\) is the associated Levi Civita connection in \(TM\). In holomorphic coordinates this means
\[
\frac{\partial g_{ij}}{\partial z_k} = \frac{\partial g_{kj}}{\partial z_i}, \quad \frac{\partial g_{ij}}{\partial \bar{z}_k} = \frac{\partial g_{kj}}{\partial \bar{z}_i}.
\]
We can extend the Levi Civita connection \(\nabla\) \(\mathbb{C}\)-linearly to \(TM \otimes \mathbb{C}\) and so we obtain for the Christoffel symbols in complex coordinates
\[
\nabla_{\partial z_i} \partial z_j = \Gamma^k_{ij} \partial z_k = g^{kl} \partial z_l g_{kj}, \quad \nabla_{\partial z_i} \partial \bar{z}_j = \Gamma^k_{ij} \partial z_k = g^{lk} \partial z_l g_{kj}, \quad \nabla_{\partial z_i} \partial \bar{z}_j = 0, \quad \nabla_{\partial \bar{z}_i} \partial z_j = 0.
\]
This shows that \(\nabla = \nabla^C\) can be split as \(\nabla = \nabla^{1,0} + \nabla^{0,1}\) where \(\nabla^{1,0}\) is \(\nabla\) restricted to \(TM^{1,0}\) and \(\nabla^{0,1}\) is \(\nabla\) restricted to \(TM^{0,1}\), i.e. \(\nabla^{1,0}_X = \nabla^{1,0}_0 X\) and \(\nabla^{0,1}_X = \nabla^{0,1}_Y X\). In the following we will write \(\nabla = \nabla^{(1,0)}\) for the \((1, 0)\)-part of \(\nabla\) and \(\nabla = \nabla^{(0,1)}\) for the \((0, 1)\)-part of \(\nabla\) unless otherwise indicated. Note that in particular for any smooth real valued function \(f\)
\[
\nabla_i \nabla_j f = \nabla_j \nabla_i f
\]
does not depend on the choice of \(g\). Moreover,
\[
\frac{1}{4} \Delta_g f = \Delta_\partial f = \partial^* \partial f = g^{ij} \nabla_i \nabla_j f = g^{ij} \frac{\partial^2 f}{\partial z_i \partial z_j} = \frac{n}{2} \sqrt{-1} \partial \bar{\partial} f \wedge \omega_g^{n-1}.
\]

**Remark 2.1.4.** We will write \(\Delta_\partial\) (when \(g\) is a Kähler metric) for \(\Delta_\partial\) when acting on functions.

More generally, for a \((1, 1)\)-form \(\alpha\)
\[
\text{tr}_g(\alpha) = g^{ij} \alpha_{ij} = \frac{n}{2} \frac{\alpha \wedge \omega_g^{n-1}}{\omega_g^n}.
\]

For the curvature tensor on \(TM \otimes \mathbb{C}\) associated to \(\nabla\) we have in local holomorphic coordinates
\[
R_{ijkl} = \text{Rm}(e_i, e_j, e_k, e_l) = -\frac{\partial^2 g_{kl}}{\partial z_i \partial z_j} + g^{mp} \frac{\partial g_{kp}}{\partial z_i} \frac{\partial g_{ml}}{\partial z_j}
\]
where we wrote \(e_j = \partial z_j\) and \(e_j = \partial \bar{z}_j\), \(j = 1, ..., n\). And the complex Ricci curvature can be expressed as
\[
R_{ij} = g^{kl} R_{ijkl} = -\frac{\partial}{\partial \bar{z}_j} \left( g^{kl} \frac{\partial g_{kl}}{\partial z_i} \right) = -\frac{\partial^2}{\partial z_j \partial z_i} \log (\det(g_{kl})).
\]

\(^1\)Note that with this convention \(\omega_{Eucl,1}\), the Kähler form associated to the euclidean metric \(h = \sum_i dx_i d\bar{x}_i\) on \(\mathbb{C}^n\) is of the form \(\omega = 2 \sum_{i=1}^n dz_i \wedge d\bar{z}_i\).
It can be checked that \((R_{ij})\) is a hermitian tensor on \((M,J)\) and it splits into a real part \(Rc_g\) (the Ricci curvature of \(g\)) and an imaginary part \(\rho_g(\cdot,\cdot) = Rc_g(J\cdot,\cdot)\), the associated Ricci form:

\[
R(\cdot,\cdot) = Rc_g(\cdot,\cdot) - \sqrt{-1}\rho_g(\cdot,\cdot).
\]

In local coordinates

\[
\rho_g = \sqrt{-1}\sum_{i,j} Rc_{ij}dz_i \wedge dz_j = -\sqrt{-1}\sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(\det(g_{ij})) dz_i \wedge d\bar{z}_j.
\]

Abusing notation we will also write \(\rho(\omega) = -\sqrt{-1}\partial\bar{\partial} \log(\omega^n)\).

**Lemma 2.1.5.** Let \(h\) be a hermitian metric on the canonical line bundle \(K_M\) of a Kähler manifold \((M,J,g)\) and \(\Theta_h\) its curvature. Then there exists a real valued function \(f \in C^\infty(M)\) such that \(2\pi \Theta_h + \rho(g) = \sqrt{-1}\partial\bar{\partial} f\).

**Proof.** Using \(g\) we can define a hermitian metric \(\tilde{h}\) on \(K_M\) as follows. Let \(\sigma \in \Gamma(K_M)\) be a section of the canonical line bundle. Then \(\tilde{h}\) is defined by

\[
|\sigma|_{\tilde{h}}^2 := \frac{\sigma \wedge \bar{\sigma}}{\omega^n_g}.
\]

One can check that the definition of \(\tilde{h}\) does not depend on the choice of the local section \(\sigma\) and that \(\Theta_{\tilde{h}} = -\frac{1}{2\pi}\rho(g)\). Now

\[
\Theta_h - \Theta_{\tilde{h}} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\tilde{h}^{-1} h),
\]

where \(f := \log(\tilde{h}^{-1} h)\) is a well defined smooth real valued function on \(M\).

It follows that \(\rho(\omega)\) is a representative of \(2\pi c_1(M)\). Moreover, even if the \(\partial\bar{\partial}\)-lemma below does not hold in general the difference between two Ricci forms is always exact in the Bott Chern cohomology.

### 2.1.4. Holomorphic vector fields.

**Definition 2.1.6.** Let \((M,J)\) be a complex manifold. A real holomorphic vector field is a smooth section \(\zeta \in \Gamma(TM)\) such that \(L_\zeta J = 0\). A complex holomorphic vector field is a section \(X \in \Gamma(T^{1,0}M)\) such that \(\nabla X = \partial X = 0\).

Note that if \(X\) is a complex holomorphic vector field then it is of the form \(\zeta = \sqrt{-1}J\zeta\) where \(\zeta\) is a real holomorphic vector field. Given a complex holomorphic vector field \(X\) we write \(\zeta := \Im(X) = J\zeta\).

We will need to following two basic observations later:

**Proposition 2.1.7.** If \(M\) is simply connected, \(X\) a complex holomorphic vector field, and \(L_\zeta \omega_g = 0\). Then \(\zeta\) is a gradient vector field: there exists a real valued function \(f\) such that \(X = (\nabla f)^{(1,0)}\).

**Proof.** Let \(e_1,\ldots,e_{2n}\) be a local frame for \(TM\). Since \(X\) is holomorphic we have \(L_\zeta J = 0\) (and \(L_\zeta J = 0\)) and so (using Einstein summation convention)

\[
0 = \langle L_\zeta J e_i, e_j \rangle = \langle -\nabla_{J e_i} \zeta + J \nabla e_i \zeta, e_j \rangle = -J^k \nabla_{e_k} \zeta_j + J^k \nabla_{e_k} \zeta_j
\]

and hence

\[
-\nabla_i \zeta_j = J^k_i J^j_k \nabla_{e_k} \zeta_j = J^k_j \nabla_i \zeta + J^j_k \nabla_i \zeta_j + \nabla J e_i, J \zeta_j.
\]

On the other hand

\[
0 = L_{J \zeta} g(e_i, e_j)
\]

\[
= L_{J \zeta} (g(e_i, e_j)) - \langle L_{J \zeta} e_i, e_j \rangle - \langle e_i, L_{J \zeta} e_j \rangle
\]

\[
= \langle \nabla_{J \zeta} e_i, e_j \rangle + \langle e_i, \nabla_{e_j} J \zeta \rangle
\]

\[
= J^k_i \nabla_{e_k} \zeta + J^k_i \nabla_{e_k} \zeta_j
\]

\[
= J^k_i \nabla_{e_k} \zeta + J^k_i \nabla_{e_k} \zeta_j
\]
So $\nabla_k \zeta_j = \nabla_j \zeta_k$ and since $\pi_1(M) = 0$ it follows that $\zeta = \nabla f$.  

\[ \square \]

**Remark 2.1.8.** Of course $\pi_1(M) = 0$ is not necessary, in fact, $H^{0,1}(M, \mathbb{C}) = 0$ would suffice.

Conversely

**Proposition 2.1.9.** Suppose $X = (\nabla f)^{(1,0)}$ is a holomorphic vector field on $M$ and $f$ some real-valued function on $M$. Then $\xi = J\nabla f$ is a Killing vector field.

**Proof.** First note that

\[
\begin{align*}
0 &= (L_J \nabla f) e_j \\
&= L_J \nabla f(J e_j) - JL_J \nabla f e_j \\
&= -\nabla_J e_j(J \nabla f) + J \nabla_J (J \nabla f) \\
&= -J^k_j J^m_k g^{mn} \nabla_k \nabla_m f + J^k_j g^{mn} \nabla_k \nabla_m f \\
&= -J^k_j g^{mn} \nabla_k \nabla_m f - g^{mn} \nabla_k \nabla_m f
\end{align*}
\]

which implies

\[-J^k_j \nabla_k \nabla_m f = J^a_i \nabla_i \nabla_m f\]

And so we conclude

\[
\begin{align*}
L_J \nabla f (e_i, e_j) &= L_J \nabla f (g(e_i, e_j)) - g(L_J \nabla f e_i, e_j) - g(e_i, L_J \nabla f e_j) \\
&= g(\nabla_e_i J \nabla f, e_j) + g(e_i, \nabla_e_j J \nabla f) \\
&= g e_k J^k_j g^{mn} \nabla_i \nabla_m f + g e_k J^k_j g^{mn} \nabla_j \nabla_m f \\
&= -J^a_i \nabla_i \nabla_m f - J^a_j \nabla_j \nabla_m f \\
&= 0.
\end{align*}
\]

\[ \square \]

### 2.1.5. The $\partial \bar{\partial}$-Lemma. Let $f$ be a real valued function. Then we can define an exact real $(1,1)$-form by

\[
\sqrt{-1} \partial \bar{\partial} f = \sqrt{-1} \frac{\partial^2 f}{\partial z_i \bar{z}_j} dz^i \wedge d\bar{z}^j.
\]

The following shows that in the compact case the converse is also true: any exact real $(1,1)$-form is of the form described above.

**Lemma 2.1.10 ($\partial \bar{\partial}$-Lemma).** Let $(M, J, g)$ be a compact Kähler manifold. If $\alpha$ is an exact real $(1,1)$-form, i.e. $\alpha = d\eta$, then there exists a real-valued smooth function $f$ on $M$ such that

\[
\alpha = \sqrt{-1} \partial \bar{\partial} f.
\]

Note that this is not true if $M$ is non-compact as the following example shows:

**Example 2.1.11.** Let $M = \Delta(1,2) \cup \Delta(2,1) = \{ |z_1| < 1, |z_2| < 2 \} \cup \{ |z_1| < 2, |z_2| < 1 \} \subset \mathbb{C}^2$. Let $\eta$ be a $\bar{\partial}$-closed $(0,1)$-form which is not $\bar{\partial}$-exact. Then let $\alpha := \sqrt{-1} \partial \eta$. Obviously $\alpha$ is exact ($\alpha = d\eta$). However if $\alpha = \sqrt{-1} \partial \bar{\partial} f$ then $\partial (\partial f - \eta) = 0$ but also $\bar{\partial} (\partial f - \eta) = 0$. Hence $\partial (\partial f - \eta)$. Since the domain is contractible there exists a function $h$ such that $dh = \partial f - \eta$ and in particular $dh = 0$. But this implies that $\eta = \partial (f - h)$ which is a contradiction. One can check that

\[
\eta := \frac{(\bar{z}_2 - a) d\bar{z}_1 - (\bar{z}_1 - a) d\bar{z}_2}{(|z_1 - a|^2 + |z_2 - a|^2)^2}
\]

for $a \in (1, \sqrt{2})$ does the job\footnote{Note that any holomorphic function $f$ on $M$ extends to a holomorphic function on $M' := \Delta(2,2) \cap \{ |z_1, z_2| < 2 \}$.}.
Let \((M, J)\) be a complex manifold (not necessarily compact). We call a class \([\alpha] \in H^{1,1}(M, \mathbb{R})\) a Kähler class if there exists a representative \(\omega\) such that \(\omega(\cdot, \cdot) > 0\). On a compact complex manifold by the \(\partial\bar{\partial}\)-lemma two Kähler forms \(\omega_1, \omega_2\) in the same Kähler class \([\alpha]\) differ by \(\sqrt{-1}\partial\bar{\partial}c\) for some smooth real valued function. And so the cone

\[\mathcal{H}_\omega := \{ \varphi \in C^\infty(M) : \omega + \sqrt{-1}\partial\bar{\partial}\varphi \}\]

parametrizes the space of Kähler metrics in the same Kähler class \([\omega]\).

### 2.2. Notions of Positivity

In algebraic geometry there are different notions of positivity and all of them also play an important role in Kähler Ricci flow and related elliptic problems. The first notion is “ample”: a holomorphic line bundle over a complex manifold \((M, J)\) is called ample if for some integer \(p\), sufficiently large, \(H^0(M, L^p)\) gives an embedding of \(M\) into \(\mathbb{C}P^N\). Kodaira showed that if \(M\) is compact\(^4\) this is equivalent to the property that \(c_1(L)\) can be represented by a positive (1, 1)-form. In other words there exists a Kähler metric \(\omega\) on \(M\) such that \([\omega] = c_1(L)\). In particular if the (anti-)canonical line bundle is ample then \(\pm c_1(M)\) can be represented by a positive (1, 1)-form, i.e. \(c_1(M)\) has a sign. If the anti-canonical line bundle is ample then the manifold \((M, J)\) is called a Fano manifold. Note that any representative of \(c_1(L) \in H^{1,1}(M, \mathbb{R})\) is also the curvature form of a hermitian metric on \(L\). This is not true in general, for example if the \(\partial\bar{\partial}\)-lemma fails in the non-compact case. There are several weaker notions of positivity: a holomorphic line bundle is called nef if \(\langle c_1(L), \Sigma \rangle \geq 0\) for every compact complex curve \(\Sigma \subset M\). Note that an ample line bundle is automatically nef. If the canonical line bundle is nef then \(M\) is called minimal. A good reference for different notions of positivity in algebraic geometry is [Laz04].

### 2.3. Monge Ampère Equations and the Calabi Yau theorem

One major achievement in Kähler geometry is the proof of the Calabi conjecture by Aubin and Yau (Yau78, see also Tia00 and Loy00): given a compact Kähler manifold \((M, J, \omega)\) and a representative \(\beta\) of \(c_1(M) \in H^{1,1}(M, \mathbb{R})\) one can always find a Kähler metric in the same Kähler class as \(\omega\) such that its Ricci form is exactly \(2\pi\beta\). In particular, if \(c_1(M) = 0\) there exists a (unique) Ricci flat Kähler metric in every Kähler class of \(M\). Yau’s proof consists of an existence result for an elliptic Monge Ampère equation which is derived as follows: given \(\beta \in c_1(M)\) then by the \(\partial\bar{\partial}\)-lemma there exists a smooth real-valued function \(f\) such that \(2\pi\beta - \rho(\omega) = \sqrt{-1}\partial\bar{\partial}f\). And again using the \(\partial\bar{\partial}\)-lemma a solution in the same Kähler class will be of the form \(\omega_\beta := \omega + \sqrt{-1}\partial\bar{\partial}\varphi\), i.e. \(\rho(\omega_\beta) = 2\pi\beta\). Subtracting the previous equation we have as an equivalent equation

\[\rho(\omega_\beta) - \rho(\omega) = \sqrt{-1}\partial\bar{\partial}f,\]

which, considering the local expression for \(\rho\), becomes:

\[-\sqrt{-1}\partial\bar{\partial}\log\left(\frac{\omega_\beta^n}{\omega^n}\right) = \sqrt{-1}\partial\bar{\partial}f.\]

Now one observes that the left hand side is in fact a global potential and so one gets the Monge Ampère equation

\[\omega^n_\beta = e^{-f}\omega^n,\tag{2.2}\]

where we absorbed the free constant in \(f\) (recall that \(f\) is determined up to a constant). Yau showed that this equation always has a solution if \(M\) is compact. His proof works roughly like this. Fix \(f\) such that

\[\int_M e^{-f}\omega^n = V = \int_M \omega^n.\]

Then consider the set \(S\) of \(t \in [0, 1]\) such that

\[\omega^n_t = e^{-tf}\omega^n,\tag{2.3}\]

\(^3\)By \(\omega > 0\) we mean that the symmetric 2-tensor \(\omega(x)(\cdot, \cdot)\) is positive for every \(x \in M\).

\(^4\)If \(M\) is not compact positivity of \(L\) does not imply ampleness in general, see [Ohs79].
2.3. MONGE AMPÈRE EQUATIONS AND THE CALABI-YAU THEOREM

has a solution. Obviously $0 \in S$ hence $S \neq \emptyset$. The idea is to show that $S$ is open and closed in $[0,1]$. This proves in particular that $1 \in S$ which means that there exists a solution to equation (2.3). The linearization of the left hand side of the continuity equation (2.3) at a point $t \in S$ is just $\Delta_\varphi$ which is an isomorphism between the function spaces in question. And so the implicit function theorem tells us that $S$ is open. The more difficult part of the proof is to show that $S$ is closed. This requires uniform estimates of solutions $\varphi_t$, $t \in S$, in $C^{2,\alpha}$. Once one has these estimates a bootstrapping argument gives all the higher order estimates and then it is an easy consequence of Arzela-Ascoli that $S$ is closed. Yau’s original proof of these a priori estimates goes along these lines: a Moser iteration argument shows uniform $C^0$-estimates (using previously established $L^1$ estimates). This first estimate is Yau’s main contribution to the proof of the Calabi conjecture and so historically it is actually the last. It is a surprising feature of the equation (or the proof) that one can skip the estimates for the gradient and go ahead to prove estimates for $\Delta_\varphi$ which imply by subharmonicity (with respect to $\omega$) uniform estimates for $\partial\bar\partial\varphi_t$. The proof relies on the uniform estimates in $C^0$ and uses the maximum principle. This part uses the compactness of $M$ only in the sense that the bisectional curvature of the complex manifold $(M,J,\omega)$ is bounded from below. Note that the non-mixed derivatives $\nabla^2\varphi_t$ are not estimated. The estimate for $\Delta_\varphi$ then gives also a uniform bound for $\nabla\varphi_t$. In the original proof Yau then goes on to show that the first covariant derivatives of $\partial\bar\partial\varphi_t$ are uniformly bounded. The proof is based on previous work of Calabi and Nirenberg. This part of the proof can be adapted to the non-compact case with almost no modifications. Together these estimates are sufficient to apply Schauder theory and obtain the required $C^{2,\alpha}$-estimates.

The Calabi-Yau theorem has several interesting consequences. For example, it implies that any connected Fano manifold $M$ (a compact complex manifold with $c_1(M) > 0$) is simply connected. Indeed, choosing a Kähler metric such that its Ricci form is a positive representative of $\omega c_1(M)$, the Bochner formula for holomorphic $(p,0)$-forms, $p \geq 1$, implies that $h^{p,0}(M) = 0$ and hence $h^{0,p}(M) = 0$. By Bonnet-Myers the fundamental group $\pi_1(M)$ is finite and by Riemann-Hirzebruch-Roch the universal cover $\tilde{M}$ is compact and satisfies $\chi(\tilde{M},\mathcal{O}) = |\pi_1(M)|\chi(M,\mathcal{O})$. However, using the same arguments for $\tilde{M}$ we have $\chi(\tilde{M},\mathcal{O}) = \sum_p h^{0,p}(M) = 1$ which implies that $|\pi_1(M)| = 1$ and hence $M$ is simply connected (See [Bal06]).

Remark 2.3.1. Ever since Yau’s work appeared in 1978 various steps in the proof have been simplified. Most notably Evans, Krylov, and Safonov developed a general theory of non-linear convex elliptic equations. In particular, their methods allow to obtain interior $C^{2,\alpha}$-estimates (for some $\alpha \in (0,1)$) for the solution $\varphi_t$ once estimates for the second derivative are known. As mentioned above the $C^2$-estimates in Yau’s proof do not really give bounds on all second derivatives of $\varphi_t$, however one can show that the theory of Krylov and Evans can still be applied to obtain uniform $C^{2,\alpha}$-estimates. While the $C^3$-estimates of Yau are global the estimates using Krylov-Evans-Safonov theory are purely local which seems to be helpful in more general settings. The proof of the $C^0$-estimate was modified for more general settings by Kolodziej [Kol98] using pluripotential theory.

Apart from the improvements of Yau’s proof which resulted from new developments in the theory of non-linear elliptic PDEs a lot of new results for the complex Monge Ampere equation have been obtained. First of all Yau and Tian ([TY90, TY91]) studied the equation in the non-compact case, Cheng and Yau ([CY80]) considered the Monge Ampere equation related to the existence problem of complete Kähler Einstein metrics on domains in Kähler manifolds (i.e. with infinite boundary condition), and, more recently, a whole theory of generalized solutions has been developed which deals with the Monge Ampere equation when the right hand side is degenerate (i.e. vanishing at some points) or unbounded (i.e. bounded in a weaker sense; for example in $L^p$). The geometric motivation to study these problems is, for example, the question whether a Kähler manifold admits a Kähler Einstein metric on the complement of a divisor $D$ with prescribed singular behavior near $D$ (i.e. a Kähler Einstein metric with edges). Other questions consider the Kähler Einstein problem in singular varieties or the existence of singular Kähler Einstein metrics on compact complex manifolds whose canonical bundle satisfies a weaker positivity condition (for example nef and big or just nef) than the one (ample) necessary for the existence of smooth Kähler Einstein metric (see for example [Tsu88]).
2.4. Kähler Ricci flow

Hamilton ([Ham86]) showed that the Ricci flow
\[ \frac{\partial g_{ij}}{\partial t} = -2Rc_{ij}(g(t)) \]
on a compact manifold preserves the holonomy group of the starting metric (in the sense that the holonomy group of the evolving metric will be contained in the initial one in the future) and in particular if the initial metric is Kähler (i.e. Hol(g) \subset U(n)) then the evolved metric g(t) will also be Kähler as long as it exists. For the noncompact case this was shown by Shi in [Shi89, Shi97] under the assumption of completeness and bounded curvature. So let ω(t) be the Kähler form of the Kähler Ricci flow g(t). In terms of classes we have
\[ [\omega(t)] = [\omega(0)] - 2\pi c_1(M). \]
If M is compact it was shown by Cao [Cao85] that the (renormalized) flow exists for all times if c_1(M) is positive, zero, or negative and converges in the last two cases as t \to \infty to a Kähler Einstein metric (i.e. ρ(ω) = λω, λ \in \{0, 1\}). As before the Ricci flow equation reduces to a scalar parabolic Monge Ampère equation: let ω_t be defined as
\[ ω_t := ω_0 - tρ(ω_0) \]
for some initial Kähler metric ω_0 with bounded curvature. Then the evolution equation satisfied by the Kähler potential φ of the Ricci flow with respect to ω_t, i.e. ω(t) = ω_t + √{-1}∂\bar{∂}φ is
\[ \frac{\partial φ}{\partial t} = \log \left( \frac{ω^n_t}{ω^n_0} \right) + c(t), \]
where c(t) only depends on t. Conversely any solution of the equation above with c(t) = 0 and φ(\cdot, 0) = 0 gives rise to a Ricci flow with initial condition ω_0. It was later shown by Tian and Zhang [TZ06] that the flow exists on a compact manifold as long [ω(t)] is positive, so in particular, if K_M is nef (i.e. c_1(M) non-positive) the flow exists for all t \in [0, \infty). This can be generalized to the non-compact case in the following way. Recall that here we cannot assume that a positive or negative representative of c_1(M) is given by the Ricci curvature of some Kähler metric.

**Theorem 2.4.1.** Let (M, J, g) be a complete Kähler manifold with bounded geometry. Assume that K_M is non-negative in the sense that there exists a hermitian metric h on K_M such that the associated curvature form Θ_h is non-negative and bounded in all derivatives with ρ(ω) + 2πΘ_h = √{-1}∂\bar{∂}F, for some bounded smooth F. Then the Kähler Ricci flow on M exists for all times.

**Proof.** Let
\[ T := \sup \{ \tau : g(t) \cong g, |Rm|(t) < \infty, \forall t \in [0, \tau] \}. \]
Assume that T < \infty. Now set ω_t := ω + 2πtΘ_h. Obviously [ω_t] = [ω] - 2πtc_1(M). One can check that the Ricci flow starting with ω is of the form ω_t + √{-1}∂\bar{∂}φ where φ(x, t) is a real valued function that solves the initial value problem
\[ \frac{\partial φ}{\partial t} = \log \left( \frac{(ω_t + √{-1}∂\bar{∂}φ)^n}{ω^n_0} \right) - F, \quad φ(x, 0) = 0. \]
Taking one more derivative in t we get
\[ \frac{\partial^2 φ}{\partial t^2} = \Delta φ + 2πr_{g(t)}(Θ_h) ≥ Δ φ. \]

---

5This was also discovered by Cao earlier in [Cao85].
where $\Delta'$ is the Laplacian with respect to the metric $g(t)$ evolving by the Ricci flow. On the other hand
\[
\frac{\partial^2 \varphi}{\partial t^2} = \Delta' \frac{\partial \varphi}{\partial t} + 2\pi \text{tr}_{g(t)}(\Theta_h) \\
= \Delta' \frac{\partial \varphi}{\partial t} + \frac{n}{t} - \frac{1}{t} \text{tr}_{g(t)}(\omega) - \frac{1}{t} \Delta' \varphi \\
\leq \Delta' \left( \frac{\partial \varphi}{\partial t} - \frac{1}{t} \varphi \right) + \frac{n}{t}.
\]
And so
\[
\frac{\partial}{\partial t} \left( t \frac{\partial \varphi}{\partial t} - \varphi - nt \right) \leq \Delta' \left( t \frac{\partial \varphi}{\partial t} - \varphi - nt \right).
\]
The parabolic maximum principle plus the known short time existence theorem gives a uniform bound (only dependent on $T$ and the initial conditions) for $\varphi$ on $M \times [0, \tilde{T}]$ for all $\tilde{T} < T$ and hence we know by the lemma below
\[
\|\varphi(\cdot, t)\|_{C^{k,\alpha}(M)} < C(T, g, h, k, \alpha)
\]
for some constant $C(T, g, h, k, \alpha)$ which only depends on the initial condition, $T$, the curvature form $\Theta_h$, and $k + \alpha$. But this means that the conditions of the theorem are satisfied even up to $T$ and hence beyond which is a contradiction. This proves that $T = \infty$.

**Lemma 2.4.2.** Let $(M, J)$ be a complex manifold of complex dimension $n$ and $\omega_t = \omega_0 + t_\chi$ a smooth path of complete and equivalent Kähler metrics with bounded geometry for all $t > 0$. Suppose that $\varphi_t$ is a smooth solution on $M \times [0, T)$ of
\[
\frac{\partial \varphi}{\partial t} = \log \left( \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_0^n} \right) + F
\]
for some function $F \in \cap_k C^{k,\alpha}(M \times [0, T])$. Assume further that $\varphi$ is uniformly bounded on $M \times [0, T]$. Then $\varphi$ is uniformly (independent of $t$) bounded in $C^{k,\alpha}(M)$ for each $k$ and $\alpha \in (0, 1)$.

**Proof.** The arguments are basically the same as in [Cha04] with the exception that $\omega$ is replace by $\omega_t$. Indeed, let
\[
A := \log(n + \Delta_t \varphi) - \kappa \varphi,
\]
where $\Delta_t$ is the Laplacian with respect to $\omega_t$ and $\kappa \in \mathbb{R}$ a constant depending on $\omega_t$. Then following [Cao85] we can find uniform constants $c_1 = c_1(\omega_t, \omega)$ and $c_2(\omega_t, \omega)$ such that
\[
\frac{\partial A}{\partial t} - \Delta' A \leq -c_1 e^{A/(n-1)} + c_2.
\]
It follows from the maximum principle that $A$ is uniformly bounded on $M$ by a constant $C(\omega_t)$. The higher order estimates follow from the arguments in [Cha04] similarly. Note that in the non-compact setting in order to apply parabolic Schauder estimates one needs in addition to the third order estimates fourth and fifth order estimates as pointed out in [Cha04].

**Remark 2.4.3.** Theorem 2.4.1 also follows from theorem 4.1 in [LZ11]. The proof described above is almost the same as in [LZ11] as it is based on the arguments in [TZ06] for the compact case and [Cha04] which combined need almost no modification.

### 2.5. Sasakian Geometry

**Definition 2.5.1.** A Sasaki manifold is an odd-dimensional manifold $S$ together with a quadruple $(\eta, \xi, \Phi, g)$ (a Sasaki structure) consisting of

1. a 1-form $\eta$ such that $\eta \wedge d\eta^{n-1} \neq 0$ (a contact form),
2. a vector field $\xi$ such that $\eta(\xi) = 1$ and $\iota_\xi dg = 0$ (a Reeb vector field),
3. an endomorphism $\Phi \in \Gamma(\text{End}(TM))$ such that $\Phi^2 = -1 + \xi \otimes \eta$,
4. a Riemannian metric $g$ such that $g(\Phi \cdot, \Phi \cdot) = g(\cdot, \cdot) - \frac{1}{4} \eta \otimes \eta$ (compatibility) and $(\nabla_X \Phi)Y = g(X, Y)\xi - \frac{1}{4} \eta(Y)X$ (integrability).
It follows that $\Phi(\xi) = 0$ and $D := \text{Ker}(\eta)$ together with the restriction $J_D := \Phi_\mathcal{D}$ is a complex vector bundle over $S$. Moreover, $d\eta = 2g(\Phi)$ and hence $(D, J_D, d\eta)$ is a strictly pseudoconvex CR structure on $S$. The Reeb vector field induces a foliation $\mathcal{F}_\xi$ of one dimensional leaves, i.e. each leaf is an integral curve of $\xi$. The space of leaves which is the quotient space of this foliation is usually not a manifold (see comments below). However, it can be given a so-called transverse complex structure as follows. The manifold $S$ can be covered by a foliation atlas $\{U_\alpha, (x_\alpha, z_\alpha)\}$, $x_\alpha : U_\alpha \to I \subset \mathbb{R}$ where $z_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^{n-1}$ are surjective submersions such that

\[(1) \quad \xi|_{U_\alpha} = \frac{\partial}{\partial z_\alpha},
\]

\[(2) \quad \text{the transition maps } z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\beta \cap U_\alpha) \text{ are biholomorphisms.}
\]

Since the Reeb vector field $\xi$ is a Killing vector field ($L_\xi g = 0$) the metric $g$ induces a hermitian metric $g_T^\mathcal{D} := (z_\alpha)\ast g_{U_\alpha}$ on each patch $V_\alpha$ with the associated form $\omega^T_\alpha$. One can check now that $z_\alpha^\ast g^T_\alpha$ is just the restriction of $g$ to $D$ and $z_\alpha^\ast \omega^T_\alpha$ is equal to the restriction of $\frac{1}{2}d\eta$ to $\{x \in U_\alpha : x_\alpha = \text{const}\}$. And so it follows in particular that $\omega^T_\alpha$ is closed and hence $g^T_\alpha$ is Kähler. The collection $\{\omega^T_\alpha, g^T_\alpha\} = (\omega^T, g^T)$ is called a transversal Kähler structure on $S$. The transversal Kähler metric $g^T$ can be identified with the restriction of $g$ to $D$ and hence the definition of the transverse Kähler structure does not depend on the exact choice of holomorphic foliation atlas.

The picture becomes complete with the basic cohomology of the Reeb foliation. A $p$-form $\beta$ on a Sasaki manifold is called basic if $L_\xi \beta = 0$ and $\iota_\xi \beta = 0$. This condition implies that $d\beta$ is also basic. Indeed

\[\iota_\xi d\beta = L_\xi \beta - dt_\xi \beta = 0\]

and

\[L_\xi d\beta = d\iota_\xi d\beta = 0.\]

And so we have a cochain complex

\[
\Omega^0_B \xrightarrow{d_B} \Omega^1_B \xrightarrow{d_B} \ldots \xrightarrow{d_B} \Omega^p_B \xrightarrow{d_B} \Omega^{p+1}_B \xrightarrow{d_B} \ldots
\]

where $d_B$ is the restriction of $d$ to the set of global sections $\Omega^p_B$ of the sheaf of basic $p$-forms. The corresponding cohomology is called basic de Rham cohomology and denoted by $H_B^p(S, \xi)$. Similar to the complex case $\Omega^p_B$ can be split as

\[\Omega^p_B = \bigoplus_{k+l=p} \Omega^{k,l}_B\]

where a basic $p$-form $\beta$ is of type $(k,l)$ if for each chart $\{U_\alpha, (x_\alpha, z_\alpha)\}$ $\beta$ is of the form

\[\beta = \sum_{i_1 < \ldots < i_k} \sum_{j_1 < \ldots < j_l} \beta_{i_1 \ldots i_k j_1 \ldots j_l} dz_{i_1}^1 \wedge \ldots \wedge dz_{i_k}^k \wedge d\bar{z}_{j_1}^1 \wedge \ldots \wedge d\bar{z}_{j_l}^l.
\]

One can check that this definition does not depend on the exact choice of foliation atlas. This gives rise to Dolbeault operators

\[\partial_B : \Omega^k_B \to \Omega^{k+1,l}_B\]

\[\bar{\partial}_B : \Omega^k_B \to \Omega^{k,l+1}_B\]

which satisfy $\partial_B^2 = 0$ and $\bar{\partial}_B^2 = 0$. The associated basic Dolbeault cohomology groups we denote by $H^{p,q}_B(S, \xi)$. On each leaf $V_\alpha$ we can define a transversal Ricci curvature $\text{Rc}_T^\alpha$ and together with the complex structure on $V_\alpha$ a transversal Ricci form $\rho_T^\alpha$. Just as in the Kähler case we have

\[\rho_T^\alpha = -\sqrt{-1}\partial\bar{\partial} \log(\det(g^T_\alpha)).\]

This gives a globally defined basic and closed $(1,1)$-form $\rho_T$ and a hermitian tensor $\text{Rc}_T$ on $S$. The basic cohomology class $\frac{1}{2\pi}[\rho_T]$ in $H^{1,1}_B$ is called basic first Chern class $c^B(S, \xi)$ (each representative of this class is also a representative of the first Chern class $c_1(D)$ of the complex vector bundle $D$) also denoted by $c_1(\mathcal{F}_\xi)$. Again similar to the Kähler case $c^B_1$ does not depend on the choice transverse Kähler metric, i.e. any Sasaki structure on $S$ with a Reeb vector field proportional to $\xi$ that gives the same transverse complex structure (in particular the basic (Dolbeault)-cohomology remains unchanged) will also give back the same basic first Chern class. As for Kähler manifolds there exists a Sasaki analogue of
the \( \partial \bar{\partial} \)-lemma: if \( \gamma \) and \( \delta \) are two real closed basic \((1,1)\)-forms such that \([\gamma] = [\delta]\) in \( H^{1,1}_B(S) \) then there exists a real valued basic function \( \varphi \) on \( S \) (i.e. \( \nabla \varphi = 0 \)) such that \( \gamma = \delta + \sqrt{-1} \partial \bar{\partial} \varphi \). Conversely, fixing the transverse complex structure and \( \xi \), any deformation of the transverse Kähler form \( \omega^T \) of the form \( \omega^T + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \), where \( \varphi \) is a basic function, gives rise to a Sasaki structure on \( M \) with the same transverse complex structure and the same \( \xi \). In fact, this Sasaki structure is unique if one also requires that the complex structure on the cone is the same. This kind of deformation is often called transverse Kähler deformation. The now obvious analog of the Calabi conjecture for Sasaki manifolds was proven by El Kacimi-Alaoui [EKA90]: suppose \( \beta \) is basic real closed \((1,1)\)-form \( \beta \) is a representative of \( 2\pi c_1(F) \). Then in any transverse Kähler class \( [\omega^T] \) there exists a unique transverse Kähler form, i.e. a transverse Kähler deformation \( \tilde{\omega}^T \) of \( \omega^T \), such that \( \beta \) is the transverse Ricci form \( \rho^T \) of \( \tilde{\omega}^T \).

**Example 2.5.2.** A first example of a Sasakian manifold is the round sphere \( S^{2n-1} \) in \( \mathbb{C}^n \) with the induced Sasaki structure. This is not the only Sasaki structure on \( S^{2n-1} \). In fact, there are many other Sasaki Einstein structures \( S^{2n-1} \) (see [BG98]). More generally, for any finite subgroup \( \Gamma \subset U(n) \) that acts freely on \( S^{2n-1} \) the quotient \( S^{2n-1}/\Gamma \) has a canonical Sasaki structure.

**Definition 2.5.3 (Regularity).** A Sasaki manifold is called regular if all orbits generated by \( \xi \) are compact and of equal length. It is called quasiregular if all orbits are compact. And it is called irregular if \( \xi \) has noncompact orbits.

It can be shown that a regular Sasaki manifold is a \( U(1) \)-bundle over a smooth Kähler manifold whereas a quasi-regular Sasaki manifold is a \( U(1) \)-bundle over a Kähler orbifold. More precisely

**Theorem 2.5.4 (from [BG98]).** Let \( S \) be a compact regular (quasi-regular) Sasaki manifold. Then the space of leaves of the Reeb foliation \( F_\xi \) is a compact Kähler manifold (orbifold) \((Z,\omega_Z,J_Z)\). And the projection \( \pi : S \to Z \) is a (orbifold) Riemannian projection, such that the fibres are totally geodesic circles. The cohomology class \([\omega_Z]\) is proportional to an integer class in the (orbifold) cohomology \( H^2_\mathbb{Z}(Z,Z) \).

**Example 2.5.5.** \( S^{2n-1} \) with the standard Sasaki structure is a regular Sasaki manifold as a \( U(1) \) bundle over \( \mathbb{C}P^{n-1} \). An example of a quasi-regular Sasaki manifold is \( S^3/\Gamma \) where \( \Gamma \subset U(n) \) is a finite cyclic subgroup that is not generated by a multiple of the identity. We will see later that a manifold can admit several Sasaki structures. For example the circle bundle associated to the canonical line bundle over \( \mathbb{C}P^2 \# 2\mathbb{C}P^2 \) admits an irregular Sasaki Einstein structure but not a regular one (see remarks below).
over \( \mathbb{C}P^2 \# k\mathbb{C}P^2 \) as above for \( k = 1, 2 \). However, there still exist irregular Sasaki Einstein structures in both cases (see [FOW09]). For more details we refer to the survey of J. Sparks [Spa11].

2.7. Kähler Cones

A Riemannian cone \((C(S), g_C)\) over a compact manifold Riemannian manifold \((S, g_S)\) is called a Kähler cone if there exists a compatible integrable almost complex structure \(J\) on \(C\) such that \(\xi := Jr\partial_r/2 \in T_{r,s}S \subset T_{r,s}C\). The Euler vector field \(r\partial_r\) is a holomorphic vector field on \(C(S)\):

\[
(L_r \xi, J) X = [r\partial_r, JX] - J[r\partial_r, X] = J\nabla^g_{r\partial_r} X - \nabla^g_{JX} r\partial_r - J\nabla^g_{r\partial_r} X + J\nabla^g_{JX} r\partial_r = 0.
\]

where \(X\) is any vector field on \(C\) and \((X)_t\) and \((X)_r\partial_r\) denote the part of \(X\) which is tangential to \(S\) and the radial part of \(X\) respectively. The characteristic vector field \(\xi\) is also holomorphic and satisfies in addition

\[
L_\xi g_C = 0.
\]

The Kähler structure on \(C(S)\) induces a Sasaki structure on \(S\) as follows: first let

\[
\eta := 2\sqrt{-1} \log(r) = 2d\log(r) \circ J^{-1}.
\]

Then one can check that \(\eta\) restricted to \(\{1\} \times S\) satisfies \(\eta \wedge d\eta^{n-1} \neq 0\), and \(\eta(\xi) = 1\). Moreover, \(\Phi\) can be defined on \(S\) by \(\Phi := J_\xi r - \eta \otimes \xi\). It follows that \(\Phi\) and \(g_S\) satisfy the compatibility and the integrability condition (see [BG08] for details). Altogether \((\eta, \xi, \Phi, g)\) gives a Sasaki structure on \(S\). Conversely a Sasaki structure \((\xi, \eta, \Phi, g_S)\) on \(S\) gives rise to a complex structure on the cone \(C(S)\) and a Kähler structure defined by

\[
\omega_C := \sqrt{-1} \partial \bar{\partial} r^2 = \frac{r^2 \eta}{2} \quad \text{and} \quad g_C := \sqrt{-1} \partial \bar{\partial} (\cdot, J \cdot) = r \partial \eta \wedge (\cdot, J \cdot) + \frac{r^2}{2} d\eta(\cdot, J \cdot) + d\eta(\cdot, J \cdot).
\]

\textbf{Theorem 2.7.1.} [vC10] Let \((C(S), J_C, \omega_C, g_C)\) be the Kähler cone over a Sasaki manifold \((S, \Phi, \eta, \xi, g)\). Then \(C(S) \cup \{\rho\}\) is a normal complex analytic space, in fact it is an affine normal algebraic variety in \(\mathbb{C}^N\) for some \(N > n\).

