Master Thesis

Single viscous layer fold interplay and linkage a 3D-fem modelling approach

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Master Thesis

SINGLE VISCOUS LAYER FOLD INTERPLAY AND LINKAGE:
A 3D-FEM MODELLING APPROACH

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Abstract

Recent fieldwork observations and numerical experiments have demonstrated that large fold-belt systems do not necessarily grow uniformly in a cylindrical manner but arise from the lateral connection (parallel to the fold axis) of smaller embryonic folds. The mechanical feasibility of the linkage of two isolated embryonic folds has already been studied. In this context, the mechanical feasibility of the fork-linkage or more generally the triple-linkage (three isolated embryonic folds linking laterally together) is studied. To address this issue, a template for modelling the triple-linkage is introduced, which consists of a solitary embryonic fold opposite to a binary perturbation. A new terminology stemming from the observed patterns is introduced and a phase diagram highlighting the various linkage structures as a function of the geometric parameters is presented. The folding and linkage process is then tackled considering the vorticity field $\omega = \frac{1}{2} \nabla \times u$, where $u$ is the velocity field. It turns out to be a very interesting and fruitful framework that makes the linkage patterns and embryonic fold interplays simple to understand. Based on the 3D analytical solution for the finite amplitude folding of a single viscous layer embedded in a matrix, the planar-vorticity dominant wavelength (in the viscous layer plane) is computed numerically. This planar-vorticity dominant wavelength turns out to be distinct from the dominant amplification wavelength and it appears to be the characteristic length controlling the linkage process. In the light of these observations, a new interpretation and explanation is given for the simple-linkage previously studied and the perspective for the general case is finally discussed. The fieldwork carried out in the French Prealps south of the Geneva Lake is presented as well as a new tectonic interpretation. It is observed that the folds are asymmetric with an apparent vergence pointing to the NW. More surprising, the massive Malm limestone accommodates the large deformation by forming well localised tight kink-like structures whereas the rest of the layer is gently deformed.
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Chapter 1

Introduction

In nature, and specially in field geology, the information available at the outcrops allows generally only a 2D spatial reconstruction of the structures in place (e.g. folds, faults). First of all, the large scale spatial variations are de facto difficult to estimate and it is a reason why the folding mechanical process were often treated only in two dimensions assuming no shape variation of the fold along the lateral direction (parallel to the fold hinge) [Grasemann and Schmalholz, 2012]. Another reason is that this simplification makes the study on the mechanical and mathematical point of view easier and allow to get a well developed understanding of the folding process in terms of mechanical instability [Biot, 1957, Biot, 1961, Fletcher, 1974, Fletcher, 1977, Johnson, 1994].

Eventhough the two dimensional approach is a good way to understand the local picture of a fold, the cylindrical assumption has to be relaxed to gain a better comprehension of the global pattern at a fold-belt scale. Indeed, clear evidence of lateral fold propagation has been observed in soft Miocene sediments of the Zagros fold-and-thrust belt [Ramsey et al., 2008]. These observations show that folds do not grow laterally at the same time forming cylindrical structure but develop laterally from embryonic folds. In this context, detailed geomorphological studies have shown that long anticlines >100 km don’t originate from a single embryonic fold but are the result of the lateral linkage of several smaller syngenetic folds [Ramsey et al., 2008, Bretis et al., 2011]. From the lateral linkage concept arises naturally new patterns that are explained without invoking a regional change in the stress field or a blind fault system. For example, the oblique-linkage (or en-echelon link-
age) which resembles to a refold (type II) structure [Bretis et al., 2011] is explained without introducing any changes in the stress field or the presence of a blind strike-slip fault passing through the oblique hinge. It results in the linkage of the anticline growing laterally towards each other.

The mechanical feasibility of the linkage process has been demonstrated by [Grasemann and Schmalholz, 2012] by a 3D Finite Element Method (FEM) study. They modelled a single viscous layer embedded in a viscous matrix assuming a viscous power-law rheology and have treated the case of two embryonic opposite perturbations growing and interfering laterally. Added to this are other works that have proved the potential of fold linkage in the case of randomly perturbed viscous layer interface [Schmid et al., 2008, Fernandez and Kaus, 2013]. The results highlight the various linkage patterns that are mechanically stables and in particular the fork pattern.

In the present work, the ability of the marker in cell method (MAC) coupled with the finite element method (FEM) is tested to model accurately the folding process for large strain. The three dimensional folding analytical solution of [Fletcher, 1991] is revisited considering punctual perturbations in the Fourier space. The fork-linkage or more generally the triple-linkage (three embryonic folds linking together) mechanical feasibility is then studied. The folding and linkage process is finally tackled considering the vorticity field. The idea of the numerical work was initially motivated by the fieldwork carried out in the French Prealps south of the Geneva Lake. The results are presented as well as a new tectonic interpretation of the area. The region is too deformed to explicitly affirm the presence of fork linkage type-structures, but other issues are raised stemming from the field observations.

1.1 Physical model

To model the viscous layer folding, the classical abstraction of a single perturbed viscous layer embedded in a matrix is used. In the framework of the continuum mechanics, the conservation of mass and the momentum is imposed yielding to the well-known Navier-Stokes equations. In geodynamics, the processes considered are generally slow, of large scale and with high viscous materials. It results that the time derivative and the non-linear term (in velocity) of the Navier-Stokes equations can be neglected and rewritten in
the form of the Stokes equations given by (Eq. 1.1) (see for example [Gerya, 2010]).

\[
\begin{align*}
\sigma_{ij,j} + b_i &= \sigma'_{ij,j} - p_i - f_i = 0, \quad \mathbf{x} \in \Omega \\
u_{k,k} &= 0, \quad \mathbf{x} \in \Omega
\end{align*}
\]

(1.1)

Under the assumption of isotropic viscous flow, \( \sigma'_{ij} = 2\eta \dot{\epsilon}'_{ij} \) that gives \( \sigma'_{ij,j} = (\eta(u_{ij} + u_{ji}))_{,j} \) where the deviatoric strain rate tensor is defined as \( \dot{\epsilon}'_{ij} := \frac{1}{2}(u_{ij} + u_{ji}) - \frac{1}{3}u_{k,k}\delta_{ij} \). \( u_{k,k} \) vanishes for incompressible fluids and it follows that \( \dot{\epsilon}'_{ij} = \dot{\epsilon}_{ij} \). The

\[
\begin{align*}
(2\eta \dot{\epsilon}'_{ij})_{,j} - p_i - f_i &= 0, \quad \forall \mathbf{x} \in \Omega \\
u_{k,k} &= 0, \quad \forall \mathbf{x} \in \Omega \\
\mathbf{u} &= \bar{\mathbf{u}}, \quad \forall \mathbf{x} \in \Gamma_D \\
\sigma_{ij}n_j &= \bar{t}_j, \quad \forall \mathbf{x} \in \Gamma_N
\end{align*}
\]

(1.2)

where \( \Gamma_D \) are the Dirichlet boundaries and \( \Gamma_N \) Neumann boundaries such that \( \partial \Omega = \Gamma_N \cup \Gamma_D \) and \( \Gamma_N \cap \Gamma_D = \{0\} \). \( n_j \) is the outwards pointing normal.

To describe completely the system, it is also impose that the material is only advected and no material diffusion occurs in time:

\[
\frac{D\eta(\mathbf{x})}{Dt} = 0, \quad \frac{Df(\mathbf{x})}{Dt} = 0
\]

(1.3)

1.2 Model numerical solution

The Finite Element Method (FEM) is applied to discretize and solve (Eq. 1.2) with the help of the \( Q_2P_1 \) elements. The resulting linear system is solved using an augmented Lagrangian approach in 2D. For the 3D modelling, the pTatin code is used with a multigrid solver. The code was run in parallel using the Brutus cluster from ETHZ that reduced drastically the computation time or the present work would not have been possible.

The report is divided in four parts:

1. The first part describes the numerical method in detail and presents some numerical tests of the 2D code and the pTatin code in the case of folding experiments.

2. The second part describes the analytical solution of a 3D viscous layer folding in the case of infinitesimal amplitude for local perturbation. The analytical solution is largely inspired from [Fletcher, 1991].
3. The third part introduces the simple and triple fold linkage as well as the folding process and linkage in the vorticity context.

4. In the last part the fieldwork results conducted in the Prealps are presented. More precisely, the area of investigation is situated south of the Geneva Lake in France at Pointe de Pavis. The fieldwork motivated the numerical simulations, but no linkage were observed in this area because the place underwent a too high shortening accommodated by thrusts which eclipses the linkage indices.
Chapter 2

Numerical methods and folding tests

The numerical code pTatin3D has been successfully tested in different case for 3D geodynamic modelling. It is intended first to carry out some preliminary tests to get confident with the code to model folding processes. The approach is to use a FEM serial code for 2D folding and compare the results produced by the pTatin code for 3D cylindrical folds. For that purpose, a Lagrangian FEM code has been written with \( Q_2 P_1 \) elements solving Stokes problems for Newtonian viscous material in a force field. For large strain, people generally remesh the domain with triangular elements falling to an increase of the computation time and a more complicated strategy for the solver. In the present work, a fully Lagrangian mesh method with markers is implemented and tested for folding simulation.

In this chapter, the finite element method and its numerical implementation in 2D is presented along with the marker approach for large strains. Some results for the comparisons between the numerics and the linear theory of Fletcher and Biot [Biot, 1957, Biot, 1961, Fletcher, 1974, Fletcher, 1977] are discussed and we end up with the comparison between the 2D and 3D codes.
2.1 Numerical methods

2.1.1 Pure FEM

Discretization of the Stokes problem

The Stokes problem for incompressible Newtonian viscous material in a gravity field is given by (Eq. 1.2) where \( f_i = -\rho g z_i \). In the finite element approach, the strong form of the partial differential equation is transformed to a weak formulation. This allows to use the concepts of linear algebra to solve the problem and to have alternative descriptions of problems with discontinuous boundary conditions not solvable in other cases. The FEM provides a numerical solution approximating the weak solution with respect to some test functions. Before going further, let us define two functional spaces useful in the finite element theory: the function space of square-integrable function \( L^2(\Omega) \) and the Sobolev space of order one, \( H^1(\Omega) \), defined on \( \Omega \subset \mathbb{R}^d \) an open space of \( \mathbb{R}^d \).

\[
L^2(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ Lebesgue measurable} \mid \int_\Omega |f(x)|^2 dx < \infty \right\} \tag{2.1}
\]

\[
H^1(\Omega) := \left\{ f \in L^2(\Omega) \mid \partial_k f(x) \in L^2(\Omega), \ \forall k \leq d \right\} \tag{2.2}
\]

We are concerned here only with \( d = 2 \) or 3. These function spaces are meaningful because the weak problem consists in finding the solution in \( L^2(\Omega) \), and more precisely in \( H^1(\Omega)^d \) for the velocity of the Stokes problem, whereas the strong problem consists in finding a solution in \( C^2(\Omega) \) which is more restrictive in terms of admissible boundary conditions (no permitted discontinuity).

By transforming the problem (Eq. 1.2) into its weak form (see appendix A.2), we look for the solution \( (\mathbf{u}, p) \) in the infinite space function \( \mathbb{V} \times Q \subset [H^1(\Omega)]^3 \times L^2(\Omega) \) satisfying (Eq. 2.5). Only Dirichlet type and the free surface \((\sigma_{ij} = 0)\) boundary conditions are considered here and it motivates the definition of the solution space for the velocity \( \mathbb{V} \) and the test space \( \mathbb{V}_0 \) (according to the definition of [H. C. Elman, 2005]).

\[
\mathbb{V} := \left\{ \mathbf{f} \in H^1(\Omega)^3 \mid \mathbf{f}(x) = \mathbf{w}(x) \text{ on } \partial \Omega \right\} \tag{2.3}
\]

\[
\mathbb{V}_0 := \left\{ \mathbf{f} \in H^1(\Omega)^3 \mid \mathbf{f}(x) = \mathbf{0} \text{ on } \partial \Omega \right\} \tag{2.4}
\]
For the pressure, the solutions space is \( Q := L^2(\Omega) \). The weak problem is then to find \((u, p) \in V \times Q\) satisfying (2.5).

\[
\begin{aligned}
\{ a(u,v) + b(v,p) &= f(v) \\
b(u,q) &= 0 \}
\end{aligned}
\quad \forall (v,q) \in V_0 \times Q
\]

where the bilinear forms \( a(.,.) \), \( b(.,.) \) and \( f(.) \) are defined in the appendix (A.2).