\textbf{Proof.} See appendix B. \qed

\textbf{Remark 2.7.2.} On a singular variety \(V\) the canonical line bundle \(K_V\) still has a meaning. However, it is not a holomorphic line bundle over \(V\) anymore. Details are described in appendix B. In the case of a Kähler cone (or more generally a complex variety with sufficiently mild singularities) \(K_{C(S)\cup\{\rho\}}\) can be defined as a more general object. More precisely, if \(L\) is a holomorphic line bundle over a non-singular variety \(V\) then the sheaf of holomorphic sections \(\mathcal{O}_V(L)\) is what is called an invertible sheaf (a sheaf of \(\mathcal{O}_V\)-modules that is locally isomorphic to \(\mathcal{O}_V\), the sheaf of holomorphic functions on \(V\), see appendix B). Conversely, it is also true that every invertible sheaf \(\mathcal{F}\) on a non-singular variety gives rise to a holomorphic line bundle (i.e. \(\mathcal{F} = \mathcal{O}_V(L)\)). On a singular variety invertible sheaves can still be defined and are therefore the natural extension of the definition of holomorphic line bundles to (normal) singular varieties. Invertible sheaves have the following additional property: Suppose \(\mathcal{L}\) is an invertible sheaf on a (normal) singular variety \(V\). If \(\pi : \hat{V} \to V\) is a resolution then the inverse image of the sheaf \(\pi^*\mathcal{L}\)
is also an invertible sheaf and hence the sheaf of holomorphic sections $\mathcal{O}_V(\pi^*L)$ of a holomorphic line bundle on $\hat{V}$ which we will denote by $\pi^*L$. If the singularities are for example Gorenstein then $K_V$ can be understood as an invertible sheaf.\footnote{If it is clear what is meant we will write $K_V$ for the invertible sheaf.} In case of a Ricci flat Kähler cone (see below) the singularities are $\mathbb{Q}$-Gorenstein (\cite{vC10}, see theorem B.2.1) which means that for some sufficiently large positive integer $p \in \mathbb{N}$ the sheaf $\pi_\ast \mathcal{O}_{C(S)}(K_{C(S)}^p)$, where $\pi : C(S) \to C(S) \cup \{\emptyset\}$, is invertible. For such a cone $\pi_\ast \mathcal{O}_{C(S)}(K_{C(S)}^p)$ replaces $K_{C(S) \cup \{\emptyset\}}^p$.

### 2.8. Ricci flat Kähler cones

The Ricci curvature of an $n$-dimensional Riemannian cone $(C(S), dr^2 + r^2 g_S)$ is given by

$$\text{Re}(g_C) = \text{Re}(g_S) - (n - 2)g_S.$$ 

It follows that the cone $(C(S), g_C)$ is Ricci flat if and only if $\text{Re}(g_S) = (n - 2)g_S$. In particular, Ricci flat Kähler cones are Riemannian cones over Sasaki Einstein manifolds. The following proposition of van Coevering gives a necessary condition for $C(S)$ to admit a Ricci flat Kähler cone metric.

**Proposition 2.8.1**\footnote{\cite{vC10}.} If $(C(S), g_C)$ is a Ricci flat Kähler cone then for some positive integer $p$ the $p$-th power of the canonical line bundle $K_{C(S)}^p$ admits a nowhere vanishing section $\Omega$ with $L_\Omega = \sqrt{-1}n p \Omega$.

We call $C(S)$ a Calabi Yau cone if $K_{C(S)}$ (i.e. $p = 1$) admits a global non-vanishing section. This implies in particular $K_{C(S) \cup \{\emptyset\}} := \pi_\ast \mathcal{O}_{C(S)}(K_{C(S)})$, where $\pi : C(S) \to C(S) \cup \{\emptyset\}$, is not only an invertible sheaf but also trivial or, equivalently, it admits a global non-vanishing section (it follows that $(C(S) \cup \{\emptyset\}$ is Gorenstein, which is the case if $S$ is simply connected). The holonomy group of $(C(S), J_C, g_C)$ is contained in $SU(n)$ if and only if $C(S)$ is Calabi Yau.

**Example 2.8.2.** Let $\Gamma$ be a finite subgroup of $U(n)$ acting freely on $\mathbb{C}^n \setminus \{0\}$. Then $S^{2n-1}/\Gamma$ can be given a canonical Sasaki Einstein structure (the standard one). And the corresponding Kähler cone $(C(S), g_C) := (\mathbb{C}^n / \Gamma; g_{\text{eucl}})$ is Ricci flat (in fact, it is flat). The canonical divisor is not Cartier (see Appendix B) in $C(S) \cup \{\emptyset\}$ (i.e. the sheaf $\pi_\ast \mathcal{O}_{C(S)}(K_{C(S)})$ is not invertible) in general. However, if $\Gamma \subset SU(n)$ (more generally $\Gamma \subset SL(n, \mathbb{C})$) then the quotient $\mathbb{C}^n / \Gamma$ is indeed Calabi Yau in the sense above.

#### 2.8.1. Resolution of Ricci flat Kähler cones

Since $C(S) \cup \{\emptyset\}$ is a normal algebraic variety it can, by Hironaka’s theorem (see \cite{EV03}), always be resolved and so for any given Kähler cone there exists a smooth complex manifold $M$ which is biholomorphic to $C$ outside a compact set.\footnote{The converse is also true: If we assume that $M$ is biholomorphic to $C(S)$ outside a compact set then this biholomorphism $\phi : M \setminus K \to (C(S) \setminus \overline{B(0, R)})$ can be extended across the compact set $K$ to become a resolution $\tilde{\phi} : M \to C(S) \cup \{\emptyset\}$ of the complex space $C(S) \cup \{\emptyset\}$ described above (see \cite{vC12}).} Since $H^1(S, \mathbb{R}) = 0$ ($S$ has positive Ricci curvature) we know that $\pi : H^2(M, \mathbb{R}) \to H^2(M, \mathbb{R})$ is a well defined injection and if we assume that the Kähler metric $\omega$ on $M$ in the Kähler class $[\omega] \in H^2(M, \mathbb{R})$ (if it exists) approaches the cone metric sufficiently fast then $[\omega] \in i(H^2(M))$.

**Proposition 2.8.3.** Let $\pi : M \to C(S) \cup \{\emptyset\}$ be a resolution of a Kähler cone $C(S)$. Suppose $\omega$ is a Kähler form on $M$ such that $\omega_C - \pi_\ast \omega = O(r^{-\tau})$, $\tau > 2$. Then $[\omega] \in i(H^2(M))$.

**Proof.** Let $\Sigma$ be a compact 2-cycle in $M \setminus K$. Then $\Sigma$ is a 2-cycle in $S \times \{1\}$ and by Stokes’ theorem

$$\int_{\Sigma \times \{1\}} \pi_\ast \rho - \rho_C = \int_{\Sigma \times \{r\}} \pi_\ast \rho - \rho_C = \text{Area}_g(\Sigma)(r^{2-\tau}) \to 0, \quad \text{as } r \to \infty$$

hence $[\pi_\ast \omega - \omega_C] = 0$ in $H^2(C, \mathbb{R})$ and so $[\omega] \in i(H^2(M))$. $\square$

It follows that any Kähler metric on $M$ asymptotic to the cone can be found in $i(H^2(M, \mathbb{R}))$. Similarly one can prove that $c_1(M) \in i(H^2(M, \mathbb{R}))$. From now on we will write $H^2(M, \mathbb{R})$ for $i(H^2(M, \mathbb{R}))$. 
As we have seen above the cone $C(S) \cup \{0\}$ can be understood as an affine variety $V := \overline{C(S)}$ in $\mathbb{C}^N$ with an isolated singularity at the origin (even if $C(S)$ is not Ricci flat). The radial holomorphic vector field is then a tangent vector field on the regular part $V_{reg}$ of $V$. Moreover, it extends to a holomorphic vector field on $\mathbb{C}^N$. Namely, it corresponds to the action of $C^* : \mathbb{C}^N \rightarrow \mathbb{C}^N, z_j \mapsto e^{a_jt}z_j$, $a_j \in \mathbb{R}$, $t \in \mathbb{C}$. Now this group action of $C^*$ has a unique lift to a minimal resolution $M$ of the singular variety by theorem 1.5 in [EV03]. We will call $M$ an equivariant resolution. This gives rise to a holomorphic vector field on the nonsingular variety $M$ which is a lift of the original radial holomorphic vector field. In other words a Kähler cone always has a minimal resolution which admits a unique lift of the radial holomorphic vector field, i.e. an equivariant resolution.

**Proposition 2.8.5.** Let $\pi : M \rightarrow C(S) \cup \{0\}$ be an equivariant resolution of a Ricci flat Kähler cone, $X$ a holomorphic vector field on $M$ and $\omega$ a Kähler form on $M$. Then $X$ is a gradient vector field with respect to $\omega$ meaning that there exists a complex valued function $\theta_X$ such that $\text{grad}_{\omega} \theta_X = X$. Moreover, if $\xi := \Im(X)$ is a Killing vector field ($L_{\xi}g = 0$) then $\theta_X$ is real valued.

**Proof.** Note that $\alpha := i_X\omega$ is a holomorphic $(0,1)$-form. And by the arguments in the proof of proposition [B.2.2] $H^1_\omega(X) = 0$ which implies that $\alpha = \sqrt{-1}\partial\bar{\partial}\theta_X$ for some complex-valued function. If $L_\xi\omega = 0$ then $\Im(\partial\alpha) = \Im(\sqrt{-1}\partial\bar{\partial}\theta_X) = 0$ and hence (up to a pluriharmonic function) $\theta_X$ can be chosen to be real-valued. □

**Proposition 2.8.6.** [VC10] Suppose that $\pi : M \rightarrow C \cup \{0\}$ is a resolution of a Ricci flat Kähler cone $C$. Let $E := \pi^{-1}(C)$ be the exceptional set and let $\omega_1$ be a complete Kähler metric on $M$ such that $\omega_1 = \beta + \sqrt{-1}\partial\bar{\partial}f \in A^{1,1}(M, \mathbb{C})$ for some smooth function $f \in C^\infty(M, \mathbb{R})$ and $\beta \in A^1(M, \mathbb{C})$ a closed $(1,1)$-form with compact support. Then there exists a complete Kähler metric $\omega_0$ on $M$ and compact subsets $K_1 \subset K_2 \subset M$ such that $\omega_0 = \omega_1$ on $K_1$ and $\omega_0 = \pi^* \omega_C$ on $M \setminus K_2$.

**Proof.** See appendix B.

**Corollary 2.8.7.** Let $\pi : M \rightarrow C \cup \{0\}$ be an equivariant resolution of $C \cup \{0\}$. And suppose $\omega_1$ and $\beta$ are as above with the additional assumption that $L_\xi\beta = L_\xi\omega_1 = 0$. Then there exists a Kähler metric $\omega_0$ such that $\omega_0 = \omega_1$ inside a compact set and $\omega_0 = \pi^* \omega_C$ outside a large compact set and $L_\xi\omega_0 = 0$.

Let $G$ be the closure of the one-parameter subgroup generated by $\xi$ in $\text{Aut}_0(M)$ (The connected component of the identity). This is a compact subgroup and we claim that in any Kähler class $[\alpha]$ we can find a Kähler form $\omega$ which is $G$-invariant. Indeed, let $\omega_0$ be an arbitrary Kähler form in $[\alpha]$ then $\phi_{t}\omega_0$ lies in the same Chern-Bott class as $\omega$ by the lemma below and since $G$ is compact we can average over $G$ to obtain the desired invariant Kähler form.

**Lemma 2.8.8.** Let $\phi_{t}$ be an element of the identity component of $\text{Aut}(M)$. Then $\phi_{t}^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}f$ for some real valued function $f$.

**Proof.** Let $\phi_{t}$, $t \in [0,1]$ be a smooth path in $G$ connecting the identity with $g$ and let $\zeta := \zeta_0$ be the corresponding holomorphic vector field. Then $Y := \zeta - \sqrt{-1}\partial\bar{\partial}\zeta$ is a complex holomorphic vector field on $M$. And since $H^2_{\omega}(M, \mathbb{C}) = 0$ we have $\tau_Y\omega = \sqrt{-1}\partial\bar{\partial}\theta_Y$ for some complex valued function $\theta_Y$. Similarly $\tau_Y\omega = \partial\bar{\partial}(\theta_Y - \bar{\theta}_Y) = -2\sqrt{-1}\partial\bar{\partial}\zeta(\theta_Y)$. And so

$$\phi_t^*\omega - \omega = \int_0^1 \phi_t^*L_\xi\omega dt = -2\sqrt{-1}\partial\bar{\partial} \int_0^1 \zeta(\theta_Y) \circ \phi_t dt. \quad (2.5)$$

It follows that in each Kähler class $H^2_{\omega}(M, \mathbb{R})$ we can find a Kähler metric $\omega$ which is $G$-invariant and proportional to the pull-back of the cone metric outside of a compact subset.

### 2.8.2. 2-dimensional Ricci flat Kähler cones.

If $\dim_{\mathbb{C}}(C) = 2$ and $Rc(g_C) = 0$ then $Rc(g_S) = 2g_S$ and so $S$ is a 3-dimensional space form: a quotient of a round sphere by a finite subgroup $\Gamma$ of $SO(4)$ and $C$ is of the form $\mathbb{R}^4/\Gamma$. Since the standard Sasaki structure is the only Sasaki Einstein structure on $S^3/\Gamma$ it follows that $C = \mathbb{C}^2/\Gamma$ and $\Gamma \subset U(2)$ if we assume that $C$ is a Ricci flat Kähler
cone. There are basically two different ways to obtain a smooth Kähler manifold asymptotic to \( C \) (see next section for the definition of asymptotically conical). The first method is to resolve the singularity by adding in an exceptional divisor resulting from a blow up. Such a resolution always exists and is unique. The second method is to deform the singularity to a smooth manifold (this procedure is called smoothing). See [Joy00] for a definition of deformation. Both these methods have been studied extensively for \( \Gamma \subset SU(2) \). Such singularities are called rational double points and are completely classified. They are commonly known as ADE-singularities (A for \( A_k \), D for \( D_k \), E for \( E_6, E_7, \) and \( E_8 \) which are the Dynkin diagrams of the corresponding subgroups of \( SU(2) \)) and the intersection matrix of the rational curves that make up the exceptional set is completely determined by this combinatorial data.

A minimal resolution of such a singularity always admits an ALE hyperkähler structure as shown in [EH79], [GH79], [LeB91], [Hil79], [Kro89b], [Kro89a], and [Joy00]. In particular, these resolutions are ‘crepant’ meaning that the canonical line bundle is trivial. These type of singularities also admit smoothings which are (always) diffeomorphic to the corresponding minimal resolutions. In [Kro89a] it is shown that these manifolds admit ALE hyperkähler structures as well. If \( \Gamma \) is a subgroup of \( U(2) \) which is not contained in \( SU(2) \) then \( \mathbb{C}^2/\Gamma \) does not admit a crepant resolution meaning that the unique minimal resolution has a non-trivial canonical line bundle. If \( \Gamma \) is a cyclic subgroup of \( U(2) \) then the minimal resolution is a toric resolution. Again the intersection matrix of such a minimal resolution can be determined by the combinatorial data of \( \Gamma \) and it is shown in [CS04] that each such resolution admits a scalar flat ALE Kähler metric. Ricci flat Kähler metrics on smoothings of \( \mathbb{C}^2/\Gamma, \Gamma \subset U(2) \), which are not necessarily hyperkähler, have been studied in [Suv12]. The only such singularities that admit a smoothing are cyclic (not necessarily of type \( A_k \)) and the rational double points (ADE) mentioned above. The smoothings of these cyclic singularities are not simply connected and hence in particular not diffeomorphic to the corresponding toric resolutions (which are simply connected). Note that in complex dimension greater than 2 there do not exist any non-trivial deformations of \( \mathbb{C}^n/\Gamma, \Gamma \subset U(n) \), \( (\mathbb{C}^n/\Gamma \) is rigid) [Sch71] and so for \( n \geq 3 \) the ALE Kähler manifolds can assumed to be resolutions (see [Joy00]).

2.8.3. Higher dimensional Kähler cones. The class of 5 and higher dimensional Sasaki Einstein manifolds is much larger. Higher dimensional Ricci flat Kähler cones are not necessarily quotient singularities. In [Joy00] Joyce investigates the case \( n \geq 3 \) when \( C = \mathbb{C}^n/\Gamma, \Gamma \subset SU(n) \). He shows that on any crepant resolution \( M \) of \( \mathbb{C}^n/\Gamma \) there exists a unique ALE Ricci flat Kähler metric in each Kähler class. Since \( \mathbb{C}^n/\Gamma \) is rigid this is indeed the only interesting case to study. These results have been generalized crepant resolutions of Calabi Yau cones by van Coevering [vC08, vC09, vC10, vC11], Goto [Got12], Santoro [San09], and deformations of Calabi Yau cones in [CH12].

2.9. Asymptotically conical manifolds

In this section we define asymptotically conical Riemannian and in particular asymptotically conical Kähler manifolds. We start out with an arbitrary smooth Riemannian cone \( C := \mathbb{R}_+ \times S \) over a compact Riemannian manifold \( (S, g_S) \) with the cone metric \( g_C := dr^2 + r^2 g_S \).

**Definition 2.9.1.** A Riemannian manifold \( (M, g) \) is called asymptotically conical to \( (C, g_C) \) if there exists a compact subset \( K \subset M \), a positive number \( R \) and a diffeomorphism \( \Phi : M \setminus K \rightarrow C \setminus B_R(0) \) such that for \( k \in \mathbb{N} \)

\[
\sup_{x \in C \setminus B_r(0)} |\nabla^k (\Phi_* g - g_C)(x)|_{g_C} = O(f_{k,0}(r)), \text{ as } r \rightarrow \infty, \tag{2.6}
\]

where \( \nabla \) is the Levi Civita connection with respect to \( g_C \), and for \( \alpha \in (0, 1) \)

\[
\sup_{x \in C \setminus B_r(0)} \sup_{d(x,y) < \delta} \frac{|\nabla^k (\Phi_* g - g_C)(x) - \nabla^k (\Phi_* g - g_C)(y)|}{d(x,y)^\alpha} = O(f_{k,\alpha}(r)) \text{ as } r \rightarrow \infty, \tag{2.7}
\]

where \( \delta \) is the injectivity radius of \( g_C \) at \( x \), \( \nabla^k (\Phi_* g - g_C)(x) - \nabla^k (\Phi_* g - g_C)(y) \) is defined via parallel transport along the geodesic joining \( x \) and \( y \), and \( \lim_{r \rightarrow \infty} f_{k,\alpha}(r) = 0 \) for \( k \in \mathbb{N}_0, \alpha \in [0, 1) \), and \( r > R \).

\[\text{This also follows from the results of Tian and Yau in } [TY90] \text{ as Joyce points out.}\]
Remark 2.9.2. Typically $f_{k,\alpha}(r) = r^{-\tau - k - \alpha}$ for some $\tau > 0$. And if in addition $(S, g_S) = (S^n/\Gamma, g_{S^n})$ for some finite subgroup $\Gamma \subset SO(n)$ acting freely on $S^n$ then $(M, g)$ is called ALE (asymptotically locally euclidean). Such manifolds have been studied extensively in various contexts. Here we are mostly interested in the case where $\tau > 2$ and the case where $f_{k,\alpha}(r) = r^{k+2+\alpha}e^{-r^2/2}$ (or $f_{k,\alpha}(r) = C_k e^{-\mu r^2/2}$, $\mu > 0$). In the first case we say that $(M, g)$ is asymptotic to $(C,g_C)$ at rate $\tau$ while in the second case we say that $(M, g)$ is asymptotic to $(C, g_C)$ at exponential rate.

We also define the following function spaces:

**Definition 2.9.3.** Let $(M, g)$ be an asymptotically conical manifold. And let $C^{k,\alpha}_{w}(M)$ the space of $k$ times continuously differentiable functions with $\alpha$-Hölder continuous $k$-th derivative on $M$. Let $w : M \rightarrow \mathbb{R}_+$ be some smooth weight function then

$$C^{k,\alpha}_{w}(M) := \{ u \in C^{k,\alpha}_{loc}(M) : \| u \|_{C^{k,\alpha}_{w}(M)} < \infty \},$$

where

$$\| u \|_{C^{k,\alpha}_{w}(M)} := \sum_{k=0}^{\infty} \sup_{x \in M} \left| \nabla^j(wu)(x) \right| + \sup_{x \in M} \sup_{y, d(x,y) < \delta} \frac{|\nabla^k(wu)(x) - \nabla^k(wu)(y)|}{d(x,y)^\alpha},$$

where $\delta$ is the injectivity radius of $g_C$ at $x$ and $\nabla^k(wu)(x) - \nabla^k(wu)(y)$ is defined via parallel transport along a geodesic joining $x$ and $y$.

**Remark 2.9.4.** Note that these function spaces are different from the ones that are usually defined in order to study the Calabi Yau problem on AC manifolds. In chapter 5 $w$ will be $\exp(\theta_X)$.

### 2.9.1. Asymptotically conical Kähler manifolds.

**Definition 2.9.5.** A Kähler manifold $(M, J, g)$ is complex asymptotically conical to a Kähler cone $(C, J_C, g_C)$ if $(M, g)$ is asymptotic conical to $(C, g_C)$ and in addition

$$\sup_{x \in C \setminus \{0\}} \left| \nabla^k(\Phi \ast J - J_C)(x) \right|_{g_C} = O(g_{k,0}(r)), \text{ as } r \rightarrow \infty,$$

and for $\alpha \in (0, 1)$

$$\sup_{x \in C \setminus \{0\}} \sup_{y, d(x,y) < \delta} \frac{|\nabla^k(\Phi \ast J - J_C)(x) - \nabla^k(\Phi \ast J - J_C)(y)|}{d(x,y)^\alpha} = O(g_{k,\alpha}(r)), \text{ as } r \rightarrow \infty,$$

where $\delta$ is the injectivity radius of $g_C$ at $x$, $\nabla^k(\Phi \ast g - g_C)(x) - \nabla^k(\Phi \ast g - g_C)(y)$ is defined via parallel transport along the geodesic joining $x$ and $y$, and $\lim_{r \rightarrow \infty} g_{k,\alpha}(r) = 0$ for $k \in \mathbb{N}_0$, $\alpha \in [0, 1)$, and $r \in \mathbb{R}$.

**Definition 2.9.6.** A Kähler manifold $(M, J, g)$ is called asymptotic to a Kähler cone $(C, J_C, g_C)$ if $g_{k,\alpha} = 0$, in other words $\Phi$ is a biholomorphism.

One can show that in the second case $M$ is a resolution of $C$ and $[\omega_g] \in H^2(M, \mathbb{R})$ under sufficient decay assumptions on $g_{k,\alpha}$. Moreover, the first definition reduces to the second one in case $M$ is ALE.
CHAPTER 3

Ricci Flows attaining the Cone as an initial condition

3.1. Introduction

In general the Ricci flow develops singularities meaning that there are regions where the curvature blows up. Typically such a singularity is of conic type (see [FIK03]). In order to continue the flow one has to understand the Ricci flow with more general metric spaces as initial conditions. Such flows have been studied by Miles Simon. In [Sim09] Miles Simon studies Ricci flows of metric spaces which are Gromov Hausdorff limits of sequences of smooth and complete manifolds which are non-collapsed and whose Ricci curvature is bounded from below. In the case dim(M) = 3 he shows that for any such limit (X, d) there exists a smooth Ricci flow (M, g(t)), t ∈ (0, T] such that (M, g(t)) converges to the initial metric space in the Gromov Hausdorff sense as t ↘ 0. In particular it follows that M is homeomorphic to X and hence X is actually smooth with a non-smooth metric at t = 0. In higher dimension this is no longer the case as the following example shows.

Example 3.1.1 (Eguchi Hansen). Let $M := L^2 → \mathbb{CP}^1$ be the total space of the canonical line bundle of $\mathbb{CP}^1$ equipped with the Ricci flat Eguchi Hansen metric $g$. Then $(M, x, g_j = j^{-1}g)$, $j \in \mathbb{N}$, $x \in \mathbb{CP}^1$ is a non-collapsed sequence Ricci flat manifolds and the associated Ricci flows $g_j(t) = g_j$ converge to a cone $(\mathbb{C}^2/\mathbb{Z}_2, \delta)$ as $j → \infty$. In other words, the limit is not smooth.

Simon constructs Ricci flows with metric initial conditions by taking the limits of the Ricci flows on each manifold $(M_i, g_i(t))$ in the sequence. While the limit might not be smooth in higher dimensions even for positive times there still might be a smooth Ricci flow with $(X, d) = \lim_{i→∞}(M_i, g_i)$ as an initial condition:

Example 3.1.2 (4-dimensional expanders in [FIK03]). Let $M_k$, $k > 2$, be a crepant resolution of a complex 2-dimensional cyclic quotient singularity equipped with an ALE hyper-Kähler metric $g$ (see [Kro89a]). Again the sequence $(M_k, x, g_j = j^{-1}g)$ is non-collapsed and $\text{Re}(g_j) = 0$, $j \in \mathbb{N}$. The limit is a cone $(\mathbb{C}^2/\mathbb{Z}_k, g_{\text{eucl}})$. This cone is isometric (although not biholomorphic) to the initial cones in [FIK03]. But there it is shown that there exist smooth flows $(M_k, g_k(t))$ which converge to the original cones as $t → 0$. However, the Ricci curvature is not bounded from below for these flows as $t \downarrow 0$.

Now assume that $(M, g(t))$ is a smooth Ricci flow which has a metric space $(X, d)$ with conical singularities as an initial condition. This means that $(M, g(t))$ converges in the Gromov Hausdorff sense to $(X, d)$ as $t → 0$ and smoothly away form the singular set to a smooth incomplete Riemannian manifold $(X \setminus X_{\text{sing}}, g_X)$ such that $(X, d)$ is the metric completion of $(X \setminus X_{\text{sing}}, d_{g_X})$. Let $λ_j$ be a sequence of positive real numbers converging to 0. Then, ideally, the blow up limit $(M, x_j, λ_j^{-1}g(λ_j t))$ will converge to a smooth Ricci flow coming out of a cone (see next section for the definition).

Example 3.1.3 (compact Ricci flows with singular initial condition). Let $M_k := P(L_{\mathbb{CP}^1}^k \oplus C)$ be the Hirzebruch surfaces with $k > 2$. Then one can construct a sequence of cohomogeneity one (rotationally symmetric) Ricci flows on $M_k$ such that the initial conditions converge to a compact singular metric space which is obtained by the contraction of the zero section of the $\mathbb{CP}^1$-bundle that is $M_k$. However, for positive times the injectivity radius and the curvature can be controlled and so one obtains in the limit a smooth Ricci flow with a singular initial condition.

Another way to look at the problem of Ricci flows of cones is to consider only the smooth part of the cone $(C, g_C)$ as an incomplete Riemannian manifold with boundary and to require the Ricci flow to...
Theorem 3.2.1 (Perelman’s Pseudolocality Theorem, [GT10]). Let $D$ be the open unit disc in $\mathbb{R}^2$ with the euclidean metric $g_0$. Clearly $(D, g_0)$ is not complete. Now let $g_j$ be a sequence of complete metrics on $g$ with negative curvature that converges to the euclidean metric on compact subset of $D$. Taking the associated Ricci flows $g_j(t)$ on $D$ it is shown in [GT10] that these flows $(D^2, g_j(t))$ converge smoothly for $t > 0$ and the limit is a instantaneously complete Ricci flow.

Example 3.1.4 (instant completeness, [GT10]). Let $D$ be the open unit disc in $\mathbb{R}^2$ with the euclidean metric $g_0$. Clearly $(D, g_0)$ is not complete. Now let $g_j$ be a sequence of complete metrics on $g$ with negative curvature that converges to the euclidean metric on compact subset of $D$. Taking the associated Ricci flows $g_j(t)$ on $D$ it is shown in [GT10] that these flows $(D^2, g_j(t))$ converge smoothly for $t > 0$ and the limit is an instantaneously complete Ricci flow which converges to $g_0$ as $t \nearrow 0$ on compact subset of $D^2$.

Example 3.1.5 (instant topological change, [GT10]). Let $T^2$ be the flat torus and $p \in T^2$. Then $M := T^2 \setminus \{p\}$ can be equipped with a complete metric $g$ with constant curvature $-1$. In [GT10] the authors construct a sequence of metric $g_j$ on $T^2$ which converges on compact subset of $T^2 \setminus \{p\}$ to $g$. They show that the sequence of associated Ricci flows $g_j(t)$ converge smoothly for positive times giving a limiting Ricci flow on $T^2$ which converges to $g$ as $t \nearrow 0$ on compact subsets of $T^2 \setminus \{p\}$. This Ricci flow changes the topology instantly by adding in the point $\{p\}$.

The following example shows how this applies to Ricci flows coming out of cones.

Example 3.1.6 (FIK solitons). The examples constructed in [FIK03] combine both of these phenomena. Here the initial manifold can be seen as an incomplete smooth manifold with boundary. The smooth forward flows constructed in [FIK03] are instantly complete and topologically different from the initial cone as they add in a $\mathbb{C}P^{n-1}$.

Other Ricci flows coming out of cones have been constructed in [Cao85, GK04, DW11, SS10].

Here we first try to give an appropriate definition of such a Ricci flow. This definition may seem a bit ad hoc which is due to the lack of a general definition for a weak Ricci flow. Then we go on to show that the quadratic curvature decay of the cone as $r \to \infty$ is preserved under the Ricci flow. This implies in particular that the Ricci curvature of the evolving metric decays exponentially to 0 if the initial cone is Ricci flat.

### 3.2. Ricci flows coming out of cones

Let $(C(S), g_C)$ be a Riemannian cone over a compact Riemannian manifold $S$ which is smooth except at the vertex $o$. Assume that there exists a smooth manifold $M$ and a family of smooth and complete Riemannian metrics $\{g(t) : t \in [0, T]\}$ such that

1. there exists a diffeomorphism $\varphi : C \to M \setminus K$ where $K$ is a compact subset of $M$,
2. the metric spaces $(M, p, d_{g(t)})$ converge to $(C \cup \{o\}, o, d_{g_C})$ in the pointed Gromov Hausdorff topology, for some $p \in K$,
3. $\varphi^*g(t)$ converges smoothly to $g_C$ away from the vertex,
4. for each $t \in (0, T]$ the Riemannian curvature of $g(t)$ is bounded on $M$,
5. $g(t)$ satisfies the Ricci flow equation

$$\frac{\partial g(t)}{\partial t} = -2\text{Rc}(g(t)),$$

then we say that $(M, g(t))$ is a complete Ricci flow with bounded curvature starting from the Riemannian cone $(C, g_C)$. If in addition $(M, J, g(t))_{t \in [0, T]}$ is a Kähler Ricci flow, $(C, J_C, g_C)$ is a Kähler cone, and $\varphi$ is a biholomorphism then we say that $(M, J, g(t))$ is a Kähler Ricci flow coming out of the cone. To prove the main theorem of this section we will need Perelman’s famous pseudolocality theorem:

**Theorem 3.2.1 (Perelman’s Pseudolocality Theorem, [Per02]).** Let $(M, g(t)), t \in [0, T]$ be a complete Ricci flow with bounded curvature. There exist $\epsilon(n)$ (depending only on the dimension $n$ of $M$) such that the following is true. Suppose that for some $x_0 \in M$

$$\text{Vol}_g(B(x_0, r_0)) \geq (1 - \epsilon)\omega_n r_0^n$$

become complete instantly while allowing for topological change. The following related examples have been constructed by Giesen and Topping [GT10].
where \( \omega_n \) is the volume of the Euclidean \( n \)-ball. Then, if
\[
|\text{Rm}|(x) \leq \frac{1}{r_0^2}, \quad \forall x \in B_{g(0)}(x_0, r_0)
\]
it follows that
\[
|\text{Rm}|(x) \leq \frac{1}{(e r_0)^2}, \quad \forall (x, t) \in B_{g(t)}(x_0, e r_0) \times [0, \min\{T, (e r_0)^2\}].
\]

**Proof.** Due to Perelman \cite{Per02} and in the non-compact version in \cite{CTY11}. \(\square\)

In combination with Shi’s curvature estimates we have the following

**Corollary 3.2.2.** In addition to the assumptions above suppose that
\[
|\nabla^k \text{Rm}|(x) \leq \frac{1}{r_0^{2+k}}
\]
for all \( x \in B(x_0, r_0) \) then there exists a \( C_k > 0 \) such that
\[
|\nabla^k \text{Rm}|(x, t) \leq \frac{C_k}{r_0^{2+k}}, \quad \forall (x, t) \in B_{g(t)}(x_0, e r_0) \times [0, \min\{T, (e r_0)^2\}]
\]

**Proof.** See \cite{Top10}. \(\square\)

**Theorem 3.2.3.** Let \( (M, g(t))_{t \in [0,T]} \) be a complete Ricci flow with bounded curvature starting form the Riemannian cone \( (C, g_C) \). Then \( |\text{Rm}|_{g(t)} \) decays quadratically. In other words, let \( r(x) \) be the distance function on \( C \) to the vertex. Then there exist constants \( C \) and \( \lambda \) independent of \( t \) such that for \( (x, t) \in M \setminus K \times (0, T] \) with \( r(x)^2/4t > \lambda \)
\[
|\text{Rm}|(x, t) \leq C r(x)^{-2}.
\]

Moreover, for each \( k \in \mathbb{N} \) there exists a constant \( C_k \) such that
\[
|\nabla^k \text{Rm}|(x, t) \leq C_k r(x)^{-2-k}.
\]

for \( r(x)^2/4t > \lambda \).

**Remark 3.2.4.** The decay property can be formulated as
\[
\sup_{x \in M} |\text{Rm}|(x, t) d_{g(t)}(p, x)^2 < \infty
\]
for all \( t \in (0, T] \) and
\[
\sup_{x \in M} |\nabla^k \text{Rm}|(x, t) d_{g(t)}(p, x)^{2+k} < \infty.
\]

**Proof.** Note first that on the cone we can choose a \( \delta > 0 \) small enough such that for every \( x_0 \in C \setminus \{o\} \) we have
\[
|\text{Rm}|(x, 0) \leq \frac{1}{(2\delta r(x_0))^2}
\]
for any \( x \in B_{g_C}(x_0, 2\delta r(x_0)) \) and
\[
\text{Vol}_{g(0)}(B(x_0, 2\delta r(x_0))) \geq (1 - \epsilon/2) \omega_n (2\delta r(x_0))^n
\]
where \( \epsilon \) is the universal constant in the pseudolocality theorem in \cite{Per02}.
Now pick any point \( x_0 \in M \setminus K \). From the convergence of \( g(t) \) it follows that we can find \( s_0 = s(x_0) > 0 \) small enough such that
\[
|\text{Rm}|(x, s) \leq \frac{1}{(\delta r(x_0))^2}
\]
for \( x \in B_{g(s)}(x_0, \delta r(x_0)) \), all \( s \leq s_0 \) and
\[
\text{Vol}_{g(s)}(B(x_0, \delta r(x_0))) \geq (1 - \epsilon) \omega_n (\delta r(x_0))^n
\]
for all \( s \leq s_0 \).

By the Pseudolocality theorem it follows that
\[
|Rm|(x, s_0 + t) \leq \frac{1}{(\delta r(x_0))^2}
\]
for \( x \in B_\rho(x_0, \delta r(x_0)) \) and \( t \in [0, \min\{0, T - s_0, (\epsilon \delta r(x_0))^2\}] \). Choosing \( s_0 \) smaller if necessary we may assume that \( T - s_0 > 0 \). In particular
\[
|Rm|(x_0, s_0 + t) \leq \frac{1}{(\delta r(x_0))^2}
\]
for \( t \in [0, T - s_0] \) if \( r(x_0)^2 \geq T/(\epsilon \delta)^2 \). After sending \( s_0 \) to 0 it follows that
\[
|Rm|(x, t) \leq \frac{C}{r(x)^2}
\]
for \( t \in [0, T'] \) if \( r(x) \geq \sqrt{T'/\epsilon \delta} \), \( T' \leq T \). The estimates for the higher derivatives of the curvature follow from lemma A.4 in [Top10].

3.3. Ricci flows coming out of Ricci flat cones

Now we assume that the starting cone is Ricci flat. Of course in this case we expect the solution of the Ricci flow to stay close to the original cone. The intuition comes from mean curvature flow starting with stationary hypercones which are not area-minimizing cones. In the context of mean curvature flow the Ricci flow to stay close to the original cone. The intuition comes from mean curvature flow starting with stationary hypercones which are not area-minimizing cones. In the context of mean curvature flow.

One can check that for \( x \) near \( \delta r(x_0) \) it follows that
\[
K_g(x, t) = \frac{1}{t} \left( \frac{1}{F(r)} - \frac{1}{2} \right),
\]
where
\[
F(r) = \frac{2}{W\left(\frac{2}{F(0)} - 1\right) \exp\left(\frac{2}{F(0)} - 1 - r^2\right)} + 1,
\]
\( W \) being the Lambert function.

One can check that for \( x \in \mathbb{R}^2 \) and \( t > 0 \)
\[
|x| = \sqrt{\int_0^{r(x)} F(r)dr}
\]
and so for large \( r \) we have \( |x| \sim 2\sqrt{r} \) (where \( |x| \) is the distance from the origin in the cone metric). Studying the asymptotics of \( W \) near 0 it follows that \( K_g(x, t) \sim t^{-1} \exp(-|x|^2/4t) \). These expanding Ricci solitons have negative scalar curvature if \( F(0) > 2 \) and positive scalar curvature if \( F(0) \in (0, 2) \). If \( F(0) = 2 \) then \( F(r) = 2 \) for all \( r \) and the resulting expanding Ricci soliton is the Gaussian soliton on \( \mathbb{R}^2 \).

---

1. \( g(t) = t(F(r)^2 dr^2 + r^2 d\theta^2), \theta \in \mathbb{R}/F(0)2\pi \) and \( \mathbb{R}^2 \) is parametrized as \((0, \infty) \times S^1(F(0))\).

2. \( W(x) \exp(W(x)) = x \)
Another example that was already described before is the following

**Example 3.3.3.** The expanding Kähler Ricci solitons constructed in [FIK03] are Ricci flows coming out of Ricci flat Kähler cones. These expanding Ricci solitons are rotationally symmetric and so the expander equation can be reduced to an ODE at \( t = 1/4 \) for the potential \( P \) of the expander metric.

More precisely, the solution \( \omega \) is of the form \( \sqrt{-1} \partial \bar{\partial} P \) where \( P = P(\log(|z|^2)) \) is a smooth function defined on \( \mathbb{C}^n / \mathbb{Z}_k \setminus \{0\} \) \((k > n, k \in \mathbb{N})\), where \( \mathbb{Z}_k \) acts diagonally on \( \mathbb{C}^n \) sending \( z \) to \( \exp(2\pi \sqrt{-1} j/k)z \) for \( j = 0, \dots, k-1 \). An appropriate choice of the free parameters gives a complete metric by gluing in a \( \mathbb{C}P^{n-1} \) at 0. For the Ricci flat cone the ODE for \( P \) is

\[
P_{rr} = -a^n e^\phi e^{-P} + P_r
\]

where \( a = k - n > 0 \) and \( r = \log(|z|^2) \). Setting \( \phi = P_r \) one can check that the eigenvalues of the resulting Ricci curvature \( \text{Rc} \) with respect to the expander metric \( g \) are

\[
\lambda_n = a^n e^\phi e^{-n}(\phi + (n-1))
\]

and

\[
\lambda_1 = \cdots = \lambda_{n-1} = \frac{-a^n e^\phi}{\phi^n}.
\]

Finally it is shown in [FIK03] that \( \phi \) behaves like \( e^r = |z|^2 \) near infinity and so it follows that

\[
|\text{Rc}|(z,1/4) = O(\exp(-|z|^2))
\]
on \( \mathbb{C}^n / \mathbb{Z}_k \setminus \{0\} \) as \( z \to \infty \). And for arbitrary \( t > 0 \)

\[
|\text{Rc}|(z,t) = O(\exp(-|z|^2/4t))
\]
as \( z \to \infty \).

**Remark 3.3.4.** Note that in the example above the scalar curvature is positive (unlike the Ricci curvature). In fact

\[
R_g = \frac{a^n e^\phi}{\phi^n - 1}.
\]

Before we start with the proof we mention one more example from the world of mean curvature flow. As explained in the introduction there exist self-similar solutions of the mean curvature flow coming out of the unstable Simons cones. And these solutions are exponentially asymptotic to the original cone as shown in [Ilm95].

**Example 3.3.5.** In [Ilm95] expanders with a \( SO(p) \times SO(q) \) symmetry coming out of the unstable Simons cones \( C_{p,q} \) are constructed. These solutions are described as Graphs over \( \mathbb{R}^2 \):

\[
M_1 = \{ x = (y,z) \in \mathbb{R}^p \times \mathbb{R}^q : |z| = u(|y|) \}
\]

where the mean curvature \( H \) satisfies

\[
H - \frac{\langle x, \nu \rangle}{2} = 0,
\]
\( \nu \) being the unit normal vector. It follows from the calculations in [Ilm95] that near infinity the mean curvature behaves like

\[
H(x) \approx C_1 \exp \left( -\frac{|z|^2}{4} \right).
\]

**Remark 3.3.6.** This example should also be compared with the expanders in [JLT10]. In [LN12] it is shown that these expanders approach the original cone exponentially fast.

**Proof of the theorem.** First recall the evolution equation for the Ricci curvature (see [CLN06])

\[
\frac{\partial}{\partial t} \text{Rc}(g(t)) = \Delta L \text{Rc}(g(t)).
\]

Throughout the proof will write \( g \) instead of \( \varphi^* g \).
where $\Delta_L$ is the Lichnerowicz Laplacian acting on symmetric 2-tensors with respect to the evolving metric

$$(\Delta_L h)_{ij} = (\Delta h)_{ij} + 2Rm_{ik}h_{kj} - Rc_{ik}h_{kj} - Rc_{kj}h_{ik}.$$ 

In particular

$$\frac{\partial}{\partial t}|Rc|^2 = 2(\Delta Rc, Rc) + 4(Rm * Rc, Rc)$$

$$\leq |\Delta Rc|^2 - 2|\nabla Rc|^2 + 4|Rm||Rc|^2.$$ 

Then for any $\epsilon > 0$

$$\frac{\partial}{\partial t}\sqrt{\epsilon t^2 + |Rc|^2} \leq \frac{\Delta |Rc|^2 + 4|Rm||Rc|^2 - 2|\nabla Rc|^2 + 2\epsilon t^2}{2\sqrt{\epsilon t^2 + |Rc|^2}}$$

where $\sqrt{\epsilon t^2 + |Rc|^2}$ is smooth in $M \times (0, T]$ and continuous on $M \setminus K \times [0, T]$. And so

$$\frac{\partial}{\partial t}\sqrt{\epsilon t^2 + |Rc|^2} - \Delta \sqrt{\epsilon t^2 + |Rc|^2} \leq 2|Rm|\sqrt{\epsilon t^2 + |Rc|^2} + \frac{\epsilon t^2}{\sqrt{\epsilon t^2 + |Rc|^2}}.$$ 

Indeed

$$\Delta \sqrt{\epsilon t^2 + |Rc|^2} = \frac{\Delta |Rc|^2}{2\sqrt{\epsilon t^2 + |Rc|^2}} - \frac{|\nabla |Rc||^2}{4\sqrt{\epsilon t^2 + |Rc|^2}}$$

$$\geq \frac{\Delta |Rc|^2}{2\sqrt{\epsilon t^2 + |Rc|^2}} - \frac{|\nabla Rc|^2 |Rc|^2}{\sqrt{\epsilon t^2 + |Rc|^2}}$$

$$\geq \frac{\Delta |Rc|^2}{2\sqrt{\epsilon t^2 + |Rc|^2}} - \left( \frac{|\nabla Rc|^2}{\sqrt{\epsilon t^2 + |Rc|^2}} \right) \left( \frac{|Rc|^2}{\epsilon t^2 + |Rc|^2} \right)$$

$$\geq \frac{\Delta |Rc|^2}{2\sqrt{\epsilon t^2 + |Rc|^2}} - \frac{|\nabla Rc|^2}{\sqrt{\epsilon t^2 + |Rc|^2}}.$$ 

Let $\sigma_R(x, t) := (r(x) - R)^2/4t$, $x \in M \setminus K$, and for $\lambda > 0$ define

$$\Omega_\lambda := \{(x, t) \in M \setminus K \times [0, T) : \sigma_R(x, t) > \lambda, r(x) > R\}.$$ 

Recall that on $\Omega_\lambda$ (for $\lambda$ sufficiently large) we can bound the curvature by

$$|Rm|(x, t) \leq C \frac{4\lambda}{4\lambda + R^2}.$$ 

where $C$ does not depend on $t$. Let $\eta := C/\lambda < 1$ (i.e. $\lambda$ sufficiently large), then for

$$u_\epsilon := (4\lambda + R^2)^{-\eta} \left( \sqrt{|Rc|^2 + \epsilon t^2} - \frac{\epsilon t}{1 - \eta} \right)$$ 

we have

$$\frac{\partial}{\partial t} u_\epsilon - \Delta u_\epsilon \leq (4\lambda + R^2)^{-\eta} \left( \left( 2|Rm| - \frac{4\lambda \eta}{4\lambda + R^2} \right) \sqrt{|Rc|^2 + \epsilon t^2} \right.$$

$$\left. + \left( \frac{\epsilon t^2}{\sqrt{|Rc|^2 + \epsilon t^2}} - \frac{\epsilon}{1 - \eta} + \frac{4\lambda \eta t}{(4\lambda + R^2)(1 - \eta)} \right) \right)$$

$$\leq (4\lambda + R^2)^{-\eta} \left( \left( \frac{2C}{4\lambda + R^2} - \frac{4C}{4\lambda + R^2} \right) \sqrt{|Rc|^2 + \epsilon t^2} \right.$$

$$\left. + \left( \epsilon - \frac{\epsilon}{1 - \eta} + \frac{\eta \epsilon}{1 - \eta} \right) \right)$$

$$\leq 0.$$ 

Now define

$$v(x, t) := \frac{1}{(4\lambda + R^2)^{1+\eta}} u_\epsilon^{-\eta},$$
with \( \mu > 0 \) to be determined later. Further note that (using \( g(0) = g_C \))
\[
\Delta_{g(0)}f(\sigma_R) = f'(\sigma_R)\Delta_{g(0)}\sigma_R + f''(\sigma_R)\nabla_{\sigma_R}^2 g(0) = f'(\sigma_R)\left( \frac{n}{2t} - \frac{(n-1)R}{2rt} \right) + f''(\sigma_R)\frac{(r-R)^2}{4t^2}.
\]
So if \( f(\sigma_R) = \sigma_R^n \exp(-\sigma_R) \) then
\[
\Delta_{g(0)}f(\sigma_R) = \left( \frac{\mu}{\sigma_R} - 1 \right) \left( \frac{n}{2t} - \frac{(n-1)R}{2rt} \right) + \left( \frac{\mu(\mu-1)}{\sigma_R^2} - \frac{2\mu}{\sigma_R} + 1 \right) \frac{\sigma_R}{t} f(\sigma_R)
\]
and
\[
\frac{\partial}{\partial t} f(\sigma_R) = f'(\sigma_R) \frac{\partial \sigma_R}{\partial t} = f(\sigma_R) \left( -\frac{\mu}{t} + \frac{\sigma_R}{t} \right).
\]
Then (note that \( g(x, t) \) is a smooth metric in \( x \) and \( t \) on each \( \Omega \cap M \times \{ t \} \))
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) v = \left( \frac{\partial}{\partial t} - \Delta_{g(0)} \right) v + \underbrace{\left( \Delta_{g(0)} - \Delta_{g(t)} \right) v}_{=: w} = \left( \mu - (1 + \eta)(4\lambda t)/(4\lambda t + R^2) + (n/2) - R(n-1)/(2r) \right) v + w.
\]
So if \( \mu = \eta + \gamma > 0 \) for some \( \gamma > 0 \), then on \( \Omega \)
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) v \geq \left( \frac{\gamma}{t} - \frac{\mu(\mu-1) + \mu(n/2) - R(n-1)/r}{t\lambda} \right) v + w.
\]
And if \( \lambda \) (independent of \( R \)) is large enough
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) v \geq \left( \frac{\gamma}{2t} \right) v + w.
\]
Next we can estimate \( w \):

**Proposition 3.3.7.** On \( \Omega \) the function \( w \) can be estimated by
\[
|w| \leq \left( \frac{C_1}{t} + C_2 s^{-1}_R \right) v
\]
where \( C_1 \) and \( C_2 \) depend only on \( T \) and a lower bound for \( \lambda \).

**Proof.** Note first that
\[
\Delta_{g(t)} v = ((\mu\sigma_R^{-1} - 1)^2 - \mu\sigma_R^{-2})\nabla_{\sigma_R}^2 g(t) + (\mu\sigma_R^{-1} - 1)\Delta_{g(t)} \sigma_R v
\]
\[
= (\mu - 1)\sigma_R^{-2} - 2\mu\sigma_R^{-1} + 1)\nabla_{\sigma_R}^2 g(t) + (\mu\sigma_R^{-1} - 1)\Delta_{g(t)} \sigma_R v
\]
Hence it remains to investigate \( |\nabla_{\sigma_R}^2 g(t)| - |\nabla_{\sigma_R}^2 g(0)| \) and \( \Delta_{g(t)} \sigma_R - \Delta_{g(0)} \sigma_R \). So let \( (x, t) \in \Omega \)
\[
\frac{d}{ds} |\nabla_{\sigma_R}(x, t)|^2_{g(s)} \leq 2|Rc(x, s)|_{g(s)} |\nabla_{\sigma_R}(x, t)|^2_{g(s)} \leq \frac{C}{r(x)^2} |\nabla_{\sigma_R}(x, t)|^2_{g(s)}.
\]
It follows that
\[
|\nabla_{\sigma_R}^2 g(t)| - |\nabla_{\sigma_R}^2 g(0)| \leq \frac{C}{t}.
\]
Indeed for some constant \( C_1 \) depending only on a lower bound for \( \lambda \)
\[
|\nabla_{\sigma_R}^2 g(t)| - |\nabla_{\sigma_R}^2 g(0)| \leq \left( \frac{C_1}{\sigma_R} - 1 \right) |\nabla_{\sigma_R}^2 g(0)| \leq \frac{C_1 \sigma_R}{\sigma_R} \frac{C_1}{t}.
\]

(3.1)
Similarly we can find constants $C_2$, $C_3$, and $C_4$ depending only on a lower bound for $\lambda$ 
\begin{equation}
\frac{(\Delta g(t) - \Delta g(0))\sigma_R(x,t)}{\norm{\nabla \sigma_R(x,t)}_{g(0)}} \leq \frac{n^2}{2(t - g(0))^{-1} - g(0)^{-1}g(0) + |\Gamma(x,t) - \Gamma(x,0))|_{g(0)} |\nabla \sigma_R(x,t)|_{g(0)}} 
\end{equation}
\begin{equation}
\leq \frac{C_3}{r(x)^2} + \frac{C_4 t}{r(x)^3} \frac{(r(x) - R)}{t} 
\end{equation}
\begin{equation}
\leq \frac{C_2}{t \sigma_R(x,t)}.
\end{equation}

So
\begin{equation}
|w| \leq (C_1/t + C_2 \sigma_R^{-1})v.
\end{equation}

In particular $C_1$ and $C_2$ do not depend on $\mu$.

So if we choose $\lambda$ and $\gamma$ large enough (again independent of $R$) we get
\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) v \geq 0
\end{equation}
on $\Omega_{\lambda}$.

**Lemma 3.3.8.** For sufficiently large $\lambda$ there exists a uniform $C_{\lambda}$ such that 
\begin{equation}
|\text{Rc}|(x,t) \leq C_{\lambda}(4\lambda t + R^2)^{\eta}v = C_{\lambda}(4\lambda t + R^2)^{-1} \sigma_R^{1+\gamma+\eta} e^{-\sigma R}
\end{equation}

**Proof.** Note first that
\begin{equation}
h(x,t) := C_{\lambda} v(x,t) - u_\epsilon(x,t) \geq 0, \quad \text{on } \{0\} \times \{r(x) > R\} \cup \{(x,t) : t \in (0,T], \sigma_R = \lambda\}
\end{equation}
where
\begin{equation}
C_{\lambda} = C\lambda^{-1-\gamma-\eta} e^{\lambda}.
\end{equation}
and
\begin{equation}
\frac{\partial h}{\partial t} - \Delta_{g(t)} h \geq 0, \quad (x,t) \in \Omega_{\lambda}.
\end{equation}

We claim that $h \geq 0$ in $\Omega_{\lambda}$. If the infimum is attained in the interior of $\Omega_{\lambda}$ then it follows by the maximum principle that $h \geq 0$. Now suppose that $\inf h < 0$ and let $(x_j, t_j)$ be a sequence in $\Omega_{\lambda}$ such that $\lim j \rightarrow \infty h(x_j, t_j) = \inf_{\Omega_{\lambda}} h$. By the observations above it follows that $t_j \rightarrow 0$ and $r(x_j) \rightarrow R$. But since $u_\epsilon$ is continuous in $\Omega_{\lambda}$ we have $\lim j \rightarrow \infty u_\epsilon(x_j, t_j) = 0$ and $\lim j \rightarrow \infty v(x_j, t_j) \geq 0$. This is a contradiction.

Letting $R \rightarrow 0$ and $\epsilon \rightarrow 0$ we get
\begin{equation}
|\text{Rc}|(x,t) \leq \tilde{C}_{\lambda} t^{-1} \sigma_R^{1+\gamma+\eta} e^{-\sigma R},
\end{equation}
and so after integrating
\begin{equation}
|g(x,t) - g(x,0)|_{gc} = O(\sigma_R^{\mu-1}) e^{-\sigma R}.
\end{equation}

We can repeat the process above. Namely, from the results above we have an even better bound for $w$, i.e. $|w|$ tends to 0 as $\lambda$ increases. More precisely, the right hand side of \(3.1\) can be improved such that it is of the form $(C'/t)\sigma_R^{\eta} \exp(-\sigma R)$. So for any $\gamma > 0$ we can always find a sufficiently large $\lambda$ such that
\begin{equation}
\left( \frac{\gamma}{t} - \frac{\mu(n-1)}{t\lambda} \mu(n/2) - R(n-1)/r \right) \geq \frac{\gamma}{2t}.
\end{equation}
and such that on $\Omega_{\lambda}$ we have $|w| \leq \frac{\gamma}{2t}$. Hence for such a $\lambda$
\begin{equation}
\frac{\partial v}{\partial t} - \Delta v \geq 0.
\end{equation}

And as before we can conclude that $|\text{Rc}| \leq C_{\lambda} v$. Obviously we can choose $\gamma > 0$ and $\eta > 0$ as small as we want and hence
\begin{equation}
|\text{Rc}| = O(\sigma_R^{\kappa+1}/te^{-\sigma R}),
\end{equation}
for every $\kappa > 0$ and so
\begin{equation}
|g(x,t) - g(x,0)|_{gc} = O(\sigma_R^\kappa e^{-\sigma R})
\end{equation}
for every $\kappa > 0$. Note however, that we cannot let $\kappa \to 0$ since the constant factor definitely depends on $\kappa$ (but not on $R$). And again, letting $R \to 0$, we have

$$|\text{Re}| = O\left(\frac{e^{\sigma^1_1}}{t} \exp(-\sigma)\right)$$

and

$$|g(x,t) - g(x,0)|_{gc} = O(\sigma^n e^{-\sigma}).$$

□

As a corollary we can conclude that the Kähler class of any Kähler Ricci flow coming from a cone must be proportional to $c_1(M)$. In particular $[\omega(t)] \in H^2_c(M)$ for all $t$ (this also follows from proposition 2.8.3 in the previous chapter).

**Corollary 3.3.9.** Assume that $(M, J, g(t))$ is a Kähler Ricci flow emerging from a (Ricci flat) Kähler cone (as defined above). Then $[\omega(1)] = -c_1(M)$ and $[\omega(1)] \in H^2_c(M)$.

**Proof.** Take a compact 2-cycle $\Sigma \in H_2(M)$. Then

$$0 = \lim_{t \to 0} \int_{\Sigma} [\omega(t)] = \int_{\Sigma} [\omega(1)] - \lim_{t \to 0} (1-t) \int_{\Sigma} 2\pi c_1(M) = \int_{\Sigma} [\omega(1)] - \int_{\Sigma} c_1(M).$$

And if $\Sigma$ is in $M \setminus K$ then we can find $[\tilde{\Sigma}] \in H_2(S)$ such $[\varphi(1) \times \tilde{\Sigma}] = [\Sigma]$. So by Stokes' theorem and the fact proven above that $\varphi^*\rho$ decays exponentially compared to $\omega_C$ it follows that

$$\int_{\Sigma} [\omega(1)] = \int_{\{1\} \times \Sigma} [\varphi^*\omega(1)] = \lim_{r \to \infty} \int_{\{r\} \times \Sigma} [\varphi^*\rho] = 0,$$

i.e. $\omega(1) \in H^{1,1}_c(M, \mathbb{R})$
CHAPTER 4

Expanding Ricci Solitons

4.1. Introduction

A solution $g(t)$ of the Ricci flow

$$\frac{\partial g}{\partial t} = -2\text{Rc}(g(t))$$

is called an expanding gradient Ricci soliton if it is of the form

$$g(t) = t\varphi_t^*g(1),$$

where $\varphi_t$ is a family of diffeomorphisms generated by a (gradient) vector field $X = -\nabla f/t$ with $\varphi_1 = \text{id}$. It follows that the metrics $g(t), \ t > 0$, solve the elliptic partial differential equation (note that $f$ depends on $t$)

$$\text{Rc}(g(t)) + \nabla^2 f + \frac{g(t)}{2t} = 0. \quad (4.1)$$

As such, expanding Ricci solitons provide local models for Ricci flows coming out of singularities. They also appear as blow down limits of eternal solutions of the Ricci flow. In particular, expanding Ricci solitons are the prototypes of Ricci flows that smooth out conical singularities. We should point out that an important part of the proof (the Gromov Hausdorff convergence) has already been provided in [CD11]. Expanding Ricci solitons show up in the following two situations.

4.1.1. Ricci flows with singular initial conditions. Let us assume that $(M, g(t))_{t \in [0,T]}$ is a, yet to be defined, weak Ricci flow with a singular initial condition such that the flow is smooth for $t \in (0,T]$ and such that $\sup_{t \in [0,T]} t\|\text{Rm}\|_{\infty} < \infty$. Then, ideally, the sequence $(M, x_j, \lambda_j^{-1}g(\lambda_j^{-1}t))_{t \in [0,\lambda_j^{-1}T]}, \lambda_j \searrow 0$, subconverges to an expanding Ricci soliton that flows out of the tangent cone with $\lim_{j \to \infty} x_j$ as its tip. Hence expanding Ricci solitons model the behavior of the Ricci flow after a singularity.

4.1.2. Blow down limits of the Ricci flow. A Ricci flow $(M, g(t))_{t \in [0,\infty)}$ is called a type III solution if $\sup_{t \in [0,\infty)} t\|\text{Rm}\|_{\infty} < \infty$. It is expected that the sequence $(M, x_j, \lambda_j^{-1}g(\lambda_j^{-1}t)), \lambda_j \searrow 0$, subconverges to an expanding Ricci soliton. This has been proven under various positivity assumptions. Cao showed in [Cao97] that for a non-compact type III solution for the Kähler Ricci flow with non-negative bisectional curvature and positive Ricci curvature such that the maximum of $tR(x,t)$ is assumed in space time the sequence $(M, x, t^{-1}g(t))$ converges to an expanding Kähler Ricci soliton. Schulze and Simon show in [SS10] that the same is true for type III Ricci flows with non-negative curvature operator and positive asymptotic volume ratio giving an expanding Ricci soliton coming out of the unique asymptotic cone of $M$. Note that without the assumption of nonnegative sectional curvature the asymptotic cone (if it exists) is usually not unique (for a given manifold $(M,g)$).

4.2. Basic properties of expanding Ricci solitons

Taking the divergence of equation (4.1) it follows that

$$R(g(t)) + |\nabla f|^2 - \frac{f}{t} = C_1(t), \quad (4.2)$$

We denote by $R(g)$ and $R_g$ the scalar curvature with respect to $g$. 

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for some constant $C_1(t)$ depending only on $t$. More precisely, the constant $C_1(t)$ is of the form $C_1/t$, $C_1 \in \mathbb{R}$. Since

\[
\frac{\partial}{\partial t} R(g(t)) = \Delta_{g(t)} R(g(t)) + 2 |\text{Rc}(g(t))|^2_{g(t)} \geq \Delta_{g(t)} R(g(t)) + \frac{2}{n} R(g(t))^2
\]

we also have

\[
0 \geq \Delta R(g(t)) - X(R(g(t))) + \frac{2}{n} R(g(t))^2 + \frac{R(g(t))}{t}.
\]

And so if the maximum principle holds for $(M, g)$ (if $(M, g)$ has bounded curvature) it follows that $\inf_M R(g(t)) \geq -\frac{n}{2}$ and hence $\Delta f \geq 0$ and $f$ is bounded from below for $t > 0$.

**Lemma 4.2.1.** Let $(M, f, g)$ be an expanding gradient Ricci soliton and suppose that

\[
\sup_{x \in M} |\text{Rm}|(x)d(p, x)^2 := A_0(g) < \infty
\]

and let

\[
c_0 := \sup_{x \in B(p, 1)} |\text{Rm}|(x).
\]

then

\[
\frac{1}{4} d(p, x)^2 + |\nabla f|(p)d(p, x) + \pi(A_0(g) + c_0)(n - 1)d(p, x) \geq f(x) - f(p) \geq \frac{1}{4} d(p, x)^2 - |\nabla f|(p)d(p, x) - \pi(A_0(g) + c_0)(n - 1)d(p, x).
\]

**Proof.** Let $\gamma(t)$ be a geodesic joining $p$ and $x$. Then

\[
\frac{d^2}{dt^2} f(\gamma(t)) = \nabla^2 f(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t)) = \text{Rc}(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t)) + \frac{1}{2}.
\]

Now integrating by parts gives

\[
\int_0^{d(p, x)} (d(p, x) - \tau) \frac{d^2}{d\tau^2} f(\gamma(\tau))d\tau = d(p, x)(\nabla f(p), \dot{\gamma}(0)) + f(x) - f(p),
\]

while

\[
\int_0^{d(p, x)} (d(p, x) - \tau) \frac{d^2}{d\tau^2} f(\gamma(\tau))d\tau = \int_0^{d(p, x)} (d(p, x) - \tau)\text{Rc}(\gamma(\tau))(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) + \frac{d(p, x) - \tau}{2} d\tau
\]

\[
\geq \frac{d(p, x)^2}{4} - \int_0^{d(p, x)} \frac{2(n - 1)(c_0 + A_0(g))d(p, x) - \tau}{1 + \tau^2} d\tau
\]

\[
\geq \frac{d(p, x)^2}{4} - 2 \arctan(d(p, x))(2(n - 1)(c_0 + A_0(g)))d(p, x)
\]

\[
\geq \frac{d(p, x)^2}{4} - \pi(2(n - 1)(c_0 + A_0(g)))d(p, x)
\]

which proves the second inequality in the lemma. The first one works similarly. □

It follows that, under the assumptions above, $f$ attains its minimum in $M$. And if we take $p$ to be that point, i.e. $f(p) = \inf_{x \in M} f(x)$ then $\nabla f(p) = 0$ and hence $f(p) = C_1 - R_0(p)$. Without loss of generality we can assume (after adding a constant to $f$) that $C_1 = R_0(p)$ and so $\inf_{x \in M} f(x) = f(p) = 0$. Moreover

\[
\frac{1}{4} d(p, x)^2 + c_1 d(p, x) \geq f(x) \geq \frac{1}{4} d(p, x)^2 - c_1 d(p, x),
\]

where

\[
c_1 := \pi(2(n - 1)(c_0 + A_0(g))).
\]
4.3. Convergence of Ricci solitons

Before we start a word on notation: let \( g(x,t) \) be a time dependent metric on \( M \) and \( t \in (0,T] \). When we write \( |\nabla^k Rm|(x,t) \) we mean the \( k \)-th covariant derivative with respect to the metric \( g(x,t) \) of the Riemannian curvature of \( g(x,t) \) at \( x \) and \( t \). On the other hand, if we write \( |\nabla^k Q(x,t)|_{g(x)} \) for some time dependent tensor field \( Q(x,t) \) on \( M \), \( t \in [0,T] \) then we mean the \( k \)-th covariant derivative of \( Q \) with respect to \( g(x) \) measured in \( g(x) \).

Suppose that \((M,f,g)\) is an expanding gradient Ricci soliton. Then using the observations of the previous section one can prove the following counterpart to the theorem 3.2.3 we proved earlier:

**Theorem 4.3.1.** Let \((M,f,g)\) be a non-compact expanding gradient Ricci soliton such that for some point \( p \in M \) and all \( k \in \mathbb{N}_0 \)

\[
A_k(g) := \sup_{x \in M} |\nabla^k Rm|(x)d(p,x)^{2+k} < \infty
\]

then the associated Ricci flow \((M,g(t))\) converges subsequentially in the Gromov Hausdorff topology and merely spell out the details of remark 1.5 in [CD11].

In particular, we show that the cone is unique and the convergence is actually smooth.

**Proof.** We already know from [CD11] that \((M,g(t))\) converges subsequentially in \( C^{1,\alpha} \) to a cone (away from the vertex) and in the Gromov Hausdorff sense as a metric space if we have quadratic curvature decay. And the proof can be modified to show that \((M,g(t))\) converges subsequentially smoothly (away from the vertex) under the curvature assumptions above. It remains to show that the limit is unique, i.e. that \((M,g(t))\) converges to the same cone independent of the choice of subsequence.

Unless we have non-negative sectional curvature we cannot assume in general that the tangent cone at infinity of \( M \) is unique. \(^2\)

So let

\[
M_c := \{ x \in M : \liminf_{t \to 0} tf(x,t) > c \},
\]

\[
M_0 := \{ x \in M : \limsup_{t \to 0} tf(x,t) > 0 \},
\]

where \( f(x,t) \) is defined as

\[
\frac{\partial}{\partial t} f(x,t) = -|\nabla f|_{g(t)}^2(x,t)
\]

\[
f(x,1) = f(x).
\]

Note that \( f(x,t) = f(\varphi_t(x),1) \) where \( \varphi_t \) is the one parameter group of diffeomorphisms

\[
\frac{d}{dt} \varphi_t(x) = -\frac{\nabla f(\varphi_t(x))}{t},
\]

and \( \nabla f \) is the gradient of \( f = f(1) \) with respect to \( g = g(1) \), i.e. the metric at \( t = 1 \).

**Lemma 4.3.3.** Let \((M,g(t))\) be as above. Then there exists a constant \( B(0,c) \) such that on \( M_c \)

\[
|Rm|(x,t) \leq B(0,c),
\]

for \( c > 0 \) independent of \( t \).

**Proof of the lemma.** Note first that

\[
|Rm|(x,t) = t^{-1}|Rm|(\varphi_t(x)),
\]

By the observations above

\[
\frac{\partial}{\partial t} f(\varphi_t(x)) = -\frac{|\nabla f|_{g(t)}^2(\varphi_t(x))}{t},
\]

\(^2\)of course it follows from the theorem that the tangent cone exists and is unique.
and so
\[
\frac{\partial}{\partial t} tf(\varphi_t(x)) = (f - |\nabla f|^2_g)(\varphi_t(x)) = R_g(\varphi_t(x))
\]
and integrating gives
\[
 tf(\varphi_t(x)) = sf(\varphi_s(x)) + \int_s^t R_g(\varphi_\tau(x)) d\tau.
\]
Moreover, for \( x \in M_c \),
\[
 tf(\varphi_t(x)) \geq \liminf_{s \to 0} sf(\varphi_s(x)) - t \sup_{y \in M} |R_g(y)| > c - t \sup_{y \in M} |R_g(y)|
\]
i.e.
\[
 f(\varphi_t(x)) > \frac{c}{t} - \sup_{y \in M} |R_g(y)|.
\]
And so, letting \( r(x) := d(p, x) \)
\[
 \frac{1}{4} r(\varphi_t(x))^2 + c_1 r(\varphi_t(x)) \geq \frac{c}{t} - \sup_{y \in M} |R_g(y)|.
\]
If
\[
 t < \frac{c}{\sup_{y \in M} |R_g(y)| + c_1^2}
\]
we have
\[
 r(\varphi_t(x)) \geq \frac{1}{2c_1} \sqrt{\frac{c}{t} - \sup_{y \in M} |R_g(y)|}.
\]
This means that for \( x \in M_c \),
\[
 |\text{Rm}](x, t) = t^{-1}|\text{Rm}|(\varphi_t(x)) \leq \sup_{y \in M} |\text{Rm}|(y)t_0^{-1} + \frac{8A_0(g)c_1^2}{c} \leq \frac{C(g)}{c} =: B(0, c),
\]
where
\[
 t_0 := \frac{c}{2(\sup_{y \in M} |R_g(y)| + c_1^2)},
\]
and \( C(g) \) is a constant that depends only on the metric and its derivatives.

**Lemma 4.3.4.** Let \( (M, g(t)) \) be as above. Then on \( M_c \) we have
\[
 \sup_M |\nabla^k \text{Rm}|(x, t) \leq B(k, c)
\]
for some uniform constant \( B(k, c) \) depending only on \( k \) and \( c \).

**Proof.** Arguing as above we have
\[
 |\nabla^k \text{Rm}|(x, t) = t^{-1-k/2}|\nabla^k \text{Rm}|(\varphi_t(x)) \leq \sup_{y \in M} |\nabla^k \text{Rm}|(y)t_0^{-1-k/2} + \frac{8A_k(g)c_1^2}{c} \leq B(k, c).
\]

**Lemma 4.3.5.** Let \( f(x, t) \) and \( (M, g) \) be as above. Then \( tf(x, t) \) converges uniformly in \( C^0(M) \) and locally in \( C^\infty(M_0) \) with respect to \( g \) to a smooth function \( f_0 \) on \( M_0 \). Moreover \( \nabla f_0(x) \neq 0 \) for all \( x \in M_0 \).

**Proof.** By the proof of the previous lemma we have
\[
 tf(x, t) - sf(x, s) = \int_s^t R_g(\varphi_\tau(x)) d\tau
\]
which proves that \( tf(x, t) \) is a Cauchy sequence in \( C^0(M) \) and hence converges uniformly as \( t \to 0 \). Now we compare \( g \) and \( g(t) \) on \( M_0 \). First let \( x \in M_0 \) i.e. \( \lim_{t \to 0} tf(x, t) > c \) for some \( c > 0 \). Then for \( 0 < t < 1 \)
\[
 C(0, c)g(x) = \exp(2(n - 1)B(0, c)g(x) \geq g(x, t) \geq \exp(-2(n - 1)B(0, c)g(x) = C(0, c)^{-1}g(x)
\]

and by induction (see lemma below) it can be shown that for each $k \geq 1$ there exists a constant $C(k, c)$ such that

$$|\nabla^k g(x, t)|_{g(x)} \leq C(k, c).$$

and a constant $L(k, c)$ such that

$$|\nabla^k \text{Rm}(x, t)|_{g(x)} \leq L(k, c)$$

So again by induction (see lemma below) it follows that for each $k \geq 2$ there exists a constant $K(k, c)$ such that

$$|\nabla^k t f(x, t)|_{g(x)} \leq K(k, c).$$

It follows that $tf(x, t)$ converges in $C^\infty_{loc}(M_0)$. And hence $f_0(x) = \lim_{t \to 0} tf(x, t)$ is a smooth function on $M_0$. Finally the claim that $|\nabla f_0(x)|_g \neq 0$ for $x \in M_0$ follows from equation (1.2). \hfill \Box

**Lemma 4.3.6.** For each $k \geq 0$ there exist constants $C(k, c)$ and $L(k, c)$ such that for $t \in (0, 1]$ 

$$|\nabla^k g(x, t)|_{g(x)} \leq C(k, c)$$

and 

$$|\nabla^k \text{Rm}(x, t)|_{g(x)} \leq L(k, c).$$

**Proof.** We prove this lemma by induction. The case $k = 0$ has already been treated above so assume that the lemma is true for all $t < k$. We denote by $\nabla$ the Levi Civita connection with respect to $g(x, t)$. Then

$$\frac{\partial}{\partial t} |\nabla^k g(x, t)|_{g(x)} \leq 2 |\nabla^k \text{Rc}(x, t)|_{g(x)}$$

$$\leq 2 \left( |\nabla^k - \nabla^k| \text{Rc}(x, t)|_{g(x)} + C(0, c)k^{1/2+1} |\nabla \text{Rc}|(x, t) \right)$$

$$\leq 2 \left( \sum_{j=0}^{k-1} |\nabla^{k-j} g(x, t)|_{g(x)} |\nabla^j \text{Rc}(x, t)|_{g(x)} + C(0, c)^{k/2+1} |\nabla \text{Rc}|(x, t) \right)$$

$$\leq 2 \left( \sum_{j=0}^{k-1} C(k-j, c)L(j, c) + C(0, c)^{k/2+1} B(k, c) \right) + 2 |\nabla^k g(x, t)|_{g(x)} L(0, c).$$

Integrating implies that 

$$|\nabla^k g(x, t)|_{g(x)} \leq C(k, c),$$

for some constant $C(k, c)$. Similarly 

$$|\nabla^k \text{Rm}(x, t)|_{g(x)} \leq |(\nabla^k - \nabla^k) \text{Rm}(x, t)|_{g(x)} + C(0, c)^{k/2+2} |\nabla^k \text{Rm}|(x, t) \leq L(k, c).$$

We continue with the proof of the theorem. By the observations above we have for $s < t \leq 1$

$$\| (M_c, g(t)) - (M_c, g(s)) \|_{(C^k(M_0), g(t))} \leq \left\| \int_s^t \text{Rc}(x, \tau) \, d\tau \right\|_{(C^k(M_0), g)}$$

$$\leq \sup_{j=0}^k \int_s^t |\nabla^j \text{Rc}(x, \tau)|_g \, d\tau$$

$$\leq |t - s| \sum_{j=0}^k L(j, c)$$

and it follows that $g(t)$ converges uniformly in $C^k(M_0)$ for every $k \geq 0$ and $c > 0$ to a unique metric $g(0)$ on $M_0$. Let $f_0(x) := \lim_{t \to 0} tf(x, t)$ which is a smooth function on $M_0$ with $\nabla f_0(x) \neq 0$ for all $x \in M_0$ (by the observations above) and let $L := \{ x \in M_0 : f_0(x) = 1/4 \}$. Now note that $(g_0 := g(0)$ on $M_0$ taking the limit in (4.1) it follows that

$$\nabla^2 f_0(x) = \frac{g_0(x)}{2}$$
on $M_0$ where $\nabla^2 f_0$ is the Hessian with respect to $g_0$. Set
\[ \tilde{g}(x) := (4f_0(x))^{-1}g_0(x) \]
and
\[ \tilde{f}(x) := \log(4f_0(x)) \]
for $x \in M_0$. Then $(M_0, \tilde{g})$ is a complete manifold and
\[ \tilde{\nabla}^2 \tilde{f}(x) = 0 \]
and so by the de Rham splitting theorem $M_0$ is isometric to $(\mathbb{R} \times \tilde{L}, ds^2 + \tilde{g}_{\tilde{L}})$ where $(\tilde{L}, \tilde{g}_{\tilde{L}}) = \tilde{f}^{-1}(0)$.

It follows that $(M_0, g_0)$ is isometric to a cone $(\mathbb{R} \times [0, \infty), ds^2 + r^2 g_L)$ where $g_L = \iota_L^* g_0$. Finally $M \setminus M_0$ is a compact (non-empty since $p \in M \setminus M_0$) set in the original topology since $f_0^{-1}(0)$ is a closed subset of a bounded set:
\[ \{ x \in M : f_0(x) = 0 \} \subset \{ x \in M : f(x) \leq \sup_{x \in M} |R_\phi(x)| \}. \]
CHAPTER 5

Existence of Expanding Kähler Ricci Solitons

5.1. Introduction

As explained in the previous chapter we are interested in self similar solutions of the Ricci flow in order to understand its behavior near singularities. These singularities are particularly well understood in the Kähler case. In an ideal case the Kähler Ricci flow develops a singularity along a subvariety (if it does not collapse to a lower dimensional variety) as $t$ approaches the singular time and converges in the Gromov Hausdorff sense to a singular space and smoothly away from the subvariety. The limit should be a possibly singular variety. To restart the flow on a singular variety it is necessary to understand the Kähler Ricci flow with singular initial conditions. A particularly simple example of a singularity is a Ricci flat Kähler cone (which, including the vertex, is itself an affine algebraic variety). Here we construct solutions of the Ricci flow that come out of such cones and smooth out the singularity by adding in several subvarieties (resolving the singularity). In particular these solutions are topologically different for positive times but birationally equivalent to the original cone.

Tian and Song laid out a conjectural program in \cite{ST09} to show that on a compact algebraic variety the Kähler Ricci flow does not have to be stopped before the singularity (as in the Perelman program) but can actually be extended to flow through a singularity performing algebraic surgeries such as divisorial contractions or flips. In other words, one gets a sequence of varieties $X_1, X_2, X_3, \ldots$ and times $T_1 < T_2 < T_3, \ldots$ such that $X_j$ is related to $X_{j+1}$ by one of the algebraic surgeries mentioned above and Kähler Ricci flows $g_j(t)$ on $X_j \times (T_j-\lambda, T_j)$ such that $\lim_{t \to T_j+\lambda}(X_j, g_j(t)) = \lim_{t \to T_{j+1}}(X_{j+1}, g_{j+1}(t))$ in a suitable sense.

This program allows the varieties to have log terminal singularities (not only at the singular times where the singularities can be much more complicated). A typical example of such a singularity is a quotient of $\mathbb{C}^n$ by a finite subgroup of $U(n)$. In dimension 2 log terminal singularities will always be of this form. The Ricci flows constructed here (self-similar expanding solutions) will smooth out such singularities therefore creating an ambiguity in the continuation of the Ricci flow (see remark 5.3.9).

We begin by recalling some well known facts about Kähler Ricci solitons. In the following section we reduce the soliton equation to a Monge Ampere equation and prove that a unique solution of this equation exists on resolutions $M$ of Ricci flat Kähler cones $C$ with $K_M > 0$. Finally we list a number of old and new examples to which this theorem applies.

5.2. Kähler Ricci solitons

In this section we discuss some general properties of Kähler Ricci solitons. Let $(M, J, g)$ be a Kähler manifold, $h := g - \sqrt{-1} \omega_g$ its associated hermitian metric, and $R(h) := \text{Rc}(g) - \sqrt{-1} \rho(\omega_g)$ the hermitian Ricci tensor. $(M, J, g)$ is a Ricci soliton as a real Riemannian manifold if for some vector field $\zeta$ on $M$

$$-L_\zeta g + \lambda g = -\text{Rc}(g),$$

(5.1)

for $\lambda \in \{-1, 0, 1\}$. We say that $(M, J, g)$ is a Kähler Ricci soliton if in addition

$$-L_\zeta \omega_g + \lambda \omega_g = -\rho(\omega_g).$$

(5.2)

This makes $(M, J, g)$ a hermitian soliton in the sense that

$$-L_\zeta h + \lambda h = -R(h).$$

(5.3)

Since $g = \omega(\cdot, J \cdot)$ subtracting (5.1) from (5.2) leaves $L_\zeta J = 0$ and so it follows that $\zeta$ has to be a holomorphic vector field. Equation (5.1) alone does not imply that $\zeta$ is holomorphic, however if $\zeta$ is a
5.2.1 Necessary conditions. Let \((M, J, g)\) be a Kähler Ricci soliton and let \(\Sigma\) be a compact complex curve in \(M\) and \([\Sigma]\) its associated 2-cycle. Recall that \(\rho(\omega_g)\) is a representative of \(2\pi c_1(M)\). So integrating equation (5.2) over \(\Sigma\) gives

\[
\int_{\Sigma} -L_X \omega_g + \lambda \Vol_g(\Sigma) = -2\pi \langle c_1(M), [\Sigma] \rangle.
\]

Since \(\omega_g\) is closed \(L_X \omega_g\) is exact and so

\[
\lambda \Vol(C) = -2\pi \langle c_1(M), [C] \rangle.
\]

In particular \(-K_M (K_M)\) is numerically positive if \(\lambda = -1\) (\(\lambda = 1\)). Moreover, it follows from equation (5.2) that \(\lambda [\omega_g] = -2\pi c_1(M)\). If \(M\) is compact the \(\partial\bar{\partial}\)-Lemma implies that \(\lambda \omega_g = -2\pi \rho(\omega_g) + \sqrt{-1} \partial \bar{\partial} f\) for some smooth real valued function \(f\). In particular, every expanding Kähler Ricci soliton on a compact Kähler manifold is Einstein since compact Kähler manifolds with negative first Chern class do not admit non-trivial holomorphic vector fields.

For more background on Kähler Ricci solitons see [CH00], [Cao97], [Cao96], [Bry08], and [FIK03].
5.3. Statement of the Existence Theorem

The goal of this chapter is summarized in the following theorem:

**Theorem 5.3.1 (Main Theorem).** Let \((C, J_C, g_C)\) be a Ricci flat Kähler cone. Suppose that \(\pi : M \rightarrow C\) is a simply connected equivariant resolution\(^1\) of \(C\) with exceptional set \(E\) and \(X\) the lift of the radial holomorphic vector field, i.e. \(X|_{\omega |_{\mathbb{C}}^n} = \frac{1}{2} \pi^{*}(r\partial_r - \sqrt{-1}Jr\partial_r)\). Then there exists an expanding (gradient) Kähler Ricci soliton metric on \(M\) with the soliton vector field \(X\) if there exists a hermitian metric \(h\) on \(K_M\) such that the associated curvature form \(\Theta_h\) is positive and \(L_{\Omega(X)}\Theta_h = 0\).

**Remark 5.3.2.** Roughly speaking, what the theorem says is that any equivariant resolution \(\pi : M \rightarrow C\) of a Ricci flat Kähler cone \(C\) such that \(c_1(M) < 0\) admits an expanding Kähler Ricci soliton metric. This should be compared to the results of [Got12] that any crepant resolution \((C_1(M) = 0)\) of a Ricci flat Kähler cone admits a unique Ricci flat Kähler metric in every Kähler class.

**Remark 5.3.3.** By theorem 4.14 in [VC12] the link of such a cone \(C\) which is a Sasaki Einstein manifold cannot be simply connected unless it is the round sphere \(S^{2n-1}\) with the standard CR-structure, i.e. \(C = \mathbb{C}^n\) with the euclidean metric.\(^2\)

Here we will prove the following theorem which implies the main theorem stated above but has the advantage of using maybe more familiar terminology from Riemannian geometry.

**Theorem 5.3.4 (Existence of expanding Kähler Ricci Solitons, 1st Version).** Let \((M, J, g)\) be a Kähler manifold asymptotic to a Ricci flat Kähler cone \((C, J_C, \omega_C)\) at rate \(r^{-\tau}, \tau > 2\). Moreover let \(X\) be a holomorphic vector field on \(M\) such that

1. \(X = \frac{1}{2}(r\frac{\partial}{\partial r} - \sqrt{-1}Jr\frac{\partial}{\partial r})\) outside a compact set.
2. \(\Re(X) = \nabla\theta_X\) for some real valued function \(\theta_X\).
3. \(L_{\Omega(X)}\omega = 0\).

If there exists a function \(F \in C^\infty(M)\) with \(L_{\Omega(X)}F = 0\) such that

\[\rho(\omega_g) + \omega_g = L_X\omega_g + \sqrt{-1}\partial\bar{\partial}F,\]

then there exists a complete Kähler metric \(g'\) with \(\omega_g' = \omega_g + \sqrt{-1}\partial\bar{\partial}\varphi\) for some \(\varphi \in \mathcal{U}\) such that \(\omega_g' + \rho(\omega_g') = L_X\omega_g'\).

**Proof of the Main Theorem 5.3.1 from Theorem 5.3.4.** Suppose there exists a hermitian metric \(h\) on \(K_M\) as in theorem 5.3.1 and set \(\omega_0 := 2\pi\Theta_h\). Since \([\Theta_h] \in H^2(M)\) using proposition 5.2.3 we can find a complete Kähler metric \(g\) on \(M\) such that \(\omega_g = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi, L_{\Omega(X)}\omega_g = 0\), and \(\omega_g = \pi^*\omega_C\) outside a compact set. From chapter 2 we know that there exists a smooth function \(f\) such that \(\rho(\omega_g) + 2\pi\Theta_h = \sqrt{-1}\partial\bar{\partial}f\). And we can conclude that

\[\omega_g + \rho(\omega_g) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi - 2\pi\Theta_h + \sqrt{-1}\partial\bar{\partial}f = \sqrt{-1}\partial\bar{\partial}(f + \varphi)\]

Let \(F := f + \varphi - \theta_X\), where \(\theta_X\) is the potential of \(X\) with respect to \(g\) then

\[\omega_g + \rho(\omega_g) = L_X\omega_g + \sqrt{-1}\partial\bar{\partial}F\]

and the existence of the soliton Kähler metric follows from theorem 5.3.4. \(\square\)

**Remark 5.3.5.** The two theorems are in fact equivalent. Suppose there exists a Kähler metric \(\omega_g\) and a function \(F\) as in theorem 5.3.4. From the discussion in [VC12] it follows that \(M\) is in fact a resolution of \(C \cup \{a\}\) (see remarks below). Now \(\omega_g\) induces a hermitian metric \(h_0\) on \(K_M\) such that \(\rho(\omega_g) = -2\pi\Theta_{h_0}\). Hence for \(h = \exp(\theta_X + F)h_0\)

\[-2\pi\Theta_h = \rho(\omega_g) - L_X\omega_g - \sqrt{-1}\partial\bar{\partial}F = -\omega_g\]

meaning that \(h\) satisfies all the assumptions of theorem 5.3.1.