The idea of the FEM based on the Galerkin method, is to replace the infinite dimension space \( V \times Q \) by a subspace \( V^h \times Q^h \subset V \times Q \) and \( V_0^h \subset V_0 \) of finite dimension. It results to a problem of finite dimension reading: find \((u_h, p_h) \in V^h \times Q^h\) satisfying (Eq. 2.6).

\[
\begin{aligned}
\{ a(u_h,v_h) + b(v_h,p_h) &= f(v_h) \\
b(u_h,q_h) &= 0 \}
\end{aligned}
\quad \forall (v_h,q_h) \in V_0^h \times Q^h.
\]

Let \( \{ \varphi_1, \ldots, \varphi_N \} \) be a base of \( V_0^h \) (dim \( V_0^h = N = N(h) \)) and \( \{ \psi_1, \ldots, \psi_M \} \) a base of \( Q_h \) (dim \( Q_h = M = M(h) \)). The \( \{ \varphi_j \}_{j=1}^N \) does not ensure the Dirichlet conditions to be satisfied. For this reason, it is extended with the functions \( \{ \varphi_{N+1}, \ldots, \varphi_{N+N_0} \} \) and selected fixed value \( u_j \) such that \( \sum_{j=N+1}^{N+N_0} u_j \varphi_j \) interpolates the prescribed Dirichlet boundary conditions. The general solution of the problem in \( V^h \times Q^h \) can be expand as

\[
\begin{aligned}
u_h(x) &= \sum_{j=1}^{N} u_j \varphi_j(x) + \sum_{m=N+1}^{N+N_0} u_m \varphi_m(x), \quad p_h(x) = \sum_{k=1}^{M} p_k \psi_k(x), \quad \text{for } x \in \Omega
\end{aligned}
\]

In the Galerkin approximation method (Budnov-Galerkin), the test functions are chosen such that \( v_h = \varphi_i \) for all \( i = 1, \ldots, N \) and \( q_h = \psi_m \) for all \( m = 1, \ldots, M \). Due to the linearity of the Stokes problem, the solution of the problem (Eq. 2.6) consists in finding the coefficients \( \{ u_j \}_{j=1}^N \) and \( \{ p_k \}_{k=1}^M \) such that (Eq. 2.7) is verified.

\[
\begin{aligned}
\sum_{j=1}^{N} u_j a(\varphi_j, \varphi_i) + \sum_{k=1}^{M} p_k b(\varphi_i, \psi_k) &= f(\varphi_i) - \sum_{n=N+1}^{N+N_0} u_n a(\varphi_n, \varphi_i), \quad \forall i = 1, \ldots, N \\
\sum_{j=1}^{N} b(\varphi_j, \psi_m) &= 0, \quad \forall m = 1, \ldots, M
\end{aligned}
\quad (2.7)
\]
Defining $U = (u_1, \ldots, u_N)^T$ and $P = (p_1, \ldots, p_M)^T$, we end up with the compact form given by (Eq. 2.8) which is a sparse linear system to solve.

\[
\begin{pmatrix}
A & B \\
B^T & 0
\end{pmatrix}
\begin{pmatrix}
U \\
P
\end{pmatrix}
= 
\begin{pmatrix}
F \\
0
\end{pmatrix}
\tag{2.8}
\]

where $A_{ij} = a(\varphi_j, \varphi_i)$, $B_{ik} = b(\varphi_i, \psi_k)$ and $F_i = f(\varphi_i) - \sum_{m=N+1}^{N+N_0} u_m a(\varphi_m, \varphi_i)$ for all $i, j = 1, \ldots, N$ and $k = 1, \ldots, M$.

With the FEM, we have a technique to construct the subspace $V_h$ and $Q_h$ considering the problem locally or elementwise. The idea is to provide a partition $\mathcal{T}_h$ of $\Omega$ in $n$ subdomains $\mathcal{E}_k$ such that $\Omega = \bigcup_{k=1}^{n} \mathcal{E}_k$ and $\mathcal{E}_k \cap \mathcal{E}_{k'} = \emptyset$, $\forall k \neq k'$. $\mathcal{E}_k$ defines the element (generally triangle or rectangle in 2D) subdomain. We introduce the Lagrangian\(^1\) finite element space of degree $r \geq 1$.

\[
X^d_h = \{v_h \in \mathcal{C}^0(\Omega) | v_h|_{\mathcal{E}_k} \in \mathcal{Q}^d_r, k = 1, \ldots, n\}
\]

where $\mathcal{Q}^d_r$ is the set of polynomials of degree $r$ in $d$ variables defined for the $Q$ elements (quadrilateral elements).

\[
\mathcal{Q}^d_r := \{p(x) = \sum_{i_1, \ldots, i_d | \sum_{k} i_k \leq r} a_{i_1 \ldots i_d} x_1^{i_1} \ldots x_d^{i_d}\}
\]

If one wants to use triangular elements or $P$ elements, the $\mathcal{P}^d_r$ polynomials are used, where

\[
\mathcal{P}^d_r := \{p(x) = \sum_{i_1, \ldots, i_d | \sum_{k} i_k \leq r} a_{i_1 \ldots i_d} x_1^{i_1} \ldots x_d^{i_d}\}
\]

We can then choose the subspace $V^*_0 = X^r_h \cap V_0$ to be the space of piece-wise polynomial continuous function of degree $r$ vanishing at the boundaries. Following the same approach, we have $Q^h = X^*_h \cap Q_0$.

The idea is to choose the basis function $\{\varphi_j\}_{j=1}^{N}$ to be non-zero locally at the element but vanishing elsewhere. The nodes are defined at the edges, apex and midpoint of the elements and the global shape function $\varphi_j$ is chosen

\(^1\) Lagrangian because the basis functions stem from the Lagrange basis polynomials in the theory of interpolation.
to be non-zero at the node \( x_j \) (generally being unity) but identically zero at the others:

\[
\varphi_j(x_j) = 1, \quad \varphi_j(x_i) = 1, \quad \forall i \neq j
\]

It results that the values \( u_j \) and \( p_k \) are precisely the values of the velocity and pressure at the nodes of the mesh domain themselves. For a piecewise-polynomial interpolation (elementwise support), the number of nodes at the element is constrained by the degree of the polynomial interpolation. For example if one wants to interpolate a field on the element with quadratic functions of the form: \( ax + by + cxy + dx^2 + ey^2 + f \), six nodes are necessary because there are six unknown coefficients. The piecewise basis functions are constructed at the element scale using product of one-dimensional Lagrange polynomials satisfying the conditions aforementioned at the nodes and with a elementwise support. If a node is shared across more than one element, the support of the basis function at the node is the subdomain, union of the elements sharing this node.

Supposing that the element \( k \) has \( n_k \) nodes, the element shape functions (or basis functions) are given by \( \{ \phi^k_1, \ldots, \phi^k_{n_k} \} \) and the solution within the element \( k \) is

\[
\mathbf{u}_k |_k = \sum_{j=1}^{n_k} u^k_j \phi^k_j
\]

The solutions for the pressure is similar with the appropriate basis.

**Numerical implementation and the isoparametric elements**

In practice, the matrices \( A \) and \( B \) (Eq. 2.8) are constructed using the expanded basis \( \{ \varphi_i \}_{i=1}^{N+N_\theta} \); the \( i \) rows and columns for all \( i = N + 1, \ldots N + N_\theta \) are set to zero and the Dirichlet conditions are imposed by setting \( F_i = u_i \) for the corresponding nodes \( i \). On each subdomain \( \mathcal{E}_k \), the element matrices \( A_k \) and \( B_k \) are constructed using the local element basis \( \{ \phi^k_1, \ldots, \phi^k_{n_k} \} \) and integrating on \( \mathcal{E}_k \) and not on \( \Omega \). The element matrices are assembled together to form the global matrices \( A \) and \( B \).

The partition of \( \Omega \) is done using well-known reference elements (triangle, square, brick etc.). These reference elements and their related shape functions \( N_i(\xi) \) are defined in a given local system of coordinates \( \xi \). In practice
the elements can be deformed and rotated such that the reference coordinate system is not collinear to the global coordinate system. The local coordinates are mapped to the global system of coordinate $x$ using bijective and differentiable transformations. The isoparametric elements are one class of elements for which the shape functions verify (Eq. 2.9)

$$x = \sum_{i=1}^{n_k} N_i^k(\xi)x_i^k, \quad \sum_{i=1}^{n_k} N_i^k(\xi) = 1$$

(2.9)

where $x_i^k$ are the global coordinates of each nodes at the element $k$. The transformation from the local to the global coordinates is bijective (invertible) and differentiable and uses the shape functions of the unknown field. It insures continuity through edges shared by different elements.

For a two dimensional domain, $d = 2$, the quadrilateral $Q_2P_1$ isoparametric elements has been chosen because they suit well for the Stokes problem [Boffi and Gastaldi, 2002, H. C. Elman, 2005]. It follows a quadrilateral partition of the domain for the velocity and the use of quadratic polynomial in two variables for the interpolation\(^2\) (in $Q_2^2$). For the pressure, not connected triangular elements are used with linear polynomials in $P_1^1$ for the interpolation. For more details about the basis function and the construction of the matrix $A$ and $B$, refer to the appendix.

2.1.2 The solver

Iterative penalty method

To solve the Stokes problem, an iterative penalty method has been used following the scheme of Codina [Codina, 1993] which is an Uzawa-like algorithm.

The incompressibility constraint is resolved by a penalty-function formulation [Hughes, 2000, Temam, 2001] and was first introduced by R. Temam for the Navier-Stokes equation [Temam, 1968]. This approach can be viewed as an optimization problem, where the Stokes problem corresponds to the minimization of a functional with the constraint $\nabla \cdot u = 0$. The pressure

\[u_h|k = a + bx + cx + dx^2 + ey^2 + fxy + gxy^2 + hx^2y + ix^2y^2\]

that is 9 unknowns, hence 9 nodes per element.
is interpreted as a Lagrange-multiplier and the problem is reformulated for slightly compressible fluid with the constrain of incompressibility [Temam, 2001]. Instead of solving (Eq. 1.2), the problem (Eq. 2.10) is solved.

\[
\eta \left( u_{i,j} + u_{j,i} \right) \epsilon_{i,j} - \rho \dot{g} \epsilon_i = 0, \quad i, j = 1, 2, 3 \text{ and } \mathbf{x} \in \Omega \quad (2.10)
\]

One can show that when \( \epsilon \to 0 \), \( v^{(\epsilon)} \to u \) in \( H^1(\Omega) \) and \( p^{(\epsilon)} \to p \) in \( L^2(\Omega) \) with \( p \) satisfying \( \int_\Omega p \, d\Omega = 0 \). After [Codina, 1993], the solution can be found using a penalty method applied iteratively (it is proved only in the isoviscous case). If the discretized formulation of (Eq. 2.10) is used according to (Eq. 2.8), the algorithm reads:

1. Find the velocity \( U^{k+1} \) for:

\[
AU^{k+1} = F - BP^k
\]

2. Find the pressure \( P^{k+1} \) for:

\[
P^{k+1} = P^k - \frac{1}{\epsilon} B^T U^{k+1}
\]

where \( \epsilon \in \mathbb{R}^+ \) is small enough (\( \epsilon \sim 10^{-7} \)).

3. Iterate step 1 and 2 until the divergence of \( U^{k+1} \) is small enough:

\[
B^T U^{k+1} < \alpha, \quad \text{for } \alpha \in \mathbb{R} \text{ small (} \alpha \sim 10^{-7} \)
\]

**Time stepping**

The Stokes problem is stationary and no time dependence appears explicitly in (Eq. 1.2). The dependency comes with (Eq. 1.3) i.e. the constraint on the evolution of the viscosity and the body force fields. This is solved by the equation (Eq. 2.11) applied at the finite element nodes where the velocity is computed.

\[
\frac{d\mathbf{x}}{dt}(t) = \mathbf{u}(t) \quad (2.11)
\]

The procedure is the following:

1. For a given mesh configuration \( \{X_i^t\}_{i=1}^N \) at time \( t \), the Stokes problem is solved for the given boundary conditions and the solutions \( \{U_i^t\}_{i=1}^N \) and \( \{P_k^t\}_{k=1}^M \) are provided.
2. The solution for the velocity at the nodes is used to solve (Eq. 2.11) using the Euler explicit discretization in time. The position of the nodes are then updated according to:

\[ X_{i}^{t+1} = X_{i}^{t} + dtU_{i}^{t} \]

3. Iterate step 1 and 2 until \( t \geq t_{\text{max}} \).

### 2.1.3 Large strain deformation and the marker in cell methods

For large strain deformation with Lagrangian grid, we generally need to remesh the entire domain to avoid element entanglement, overlap or simply high deformation which can give rise to erroneous numerical results and are detrimental for the stability of the solver.

For a multiphase problem, to remesh with quadrilateral elements while conserving a structured grid, is not a simple task and sometimes it is impossible. In such cases, the unstructured grid with triangular elements are more appropriate but need a different solving strategy in 3D to reduce the CPU time. For example, the geometric multigrid method commonly used for 3D FEM is a tedious work to implement with unstructured grid and it makes the triangular elements difficult to use in 3D. Another option is to use marker methods [Harlow and Welch, 1965, Poliakov and Podladchikov, 1992, Sulsky et al., 1994, Sulsky et al., 1995, Wieckowski, 2004] in which marker points are introduced. The markers are scattered through the domain and carry all the physical information related to the material lithology (eg. \( \eta \), \( \rho \)).

In the present work, we follow the method described by [Poliakov and Podladchikov, 1992] consisting in using markers and a deformable Lagrangian mesh. The markers are distributed locally in the quadrilateral elements and contain the local physical information (for us the density and the viscosity), the element index to which it belongs and its local coordinates in the element. The markers are used locally at the element level to interpolate the physical properties on the quadrature points. These interpolated values are then used to construct the local stress operators. The interpolation on the quadrature points changes with the problem treated. If the physical field can be projected locally on a polynomial base, a bilinear or bicubic interpolation
can be used. In the case of a discontinuous physical field, like for folding, the value of the nearest marker or a the arithmetic mean on the nearest markers can be taken as the interpolated value for a given quadrature point.

At each time step, only the numerical mesh is updated because the markers are fixed in the local coordinate of their elements and there is no need to know their global coordinates or velocity for every steps. However to produce output or before a remeshing, it is necessary to know the global coordinate and the velocity of the markers. These values are interpolated via the element shape functions. For example the global coordinate $\mathbf{x}_g$ of a marker point are given in (Eq. 2.12).

$$\mathbf{x}_g = \sum_{j=1}^{n_k} \phi_{k,j}^E(\xi) \bar{\mathbf{x}}_j$$

(2.12)

where $\phi_{k,j}(\cdot)$ are the element shape functions of element $k$, $\xi$ the local coordinates of the marker, $n_k$ the number of nodes of element $k$ and $\bar{\mathbf{x}}_j$ are the global coordinates of the node $j$.

If the mesh is too deformed, the domain is entirely remeshed conserving the same number of quadrilateral elements in each direction of space. The new mesh is horizontally equally spaced, whereas the vertical direction is equally spaced according to the free-surface. This method works only if there exists a projection map from the free surface to the bottom boundary of the domain, i.e. the horizontal line $[0, L_x]$. Because the material properties are carried by the markers, there is no need to remesh by shaping the different phases.

After remeshing, for each markers, the new element containing the marker has to be found and its local coordinate in the new element has to be computed. The method is to solve the non-linear system (Eq. 2.12) for $\xi$. In the present work, the Newton’s method is used to solve the non-linear system with a precision of $10^{-15}$.

**Need of higher grid resolution**

With the FEM, the sharp viscosity and density contrasts we deal with can be easily captured by the elements: the material properties are constant
over one finite element. The mesh is constructed according to the geometry of the lithology, where we use deformed finite elements of constant viscosity and density as building block. In that case, the phase contrasts are well defined and it results a well resolved pressure field (discontinuous through different lithologies).

Using FEM with markers, the material properties are not constant over one finite element and it results a smeared/blurred pressure field in the neighbourhood of the interface between two lithologies. The pressure field on the elements crossing an interface is a smooth approximation of the pressure sharp contrast we should physically have. In order to reduce this error, the mesh resolution has to be increased around the interfaces (the mesh resolution is actually increased globally over the entire domain).

2.2 Numerical tests

2.2.1 The 2D linear theory of folding

Without going into details, the linear theory of folding of a viscous layer, developed mainly by Biot and later Fletcher [Biot, 1961, Fletcher, 1977], gives an linear analytical solution for folding with an initial sinusoidal waveform perturbation of an infinitesimal amplitude. The two main results are the initial growthrate $(1 + q)(-\ddot{\epsilon}_{xx})$ and the dominant wavelength $\lambda_d$.

In [Biot, 1957], the infinitesimal linear amplitude equation is given (Eq. 2.13). For uniform compression, the amplitude $A$ grows exponentially in the framework of the linear theory, it is an unstable point. For larger amplitude, the non-linear terms have to be taken into account and they introduce a damping of the fold amplification [Johnson, 1994].

$$\frac{dA}{dt} = (1 + q)(-\ddot{\epsilon}_{xx})A$$  \hspace{1cm} (2.13)

The geometrical and material dependence of the growth rate is contained only in the term $q(k, R)$ and is expressed in (Eq. 2.14) where $k = 2\pi H/\lambda$ with $H$ the viscous layer thickness, $\lambda$ the wavelength and $R = \eta/\eta'$ the viscosity ratio ($\eta$ is the viscosity of the viscous layer and $\eta'$ is the viscosity of the matrix).
\[ q(k, R) = \frac{8(1 - R)Rk}{4k(R^2 - 1) - [(1 + R)^2e^k - (1 - R)^2e^{-k}]} \]  \hspace{1cm} (2.14)

The maximum of \( q(k, R) \), gives the wavelength with the largest growth rate. This wavelength is expected to dominate all the others. No analytical solution exists for the general case but in the thin-plate approximation \((H \ll \lambda)\) supposing that \( R^2 \gg 1 \), it is expressed by (Eq. 2.15) [Biot, 1961, Johnson, 1994].

\[ \lambda_d(H, R) = 2\pi H(\eta/6\eta')^{\frac{1}{2}} \]  \hspace{1cm} (2.15)

It is obvious that the dominant wavelength is independent of the uniform shortening \( \bar{\epsilon}_{xx} \) and depends only on the viscosity ratio and the thickness of the viscous layer.

It is important to notice that as the deformation gone, the function \( q(k, R) \) changes in time [Adamuszek et al., 2013]. It means that the growth spectrum changes in time. Moreover, the infinitesimal theory developed by Biot and Fletcher is only valid in the case of infinitesimal sinusoidal perturbation where the different modes grow independently of each other. It is not the case when we consider larger deformations [Johnson, 1994, Adamuszek et al., 2013].

The results (Eq. 2.14) and (Eq. 2.15) are used for infinitesimal sinusoidal deformation to compare with the numerical results and validate the numerical code used for the simulations.

2.2.2 Boundary conditions for the specific problem

For the case of folding in 2D which are performed here, there are two choices for the boundary conditions. Considering an initial rectangle with a horizontally layered viscosity, in any case, the top boundary is taken as a free surface and the bottom boundary as a free slip boundary. The domain is uniformly and laterally compressed: the velocity vector field at the lateral boundary point inwards with a uniform norm. If a constant velocity field is applied, the value of the velocity is determined at the beginning and \( |v_{lateral}| = v_0 \). If the domain is compressed with a constant strain rate, the
applied velocity changes in time according to $|v_{\text{lateral}}| = \bar{\varepsilon}_{xx} \cdot L_x / 2$, where $L_x$ is the instantaneous width of the domain.

2.2.3 Benchmark

In this subsection, we will use the analytical solutions (Eqs. 2.14 and 2.15) to test the numerical code in the framework of the linear theory.

The dominant wavelength

The analytical solution of the dominant wavelength given by (Eq. 2.15) allows to test the numerical results for the two parameters controlling the folding process (at least during the earliest phase): the viscosity ratio $R$ and the viscous layer thickness $H$.

The results are obtained by initially perturbing the interfaces between the
high and low viscous regions with a cosine series of the form of (Eq. 2.16).

\[ y = \text{const} + \sum_{m=1}^{n_m} \left( \frac{1}{n_m} \right) \cos\left( \frac{2\pi x}{m\lambda_d/10} \right) \]  

(2.16)

where \( \lambda_d \) is the estimated dominant wavelength of the viscous layer computed from (Eq. 2.15). After one time step, the dominant wavelength grows the fastest. By comparing the Fourier power-spectrum between both, the initial state and after the first time step, the dominant wavelength happens to be the one with the largest difference i.e. the largest growth. In Fig. 2.2, the numerical estimation of the dominant wavelength is compared with the linear theory. The numerical wavelength is slightly smaller than the theoretical ones which is not surprising: after one time step, the domain is compressed as well as the dominant wavelength which is also shortened.

![Figure 2.2: Initial dominant wavelength after one time step for a cosine series perturbation at the interfaces. The stars represent the numerical results compared with theoretical values (full-line)](image)

Simulation parameters: \( dt = 0.001, L_x = 2\lambda_d, L_y = 87, H = 3, \bar{\epsilon}_{xx} = 0.01, n_m = 50.0. \)
The amplification

The growth rate given by (2.14) is compared with the numerical results. For the simulation, the viscous layer has a cosine shape with a well defined period and we look at the growth rate of the cosine crest after one time step. This procedure is repeated for different periods for which the value $q$ is extracted and compared with the theory taking $\tilde{k} = 2\pi H/(\text{cosine period})$. The results are represented in Fig. 2.3 for different $R$ fixing the viscous layer thickness.

Figure 2.3: Initial growth rate after one time step for sinusoidal perturbations of different periods. The stars represent the numerical results in the color of their corresponding theoretical values (full-lines) Simulation parameters: $dt = 0.001$, $L_x = 2\lambda_d$, $L_y = 87$, $H = 3$, $\bar{\epsilon}_{xx} = 0.01$.

2.2.4 Pure-FEM vs. MAC-FEM

Some tests have been performed to compare the pure-FEM method and the FEM method using markers (MAC-FEM) as soon as the mesh is too deformed (according to some geometric criteria). For the coarse grid, the markers are introduced after 3700 steps and 3800 with the fine grid and it corresponds to the first remeshing. The subsequent remeshings are more closely spaced in time with increasing deformation. Constant velocity $v_B$ is
set domain-inwards at the East and West boundaries and the simulations are stopped when the solver or the program crashes for any reasons. The MAC-FEM crashed first after 6100 time steps because the free-surface folded over itself and it was not possible any more to project the surface line on the bottom boundary which is a necessary condition for the employed remeshing method.

Fig. 2.4 shows a qualitative comparison of the two methods at two different time steps. For large deformation, with a coarse grid, the pinched shape of the free surface is not well resolved using markers. After 6100 time steps, the folded viscous layer from the pure FEM and the MAC-FEM are not nicely superposed. This is improved by using a finer grid (right pictures). The free-surface shape from the MAC-FEM code is better resolved together with the viscous layer superposition compared with the pure-FEM code.

Fig. 2.5 shows the constant component of the pressure\(^3\) at the finite element cells. The intradoses undergo compression \((p_0 > 0)\) whereas the extradoses undergo dilatation \((p_0 < 0)\) as expected. Fig. 2.6 shows the same parameter with the MAC-FEM method. For the coarse grid, the pressure is badly resolved for the elements crossing the lithology interfaces because it tries to represent a sharp discontinuous pressure field with linear functions \((P_1\) triangular elements are used for the pressure). The coarser the mesh is across the interfaces, the coarser is the pressure solution. If the mesh is finer, the pressure solution is improved in the neighbourhood of the interfaces but still rough for the elements crossing the interfaces.

To obtain consistent results between pure-FEM and MAC-FEM, we advocate using four times more elements in each directions of space for the MAC-FEM mesh-grid.

2.2.5 The pTatin code

The pTatin code has been developed by D.A. May to model 3D-multiphase Stokes flows for geodynamical applications. The code is written in C and takes advantage of the PETSc library. This is aimed at launching high res-

---

3. The pressure with the P1 element is projected on \(P_2\) and then described by the a function of the form \(p_0 + p_{11}x + p_{12}y\). The constant component corresponds to \(p_0\).
olution problem in parallel on clusters.
This is a $Q_2P_1$ FE code using the Lagrangian marker methods highlighted above with an iterative solver based on the multigrid algorithms.

2.2.6 Test of the code for 3D processes (pTatin)

Results from the 3D code have been compared with the 2D code in the case of cylindrical folding. For the same boundary conditions, geometry and initial perturbation (cylindrical perturbations in 3D), the same pattern comes out in 3D and 2D. Qualitatively, the folding pattern are superimposed and no differences are observed after 5400 time steps for a $dt = 0.05$ (Fig. 2.7). If the lithology pattern is the same after the same number of time steps and for the same boundary conditions, it is clear that the velocity field evolution must be the same.

From this and from the fact that the 2D code matches the linear theory for small deformations, it is accepted that the pTatin 3D code is able to capture accurately folding processes. It should be pointed out that a fully 3D test could be done by confronting pTatin with the 3D linear theory of folding from Fletcher [Fletcher, 1991] using non cylindrical perturbations. The Large amplitude folding model of Adamuszek [Adamuszek et al., 2013] in the case of cylindrical folding could be also used to test the code for large deformations. To be within the time limit of this work, we will take this simple comparison as a sufficient proof of the ability of the pTatin code to model folding.
Figure 2.4: Comparison of a pure-FEM simulation with a MAC-FEM simulation. The red folded viscous layer is obtained with a pure-FEM code whereas the black points are markers of high viscous material outlining the viscous layer in the case of MAC-FEM simulation (the markers are downsampled for the fine grid). Simulation parameters: $dt = 0.05$, $R = 400/1$, $L_x = 50$, $L_y = 9$, $H = 3$, $v_B = 0.05$. 

Coarse grid (50 x 9) Fine grid (200 x 36)
Figure 2.5: Pure-FEM simulation at two different time steps. The constant part $p_0$ of the pressure field is represented at the finite element cell for two mesh resolutions. Simulation parameters: $dt = 0.05$, $R = 400/1$, $L_x = 50$, $L_y = 9$, $H = 3$, $v_B = 0.05$. 
Figure 2.6: MAC-FEM simulation at two different time steps. The constant part $p_0$ of the pressure field is represented at the finite element cell for two mesh resolutions. The black points represent the matrix material markers and they outline the viscous layer folded shape (the markers are down-sampled for the fine grid). Simulation parameters: $dt = 0.05$, $R = 400/1$, $L_x = 50$, $L_y = 9$, $H = 3$, $v_B = 0.05$. 
Figure 2.7: Comparison between the 2D code (coloured wireframe) and 3D code (transparent coloured surface) after 5400 time steps. Qualitatively, no differences are observed neither at the free surface nor at the viscous interfaces. Simulation parameters: $dt = 0.05$, $R = 400/1$, $L_x = 50$, $L_y = 9$, $H = 3$ $v_B = 0.05$. 
Chapter 3

3D linear theory of folding of a viscous layer embedded in a matrix

The linear theory of folding of viscous layer, developed mainly by Biot and later Fletcher [Biot, 1961, Fletcher, 1977, Fletcher, 1991], gives an infinitesimal-amplitude analytical solution for folding with a sinusoidal waveform perturbation. We present here the three-dimensional analytical solution for folding of an embedded viscous layer in pure shear. The solution has been published for the first time by [Fletcher, 1991] where he considers periodic perturbations which can be projected on a Fourier base. Here new results for the case of an infinite infinite $x - y$ plane for local perturbations are presented. Whilst the scenario considered was slightly different to that of [Fletcher, 1991], the conclusions obtained are the same; namely that the initial growth rate and its associated dominant wavelength are independent of the form of the initial perturbation.

3.1 Preliminary considerations

The onset of the problem are the Stokes equations describing the dynamic of each isoviscous layer individually. In this case we consider the Stokes equation for an isotropic linear viscous material without body forces (Eq. 3.1). If one wants to tackle the problem as a whole, it must consider a heterogeneous viscosity and include the viscosity tensor behind the divergence
operator of (Eq. 3.1). The method to decouple the dynamics in each layer allows to overcome this difficulty. In this way each layer is taken as being iso-viscous.

\[
\begin{align*}
\begin{cases}
\eta \nabla^2 \mathbf{u} - \nabla p &= 0 \\
\nabla \cdot \mathbf{u} &= 0.
\end{cases}
\end{align*}
\] (3.1)

From this equation we see that the pressure is a harmonic function: \( \nabla^2 p = 0 \). In the case of viscous layered media, one can solve (Eq. 3.1) in each isoviscous layers and by applying the interface continuity conditions between the layers. Actually, the general solution of (Eq. 3.1) within some constants is found in the case of a single perturbed layer in compression. The constants for the different viscous layers are then determined by applying the interface continuity conditions between the layers. Within this framework, we prescribe that the velocity field \( \mathbf{u} \), as well as the traction vector field \( \mathbf{T} = n^t \sigma \) should be continuous at the interface between two viscous layers.