\(^1\)recall that by equivariant resolution we mean that radial vector field has a holomorphic lift

\(^2\)More precisely, the cone cannot be Gorenstein (which is implied by simply connectedness of \(S\) but in fact a weaker condition), unless it is \(\mathbb{C}^n\).
5.3.1. Reduction to a Monge Ampere equation. Assume that we can find a Kähler Ricci soliton metric \( \omega' := \omega_g \) such that \( L_{\partial \bar{\partial}} \omega' = 0 \) and \( \omega := \omega_g \) as in theorem 5.3.4. Then \( \omega' = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \) i.e. \( \omega' \) is in the same Chern Bott class as \( \omega \) (see below). We can subtract the equations for \( \omega' \) and \( \omega \) to obtain

\[
\omega' - \omega + \rho(\omega') - \rho(\omega) = L_X \omega' - L_X \omega - \sqrt{-1} \partial \bar{\partial} F.
\]

And so

\[
\sqrt{-1} \partial \bar{\partial} \varphi - \sqrt{-1} \partial \bar{\partial} \log \left( \frac{\omega''}{\omega'} \right) = \sqrt{-1} L_X \partial \bar{\partial} \varphi - \sqrt{-1} \partial \bar{\partial} F.
\]

By lemma 5.4.23 below we have \( \sqrt{-1} L_X \partial \bar{\partial} \varphi = \sqrt{-1} \partial \bar{\partial} X(\varphi) \) and the equation becomes

\[
\sqrt{-1} \partial \bar{\partial} \varphi - \sqrt{-1} \partial \bar{\partial} \log \left( \frac{\omega''}{\omega'} \right) = \sqrt{-1} \partial \bar{\partial} X(\varphi) - \sqrt{-1} \partial \bar{\partial} F.
\]

Up to a pluriharmonic function \( \varphi \) satisfies a scalar partial differential equation

\[
-\varphi + \log \left( \frac{\omega''}{\omega'} \right) + X(\varphi) = F.
\]

Conversely if the equation

\[
-\varphi + \log \left( \frac{\omega''}{\omega'} \right) + X(\varphi) = F.
\]

has a solution then clearly \( \omega' = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \) is a soliton metric if \( \omega = \omega_g \) solves (5.4).

5.3.2. Reduction of the assumptions of theorem 5.3.4. To prove the theorem we show first that we can make the following assumptions without loss of generality:

1. The function \( F \) can be assumed to have compact support
2. \( \varphi, \omega = \omega_C \) outside of a compact set
3. \( \theta_X = r^2 \) outside of a compact set.

We first show that the assumptions of the theorem imply that we can choose a background Kähler metric as in the theorem that is equal to the cone metric outside of a compact set (assumption (2)) and that \( F \) has compact support (assumption (1)). This can be seen as follows. Note first that \( \varphi \) can be extended across the compact set. Finally \( \omega' \) has a solution then clearly \( \omega' = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \) is a soliton metric if \( \omega = \omega_g \) solves (5.4).

5.3.3. Reduction of the assumptions of theorem 5.3.4. To prove the theorem we show first that we can make the following assumptions without loss of generality:

1. The function \( F \) can be assumed to have compact support
2. \( \varphi, \omega = \omega_C \) outside of a compact set
3. \( \theta_X = r^2 \) outside of a compact set.

We first show that the assumptions of the theorem imply that we can choose a background Kähler metric as in the theorem that is equal to the cone metric outside of a compact set (assumption (2)) and that \( F \) has compact support (assumption (1)). This can be seen as follows. Note first that \( \varphi \) can be extended across \( K \) to become a degree one morphism \( \tilde{\varphi} : M \to C \cup \{ o \} \) making \( (M, C, \tilde{\varphi}) \) a resolution of \( C \cup \{ o \} \) (see Appendix B). We have the following expression for the canonical divisor

\[
K_M = \tilde{\varphi}^* K_C + \sum_j a_j E_j, \quad a_j \in \mathbb{R}
\]

where \( K_C \) is in fact \( \mathbb{Q} \)-Cartier in \( C(S) \cup \{ o \} \) (i.e. \( \mathcal{O}_{C(S)}(K_C^p) \) is an invertible sheaf for some \( p \in \mathbb{N} \)) and, for some \( p \in \mathbb{N} \), \( pK_C \) is trivial. It follows that \( K_M \) is generated by the prime elements \( E_i \) of the exceptional divisor \( E \) which implies that \( \omega(M) \in H^2_c(M) \). Since we assume that \( \omega = -2\pi c_1(M) \) we have \( \omega \in H^2_c(M) \) and so by proposition 2.2.3 there exists a Kähler form \( \tilde{\omega} \) with \( L_{\partial \bar{\partial}} \tilde{\omega} = 0 \) such that \( \tilde{\Phi}_* \tilde{\omega} = \omega_C \) outside of a compact set. Here is another way to see that the existence of \( \omega \in H^2_c(M) \) is necessary. As \( g' \) is a solution of the soliton equation and the curvature decays quadratically it is by theorem 4.3.1 also a solution of the Ricci flow starting from the cone \( C \). But since the initial cone \( C \) is Ricci flat the solution, \( g' \), by theorem 4.3.1 approaches the cone metric in fact exponentially fast. By proposition 2.8.3 this means that \( \omega' \in H^2_c(M) \) which again by proposition 2.2.3 implies the existence of \( \tilde{\omega} \) with \( [\tilde{\omega}] = [\omega'] \). If the initial cone is not Ricci flat we cannot assume that \( c_1(M) \in H^2_c(M) \) in general.

To show that we can assume (1) note first that (2) implies that \( F \) is pluriharmonic outside some compact set. In particular \( F \) is pluriharmonic on \( \overline{C \setminus B_R} \) for some large \( R > 0 \). This implies that \( X(F) \) is holomorphic. But since \( X(F) \) has real valued \( X(F) = c_0 \) for some constant \( c_0 \) outside of \( B_R \). Hence, on \( \partial C \setminus B_R, F = c_0 \log(r) + c_1 \) where \( c_1 \) is a function that depends only on \( x \in S \). But we also have \( \Delta \omega C F = 0 \) which implies that \( (2n - 2)r^{-2}c_0 + r^{-2}\Delta \omega C c_1 = 0 \). Integrating over \( S \) shows that \( c_0 = 0 \) and hence \( F = c_1 \) for some constant \( C \). Clearly \( F - C \) can be extended across the compact set. Finally we can replace \( F \) by \( F - C \) to get a smooth function on \( M \) with compact support that satisfies (5.3).

Assumption (3) can be shown similarly.
5.3.3. **Weighted spaces.** From now on we will assume that \((M, J, g)\) is as in theorem [5.3.4](#) and in addition \(g = \Phi^* g_C\) outside of a compact set and \(F\) is compactly supported. To show that the Monge Ampere equation derived above has a solution we will employ the continuity method. Part of this method involves showing that the linearization of a non-linear operator is an isomorphism between certain function spaces \(X\) and \(Y\). This motivates definition of the following spaces:

1. The weighted Hölder space on \(M\)
   \[
   C^{k,\alpha}_w(M) := \{u \in C^{k,\alpha}_{\text{loc}}(M) : \|u\|_{C^{k,\alpha}_w(M)} < \infty\}, \quad \|u\|_{C^{k,\alpha}_w(M)}(M) := \|\exp(\theta_X)u\|_{C^{k,\alpha}(M)}.
   \]
   Note that \((C^{k,\alpha}_w(M)), \|\cdot\|_{C^{k,\alpha}(M)}\) = \((e^{-\theta_X}C^{k,\alpha}_w(M)), \|\cdot\|_{C^{k,\alpha}(M)}\).

2. The space of decaying \((k, \alpha)\)-regular Kähler potentials
   \[
   K^{k,\alpha} := \{u \in C^{k,\alpha}_w(M) : \omega + \sqrt{-1}\partial\partial u > 0\}
   \]
   and the space of smooth decaying Kähler potentials \(K := \bigcap_k K^{k,\alpha}(M)\).

3. The space of \(J \zeta = 3(X)\)-invariant \(C^{k,\alpha}\)-functions
   \[
   C^{k,\alpha}_X(M) := \{u \in C^{k,\alpha}(M) : L_{J\zeta} u = 0\}.
   \]

4. The space of \(J \zeta\)-invariant decaying functions
   \[
   C^{k,\alpha}_{w,X}(M) := C^{k,\alpha}_X(M) \cap C^{k,\alpha}_w(M).
   \]

5. The space \(X^{k,\alpha}, \|\cdot\|_{X^{k,\alpha}}\)
   \[
   X^{k,\alpha} := \{u \in C^{k,\alpha}_{w,X}(M) : \|X(e^{\theta_X} u)\|_{C^{k-2,\alpha}} < \infty\}
   \]
   together with the norm
   \[
   \|u\|_{X^{k,\alpha}} := \|u\|_{C^{k,\alpha}_w(M)} + \|X(e^{\theta_X} u)\|_{C^{k-2,\alpha}(M)}.
   \]

6. The space \(\tilde{X}^{k,\alpha}, \|\cdot\|_{\tilde{X}^{k,\alpha}}\)
   \[
   \tilde{X}^{k,\alpha} := \{u \in C^{k,\alpha}_{w,X}(M) : \|X(u)\|_{C^{k-2,\alpha}} < \infty\} = e^{\theta_X}X^{k,\alpha}
   \]
   together with the norm
   \[
   \|u\|_{\tilde{X}^{k,\alpha}} := \|u\|_{C^{k,\alpha}(M)} + \|X(u)\|_{C^{k-2,\alpha}(M)}.
   \]

7. The space of \(J \zeta\)-invariant decaying Kähler potentials in \(X^{k,\alpha}\):
   \[
   U^{k,\alpha} := X^{k,\alpha} \cap K^{k,\alpha}
   \]
   and
   \[
   U := \bigcap U^{k,\alpha}.
   \]

**Remark 5.3.6.** The reason that we require \(X(e^{\theta_X} u) \in C^{k-2,\alpha}\) will become clear later. We will consider second order operators on \(C^{k,\alpha}_w(M)\) (which linearization is) of the form \(L = \Delta + X\) (\(\Delta\) being the Laplacian with respect to \(g\)). The idea is to conjugate \(L\) with \(e^{\theta_X}\) to obtain an operator \(\tilde{L} := e^{\theta_X} \circ L \circ e^{-\theta_X}\) which acts on the unweighted space \(C^{k,\alpha}\). In particular we will have a term of the form
   \[
   e^{\theta_X} X(e^{-\theta_X} \tilde{u}) = -|X|^2 \tilde{u} + X(\tilde{u}).
   \]
   On the other hand
   \[
   e^{\theta_X} \Delta(e^{-\theta_X} \tilde{u}) = \Delta \tilde{u} + (|X|^2 - \Delta \theta_X) \tilde{u} - 2X(\tilde{u}).
   \]
   So
   \[
   \tilde{L}(\tilde{u}) = \Delta \tilde{u} - X(\tilde{u}) - (\Delta \theta_X) \tilde{u}.
   \]
   Now it is clear that the image of \(u\) under \(L\) lies in \(C^{k-2,\alpha}_w\) if and only if the image of \(\tilde{u} := e^{\theta_X} u\) under \(\tilde{L}\) lies in \(C^{k-2,\alpha}\). But this is quite obvious for the first and third term above which means that \(X(\tilde{u}) = X(e^{\theta_X} u)\) has to be an element of \(C^{k-2,\alpha}\).
5.3.4. Restatement of the main theorem. Using the reduction to a Monge-Ampere equation above we can now state the existence theorem as scalar partial differential equation:

**Theorem 5.3.7** (Existence of expanding Kähler Ricci Solitons, PDE Version). Let $(M,J,g)$ be a Kähler manifold which is biholomorphically isometric to a Ricci flat cone $(C,J_C,ω_C)$ outside a compact set. Moreover let $X$ be a holomorphic vector field on $M$ such that

1. $X = \frac{1}{2} (\mathbb{J} \frac{\partial}{\partial \varphi} - \sqrt{-1} J_C r \frac{\partial}{\partial \varphi})$ outside a compact set.
2. $\mathfrak{R}(X) = \nabla \theta_X, \theta_X = r^2$ outside a compact set.
3. $L_{3\lambda(X)}\omega = 0$.

Then for any $F \in C^\infty(M)$ there exists a unique $\varphi \in \mathcal{U}$ such that $\varphi$ satisfies the equation

$$\omega^n = e^{\varphi - X(\varphi) + F} \omega^n. \quad (5.5)$$

**Remark 5.3.8.** The assumptions in the theorem actually imply that $M$ is an equivariant resolution of $C \cup \{a\}$.

**Remark 5.3.9.** Note that the Kähler Ricci flow associated to our expanding Ricci soliton converges smoothly on complements of open neighborhoods of the mentioned compact set to the original Ricci flat Kähler cone. Moreover, it converges in the Gromov Hausdorff sense as $t \to 0$ to the metric completion of $(C,g_C)$ as a metric space. This flow is different from the one constructed in [ST09] (see also [BCG+12]). There Ricci flows on compact varieties with log terminal singularities are constructed. In particular, following the argument in [ST09] and [BCG+12] the corresponding Kähler Ricci flow would be constructed as follows: suppose $C \cup \{a\}$ is a log terminal singularity and $\pi : M \to C \cup \{a\}$ a minimal resolution of $C$. Then there exists a unique Ricci flow $\omega_\varphi$ on $M$ such that $\omega_\varphi = \pi^* \theta_0 + t \tau^* \chi + \sqrt{-1} \partial \bar{\partial} \varphi(...,t)$, where in our case $\theta_0 = 0$ and $\pi^* \tau^* \chi$ is a curvature form corresponding to a hermitian metric $h$ on the well-defined line bundle $\pi^* K_C^p$ ($p \in \mathbb{N}$ sufficiently large) on $M$ which in our case is trivial and hence can be chosen to be equal to zero. Finally $\varphi(...,t)$ solves

$$\frac{\partial \varphi}{\partial t} = \log \left( \frac{\omega^n}{\mu_h} \right), \quad (x,t) \in C \times (0,T]$$

where $\log(\mu_h)$ is a potential for $\chi$ (in the terminology of [BCG+12], $\mu_h$ is called the associated adapted measure) and $\varphi_t \to \varphi_0 = r^2$ as $t \to 0$ uniformly in complements of open neighborhoods of $\{a\}$. It follows that the unique solution $\varphi$ is the stationary solution on the cone. Of course the cone is not compact as required in [ST09]. However, it is not hard to give compact examples (compactifications of the examples constructed in [FIK03]) with log-terminal singularities modeled on Ricci flat Kähler cones which have solutions of type described in [ST09] as well as solutions on topologically different manifolds.

Note that in the program of Song and Tian the singularities that form during the Kähler Ricci flow are not necessarily log-terminal. In particular the limit might not be a variety with $\mathbb{Q}$-Gorenstein singularities (for example for a small, i.e. higher codimension contraction) and hence their tangent cones are certainly not Ricci flat. So to study such initial conditions one would have to investigate general Kähler cones where the canonical line bundle is not well-defined as above. In this case it is conjectured that the Ricci flow can be continued only after a flip and the blow-up limit of the resulting flow should be an expanding Kähler Ricci soliton coming out of a cone. While these expanding Kähler Ricci solitons are also topologically different from the initial cone their exceptional locus has higher codimension and so the curvature can decay at most quadratically. In particular, these expanding Kähler Ricci solitons do not fall into the class of expanding Kähler Ricci solitons constructed here. In conclusion it seems likely (at least on complex surfaces) that the static solution of the Ricci flow on the Ricci flat cone is indeed the preferred solution while the solutions presented here seem to correspond to Toppings reverse bubbling examples for the harmonic map heat flow (i.e. the entropy jumps down at time $t$).

### 5.4. Proof of the existence theorem

We begin by stating four theorems which we then combine to prove theorem 5.3.7.

Note however that $\lambda_{nc}(g_{FIK}) > 0$ while $\lambda_{nc}(g_{eucl}) = \lambda_{nc}(g_{eucl}) = 0$. 
Theorem 5.4.1. [a priori estimates I] Let \((M, J, g)\) and \(X\) as in theorem [5.3.7]. For any \(D_1 > 0\) there exists \(D_2, D_3, D_4, D_5\) such that if \(F \in C_c^\infty(M) \cap C_X^\infty(M)\), \(\|F\|_{C^1(M)} \leq D_1\), \(\text{diam}(\text{supp}(F)) \leq D_1\), and \(\varphi \in \mathcal{U}^{3,\alpha}\) a solution of
\[
\omega^n_\varphi = e^{\varphi - X(\varphi) + F} \omega^n
\]
then \(\varphi \in C_c^\infty(M)\), \(\|\varphi\|_{C^0(M)} \leq D_2\), \(\|X(\varphi)\|_{C^0(M)} \leq D_3\), \(\|\partial \varphi\|_{C^0(M)} \leq D_4\), and \(\|\nabla \partial \varphi\|_{C^0(M)} \leq D_5\).

Theorem 5.4.2. [a priori estimates II] Let \((M, J, g)\) and \(X\) as in theorem [5.3.7]. For any \(D_1, D_2, D_3, D_4, D_5 > 0\) there exists \(D_6, D_7\) such that if \(F \in C_c^\infty(M) \cap C_X^\infty(M)\), \(\|F\|_{C^1(M)} \leq D_1\), \(\text{diam}(\text{supp}(F)) \leq D_1\), and \(\varphi \in \mathcal{U}^{3,\alpha}\) a solution of
\[
\omega^n_\varphi = e^{\varphi - X(\varphi) + F} \omega^n
\]
then \(\varphi \in \mathcal{U}^{4,\alpha}\) for all \(k \in \mathbb{N}, \alpha \in (0, 1)\).

Theorem 5.4.3. Let \((M, J, g)\) and \(X\) be as in theorem [5.3.7]. Suppose that \(\varphi \in \mathcal{U}^{3,\alpha}\) is a solution of
\[
\omega^n_\varphi = e^{\varphi - X(\varphi) + F} \omega^n
\]
then \(\varphi \in \mathcal{U}^{4,\alpha}\).

Theorem 5.4.4. Let \((M, J, g)\) and \(X\) be as in theorem [5.3.7]. Suppose that \(\tilde{F} \in C_c^\infty(M)\). And \(\tilde{\varphi} \in \mathcal{U}^{3,\alpha}\) is a solution of
\[
\omega^n_{\tilde{\varphi}} = e^{\tilde{\varphi} - X(\tilde{\varphi}) + \tilde{F}} \omega^n.
\]
Then for any \(F \in C_c^\infty(M)\) such that \(\|F - \tilde{F}\|_{C^1(M)}\) is sufficiently small there exists a solution \(\varphi \in \mathcal{U}\) of
\[
\omega^n_\varphi = e^{\varphi - X(\varphi) + F} \omega^n
\]

Theorem 5.4.5. Let \((M, J, g)\) and \(X\) be as in theorem [5.3.7] and \(F \in C_c^\infty(M)\). Then there exists at most one solution \(\varphi \in \mathcal{U}^{3,\alpha}\) of
\[
\omega^n_\varphi = e^{\varphi - X(\varphi) + F} \omega^n.
\]

The partial differential equation [5.5] we consider is a very familiar one. In fact, it is almost the same as the Monge Ampere equation associated with the Calabi conjecture (up to the linear term \(X(\varphi) - \varphi\), the problem of finding a Kahler Einstein metric with negative scalar curvature (up to the term \(X(\varphi)\)), or shrinking Kahler Ricci solitons (up to the sign in front of \(\varphi\)). If \(M\) is compact the existence of solutions in the first two cases has been shown by Yau and Aubin. In principle our partial differential equation [5.5] is most similar to the Kahler Einstein problem (although the asymptotic behavior of the solution is very different). The essential part of the proof is to find uniform a priori estimates for \(\varphi\) and just like in the Kahler Einstein problem we can easily find uniform \(C^0\)-estimates for \(\varphi\) using the maximum principle. However, here we have to deal with two additional problems: the non-compactness of \(M\) and the fact that the vector field \(X\) is unbounded. The first problem can be resolved by using the weighted function spaces introduced earlier. Here these weighted spaces are much simpler than those considered for example in the partial differential equation associated to the non-compact Calabi Yau problem. The second obstacle (unbounded coefficients) requires a bit more work and basically uses results for linear elliptic operators with unbounded coefficients and global Schauder estimates developed there.

So here are the basic ideas of the proof: as usual we show the existence of a solution using the continuity method. Openness (theorem [5.4.4] will follow from the fact that the linearized operator of \(\Delta - X - \lambda\) acting on weighted spaces is invertible for \(\lambda > 0\). In the compact case this is easy since the choice of function spaces is quite obvious. In the non-compact case the function spaces have to chosen a bit more carefully. If the underlying cone is Ricci flat then it turns out that there exists a simple choice (the weighted spaces from above) and the problem becomes very similar to the compact case. The closedness will be shown by establishing the following a priori estimates (theorem [5.4.1]): first the unweighted \(C^0\)-estimates follow directly from the maximum principle. The weighted \(C^0\)-estimates are also obtained.

\[\text{Note that there is a minus sign in front of } X.\] In fact, this operator is not the linearization of the Monge Ampere equation associated with [5.5] but its conjugation with \(\exp(-\theta_X)\) as described in the previous subsection.
using the maximum principle by considering the function $\exp(\theta X)\varphi$ on the conical end. In combination with the $C^0$-estimates in the interior set this gives the desired $C^0_w$-estimates. Before we can proceed to the so-called $C^2$-estimates we have to establish uniform estimates for $X(\varphi)$. These follow from the weighted $C^0$-estimates and the the fact that $\varphi$ is a Kähler potential, i.e. $\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$. Next we find uniform estimates for $|\partial\bar{\partial}\varphi|$. This is done exactly like in Aubin’s and Yau’s work (up to some additional terms), i.e. again using the maximum principle. Note that this does not yet give a $C^2$-estimate for $\varphi$ directly. However, together with the estimates for $X(\varphi)$ and $\varphi$ it follows that the metric $g'$ associated to $\omega_\varphi$ is uniformly equivalent to $g$. The estimates for $\nabla\partial\bar{\partial}\varphi$ are again due to Yau. This implies in particular uniform estimates for $\|g^{-1}\omega\|_{C^{0,\alpha}}$. Next the maximum principle establishes decay estimates for $X(\varphi)$ which are slightly weaker than $C^0_w$. This, in turn, gives us decay estimates for $\partial\bar{\partial}\varphi$ (at the same rate). And finally the global Schauder estimates for the conical end provide us with the $C^{2,\alpha}$-estimates and a bootstrapping argument delivers the $C^{k,\alpha}_w$-estimates for $k > 2$ (theorems 5.4.2 and 5.4.3).

**Proof of theorem 5.3.7.** For $t \in [0,1]$ denote by $\varphi_t \in \mathcal{U}^{\beta,\alpha}$ a solution (if one exists) of

$$ \omega_\varphi^n = e^{\varphi - X(\varphi) + tF} \omega^n. $$

And let

$$ S := \{ s \in [0,1]: \text{there exists a solution } \varphi_s \in \mathcal{U}^{\beta,\alpha} \text{ for all } t \in [0,s]\}. $$

**Theorem 5.4.6.** The set $S$ is open in $[0,1]$.

**Proof.** Let $t_0 \in S$ and choose $\epsilon > 0$ small enough such that $\epsilon \|F\|_{C^{\beta,\alpha}_w(M)}$ is sufficiently small to apply theorem 5.4.4. Then by theorem 5.4.4 we get a solution $\varphi_t \in \mathcal{U}^{\beta,\alpha}$ for any $t_0 \leq t < t_0 + \epsilon$ and so it follows that $s \in S$ for any $s < t_0 + \epsilon$, in other words $S$ is open.

**Theorem 5.4.7.** The set $S$ is closed in $[0,1]$.

**Proof.** We have to show that for any sequence $t_j \in S$ which converges in $[0,1]$ the limit $t_j \to \infty t_j = t$ is also contained in $S$. So assume that $\varphi_j = \varphi_t$ is a sequence of solutions of

$$ \omega_\varphi^n_j = e^{\varphi_j - X(\varphi_j) + F_j} \omega^n, $$

where $F_j = t_j F$. Clearly $F_j \in C^\infty_c(M) \cap C^\infty_\omega(M)$ and there is a positive number $D_1$ such that $\|F_j\|_{C^{\beta,\alpha}_w(M)} \leq D_1$ and diam($\text{supp}(F_j)$) $\leq D_1$ for all $j$. By theorems 5.4.1 and 5.4.2 we get a uniform bound $D_0$ for $\|\varphi_j\|_{C^{\alpha,\alpha}_w(M)}$. By Arzela Ascoli we can find a subsequence $\varphi_{j_n}$ that converges in $C^{\beta,\alpha}_w(M)$ to a function $\varphi \in C^{\alpha,\alpha}_w(M)$, and since the sequence is bounded in $X^{\beta,\alpha}$ we have $\varphi \in X^{\beta,\alpha}$. Moreover, $\varphi \in \mathcal{U}^{\beta,\alpha}$. Indeed, assume to the contrary that $\varphi \not\in \mathcal{U}^{\beta,\alpha}$. Then there exists a point $x \in M$ such that $\omega_\varphi(x^n) = 0$ (since $\omega_\varphi > 0$ outside a large compact set by the decay property of $\varphi$). But this means that $\lim_{t \to \infty} \varphi_t(x) - X(\varphi_t)(x) + t_j F = -\infty$ which contradicts the uniform boundedness of $\varphi_t$ and $X(\varphi_t)$. Therefore $\varphi$ satisfies

$$ \omega_\varphi^n = e^{\varphi - X(\varphi) + tF} \omega^n. $$

And so $t \in S$ which proves that $S$ is closed.

Now we can finish the proof of theorem 5.3.7. Since $S$ is non-empty as $0 \in S$ (with $\varphi_0 = 0$) and $S$ is closed and open in $[0,1]$, we have $S = [0,1]$. This means that there exists a solution $\varphi_1 \in \mathcal{U}^{\beta,\alpha}$ which is exactly the desired solution $\varphi$ of (5.5).

**5.4.1. Proof of theorems 5.4.1-5.4.5.**

**Proof of theorem 5.4.1.** We will now prove theorem 5.4.1. Parts of this proof are very similar to the compact case. In particular the proofs of the estimates for $\varphi$, $\partial\bar{\partial}\varphi$ and $\nabla\partial\bar{\partial}\varphi$ are almost exactly the same as in Yau78. However, we will need in addition uniform estimates for $\varphi$ in $C^0_w(M)$ and $X(\varphi)$ in order to carry out the second and third order estimates of Yau and Aubin.

So in the following five subsections we assume that $\varphi \in \mathcal{U}^{\beta,\alpha}$ is a solution of

$$ \omega_\varphi^n = e^{\varphi - X(\varphi) + F} \omega^n, $$

with $F \in C^\infty_c(M)$ and $\|F\|_{C^{0,\alpha}_w(M)} \leq D_1$ and diam($\text{supp}(F)$) $\leq D_1$. 
5.4.1.1. Local regularity. In order to start with the proof of theorem 5.4.1 we will need the following proposition.

**Proposition 5.4.8.** Let \((M,J,g)\) be as in theorem 5.4.1, \(k \geq 1\), and \(\varphi \in \mathcal{U}^{k,\alpha}\) a solution of \((5.5)\) with \(F \in C^{k,\alpha}_{loc}(M)\). Then \(\varphi \in C^{k+2,\alpha}_{loc}(M)\).

**Proof.** By the local \(\partial \bar{\partial}\)-lemma \([Joy00]\) \(\omega\) can be locally expressed as \(\sqrt{-1} \partial \bar{\partial} u_0, u_0 \in C^\infty_{loc}(U), U \subset M\) open, we can set \(u := u_0 + \varphi\) and study the equation in local holomorphic coordinates on a possibly smaller subset \(\Omega \subset U\)

\[
\log(\det(u_{ij})) + X(u) - u = \tilde{F}
\]

for \(\tilde{F} \in C^{k,\alpha}_{loc}(\Omega), \Omega \subset \mathbb{C}^n\). Now \(u \in C^{3,\alpha}(\Omega)\) by assumption and \(u_k := \partial_{z_k} u\) satisfies a linear elliptic partial differential equation with coefficients in \(C^{1,\alpha}(\Omega)\). Indeed

\[
\frac{\partial F}{\partial z_k} = g^{ij}_k \frac{\partial^2 u_k}{\partial z_i \partial \bar{z}_j}.
\]

And since \(\tilde{F} \in C^{2,\alpha}(\Omega)\) it follows form standard regularity theory that \(\nabla u \in C^{3,\alpha}(\Omega')\) for \(\Omega' \subset \Omega\) and hence \(\varphi \in C^{2,\alpha}_{loc}(M)\). The case \(k > 1\) works similarly. \(\square\)

5.4.1.2. \(C^0\)-estimates. We start with the uniform estimates for \(\varphi\). These are very easy to obtain as the sign in front of the zeroth order term in the equation helps us to apply the maximum principle in a straightforward fashion.

**Lemma 5.4.9.** Let \(\varphi \in \mathcal{U}^{3,\alpha}\) be a solution of \((5.5)\). Then there exists a constant \(C_1 = C_1(g)\) such that

\[
\|\varphi\|_{C^0} \leq C_1.
\]

**Proof.** By assumption \(\lim_{x \to \infty} \varphi(x) = 0\). So if \(\sup \varphi > 0, \inf \varphi < 0\) then there must be a point \(x_\text{max} \in M, (x_\text{min} \in M)\) such that \(\varphi(x_\text{max}) = \sup \varphi \) and \(\varphi(x_\text{min}) = \inf \varphi\). But at \(x_\text{max}, \varphi(x_\text{min})\) we have \(\omega^a_\varphi \leq \omega^a (\omega^a_\varphi \geq \omega^a)\) and \(X(\varphi)(x_\text{max}) = 0(X(\varphi)(x_\text{min}) = 0)\). And so

\[
\begin{align*}
\sup \varphi & \leq -F(x_\text{max}) \leq -\inf F \leq \sup |F| \\
\inf \varphi & \geq -F(x_\text{min}) \leq -\sup F \leq -\sup |F|,
\end{align*}
\]

and hence

\[
\|\varphi\|_{C^0} \leq C_1.
\]

5.4.1.3. \(C^0_u\)-estimates. Unlike in the compact case and in the non-compact ALE case of the Calabi-Yau theorem described in \([Joy00]\) we have to find weighted \(C^0\) estimates for \(\varphi\) first in order to proceed to the higher order estimates. In particular, we will need the decay estimates to get uniform bounds for \(X(\varphi)\).

**Lemma 5.4.10.** Let \(\varphi \in \mathcal{U}\) be a solution of \((5.5)\). Then there exists a uniform constant \(C_2 = C_2(D_1, g)\) such that

\[
\|\varphi\|_{C^2_u} \leq C_2.
\]

**Proof.** We split the proof into two parts. First we prove uniform upper bounds for \(\exp(\mu \theta_X) \varphi\). This uses the fact that \(\exp(\mu \theta_X) \varphi\) is actually a subsolution of a linear elliptic partial differential equation for any \(\mu \in (0,1)\). Again the maximum principle gives us uniform estimates for \(\sup \exp(\mu \theta_X) \varphi\) for any \(\mu \in (0,1)\). As these estimates do not depend on \(\mu\) we obtain the desired upper bound. To prove that \(\inf \exp(\alpha \theta_X) \varphi\) is bounded from below for any \(\alpha \in (0,1)\) we first construct a barrier function \(\chi\) (independent of \(\varphi\)) on the cone end which decays at the rate \(\exp(-\mu \theta_X)\) for some small \(\mu > 0\) that bounds \(\varphi\) from below outside a uniformly large compact set. Here we will need the \(C^0\) estimates for \(\varphi\) from before. In a second step we apply the minimum principle to the function \(\exp(\alpha \theta_X) \varphi\) for any \(\alpha \in (0,1)\) which again gives us a uniform lower bound independent of \(\alpha\). Arguing as above it follows
that \( \exp(\theta_X)\varphi \) is uniformly bounded form below and hence \( \|\varphi\|_{C^2(M)} \leq C_2 \) for some uniform constant \( C_2 \). Note first that

\[
F = \log \left( \frac{\omega^n}{\omega^n} \right) + X(\varphi) - \varphi = \Delta \varphi - \int_0^1 \int_0^\sigma |\partial \bar{\partial} \varphi\omega_{\omega_{\omega}}| \, d\tau d\sigma + X(\varphi) - \varphi
\]

\[
\leq \Delta \varphi + X(\varphi) - \varphi.
\]

In other words \( \varphi \) is a subsolution of a linear elliptic equation. This uses the concavity of the \( Q \mapsto \log(\det(1 + Q)) \). Let \( \mu \in (0, 1) \) and \( \psi_\mu := e^{\mu \theta X} \varphi \). Then \( \psi_\mu \) satisfies

\[
F e^{\mu \theta X} \leq \Delta \psi_\mu + (1 - 2\mu) X(\psi_\mu) - (\mu \Delta \theta_X + (\mu - \mu^2)|X|^2 + 1) \psi_\mu.
\]

Since \( F \) has compact support we can find a uniformly large compact set \( K \) such that on \( M \setminus K \) (using \( \Delta \theta_X = n = n \setminus K \))

\[
0 \leq \Delta \psi_\mu - X(\psi_\mu) - (\mu + (\mu - \mu^2)|X|^2 + 1) \psi_\mu
\]

and since \( \rho (\mu) + (\mu - \mu^2)|X|^2 + 1 \geq 1 \) and \( \psi_\mu(x) \to 0 \) as \( x \to \infty \) (here we use that \( \mu < 1 \)) it follows by the maximum principle that

\[
\sup_M \psi_\mu \leq C'_2(K).
\]

And since \( C'_2(K) \) does not depend on \( \mu < 1 \) we have, letting \( \mu \to 1 \),

\[
\sup_M e^{\theta X} \varphi \leq C'_2(K).
\]

To show the lower bound we have to work a bit harder. First recall that outside of a compact set \( K_0 \) (i.e. where \( \omega \) is exactly the cone metric and \( \theta_X = r^2 \)) we have

\[
\omega = \sqrt{-1} \bar{\partial} \partial \theta_X \quad \text{and} \quad |X|^2 = \theta_X.
\]

So let us consider the set

\[
\Omega_\kappa := \{ x \in M : \theta_X > \kappa \}.
\]

We know that there exists a uniformly large \( \kappa_0 \) such that \( \Omega_\kappa \subset M \setminus K_0 \) and \( F = 0 \) on \( \Omega_\kappa \) for \( \kappa \geq \kappa_0 \). On such a set \( \Omega_\kappa \) we will to construct a function \( \chi \) such that

\[
\log \left( \frac{\omega^n}{\omega^n} \right) + X(\chi) - \chi \geq 0
\]

on \( \Omega_\kappa \) and \( \chi \leq \varphi \) on \( \partial \Omega_\kappa \). We use the following ansatz

\[
\chi := x_{\rho, \mu} = -\rho e^{-\mu \theta X}
\]

where \( \rho > 0, \mu \in (0, 1) \). We cannot expect that \( \chi \in \mathcal{U} \) (i.e. \( \omega_\chi > 0 \)) on all of \( M \). However, we show that for an appropriate choice of \( \kappa \) we will have \( \omega_\chi > 0 \) on \( \Omega_\kappa \). This works for example for the following choices of \( \rho, \mu, \) and \( \kappa \) (\( C_1 \) being the constant from the the previous lemma):

\[
\kappa := \max\{4C_1 e^{-1/2} , \kappa_0\}, \quad \mu := \frac{3}{2\kappa}, \quad \rho := C_1 \exp(3/2).
\]

First note that on \( \partial \Omega_\kappa \) we have \( \theta_X = \kappa \) and so on \( \partial \Omega_\kappa \)

\[
\chi = -C_1 \exp(3/2) \exp(-3/2) = -C_1 \leq \varphi.
\]

The eigenvalues of the hermitian 2-tensor associated to

\[
\omega_\chi = \omega + \sqrt{-1} \bar{\partial} \partial \chi = (1 + \rho \mu e^{-\mu \theta X}) \omega - \sqrt{-1} \rho \mu^2 e^{-\mu \theta X} \partial \theta_X \land \bar{\partial} \theta_X
\]

are \( (1 + \rho \mu \exp(-\mu \theta_X)) \) with multiplicity \((n-1)\) (the base eigenvalue i.e. the ones whose eigenvectors are transversal to \( X \)) and \((1 + \rho \mu^2|X|^2 - \mu) \exp(-\mu \theta_X)\). Clearly the transversal eigenvalues are bounded.
from below by 1. As for the remaining eigenvalues we have in $\Omega_\kappa$

\[(1 - \rho(\mu^2|X|^2 - \mu) \exp(-\mu\theta_X)) = 1 - C_1 \exp(3/2)\mu(\mu|X|^2 - 1) \exp(-\mu\theta_X) \geq 1 - C_1 \exp(3/2)\mu \sup_{y} (y - 1) \exp(-y) \]

\[= 1 - C_1 \exp(3/2) \frac{3}{2\kappa} \exp(-2) = 1 - C_1 \frac{3}{2\kappa} e^{-1/2} \geq 1 - \frac{3}{8} > \frac{1}{2}.\]

So on $\Omega_\kappa$ we have

\[\log \left( \frac{\omega_{n_{X}}}{\omega_{n}} \right) \geq \log(1 - \rho(\mu^2|X|^2 - \mu) \exp(-\mu\theta_X)).\]

By the estimates above we know that on $\Omega_\kappa$

\[\rho(\mu^2|X|^2 - \mu) \exp(-\mu\theta_X) \in (0, 1/2).\]

Now we can use that

\[\log(1 - x) \geq -2x, \text{ if } x \in (0, 1/2)\]

to conclude that in $\Omega_\kappa$

\[\log(1 - \rho(\mu^2|X|^2 - \mu) \exp(-\mu\theta_X)) \geq -2\rho(\mu^2|X|^2 - \mu) \exp(-\mu\theta_X).\]

We also have in $\Omega_\kappa$

\[X(\chi) - \chi = \rho(\mu|X|^2 + 1) e^{-\mu\theta_X}.\]

Putting everything together

\[\log \left( \frac{\omega_{n_{X}}}{\omega_{n}} \right) + X(\chi) - \chi \geq -\rho((2\mu^2 - \mu)|X|^2 - 2\mu) e^{-\mu\theta_X},\]

So on $\Omega_\kappa$ (if $\mu < 1/2$, which we can assume without loss of generality by choosing $C_1$ large if necessary)

\[\log \left( \frac{\omega_{X}}{\omega_{n}} \right) + X(\chi) - \chi \geq 0.\]

and

\[\log \left( \frac{\omega_{\varphi}}{\omega_{n}} \right) + X(\varphi) - \varphi = 0.\]

Letting $\psi := \chi - \varphi$ we have on $\Omega_\kappa$

\[\log \left( \frac{(\omega_{\varphi} + \sqrt{-1}\partial\bar{\partial}\psi)^n}{\omega_{\varphi}^n} \right) + X(\psi) - \psi \geq 0.\]

And so by the maximum principle

\[\sup_{\Omega_\kappa} \psi \leq \max\{0, \sup_{\partial\Omega_\kappa}(\chi - \varphi)\} \leq 0.\]

It follows that $\chi \leq \varphi$ in $\Omega_\kappa$ and by the observations above we have a lower bound for $\varphi$ of the form $-\rho \exp(-\theta X)$.

It remains to show that we can find a uniform constant $C_2^\alpha$ such that $\varphi \geq -C_2^\alpha \exp(-\theta X)$. Again let

$\psi_\alpha := \exp(\alpha\theta_X)\varphi$, $\alpha \in (0, 1)$. We can choose a uniformly large compact set $K$ outside of which

$\omega - \sqrt{-1}\alpha\varphi \partial\bar{\partial}\psi + \sqrt{-1}\alpha^2 \varphi \partial\theta_X \wedge \bar{\partial}\theta_X > 0$.\]

So if $\psi_\alpha$ has a minimum in $\Omega \setminus K$ then at this point (call it $p_\alpha$ and $\varphi(p_\alpha) = m_\alpha$ and $\psi(p_\alpha) = \inf \psi_\alpha = m_\alpha'$)

$\omega(p_\alpha) + \sqrt{-1}\partial\bar{\partial}\psi(p_\alpha) \geq \omega - \sqrt{-1}\alpha m_\alpha \partial\theta_X(p_\alpha) + \sqrt{-1}\alpha^2 m_\alpha \partial\theta_X(p_\alpha) \wedge \bar{\partial}\theta_X(p_\alpha) > 0$.
and so
\[
\log \left( \frac{a - \sqrt{1-a} \alpha(\theta X + \sqrt{-1}a \alpha \partial \theta X \wedge \partial X)^n}{\omega^n} \right) - \alpha |X|^2 m_\alpha - m_\alpha \leq 0.
\]
Again it follows that (calculating the eigenvalues)
\[
0 \geq (n-1) \log(1 - \alpha m_\alpha) + \log(1 - \alpha m_\alpha + \alpha^2 |X|^2 m_\alpha) - \alpha |X|^2 m_\alpha - m_\alpha
\]
\[
\geq \log(1 - \alpha m_\alpha + \alpha^2 |X|^2 m_\alpha) - \alpha |X|^2 m_\alpha - m_\alpha
\]
where we assume that \( m_\alpha \leq 0 \). Moreover without loss of generality we can assume that
\[
(\alpha^2 |X|^2 - \alpha)m_\alpha \geq -\frac{1}{2}
\]
on \( M \setminus K \) and since
\[
\log(1 - x) \geq -x - x^2, \quad x \in (0, 1/2)
\]
we have (again for a sufficiently large choice of \( K \) independent of \( \alpha \))
\[
0 \geq (\alpha^2 |X|^2 - \alpha)m_\alpha - (\alpha^2 |X|^2 - \alpha)^2 m_\alpha^2 - \alpha |X|^2 m_\alpha - m_\alpha
\]
\[
\geq (\alpha^2 |X|^2 - \alpha)m_\alpha + (\alpha^2 |X|^2 - \alpha)^2 C \exp(-\mu \theta X) m_\alpha - \alpha |X|^2 m_\alpha - m_\alpha.
\]
This implies
\[
0 \geq (\alpha^2 |X|^2 - \alpha)m_\alpha' - \frac{1}{2} m_\alpha' - \alpha |X|^2 m_\alpha'
\]
hence \( \psi_\alpha \geq 0 \) on \( M \setminus K \) and we can find a constant \( C_2'' = C_2''(K) \) such that
\[
\varphi \geq -C_2'' \exp(-\alpha \theta X).
\]
Since \( C_2'' \) and \( K \) do not depend on \( \alpha \) we also have
\[
\varphi \geq -C_2'' \exp(-\theta X).
\]
\[
5.4.1.14. C^0\text{-estimates for } |X(\varphi)|. \text{ Now we estimate } |X(\varphi)|. \text{ Unlike in the compact case this estimate has to be done separately since } X \text{ does not have bounded coefficients. We start with some basic observations. Recall that } \Im(X)(\varphi) = 0 \text{ and we denote } \zeta := \Re(X).\]

**Lemma 5.4.11.** Let \( \varphi \) be a smooth function such that \( J\zeta(\varphi) = 0 \). Then \( J\zeta(\zeta(\varphi)) = 0 \).