Suppose an upper and a lower isoviscous medium, indexed by the prime and second superscript, embedding a third isoviscous medium (no superscript) with a different viscosity (see Fig. 3.1). For the reason of simplification, it supposed that the physical properties and the geometry of the upper and lower matrix are identical such that the problem is symmetric. The continuity of the velocity and traction fields are given by

\[
\begin{align*}
\begin{cases}
\mathbf{u} - \mathbf{u}' \bigg|_{\xi'} &= 0, \\
\mathbf{u} - \mathbf{u}'' \bigg|_{\xi''} &= 0, \\
n^t(\sigma - \sigma') \bigg|_{\xi'} &= 0, \\
n^t(\sigma - \sigma'') \bigg|_{\xi''} &= 0,
\end{cases}
\end{align*}
\] (3.2)

where the two interfaces are parametrized by the function graphs \( \xi' = (x, y, h + \delta \xi(x, y))^t \) and \( \xi'' = (x, y, -h + \delta \xi(x, y))^t \). From the symmetry of the problem, we can consider only the interface between the upper medium and the middle one. The normal vector at the interface is given by

\[
n = \frac{\partial_x \xi' \times \partial_y \xi'}{\left| \partial_x \xi' \times \partial_y \xi' \right|} = (-\partial_x \xi e_x - \partial_y \xi e_y + e_z)N, \quad N = ((\partial_x \xi)^2 + (\partial_y \xi)^2 + 1)^{-1/2},
\] (3.3)
and the traction vector is expressed by

\[
T = \begin{pmatrix}
n_x \sigma_{xx} + n_y \sigma_{yx} + n_z \sigma_{zx} \\
n_x \sigma_{xy} + n_y \sigma_{yy} + n_z \sigma_{zy} \\
n_x \sigma_{xz} + n_y \sigma_{yz} + n_z \sigma_{zz}
\end{pmatrix}
= \begin{pmatrix}
-\sigma_{xx} \delta \xi - \sigma_{yx} \delta \eta + \sigma_{zx} \\
-\sigma_{xy} \delta \xi - \sigma_{yy} \delta \eta + \sigma_{zy} \\
-\sigma_{xz} \delta \xi - \sigma_{yz} \delta \eta + \sigma_{zz}
\end{pmatrix}
\]  \hspace{1cm} (3.4)

The equations (3.3) and (3.4) are non-linear in terms of the slope of the interface along the x and y axis. The first order solution consist in linearizing these equations.
3.2 The solution of the Stokes problem in viscous layered media undergoing pure shear

We consider the folding process of the viscous layer embedded in a low viscous matrix extended to infinity. During the initial phase of folding, the flow can be decomposed into a uniform shortening flow and a superimposed perturbed flow. The former corresponds to the mean flow obtained without any perturbed layer; i.e. in pure shear. The secondary flow results from the perturbed interface between two layers of different viscosity. The mechanical instability gives rise to the amplification of the initial perturbation yielding to folding. We present here the solution for an unperturbed interface followed by the linear solution for a perturbed interface.

3.2.1 Pure shear solution of non-perturbed embedded isotropic linear viscous layer

We first want the solution in the case of an unperturbed horizontal layer subjected to constant strain rate along the \( x \) and \( y \) axis. Under the incompressibility assumption, we can express the vertical strain rate as a function of the two other directions:

\[
\bar{\epsilon}_{zz} = -(\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy}).
\]  

(3.5)

The solution for the velocity is then given by

\[
\bar{u}_x = x\bar{\epsilon}_{xx}, \quad \bar{u}_y = y\bar{\epsilon}_{yy}, \quad \bar{u}_z = z\bar{\epsilon}_{zz}.
\]  

(3.6)

It is interesting to note that in the case of unperturbed interface, the flow is irrotational. The stress tensor is given by

\[
\begin{align*}
\bar{\sigma}_{xx} &= 2\eta\bar{\epsilon}_{xx} - \bar{p}, \\
\bar{\sigma}_{yy} &= 2\eta\bar{\epsilon}_{yy} - \bar{p}, \\
\bar{\sigma}_{zz} &= 2\eta\bar{\epsilon}_{zz} - \bar{p}, \\
\bar{\sigma}_{xy} &= \bar{\sigma}_{yx} = \bar{\sigma}_{xz} = \bar{\sigma}_{zy} = 0 \\
\bar{p} &= \bar{p}_0.
\end{align*}
\]  

(3.7)

Finally the continuity of the traction vector at the interface yields

\[
\bar{p}' - \bar{p} = 2(\eta - \eta')(\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy}).
\]  

(3.8)

This flow derives from a scalar potential \( \bar{\phi} = \frac{1}{2}(\bar{\epsilon}_{xx}x^2 + \bar{\epsilon}_{yy}y^2 + \bar{\epsilon}_{zz}z^2) \) such that \( \bar{u} = \nabla \bar{\phi} \).
3.2.2 Pure shear solution of perturbed embedded isotropic linear viscous layer

The starting point for the analytical solution closely follows Fletcher [Fletcher, 1974, Fletcher, 1977, Fletcher, 1991, Johnson, 1994]. The idea is quite common in fluid instability theory where the solutions are sought by a linearisation of the Navier-Stokes equations. Here, we deal with the linear Stokes equations and the linearisation comes into play at the perturbed interface between two viscous layers when one wants to apply the interface non-linear continuity conditions (Eqs. 3.3, 3.4). The solution is expressed in terms of the sum of the solution of the unperturbed interface (linear solution $\bar{u}$) and the perturbed solution of first order ($\tilde{u}$):

$$u = \bar{u} + \delta \tilde{u}. \quad (3.9)$$

Only the lower order terms of $O(\delta)$ are considered. Higher order terms are neglected. The interface is given by the expression

$$z|_{\xi'} = h + \delta \xi(x, y) \quad (3.10)$$

and the non-deviatoric stress tensor by

$$\sigma = \bar{\sigma} + \delta \tilde{\sigma} \quad (3.11)$$

and the pressure

$$p = -\frac{1}{3} \sigma_{kk} = \bar{p} + \delta \tilde{p}. \quad (3.12)$$

In [Fletcher, 1991] the solution for a periodic perturbation in a bounded domain is considered but here we consider the solution of an unbounded domain with punctual perturbations vanishing at infinity. In this case, the perturbed velocity vanishes at infinity and can be decomposed into a curl-free component and a divergence-free component according to the Helmholtz decomposition theorem [Pujol, 2003]. Without any perturbations we have seen that the linear flow is irrotational and thus we can express the solution in terms of a scalar potential $\bar{\phi}$. The perturbation of the interface introduces a rotational component in the solution of the velocity which is crucial for the folding process. We suppose that the perturbed flow is purely rotational and hence can be written in the form of (Eq. 3.13) where $\Psi$ is a potential vector defined with respect to a gradient.

$$\tilde{u} = \nabla \times \Psi = \nabla \times \Psi^P + \nabla \times \Psi^T, \quad (3.13)$$
where $\Psi^P$ and $\Psi^T$ are respectively the poloidal and toroidal potential vector component of the velocity field (see the next sub-sections).

**The poloidal flow**

A priori, when one applies a constant strain in only one direction, let’s say along the $x$-axis, we expect that for a punctual perturbation, the vorticity $z$-component is negligible at first order. We will see later that the vertical component of the vorticity is zero in the case of cylindrical perturbations (symmetry axis normal to the main strained direction) but it is generally not true for the other cases. The poloidal flow is the solution for which the vorticity vanishes in the vertical direction. We can choose $\Psi^P$ to be solenoidal so it exists a potential vector $M$ such that $\Psi^P = \nabla \times M$. If one choose $M = F(x, y, z)e_z$. The development and idea here is taken from [Selvadurai, 2000] and we find:

\[
\begin{align*}
\Psi^P &= \partial_y F e_x - \partial_x F e_y \\
\tilde{u}^p &= \partial_{zz} F e_x + \partial_{yy} F e_y - (\partial_{xx} + \partial_{yy}) F e_z \\
p^p &= \eta\partial_z \nabla^2 F \\
\nabla \times \tilde{u}^p &= -(\partial_{xzy} + \partial_{zyz} + \partial_{yyz}) F e_x + (\partial_{xyy} + \partial_{zzz} + \partial_{zzx}) F e_y.
\end{align*}
\]

It is a poloidal flow with no spinning about the vertical axis: $\omega_z = (\nabla \times u)_z = 0$. Computing $\nabla \times \nabla^2 \tilde{u}^p = 0$ one can show that $F$ is a biharmonic function, solution of (Eq. 3.15) [Selvadurai, 2000], i.e

\[
\nabla^4 F := (\partial_{xxxx} + \partial_{yyyy} + \partial_{zzzz} + 2\partial_{xxyy} + 2\partial_{xxzz} + 2\partial_{yyyy}) F = 0. \tag{3.15}
\]

The classical way to solve this equation is the separation of variables. We are looking solutions of the form $F(x, y, z) = \Phi^p(x, y) W(z)$ which is motivated by the symmetry of the problem. In order to be able to isolate a partial differential equation for $W(z)$, we transform the partial derivative in term of the $x$ and $y$ variable to an algebraic form through Fourier transforms. The Fourier decomposition of $\Phi^p(x, y)$ is given by (Eq. 3.16)

\[
\Phi^p(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Phi}^p(k_x, k_y) e^{-i(xk_x + yk_y)}.
\]

Inserting (Eq. 3.16) into (Eq. 3.15) yields

\[
W^{(IV)} - 2k^2 W^{(II)} + k^4 W = 0, \tag{3.17}
\]
where \( k := (k_x^2 + k_y^2)^{1/2} \) and because \( k_x^2 + k_y^2 \geq 0, k \in \mathbb{R}^{0+} \). One form of the general solution of (Eq. 3.17) is

\[
W(z) = (A + Bz)e^{kz} + (C + Dz)e^{-kz},
\]

where \( A, B, C, D \) depend on \( k_x \) and \( k_y \) and are determined through the interface velocity and stress boundary conditions.

If we take advantage of the symmetry and choose the origin of the coordinate system for the viscous layer in the mean plane of the layer [Fletcher, 1974, Fletcher, 1991, Johnson, 1994], the velocity should satisfy \( \tilde{u}_z(x, y, z) = \tilde{u}_z(x, y, -z) \) which implies \( A = C \) and \( C = -D \). In the upper medium, the velocity has to vanish when \( z \to \infty \) which implies that \( A' = B' = 0 \) and yields that

\[
\begin{align*}
\hat{F}(k_x, k_y, z) &= 2\hat{\Phi}^{P}(A \cosh(kz) + zB \sinh(kz)) \\
\hat{F}'(k_x, k_y, z) &= \hat{\Phi}^{P}(C' + D'z)e^{-kz}.
\end{align*}
\]

The problem is overdetermined because there are four unknowns \( (A, B, C', D') \) for six interface continuity conditions. This problem is overcame in [Fletcher, 1991] by superimposing a toroidal flow having a non-zero vorticity about the \( z \)-axis. This introduces two additional constants which, with the remaining constants closes the system.

**The toroidal flow**

The simplest way to introduce an additional flow with a non-zero vertical vorticity component is to consider a \( x-y \) planar flow defined by the stream line function \( \Psi^T \) which depends on \( x, y \) and \( z \), e.g.

\[
\Psi^T = \Psi^T e_z
\]

This yields

\[
\begin{align*}
u_x^T &= \partial_y \Psi^T, & v_y^T &= -\partial_x \Psi^T, & \omega_z &= -(\partial_{xx} + \partial_{yy}) \Psi^T
\end{align*}
\]

With such flow, the continuity equation is satisfied and if one requires the \( z \)-dependence and \( x, y \)-dependence for \( \Psi^T = \Phi^T(x, y)W^T(z) \), inserting (Eq. 3.21) in (Eq. 3.14) we obtain the similar results obtained by Fletcher [Fletcher, 1991]:

\[
\hat{\Psi}^T = (Me^{kz} + Ne^{-kz})\hat{\Phi}^T
\]
For the a given initial perturbation, the component of the vorticity about the $z$-axis has to be symmetric, so the $\omega_z$ must be an even function of the $z$ variable, and hence $M = N$ in the viscous layer. The vorticity has also to cancel when $z \to \infty$ then $M' = 0$.

It results in six unknowns $A, B, C', D', M, N'$ and six equations. The system is now well-determined and can be solved. For the reason of symmetry, we will chose $\hat{\Phi}^T \propto \hat{\Phi}^P$ where the proportionality factor will depend on $k$. This dependency is absorbed into the unknown variables such that we can choose, $\hat{\Phi}^T = \hat{\Phi}^P$.

\[
\begin{align*}
\hat{\psi}^T(k_x, k_y, z) &= -2k^2 M \cosh(kz) \hat{\Phi}^P,
\hat{\psi}'^T(k_x, k_y, z) &= -k^2 N' e^{-kz} \hat{\Phi}^P.
\end{align*}
\] (3.23)

The resulting velocity field for the viscous layer is then

\[
\begin{align*}
\hat{u}_x &= 2i \hat{\Phi}^P[-k_x(Ak \cosh(kz) + zBk \cosh(kz) + B \sinh(kz)) + Mk^2 k_y \cosh(kz)], \\
\hat{u}_y &= 2i \hat{\Phi}^P[-k_y(Ak \cosh(kz) + zBk \cosh(kz) + B \sinh(kz)) - Mk^2 k_x \cosh(kz)], \\
\hat{u}_z &= 2k^2 \hat{\Phi}^P(A \cosh(kz) + zB \sinh(kz))
\end{align*}
\] (3.24)

and for the upper medium,

\[
\begin{align*}
\hat{u}'_x &= i \hat{\Phi}^P[-k_x(D' - k(C' + zD')) + N' k^2 k_y e^{-kz}], \\
\hat{u}'_y &= i \hat{\Phi}^P[-k_y(D' - k(C' + zD')) - N' k^2 k_x e^{-kz}], \\
\hat{u}'_z &= k^2 \hat{\Phi}^P(C' + zD') e^{-kz}.
\end{align*}
\] (3.25)

**Linearization of the interface continuity conditions**

At the interfaces, the continuity conditions given by (3.2) have to be satisfied. From [Fletcher, 1977, Johnson, 1994] we know that at the interface, at the first order, those conditions can be given by $\hat{u}(k_x, k_y, h + \delta \xi) \approx \hat{u}(k_x, k_y, h)^1$. But if now, for the upper medium, we change the origin of the coordinate system to lay in the middle plane of the interface, the velocity field for the upper medium is given to first order by $\hat{u}'(k_x, k_y, \delta \xi) \approx \hat{u}'(k_x, k_y, 0)$. This yields

---

1. This is obtained by using the Taylor expansion of $u$ around $z = h$ and eliminating the terms of the order greater than one in $\delta$. 
\begin{equation}
\begin{aligned}
\hat{u}_x(k_x, k_y, h) - \hat{u}'_x(k_x, k_y, 0) &= 0, \\
\hat{u}_y(k_x, k_y, h) - \hat{u}'_y(k_x, k_y, 0) &= 0, \\
\hat{u}_z(k_x, k_y, h) - \hat{u}'_z(k_x, k_y, 0) &= 0.
\end{aligned}
\tag{3.26}
\end{equation}

The traction vector to the first order in \(\delta\) is given by

\[ T \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -\bar{\sigma}_{xx}\partial_x\xi + \bar{\sigma}_{xx} \\ -\bar{\sigma}_{yy}\partial_y\xi + \bar{\sigma}_{zy} \end{pmatrix}. \tag{3.27} \]

In the \(k_x-k_y\) Fourier space it gives the following conditions

\begin{equation}
\begin{aligned}
\bar{\sigma}_{xx}(k_x, k_y, h) - \bar{\sigma}'_{xx}(k_x, k_y, 0) &= -ik_x2\eta'(1 - R)(2\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy})\hat{\xi}, \\
\bar{\sigma}_{zy}(k_x, k_y, h) - \bar{\sigma}'_{zy}(k_x, k_y, 0) &= -ik_y2\eta'(1 - R)(2\bar{\epsilon}_{yy} + \bar{\epsilon}_{xx})\hat{\xi}, \\
\bar{\sigma}_{zz}(k_x, k_y, h) - \bar{\sigma}'_{zz}(k_x, k_y, h) &= 0,
\end{aligned}
\tag{3.28}
\end{equation}

where we used the fact that \(-\bar{\rho} + \bar{\rho}' = 2(\eta - \eta')(\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy})\) from (Eq. 3.8)

**Solution of the closed system**

The six equations (Eqs. 3.26, 3.28) constitute a well-determined system for the variables \(A, B, C', D', M, N'\). The linear system to solve is given in (Eq. 3.29).

\[
\begin{pmatrix}
-2kk_xS & -4k_x(S + hS) & kk_x & 0 & 2k_yC & -k_y \\
-2kk_yS & -4k_y(S + hS) & kk_y & 0 & 2k_xC & -k_x \\
2k^2C & 4hk^2S & k^2 & k^2 & 0 & 0 \\
4Rk^3C & 8Rhk^3C & -2k^3 & -2k^3 & 0 & 0 \\
-4Rk^3k_xC & -8Rkk_x(C + hkS) & -2k^2k_x & 0 & 2Rkk_yS & kk_y \\
-4Rk^3k_yC & -8Rkk_y(C + hkS) & -2k^2k_y & 0 & -2Rkk_xS & kk_x
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C' \\
D' \\
M \\
N'
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
2k_x(R - 1)(2\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy})\hat{\xi}/\hat{\Phi}^P \\
2k_y(R - 1)(2\bar{\epsilon}_{yy} + \bar{\epsilon}_{xx})\hat{\xi}/\hat{\Phi}^P
\end{pmatrix},
\tag{3.29}
\]

where \(S := \sinh(hk)\) and \(C := \cosh(hk)\) and the perturbation is taken in the Fourier space : \(\hat{\xi}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\xi}(k_x, k_y) e^{-i(k_x x + k_y y)}\). If we make the ansatz that \(\hat{\Phi}^P = \hat{\xi}\), the system is easily solved by inverting the matrix with symbolic calculation. It yields the following solutions

\[
\begin{pmatrix}
A \\
B \\
C' \\
D' \\
M \\
N'
\end{pmatrix}
= 
\begin{pmatrix}
-2h(R - 1)(S + RC)E/(2kD) \\
2(R - 1)(C + RS)E/(4kD) \\
2(R - 1)(RC^2 + SC - R + hk)E/(k^2D) \\
-2(R - 1)(RC^2 + SC - R + hk(1 - R))E/(k^2D) \\
k_xk_y(\bar{\epsilon}_{xx} - \bar{\epsilon}_{yy})(R - 1)/(k^3(C + RS)) \\
2Ck_xk_y(\bar{\epsilon}_{xx} - \bar{\epsilon}_{yy})(R - 1)/(k^3(C + RS))
\end{pmatrix},
\tag{3.30}
\]
where \( E := \bar{\epsilon}_{xx}((k_x/k)^2 + 1) + \bar{\epsilon}_{yy}((k_y/k)^2 + 1) \) and \( D := -\sinh(2hk)(R^2 + 1) + 2hk(R^2 - 1) - 2R \cosh(2hk) \). The poloidal and toroidal potentials depend on the initial interface perturbation through the fact that the perturbation scalar function verifies \( \hat{\xi} = \hat{\Phi}^P = \hat{\Phi}^T \).

**Discussion**

We have seen that the toroidal flow is totally decoupled from the poloidal flow and it is required to get the general solution. The toroidal flow introduces a non-zero vorticity component about the \( z \)-axis and this quantity is given by (Eq.3.31).

\[
\omega_z = - (\partial_{xx} + \partial_{yy})\Psi^T.
\] (3.31)

If \( \omega_z \) vanishes, the toroidal flow vanishes as well. From equation (3.30), the only case of pure poloidal flow happens when \( \bar{\epsilon}_{xx} = \bar{\epsilon}_{yy} \). This means that the toroidal flow takes it source from the background differential strain rate \( \bar{\epsilon}_{xx} - \bar{\epsilon}_{yy} \) along the \( x \) and \( y \) axis. From the numerical simulations presented in chapter 4, it is observed that the vorticity flow in the case of buckling dominantly lays in the \( x-y \) plane. We are hence interested in the quantity (3.32).

\[
(\hat{\omega}_x^2 + \hat{\omega}_y^2)^{1/2} = |4\delta \hat{\xi}k(C + RS)(R - 1)E/D|.
\] (3.32)

In the Fourier space, the directional variation of this quantity is ruled by \( E \) i.e. the background strain rate. It cancels out when \( k = 0 \) and when \( k \to \infty \). In between, this function reaches only one maximum in all the directions of the Fourier space and it means there is at least one mode for which, the plane vorticity is maximal.

### 3.2.3 The perturbation growth rate

The interface dynamic equation is given by many authors [Fletcher, 1977] and is expressed by

\[
\frac{D}{Dt} (z - \xi') = 0.
\] (3.33)

Let’s suppose that the perturbation is controlled and scaled by two characteristic lengths \( \lambda^{-1}_x \) and \( \lambda^{-1}_y \) in both directions of the \( x-y \) plane such that \( \xi = \xi(x\lambda_x, y\lambda_y) \). To the first order in \( \delta \) at the interface this yields

\[
\partial_t \xi' = \partial_t h + \xi \partial_t \delta + \delta \partial_t (\lambda_x) \partial_{\lambda_x} \xi + \delta \partial_t (\lambda_y) \partial_{\lambda_y} \xi
= \bar{u}_z + \delta \bar{u}_z + \bar{u}_x \delta \partial_x \xi + \bar{u}_y \delta \partial_y \xi,
\] (3.34)
however

\[
\begin{align*}
\partial_t h &= h \epsilon_{zz}, \\
\partial_t \lambda_x &= \bar{\epsilon}_{xx} \lambda_x, \\
\partial_t \lambda_y &= \bar{\epsilon}_{yy} \lambda_y, \\
\bar{u}_z\big|_\xi &= (h + \delta \xi) \bar{\epsilon}_{zz}, \\
\bar{u}_x\big|_\xi &= x \bar{\epsilon}_{xx}, \\
\bar{u}_y\big|_\xi &= y \bar{\epsilon}_{yy}, \\
\end{align*}
\]

(3.35) (3.36) (3.37) (3.38) (3.39) (3.40)

which leads to

\[
\begin{align*}
\partial_t \delta &= \frac{(\bar{\epsilon}_{zz} \xi + \bar{u}_z) \delta}{\xi} + \frac{(x \bar{\epsilon}_{xx} \delta \partial_x \xi - \bar{\epsilon}_{xx} \lambda_x \delta \partial_{\lambda_x} \xi)}{\xi} \\
& \\
& + \frac{(y \bar{\epsilon}_{yy} \delta \partial_y \xi - \bar{\epsilon}_{yy} \lambda_y \delta \partial_{\lambda_y} \xi)}{\xi}.
\end{align*}
\]

(3.41)

Note the two last terms vanish because \(\xi = \xi(x \lambda_x, y \lambda_y)\) and yields to the final expression (which can be taken in the \(x-y\) Fourier space for the velocity) given by

\[
\left(\frac{\partial \delta}{\partial t}\right) / \delta = \bar{\epsilon}_{zz} + \bar{u}_z / \xi
\]

\[
= -\bar{\epsilon}_{xx} - \bar{\epsilon}_{yy} - E / D',
\]

(3.42)

where we define \(D' := 2hkR(1-R)/\{2hk(R^2 - 1) - [\sinh(2hk)(R^2 + 1) + 2R \cosh(2hk)]\}\).

Discussion

We have defined \(k = (k_x^2 + k_y^2)^{1/2} \geq 0\) and it follows that \(D'\) reaches always a maximum which corresponds to the dominant wavenumber in each direction of the \(x-y\) plane when \(\bar{\epsilon}_{yy} = \bar{\epsilon}_{xx}\) because \(E = 3\) (Fig. 3.2-A). In the other cases, the growth rate is modulated by \(E\) which gives the relative importance of the axial modes \(k_x\) and \(k_y\) weighted by the axial strain rates \(\bar{\epsilon}_{xx}\) and \(\bar{\epsilon}_{yy}\).

Some remarkable cases can be distinguished:

1. In compression, the axial strain rates are negative and it results that \(E\) is always negative and it results a positive growth rate for all modes (Fig. 3.2-A,E).
2. If we compress along the $x$-axis and stretch along the $y$-axis, $E$ can be positive for $\frac{\bar{\epsilon}_{yy}}{\bar{\epsilon}_{xx}} > k_y^2 + k_x^2$ and we a have a negative growth rate for these modes (Fig. 3.2-C). However by stretching along the $y$ axis a positive growth rate is still possible along this direction (Fig. 3.2-D).

3. If $\bar{\epsilon}_{yy} = 0$, there is still a dominant mode along the $y$-axis but it is overshadowed by the dominant mode along the $x$-axis (Fig. 3.2-B).

Back to 2D

By setting $k_y = 0$ and $\bar{\epsilon}_{yy} = 0$ we retrieve the results for 2D folding given by Biot and Fletcher and used in (Eq. 2.14) and (Eq. 2.15) for the numerical tests.
Figure 3.2: Growth rate variation according to the directions $k_x$ and $k_y$ for different strain states. The explanations are given in the text. $R = 10$, $H = 1.8$.
3.2.4 Some remarks on the fold initiation

The continuity of the tangential stress vector at the interface furnishes the follow two equations which are meaningful to understand the fold initiation.

\[
\begin{align*}
(\tilde{\sigma}_{zx} - \tilde{\sigma}'_{zx})|_{\xi'} &= 2\eta'(1 - R)(2\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy})\partial_x \xi \\
(\tilde{\sigma}_{zy} - \tilde{\sigma}'_{zy})|_{\xi'} &= 2\eta'(1 - R)(2\bar{\epsilon}_{yy} + \bar{\epsilon}_{xx})\partial_y \xi,
\end{align*}
\]

which are meaningful to understand the fold initiation. If we consider a major compression along the \(x\) axis such that \(2\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy} \neq 0\) and \(2\bar{\epsilon}_{xx} + \bar{\epsilon}_{yy} \neq 0\), the stress along the \(x\) and \(y\) axis in the plane normal to \(z\) is discontinuous if the slope of the interface respectively along the \(x\) and \(y\) direction is non-zero.

![Figure 3.3: Representation of an arbitrary volume undergoing a differential shear stress across the interface.](image)

At the points of the interface where the interface slope doesn’t vanish, a mechanical instability will develop through the difference of shear stress \(\Delta \tilde{\sigma}_{xx}\) and \(\Delta \tilde{\sigma}_{zy}\) between the two adjacent phases. It results the creation of vorticity along the inter-phase line initiating the rotation of the material in the sense given by the sign of the slope (Fig. 3.3). In the case of random perturbation, these centres of vorticity coalesce later to form a coherent pattern.
Explained in an other manner: taking an arbitrary volume at the interface (Fig. 3.3), we see that the shear force applied at the upper edge would be different from the force applied to the lower edge due to the discontinuity of $\sigma_{xx}$ across the interface. This makes this representative volume rotate in a sense depending on the interface slope.
Chapter 4

Numerical results and interpretation

In nature, we observe different fold patterns emerging from the complexity of the terrain, the variations in lithology and the regional and local stresses. It has been shown that the viscosity ratio $R$, the layer thickness $H$ and the background strain rate $\bar{\varepsilon}_{xx}/\bar{\varepsilon}_{yy}$ control the dominant amplification wavelength and the growth rate in the case of a single viscous layer. In this chapter, the folding process is analysed taking advantage of the vorticity flow point of view, which allows a clearer understanding of the linkage process. It is shown that the vorticity flow controls the fold linkage through a characteristic length being the planar-vorticity dominant wavelength, different from the fold amplification wavelength.

4.1 The lateral fold interplay

The majority of the investigations on folding processes have been performed in 2D, supposing no variation in shape along the third direction (cylindrical folds). From natural observations, it is clear that, folds are not absolutely cylindrical and can have a curved hinge line, in some cases, due to the folds lateral growth [Schmid et al., 2008]. However, the lateral fold growth has not been extensively studied before, and we note the initial work of Grasemann and Schmalholz [Bretis et al., 2011, Grasemann and Schmalholz, 2012] where they consider two local or punctual perturbations, located laterally opposite one another and growing laterally. Depending on the hinge
line separation, they observe linkage or no-linkage folds.

Here, it is proposed to investigate the case of one local perturbation growing laterally and interacting with two other opposite embryonic folds. The different observed patterns are described and explained in terms of the vorticity flow. The study of the vorticity flow led us to reconsider the simple-linkage analysis of [Grasemann and Schmalholz, 2012] and introduce a new terminology to describe this phenomena.