**Proof.** Note that
\[
J\zeta(\zeta(\varphi)) = [J\zeta, \zeta](\varphi) + \zeta J(\zeta(\varphi)) = -L_\zeta(J\zeta)(\varphi) = -L_\zeta J\zeta + JL_\zeta \zeta(\varphi) = 0,
\]
where we used that \( L_\zeta J = 0 \) since \( \zeta \) is a real holomorphic vector field. \( \square \)

**Lemma 5.4.12.** Let \( \varphi \in \mathcal{U}^{3,\alpha} \). Then for any vector field \( Y \) on \( M \) we have \( \sqrt{-1} \partial \bar{\partial} \varphi(Y, JY) > -|Y|^2 \).

**Proof.** By assumption \( \omega_\zeta(Y, JY) > 0 \) and so
\[
0 < \omega(Y, JY) + \sqrt{-1} \partial \bar{\partial} \varphi(Y, JY) = |Y|^2 + \sqrt{-1} \partial \bar{\partial} \varphi(Y, JY)
\]
which proves the lemma. \( \square \)

**Lemma 5.4.13.** Suppose \( \varphi \in \mathcal{U}^{3,\alpha} \). Then
\[
X(X(\varphi)) = \sqrt{-1} \partial \bar{\partial} \varphi(X, \bar{X})
\]
and in particular
\[
X(X(\varphi)) \geq -|X|^2.
\]
5.4. Proof of the Existence Theorem

Proof. Note first that by the previous lemma
\[ X(X(\varphi)) = \zeta(\zeta(\varphi)) = \bar{X}(X(\varphi)) \]
where \( \bar{X} \) is the complex conjugate of \( X \). Now as usual we denote
\[ dz_i = dx_i + \sqrt{-1} dy_i, \quad dz_j = dx_j - \sqrt{-1} dy_j, \quad \frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} - \sqrt{-1} \frac{\partial}{\partial y_i} \right), \quad \frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + \sqrt{-1} \frac{\partial}{\partial y_j} \right), \]
moreover
\[ \zeta = \sum_{i=1}^{n} \zeta^i \frac{\partial}{\partial x_i} + \zeta^{n+i} \frac{\partial}{\partial y_i} \]
and so
\[ X^i = \zeta^i + \sqrt{-1} \zeta^{n+i}, \quad \bar{X}^i = \zeta^i - \sqrt{-1} \zeta^{n+i}. \]
To make notation easier we let
\[ A_{ij} := \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \frac{\partial^2 \varphi}{\partial y_i \partial y_j}, \quad B_{ij} := \frac{\partial^2 \varphi}{\partial x_i \partial y_j} - \frac{\partial^2 \varphi}{\partial y_i \partial x_j}. \]
Then
\[ \sqrt{-1} \partial \bar{\partial} \varphi(\zeta, J \zeta) = A_{ij} \left( \zeta^i \zeta^j + \zeta^{n+i} \zeta^{n+j} \right) + B_{ij} \left( \zeta^i \zeta^{n+j} + \zeta^{n+i} \zeta^j \right). \]
On the other hand
\[ X^i \bar{X}^j \frac{\partial^2 \varphi}{\partial z_i \partial z_j} = (\zeta^i + \sqrt{-1} \zeta^{n+i})(\zeta^j - \sqrt{-1} \zeta^{n+j})(A_{ij} + \sqrt{-1} B_{ij}) \]
\[ = A_{ij} \left( \zeta^i \zeta^j + \zeta^{n+i} \zeta^{n+j} \right) + B_{ij} \left( \zeta^i \zeta^{n+j} - \zeta^{n+i} \zeta^j \right). \]
But since
\[ \bar{X}(X(\varphi)) = X^i \bar{X}^j \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \]
it follows that \( X(X(\varphi)) = \sqrt{-1} \partial \bar{\partial} \varphi(\zeta, J \zeta) \). By the previous lemma this implies that
\[ X(X(\varphi)) > -|X|^2. \]

Now we go on to proof the uniform estimates for \( X(\varphi) \) which is a real valued function by assumption. The estimate for \( X(\varphi) \) follows from the fact that \( \varphi \) decays very fast as \( x \to \infty \) while (as we know form the lemma above) \( X(X(\varphi)) \) is bounded from below by function that grows much slower compared to \( \exp(\theta X) \). The bounds on the first derivative are then achieved via interpolation.

Lemma 5.4.14. Let \( \varphi \in U^{3,\alpha} \) be a solution of \( 5.3 \). Then there exists a uniform constant \( C_3 \) such that
\[ \|X(\varphi)\|_{C^\alpha} \leq C_3. \]

Proof. Let \( x \in M \). Let \( \gamma_x(t) := \Phi^t_\zeta(x) \) (where \( \Phi^t_\zeta \) is the flow of the (complete) vector field \( \zeta \), \( t \in (-\infty, \infty) \), \( \varphi_x(t) := \varphi(\Phi^t_\zeta(x)) \).

(1) First we show that \( X(\varphi) \) is bounded from above: Let \( \epsilon > 0 \) to be fixed later and \( \alpha : [0, \infty) \to \mathbb{R} \) such that
\[ \begin{align*}
&\bullet \quad 0 \leq \alpha(t) \leq 1, \quad t \in [0, \infty), \\
&\bullet \quad \alpha(0) = 1, \quad \alpha(t) = 0, \quad t > \epsilon, \\
&\bullet \quad \alpha'(0) = 0 = \alpha'(\epsilon), \\
&\bullet \quad \sup |\alpha'(t)| < 2/\epsilon, \quad \sup |\alpha''(t)| < 2/\epsilon^2.
\end{align*} \]
Then
\[ -\int_0^\infty |\alpha''(t)|C(x)dt \leq -\int_0^\infty \alpha''(t)\phi_x'(t)dt = \int_0^\infty \alpha'(t)\phi_x''(t)dt \]
\[ = -\phi_x'(0) - \int_0^\epsilon \alpha(t)\phi_x''(t)dt \]
\[ = -\phi_x'(0) - \int_0^\epsilon \alpha(t)X(X(\phi))(\gamma_x(t))dt \]
\[ \leq -\phi_x'(0) + \int_0^\epsilon |X|^2(\gamma_x(t))dt \]
where \( C(x) := \sup_{t \in [0, \infty)} |\phi_x(t)| \). And so
\[ \phi_x'(0) \leq \int_0^\epsilon C(x)|\alpha''(t)| + \int_0^\epsilon |X|^2(\gamma_x(t))dt \leq \frac{2C(x)}{\epsilon} + C_1(e^{\lambda_x} - 1)|X|^2(x), \]
where we used that \( |X|^2(\gamma(t)) \leq e^{\lambda_x}|X|^2(\gamma_x(0)) \),
where
\[ \lambda := 2\sup_M |\nabla X|. \]
This follows from
\[ \frac{d}{dt}|X|^2(\gamma_x(t)) \leq 2|\nabla X|(\gamma_x(t))|X|^2(\gamma_x(t)) \leq 2\lambda |X|^2(x). \]
Setting
\[ \epsilon := \frac{1}{\lambda} \log \left(1 + \frac{1}{1 + |X|^2(x)}\right) \]
it follows that
\[ X(\phi)(x) \leq \frac{C(x)\lambda}{\log \left(1 + \frac{1}{1 + |X|^2(x)}\right)} + 1. \]
But since \( C(x)\lambda(\log(1 + (1 + |X|^2(x))^{-1}))^{-1} \) can be uniformly bounded by the previous observations we have a uniform upper bound \( C'_x \) for \( X(\phi) \). Indeed, we know that \( \zeta \) is just the radial vector field on the set \( \Omega_\kappa \) for some sufficiently large \( \kappa \). And on the complement of \( \Omega_\kappa \) we have \( |X|^2 < \kappa \) which gives along with the uniform bound for \( C(x) \) on \( M \) a uniform upper bound for \( X(\phi) \) on \( \Omega_\kappa \), namely
\[ \sup_{\Omega_\kappa} X(\phi) \leq \frac{C_1}{\log(1 + \kappa^{-1})}. \]
In \( \Omega_\kappa \) we know that
\[ \phi(\Phi_x^\kappa(x)) \leq C_2 \exp(-\theta_X(\Phi_x^\kappa(x))) = C_2 \exp(-\theta_X(x)), \]
since
\[ \frac{d}{dt} \theta_X(\Phi_x^\kappa(x)) = \langle \zeta, \nabla \theta_X \rangle = |\zeta|^2 \geq 0. \]
So (using \( |X|^2 = \theta_X \) on \( \Omega_\kappa \))
\[ \sup_{x \in \Omega_\kappa} X(\phi)(x) \leq \frac{C_2 \exp(-\theta_X(x))}{\log \left(1 + \frac{1}{1 + |X|^2(x)}\right)} \leq \sup_{y > \kappa} \frac{C_2 \exp(\theta_X(y))}{\log(1 + 1/(1 + y))} \leq C'_3. \]

(2) Secondly we establish a lower bound \( C'' \) for \( X(\phi) \). Let \( \beta : (-\infty, 0] \to \mathbb{R} \) be a function such that
- \( 0 \leq \beta(t) \leq 1, t \in (-\infty, 0] \),
- \( \beta(0) = 1, \beta'(0) = 0 \),
- \( \beta(t) = 0, t \leq -\epsilon \), \( \beta'(0) = \beta'(-\epsilon) = 0 \),
- \( \sup |\beta'(t)| \leq 2/\epsilon, \sup |\beta''(t)| \leq 2/\epsilon^2 \).
and Yau's previous work. The only difference are the additional terms involving $X$ whose heart are the so-called $C$.

5.4.1.6. Lemma

All together we have the existence of holomorphic normal coordinates at a point with respect to a given Kähler metric $g$ such that at each point $p$ we have

$$
\varphi'(0) = \int_{-\epsilon}^{0} \frac{d}{dt}(\beta(t)\varphi'_x(t)) = \int_{-\epsilon}^{0} \beta'(t)\varphi'_x(t) + \beta(t)\varphi''_x(t)dt
$$

$$
\geq \int_{-\epsilon}^{0} -\beta''(t)\varphi_x(t) - \beta(t)|X|^2(\gamma_x(t))
$$

$$
\geq -\frac{2}{\epsilon} C(x) - (e^{\lambda \epsilon} - 1)|X|^2(x),
$$

where $C(x) := \sup_{t \in (-\epsilon, \epsilon)} |\varphi_x(t)|$ and we use that

$$
|X|^2(\gamma_x(t)) \leq (e^{\lambda \epsilon} - 1)|X|^2(\gamma_x(0)).
$$

So choosing

$$
\epsilon := \min \left\{ \frac{1}{\lambda} \log \left( 1 + \frac{1}{1 + |X|^2(\epsilon)} \right), \frac{1}{2} \right\}
$$

we get

$$
X(\varphi)(x) \geq -\frac{2\lambda C(x)}{\log \left( 1 + \frac{1}{1 + |X|^2(\epsilon)} \right)} - 1
$$

which gives a uniform lower bound for $X(\varphi)$. Indeed,

$$
\tilde{C}(x) = \sup_{t \in (-\epsilon, \epsilon)} |\varphi(t)| \leq C_1 \exp(-\inf_{t \in (-\epsilon, \epsilon)} \theta_x(\gamma_x(t))) \leq C_1 \exp(\lambda \epsilon \theta_x(x) + C)
$$

All together we have

$$
|X(\varphi)| \leq C_4.
$$

5.4.1.5. Calculations at a point. Before we continue with the higher order estimates we recall the existence of holomorphic normal coordinates at a point with respect to a given Kähler metric $g$

**Lemma 5.4.15.** [Joy00] Let $(M, J, g)$ be as above. Then there exists a positive number $r > 0$ and $C > 0$ such that at each point $p \in M$ there exist an open neighborhood $U$ of $p$ and holomorphic coordinates $z_1, \ldots, z_n : U \to \mathbb{B}(0, 2r) \subset \mathbb{C}^n$ such that $g_{ij}(p) = \delta_{ij}$, $C^{-1} \delta \leq (g_{ij}) \leq C \delta$ in $B_r(0)$, and $\Gamma_{ij}^k(p) = 0$. Moreover, if $g'$ is the metric associated to $\omega_\varphi$, $\varphi \in \mathcal{U}_{3, \alpha}$ then the coordinates the coordinates can be chosen such that they satisfy in addition $g_{ij} = (1 + \varphi_{ij}(p))\delta_{ij}$.

It follows that in holomorphic normal coordinates we have at $p$

$$
\omega^n(p) = dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n,
$$

$$
\omega_\varphi(p)^n = \prod_{i=1}^n (1 + \varphi_{ii}(p))\omega(p)^n,
$$

$$
g_{ij}^\varphi = \frac{1}{1 + \varphi_{ii}(p)} , \text{ and }
$$

$$
\Delta \varphi(p) = \sum_{i=1}^n (1 + \varphi_{ii}(p)) - n.
$$

5.4.1.6. Second order estimates estimates. Now we can return to Yau’s and Aubin’s original proof whose heart are the so-called $C^2$-estimates. The next lemma is proven almost exactly like in Aubin’s and Yau’s previous work. The only difference are the additional terms involving $X$.

**Lemma 5.4.16.** Let $\varphi \in \mathcal{U}_{3, \alpha}$ be a solution of (5.3) then we can find a uniform constant $C_4$ such that

$$
\|n + \Delta \varphi\|_{\mathcal{C}^0} \leq C_4.
$$
Proof. We follow [Tia00]. Consider

\[ \log \left( \frac{\omega^\varphi}{\omega^n} \right) + X(\varphi) - \varphi = F. \]

Differentiating by \( \partial_z \) we get

\[ g^i_j \left( \frac{\partial g_{ij}}{\partial z} + \frac{\partial^3 \varphi}{\partial z^3 \partial z_i \partial z_j} \right) - g^i_j \frac{\partial g_{ij}}{\partial z} = \frac{\partial \tilde{F}}{\partial z}. \]

where

\[ \tilde{F} = F - X(\varphi) + \varphi. \]

And differentiating again by \( \partial_z \):

\[ \frac{\partial^2 \tilde{F}}{\partial z \partial z} = g^i_j \left( \frac{\partial^2 g_{ij}}{\partial z \partial z} + \frac{\partial^4 \varphi}{\partial z^4 \partial z_i \partial z_j} \right) + g^i_j g^k_l \frac{\partial g_{ik}}{\partial z} \frac{\partial g_{lj}}{\partial z} - g^i_j \frac{\partial^2 g_{ij}}{\partial z \partial z} \]

So in normal coordinates at a point \( p \) with respect to \( g \) we have after taking the trace (in \( k, l \)) with respect to \( g \):

\[ \Delta \tilde{F} = g^i_j g^{kl} \left( \frac{\partial^4 \varphi}{\partial z^4 \partial z_i \partial z_j} + \frac{\partial^2 g_{ij}}{\partial z \partial z} \right) - g^k_l g^{ij} \frac{\partial^2 g_{ij}}{\partial z \partial z} - g^k_l g^i_j g^{kl} \left( \frac{\partial^3 \varphi}{\partial z^3 \partial z_i \partial z_j} \right) \left( \frac{\partial^3 \varphi}{\partial z^3 \partial z_i \partial z_j} \right). \]

Using that

\[ \Delta' \Delta \varphi = g^i_j \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \]

we get

\[ \Delta \tilde{F} = \Delta' \Delta \varphi - g^k_l g^{ij} \frac{\partial^2 \varphi}{\partial z^2 \partial z_i \partial z_j} - g^k_l g^{ij} \frac{\partial^2 g_{ij}}{\partial z \partial z} - g^k_l g^{ij} \frac{\partial^2 g_{ij}}{\partial z \partial z} \]

Now

\[ \frac{\partial^2 g_{ij}}{\partial z^2 \partial z} = -R_{ijkl} \quad \text{and} \quad \frac{\partial^2 g^{ij}}{\partial z \partial z} = g^{ij} g^{mn} R_{mijkl}. \]

So

\[ \Delta \tilde{F} = \Delta' \Delta \varphi - g^k_l g_{ijkl} \frac{\partial^2 \varphi}{\partial z^2 \partial z_i} - g^k_l g^{ij} R_{ijkl} + g^k_l g^{ij} \frac{\partial^2 g_{ij}}{\partial z \partial z} - g^k_l g^{ij} g^{kl} \left( \frac{\partial^3 \varphi}{\partial z^3 \partial z_i \partial z_j} \right) \left( \frac{\partial^3 \varphi}{\partial z^3 \partial z_i \partial z_j} \right). \]

We can re-choose the coordinates such that both \( g_{ij} = \delta_{ij} \) and \( (g_\varphi)_{ij} = (1 + \varphi_{ij}) \delta_{ij} \) at the fixed point \( p \in M \). So the above equation simplifies to

\[ \Delta \tilde{F} = \Delta' \Delta \varphi - \frac{1}{1 + \varphi_{kk}} R_{ijkl} \varphi_{kl} = \frac{1}{1 + \varphi_{ii}} R_{ijkl} + R_{iikk} - \frac{1}{1 + \varphi_{ii}} \varphi_{ii} + \frac{\varphi_{ii}}{1 + \varphi_{kk}}. \]

Letting \( C = \text{inf}_{\varphi_{jk}} R_{iikk} \) we get

\[ R_{iikk} \left( -1 + \frac{1}{1 + \varphi_{ii}} + \frac{\varphi_{ii}}{1 + \varphi_{kk}} \right) = \frac{1}{2} R_{iikk} \left( \frac{(\varphi_{kk} - \varphi_{ii})^2}{(1 + \varphi_{ii})(1 + \varphi_{kk})} \right) \geq C \left( \frac{(1 + \varphi_{kk}) - (1 + \varphi_{ii})^2}{(1 + \varphi_{ii})(1 + \varphi_{kk})} \right) = C \left( \frac{1 + \varphi_{ii}}{1 + \varphi_{kk}} - 1 \right), \]
Hence
\[ \Delta'(\Delta \varphi) \geq \frac{1}{(1 + \varphi_{ii})} \varphi_{ijk} \varphi_{ijk} + \Delta \dot{F} + C \left( n + \Delta \varphi \right) \left( 1 \right) \frac{1}{1 + \varphi_{ii} - n^2}. \]

Now let \( u := e^{-\lambda \varphi}(n + \Delta \varphi) \) where \( \lambda \) is to be determined later. Then
\[
\Delta'u = e^{-\lambda \varphi} \Delta' \Delta \varphi + 2(\nabla' e^{-\lambda \varphi}, \nabla'(n + \Delta \varphi))_{g_{ij}} + \Delta' e^{-\lambda \varphi}(n + \Delta \varphi)
\]
\[
= e^{-\lambda \varphi} \Delta' \Delta \varphi - 2\lambda e^{-\lambda \varphi} (\nabla' \varphi, \nabla' \Delta \varphi)_{g_{ij}} - \lambda e^{-\lambda \varphi} \Delta^2 \varphi(n + \Delta \varphi) + \lambda^2 |\nabla' \varphi|^2_{g_{ij}} e^{-\lambda \varphi}(n + \Delta \varphi)
\]
\[
\geq e^{-\lambda \varphi} \Delta' \Delta \varphi - e^{-\lambda \varphi} g^i_{jij}(n + \Delta \varphi)^{-1} (\Delta \varphi)_{ij}(\Delta \varphi)_{ji} - \lambda e^{-\lambda \varphi} \Delta' \varphi(n + \Delta \varphi),
\]
using the Schwarz inequality. Finally (again using the Cauchy-Schwarz inequality)
\[
g^i_{jij}(n + \Delta \varphi)^{-1} (\Delta \varphi)_{ij}(\Delta \varphi)_{ji}
\]
\[
= (n + \Delta \varphi)^{-1} \frac{1}{1 + \varphi_{ii}} |\varphi_{kk}|^2
\]
\[
= (n + \Delta \varphi)^{-1} \frac{1}{1 + \varphi_{ii}} \sum_k |\varphi_{kk}|^2 (1 + \varphi_{kk})^{-1} (1 + \varphi_{ii})
\]
\[
\leq (n + \Delta \varphi)^{-1} \frac{1}{1 + \varphi_{ii}} \sum_k |\varphi_{kk}|^2 (1 + \varphi_{kk})^{-1} (1 + \varphi_{ii})
\]
\[
\leq \frac{1}{1 + \varphi_{ii}} \frac{1}{1 + \varphi_{kk}} \varphi_{ji} \varphi_{jk}.
\]

Hence
\[
\Delta'(e^{-\lambda \varphi}(n + \Delta \varphi)) \geq e^{-\lambda \varphi} \left( \Delta \dot{F} + C(n + \Delta \varphi) \right) \left( 1 \right) \frac{1}{1 + \varphi_{ii} - n^2} - \lambda e^{-\lambda \varphi} \Delta' \varphi(n + \Delta \varphi)
\]
\[
\geq e^{-\lambda \varphi} \left( \Delta \dot{F} + \Delta \varphi - \Delta X(\varphi) + C(n + \Delta \varphi) \right) \left( 1 \right) \frac{1}{1 + \varphi_{ii} - n^2} - \lambda e^{-\lambda \varphi} \Delta' \varphi(n + \Delta \varphi).
\]

Now
\[
\Delta X(\varphi) = \nabla_i X^k \varphi_{ik} + X^k \nabla_i \nabla_k \varphi
\]
\[
= \nabla_i X^k \varphi_{ik} + X^k \nabla_k \nabla_i \varphi
\]
\[
= \nabla_i X^k \varphi_{ik} + X(n + \Delta \varphi)
\]
\[
= \nabla_i X^k (\delta_{ik} + \varphi_{ik} - \delta_{ik}) + X(n + \Delta \varphi)
\]
\[
\leq C'(n + \Delta \varphi) - \nabla_i X^k \delta_{ik} + X(n + \Delta \varphi)
\]
where we use \( |\nabla X| \leq C' \) and
\[
e^{-\lambda \varphi} \Delta X(\varphi) \leq X(e^{-\lambda \varphi}(n + \Delta \varphi)) + C'(n + \Delta \varphi) e^{-\lambda \varphi} + C',
\]
where we use the uniform upper bounds for \( ||\varphi||_{C^0(M)} \) and \( ||X(\varphi)||_{C^0(M)} \).

It follows that
\[
\Delta'u \geq e^{-\lambda \varphi} \left( \Delta \dot{F} + C(n + \Delta \varphi) \right) \left( 1 \right) \frac{1}{1 + \varphi_{ii}} - C' - X(u) - C'u + u - \lambda \Delta' \varphi u.
\]
Setting \( \lambda := 1 - C \geq 0 \) (where we can assume without loss of generality that \( C \leq 1 \)) and using
\[
\Delta' \varphi = n - \sum_i 1 \frac{1}{1 + \varphi_{ii}}
\]
we get for some uniform constants $c_0$, $c_2$

$$\Delta' u \geq -c_0 - X(u) - c_2 u + e^{-\lambda \varphi} \sum_i \frac{1}{1 + \varphi_{ii}} (n + \Delta \varphi).$$

But

$$\sum_i \frac{1}{1 + \varphi_{ii}} \geq \left( \frac{\sum (1 + \varphi_{ii})}{\prod (1 + \varphi_{ii})} \right)^{\frac{1}{n-1}} = e^{-\frac{1}{n-1}(F - X(\varphi)) \frac{1}{n-1}},$$

so

$$e^{-\lambda \varphi} \sum_i \frac{1}{1 + \varphi_{ii}} (n + \Delta \varphi) \geq e^{-\frac{1}{n-1}(F - X(\varphi)) + \frac{\lambda}{n-1} u \frac{1}{n-1}}$$

which implies (using the estimates of $X(\varphi)$ and $\varphi$) for some uniform constants $c_1$ and $c_3$

$$\Delta' u \geq -c_1 - c_2 u - X(u) + c_3 u \frac{1}{n-1}.$$

So by the maximum principle it follows that $u$ and $|\Delta \varphi|$ is uniformly bounded.

\textbf{Corollary 5.4.17.} Let $\varphi \in \Phi^{\beta}$ be a solution of (5.5) then there exists a uniform constant $C_5 = C_5(g)$ such that $\|\partial \varphi\|_{C_5(M)} \leq C_5$.

\textbf{Proof.} Since $\omega_\varphi$ is positive we know that $\sqrt{-\implies} \Delta \varphi > -\omega$. And so

$$\|\omega + \sqrt{-\implies} \Delta \varphi\|^2 \leq (n + \Delta \varphi)^2 \leq C_4^2.$$  

\hfill \Box

5.4.17. Third order estimates. We go on to prove the so-called $C^3$-estimates. Of course we do not need the full $C^3$-estimates of $\varphi$. What we need to start the Schauder machine is simply $C^{3,\alpha}$-estimates for $\partial \varphi$. However, even though Yau’s calculation (which we reproduce here) is quite long and tedious it is self-contained and uses, apart from the maximum principle, only algebraic identities. An alternative way to prove $C^{2,\alpha}$-estimates of $\varphi$ (avoiding any Schauder) directly from the $C_5$-estimates for $\partial \varphi$ (or $\Delta \varphi$) is to use results obtained after Yau’s proof of the Calabi conjecture by Evans, Krylov and Safonov among others. Namely the $C_5$-estimates for $\Delta \varphi$ and $\varphi$ give us (via Morrey) a uniform $C^{1,\alpha}$ estimate for $\varphi, \alpha \in (0,1)$. It follows from Krylov theory using the uniform ellipticity and concavity of the operator in question that there exists a uniform bound for the $C^{2,\alpha}$-norm of $\varphi$ where $\gamma \in (0,\alpha)$. More precisely, the following theorem is used:

\textbf{Theorem 5.4.18.} [CW98] Assume that $\Omega \subset \mathbb{R}^n$ has a smooth boundary, and the boundary is smooth. Assume $G(x,u,Du,D^2u)$ is smooth in all variables $(x,z,p,r) \in \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^+_1$ (where $S^+_1$ are the positive symmetric matrices), uniformly elliptic and concave (or convex) in $D^2u$, and assume that $\|u\|_{C^{1,\alpha}(\Omega)}$ is bounded for some $0 < \alpha \leq 1$. Then there are constants $0 < \beta < \alpha$ and $C$ so that for any $0 < \gamma < \beta$, and $\Omega' \subset \Omega$ we have for any solution $u$ of $G(x,u,Du,D^2u) = 0$

$$\|u\|_{C^{2,\gamma}(\Omega')} \leq C,$$

where $\beta$ depends on $\|D^2u\|_{C^{1,\alpha}(\Omega)}, \|u\|_{C^{1,\alpha}(\Omega)},$ and $\|G\|_{C^{1,\alpha}(\Omega)}$, the ellipticity $\Lambda/\lambda$ of $G$ and $C$ depends on $\text{dist}(\Omega',\partial\Omega)$.

\textbf{Remark 5.4.19.} By a bound $C$ for $\|G\|_{C^{1,\alpha}}$ we mean that $G = G(x,z,p,r)$ satisfies

1. $\sup_T (|\partial_x G| + |\partial_z G| + |\partial_p G|) < C(1 + |r|)$
2. $\sup_T (|\partial_x G|_{\alpha,z} + |\partial_z G|_{\alpha,p} + |\partial_p G|_{\alpha,z}) < C$
3. $\sup_T (|\partial_p G|_{\alpha,r} + |\partial_z G|_{\alpha,r} + |\partial_x G|_{\alpha,r}) < C$ and
4. $\sup_T (|\partial_p G|_{\alpha,p} + |\partial_p G|_{\alpha,z} + |\partial_p G|_{\alpha,z} + |\partial_z G|_{\alpha,z} + |\partial_p G|_{\alpha,x} + |\partial_x G|_{\alpha,x}) < C(1 + |r|)$,

where the notation $|\cdot|_{\alpha,y}$ just refers to the Hölder seminorm in the corresponding variable $y$. 

In our case

\[ G(x, u, Du, D^2u) = \log \left( \frac{\omega_n}{\omega_n^u} \right) - u + X(u) - F(x), \]

i.e.

\[ G(x, z, p, r) = \log \left( \frac{(\omega + \pi(r))^n}{\omega_n} \right) + \langle X, p \rangle - z - F(x), \]

where \( \pi \) maps \( r \) to \( \frac{1}{2}(r + Jr^I) \). Unfortunately we do not have uniform bound for \( \| \partial_p G \|_{C^{0,\alpha}(\Omega)} \). Namely we have

\[ \| \partial_p G \|_{C^{0,\alpha}(\Omega)} = \| X \|_{C^{0,\alpha}(\Omega)} \]

which is not bounded as \( \Omega \) moves towards infinity. This means that we either have to find a good decay estimate for \( X(\varphi) \) first or show that the condition on \( G \) can be relaxed. The second method would be to go through the details of Evans’ and Krylov’s proof which would probably take much longer than the \( C^3 \)-estimates of Yau which can be applied without any major changes.

**Lemma 5.4.20.** Let \( \varphi \in U^{1,\alpha} \) be a solution of (5.5). Then there exists a uniform constant \( C_6 \) such that

\[ \| \nabla \partial \bar{\partial} \varphi \|_{C^{0}(M)} \leq C_6. \]

**Proof.** The proof works just like in [Yau78]. We will omit some of the point-wise calculations since they are almost the same as in [Yau78]. Set

\[ S^2 := |\nabla \partial \bar{\partial} \varphi|_{g_\varphi}^2. \]

Then the following expression can be found in [Yau78]:

\[ \Delta' S^2 \cong \sum (1 + \varphi_{aa})^{-1}(1 + \varphi_{ii})^{-1}(1 + \varphi_{jj})^{-1}1 + \varphi_{kk}^{-1} \]

\[ \times \left\{ \left| \varphi_{ij} - \sum_p \varphi_{ipk}\varphi_{pja}(1 + \varphi_{pp})^{-1} \right|^2 \right. \]

\[ \left. + \left| \varphi_{ij} - \sum_p (\varphi_{pia}\varphi_{pjk} + \varphi_{pik}\varphi_{pja})(1 + \varphi_{pp})^{-1} \right|^2 \right\} \]

\[ + 2\Re \left( (1 + \varphi_{ii})^{-1}(1 + \varphi_{jj})^{-1}(1 + \varphi_{kk})^{-1} \varphi_{ij} \bar{F}_{ijk} \right) \]

\[ - 2(1 + \varphi_{ii})^{-1}(1 + \varphi_{jj})^{-1}(1 + \varphi_{kk})^{-1}(1 + \varphi_{rr})^{-1} \bar{F}_{ij} \varphi_{ijk} \varphi_{rjk} \]

\[ - (1 + \varphi_{ii})^{-1}(1 + \varphi_{jj})^{-1}(1 + \varphi_{kk})^{-1}(1 + \varphi_{rr})^{-1} \bar{F}_{ij} \varphi_{ijkl} \varphi_{lk} \]

where

\[ \bar{F} = F + \varphi - X(\varphi) \]

and \( A \cong B \) means that there exist constants \( B_1, B_2, \) and \( B_3 \) such that

\[ |A - B| \leq B_1S^2 + B_2S + B_3. \]
We have one additional term:

\[ X(S^2) = \nabla_X^2 \nabla \partial \bar{\partial} \varphi |_{\varphi}^2 \]

\[ = g_{\varphi} g_{\varphi} g_{\varphi} m_{\varphi} X^p \nabla_{\varphi}^l (\varphi_{ilm} \varphi_{jk}) \]

\[ = g_{\varphi} g_{\varphi} g_{\varphi} m_{\varphi} X^p (\varphi_{ilm} \varphi_{jk} + \varphi_{ilm} \varphi_{jk}) - (\Gamma^q)_{ilk} \varphi_{qlm} \varphi_{jk} - (\Gamma^q)_{ilk} \varphi_{qlm} \varphi_{jk} - (\Gamma^q)_{ilk} \varphi_{qlm} \varphi_{jk} \]

\[ = g_{\varphi} g_{\varphi} g_{\varphi} m_{\varphi} X^p (\varphi_{ilm} \varphi_{jk} + \varphi_{ilm} \varphi_{jk} - (\Gamma^q)_{ilk} \varphi_{qlm} \varphi_{jk} - (\Gamma^q)_{ilk} \varphi_{qlm} \varphi_{jk} - (\Gamma^q)_{ilk} \varphi_{qlm} \varphi_{jk} \]

\[ = (1 + \varphi_{jk})^{-1} (1 + \varphi_{kk})^{-1} \]

\[ \times (1 + \varphi_{mm})^{-1} (1 + \varphi_{qq})^{-1} \]

\[ \times \left( (X(\varphi)_{ijkl} \varphi_{ikjm} + X(\varphi)_{ijkl} \varphi_{ikjm} + X(\varphi)_{ijkl} \varphi_{ikjm}) \right) \]

\[ = (1 + \varphi_{kk})^{-1} (1 + \varphi_{mm})^{-1} (1 + \varphi_{qq})^{-1} \]

\[ \times \left( (X(\varphi)_{ijkl} \varphi_{ikjm} + X(\varphi)_{ijkl} \varphi_{ikjm} + X(\varphi)_{ijkl} \varphi_{ikjm}) \right) \]

\[ \approx - (1 + \varphi_{kk})^{-1} (1 + \varphi_{mm})^{-1} (1 + \varphi_{qq})^{-1} \]

\[ \left( \tilde{F}_{ijkl} \varphi_{ikjm} + \tilde{F}_{ijkl} \varphi_{ikjm} + \tilde{F}_{ijkl} \varphi_{ikjm} \right) \]

So

\[ \Delta' S^2 + X(S^2) \equiv \sum (1 + \varphi_{aa})^{-1} (1 + \varphi_{ii})^{-1} (1 + \varphi_{jj})^{-1} (1 + \varphi_{kk})^{-1} \]

\[ \times \left\{ \varphi_{ijkl} - \sum_p \varphi_{ik} \varphi_{pjm} (1 + \varphi_{pp})^{-1} \right\} \]

\[ \left. \left. + \sum_p \varphi_{ijkl} - \sum_p (\varphi_{ijkl} \varphi_{pjk} + \varphi_{pkm} \varphi_{m}) (1 + \varphi_{pp})^{-1} \right\} \right. \]

On the other hand (by the previous estimates)

\[ \Delta' (\Delta \varphi) + X(\Delta \varphi) \geq (1 + \varphi_{kk})^{-1} (1 + \varphi_{ll})^{-1} | \varphi_{kl} |^2 - C \]

for some uniform constant \( C \). So we can find a uniform constant \( C' \) such that

\[ \Delta' (S^2 + C' \Delta \varphi) + X(S^2 + C' \Delta \varphi) \geq C'' S^2 - C''' \]

For some positive constants \( C'' > 0 \) and \( C''' > 0 \). By the maximum principle we obtain a uniform estimate for \( S \).

This finishes the proof of theorem \( \ref{thm:5.4.1} \). Indeed setting \( D_2 = C_2, D_3 = C_3, D_4 = C_5, \) and \( D_5 = C_6 \) we get

\[ \| \varphi \|_{C^0(M)} \leq D_2, \| X(\varphi) \|_{C^0(M)} \leq D_3, \| \partial \bar{\partial} \varphi \|_{C^0(M)} \leq D_4, \] and \( \| \nabla \partial \bar{\partial} \varphi \|_{C^0(M)} \leq D_5. \)

\[ \square \]

**Proof of theorem \( \ref{thm:5.4.2} \)** We now use theorem \( \ref{thm:5.4.1} \) to show \( \ref{thm:5.4.2} \). First we make the following simple observation:
Lemma 5.4.21. Let \( \varphi \in \mathcal{U}^{3,\alpha} \) be a solution of \((5.5)\). Then for any \( \alpha \in (0,1) \) there exists a uniform constant \( D_\varphi \) such that

\[
\|g^{-1}\varphi\|_{C^{0,\alpha}(M)} \leq D_\varphi.
\]

Proof. Let \( h_\varphi := g^{-1}\varphi \) then \( \|h_\varphi\|_{C^{0,\alpha}(M)} \leq D_\varphi \) by theorem 5.4.1. In particular the positive eigenvalues \( \lambda_i(h) \) of \( h_\varphi \) are uniformly bounded. Now since \( h_\varphi \) also solves

\[
det(h_\varphi) = \exp(\varphi - X(\varphi) + F)
\]

and \( \varphi \) and \( X(\varphi) \) are uniformly bounded it follows that there exists a uniform positive constant \( c > 0 \) such that \( \prod_{i=1}^n \lambda_i \geq c \). But this implies that each \( \lambda_i \) is uniformly bounded from below and hence \( \|h_\varphi^{-1}\|_{C^{0,\alpha}(M)} \leq C \) for some uniform constant. The estimates of \( \|h_\varphi^{-1}\|_{C^{0,\alpha}(M)} \) can be obtained similarly.

\[ \square \]

5.4.1.8. \( C^2_w,\alpha \)-estimates. We are now finally ready to prove that a solution \( \varphi \in \mathcal{U}^{3,\alpha} \) of \((5.5)\) is uniformly bounded in the \( C^2_w,\alpha \)-norm. The first step is to show that for any \( \mu \in (0,1) \) we can find a constant \( C_7 = C_7(g) \) such that \( \|X(\varphi)\exp(\mu\theta X)\|_{C^{0,\alpha}(M)} \leq C_7 \). The function \((X(\varphi) - \varphi)\) satisfies an elliptic equation and the maximum principle gives the desired estimate. In a second step we show that for any \( \mu \in (0,1) \) there exists a constant \( C_7' \) such that \( \|\varphi\exp(\mu\theta X)\|_{C^{0,\alpha}(M)} \leq C_7' \). This is done using the local Morrey-Schauder estimates \[D.2.1\] theorem 5.4.1 and the previously obtained estimates for \( X(\varphi) \). And again using Schauder estimates \[D.2.1\] we show that for any \( \mu \) we can find a constant \( C_7'' \) such that \( \|\varphi\exp(\mu\theta X)\|_{C^{0,\alpha}(M)} \leq C_7'' \). The last step is to apply the global Schauder estimates in appendix A to the linear elliptic partial differential equation (the linear part of the Taylor expansion of the nonlinear Monge Ampere operator) whose right hand (the quadratic remainder term of the expansion) decays sufficiently fast. This gives the uniform \( C^2_w,\alpha \)-estimates for \( \varphi \).

We start with three lemmas that will help us to set up the equation that \( X(\varphi) \) satisfies and to get estimates for the coefficients in this equation.

Lemma 5.4.22. Let \( \varphi \) be a solution of \((5.5)\) then outside of a compact set it satisfies

\[
\Delta'(X(\varphi) - \varphi) + X(X(\varphi) - \varphi) = 0,
\]

where \( \Delta' = \Delta_{g_\varphi} \).

Proof. Note that outside of a compact set \( \omega = \sqrt{-1}\partial \bar{\partial} \theta_X \) and \( F = 0 \). So

\[
X(-X(\varphi) + \varphi) = \nabla \omega \log \left( \frac{\omega^g}{\omega^n} \right)
\]

\[
= g_{ij}^X \nabla_i \nabla_j \varphi
\]

\[
= g_{ij}^X \nabla_i \nabla_j \varphi
\]

\[
= \Delta' X(\varphi) - g_{ij}^X (g^{kl} \nabla_i \nabla_j \theta_X) \nabla_k \nabla_l \varphi
\]

\[
= \Delta' X(\varphi) - g_{ij}^X \left( g^{kl} \frac{\partial^2 \varphi}{\partial z_i \partial z_l} \right)
\]

\[
= \Delta' X(\varphi) - g_{ij}^X \left( \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \right)
\]

\[
= \Delta' X(\varphi) - g_{ij}^X \left( \frac{\partial^2 \varphi}{\partial z_i \partial z_j} \right)
\]

\[
= \Delta' X(\varphi) - \Delta' \varphi.
\]

where we used the fact that \( \nabla_j X = 0 \) and \( \nabla_{X(\varphi)} = \nabla_{R(X)} \varphi \). It follows that

\[
\Delta'(X(\varphi) - \varphi) = X(\varphi - X(\varphi))
\]

\[ \square \]
Lemma 5.4.23. If $\theta_X$ is a potential for $X$ with respect to $g$ then $\theta_X + X(\varphi)$ is a potential for $X$ with respect to $g_{\varphi}$.

Proof. Note first that
\[ i_X \omega = \sqrt{-1} g_{ij} d\xi^i = \sqrt{-1} \partial \theta_X. \]
Hence
\[ i_X \omega_{\varphi} = i_X \omega + \sqrt{-1} i_X \partial \varphi = \sqrt{-1} \partial \theta_X + \sqrt{-1} \partial_i \partial \varphi = \sqrt{-1} \partial (\theta_X + X(\varphi)) \]
in other words
\[ \text{grad}_{g_{\varphi}}(1,0)(\theta_X + X(\varphi)) = \text{grad}_{g}(1,0)\theta_X = X \]
\[ \blacksquare \]

Lemma 5.4.24. Let $\varphi$ be a solution of \([5.5]\). Then $\Delta'(\theta_X + X(\varphi))$ grows at most linearly.