4.1.1 Geometric setting

We consider the geometry of a single linear isoviscous layer embedded in a matrix like in Fig. 3.1. In a horizontal uni-axial compression regime (say in x-direction) it is observed that an initial local punctual perturbation amplifies and forms an embryonic fold. With increased strain, a fold train develops outwards from the instability parallel to the main compression direction with a characteristic wavelength given by the dominant wavelength of the viscous layer.

At the same time, the perturbation and the resulting fold train grows laterally, normal to the main compression direction: it is an axis parallel elongation. The fold axis, and the hinge line are oriented in this same direction. If two nearby perturbations have a close hinge line, they can interact with each other. The embryonic folds can be deviated, linked, merged and/or locked in their lateral growth (see Figs. 4.2, 4.3) depending on the surrounding deformation state.

All parameters are dimensionless and we consider three characteristic scales: a characteristic length, a characteristic viscosity and a characteristic stain rate. As mentioned before, we are interested in the lateral interaction of a solitary local perturbation with an opposite binary perturbation consisting of two similar perturbations as illustrated in Fig. 4.1. We use a Gaussian perturbation with a characteristic standard deviation $\sigma$ and a maximum height of 10 % of the viscous layer thickness $H$. The solitary perturbation has a standard deviation $\sigma_1$ and the binary perturbation has a standard deviation of $\sigma_2$. The upper and the lower interface of the viscous layer are perturbed in a parallel way.
The domain undergoes a constant background strain rate given by \( \bar{\epsilon}_{xx} = -0.1 \) and \( \bar{\epsilon}_{yy} = 0.0 \). Its size is given by \( L_x \times L_y \times L_z \). In the binary perturbation, the two embryonic folds are separated by a distance \( 2L \) proportional to \( \sigma_2 \) and the opposite solitary perturbation has a hinge shifted relative to the centre of the binary with an offset given by \( \delta \) (Fig. 4.1). The control parameters are the perturbation separation \( 2L \), the offset \( \delta \), the standard deviations \( \sigma_1 \) and \( \sigma_2 \). By varying these four parameters, different folding patterns emerged for which we will now explain in detail.

Figure 4.1: Geometric setting of the lateral fold interplay simulations. The upper image gives an example of the initial geometry with two punctual perturbations on one side and the single one to the other side. The lower image identifies the different geometrical parameters for an exaggerated initial perturbation. The topographic height is indicated by the colour contrasts.
Figure 4.2. Two different deformation styles of simple-linkage: linear-linkage (upper), oblique-linkage (lower). The grey ticks indicate the position of the initial perturbation. The domain undergoes a main compression about the $x$-axis whereas the velocity at the boundaries normal to the $y$-axis is kept to zero. The parameters $\sigma_1, \sigma_2, L$ and $\delta$ are geometric parameters defined in Fig. 4.1. The colours indicate the relative topography. Parameters of simulation: $R = 40, H = 0.05$. 

Almost linear-linkage and en-échelon linkage:

$\sigma_1 = 0.1, \sigma_2 = 0.05$
$L = 8\sigma_1, \delta = 0.1L$

En-échelon-linkage:

$\sigma_1 = 0.1, \sigma_2 = 0.05$
$L = 16\sigma_1, \delta = 0.04L$
Figure 4.3: Four different deformation styles of triple-linkage: merged-embryonic fold (upper), fork-linkage (upper-middle), laterally locked anticline or anti-triple-linkage (lower-middle), laterally-locked (lower). The grey ticks indicate the position of the initial perturbation. The domain undergoes a main compression about the $x$ axis whereas the velocity at the boundaries normal to the $y$ axis is held to zero. The parameters $\sigma_1$, $\sigma_2$, $L$ and $\delta$ are geometric parameters defined in Fig. 4.1. The colours indicate the relative topography. Parameters of simulation: $R = 40$, $H = 0.05$. 

Merged-embryonic fold:
$\sigma_1 = 0.1$, $\sigma_2 = 0.025$
$L = 8\sigma_1$, $\delta = 0$

Fork-linkage:
$\sigma_1 = 0.1$, $\sigma_2 = 0.025$
$L = 16\sigma_1$, $\delta = 0$

Laterally locked anticline or anti-fork-linkage:
$\sigma_1 = 0.1$, $\sigma_2 = 0.05$
$L = 16\sigma_1$, $\delta = 0$

Linear-linkage and laterally-locked anticline:
$\sigma_1 = 0.1$, $\sigma_2 = 0.05$
$L = 8\sigma_1$, $\delta = 0.04L_2$
4.1.2 Simple linkage pattern

All linkages consisting in the interaction of only two embryonic fold hinges are grouped under the term simple linkage. It corresponds to the linkage class observed and described in the field by [Bretis et al., 2011] and investigated numerically in [Grasemann and Schmalholz, 2012]. For this class we used the same terminology listed here:

**linear-linkage** This linkage occurs when two separated embryonic folds are linearly aligned according to their fold axis normal to the direction of shortening. The folds join laterally to form a subcylindrical fold with a saddle at the location where the two folds have linked [Grasemann and Schmalholz, 2012] (see Fig. 4.2 upper panel).

**en-échelon/oblique-linkage** This linkage results when two separated embryonic folds are aligned en-échelon according to their fold axis normal to the direction of shortening. The two separated folds link through a curved saddle similar to the type II refold structure from Ramsay [Ramsay and Huber, 1987, Bretis et al., 2011] (see Fig. 4.2 lower panel).

**oblique- and linear-no-linkage** This pattern occurs when two parallel folds have a fold axis plunging in the opposite direction without linkage of the two embryonic folds. In this case, the primary folds link to the opposite fold train and in that sense it is also a linkage (later we will introduce the term of a first order linkage). The primary fold axis can be deviated and in this situation, we speak of oblique-no-linkage and of linear-linkage in the other (linear fold axis).
4.1.3 Triple-linkage pattern

All linkages consisting in the interaction of three embryonic fold hinges is grouped under the term **triple-linkage**. We introduce here three subclasses resulting in the different patterns observed in the numerical simulation.

**fork-linkage** This pattern occurs when one embryonic fold hinge join laterally two other opposite embryonic fold hinges forming a fork-like structure (see Fig. 4.3 upper middle panel). The fold axis of the initially solitary embryonic fold plunges in the opposite direction of the opposite binary embryonic folds. The junction region is larger than the remaining with a quite flat topography like a double hinge fold (Fig. 4.4). Two saddles corresponding to the two fold branches are also present. In the ideal case, the overall fork structure has an amphora-shape in the plan-view. The term anti-fork-linkage is used when we talk about synclinal connecting in a fork-manner.

**laterally-locked** This structures occur when the lateral growth of an embryonic fold is locked by the surrounding folds or deformation state. Generally, this fold is surrounded by two oblique linkages or an oblique and a linear linkage and it actually corresponds to a syncline triple-linkage around the locked anticline structure. (see Fig. 4.3 lower middle and lower panel).

**merged embryonic fold** This structures occurs when the binary perturbations are so close to each other that they form a unique or a double hinge structure (see Fig. 4.3 upper panel). The solitary fold can be linked to this wide structure as in the simple-linkage case. At the location of the binary perturbation, a very flat topography is often observed forming a double hinge fold (box fold).

In the case of wide enough binary perturbation, no triple linkage pattern is observed (neither anticline nor syncline triple-linkage) and it falls within the simple-linkage class (see Fig. 4.2).

The parameter study of the triple-linkage is presented in Section 4.2.2 after introducing the vorticity framework for folding.
Figure 4.4: Triple-linkage: slices of a fork-structure. At the junction, we observe clearly the double hinge fold section with a flat topography. The colours indicate the relative topography. The simulation parameters for this figure are: $\sigma_1 = 0.1$, $\sigma_2 = 0.025$, $L = 16\sigma_2$ and $\delta = 0$.

Triple linkage pattern in nature

Surprisingly no known works mention the observation of fork-like structure in nature. This could be because the concept of fold-linkage is quite young and that no such structures have been envisaged it before. As a reminder, the linkage process has been identified first in the Zagros mountains [Ramsey et al., 2008, Bretis et al., 2011] and a detailed study of this area considering the fork-linkage could furnish more natural examples. However, a potential candidate for such linkage in Switzerland could be the Mont-Chasseral between Neuchâtel and La Chaux-de-Fonds in the Jura belt-mountain (see Fig. 4.5). The Mont-Chasseral would correspond to the solitary fold of our model whereas the two anticlines embracing the Val de Ruz, would be the binary fold. At the Mont-Chasseral, a thrust is observed and could result in a later brittle deformation at the linkage point.
Figure 4.5: Map view of the Chasseral area between Neuchâtel and La Chaux-de-Fonds in the Jura mountain-belt in Switzerland. The mount Chasseral and its terminal branch embracing the Val de Ruz, is a potential candidate for a fork-like structure. The anticline hinge lines are indicated by the black dashed lines whereas the thrust fault is indicated by the red line. Map adapted from [federal office for Water and Geology, 2009]

It could also be argued that this structure is an anticline thrusting over a second anticline. It would not be a result of a fork-linkage process. A fieldwork there regarding the linkage possibility would help to support one idea rather the other.

4.2 The vorticity in folding

In the case of layered media undergoing parallel shortening to the bedding, it has been shown in Section 3.2.2 that the introduction of an interface perturbation results in a rotational flow superimposed upon the background irrotational flow. This rotational flow is responsible for the folding process: it amplifies the perturbation and develops the fold train emerging from the initial perturbation. Actually, the rotational flow dominates the folding process and it constitutes a key element to understand the folding pattern observed
in 3D models. Since the irrotational flow has zero vorticity, the simplest way to study the rotational flow out of the total flow, is to consider the vorticity flow. This makes the understanding of the fold linkage mechanism easier.

### 4.2.1 The vorticity flow pattern of a single local perturbation

In Section 3.2.4 it was demonstrated that if the gradient of the perturbed interface (or the slope) in the $x$ direction is negative, a positive vorticity along the $y$-axis results and vice versa (Fig. 4.6).

![Figure 4.6: Illustration of the non-zero vorticity flow in the case of perturbed interface. The opposite arrows show the shortening direction, the curved arrows show the movement of the viscous layer delimited by the blue lines. In this configuration, the perturbed interface geometry generates a positive vorticity along the $y$-axis.](image)

In three dimensions, considering a single Gaussian perturbation growing laterally, the vorticity pattern is illustrated in Figs. 4.7 and 4.8. When the fold limb is well developed, the vorticity flow is oriented along the $y$-axis. Its sense depends on the dip direction of the layer relative to the shortening direction (see Fig. 4.7 - d). The periclinal terminations are characterized by a vorticity with a main orientation in the $x$ direction with a sense depending on

---

1. One can write the velocity as $\mathbf{u} = \nabla \Phi + \nabla \times \Psi$ according to the Helmholtz decomposition where $\nabla \Phi$ represents the background shortening flow for small deformation and $\nabla \times \Psi$ the flow resulting from the layer perturbation. Taking the vorticity of $\mathbf{u}$, the background flow component is removed, only the flow due to the layer perturbations remains.
the axial lateral growth direction (see Fig. 4.7 - c). At the lateral termination front, the rotational flow makes the viscous layer spin about the $x$-axis which makes grow the fold laterally. This $x$-spinning is mainly accommodated by a $y$-spinning which increases the layer dip. When the fold is locked (a limb dip near $90^\circ$), the vorticity flow is almost zero (see Fig. 4.7 - b). In Fig. 4.7 - a, the upper part (north) of the central fold is locked and the fold tightness decreases to the lower part (south). The tightness is increased continuously through the vertical component of the vorticity (see Fig. 4.7 - e), which makes the main fold at the end cylindrical (when the fold locks). Indeed, we observe that the secondary fold (train-folds) are laterally deviated from the $y$-axis (see Fig. 4.7 - a). By increasing the shortening, the vertical vorticity will make this secondary folds to become cylindrical and parallel to the $y$-axis.
Figure 4.7: Representation of the vorticity vector for a single perturbation growing laterally. The colours indicate the different component of the vorticity vector and are taken at the upper viscosity interface for the plan-views. a) the topographic height - b) the vorticity magnitude - c) the vorticity along the $x$-axis - d) the vorticity along the $y$-axis and e) the vorticity in $z$-direction.
Figure 4.8: Representation of the vorticity vector for the same configuration given in (Fig. 4.7). The colours indicate the different component of the vorticity vector and are taken at the upper viscosity interface for the plan-views. a) colours representing the $x$-component of the vorticity - b) colours representing the $y$-component of the vorticity and c) colours representing the $z$-component of the vorticity.
As a consequence, domes and basin-like patterns are delimited by a closed loops of vorticity. If the delimiting closed vorticity-line is anticlockwise oriented, it is a dome, if it is oriented clockwise, it is a basin-like structure (Fig. 4.8). As already mentioned by Johnson and Fletcher [Johnson, 1994], at the place where the initial perturbation has locally a zero spatial derivative i.e. is flat (zero-slope), it will remain flat with no-vorticity for a larger strain. This zero-vorticity line corresponds to the hinge line of the first perturbation in Fig. 4.7 - b. This band grows linearly normal to the shortening direction and can be deviated by an opposite growing embryonic folds through the linkage process. If there is no interaction with other folds, the hinge line remains straight and parallel to the $y$-axis.

The vorticity dominant wavelength

In Section 3.2.2 it has been noted that the quantity $(\omega_x^2 + \omega_y^2)^{1/2}/\delta\xi$ reaches a maximum in the Fourier space. There is then a dominant wavelength $\lambda_\omega$ for which, the planar vorticity is maximum, independent of the initial perturbation. This vorticity wavelength doesn’t correspond exactly to the fold amplification dominant wavelength $\lambda_d$ and their ratio $\lambda_\omega/\lambda_d$ asymptotically converges to a value close to 0.63 when $R$ goes to infinity (Fig. 4.9). The estimation of the dominant fold amplification wavelength and the dominant planar vorticity wavelength were determined numerically estimating the maximum in the Fourier-space of the analytical solutions (Eq. 3.42) and (Eq. 3.32) for the fold amplification and the planar vorticity respectively in the case of small perturbation.

In order to investigate the lateral fold interaction, the lateral front of the growing fold in the continuous media has to be defined. This can be done using the spatial variation of the vorticity. For example, the quantity $\partial_x \omega_y$ highlights the variation of $\omega_y$ along the $x$-axis. The resulting pattern of a single perturbation growing laterally is similar to the fingers of a hand or tongues representing the anticlines and synclines (see Fig. 4.10). The apparent periodicity of this tongues reflects the periodicity of $\omega_y$ along the $x$-axis and by the same time the periodicity of the vorticity loops. The distance between the tips of the tongues is found to be close to $\lambda_\omega$ and not $\lambda_d/2$ (see Fig. 4.11). The boundary where the quantity $\partial_x \omega_y$ drops to a value close to zero is taken as the definition of the growing lateral fold front. In Fig. 4.10,
Figure 4.9: Dominant wavelength for the fold amplification, the planar vorticity (3.32) obtained from the analytical solution maximum solved numerically. The theoretical fold amplification dominant mode in the case of the thin plate approximation (2.15) which is valid when $R^2 \gg 1$ is also plotted as a comparison. The ratio of $\lambda_\omega$ over $\lambda_d$ is given in the lower graph. The viscous layer thickness is $H = 0.05$. 
Figure 4.10: **Tongue pattern representing the absolute variation of the y-component of the vorticity along the x-axis for a single initial perturbation in the middle of the figure at y = 0.** The tongue represents the developing anticlines and synclines. Parameters of the simulation: $R = 400$, $H = 0.05$.


The dark blue area represents the region of zero vorticity. It is important to remark that near a second lateral perturbation (for example opposite to it), the distance between the tongues changes according to the interaction between the two vorticity flows present.

Two characteristic lengths can then control the linkage: one related to the vorticity flow and the other to the fold amplification.
4.2.2 The vorticity flow pattern and linkage

In the previous section it has been shown that the vorticity vectors generally lie in the plane of the layering and are aligned normal to the direction of shortening, i.e. parallel to the $y$-axis. We now consider two embryonic folds roughly aligned in the same direction and growing laterally towards each other. The vorticity flow can be schematically represented by oriented loops growing laterally; it is illustrated in Fig. 4.12 with the black arrows. The
median loop represents the initial embryonic folds (anticline) surrounded by smaller loops of the resulting train folds (only syncline-loops here).

Figure 4.12: Representation of dome-like structure vorticity flow by oriented loops in the case of a shortening given by the blue arrows. The red arrows show the connection or the deviation of the vorticity flow which results in a fold linkage of the initial embryonic anticline folds (zero order linkage), or the cross-linkage of the closer synclines of the initial embryonic anticline (first order linkage). The dashed lines represent the anticline hinge lines, whereas the grey lines represent the synclines.

If two opposite loops have more or less the same periclinal width, they can be connected or deviated according to their lateral alignment. Indeed, if there are two lateral vorticity flows (align to the y-axis) growing towards each other, but being oriented in the opposite sense, they cannot connect to each other and will be deviated. Using this representation, we can distinguish two situations (see Fig. 4.12):

- **zero order linkage** or direct-linkage: the two initial embryonic folds connect to each other and can form a curved or a linear hinge fold.
- **first order linkage** or cross-linkage: the two initial embryonic folds do not connect to each other but they are deviated and the first lateral opposite loops of the fold trains connect to each other. The situation is illustrated in Fig. 4.12. The opposite syncline loops bordering the initial embryonic folds connect to each other across the line connecting the
two initial perturbations, whereas the initial perturbation loops will be connected to the first lateral anticline of the fold train. This situation was classified as “no-linkage” by [Grasemann and Schmalholz, 2012], but in terms of vorticity it is indeed a linkage.

Considering now two embryonic folds growing laterally towards a third embryonic fold; if the solitary loop has a periclinal width which can embrace the lateral flow of the binary loop, it will result in a triple-linkage fork-like structure because the flows are oriented in the same sense and can be connected. The connection is illustrated in Fig. 4.12 by the two lateral red arrows. The curved red lines show the closure of the new central loop which is oriented clockwise, i.e. it represents a basin-like structure or a syncline.

The different flow pattern of the linkage structures mentioned above are given in Figs. 4.13 and 4.14. In the case of linkage, the lateral vorticity flows of two folds growing laterally towards each other link if they flow in the same sense at the connection point; it results in a linkage structure. If the lateral vorticity flows of two folds flow in the opposite sense in the region of a possible connection, they tend to be deviated and further away link with other compatible structures. The lateral growth could be also locked by other surrounding flows that cannot be bound; it is shown in Fig. 4.13.

The simple-linkage in the vorticity flow context

In Grasemann and Schmalholz (2012), they provided a criteria for the linkage according to the initial separation of the perturbations along the shortening direction. They found that there is linkage when the separation is smaller than value between 0.6 and 0.8. Actually, this initial perturbation separation is not particularly meaningful for the fold linkage because as soon as the lateral perturbation separation (normal to the shortening direction and along the y-axis for us) changes, the criteria for the linkage based on the initial perturbation separation parallel to the shortening direction changes too.

This is simply explained because the “radius of interaction” of the perturbations is finite and they grow laterally at a finite velocity depending on the mechanical properties of the viscous layer. Indeed, two opposite perturbations growing towards each other start to interact only when they are close
Figure 4.13: Two different vorticity flow patterns occurring in lateral simple-linkage: linear-linkage (upper), en-échelon-linkage/oblique-linkage (lower). The grey ticks indicate the position of the initial perturbation. The corresponding topographic patterns are given in (Fig. 4.2). The vorticity is given at the viscous interface. Parameters of simulation: $R = 40$, $H = 0.05$.

enough. But if their lateral separation changes (along $y$), it results that the time before they come into contact with each other is also affected. At the same time, their parallel separation when the perturbations start to interact has also changed because there is a compressive background stress field and the distance between the perturbations become closer with time. By fixing the initial separation along the $x$-axis and varying the lateral separation along the $y$-axis, it results that a transition of linkage to “no-linkage” or first order linkage will occur.

To define a robust criterion in terms of the growing perturbation separation along the shortening direction, the periclinal front has to be clearly defined along with the distance at which the two opposite fronts start to interact. The difficulty is that all the quantities considered here are continuous in space. It implies that if one wants to define a periclinal front with the vorticity field pattern, the gradient of the vorticity field, the topographic height or the mode growth in the $x$-$k_y$ half-Fourier space; it will have to set a threshold or cutoff value. The threshold is arbitrary and is therefore not satisfactory. In the next section, it is shown how to overcome this difficulty.
Figure 4.14: *Four different vorticity flow patterns occurring in lateral triple-linkage: merged-embryonic fold (upper), fork-linkage (upper middle), antitriple-linkage (lower middle), laterally-locked-fold (lower). The grey ticks indicate the position of the initial perturbation. The corresponding topographic patterns are given in (Fig. 4.3). The vorticity is given at the viscous interface. Parameters of simulation: R = 40, H = 0.05.*
The periclinal front definition used to measure $\lambda_\omega$ in Section (4.2.2) helps however to gain intuition on the linkage process. The numerical experiments show that when the two opposite fronts have disappeared because of the lateral progress of the folds, it produces a first or higher-order linkage if the distance between the two perturbations is much larger than $\lambda_\omega$. On the contrary if the distance is smaller than $\lambda_\omega$, it links at the zero-order. When the separation is slightly greater than $\lambda_\omega$, it is undefined because the initial perturbations can start to link first with the corresponding opposite train folds, and then due to the compressive background they finally link together because the distance between them has been reduced.

**The linkage characteristic length**

It has been mentioned that the linkage of zero order, or first order, depends both on the perturbation separation along the shortening direction and the separation normal to it, in the $x$-$y$ plane. For a fixed lateral separation $L_l$, the initial perturbation separation along the shortening direction can be varied to find the transition length for which the pattern evolve from a linkage of zero order to a linkage of first order. This length is named $L_{\text{linkage}} = L_{\text{linkage}}(L_l)$. An example is given in Fig. 4.15 for $R = 400, 800$ and $H = 0.05$.

The figure shows clearly the linear dependency for the first three points. The last points tend to not follow this linear behaviour because their mode of linkage looks like different. Indeed, for large enough $L_l$, linked fold hinge line resembles to a three steps ramp in plan-view: the two embryonic folds hinges are connected through an oblique line to a third line in-between and parallel to the $y-axis$. It looks like as if a third fold had grown in the region between the two initial perturbations. There are three possibilities:

1. the observed linkage mode is actually what it happens mechanically but the reason is unclear.
2. the resolution in the $y$ direction was too small to effectively capture the linkage process and artefacts could be observed. Indeed, the same number of element along the $y$-axis has been used for larger $L_l$ or $L_y$.
3. the domain was maybe too small in the $x$-direction such that boundary
Figure 4.15: Graph showing the variation of the linkage transition initial separation $L_{\text{linkage}}$ as a function of the lateral perturbation separation. The quantities are normalized by the vorticity dominant wavelength. The star represents the expected value when $L_l = 0$. The graph is constructed as follows: for a given $L_l$, a rough upper bound and lower bound for $L_{\text{linkage}}$ is estimated. Then the interval is reduced step by step around the linkage transition. When the interval is sufficiently small, the $L_{\text{linkage}}$ is assumed to be the centre of the interval around the transition. The error given corresponds to the width of the small interval around the linkage transition. Parameters of simulations: $R = 400, 800$ and $H = 0.05$.

effects could further develop in the domain for large strain\(^2\).
The last point could also explain why the linear part for $R = 800$ in (Fig. 4.15) do not meet the ordinate axis at $L_{\text{linkage}}/\lambda_\omega = 1$: with a larger $R$, the dominant amplification wavelength, vorticity wavelength and the amplification rate are larger such that the boundary effects could grow faster domain

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2. At the boundaries normal the $x$-axis, the velocity is imposed to be parallel to $x$ and pointing inwards the domain. It results in the absence of a shear stress at the boundaries and makes the viscous layer growing faster than the remaining part of the domain. From this boundaries, grows a cylindrical fold train that develops inwards the domain as it is strained. This problem was already mentioned by [Schmid et al., 2008]
inwards if the size of the domain is not adapted.

The most interesting observation is that the $L_{\text{linkage}}$ can be extrapolated when the fold lateral separation is equal to zero. The value seems to be equal to $\lambda_\omega$ (for $R = 400$ and $H = 0.05$) and not equal to the fold amplification wavelength as it would be expected if one did not take into account the vorticity flow pattern.

This observation is evident when one assumes that the vorticity flow pattern controls the linkage process (see Fig. 4.12): when a single perturbation grows laterally, the pattern looks like a collection of loops of alternating orientation with a center separated by $\lambda_\omega$ (Fig. 4.11). By considering now two similar such sets of loops growing in the opposite direction; if their central separation is smaller than $\lambda_\omega$ when they touch each other, the central loops can connect because they have the same orientation. On the contrary, when their central separation is bigger than $\lambda_\omega$, the central loops will connect to the opposite first train loops or the closer loop with the same orientation (Fig. 4.16).

This result shows that the simple-linkage process characteristic length is not the amplification dominant wavelength, but would be rather the planar-vorticity flow dominant wavelength. In that case, the linkage process would depend clearly on the vorticity flow pattern. To comfort this argument, the Fig (4.15) has to be completed for different $R$ and $H$ using a sufficiently large domain and high resolution along the $y$-axis to avoid numerical artefacts. If all the straight lines met the ordinate axis at $L_{\text{linkage}}/\lambda_\omega = 1$, it would be obvious that $\lambda_\omega$ is the linkage characteristic length.
Figure 4.16: Illustration of the simple-linkage process for different central loop separations $L_s$. The shortening direction is given by the blue arrows. The red arrows show the connection or the deviation of the vorticity flow which results to a fold linkage of the initial embryonic anticline folds (zero order linkage) or a deviation and linkage with the first anticline of the train fold (first order linkage). The dashed lines represent the anticline hinge lines whereas the grey lines represent the synclines.

The triple-linkage in the vorticity flow context

In this section, the linkage process of three initial perturbations as illustrated in Fig. 4.1 are considered regarding the vorticity flow. The controlling parameters are $\sigma_1$, $\sigma_2$, $L$ and $\delta$ and the viscosity ratio is kept constant ($R = 40$). It was expected that the width of the initial perturbations, given by $\sigma$, would control the linkage process. This is actually not the case and solely the binary perturbation separation $L$ and the offset $\delta$ have an influence on the linkage.

First, let us define the criteria used to classify the fold patterns in terms of the vorticity field:
fork-linkage occurs when the lateral vorticity flow of the single perturbation is connected to the lateral vorticity flow of the binary folds, we define this structure as a fork-linkage (see Fig. 4.14 upper and lower middle panels). Due to the lateral offset $\delta$, the single embryonic fold can be connected to only one embryonic fold of the binary perturbation through a simple-linkage. In this case, the remaining fold can connect later to a secondary fold or be isolated and locked. The last situation is illustrated in Fig. 4.13.

laterally-locked occurs when the vorticity flow loops around the structure and cannot be connected to the surrounding flow, it is considered to be laterally locked. The situation is illustrated in Fig. 4.14 (lower middle and lower panels): the linear-linkage and laterally-locked anticline shows the isolated structure and a characteristic loop of vorticity surrounded by vorticity lines.

merged embryonic fold occurs when the binary perturbation is so tight that the two vorticity loops merge together to form a unique loop which will interact with the opposite solitary perturbation as in the simple-linkage case. The situation is illustrated in Fig. 4.14 (upper panel):

From the experiments with different values of $\sigma_2$, it was observed that at the perturbation point during the early stages of shortening, the vorticity flow loop width is of the order of the perturbation width. But after a larger deformation, away from the perturbation point, the loop widens to a value close to $\lambda_\omega$ (Fig. 4.11). From this observation, it comes out that the linkage process could not depend on the initial width of the perturbations but is totally controlled by the mechanical properties of the viscous layer when one considers a sufficiently large lateral separation (along the $y$-axis).