Proof. By the estimates for $\partial \partial \varphi$ and hence $g_{\varphi}^{-1}$ we know that $\Delta' \theta_X$ is uniformly bounded. As for the remaining term
\[ \Delta' X(\varphi) \leq |g_{\varphi}^{-1}|_g |\nabla X|_g |\nabla \nabla \varphi|_g + |g_{\varphi}^{-1}| |\nabla \nabla \varphi|^2 |X|_g, \]
which proves the lemma. \( \blacksquare \)

Lemma 5.4.25. Let $\varphi$ be a solution of \([5.5]\). Then for any $\mu \in (0,1)$ there exists a uniform constant $C_7(\mu, g)$ such that $|X(\varphi)| \leq C_7 \exp(-\mu \theta_X)$.

Proof. Let $\mu \in (0,1)$ and $\psi = (X(\varphi) - \varphi)e^{\mu(\theta_X + X(\varphi))}$. Then outside of a compact set:
\[ \Delta' \psi - (2\mu - 1)X(\psi) - \left( \mu \Delta'(\theta_X + X(\varphi)) + (\mu - \mu^2)|X|_{g_{\varphi}}^2 \right) \psi = 0. \]
Indeed using lemma 5.4.1.8
\[ \Delta' \psi = e^{\mu(\theta_X + X(\varphi))} \Delta(X(\varphi) - \varphi) + 2\mu X(X(\varphi) - \varphi)e^{\mu(\theta_X + X(\varphi))} + (\mu \Delta'(\theta_X + X(\varphi)) + \mu^2 |X|_{g_{\varphi}}^2) \psi \]
\[ = (-1 + 2\mu) X(\varphi) - \varphi)e^{\mu(\theta_X + X(\varphi))} + (\mu \Delta'(\theta_X + X(\varphi)) + \mu^2 |X|_{g_{\varphi}}^2) \psi \]
\[ = (-1 + 2\mu) X(\psi) - \mu (2\mu - 1) |X|_{g_{\varphi}}^2 \psi + (\mu \Delta'(\theta_X + X(\varphi)) + \mu^2 |X|_{g_{\varphi}}^2) \psi \]
\[ = (-1 + 2\mu) X(\psi) + (\mu \Delta'(\theta_X + X(\varphi)) + (\mu - \mu^2)|X|_{g_{\varphi}}^2) \psi. \]
Now, since $\mu < 1$ we have $\mu - \mu^2 > 0$ and by the previous estimates the term $(\mu - \mu^2)|X|_{g_{\varphi}}^2$ dominates $\Delta'(\theta_X + X(\varphi))$ and so there is a compact set $K_\mu$ which is uniformly bounded such that on $M \setminus K_\mu$:
\[ \mu \Delta'(\theta_X + X(\varphi)) + (\mu - \mu^2)|X|_{g_{\varphi}}^2 \geq 1. \]
It follows from the maximum/minimum principle that
\[ \sup_M |\psi| \leq \sup_{K_\mu} |\psi|, \]
and using that $\varphi$ is exponentially decaying and the fact that $|X(\varphi)|$ is uniformly bounded we can find a constant $C_7(\mu, g)$ such that
\[ |X(\varphi)| \leq C_7 e^{-\mu \theta_X}. \]
\[ \blacksquare \]

Throughout the rest of this section we will make use of the following lemma:

Lemma 5.4.26. Suppose for some $r > 0$ and $x \in M$ and $\mu \in (0,1)$ there exists a constant $C = C(\mu)$ such that $\|u\|_{C^{1,\alpha}(B_r(x))} \leq C \sup_{B_{2r}(x)} \exp(-\mu \theta_X)$ then for any $\mu \in (0,1)$ there exists a uniform constant $C = C(r, \mu, g)$ such that $\|\exp(\mu \theta_X)u\|_{C^{1,\alpha}(M)} \leq C$. 

PROOF. Let \( \mu \in (0,1) \). By assumption we can pick \( \mu' \in (\mu,1) \) and a constant \( C = C(\mu') \) such that \( \|u\|_{C^{0,\alpha}(B_r(x))} \leq C \sup_{B_{2r}(x)} \exp(-\mu' \theta_X) \). But then it follows that \( \sum_j |\nabla^j u(x) + [\nabla^k u]_{\alpha}(x)| \leq C \exp(-\mu' \inf_{B_{2r}(x)} \theta_X) \). Since for any \( \inf_{B_{2r}(x)} \theta_X \geq \theta_X - r \sup_{B_{r}(x)} |X| \) that
\[
\sum_j |\nabla^j u(x) + [\nabla^k u]_{\alpha}(x)| \leq C \exp(-\mu' \theta_X + \mu' r \sup_{B_{2r}(x)} |X|)
\]
And since \( |\nabla^k \theta_X| \) is uniformly bounded
\[
\sum_j |\nabla^j \exp(\mu \theta_X) u(x) + [\nabla^k \exp(\mu \theta_X) u]_{\alpha}(x) | \leq C \left( \sup_{\partial F_{r}(x)} |\nabla^2 \theta_X| \right) \exp((\mu - \mu') \theta_X + \mu' r \sup_{B_{2r}(x)} |X|),
\]
where \( p(|X|) \leq C(1 + |X|)^{k+\alpha} \) with a uniform constant \( C = C(k,g) \). The right hand side in the equation above can be bounded by a uniform constant depending on \( k \) and \( \mu \) and so the lemma follows. \( \Box \)

**Lemma 5.4.27.** Let \( \varphi \) be a solution of \( [5.3] \). Then for every \( \mu \in (0,1) \) there exists a uniform constant \( C_8 = C_8(\mu, g) \) such that \( |\partial \varphi| \leq C_8 \exp(-\mu \theta_X) \).

**Proof.** Let \( K \) be a compact set such that \( F = 0 \) on \( M \setminus K \) and let \( x \in M \setminus K \) such that \( B(x,2r) \subset M \setminus K \). Choose holomorphic coordinates on \( B(x,2r) \) then
\[
0 = \log \left( \frac{\omega^n}{\omega^n} \right) + X(\varphi) - \varphi
\]
\[
= \left( \int_0^1 g^i_j \phi \, d\bar{e} \right) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + X(\varphi) - \varphi
\]
\[
= a^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + X(\varphi) - \varphi.
\]
In particular for every \( \mu \in (0,1) \) there exists a uniform constant \( D(\mu, r, g, \theta) \) such that
\[
|X(\varphi) - \varphi| + |X(\varphi) - \varphi|_{\alpha} \leq D e^{-\mu \theta_X}.
\]
We conclude that for any \( \mu \in (0,1) \) we can find a uniform constant \( C_9(\mu, g) \) such that
\[
|\partial \partial \varphi| + |\partial \partial \varphi|_{\alpha} \leq C_9 e^{-\mu \theta_X}.
\]
\( \Box \)

**Lemma 5.4.28.** Let \( \varphi \in U^{3,\alpha} \) be a solution of \( [5.3] \). Then there exists a uniform constant \( C_9 = C_9(g) \) such that
\[
\|\varphi\|_{C^{0,\alpha}(M)} \leq C_9.
\]

**Proof.** For \( u \in K^{2,\alpha} \) let
\[
M(u) := \log \left( \frac{\omega + \sqrt{-1} \partial \bar{\partial} u}{\omega^n} \right) + X(u) - u.
\]
Then
\[
M(0) = 0
\]
\[
M'(0)v = \Delta v + X(v) - v
\]
\[
M''(u)(v, v) = g^{ij}_u v_i g^{kl}_u v_k \geq |\partial \partial v|_{\alpha}^2
\]
where $g_u$ is the Kähler metric associated to $\omega + \sqrt{-1}\partial\bar{\partial}u$. So
\[
MA(u, F) := M(u) - F = \log \left( \frac{\omega + \sqrt{-1}\partial\bar{\partial}u^n}{\omega^n} \right) + X(u) - u - F
\]
\[
= MA(0, F) + M'(0)u + \int_0^1 \int_0^\sigma M''(\tau u)(u, u) d\tau d\sigma
\]
\[
= -F + \Delta u + X(u) - u + \int_0^\sigma |\partial\bar{\partial}u|_{g_\theta}^2 d\tau d\sigma.
\]
And if $\varphi$ is a solution of (5.5) then $MA(\varphi, F) = 0$, and so
\[
L(\varphi) = F - \int_0^1 \int_0^\sigma |\partial\bar{\partial}\varphi|_{g_\theta}^2 d\tau d\sigma
\]
Note that by lemma [5.4,2] above $g_{r,\varphi}$ is uniformly equivalent to $g$ for all $\tau \in [0, 1]$. And since for any $\mu \in (0, 1)$
\[
\exp(\mu \theta_X)\partial\bar{\partial}\varphi \in C^{1,\alpha}(M) \subset C^{0,\alpha}(M)
\]
and for any $\mu \in (0, 1) \exp(\mu \theta_X)\partial\bar{\partial}\varphi$ is uniformly bounded in $C^{0,\alpha}(M)$ by a constant depending only on $\mu$ and $g$ it follows that $|\partial\bar{\partial}\varphi|^2 \exp(2\mu \theta_X) \in C^{0,\alpha}(M)$ is uniformly bounded in $C^{0,\alpha}(M)$. Hence (choosing $\mu > 1/2$)
\[
\int_0^1 \int_0^\sigma |\partial\bar{\partial}\varphi|_{g_\theta}^2 d\tau d\sigma \in C^{0,\alpha}_w(M).
\]
is uniformly bounded in $C^{0,\alpha}_w(M)$. As seen above $\varphi$ solves
\[
L(\varphi) = F - \int_0^1 \int_0^\sigma |\partial\bar{\partial}\varphi|_{g_\theta}^2 d\tau d\sigma := G.
\]
By assumption $G \in C^{1,\alpha}_w(M)$ and by the observations above $G$ is uniformly bounded in $C^{0,\alpha}_w(M)$. Consider the conjugated operator
\[
\tilde{L} := \exp(\theta_X) \circ L \circ \exp(-\theta_X)
\]
acting on $C^{1,\alpha}(M)$. Then for $u \in C^{2,\alpha}(M)$
\[
\tilde{L}(u) = \Delta u - X(u) - (\Delta \theta_X + 1)u.
\]
Recall that $\psi := \exp(\theta_X)\varphi \in C^{2,\alpha}(M)$ (even though without uniform upper bound yet) and $\psi$ satisfies
\[
\tilde{L}(\psi) = \Delta \psi - X(\psi) - (\Delta \theta_X + 1)\psi
\]
\[
= \exp(\theta_X)L(\varphi)
\]
\[
= \exp(\theta_X)G.
\]
Let
\[
\tilde{\Delta}u := \Delta u - X(u) - (n + 1)u.
\]
Then
\[
\tilde{L}u - \tilde{\Delta}u = (n - \Delta \theta_X)u,
\]
where $(n - \Delta \theta_X)$ has compact support. And $\psi$ satisfies
\[
\tilde{\Delta}\psi = \exp(\theta_X)G + (n - \Delta \theta_X)\psi := H
\]
By the global Schauder estimates in the appendix, since $\psi \in \mathcal{A}^{3,\alpha} \subset \mathcal{A}^{3,\alpha},$
\[
\|\psi\|_{C^{2,\alpha}(M)} \leq C\|\tilde{\Delta}\psi\|_{C^{0,\alpha}(M)}
\]
for some uniform $C = C(g)$. And so
\[
\|\psi\|_{C^{2,\alpha}(M)} \leq C\left(\|G\|_{C^{0,\alpha}(M)} + \|(n - \Delta \theta_X)\psi\|_{C^{0,\alpha}(M)}\right)
\]
\[
= C(\|G\|_{C^{0,\alpha}(M)} + \|(n - \Delta \theta_X)\psi\|_{C^{0,\alpha}(M)}).
\]
And so it follows that 
\[ \| \varphi \|_{C^{0,\alpha}(M)} \leq C \left( \| G \|_{C^{0,\alpha}(M)} + \|(n - \Delta \theta X)\|_{C^{0,\alpha}(M)} \| \varphi \|_{C^{0,\alpha}(M)} \right) \]
\[ \leq Cg, \]
for some uniform constant \( Cg = C_0(g) \).

5.4.1.9. \( C^0_w,\alpha \)-estimates. For the \( C^0_w,\alpha \)-estimates we proceed as follows. First from the local Morrey-Schauder estimates [D.2.1] in chapter 2, the uniform \( C^0,\alpha \)-estimates for \( g_{\varphi}^{-1}g \) and equation (5.7) we get uniform estimates on \( \| \varphi \|_{C^{3,\alpha}(M)} \). The previously established \( C^0_w,\alpha \)-estimates imply that for any \( \mu \in (0, 1) \) we can find a bound \( C(\mu, g) \) for \( \| \exp(\mu \theta X)(X(\varphi) - \varphi) \|_{C^{1,\alpha}(M)} \). And so the local Schauder estimates [D.2.1] tell us that \( \| \exp(\mu \theta X)\varphi \|_{C^{3,\alpha}(M)} \) has an upper bound depending only on \( \mu \) and \( g \). Hence \( H \) from the proof of the previous theorem is uniformly bounded in \( C^1_w,\alpha \). And finally by the global Schauder estimates \( \varphi \) is uniformly bounded in \( C^3_w,\alpha \).

**Lemma 5.4.29.** Let \( \varphi \in \mathcal{U}^{3,\alpha} \) be a solution of (5.5). Then there exists a uniform constant \( C = C(g) \) such that 
\[ \| \varphi \|_{C^{3,\alpha}(M)} \leq C. \]

**Proof.** By the second order estimates (equation (5.7)) we know that \( \varphi \) satisfies an equation of the form 
\[ \Delta'(\Delta \varphi + X(\varphi) - \varphi) = \Delta F + Rm \ast \nabla \nabla \varphi \ast g_{\varphi}^{-1} + g_{\varphi}^{-1} \ast \nabla \nabla \nabla \varphi \ast \nabla \nabla \varphi. \]
We already have a uniform \( C^0 \)-estimate for the right hand side and \( \| g_{\varphi}^{-1} \|_{C^{0,\alpha}(M)} \) is also uniformly bounded. So by the Morrey-Schauder estimates [D.2.1] there exists a uniform constant \( C = C(g) \) such that 
\[ \| \Delta \varphi + X(\varphi) - \varphi \|_{C^{1,\alpha}(M)} \leq C. \]
Moreover by the weighted estimates above we already know that \( X(\varphi) - \varphi \) are uniformly bounded in \( C^{1,\alpha}(M) \). Hence by the local Schauder estimates [D.2.1] 
\[ \| \varphi \|_{C^{3,\alpha}(M)} \leq C. \]

The previous lemma implies

**Corollary 5.4.30.** There exists a uniform constant \( C \) such that \( \| g_{\varphi}^{-1}g \|_{C^{1,\alpha}(M)} \leq C. \)

**Lemma 5.4.31.** Let \( \varphi \in \mathcal{U}^{3,\alpha} \) be a solution of (5.5). Then for any \( \mu \in (0, 1) \) there exists a uniform constant \( C = C(g, \mu) \) such that 
\[ \| \exp(\mu \theta X)\varphi \|_{C^{3,\alpha}(M)} \leq C \]

**Proof.** By the previous corollary we know that \( \| g_{\varphi}^{-1}g \|_{C^{1,\alpha}(M)} \) is uniformly bounded. And so arguing as in the lemma 5.4.27 (using the local Schauder estimates [D.2.1]) we can conclude that \( \| \exp(\mu \theta X)\varphi \|_{C^{3,\alpha}(M)} \leq C \) for some constant depending on \( g \) and \( \mu \). \( \square \)

**Corollary 5.4.32.** Let \( G \) be as above. Then there exists a uniform constant \( C = C(g) \) such that 
\[ \| G \|_{C^{1,\alpha}(M)} \leq C. \]

**Lemma 5.4.33.** Let \( \varphi \in \mathcal{U}^{3,\alpha} \) be a solution of (5.5). Then there exists a uniform constant \( C_{10} \) such that 
\[ \| \varphi \|_{C^{3,\alpha}(M)} \leq C_{10}. \]

**Proof.** From the proof of the previous lemma we can conclude that \( \psi := \exp(\theta X)\varphi \in \mathcal{X}^{3,\alpha} \) satisfies 
\[ \tilde{L}\psi = \Delta \psi - X(\psi) - (n + 1)\psi = H \]
for \( H \in C^{1,\alpha}(M) \) as before which is uniformly bounded in \( C^{1,\alpha}(M) \). It follows that \( |\nabla \psi|^2 \) satisfies 
\[ \Delta |\nabla \psi|^2 - X(|\nabla \psi|^2) - 2(n+1)|\nabla \psi|^2 = 2\Re \left( (\nabla H, \nabla \psi) \right) + |\nabla \nabla \psi|^2 + |\nabla \psi|^2 + \Re(\nabla \psi, \nabla \psi) - \langle \nabla \nabla \psi, X, \nabla \psi \rangle =: K. \]
And since the right hand side is in $C^{0,\alpha}(M)$ it follows by the global Schauder estimates that $|\nabla \psi|^2 \in C^{2,\alpha}(M)$ and hence $\psi \in C^{4,\alpha}(M)$. Moreover there exists a uniform constant $C(g)$ such that
\[
\|\psi\|_{C^{4,\alpha}(M)} \leq C(\|K\|_{C^{0,\alpha}(M)} + \|\psi\|_{C^{1,\alpha}(M)}).
\]

And by the previous estimates we have $\phi \in C^{3,\alpha}_w(M)$ and a uniform constant $C_{10} = C_{10}(g)$ such that $\|\phi\|_{C^{3,\alpha}_w(M)} \leq C_{10}$.

**5.4.35.** $X^{3,\alpha}$-estimates. It remains to show that $X(\exp(\theta_X)\phi)$ is uniformly bounded in $C^{1,\alpha}(M)$.

**Lemma 5.4.34.** Let $\phi \in U^{3,\alpha}$ be a solution of (5.5). Then there exists a uniform constant $C_{11}$ such that $\|X(\exp(\theta_X)\phi)\|_{C^{1,\alpha}(M)} \leq C_{11}$.

**Proof.** Since $\phi$ satisfies equation (5.7) and $\phi$ is uniformly bounded in $C^{1,\alpha}(M)$ as well the uniform estimate for $X(\phi)$ follows immediately. □

Combining lemmas 5.4.28, 5.4.33, 5.4.34 completes the proof of theorem 5.4.2.

**Proof of theorem 5.4.3.** Now we can obtain the higher order estimates with a bootstrapping argument. Even though this is standard we will include a prove because it involves two steps unlike in the compact case where it can be done directly.

**Lemma 5.4.35.** Let $\phi \in U^{3,\alpha}$ be a solution of (5.5) and $k \in \mathbb{N}$. Suppose that there exists a uniform constant $C_k = C_k(g)$ such that $\|\phi\|_{C^{k+2,\alpha}_w(M)} \leq C_k$. Then there exists a uniform constant $C = C(k, g)$ such that $\|\phi\|_{C^{k+3,\alpha}_w(M)} \leq C$.

**Proof.** As before $\phi$ satisfies equation (5.7). Since $g^{\alpha}_{-1}$ is bounded in $C^{k,\alpha}(M)$ and uniformly equivalent to $g$ by assumption and the right hand side of equation (5.7) is uniformly bounded in $C^{k-1,\alpha}(M)$ it follows by the local Schauder estimates D.2.1 that there exists a uniform constant $C = C(k, g)$ such that $\|\Delta \phi + X(\phi) - \phi\|_{C^{k+1,\alpha}(M)} \leq C$. By the assumption that $\|\phi\|_{C^{k+2,\alpha}_w(M)}$ is uniformly bounded it follows $X(\phi) - \phi$ is uniformly bounded in $C^{k+1,\alpha}(M)$. And so again by local Schauder estimates D.2.1 in Chapter 2 there exists a uniform constant $C = C(k, g)$ such that $\|\phi\|_{C^{k+3,\alpha}_w(M)} \leq C$. □

**Corollary 5.4.36.** Let $\phi$ be as above. Then there exists a uniform constant $C = C(k, g)$ such that $\|g^{\alpha}_{-1}\|_{C^{k+1,\alpha}(M)} \leq C$.

**Lemma 5.4.37.** Let $\phi \in U^{3,\alpha}$ be a solution of (5.5) then for any $\mu$ there exists a constant $C = C(k, g, \mu)$ such that $\|\exp(\mu \theta_X)\phi\|_{C^{k+3,\alpha}_w(M)} \leq C$.

**Proof.** By the corollary above $g^{\alpha}_{-1}$ is uniformly bounded in $C^{k+1,\alpha}$ and so we can argue as in the proof of lemma 5.4.27 that for any $\mu \in (0, 1)$ there exists a uniform constant $C = C(k, g, \mu)$ such that $\|\exp(\mu \theta_X)\phi\|_{C^{k+3,\alpha}_w(M)} \leq C$. □

**Lemma 5.4.38.** Let $\phi \in U^{3,\alpha}$ be a solution of (5.5) and $\psi := \exp(\theta_X)\phi$. Let $H := \tilde{A}\psi$ then $H \in C^{k+1,\alpha}(M)$. Moreover, assume that there exists a uniform constant $C' = C'(g)$ such that $\|\phi\|_{C^{k+2,\alpha}_w(M)} \leq C'(g)$ then there exists a constant $C_{k+1} = C_{k+1}(g)$ such that $\|H\|_{C^{k+1,\alpha}(M)} \leq C_{k+1}$.

**Proof.** Recall that $H = \exp(\theta_X)(G + (n - \Delta \theta_X)\phi) = G + F - \int |\tilde{\theta}\phi|^2$. By lemma 5.4.37 it follows that there exists a uniformly constant $C(k, g)$ such that $\|G\|_{C^{k+1,\alpha}_w(M)}$. And so for some uniform constant $C = C(k, g)$ we have $\|H\|_{C^{k+1,\alpha}} \leq C$ since $(n - \Delta \theta_X)$ has compact support. □

**Lemma 5.4.39.** Let $\phi \in U^{3,\alpha}$ be a solution of (5.5). Assume that there exists a uniform constant $C_k$ with $k \in \mathbb{N}_0$ such that $\|\phi\|_{C^{k+2,\alpha}_w(M)} \leq C_k$.

Then we can find a uniform constant $C_{k+1} = C_{k+1}(g)$ such that $\|\phi\|_{C^{k+3,\alpha}_w(M)} \leq C_{k+1}$. 

By the lemma above and the assumptions follows that there exists a constant $C$.

So consider Lemma and so we can find $C$ where that right hand side is uniformly bounded in $C$.

Lemma 5.4.41 above). The second term is in $C$.

Then by assumption the combination of the first and the last three terms is in $C^k,\alpha(M)$.

\[ |\nabla^{k+1} \psi|^2 \leq 2 \Re \left( (\nabla^{k+1} H, \nabla^{k+1} \psi) \right) + |\nabla^{k+1} \psi|^2 + |\nabla^{k+2} \psi|^2 + \Re \left( \sum_{j=0}^{k-3} \nabla^j Rm \ast \nabla^{k-1-j} \psi \ast \nabla^{k+1} \psi \right) + \Re \left( \sum_{j=0}^{k} \nabla^j X \ast \nabla^{k+2-j} \psi \ast \nabla^{k+1} \psi \right) \]

=: H_{k+1}.

By the lemma above and the assumptions $H_{k+1} \in C^{0,\alpha}(M)$. And by the global Schauder estimates it follows that there exists a constant $C = C(k, g)$ such that

\[ \|\nabla^{k+1} \psi\|_{C^{2,\alpha}(M)} \leq C \|H\|_{C^{0,\alpha}(M)} \]

and so we can find $C_{k+1}$ such that

\[ \|\phi\|_{C^{k+3,\alpha}(M)} \leq C_{k+1}. \]

\[ \Box \]

Lemma 5.4.40. Let $\phi$ be a solution in $C^{k+2,\alpha}_w(M)$ then $X(e^{\theta_X} \phi) \in C^{k,\alpha}(M)$.

Proof. Let $\phi := \exp(\theta_X) \phi$ then

\[ X(\phi) = \Delta \phi - \bar{L}(\phi) - (\Delta \theta_X + 1) \phi \]

where that right hand side is uniformly bounded in $C^{k,\alpha}(M)$.

\[ \Box \]

Proof of theorem 5.4.4. Our aim is to prove theorem 5.4.4 using the implicit function theorem. So consider

\[ MA(\phi, f) := (\phi, f) \mapsto \log \left( \frac{\omega^\alpha}{\omega^n} \right) + X(\phi) - \phi - f. \]

Lemma 5.4.41. The operator $MA : U^{3,\alpha} \times C^{1,\alpha}_w(M) \to C^{1,\alpha}_w(M)$ is well-defined.

Proof. We can write (see proof of lemma 5.4.28)

\[ MA(\phi, f) = \Delta \phi - \int_0^1 \int_0^\sigma |\partial^2 \phi|_{\omega_{\tau \phi}}^2 d\tau d\sigma + X(\phi) - \phi - f. \]

Then by assumption the combination of the first and the last three terms is in $C^{k,\alpha}_w(M)$ (see the remark above). The second term is in $C^{1,\alpha}_w(M)$ since $\omega_{\tau \phi}$ is equivalent to $\omega$. More precisely, since $\phi \in C^{k,\alpha}_w(M)$ the function $|\partial \phi| \in |X|^2 e^{-\theta_X} C^{k-2,\alpha}(M)$ and so

\[ |\partial \phi|^2 \in |X|^4 e^{-2\theta_X} C^{k-2,\alpha} \subset C^{k-2,\alpha}_w(M). \]

Note that if $\bar{\phi} \in U^{3,\alpha}$ is a solution of (5.4) then $MA(\bar{\phi}, \bar{F}) = 0$. To prove the theorem we have to show that for any $F \in C^\infty(M)$ sufficiently close to $\bar{F}$ in the $C^{1,\alpha}_w$-norm satisfying $J_\phi(F) = 0$ we can find a $\phi \in U^{3,\alpha}$ such that $MA(\phi, F) = 0$. It suffices to show that for a solution $\bar{\phi} \in U^{3,\alpha}$ of (5.6)

\[ L_{\bar{\phi}} : C^{1,\alpha}_w(M), \quad \psi \mapsto \frac{d}{dc} MA(\bar{\phi} + c\psi, \bar{F}) \big|_{c=0} = \Delta_{g,\phi} \psi + X(\psi) - \psi \]
is an isomorphism. For notational convenience we will write $L$ instead of $L_\varphi$. We define the operator $	ilde{L} = \tilde{L}_\varphi$ formally by

$$
L_\varphi \tilde{\psi} := e^{g_X + X(\varphi)}L(e^{-g_X - X(\varphi)} \tilde{\psi}) = \Delta_{g_\varphi} \psi - X(\varphi) + (\text{div}_{g_\varphi} X + 1) \psi,
$$

where we use that $\theta_X + X(\varphi)$ is a potential for $X$ with respect to $g_\varphi$.

**Lemma 5.4.42.** \(\tilde{L}_\varphi : \tilde{X}^{3,\alpha} \to C^1_{\alpha}(M)\) is well-defined.

**Proof.** First note that as $\tilde{\psi} \in C^{3,\alpha}(M)$ and $\varphi \in U^{l,\alpha}$ clearly $\Delta_{g_\varphi} \psi \in C^{1,\alpha}$. Secondly $X(\varphi) \in C^{1,\alpha}(M)$ since $\psi \in \tilde{X}^{3,\alpha}$. And $\text{div}_{g_\varphi} X = \Delta_{g_\varphi}(\theta_X + X(\varphi))$. As $\theta_X$ is smooth and all second or higher derivative of $\theta_X$ are uniformly bounded $\Delta_{g_\varphi} \theta_X \in C^{1,\alpha}(M)$. By the estimates in the proofs of theorems 5.4.1 and 5.4.2 $\varphi \in U^{k,\alpha}$ for all $k > 0$ and in particular $\Delta_{g_\varphi} X(\varphi) \in C^{1,\alpha}(M)$. It is finally easy to check $L_\varphi(\tilde{\psi}) = 0$ and so the lemma follows. \(\Box\)

**Lemma 5.4.43.** The following function spaces are equal and their norms are equivalent:

$$
\exp(-\theta_X - X(\varphi)) C^{k,\alpha}(M) = \exp(-X(\varphi)) C^{k,\alpha}_w(M) = C^{k,\alpha}_w(M)
$$

and

$$
\exp(-\theta_X - X(\varphi)) \tilde{X}^{k,\alpha} = \exp(-X(\varphi)) \tilde{X}^{k,\alpha} = \tilde{X}^{k,\alpha}
$$

**Proof.** By the previous estimates $X(\varphi) \in \exp(-\mu \theta_X) C^{l,\alpha}(M)$ for any $\mu < 1$, $l \in \mathbb{N}$, and $g$ is uniformly equivalent to $g_\varphi$. So for $u = \exp(-X(\varphi)) v$, $\tilde{\psi} \in \tilde{X}^{k,\alpha}$ we have

$$
\|u\|_{X^{k,\alpha}} = \|\exp(-X(\varphi)) v\|_{X^{k,\alpha}} \leq C \exp(-X(\varphi)) \|v\|_{X^{k,\alpha}} \leq C' \|v\|_{X^{k,\alpha}}
$$

for some uniform constant $C'$. This proves that $\exp(-\theta_X - X(\varphi)) \tilde{X}^{k,\alpha} \subset X^{k,\alpha}$. The opposite inclusion works similarly. \(\Box\)

**Corollary 5.4.44.** $L$ is an isomorphism if and only if $\tilde{L}$ is an isomorphism.

**Proposition 5.4.45.** Let $(M, J, g)$ and $\theta_X$ be as in theorem 5.4.4 and $\Omega_\alpha := \{x \in M : \theta_X > \kappa\}$. Suppose that $\tilde{g}$ is a Riemannian metric on $M$ with $\|\tilde{g} - g\|_{C^{2,\alpha}(\Omega_\alpha)} = \exp(-\mu \kappa)$ for some $\mu \in (0, 1)$. Then the operator

$$
\tilde{A} : \tilde{X}^{3,\alpha} \to C^1_{\alpha}(M),
$$

$$
u \to \Delta_{\tilde{g}} \nu - X(\nu) - (n + 1) \nu
$$

is an isomorphism.

**Proof.** See appendix.

So let us show that $\tilde{L}$ is an isomorphism. Clearly $\tilde{L}$ is injective since $L$ is injective: Indeed, if $\tilde{L} : \tilde{\psi} = 0$ then $L(e^{-\theta_X - X(\varphi)} \tilde{\psi}) = 0$ and this implies that $\tilde{\psi} = 0$ by the maximum principle. In order to show that $\tilde{L}$ is an isomorphism we write

$$
\tilde{L} \tilde{\psi} = \Delta_{\tilde{g}} \tilde{\psi} - X(\tilde{\psi}) - (n + 1) \tilde{\psi} + (n - \Delta_{\tilde{g}}(\theta_X + X(\varphi))) \tilde{\psi}.
$$

Now $\tilde{A} : \tilde{X} \to \tilde{Z}$ is an isomorphism (and in particular Fredholm) by proposition 5.4.45 and so if we can show that $K : \tilde{X}^{3,\alpha} \to \tilde{Z}^{1,\alpha}$ is compact then $\tilde{L}$ is also an isomorphism (an injective Fredholm operator with index 0). This follows from the fact that the index of an operator does not change under compact deformations.

**Lemma 5.4.46.** The operator $K : \tilde{X}^{3,\alpha} \to C^2_{(\alpha)}(M) \to C^{1,\alpha}(M)$ is compact.

**Proof.** Let $u_n$ be a sequence in $\tilde{X}^{3,\alpha}$ with $\|u_n\|_{\tilde{X}^{3,\alpha}} = 1$. Let $K(u_n) = \gamma u_n$ where $\|\gamma\|_{C^{2,\alpha}(\theta_X > \kappa)} \leq C \exp(-\theta_X/2)$. We can find a subsequence $u_{n_j} \in \tilde{X}^{3,\alpha}$ which converges in $C^{1,\alpha'}(M)$ ($\alpha' < \alpha$) to $u \in C^{1,\alpha}(M)$ and $\|u_{n_j}\|_{C^1} \leq 1$. We can choose an exhaustion $\Omega_j$ of $M$ and without loss of generality we can assume that

$$
\|u_{n_j} - u\|_{C^{1,\alpha'}(\Omega_j)} \leq \frac{1}{j}.
$$
Moreover we may also assume that $\|\gamma\|_{C^{1,\alpha}(M\setminus\Omega_j)} < 1/j$ and so
\[
\|\gamma(u_{n_j} - u)\|_{C^{1,\alpha}(M)} \leq \|\gamma\|_{C^{1,\alpha}(M\setminus\Omega_j)} (\|u_{n_j}\|_{C^{1,\alpha}(M)} + \|u\|_{C^{1,\alpha}(M)})
\]
\[
+ \|u_{n_j} - u\|_{C^{1,\alpha}(\Omega_j)} \|\gamma\|_{C^{1,\alpha}(\Omega_j)}
\]
\[
< \frac{C}{j}
\]
which proves the lemma.

Now we can apply the implicit function theorem to finish the proof of theorem 5.4.4.

**Proof of theorem 5.4.5.** Assume that there exist two solutions $\varphi_1, \varphi_2 \in U$. Set $\psi := \varphi_2 - \varphi_1$, then $\psi$ satisfies
\[
\log \left( \frac{(\omega_{\varphi_1} + \sqrt{-1} \partial\bar{\partial} \psi)^n}{\omega_{\varphi_2}{n}} \right) + X(\psi) - \psi = 0,
\]
and so by the maximum principle it follows that $\psi = 0$, i.e. $\varphi_2 = \varphi_1$.

### 5.5. Old Examples of Expanding Kähler Ricci Solitons

Examples of Kähler Ricci solitons were constructed in [Cao85], [FIK03], and [DW11]. All of the examples reduce the soliton equation to an (or a system of) ODE. Cao constructed expanding Kähler Ricci solitons on $\mathbb{C}^n$ with positive bisectional curvature. Theses generalizations also include examples of nontrivial asymptotically flat expanding Ricci solitons. The examples in the first two references were then further generalized in [DW11] to include examples on complex line and vector bundles over products of arbitrary Kähler Einstein manifolds. Here we only consider asymptotically Ricci flat expanders. So in the following we will try to identify those.

**Example 5.5.1.** On $\mathbb{C}^n$ there exists a ‘trivial’ expander: namely the standard euclidean metric along with the potential $|x|^2/2$. Clearly this expander falls into our class of asymptotically Ricci flat expanders. This is the only expanding Ricci soliton metric (up to isometries) on $\mathbb{C}^n$ which is asymptotic to the euclidean metric.

The next examples are due to [FIK03]. It also follows from the main theorem in the previous chapter:

**Example 5.5.2.** Consider the cyclic group $\Gamma \subset U(n)$ generated by the action on $\mathbb{C}^n$ which sends $z_i$ to $\exp(2\pi i/k)z_i$, $i = 1 \ldots n$, where $k \in \mathbb{N}$. Clearly this group acts freely on $\mathbb{C}^n \setminus \{0\}$ and so the quotient $\mathbb{C}^n/\Gamma$ along with the euclidean metric is a smooth cone with the origin as its vertex. This singularity can be resolved by one blow up and one obtains smooth manifolds, the total spaces of line bundles $L^{-k} \rightarrow \mathbb{C}P^{n-1}$. It follows that $L^{-k}$ admits an expanding Ricci soliton metric if and only if $k > n$.

For each such $k$ there exists a rotationally symmetric expanding Kähler Ricci soliton metric $g$ such that $(M, g)$ is asymptotic to $(\mathbb{C}^n/\Gamma, \delta)$. Note that this example follows from the theorem above. One can generalize this to line bundles over manifolds over Kähler Einstein manifolds $Z$ with positive scalar curvature as done in [DW11]. However, here one can see that the assumption that the contraction of the zero section gives a Ricci flat cone is quite restrictive. These examples are then of the form $M = K_Z^{-1}/I(Z)$, where $I(Z)$ is the Fano index of $Z$ and $m > I(Z)$. Note that the asymptotic cones of these manifolds are not flat anymore. So they provide the first examples of Kähler Ricci solitons which are asymptotically Ricci flat but not ALE.

**Remark 5.5.3.** Recall that if $C(S)$ is a cone over a regular Sasaki manifold $S$ then $C(S)$ is automatically a line bundle $L$ over a compact Kähler manifold $Z$ with $c_1(L) < 0$ (As $-d\theta$ restricted to $Z$ is a curvature form of $L$). And it follows from the remarks in the beginning of this chapter that $L$ admits an expanding Kähler Ricci soliton metric if and only if $c_1(L) = \lambda c_1(K_Z)$, $\lambda > 0$ i.e. $Z$ is a Fano manifold. The first Chern class of $M$ restricted to $Z$, of the total space of $L$ is then $c_1(M)|_Z = -c_1(K_M)|_Z = -c_1(L) + c_1(K_Z) = (-\lambda + 1)c_1(K_Z)$. And as we have seen before it follows that $\lambda > 1$.

The next example will show that a further generalization for vector bundles fails if we require that the asymptotic cone is Ricci flat and the metric is constructed as in [DW11].
Example 5.5.4. This in fact a ‘non-example’: suppose $\Sigma$ is a $U(1)$-bundle over $Z := \mathbb{C}P^{n_1} \times N^{n_2}$, where $n_1 > 0$ and $N^{n_2}$ is a Fano manifold that admits a Kähler Einstein metric. Then we can find a regular Sasaki-Einstein metric on the total space $S$ of $\Sigma \to Z$ if $Z$ is an Einstein metric with Einstein constant $2(n_1 + n_2)$ and
\[
c_1(\Sigma) = q \frac{c_1(Z)}{I(Z)}, \quad q \in \mathbb{Z}_{<0},
\]
where $I(Z)$ is the Fano index of $Z$. Since $c_1(Z) = \pi_1^* c_1(\mathbb{C}P^{n_1}) + \pi_2^* c_1(N)$ we have $I(Z) = \gcd(n_1 + 1, I(N))$. Now suppose in addition that
\[
c_1(\Sigma) = \frac{-1}{n_1 + 1} \pi_1^* c_1(\mathbb{C}P^{n_1}) + \frac{p}{I(M)} \pi_2^* c_1(N)
\]
then
\[
q = \frac{I(N)}{n_1 + 1} = \frac{p I(Z)}{I(N)}, \quad \text{i.e. } p = -\frac{I(N)}{n_1 + 1}.
\]
We would need for $K_Z^{q/I(Z)}$ to admit an expanding Kähler Ricci soliton metric $q/I(Z) < -1$ i.e. $p/I(N) < -1$ and hence $-1/(n_1 + 1) < -1$ which is a contradiction.

The assumption on $c_1(\Sigma)$ is motivated as follows: note that the line bundle associated to $\Sigma$ is not a minimal resolution. In fact, a $\mathbb{C}P^{n_1}$ can be blown down and what remains is a rank $n_1 - 1$ vector bundle $N$. However if we allow $k > 1$
\[
c_1(\Sigma) = -\frac{k}{(n_1 + 1)} \pi_1^* c_1(\mathbb{C}P^{n_1}) + \frac{p}{I(N)} \pi_2^* c_1(Z)
\]
then
\[
q = \frac{-kI(Z)}{n_1 + 1} = \frac{p I(Z)}{I(N)}
\]
which implies
\[
p = \frac{-kI(N)}{n_1 + 1}.
\]
But then it follows from $q/I(Z) < -1$ that $k > I(Z)$. So the line bundle associated to $\Sigma$ is indeed a minimal resolution.

Here is an example that does not require that the zero section admits a Kähler Einstein metric:

Example 5.5.5. Consider $Z := \mathbb{C}P^{n_2} \times \mathbb{C}P^{n_1}$ and the total space $K_Z^p$, $p > 1$ ($p \in I(Z)^{-1}\mathbb{Z}_+$. Then the blow down of this space along the zero section is a complex cone $C$ that admits a Ricci flat cone metric (see [FOW09] Corollary 1.3). Note, however, that the Sasaki Einstein metric on $S$ cannot be of the form $\pi^* h + \eta \otimes \eta$ (i.e. regular) where $\eta$ is some $U(1)$-connection on the $U(1)$-bundle associated to $K_Z^p$. Nevertheless, $C \setminus \{0\}$ is biholomorphic to $K_Z^p \setminus \{\text{zero-section}\}$. In other words $K_Z^p$ is a resolution of $C$. It can be shown that the total space $K_Z$ admits a Ricci flat Kähler metric (since $Z$ is toric, see [Fut07]). The total space $K_Z^p$ on the other hand admits an expanding Kähler Ricci soliton metric for $p \in I(Z)^{-1}\mathbb{Z}_{>1}(Z)$. The cone associated to $M := K_Z^p$ is the cone over $S/\Gamma$ where $S$ is the total space a $U(1)$-bundle over $Z$ and $\Gamma \subset U(1)$ acts on $S$ fiber-wise.

5.6. New Examples of Expanding Kähler Ricci Solitons

5.6.1. Toric Kähler Cones. To construct examples which are not rotationally symmetric (or cohomogeneous one) we consider cones $C(S)$ which are Ricci flat, Kähler and toric. A special class of such cones are quotients of $\mathbb{C}^n$ by a finite abelian subgroup $\Gamma$ of $U(n)$ which acts freely on $\mathbb{C}^n \setminus \{0\}$ equipped with the euclidean metric. Here we will first study toric Kähler cones and their resolutions in general and then construct more concrete examples of expanders on toric resolutions of cones of type $\mathbb{C}^n/\Gamma$. For background on toric manifolds please refer to the appendix. Much of the following is inspired by the work of van Coevering [vC10, vC12].
The idea to construct these examples can be summarized as follows: We start out with a Ricci flat Kähler cone. This cone is, together with the vertex, a complex variety and we look at the subclass of toric varieties which are also toric. As such they possess a toric resolution which is a smooth (non-singular) toric variety \( M \). Then, finally, we construct a toric Kähler metric on \( M \) such that it satisfies the assumptions of the theorem.

**Definition 5.6.1.** A Sasaki manifold \((S, \Phi, \eta, \xi, g)\) is called toric if the associated Kähler cone \((C(S), J, \omega, g)\) is toric Kähler manifold, i.e. there exists an effective action of an \( n \)-dimensional torus \( T^n \) on \( C \) preserving the Sasaki structure such that the Reeb vector field \( \xi \) is part of the associated Lie algebra \( t \) of \( T^n \).

Given a toric Sasaki manifold we can associate a moment map \( \mu \) to the action of \( T^n \) as follows. Let \( \zeta \in t \), the Lie algebra of \( T^n \), and \( X_{\zeta} \) its associated infinitesimal action on \( C(S) \). Let

\[
\mu : C(S) \to t^*, \quad x \mapsto (\zeta \mapsto \langle \mu(x), \zeta \rangle)
\]

where

\[
\langle \mu, \zeta \rangle := -\frac{1}{2} \eta(X_{\zeta}).
\]

Then

\[
\tau_{X_{\zeta}} \omega = \frac{1}{2} \langle X_{\zeta}, d(r^2 \eta) \rangle = \frac{1}{2} L_{X_{\zeta}}(r^2 \eta) - \frac{1}{2} dt_{X_{\zeta}} r^2 \eta = \frac{1}{2} r^2 L_{X_{\zeta}} \eta + \frac{1}{2} d\langle \mu, \zeta \rangle = \frac{1}{2} d\langle \mu, \zeta \rangle.
\]

So the action of \( T^n \) on \( C(S) \) is indeed Hamiltonian and \( \mu \) is its associated moment map. The moment cone is defined as the set

\[
C(\mu) := \mu(C(S)) \cup \{0\} \subset t^* \equiv \mathbb{R}^n.
\]

It can be shown that \( C(\mu) \) is a strictly convex polyhedral cone, i.e. \( C(\mu) \) is of the form

\[
C(\mu) := \bigcap_{j=1}^{d} \{ y \in t^* : \langle u_j, y \rangle \geq 0 \}
\]

for some \( u_j \in \mathbb{Z}T^n := \text{Ker}(\exp(2\pi r) : t \to T^n) \), \( j = 1, \ldots, d \), such that the \( u_j \)'s are primitive, i.e. not integer multiples \((p \neq \pm 1)\) of integer vectors, and the set \( \{u_j\} \) is minimal in the sense that removing one of the \( u_j \)'s changes the set \( C(\mu) \). The cone is said to be non-singular if each sub-collection \( u_{j_1}, \ldots, u_{j_k} \) of \( \mathbb{Z}T \), i.e. the real subspace spanned by \( u_{j_1}, \ldots, u_{j_k} \) intersected with \( \mathbb{Z}T \) is just the \( \mathbb{Z} \)-lattice generated by \( u_{j_1}, \ldots, u_{j_k} \). Let \( \xi \) be the Reeb vector field of \( S \). It can be identified with an element of \( t \). Then \( \mu(S) = \{ y \in C(\mu) : y(\xi) = \frac{1}{2} \} \) and \( S \) is smooth if and only if \( C(\mu) \) is non-singular. Let \( \text{Int}(C(\mu)) \) be the open interior of \( C(\mu) \). Then \( \mu^{-1}(C(\mu)) \) is an open and dense subset of \( C(S) \) on which the torus act freely and biholomorphically. The dual cone is the given by

\[
C(\mu)^* := \{ x \in t : \langle x, y \rangle \forall y \in C(\mu) \}.
\]

The Kähler cone \( C(S) \cup \{0\} \) can also be described as a (singular) toric variety. Namely, \( u_1, \ldots, u_d \) generate a fan \( F \) whose highest dimensional cone is \( C(\mu)^* \). And as shown in the appendix this fan gives rise to a toric variety \( X_F \) which is isomorphic to \( C(S) \cup \{0\} \).

**5.6.2. Resolutions of toric Kähler Cones.** As a singular toric variety a toric Kähler cone admits a toric resolution. More precisely, the fan \( F \) associated with \( C(S) \) has a subdivision \( F' \) such that \( F' \) is non-singular (see appendix C for details). Then the toric variety \( X_{F'} \) associated \( F' \) is a resolution of \( X_F = C(S) \cup \{0\} \) and the morphism \( \pi : X_F \to X_F \) is constructed from the inclusion \( i : F \to F' \). The non-singular condition on \( F' \) implies that \( M := X_{F'} \) is non-singular i.e. a smooth manifold. Moreover,
the action of the algebraic torus \((\mathbb{C}^*)^n\) on \(C(S)\) lifts to \(M\), in particular there exists a holomorphic vector field \(X\) on \(M\) such that \(\pi_\ast X = (r\partial_r - \sqrt{-1}J\partial_\theta)/2\) on the smooth part of \(C(S)\).

**Example 5.6.2.** If \(n = 2\) any toric Ricci flat Kähler cone is of the form \(\mathbb{C}^2/\Gamma\) where \(\Gamma\) is some finite commutative subgroup of \(U(2)\) acting freely on \(\mathbb{C}^2 \setminus \{0\}\) and Kähler structure is given by the standard Kähler structure on \(\mathbb{C}^2\). Here we study the case when \(\Gamma\) is of the form

\[
\Gamma = \left\{ \begin{pmatrix} e^{\frac{2\pi i}{q}} & 0 \\
0 & e^{\frac{2\pi i}{p}} \end{pmatrix} \right\}_{j = 0, \ldots, q - 1}
\]

where \(g\mathbb{D}(p, q) = 1\). The associated moment map (with an appropriate choice of basis for \(t\)) is then

\[
\mu(z_1, z_2) = \left( \frac{p}{q} |z_2|^2 + \frac{1}{q} |z_1|^2, |z_2|^2 \right).
\]

And so

\[
C(\mu) = \{ \sigma_1(p\sigma_1 + q\sigma_2) + \sigma_2 \in \mathbb{R}_{\geq 0} \}
\]

and

\[
C(\mu)^\circ = \{ \sigma_1(q\sigma_1^* - p\sigma_2^*) + \sigma_2 \in \mathbb{R}_{\geq 0} \}
\]

Since \(g\mathbb{D}(p, q) = 1\) it follows that the fan is generated by the integer vectors \((q, -p)\) and \((0, 1)\). Clearly this fan is singular. Now there exists a general procedure to refine this fan to become non-singular. We will look at the case \(p = 1\). Then adding the edge generated by \((1, 0)\) gives a fan \(\mathcal{F}'\) which is indeed non-singular. The associated toric variety \(X_{\mathcal{F}'}\) is then a smooth toric resolution of \(\mathbb{C}^2/\Gamma\), indeed it is \(L^{-q} \rightarrow \mathbb{C}P^1\). If \(q > p > 1\) then the necessary refinement requires more steps. It is determined by the continued fraction expression \(p/q\). In particular if \(p = q - 1\) then the refinement of the fan gives rise to the resolutions in Kronheimer's construction of Ricci ALE Kähler metrics in [Kro89a].

We will try to determine whether the resolution of the toric Kähler cone satisfies the condition of the existence theorem for expanding Kähler Ricci solitons. As the toric action on the singular variety \(C(S)\) is lifted to the resolution and since the radial holomorphic vector field is contained in the complex torus which acts holomorphically on \(C(S)\) it clearly has an extension on any toric resolution \(M\). And any toric Kähler metric on \(M\) will make the lift a gradient vector field via its moment map. So it remains to find a condition under which the resolution admits a toric Kähler metric \(\omega\) such that \([\omega] \in H^{1,1}_c(M)\) and there exists a smooth real-valued function \(f\) on \(M\) such that

\[
\omega + \rho(\omega) = \sqrt{-1} \partial \bar{\partial} f.
\]

Recall that the fan gives back a complex toric variety. And any associated Delzant polyhedron (or its generalization for the non-compact case) gives a toric Kähler structure on that variety. This is described in more detail in the appendix. So suppose we have a refined fan \(\mathcal{F}'\). Then an associated Delzant polyhedron is of the form

\[
P := \cap_{r=1}^N \{ y \in t^* : l_r(y) \geq \lambda_r \},
\]

where \(l_j(y) = (u_j, y)\), \(u_j \in \mathbb{Z}^n\) and \(z_1, \ldots, u_N\) generate \(\mathcal{F}'\). And in order for the associated Kähler form \(\omega_P\) to satisfy (5.6.2) we need \(\lambda_k = 2\pi + \langle c, u_j \rangle\) for all \(j \in \{1, \ldots, N\}\) and some vector \(c\) (see appendix C section 3.1). Whether \(P\) is still a Delzant polytope remains to be checked. This can be done rather easily in case of example 5.6.2. As explained in the appendix the fan \(\mathcal{F}'\) of the Hirzebruch Jung resolution of \(\mathbb{C}^2/\Gamma\) is generated by \(e_1\) and \(p_1e_1 - q_1e_2\) where \(p_0 = p\), \(q_0 = q\). And \((p_j, q_j)\) are generated according to the recursion formula above (C.2.23). It is now a combinatorial problem (see appendix C) to prove that \(P\) is a Delzant polytope if and only if \(r_k > 2\) for all \(k = 1, \ldots, j\) where \(r_k\) are the integers showing up in the continued fraction expression

\[
\frac{p}{q} = r_1 - \frac{1}{r_2 - \frac{1}{r_3 - \cdots - \frac{1}{r_j}}}
\]

These numbers correspond to the diagonal elements of the intersection matrix. In summary we have
Theorem 5.6.3. Let $p > q$ such that $\gcd(p,q) = 1$ and $r_k > 2$ for all $k = 1, \ldots, j$. Then the minimal resolution of $\mathbb{C}^2 / \Gamma(p,q)$ admits an expanding Kähler Ricci soliton metric.
CHAPTER 6

Open Problems

In this chapter we would like to discuss some further directions for research, open questions and partial results that we have not presented so far.

(1) Given any Ricci flat Kähler cone. When does it have a resolution as described in the previous chapter? Which orbifold $\mathbb{C}^2/\Gamma$, $\Gamma \subset U(2)$ (or more generally $\mathbb{R}^4/\Gamma$, $\Gamma \subset SO(4)$) admits an expanding Kähler Ricci soliton coming out of it?

We have seen various examples of resolutions of Ricci flat Kähler cones with $K_M > 0$. We can also exclude several cases. In complex dimension 2 a minimal resolution is unique. Moreover, all Ricci flat cones are of the form $\mathbb{C}^2/\Gamma$, $\Gamma \subset U(2)$ and $|\Gamma| < \infty$. For $\Gamma \subset SU(2)$ the resolutions are well known. These resolutions admit Ricci flat ALE Kähler metric and in particular none of them (except $\mathbb{C}^2$ itself) admit an expanding Kähler Ricci soliton metric as $K_M$ is trivial.

When $\Gamma$ is cyclic we already have necessary and sufficient conditions for the existence of an expanding Kähler Ricci soliton metric. It remains to investigate the cases when $\Gamma$ is not cyclic and not a subgroup of $SU(2)$. Finally there might exist expanders on deformations of $\mathbb{C}^2/\Gamma$. These manifolds are not resolutions of of the cone and in particular not biholomorphic to the cone outside a compact set.

(2) Is a Kähler Ricci flow coming out of a given Kähler cone an expanding Kähler Ricci soliton? Are general Ricci flows coming out of (non-Ricci-flat) cones asymptotically self-similar?

(3) Define a relative expander entropy (see [FIN05]) for Ricci flows coming out of cones.

Recall that the expander Ricci functional $W_+$ constructed in [FIN05] is infinite on non-compact expanding Ricci solitons. We have shown, however, that a Ricci flow coming out of a Ricci flat cone approaches the original cone at infinity at exponential rate. So in principle it seems likely that one should be able to define a relative expander entropy $\nu_+$ that compares the evolving manifold with the original cone. This has been done for the mean curvature flow [Ilm]. In [Ilm95] the existence of an expander for non-minimizing cones is shown by constructing competitors which minimize the expander functional for mean curvature flow. We believe that a similar procedure should work for the Ricci flow as well. The stability of a cone should be defined in terms of the second variation of $\nu_+^{\text{rel}}$ or its maximality respectively. It should also help to show that each flow coming out of a cone converges to an expanding Ricci soliton.

(4) Prove the existence of expanding Ricci solitons coming out of non-Ricci-flat Kähler cones. In particular on cones over Sasaki Einstein manifolds.

The construction of expanding Ricci solitons in chapter 5 should be extended to the case when the original cone is not Ricci flat. In this case we do not expect that $[\omega] \in H^{1,1}_c(M)$. 

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Moreover, the expander metric $g$ approaches the cone metric $g_C$ only at quadratic rate. Examples of such expanding Kähler Ricci solitons for general Kähler can be found in [FIK03] and [DW11]. We plan to investigate the existence of non-singular expanding Kähler Ricci solitons for general Kähler cones in the future. Futaki and Wang constructed singular expanding Kähler Ricci solitons (or rather incomplete flows with boundary) coming out of cones over Sasaki manifolds which admit a transversal deformation to a Sasaki Einstein structure but are not Einstein themselves. This is also related to the program of Song and Tian: there, expanding Ricci solitons on small resolutions are needed to model the behavior of the Kähler Ricci flow after singularities which are not log-terminal. These cones are not Ricci flat and the first Chern class of the resolution is not an element of $H^{1,1}_c(M)$ in general.

(5) Are the expanding Kähler Ricci solitons constructed in the previous chapter unique?

We have shown that on a fixed resolution $M$ of a given Ricci flat Kähler cone and a fixed holomorphic vector field the expanding Kähler Ricci soliton metric is unique. We expect that any two expanders with different holomorphic vector fields on the same resolution are related by a biholomorphism. This is true for shrinking Kähler Ricci solitons on compact Fano manifolds [TZ00]. For a given cone it should also be possible that expanding Ricci solitons live on deformations and not a resolution.

(6) Formulation of an existence theory for (Kähler) Ricci flows on spaces with conical singularities.

There is a theory of Ricci flow on orbifolds as well as Ricci flows on certain singular algebraic varieties. Motivated by the program of Tian and Song it seems desirable to develop a general theory of Kähler Ricci flows that smooth out certain singularities while allowing others to persist.

(7) Do the expanding Ricci solitons constructed in the previous chapter violate the positive mass theorem? In other words do there exist asymptotically conical metric on these manifold with nonnegative scalar curvature and negative mass.

LeBrun [LeB88] showed that on $L^{-k} \to \mathbb{C}P^1$ there exist complete ALE metrics with $m(g) < 0$ and vanishing scalar curvature. In [CS04] and [Joy95] examples of scalar flat ALE metrics on resolutions of more general quotients of $\mathbb{C}^2$ are constructed. In both cases we show that there exist expanding Kähler Ricci soliton metrics on these manifolds. This should also be compared to Lagrangian mean curvature on manifolds with singularities which are modeled on special Lagrangian cones. Special Lagrangian cones are area minimizing and therefore stable. In other words, these singularities will not be smoothed out by mean curvature flow.

(8) Which Ricci flat (Kähler) cones are singularities of smooth (possibly non-Kähler) Ricci flows?

It is unlikely that a Ricci flat cone is a finite time limit of a smooth Ricci flow. Note that in contrast the heat equation admits solutions which become stationary in finite time.

(9) Which Ricci flat (Kähler) cones have a forward solution of the Ricci flow?

Not every unstable Ricci flat cone can have a smooth forward evolution. In [HHS11] it is
pointed out that a Riemannian cone over a compact four dimensional Einstein manifold which does not bound a 5-dimensional manifold cannot have a smooth forward evolution. As seen before the link of a Kähler cone always bounds a pseudoconvex domain in a smooth complex manifold. However if $\kappa_M \leq 0$ numerically we do not expect any Kähler Ricci flows coming out of the cone which are not the cone itself. However it might be possible that either the Ricci flow is not Kähler, at least not with the complex structure from the resolution.
APPENDIX A

Schauder Estimates for Ornstein Uhlenbeck Operators

In this appendix we derive the global Schauder estimates which are essential in chapter 5 from a similar result of Lunardi [Lun98] for Hölder functions on $\mathbb{R}^n$.

A.1. Global Schauder estimates on $\mathbb{R}^n$

In [Lun98] the following operator on $\mathbb{R}^n$ are considered:

$$Au(x) := \sum_{k,l} Q_{kl}^j(x) \frac{\partial^2 u}{\partial x_k \partial x_l}(x) + \sum_k P_k(x) \frac{\partial u}{\partial x_k}(x) + r(x)u(x)$$

such that there exists a uniform constant $D > 0$ with

- $D^{-1} \delta_{ij} \leq Q_{ij} \leq D \delta_{ij}$ as matrices, $\|Q\|_{C^{0,\alpha}} < D$
- $|P_k(x)| \leq D(1 + |x|)$, $\|\nabla P_k\|_{C^{2,\alpha}(\mathbb{R}^n)} < D$,
- $\sup_{x \in \mathbb{R}^n} r(x) = r_0 < \infty$, $\|r(x)\|_{C^{0,\alpha}} < D$.

Moreover, the following assumption is made:

For any $\lambda > r_0$ there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{x \to \infty} \varphi(x) = \infty$ and $\sup_{x \in \mathbb{R}^n} \mathcal{A}\varphi(x) - \lambda \varphi(x) < \infty$.

Under these assumptions the following theorem is proven

**Theorem A.1.1 (Lun98).** For any $\alpha \in (0,1)$, $\lambda > r_0$, and $f \in C^{0,\alpha}(\mathbb{R}^n)$ there exists a unique solution $u \in C^{2,\alpha}(\mathbb{R}^n)$ of $Au - \lambda u = F$ and a uniform constant $C = C(n,\alpha,\lambda, D)$ such that

$$\|u\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C \|F\|_{C^{0,\alpha}(\mathbb{R}^n)}.$$

A.2. Set up for $M$ with a conical end

Let $(M,g)$ be a smooth Riemannian manifold and $(C,g_C)$ a Riemannian cone over $(S,g_S)$ such that there exists a compact subset $C_1 \subset M$ and a diffeomorphism $\Psi : M \setminus C_1 \to S \times (1, \infty)$ such that $\Psi_* g = g_C$. Let $\{U_j\}$ be an open covering of $S$ and $\{U_j, \varphi_j : U_j \to V_j \subset S^{n-1}\}$ an atlas on $S$ such that $U_j \subset U_j$ and $(\varphi_j)_* g_S$ is uniformly equivalent to $g_{S^{n-1}}$. Let $\Omega_j := U_j \times (1, \infty)$ and $\Phi_j : \Omega_j \to V_j \times (1, \infty)$ be defined by $\Phi_j(\xi, r) := (\varphi_j(\xi), r)$.

**Lemma A.2.1.** There exists a function $\theta_j(r, \xi) = \rho(r)\chi_j(\xi) \in C^{2,\alpha}(\Omega_j) \cap C^\infty(\Omega_j)$ such that

1. $\chi_j \in C^\infty_c(U_j, [0,1])$, $\chi_j(\xi) = 1$ for all $\xi \in U_j'$,
2. $\rho \in C^\infty_c((1,\infty), [0,1])$ such that $\rho(r) = 0$ for $r \in (1,2]$ and $\rho(r) = 1$ for $r \in [10, \infty)$,
3. $\|\rho\|_{C^{0,\alpha}} \leq 2$,

for some uniform constant $C$.

**Proof.** The existence of $\rho$ is obvious and $\chi_j$, too, as $U_j'$ is relatively compact in $U_j$. 

□
A. SCHAUER ESTIMATES FOR ORNSTEIN UHLENBECK OPERATORS

A.3. Global Schauder estimates on $M$

Let $(M, g)$ and $\Phi_j$ be as above. Then $(\Phi_j)_* g$ is uniformly equivalent to $\delta$ on $V_j \times (1, \infty)$ and $\| (\Phi_j)_* g\|_{C^{4, \infty}} < C$. Let $\bar{g}$ be a continuation of $(\Phi_j)_* g$ to $\mathbb{R}^n$ such that $\bar{g}$ is uniformly equivalent to the euclidean metric and bounded in its derivatives. Consider the following operator on $M$:

$$Au := \Delta u - X(u),$$

where $X$ is a smooth vector field on $M$ such that $\Psi_* X = \frac{1}{2} r \partial_r$ on $M \setminus C_1$. We prove

**Theorem A.3.1.** Let $\alpha \in (0, 1)$, $\mu > 0$, and $F \in C^{0, \alpha}(M)$. Then there exists a unique solution $u \in C^{2, \alpha}(M)$ to $Au - \mu u = F$ and a uniform constant $C = C(\alpha, \mu, n, g)$ such that

$$\| u\|_{C^{2, \alpha}(M)} \leq C \| F\|_{C^{0, \alpha}(M)}.$$

**Proof.** We prove the estimate first. So assume that $u \in C^{2, \alpha}(M)$ is a solution. Let $\theta_j$ be as above and let $u_j := \theta_j u$. The $u_j$ satisfies

$$\Delta u_j - X(u_j) - \mu u_j = F \theta_j + A \theta_j + 2 \langle \nabla \theta_j, \nabla u \rangle =: F_j.$$

Now let $v_j$ be the continuation of $u_j \circ \Phi_j^{-1}$ by zero to $\mathbb{R}^n$. Then $v_j$ solves

$$\tilde{A} v_j - \mu v_j = \sum_{k, l} \tilde{g}^{kl} \left( \frac{\partial^2 v_j}{\partial x_k \partial x_l} - \hat{\Gamma}_l^m \frac{\partial v_j}{\partial x_m} \right) - \sum_k x_k \frac{\partial v_j}{\partial x_k} - \mu v_j = \tilde{F}_j.$$

It can be easily checked that $\tilde{A}$ satisfies the assumption of theorem A.1.1 (see lemma below) and so

$$\| v_j\|_{C^{2, \alpha}(\mathbb{R}^n)} \leq C \| \tilde{F}_j\|_{C^{0, \alpha}(\mathbb{R}^n)}.$$

So let us estimate $\| \tilde{F}_j\|_{C^{0, \alpha}(\mathbb{R}^n)}$. Clearly

$$\| \tilde{F}_j\|_{C^{0, \alpha}(\mathbb{R}^n)} \leq C \left( \| F\|_{C^{0, \alpha}(M)} + \| u\|_{C^{1, \alpha}(U_j \times (1, 10))} \right).$$

So it remains to show that $\| u\|_{C^{1, \alpha}(U_j \times (1, 10))}$ can be uniformly bounded by $\| F\|_{C^{0, \alpha}(M)}$. Indeed, it follows from the usual Schauder estimates, that

$$\| u\|_{C^{2, \alpha}(C_{1, 2, 20} U_j \times (1, 10))} \leq C \left( \| u\|_{C^{0, \alpha}(C_{1, 2, 20} U_j \times (1, 20))} + \| F\|_{C^{0, \alpha}(M)} \right).$$

And the estimate for $\| u\|_{C^0}$ follows from the maximum principle. Summing up over $j$ we get

$$\| u\|_{C^{2, \alpha}(M)} \leq C \| F\|_{C^{0, \alpha}(M)}.$$

Uniqueness follows directly from the estimate. So let us now show existence of a solution $u \in C^{2, \alpha}(M)$. To this end, let $\sigma_k(r) = 1$ for $r < k$, $\sigma_k(r) = (k + 1)/r$ for $r > (k + 1)$ such that $|\sigma_k' r^\lambda| \leq 2$ and let $X_k := \sigma_k X$. Then it can be shown by standard theory that there exists a solution $u_k \in C^{2, \alpha}(M)$ to the equation

$$\Delta u - X_k(u) - \mu u = F.$$

By the estimates above there exists a uniform (independent of $k$) bound for $\| u_k\|_{C^{2, \alpha}(M)}$ and so $u_k$ converges in $C^{0, \beta}_{loc}(M)$ to a function $u \in C^{2, \alpha}(M)$ which is a solution of $Au - \mu u = F$. \(\square\)

**Lemma A.3.2.** Let $\tilde{A}$ be as above. Then for any $\lambda > 0$ there exists a function $\varphi$ such that $\lim_{x \to \infty} \varphi(x) = \infty$ while $\sup \tilde{A} \varphi < \infty$.

**Proof.** Take for example $\varphi = |x|^2$. Then clearly $\lim_{x \to \infty} \varphi(x) = \infty$ and

$$\sup \tilde{A} \varphi = \sup -(1 + \lambda)|x|^2 + O(|x|) < \infty.$$

\(\square\)

**Corollary A.3.3.** Let $M$, $X$, and $A$ as above and $g$ asymptotically conical at exponential rate. Then for all $\alpha \in (0, 1)$, $\mu > 0$ and $f \in C^{0, \alpha}(M)$ there exists a unique function $u \in C^{0, \alpha}(M)$ such that $A - \mu u = f$ and a uniform constant $C = C(\alpha, \mu, n, g)$ such that

$$\| u\|_{C^{2, \alpha}(M)} \leq C \| F\|_{C^{0, \alpha}(M)}.$$
Corollary A.3.4. Let \((M, g), X,\) and \(A\) be as in the previous corollary. Then \(A : D(A) \to C^{1,\alpha}(M),\)
where
\[
D(A) = \{ u \in C^{3,\alpha}(M) : X(u) \in C^{1,\alpha}(M) \}
\]
is an isomorphism and there exists a uniform constant \(C\) such that
\[
\|u\|_{C^{3,\alpha}(M)} \leq C\|F\|_{C^{1,\alpha}(M)}.
\]
APPENDIX B

Kähler cones as complex varieties

B.1. General Kähler cones

We briefly describe some concepts from algebraic geometry that we need to construct smooth varieties asymptotic to cones. For more details we refer to [GH94] and [Dem12]. We will try to expand a little bit more on anything that cannot be found there. All of this is standard knowledge for anyone familiar with algebraic geometry. The application of these results to Kähler cones is due to van Coevering, most importantly, the proof that any Kähler cone can be understood as a singular complex variety can be found in [vC10]. As such any given Kähler cone can be desingularized (resolved) and the algebraic methods involved in this procedure helps to determine whether certain obstruction for the existence of a Kähler Ricci soliton metric on a resolution for a given Kähler cone vanish or not. There are different notions of complex varieties. Here we will use the following definitions:

Definition B.1.1. We define affine algebraic (analytic) varieties, abstract algebraic varieties, complex analytic sets, and complex analytic spaces as follows:

1. An algebraic set in $\mathbb{C}^N$ is the set of common zeroes of a finite number of algebraic functions, i.e. polynomials on $\mathbb{C}^N$.
2. An affine algebraic variety is an irreducible algebraic set, meaning that it cannot be written as the union of two Zariski-open proper subsets.
3. An algebraic variety is the set of common zeroes of a finite number of polynomials on $\mathbb{C}P^N$.
4. An analytic variety (or analytic set) is a subset of $\mathbb{C}^N$ which is locally the set of common zeroes of a finite number of holomorphic functions.
5. An abstract algebraic variety $V$ is a collection of affine algebraic varieties patched together via local isomorphisms.
6. A complex analytic space $V$ is a locally compact Hausdorff space together with a sheaf of continuous functions $\mathcal{O}_V$, such that there exists an open covering $\{U_i\}$ of $V$ and homeomorphisms $\phi_j : U_j \to V_j$ onto analytic sets $V_j \subset \mathbb{C}^N_j$ such that $\phi_j^* : \mathcal{O}_{V_j} \to \mathcal{O}_V|_{U_j}$ is an isomorphism ($\mathcal{O}_{U_j}$ being the sheaf of holomorphic functions on $V_j$).

Remark B.1.2. An algebraic variety as defined above is always analytic and projective. An abstract algebraic variety is not necessarily projective though certainly a complex analytic space. The famous theorem of Chow states that any projective analytic variety is in fact algebraic (and any quasi-projective analytic variety is a Zariski-open subset of an algebraic variety). And finally any complex manifold is an algebraic variety. There are also a number of famous examples of complex manifolds which are not algebraic. Namely, certain complex tori and the famous K3 surfaces. In case of an algebraic variety $V$ the structure sheaf of $V$ is the sheaf $\mathcal{O}_V$ of algebraic functions whereas on a complex analytic space $V$ the structure sheaf $\mathcal{O}_V$ is the sheaf of holomorphic functions. In either case $(V, \mathcal{O}_V)$ is a ringed space.

In general such varieties (complex analytic spaces) are not smooth. Indeed, all kinds of singularities can appear. However, there is an important condition that gives the variety some regularity.

Definition B.1.3 (normal variety). An algebraic variety (a complex analytic space) $V$ is called normal if for each $p \in V$ the ring $(\mathcal{O}_V)_p$ is normal (see [Dem12]).

As it turns out the class of normal varieties (complex analytic spaces) is flexible enough for most purposes. For example normality guarantees that the set of singular points $V_{\text{sing}}$ (which is analytic) has codimension at least 2 (see [Dem12]).
Definition B.1.4. A complex analytic space $V$ is called holomorphically convex if for every compact subset $K$ the holomorphic convex hull
\[ K := \{ z \in V : |f(z)| \leq \sup_{x \in K} |f(x)| \text{ for all } f \in \mathcal{O}_V(V) \}. \]
is compact.

Definition B.1.5. A complex analytic space $V$ is called 1-convex if it admits a continuous function which is strictly plurisubharmonic outside a compact set.

Theorem B.1.6 ([CM85]). A complex analytic space $V$ is holomorphically convex if it is 1-convex.

Definition B.1.7. A complex analytic space $V$ is called Stein if it is holomorphically convex and any compact analytic subset in $V$ is finite.

Theorem B.1.8 (See [DemT2]). A Stein space can be holomorphically embedded into $\mathbb{C}^N$ for $N$ large enough.

Finally we need the following Cartan-Remmert-reduction theorem:

Theorem B.1.9 ([Rem56], see also [Cam13]). Let $V$ be a holomorphically convex complex analytic space. Then there exists a Stein space $W$ and a proper surjective holomorphic map $f : V \to W$ such that

1. $f_*\mathcal{O}_V = \mathcal{O}_W$.
2. $f$ has connected fibres.
3. The pair $(f,W)$ is unique up to a biholomorphism.

And

Theorem B.1.10 (Riemann Extension Theorem). Let $V$ be a normal complex analytic space and $A$ an analytic subset such that $\dim(A) \leq \dim(X) - 2$. Then every holomorphic function on $V \setminus A$ has a meromorphic extension to $V$.

The following theorem appeared in [vC10].

Theorem B.1.11. Let $(C(S),I_C,\omega_C,g_C)$ be the Kähler cone over a Sasaki manifold $(S,\Phi,\eta,\xi,g)$. Then $C(S) \cup \{0\}$ is a normal complex analytic space.

Proof. As the details of this proof can be found in [vC10] we only give a sketch. Let $\mathcal{CR}(D,J)$ be the automorphism group of $S$, i.e. diffeomorphisms $\sigma : S \to S$ such that $\sigma_* D = D$ and $\sigma_* \Phi_D = J$ for the given CR structure $(D,J)$ on $S$. One can show that for a given Sasaki structure $(\Phi,\eta,\xi,g)$ compatible with $(D,J)$ the Reeb vector field $\xi$ generates a one parameter subgroup which is contained in a maximal torus $\mathbb{T}^k \subset \mathcal{CR}(D,J)$ and so the closure of this subgroup is compact in $\mathbb{T}^k$. If the given Sasaki structure is quasiregular then the one parameter subgroup itself is already compact in $\mathbb{T}^k$ and hence 1-dimensional. So if $\mathfrak{t}^\mathbb{C}$ is the Lie algebra of $\mathbb{T}^k$ and $\mathbb{Z}_l^\mathbb{C}$ the corresponding lattice then $\xi \in \mathbb{Z}_l^\mathbb{C} \otimes \mathbb{Q}$ if the Sasaki structure is quasiregular. If the Sasaki structure is not quasiregular then we can pick $\xi \in \mathbb{Z}_l^\mathbb{C} \otimes \mathbb{Q}$ such that $\eta(\xi) > 0$ and a compatible Sasaki structure $(\Phi',\eta',\zeta,g')$. Note that there is a biholomorphic $\psi$ such that on $(C,I_C)$ which maps one Sasaki structure to the other (in particular $\psi_*\xi = \zeta$). Moreover the new cone is a cone over $S$ with a quasiregular Sasaki structure. And so there exists a $\mathbb{C}^*$ action on $S$ which acts freely on $C(S)$ up to a finite number of points which have a finite stabilizer. Hence the quotient $C(S)/\mathbb{C}^*$ is a complex orbifold $Z$ and $C(S)$ is the total space $W$ of a complex line bundle $L$ over $Z$ minus the zero section. This line bundle is negative and we can find a hermitian metric $h$ on $L$ such that $r'^2 = h(x)|z|^2$ where $r'$ is the radial coordinate of the latter cone and $z$ is the fiber coordinate of the line bundle. One can see that $W$ is swept out by strictly pseudconvex domains $W_c := \{ x \in W : r'(x) < c \}$ hence $W$ is holomorphically convex and it follows that $Z \subset W_c$ for some sufficiently large $c$ and so there exists a Remmert reduction: a complex space $Y$, a discrete subset $D$ of $Y$, and a proper holomorphic map $\pi : W \to Y$ which maps $W \setminus Z$ biholomorphically onto $Y \setminus D$. Since $Z$ is connected it follows that $D$ is connected in other words just one point $\{ p \}$ and so we can give $C(S) \cup \{ 0 \}$ the complex structure of $Y$ which agrees with the original complex structure on $C(S)$. \[\square\]
We denote the set of Weil divisors in \( V \).

**Remark B.1.12.** With some more effort it can be shown that \( C(S) \cup \{ a \} \) is actually an affine algebraic variety in \( \mathbb{C}^N \) (see [vC11] and [CH12]).

On a smooth variety \( V \) there is a well defined notion of a canonical line bundle \( K_V \). To extend this to the singular case we need to distinguish two different types of divisors.

**Definition B.1.13.** Let \( V \) be a complex analytic space of dimension \( n \). A Weil divisor is a locally finite \( \mathbb{Z} \)-linear combination of \((n-1)\)-dimensional analytic irreducible set \( D_j \):

\[
D := \sum \lambda_j D_j, \quad \lambda_j \in \mathbb{Z}. \tag{B.1}
\]

We denote the set of Weil divisors in \( V \) by \( WDiv(V) \).

**Definition B.1.14.** A Weil divisor is called principal if it is the of the form \( D = \sum \text{ord}_{D_j}(f) D_j \) for some meromorphic function where the sum is over all irreducible analytic subsets of \( V \). A Weil divisor \( D \) is called Cartier if it is locally principal.

**Remark B.1.15.** Recall that \( \text{ord}_V(f) \) for \( f \) meromorphic and \( V \) a analytic subset in an open subset \( U \) of an complex analytic space is defined as follows: let \( p \in V \) and \( V = \{ x \in U : g(x) = 0 \} \). Then \( \text{ord}_V \) is defined to be the largest integer \( a \) such that in the local ring \( (\mathcal{O}_V)_p \) at \( p \) \( f = g^a \cdot h \) for some \( h \in (\mathcal{O}_V)_p \).

These two definitions have the following natural extension.

**Definition B.1.16.** A \( \mathbb{Q} \)-Weil divisor is a \( \mathbb{Q} \)-linear combination of \((n-1)\)-dimensional irreducible analytic subsets. And \( \mathbb{Q} \)-Weil divisor \( D \) is called \( \mathbb{Q} \)-Cartier if there exists an integer \( p \) such that \( pD \) is Cartier.

On a non-singular complex analytic space the two types of divisors are actually identical. Moreover, there is a one-to-one relation between holomorphic line bundles on \( V \) and Cartier divisors. In particular, on a non-singular variety \( V \) the canonical line bundle \( K_V \) has an associated Cartier divisor which we denote by \( K_V \) as well an associated invertible sheaf \( \mathcal{O}_V(K_V) \) (the sheaf of holomorphic sections of \( K_V \)).

**Definition B.1.17.** A sheaf \( \mathcal{F} \) of \( \mathcal{O}_V \) modules over a complex analytic space \( V \) is called invertible if it is locally isomorphic to the sheaf \( \mathcal{O}_V \).

The three concepts of Cartier divisors, line bundles, and invertible sheaves are more or less interchangeable. On normal (possibly singular) varieties the canonical divisor can be defined as follows. Let \( V_{\text{reg}} \) be the regular part \( V \). On this part \( K_{V_{\text{reg}}} \) can be defined as usual. Since the complement \( V_{\text{sing}} \) has codimension at least 2 the canonical divisor \( K_V \) can be defined to be the closure of \( K_{V_{\text{reg}}} \) on \( V \) and hence becomes a Weil divisor in \( V \) (note that any divisor is uniquely determined by its restriction to the complement of a codimension 2 analytic subset). The canonical divisor does not necessarily have to be Cartier. But it can be checked whether \( K_V \) is Cartier by investigating whether the sheaf \( \mathcal{O}_{V_{\text{reg}}}(K_{V_{\text{reg}}}) \) is invertible. The following example shows that not every Weil divisor is Cartier.

**Example B.1.18.** (see [Dem12]) Let \( V \) be the singular quadric \( \{ (x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 x_2 - x_3 x_4 = 0 \} \). Then the subset \( A := \{ (x_1, 0, x_3, 0) \} \subset V \) is clearly a Weil divisor. However it cannot be defined by holomorphic function near the origin. To see this let \( B := \{ (0, x_2, 0, x_4) \} \) and note that \( A \cap B = \{ 0 \} \). However if \( A \) was a level set \( f^{-1}(0) \) near the origin then \( \text{codim}_V(f^{-1}(0) \cap B) \leq 2 \) which is a contradiction. Here we used the fact that for any \( f \in \mathcal{O}_V(U) \) \( U \subset V \) open, \( f \neq 0 \) the set \( f^{-1}(0) \) is either empty or it has dimension \( \dim(X) - 1 \) (see [Dem12]).

Recall that a bimeromorphic morphism between two complex analytic spaces \( X \) and \( Y \) is a meromorphic morphism \( F : X \to Y \) which has a meromorphic inverse. A meromorphic morphism is an equivalence class of morphisms \( F : X \to Y \) defined on the complement of a nowhere dense analytic subset such that the closure of the graph of \( F \) is an analytic subset of \( X \times Y \). (Two morphisms are equivalent if they agree on the complement of a nowhere dense analytic subset).

**Definition B.1.19.** Let \( V \) be a complex analytic space with a non-empty singular set \( V_{\text{sing}} \). A resolution \( (W, \pi) \) of \( V \) is a non-singular complex analytic space \( W \) together with a bimeromorphic morphism \( \pi : W \to V \) such that \( \pi : W \setminus \pi^{-1}(V_{\text{sing}}) \to V \setminus V_{\text{sing}} := V_{\text{reg}} \) is an isomorphism (biholomorphism) and
generated by the fundamental classes of $E$. Suppose $\omega_1$ is the Poincare dual to $\sum a_i E_i$ for $a_i \in \mathbb{R}$. By assumption there exists a compactly supported closed (1,1)-form $\alpha$ such that $[\alpha] = [\omega_1]$. Let $\beta \in \mathcal{A}^1(M)$ be a smooth 1-form with $d\beta = \omega_1 - \alpha$. Then if $\beta = \beta^0 + \beta^{1,0}$ is the decomposition of $\beta$ then $d\bar{\partial}\beta^{0,1} = 0$. So there exists a smooth complex valued function $\gamma$ with $d\bar{\partial}\gamma = \beta^{0,1}$ since by the previous lemma

$$E := \pi^{-1}(V_{\text{sing}})$$

is a simple normal crossing Weil divisor. That is, $E = \sum_i E_i$ where each $E_i$ is smooth and $E_i \cap \ldots \cap E_i$ is a transverse intersection. $E$ is called the exceptional divisor.

**Definition B.1.20.** A singularity $x \in V$ of a complex variety $X$ is called rational if for any resolution $\pi : W \to V$ all the higher direct images $(R^i \pi_* \mathcal{O}_W)_x = 0$ for all $i > 0$.

For example it is shown in [KM98] that any quotient singularity is in fact rational. A proof for the following simple criterion for rationality of an isolated singularity can be found in [KM98].

**Theorem B.1.21.** Let $V$ be a complex analytic space and let $\Omega$ be a holomorphic n-form nowhere vanishing in a deleted neighborhood $U$ of an isolated singular point $x \in V$. Then $x$ is a rational singularity if and only if for a sufficiently small neighborhood $U'$ of $x$

$$\int_{U'} \Omega \wedge \bar{\Omega} < \infty.$$  \hfill (B.2)

**B.2. Ricci flat Kähler Cones**

**Proposition B.2.1.** [VC10] Suppose $C(S)$ is a Ricci flat Kähler Cone. Then $C(S) \cup \{o\}$ is $\mathbb{Q}$-Gorenstein and $o$ is a rational singularity.

A normal complex analytic space $V$ is Gorenstein if it is Cohen-Macaulay and the canonical Weil divisor $K_V$ is Cartier. It is called $\mathbb{Q}$-Gorenstein if $K_V$ is instead $\mathbb{Q}$-Cartier. In particular, a quotient $\mathbb{C}^n/\Gamma$ is Gorenstein if and only if $|\Gamma| < \infty$ and $\Gamma \subset SL(n, \mathbb{C})$. It is $\mathbb{Q}$-Gorenstein for any finite subgroup of $GL(n, \mathbb{C})$ acting freely on $\mathbb{C}^n \setminus \{0\}$. The property of being C-M is implied by the fact that the singularity is rational (by theorem 5.10 in [KM98]). Once this is shown it remains to prove that $K_V$ is a rational singularity and it follows (by the Riemann extension theorem) if we can show that $\partial_0 \mathcal{O}(pK_V)$ is invertible for some $p \in \mathbb{N}$ and since the the singularity $\{o\}$ has codimension at least 2 this follows (by the Riemann extension theorem) if we can show that $\partial_0 \mathcal{O}(pK_V)$ is invertible. By assumption there exists a non-vanishing section $\Omega_p$ of $K^p_{\text{Cart}}(S)$ or some $p \in \mathbb{N}$ and so $\partial_0 \mathcal{O}(pK_V)$ is indeed invertible (in fact trivial). The singularity is rational for the following reason. Since $S$ is Einstein with positive scalar curvature by Meyer's theorem it has a finite fundamental group $\pi_1(S)$. So its universal cover $\tilde{S}$ is a compact Sasaki Einstein manifold and $V := C(S) \cup \{o\}$ is a normal complex variety. The cone $C(S)$ admits a nowhere vanishing holomorphic n-form and it satisfies (B.2). So $o \in V$ is a rational singularity and it follows that $o \in V$ is a rational singularity as well as it is the image of a finite unramified morphism $g : \tilde{V} \to V$. The following propositions are due to van Coevering

**Proposition B.2.2.** Let $\pi : M \to C(S) \cup \{o\}$ be a resolution of a Ricci flat Kähler cone $C(S)$. Then $H^i(M, \mathcal{O}_M) = 0$ for all $k > 0$ and in particular $H^2(M) = 0$.

As $o \in V := C(S) \cup \{o\}$ is a rational singularity $(R^i \pi_* \mathcal{O}_M)_x$ for $j > 0$ and $V$ is a Stein space it follows that $H^i(V, R^j \pi_* \mathcal{O}_M) = 0$ for $i, j > 0$. The Leray spectral sequence implies that $H^i(M, \mathcal{O}_M) = 0$ for $i > 0$.

**Proposition B.2.3.** Suppose that $\pi : M \to C$ is a resolution of a Ricci flat Kähler cone $C$. Let $\Omega := \pi^{-1}(\omega)$ be the exceptional set and let $\omega_1$ be a complete Kähler metric on $M$ such that $\omega_1 = \beta + \sqrt{-1} \partial \bar{\partial} f \in \mathcal{A}^{1,1}(M, \mathbb{C})$ for some smooth function $f \in C^\infty(M, \mathbb{R})$ and $\beta \in \mathcal{A}^{1,1}(M, \mathbb{C})$ a closed (1,1)-form with compact support. Then there exists a complete Kähler metric $\omega_0$ on $M$ and compact subset $K_1 \subset K_2 \subset M$ such that $\omega_0 = \omega_1$ on $K_1$ and $\omega_0 = \pi^* \omega_C$ on $M \setminus K_2$.

This has already been proven in [VC10]. For convenience we include the proof here. Note that it suffice to assume that $[\omega_1] \in H^2(M, \mathbb{R})$.

**Proof.** Set $M := \{x \in M : r(\pi(x)) \leq 1\}$ and let $E_i$ be the prime divisors in $E$. $H^{2n-2}(M)$ is generated by the fundamental classes of $E_i$. Suppose $[\omega_1]$ is the Poincare dual to $\sum a_i E_i$ for $a_i \in \mathbb{R}$. By assumption there exists a compactly supported closed (1,1)-form $\alpha$ such that $[\alpha] = [\omega_1]$. Let $\beta \in \mathcal{A}^1(M)$ be a smooth 1-form with $d\beta = \omega_1 - \alpha$. Then if $\beta = \beta^0 + \beta^{1,0}$ is the decomposition of $\beta$ then $d\partial\beta^{0,1} = 0$. So there exists a smooth complex valued function $\gamma$ with $d\bar{\partial}\gamma = \beta^{0,1}$ since by the previous lemma
\[ H^1_0(M) = H^1(M, \mathcal{O}_M) = 0. \] And it follows that \( \sqrt{-\bar{\partial}\partial} \Re(\gamma) = \omega_1 - \alpha. \)

Let \( f = r^2 \) be the Kähler potential of the cone metric on \( C \). Choose \( 0 < a_1 < a_2 \). And let \( \nu : \mathbb{R}_+ \to \mathbb{R} \) be a smooth function with \( \nu(x) = x \) for \( x > a_2^2 \), \( \nu'(x), \nu''(x) \geq 0 \) for all \( x \) and \( \nu(x) = c \) for \( x < a_1^2 \).

Then \( \sqrt{-\bar{\partial}\partial} \nu \circ f \geq 0 \) extends to a form on \( M \). Finally choose \( b_1 \) and \( b_2 \) with \( a_2^2 < b_1 < b_2 \). Let \( \phi \) be a non-negative function of \( r \) with \( \phi(r) = 1 \) for \( r < b_1 \) and \( \phi(r) = 0 \) for \( r > b_2 \). Then define \( u := 2\phi \Re(\gamma) \) and so

\[ \omega_0 := \alpha + \sqrt{-\bar{\partial}\partial} u + \lambda \sqrt{-\bar{\partial}\partial}(\nu \circ f) > 0 \] (B.3)

satisfies the requirement of the lemma for \( \lambda \) sufficiently large. \( \square \)
APPENDIX C

Toric Manifolds and Toric resolutions

To construct examples of expanding Kähler Ricci solitons we constructed toric resolutions of a given cone such that they admit a background metric necessary to apply the theorem about the existence of expanding Kähler Ricci solitons. In this appendix we fill in the details from the literature. Most of it is quite well known. However, the generalization of the theory of symplectic toric manifolds to the non-compact case (which we will need here) seems to be more recent and requires a bit more care.

C.1. Symplectic Toric Manifolds

In this section we discuss the famous result of Delzant which shows that toric symplectic manifolds are characterized by its moment polytope. Here we only need one aspect of Delzant’s theorem: any polytope (polyhedral set) $P$ which satisfies certain conditions gives rise to a symplectic toric manifold $(M_P, \omega_P)$. Moreover, this manifold $M_P$ carries a “canonical” complex structure $J_P$ compatible with $\omega_P$.

C.1.1. Toric action and moment maps. Definition C.1.1. An action of a Lie group $G$ on a manifold $M$ is a group homomorphism

\[ \phi : G \rightarrow \text{Diff}(M) \]

\[ g \mapsto \psi_g, \]

where Diff$(M)$ is the diffeomorphism group of $M$. It is called a symplectic action on a symplectic manifold $(M, \omega)$ if

\[ \phi : G \rightarrow \text{Symp}(M, \omega) \]

\[ g \mapsto \psi_g, \]

where Symp$(M, \omega)$ is the group of symplectomorphisms, i.e. $\phi_g^* \omega = \omega$.

Now let $(M, \omega)$ be a symplectic manifold and $G$ a Lie group with an action $\psi : G \rightarrow \text{Diff}(M)$ and let $g$ its Lie algebra and $g^*$ the dual. Let $\exp : g \rightarrow G$, then for an element $\eta \in g$ we can associate a vector field $X_\eta$ on $M$ defined by

\[ X_\eta(x) := \frac{d}{dt}_{|t=0} \phi_{\exp(t\eta)}(x). \]

Definition C.1.2. The action $\psi$ is a Hamiltonian action if there exists a map

\[ \mu : M \rightarrow g^*, \]

such that

\[ d(\mu(x), \zeta) = i_{X_\zeta} \omega(x). \]

and

\[ \mu \circ \phi_g = \text{Ad}_g^* \circ \mu, \quad \text{for all } g \in G. \]

The map $\mu$ is called a moment map.

Note that it follows that $\psi_g$ is a symplectomorphism. A special case which we will consider here is the case when $G = \mathbb{T}^n = (S^1)^n$.

Definition C.1.3. A symplectic toric manifold is a (compact) connected manifold $M$ of even dimension $2n$ with a symplectic structure $\omega$ and an effective Hamiltonian actions of the torus $\mathbb{T}^n = (S^1)^n$. 
We will denote by $\mathbb{Z}_{T^n}$ the integral lattice $\text{Ker}(\exp : \mathfrak{g} \to T^n)$ and by $\mathbb{Z}_{\mathbb{R}^n}$ its dual which is called the weight lattice of $T^n$.

**Example C.1.4.** Here are some familiar examples of toric manifolds:

- $\mathbb{C}P^1$, $\omega_{CG}$ where $S^1$ acts on $\mathbb{C}P^1$ by sending $[z_0 : z_1]$ to $[z_0, \rho^{|z_1|}z_1]$.
- Note that the torus itself is not a toric manifold.
- $\mathbb{C}P^{n-1}$, $n \geq 2$ is toric and also $\mathbb{C}P^2 \times k\mathbb{C}P^2$ is toric for $k = 1, 2, 3$.

An important observation is the following.

**Theorem C.1.5 (Atiyah, Guillemin, Sternberg).** Let $(M, \mathbb{T}^n, \omega)$ be a compact connected symplectic toric manifold and $\mu$ an associated moment map. Then

1. the level sets of $\mu$ are connected
2. the image of $\mu$ is convex
3. the image of $\mu$ is the convex hull of the images of the fixed point in $M$ of the action.

The image of $\mu$ is called moment polytope.

**C.1.2. Delzant Polytopes and construction of $(M_p, \omega_P)$.**

**Definition C.1.6.** A Delzant (or unimodular) polytope $P \subset \mathbb{R}^n$ is a polytope (polyhedral set) which satisfies

- there are $n$ edges meeting at each vertex $p \in P$ (simplicity)
- the edges at each vertex $p \in P$ are of the form $p + tv_i$, $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, n$ (rationality)
- for each vertex $p \in P$ the vectors $v_1, \ldots, v_n$ from a $\mathbb{Z}$-basis of $T^n$ (smoothness).

It is a famous theorem of Delzant that not only any moment polytope is a Delzant polytope but also that any Delzant polytope is realized by a unique (up to symplectomorphisms) symplectic toric manifold.

**Theorem C.1.7 (Delzant).** There exists a bijective correspondence between the set of compact toric symplectic manifolds $\{(M, \mathbb{T}^n, \omega)\}$ and the set of Delzant polytopes.

Here we need a version that is suited for the non-compact case. As mentioned in [KL09] in the non-compact case the image of the moment map $\mu$ does not need to be convex and the map $\mu : M/\mathbb{T}^n \to \mathfrak{t}^*$ does not need to be an embedding. In [KL09] the authors show that the right analogue of the unimodular polytopes described above are manifolds $W$ with corners along with a locally unimodular embedding $\psi : W \to \mathfrak{t}^*$. They go on to show that given such a locally unimodular embedding $\psi : W \to \mathfrak{t}^*$ there exists a bijection between the set of isomorphism classes of symplectic toric manifolds $(M, \mathbb{T}^n, \omega)$ which satisfy $\mu = \psi \circ \pi$, $(\pi : M \to M/\mathbb{T}^n = W)$ and $H^2(W, \mathbb{Z}_T \times \mathbb{R})$. Here we will need a somewhat simpler version of this very general theorem. Namely we want to show that for convex unimodular polyhedral subsets $W$ of $\mathfrak{t}$ (i.e. $\psi$ is just the inclusion and hence a global embedding) then the set of isomorphism classes of toric symplectic manifolds associated $W$ is non-empty and at least one of those can be described as a symplectic quotient of $\mathbb{C}^N$, i.e. it also carries a “canonical” complex structure.

**Proposition C.1.8.** Let $P \subset \mathfrak{t}^*$ be a unimodular polytope of the form

$$P := \bigcap_{i=1}^N \{ u \in \mathfrak{t}^* : \langle u, v_i \rangle \geq \lambda_i, v_i \in \mathbb{Z}^n \}$$

then there exists a toric symplectic manifold $(M_P, \mathbb{T}^n, \omega_P)$ with a moment map $\mu_P$ such that $\mu_P(M) = P$, $\mu_P : M/\mathbb{T}^n \to \mathfrak{t}^*$ is an embedding, and $\mu : M \to \mathfrak{t}^*$ is proper. Moreover, $(M_P, \omega_P)$ is a symplectic reduction of $\mathbb{C}^N$ with the standard symplectic structure.

**Proof.** The are numerous detailed descriptions of the proof in the literature. This part of the proof of Delzant’s theorem works in the non-compact case without any major modifications. We will sketch it here. Let $v_1, \ldots, v_N \in \mathbb{Z}^n$ be the integral vectors described above. By assumption $v_1, \ldots, v_N$ span $\mathbb{Z}^n$ and they are primitive. Now define the linear map $\pi$ by

$$\pi : \mathbb{R}^N \to \mathbb{R}^d \cong \mathbb{Z}^n \otimes \mathbb{R}, \quad e_i \mapsto v_i,$$
where \( e_i, i = 1, \ldots, N \) are standard basis vectors of \( \mathbb{R}^N \). Then \( \pi \) is surjective. And we denote its kernel by \( \mathfrak{k} \). So we have an exact sequence

\[
0 \rightarrow \mathfrak{k} \rightarrow \mathbb{R}^N \rightarrow \mathbb{R}^n \rightarrow 0.
\]

Now \( \mathbb{R}^N \) is the Lie algebra of \( T^N \) so we get an induced map \( \pi : T^N \rightarrow T^n \). Its Kernel \( K \) is a \((N - n)\)-dimensional subtorus of \( T^N \) with Lie algebra \( \mathfrak{k} \). In other words

\[
1 \rightarrow K \rightarrow T^N \rightarrow T^n \rightarrow 1
\]

is an exact sequence. Finally the associated sequence

\[
0 \rightarrow \mathfrak{t}^* \rightarrow (\mathbb{R}^N)^* \rightarrow \mathfrak{t}^* \rightarrow 0
\]

is also exact and we denote the map \((\mathbb{R}^N)^* \rightarrow \mathfrak{t}^*\) by \( \iota^* \). Now consider the standard action of \( T^N \) on \( \mathbb{C}^N \).

This is a Hamiltonian action and its moment map \( \mu \in \mathbb{C}^N \) is given by

\[
\mu_{\mathbb{C}^N}(z) = -\frac{1}{2} \left( |z_1|^2, \ldots, |z_N|^2 \right) + \text{const},
\]

where const is some constant vector in \( \mathbb{R}^N \cong \mathbb{Z}^{n} \oplus \mathbb{R} \). Choosing

\[
\text{const} := (\lambda_1, \ldots, \lambda_N)
\]

we claim that

1. \( Z := \iota^* \circ \mu_{\mathbb{C}^N}^{-1}(0) \) is a smooth submanifold of \( \mathbb{C}^N \) and
2. \( K \) acts freely on \( Z \).

Once these two claims are established the manifold \( M_P := Z/K \) is smooth and the Marsden Weinstein theorem (see [MS98]) guarantees the existence of a \( T^n \) equivariant symplectic structure \( \omega_P \) on \( M_P \) which pulls back to restriction of standard symplectic structure of \( \mathbb{C}^N \) to \( Z \).

C.1.3. The canonical complex structure \( J_P \) on \( M_P \). The construction described above not only gives us a symplectic structure on \( M_P \) but also a compatible complex structure \( J_P \) on \( M_P \). Together they determine a toric Kähler structure on \( M_P \):

**Proposition C.1.9 ([Abr03])**. There exists a canonical compatible complex structure \( J_P \) on \((M, \omega_P)\) such that \( J_P \) is \( T^n \) invariant.

C.2. Toric Varieties

Instead of fixing a symplectic structure \( \omega \) on \( M \) one could also fix a complex structure on \( M \) and require that the action of the torus \( T^n \) is holomorphic, i.e. preserves \( J \). Such a manifold (variety) is called a toric variety. More precisely

**Definition C.2.1**. A toric variety is a (normal) irreducible complex variety \( X \) together with an action of an algebraic torus \((\mathbb{C}^n)^*\) which has an open dense orbit.

**Remark C.2.2**. Irreducible means that \( V \) cannot be written as the union of two proper Zariski closed subsets. In general it is not required that the toric variety is normal. However, once \( V \) is non-singular it is automatically normal. Recall that a normal variety is defined as follows: for any point \( x \in V \) there exists an affine neighborhood \( U_x \) such that for any \( f \) in the field of rational functions \( \mathbb{C}(U) \) such that

\[
f(z)^m + \sum_{j=0}^{m-1} g_j(z)f(z)^{m-j} = 0, \quad g_j \in \mathbb{C}[U]
\]

it follows that \( f(z) \in \mathbb{C}[U] \) where \( \mathbb{C}[U] \) is the coordinate ring of \( U \), i.e. \( \mathbb{C}[U] = \mathbb{C}[z_1, \ldots, z_N]/I_U \), \( I_U \) being the generating ideal of \( U \). From a geometric point of view normality is a kind of regularity assumption. For example if \( \dim_\mathbb{C}(V) = 1 \) and \( V \) normal then \( V \) is smooth. In general any singular set in a normal variety has at least codimension 2.
C.2.1. Cones and Fans. General toric varieties can be classified by their combinatorial data. These data are called fans. In the following we show how a complex variety determines a fan and how a fan gives rise to a toric variety.

Definition C.2.3. A convex polyhedral cone is a subset $\sigma \subset \mathbb{R}^n$ of the form
$$\sigma = \{a_1v_1 + \ldots + a_nv_d : a_1, \ldots, a_d \geq 0\},$$
where $v_1, \ldots, v_d \in \mathbb{R}^n$ are called the generators of $\sigma$. It is called a rational cone if $v_1, \ldots, v_d$ can chosen to be in $\mathbb{Z}^n$. A rational cone has a minimal set of generators in $\mathbb{Z}^n$ (the edges of $\sigma$). And $\sigma$ is called regular if its set of minimal generators form a $\mathbb{Z}$ basis of $N$. Moreover $\sigma$ is called simplicial if its minimal generators are linearly independent over $\mathbb{R}$. The dual cone $\sigma^* \subset (\mathbb{R}^n)^*$ of $\sigma$ is defined by
$$\sigma^* := \{u \in (\mathbb{R}^n)^* : \langle u, v \rangle \geq 0 \forall v \in \sigma\}.$$  
A face of a cone $\sigma$ is a set $\tau = H_u \cap \sigma$ for some $u \in \sigma^*$ where
$$H_u := \{v \in \mathbb{R}^n : \langle u, v \rangle = 0\}.$$  
A cone $\sigma$ is called strongly convex if $\{0\}$ is a face of $\sigma$.

Definition C.2.4. A fan $\mathcal{F}$ is a collection of cones $\{\sigma_1, \ldots, \sigma_l\}$ such that
(1) Every $\sigma \in \mathcal{F}$ is a rational strongly convex polyhedral cone,  
(2) For every $\sigma \in \mathcal{F}$, each face of $\sigma$ is also in $\mathcal{F}$,  
(3) For all $\sigma_1, \sigma_2 \in \mathcal{F}$ the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

The support $|\mathcal{F}|$ of $\mathcal{F}$ is $\bigcup_{\sigma \in \mathcal{F}}\sigma$. And $\mathcal{F}(r)$ denotes the set of all cones of dimension $r$. A fan is called complete if $|\mathcal{F}| = \mathbb{R}^n$, it is called simplicial if every cone $\sigma \in \mathcal{F}$ is simplicial, and it is called regular if every cone $\sigma \in \mathcal{F}$ is regular.

C.2.2. Construction toric varieties from fans. In this subsection we explain how a fan $\mathcal{F}$ gives rise to a complex variety $X_\mathcal{F}$. There are various ways to do this on of them is to construct $X_\mathcal{F}$ as a quotient of $\mathbb{C}^N$, where $N$ is the number of integral generator $v_1, \ldots, v_N$ of one-dimension cones $\sigma \in \mathcal{F}(1)$ and let
$$(t_1, \ldots, t_N) \cdot (z_1, \ldots, z_N) \mapsto (t_1z_1, \ldots, t_Nz_N)$$
be the standard action of $(\mathbb{C}^*)^N$ on $\mathbb{C}^N$. As before let
$$\pi : \mathbb{C}^N \to \mathbb{Z}^n, \quad e_i \mapsto v_i.$$  
Let $\mathfrak{t}$ be the kernel of $\pi$ and let $K$ be the subgroup of $(\mathbb{C}^*)^N$ generated by the one-parameter groups
$$\mathbb{C}^* \to (\mathbb{C}^*)^n, \quad \tau \mapsto (\tau^{\lambda_1}, \ldots, \tau^{\lambda_N}),$$
with $(\lambda_1, \ldots, \lambda_N) \in \mathfrak{t}$. Now for any $I \subset \{1, \ldots, N\}$ let
$$Z_I := \{z \in \mathbb{C}^N : z_i = 0 \forall i \in I\}$$
and
$$\sigma_I := \{\sum_{i \in I} a_i v_i : a_i \geq 0\}.$$  
Set
$$U \setminus \bigcup_{\sigma_I \not\in \mathcal{F}} Z_I.$$  
Then the subgroup $K$ acts on $U$ and we set formally
$$X_\mathcal{F} := U/K.$$  
It is not clear that this is indeed a toric variety. However if $\mathcal{F}$ does not contain any toric factors (i.e. $\mathcal{F}(1)$ spans $\mathbb{R}^n$) and $\mathcal{F}$ is simple then $U/K$ is a geometric quotient, i.e. all $K$-orbits are closed. Moreover, $X_\mathcal{F}$ is a toric manifold with a toric action $(\mathbb{C}^*)^n \simeq (\mathbb{C}^*)^N/K$. For details of this construction see [CLS11].

Theorem C.2.5. Given a fan $\mathcal{F}$ there exists a normal separated (i.e. Hausdorff in the classical topology) toric variety $X_\mathcal{F}$. Conversely given a normal toric separated variety $X$ there exists a fan $\mathcal{F}$ such that $X \simeq X_\mathcal{F}$. Moreover,
(1) $X_F$ is smooth (in particular a complex manifold) if and only if $F$ is regular.
(2) $X_F$ is an orbifold if and only if $F$ is simplicial.
(3) $X_F$ is compact if and only if $F$ is complete.

C.2.3. Toric resolutions. Let $X_{\text{sing}}$ (the singular locus) denote the set of all singular points in an irreducible variety $X$ and let $X_{\text{reg}} := X \setminus X_{\text{sing}}$ be the smooth locus. Recall that

**Definition C.2.6.** Given an irreducible variety $X$, a resolution of $X$ is a variety $X'$ together with a morphism $\pi : X' \to X$ such that

1. $X'$ is smooth ($X'_{\text{sing}} = \emptyset$) and irreducible.
2. $\pi$ is proper.
3. $\pi : X' \setminus \pi^{-1}(X_{\text{sing}}) \to X \setminus X_{\text{sing}}$ is an isomorphism.

It was proven by Hironaka that such a resolution exists for any (algebraic) variety over any algebraically closed field with characteristic zero (and the proof was later extended to analytic varieties). The proof is quite difficult although it has been subsequently simplified. For toric varieties, however, things are much simpler and a resolution can be obtained by “improving” (refining) the fan $F$ associated to a toric variety $X$. A refinement of a fan $F$ is a fan $F'$ such that $F \subseteq F'$. The idea is that for any refinement $F'$ of $F$ there exists a toric morphism $\pi : X_{F'} \to X_F$ that makes $(X_F, \pi)$ a resolution of $X_F$. The singular locus of $X_F$ can easily be determined by its fan. In fact,

$$X_{\text{sing}} = \bigcup_{\sigma \text{ is not smooth}} V(\sigma).$$

Now the main theorem is the following

**Theorem C.2.7 (CLS11).** Every fan $F$ has a refinement $F'$ such that

1. $F'$ is regular.
2. $F'$ contains every cone of $F$.
3. The toric morphism $\pi : X_{F'} \to X_F$ is a projective resolution.

The proof of this theorem can be found in CLS11. To find the refinement $F'$ of $F$ there exists a general procedure which we will not describe here (see CLS11 for more details). However, we will discuss the the two-dimensional example of Hirzebruch Jung resolutions. We follow Ful93 in this description. Hirzebruch Jung strings or resolutions are resolutions of quotient singularities of the type $\mathbb{C}^2 / \mathbb{Z}_k$ where $\mathbb{Z}_k$ is a cyclic subgroup acting freely on $\mathbb{C}^2 \setminus \{0\}$. Since $\mathbb{Z}_k$ is a commutative subgroup of $U(2)$ the singularity is indeed toric and $\mathbb{C}^2 / \mathbb{Z}_k$ is a (singular) affine toric variety with the associated fan

$$F = \langle e_2, pe_1 - qe_2 \rangle,$$

where $e_2$ and $e_1$ are the standard lattice vectors of $\mathbb{Z}^2 \subset \mathbb{C}^*$ and $p$ and $q$ are determined by the group action of $\mathbb{Z}_k$ which sends $(z_1, z_2)$ to $(e^{2\pi i/p} z_1, e^{2\pi i q/p})$. Clearly the fan is not generated by a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ and hence $X_F$ is indeed singular. The first step to resolve this singularity is to refine $F$ by inserting another generator $e_1$. The cone generated by $e_1$ and $e_2$ is clearly non-singular. So if the refinement $F'$ is still singular so must be the cone generated by $e_1$ and $pe_1 - qe_2$. The new singularity is strictly less singular (meaning the order of the group in the quotient is strictly less) than the first one. Indeed, rotating the lattice by 90 degrees, i.e. sending $e_1$ to $e_2$ and $e_2$ to $-e_1$ and then applying the lattice automorphism

$$A := \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$

we can see that the new singularity correspond to the variety which belong to the fan (cone) generated by $e_2$ and $p_1 e_1 - q_1 e_2$, where $p_1 = q$ and $q_1 = cq - p$. Since $q < p$ we can choose $c$ to be the smallest integer such that such that $cq - p \geq 0$, call it $r_1$. It is easy to see that $q_1 = r_1 q - p$ is strictly smaller than $q$ and so the singularity is of the form $\mathbb{C}^2 / \mathbb{Z}(q_1, p_1)$. We can repeat this procedure inductively lowering the group order until we end up with a smooth cone (which is the case when $q_j = 0$). It follows that
the numbers $r_i$ can be determined by the continued fraction expansion of
\[
\frac{p}{q} = r_1 - \frac{1}{r_2 - \frac{1}{r_3 - \cdots - \frac{1}{r_j}}}
\]
The resulting resolution $X_{F'}$ associated to refined non-singular fan $F'$ is simply connected. Moreover, $X_{F'}$ contains $r$ holomorphic spheres $S_1, \ldots, S_r$ with the intersection matrix
\[
(S_i \cdot S_j) = \begin{pmatrix}
-r_1 & 1 & 0 & \cdots & 0 \\
1 & -r_2 & 1 & \cdots & 0 \\
0 & 1 & -r_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -r_j
\end{pmatrix}.
\]
The exceptional set of the resolution is given by $E = S_1 \cup \ldots \cup S_r$.

Remark C.2.8. Note that there is a slight ambiguity here. In fact, if the process described above is applied straightforward it does not (sometimes) seem to give a refinement of the original fan. However, it can be shown that the toric variety associated to the fan $\tilde{F}$ generated by \{\tilde{e}_2, \tilde{p}e_1 - \tilde{q}e_2\} is isomorphic to the $X_F$ if $\tilde{p} = p$ and $\tilde{q}q \equiv 1(p)$. One can check that this implies that the resolution does not depend on the order of $r_1, \ldots, r_j$. Unfortunately this simple algorithm to produce a toric resolution does not work in higher dimensions anymore.

### C.3. Toric Kähler geometry

The two concepts of toric manifolds, toric varieties on the one hand and toric symplectic manifolds on the other, come together in the theory of toric Kähler manifold as follows.

**Definition C.3.1.** Given a moment polytope $P$ of the form
\[
P := \bigcap_{i=1}^N \{u \in t^* : \langle u, v_i \rangle \geq \lambda_i\},
\]
where $v_1, \ldots, v_N \in \mathbb{Z}_{T^n}$ and $\lambda_i \in \mathbb{R}$. Then the faces of $P$ consist of the sets
\[
F_I := \bigcap_{i=1}^N \{u \in t^* : \langle u, v_i \rangle = \lambda_i, \forall i \in I\},
\]
where $I \subset \{1, \ldots, N\}$. The associated fan $\mathcal{F}(P)$ is then
\[
\mathcal{F}(P) := \{\sigma_I : F_I \text{ is a face of } P\},
\]
where $\sigma_I$ is the cone generated by $v_i, i \in I$.

It follows that if $P$ is of the form described above then $(X_P, \omega_P)$ also has the structure of a toric variety and hence a complex structure $J_{\mathcal{F}(P)}$ determined by the associated fan $\mathcal{F}(P)$. One can show that this complex structure is indeed compatible with $\omega_P$ and

**Theorem C.3.2 (Abr03).** The complex manifold $(M_P, J_P)$ is biholomorphic to the toric variety $X_{\mathcal{F}(P)}$.

To summarize: there exists a canonical Kähler structure on each toric manifold $(M_P, \omega_P)$. In general we define a toric Kähler manifold as follows

**Definition C.3.3.** A toric Kähler manifold is a connect $2n$-dimensional Kähler manifold $(M, J, \omega)$ equipped with an effective hamiltonian action
\[
\tau : T^n \to \text{Bih}(M, J, \omega)
\]
of the standard torus $T^n$. 

In particular, a toric Kähler manifold \((M,J,\omega,\tau)\) is a symplectic toric manifold. And as such it has a moment polytope \(P\) (which is the image of the moment map if for example \(M\) is compact). Now it follows from Delzant’s theorem that \((M_P,\omega_P,\tau_P)\) is symplectomorphic to \((M,\omega,\tau)\). Moreover, there exists a canonical complex structure \(J_P\) on \(M_P\) which makes \((M_P,J_P,\omega_P,\tau_P)\). As a complex manifold \((M_P,\omega_P,\tau_P)\) is biholomorphic to \((M,J,\tau_P)\) however this map is usually not the symplectomorphism which identifies \(\omega\) and \(\omega_P\), i.e. the two manifolds are not isometric as Kähler manifolds. In the following we will describe how to obtain the canonical Kähler structure explicitly using action angle coordinates.

Let \(M := (M,J,\omega,\tau) := (M_P,J,\omega_P,\tau_P)\) be a toric Kähler manifold where \((M_P,\omega_P,\tau_P)\) obtained from a polytope \(P := \bigcup_{i=1}^{N}{u \in \mathfrak{t}^* : \langle u,v_i \rangle \geq \lambda_i}\) and \(J\) is a complex structure on \(M_P\) compatible with \(\omega_P\). Let \(\mu_P\) its moment map. Let \(P^o\) denote the open interior of \(P\) and let \(M^o = \mu_P^{-1}(P^o)\). Then \(M^o_P\) is open and dense in \(M_P\) and \(T^n\) acts freely on \(M^o_P\). We can choose (symplectic action-angle coordinates)coordinates \((x,y) : \omega = \sum_{j=1}^{n}{dx_i \wedge dy_j}\).

In these coordinates the complex structure is of the form \(J = \begin{pmatrix} 0 & -G^{-1} \\ G & 0 \end{pmatrix}\) and the integrability of \(J\) implies that \(G_{ij,k} = G_{ik,j}\) and so \(G = \text{Hess}(g)\) and the Riemannian metric is \(\omega(J \cdot, \cdot) = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}\).

As mentioned above the Delzant construction induces a canonical complex structure and it was determined by Guillemin that there corresponding complex potential is \(g_P(x) = \frac{1}{2}\sum_{j=1}^{N}{l_j(x) \log(l_j(x))}\), where \(l_j(x) = \langle x,v_j \rangle - \lambda_j\). Moreover any compatible \(J\) is determined by a potential of \(g_P\) the form \(g = g_P + h\) where \(h\) is a smooth function on the closure of \(P^o\). On the other hand we can choose complex coordinates \((w,z)\) on \(M^o \cong \mathbb{C}^n/(2\pi i\mathbb{Z}) = \{w + iz : w \in \mathbb{R}, z \in \mathbb{R}/2\pi \mathbb{Z}\}\) and in these coordinates \(J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and \(\omega\) is given by its potential \(\omega = 2\sqrt{-1}\partial\bar{\partial}f\).

Since \(\omega\) is invariant under the action of \(T^n\) the function \(f\) depends only on the coordinates \(w_1,\ldots,w_n\in\mathbb{R}\), in other words \(f = f(w) \in C^\infty(\mathbb{R}^n,\mathbb{R})\). In these coordinates \(\omega = \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}\) where \(F = \text{Hess}(f)\). The two coordinate systems are related as follows: note first that the coordinate \(x\) is given by the moment map \(x = \mu(w,z)\). In complex coordinates the moment map is given by the relation \(\frac{\partial \mu(w,z)}{\partial w_j} = \frac{1}{\sigma_j} \omega = F_{ij}\).
and so up to a constant

\[ x = \mu_i(w, z) = \frac{\partial f}{\partial w_i}. \]

In other words, \((x, y)\) and \((w, z)\) are related by a Legendre transformation and

\[ x = \frac{\partial f}{\partial w}, \quad y = z \]
maps \(\mathbb{C}^n/\mathbb{Z}\) diffeomorphically onto \(\mathbb{P}^n \times \mathbb{R}^n/\mathbb{Z}\). Moreover,

\[ f(w) + g(x) = \sum_j \frac{\partial f}{\partial w_j} \cdot \frac{\partial g}{\partial x_j}. \]

It follows that the canonical Kähler potential is of the form

\[ f(w) = -\frac{1}{2} \sum_{j=1}^N l_j(x) \log(l_j(x)) + \frac{1}{2} \sum_{j=1}^N [(l_j(x) + \lambda_j) \log(l_j(x)) + (l_j(x) + \lambda_j)] \]

\[ = \frac{1}{2} \sum_{j=1}^N [\lambda_j \log(l_j(x)) + (l_j(x) + \lambda_j)]. \]

Moreover, \(F\) and \(G\) are related by

\[ G^{ij}(x) = F_{ij}(w), \]

where \(G^{ij} = (G^{-1})_{ij}\).

**Proposition C.3.4** ([Abr03]). Let \((M_P, J_P, \omega_P, \mathbb{T}^n)\) be a toric Kähler manifold with the canonical complex structure \(J_P\). Then

\[ \det(G_P) = \left[ \delta(x) \prod_{j=1}^N l_j(x) \right]^{-1}, \]

where \(\delta(x)\) is a strictly positive function which is smooth on the closure \(P\).

In complex coordinates the Ricci form is given by

\[ \rho_{ij} = -2\pi \frac{\partial^2 \log(\det(F))}{\partial w_i \partial w_j}. \]

**C.3.1. Geometric equations on toric manifolds.** With the observations above several Kähler geometric problems become non-linear scalar PDEs on polytopes in \(\mathbb{R}^n\). Also obstructions to the existence of solutions can be formulated in a combinatorial way. For example let \((M, \omega_P, J_P, \mathbb{T}^n)\) be a toric Kähler manifold. Then the question whether there exists a toric symplectic form \(\omega\) such that \([\omega] = [\omega_P]\) compatible with \(J_P\) such that for some constant \(\lambda\)

\[ \rho = \lambda \omega. \]

**Proposition C.3.5** ([Abr03]). Let \((M, J_P, \omega_P, \mathbb{T}^n)\) be a toric Kähler manifold. Suppose that \(\varphi\) is a smooth function on \(M\) which is invariant under the torus action such that \(\omega + \sqrt{-1} 2\partial \bar{\partial} \varphi\) is still positive. Then there exists a smooth function \(h : P \to \mathbb{R}\) such that the corresponding complex potential \(g\) is of the form \(g = g_P + h\).

Now in complex coordinates the problem can be reformulated as

\[ f = -\pi \lambda \log(\det(F)) + c(w), \]

where \(c(w)\) is an affine function. In symplectic coordinates this equation becomes

\[ -g + x_i \frac{\partial g}{\partial x_i} = \pi \lambda \log(\det(G)) + c(\nabla g). \]

And using that \(g - g_P =: h\) is smooth on \(P\) and

\[ \log(\det(G)) = \log(\det(G_P)) + \log(\delta_h), \]
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Figure 1. The polytope $P$ associated to $[\omega_P] = -2\pi c_1(M)$ on the resolution of $\mathbb{C}^2/\Gamma(11,4)$.

where $\delta_h$ is a positive smooth function on $P$ we get

$$-g_P + x_i \frac{\partial g_P}{\partial x_i} - h + x_i \frac{\partial h}{\partial x_i} = \pi \lambda \log(\det(G_P)) + 2\pi \log(\delta_h) + c(\nabla g).$$

And so we can conclude that a necessary condition is that

$$\tilde{h} := -g_P + x_i \frac{\partial g_P}{\partial x_i} - \pi \lambda \log(\det(G_P)) - c(\nabla g)$$

is smooth on $P$. Now using that $\det(G_P) = \prod_j (l_j(x))^{-1} \delta(x)$

$$\frac{1}{2} \sum_j \lambda_j \log(l_j(x)) + \pi \lambda \sum_j \log(l_j(x)) - \frac{1}{2} \sum_j c(u_j) \log(l_j(x))$$

is smooth on $P$ only if $\lambda_j = -2\pi \lambda + c(u_j)$ for all $j$. In particular, $[\omega] = 2\pi c_1(M)$.

C.3.2. Kähler metrics on toric varieties. Not every toric variety $X_F$ can be equipped with a toric Kähler structure. However if the fan arises from a polytope then this is possible. So given a fan $F$ generated by primitive elements $u_j \in F(1)$, i.e. $u_j \in \mathbb{Z}^n$, we have to figure out whether there exists a good choice of $\lambda_j$ such that the polyhedral set

$$P := \bigcap_{j=1}^N \{ y \in \mathbb{R}^2 : (u_j, y) \geq \lambda_j \}$$

is a Delzant polytope. Once this is the case the Kähler metric on $X_F$ is given by the Delzant construction and more concretely by Guillemin’s formula for the canonical symplectic potential.

Example C.3.6. Consider the example $\mathbb{C}^2/\Gamma(p,q)$ above with $p = 11$ and $q = 4$. It follows that $r_1 = 3$ and $r_2 = 4$. So the associated refinement $F'$ is given by $p_1 = 4$, $q_1 = 1$ and $p_2 = 1$, $q_2 = 0$. So we can construct our desired background metric on $X_{F'}$ as follows. Choose $\lambda = -1$, then we set

$$g_P(x) := (y - \lambda_0) \log(y - \lambda_0) + (11x - 4y - \lambda_1) \log(11x - 4y - \lambda_1) + (4x - y - \lambda_2) \log(4x - y - \lambda_2) + (x - \lambda_3) \log(x - \lambda_3)$$

where $\lambda_r = 2\pi$, $r = 0, \ldots, 3$ and $P := \{(x, y) \in \mathbb{R}^2 : l_r(x, y) \geq 0\}$ (see figure 1).
D.1. Maximum Principles

In this section we recall some well known maximum principles tailored for our purposes.


Theorem D.1.1 (Maximum principle on complex manifolds, [BCG+12]). Let \((M, J)\) be a complex manifold and \(f \in C^2(M, \mathbb{R})\). Assume that \(f\) attains its maximum (minimum) at \(p\). Then
\[
\sqrt{-1} \partial \bar{\partial} f(p) \leq 0 (\geq 0), \quad \text{and} \quad df(p) = 0.
\]

Corollary D.1.2. Let \((M, J, g)\) be a Kähler manifold and \(f \in C^2(M, \mathbb{R})\). If \(f\) attains its maximum (minimum) at \(p \in M\) then
\[
\Delta f(p) \geq 0 (\Delta f(p) \leq 0).
\]

Theorem D.1.3 (Elliptic Maximum principle on Riemannian manifolds, [BCG+12]). Let \((M, g)\) be a Riemannian manifold and \(f \in C^2(M, \mathbb{R})\). Let \(Lu := \Delta u + X(u)\). Assume that \(u\) attains its maximum (minimum) at \(p\) then \(Lu(p) \geq 0 (Lu(p) \leq 0)\).

Corollary D.1.4. Suppose that \((M, J, g)\) is a complete noncompact manifold, \(\lambda : M \to \mathbb{R}\) such that \(\inf \lambda = \lambda_0 > 0\), and \(\lim_{x \to \infty} u(x) = 0\). If \(Lu \geq 0\) on \(M\) then \(\sup u \leq 0\).


Theorem D.1.5 (Parabolic maximum principle, [BCG+12]). Let \(M\) be a smooth non-compact manifold, \(\Omega_T \subset (0, T] \times M\) such that \(\Omega_T = M \times (0, T] \setminus K\) for some connected compact subset \(K \subset M \times [0, T]\), \(\{g(t) : t \in [0, T]\}\) a smooth family of uniformly equivalent complete Riemannian metrics on each time slice of \(\Omega_T\) and \(u \in C^2(\Omega_T) \cap C^0(\Omega_T)\) with \(\max_{[0, T]} \limsup_{x \to \infty} u(x, t) \leq 0\) satisfies
\[
\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u
\]
on \(\Omega_T\). Let \(\partial \Omega_T := \partial K \cup (\Omega_T \cap M \times \{0\})\). Then
\[
\sup_{\Omega_T} u \leq \sup_{\partial \Omega_T} u.
\]

Proof. Let
\[
v_c(x, t) := u(x, t) - \frac{\epsilon}{T - t} - \sup_{(x, t) \in \partial \Omega_T} u(x, t).
\]
Assume that \(\sup_{\partial \Omega_T} v_c > 0\). Then the maximum has to be attained in the interior of \(\Omega_T\) in a point \((x_0, t_0)\). And there \(\partial_t v_c(x_0, t_0) = 0\) and \(\Delta_{g(t_0)} v_c(x_0, t_0) \leq 0\). At the same time
\[
\frac{\partial v_c}{\partial t} \leq \Delta_{g(t_0)} v_c - \frac{\epsilon}{(T - t_0)^2},
\]
And this implies that at \((x_0, t_0)\)
\[
0 = \frac{\partial v_c}{\partial t}(x_0, t_0) \leq \Delta_{g(t_0)} v_c(x_0, t_0) - \frac{\epsilon}{(T - t_0)^2} \leq - \frac{\epsilon}{(T - t_0)^2}
\]
which is clearly a contradiction. Hence
\[
u_c(x, t) \leq \frac{\epsilon}{T - t} + \sup_{(x, t) \in \partial \Omega_T} u(x, t),
\]
and since this is true for any \( \epsilon > 0 \) the theorem follows.

□

D.2. Schauder Estimates

In this section we first state the famous local Schauder estimates that will be needed in chapter 5.

Theorem D.2.1 (Schauder-Morrey Estimates). Let \( B_1(0) \) and \( B_2(0) \) be two open balls in \( \mathbb{R}^n \) and \( L \) be a linear elliptic operator of second order on function on \( B_2(x) \) defined by

\[
Lu := \sum_{i,j} a(x)^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_j b(x)^j \frac{\partial u}{\partial x_j}(x) + c(x)u(x).
\]

Let \( \alpha \in (0,1) \) and assume that for some \( \lambda, \Lambda > 0 \)

\[
\max\{\|a^{ij}\|_{C^{0,\alpha}}, \|b^j\|_{C^{0,\alpha}}, \|c\|_{C^{0,\alpha}}\} < \Lambda
\]

and

\[
\inf_{\xi \in \mathbb{R}^n \setminus \{0\}} \sum a^{ij} \xi_i \xi_j > \lambda
\]

for all \( x \in B_2 \) then there exist constants \( C_1 \) and \( C_2 \) depending on \( n, \alpha, \Lambda, \lambda \) such that if \( u \in C^2(B_2) \) and \( Lu \in C^{0,\alpha}(B_2) \) then \( u \in C^{2,\alpha}(B_1) \) and

\[
\|u\|_{C^{2,\alpha}(B_1)} \leq C_1 \left( \|Lu\|_{C^{0,\alpha}(B_2)} + \|u\|_{C^{0,\alpha}(B_2)} \right)
\]

and

\[
\|u\|_{C^{2,\alpha}(B_1)} \leq C_1 \left( \|Lu\|_{C^{0,\alpha}(B_2)} + \|u\|_{C^{0,\alpha}(B_2)} \right).
\]

For proofs see \[Mor66\] and \[GT01\].
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