To be convinced of this assumption, a phase diagram is given in Fig. 4.17. It illustrates the different linkage pattern obtained as a function of $L$ and $\delta$ only. These two quantities are normalized by the corresponding $\lambda_\omega$ and written as $L^*$ and $\delta^*$. Six different regions can be clearly delimited where the various patterns emerge without including a dependency on $\sigma_2$. 

Figure 4.17: Phase-diagram representing the different patterns observed as a function of the lateral separation of the binary perturbation $L$ and the solitary perturbation offset $\delta$. The stars indicate that the quantities are normalized by the corresponding $\lambda_\omega$. The grey lines bound the different regions that has been identified. Parameters of the simulation: $R = 40, H = 0.05$.

1. The simple-linkage of first order (or greater) region corresponds to the case when the width of the binary perturbations is large enough that none of the initial embryonic folds interact and link together. They are only linked to the resulting train folds.

2. The simple-linkage of zero order region occurs when the offset $\delta$ is large enough for a given separation $2L$. One of the embryonic fold of the binary perturbation connects to the opposite solitary embryonic folds. The second fold of the binary perturbation connects to a train fold of the solitary perturbation.

3. The laterally locked regions corresponds to the case of the simple-linkage of zero order but with a smaller lateral separation such that the second fold of the binary perturbation cannot link with the first train fold of the solitary perturbation. It will simply be locked and form a kind of dome structure. Actually, this case is a syncline-fork of first order as illustrated in Fig. 4.18 (middle sketch).
4. The syncline-fork region is an unstable case which happens only when the offset is zero. It is the special case when the lateral separation is large enough such that the binary fold cannot connect to the solitary one, but it is still tight such that it doesn’t allow the solitary fold to pass in between and develop further. In this situation the solitary fold is locked and it will form a syncline-fork as illustrated in (see Fig. 4.18, left sketch). This special case could be merged with the laterally locked region because both form a syncline-fork.

5. The binary merged region corresponds to the merged embryonic fold. It happens when the separation $2L$ between the two initial perturbations is of the order of $\lambda_\omega$ (it is $L^* \approx 1/2$ on the phase diagram). This can be explained by considering first the two perturbations independently: by combining the two embryonic vorticity loops with a median separation of the order of $\lambda_\omega$, the two vorticity flows will overlap in the central part. Because of their similar orientation, the flows overlap in the opposite direction and they have to cancel out. It results in a larger loop or fold, often with two hinges (box fold), which can interact with the opposite solitary perturbation as in the simple-linkage case. This linkage is actually not a lateral linkage but belongs to the class of parallel to shortening linkage. The situation is illustrated in Fig. 4.18 (right sketch).

6. The anticline-fork region corresponds to the case illustrated in Fig. 4.12. This domain is surrounded by the binary merged region, when the binary separation is too small, and by the laterally-locked region or syncline-fork of first order when the lateral separation is too large to form a triple-linkage.

It makes no sense here to consider the distribution of the region relative to the value $L^*$ and $\delta^*$ because it will depend on the initial separation of the perturbation along the $y$-axis as explained for the simple-linkage case in the previous section. If $L_y$ is changed, the regions characterising the different patterns will be rescaled but their relative distribution and transitions should remain the same. It is expected that the yellow region (simple-linkage of zero order and laterally locked) in Fig. 4.17 will be surrounded to the left by the simple linkage of first order when the separation $2L$ is so large that the simple-linkage of zero order cannot occur. Actually, a phase diagram could be determined for different lateral separation. Therefore, the phase boundaries
at $L_l = 0$ could be extrapolated as it has been done for the simple-linkage. It is expected that the transition of anticline-fork to simple-linkage (first order) occurs when $2L$ is equal to $2\lambda \omega$ when $L_l = 0$.

From the phase diagram (Fig. 4.17), it comes out that the fork-linkage (syncline and anticline) is not an unstable structure as saying it can occur for a large range of different initial geometries ($\delta$ and $L$). Moreover we note simulation using a viscous interface perturbed with red and white noise [Schmid et al., 2008, Fernandez and Kaus, 2013] produce also such triple-linkage structures.

Figure 4.18: Illustration of the vorticity flow represented by oriented loop for some patterns shown in the phase diagram (Fig. 4.17). The dashed lines represent the anticline hinge lines whereas the grey lines represent syncline hinges. The blue arrow for the merged case indicates the change in time of the binary structure. The grey rectangles give the initial perturbation and it is important to distinguish the laterally locked and syncline-fork.
Chapter 5

Fieldwork

Two months field work have been carried out in August-September 2012 at the Aiguilles de Pavis mountain in the median Prealps (more precisely in Chablais, south of Lake Léman). The surrounding area of Aiguilles de Pavis has been well studied over the last 150 years (e.g. [Badoux, 1962, Septfontaine and Lombard, 1976, Septfontaine, 1976, Mason, 1976, Septfontaine, 1984, Mosar, 1988, Mosar, 1989, Mosar, 1991, Septfontaine, 1995, Borel and Mosar, 2000] among others) and appears to have undergone different stages of transgression-regression, leading to large unconformities with important hiatus (up to 20 Ma, in Bajocian - Calovian times) [Septfontaine, 1995]. Angular and progressive unconformities unique in the Prealps have been observed at Pointe Saint-Laurent (fig. 5.1) and are supposed to be formed as the result of large-scale transpressive strike-slip movement along the North-Brianconnaissais margin during the Late Lias (i.e. Early Jurassic) [Septfontaine, 1984, Septfontaine, 1995].

Based on field observation, we proposed here a new tectonic interpretation for the Aiguilles de Pavis.

5.1 General overview

The region records high shortening and strongly deformed structures with very tight syncline and large asymmetric anticline folds often with the southern limb dipping almost vertically. The relief is particularly high due to important thrusts and is dominated by the massive limestones from Dogger and
Figure 5.1: Simplified tectonic map of the investigated region. The fieldwork took place at the Aiguilles de Pavis (adapted after [federal office for Water and Geology, 2005]).

Malm ages.

The Median Prealps nappe has been studied by Schardt, Lugeon, Peterhans, Badoux, Lombard, Septfontaine, Mosar among many others. The median Prealps, with a penninic root, has been transported above the Helvetic nappe, on wildflysch with ultrahelvetic blocs. The zone undergoes diagenetic to epicontinental metamorphism with a horizontal positive gradient from the front to the back of the nappe [Mosar, 1988]. The nappe is mainly formed by en-echelon thrust-faults, fault-propagation folds, detachment folds and large folds with kilometric-scale wavelengths [Mosar, 1991]. The main process of deformation is controlled by dissolution-precipitation, which is typical for thin bedded limestone such as in the Jura mountains [Droxler and Schaer, 1979, Mosar,
5.2 Lithology

It is possible to distinguish clear different formations in the area after the work of M. Septfontaine [Septfontaine, 1984].

**Heiti formation (Lias - Dogger)**: The “Heitischichten” formation, well known in the Prealps of Bern, outcrops in the region of Simmental as large anticlines. Its repartition is larger in the Prealps and extends to the Chablais in the anticlinal zone of Chillon - Tour d’Aï [Septfontaine, 1984] (Fig. 5.1). It is a 200 to 400 m thick stage formed of bedded calcareous marl often siliceous and alternating with calc-shales and marls (Fig. A.5).

**Sommant formation (Dogger)**: The Sommant formation is a carbonate deposit of middle to high energy shelf break. Its thickness varies between 100 m and 250 m depending on the locality, with ages ranging from Bajocian to Bathonian [Septfontaine, 1984]. The formation has large spatial distribution and can be divided into two Members: the Mieussy Member and the Langel Member [Septfontaine, 1984]. In the studied area, only the Langel Member is present at the outcrop. The member can be subdivided into an oolitic unit at the base and an oncolitic unit on the top.

**The oolitic unit**: It is an oolite limestone without any particular structures. Its microfacies is classified as oosparite made of oolites with a diameter of 400 to 600 µm. The oolites are often micritic with a bioclast seed.

**The oncolite unit**: It is a dark limestone often with an organic component, gravelly with a micritic matrix. The microfacies is classified as pelmicrite rich in lagoonal foraminifers and bioclasts. The more characteristic foraminifera are *Archaeosepta platierensis*, *Protopeneroplis striata* and *Trocholina* [Septfontaine, 1984].

**Malm Limestone**: This formation is taken in the chronological sense of the term because there are few detailed studies about this formation in the Prealps. The Formation is build up of a massive limestone
(mudstone), sometimes oncolitic, with white patina and grey, dark to light brown fracture. The microfacies is a microconcolitic lime packstone rich in open see foraminifera. In the region of interest, the base of the Malm, is characterized by a microconglomerate with dolomitic gravel and detritic quartz. The microfacies is composed by the characteristic microncoolites. The *Protoglobigerina* is typical for the lower Malm (middle Oxfordian) whereas the *Calpionella* are characteristic for the upper Malm (Tithonian).

**Couches-rouges (Cretaceous)**: It is build up of regular very thin-bedded calcareous mudstones, alternating with more or less clayey and argillaceous marls. It has a light patina from grey to yellow-greenish, but the largely red shade observed gave the name to the formation (red beds) which is characteristic of the Prealps (bottom Fig.A.4). The thickness of the formation is sometimes difficult to estimate due to strong foldings, but it can reach 500 m in some places. It is rich in planktonic foraminifera often concentrated in turbiditic currents. The formation shifts gradually towards shaly sandstone and flyschs of the Tertiary.

The Sommant Formation was before included in the Malm limestone by the ancient authors (e.g. the geological map of the 1960s from Badoux [bureau recherches geologiques et minieres, 1961]). Actually, a detailed study of the microfauna has shown that the Sommant Formation belongs to the Dogger in age [Septfontaine, 1984] and has to be distinguished from the Malm limestone. In the field both formations are very similar and can often be distinguished by a microfacies study only.

The region of the Pavis mountain belongs to the Mont-Chillon - Tour d’Aï zone, which underwent an uplift in Callovian-Oxfordian times. This is observed by the hiatus (up to 20 Ma) and the angular unconformity of the Dogger-Malm unique in the Préalpes [Septfontaine, 1995]. At the Pavis mountain, the thin-section study has shown that the Malm transgresses directly on the Heiti Formation (Lias) and the Sommant Formation (Dogger) is absent. The Malm limestone is overlain by the Couches-rouges Formation with the shaly sandstones on the top.
5.3 Tectonics

The Aiguilles de Pavis consists of an anticline in the northern part with a smooth topography whereas the southern spectacular cliff is an overthrust fold attached to the Sémy-Chillon anticline (cf. Mont Chillon peak in Fig. A.2). The core of the massif belongs to the Heiti Formation (Lias) and is roofed by the transgressive massive Malm limestone forming the cliffs (Fig. A.3).

Pinched in the tight synclines, the characteristic marls of the Couches-rouges are weak layers relative to the very competent layer of the Malm (top Fig. A.6). The Malm limestone forms open kilometric folds whereas the embedding formations are incompetent (mainly due to the thinner bedding and the presence of clay) and forms many second-order folds. This also allows the embedding formations to be partly detached from the competent layer and deformed more easily: this situation is the reminiscent of the a viscous layer embedded in a matrix even if the analogy is not strictly true.

The pinched synclines and large anticlines remind the situation of the free-surface of the numerical experiments where the viscous layer rests on a thick matrix, and is overlain by only a thin matrix. These structures are similar to the cuspatelobate folds and motivate the idea that the folding occurs at rather shallow depths. This is also supported by the observations that the Prealps underwent no or little metamorphism.

The entire region is cut mainly by strike-slip faults with a strike ranging from NW-SE to N-S. They are generally postgenetic to the folding and main thrusts.

The Pavis mountain can be separated in three units: the Pavis anticline as such, the brittle north-east unit, where the Malm limestone is completely eroded and where only the Heiti Formation crops, out and finally the Sémy-Chillon anticline.

5.3.1 The Pavis anticline

North-east, the anticline is well developed and forms a well defined antiformal structure with a culmination highlighted by the Malm limestone cliffs
It is an inclined open anticline with a width of around 300 m. The measurements of the bedding \( S_0 \) allow to construct the \( \pi \)-plane of the large fold and to determine a fold axis orientation (Fig. 5.2 and 5.3).

Figure 5.2: Stereoplot showing the \( \pi \)-poles (poles to \( S_0 \)) of the Malm formation of the Pavis anticline. The Bingham axial distribution is estimated and allows to estimate the fold axis (eigenvector V3).

- Estimated fold axis (\( \Pi \)-axis)
- \( \Pi \)-poles
- \( \Pi \)-plane

Bingham axial distribution: \( N = 7 \)

Eigenvalues:
- Eigenvector V1: Trend = 325.3 Plunge = 10.2, Eigenvalue S1 = 0.6300
- Eigenvector V2: Trend = 147.0 Plunge = 79.8, Eigenvalue S2 = 0.3454
- Eigenvector V3: Trend = 055.4 Plunge = 0.3, Eigenvalue S3 = 0.0245

It has been chosen to distinguish the Heiti formation (Fig. 5.3) from the Malm Formation (Fig. 5.2) because they clearly show two different fold geometries with an apparent inclined fold vergence:

- the competent Malm Formation has a horizontal fold axis with a SW-NE trend.
- the less competent Heiti Formation underneath has a slightly plunging fold axis (9.0 degrees to the E).

The difference can be explained in two different ways:

1. It has been shown that an angular unconformity between the Heiti Formation and the Dogger-Malm Formations exists. This unconformity was observed by Septfontaine (1995) at the Saint-Laurent [Septfontaine, 1995], which is situated 2.5 km to the NE, i.e. along the direction of the Pavis anticline fold axis (Fig. A.2). It has been observed at the Pavis mountain that the layers of the Heiti Formation are effectively
tilted regarding the overlaying Malm limestones at the top-hinge of the anticline. The effect is that, if the layers of the Heiti Formation and the ones of the Malm limestones were not perfectly parallel before the folding, two slightly different geometries of the measured fold results considering the two formations separately.

2. The Heiti Formation alternates between sandy beds and rich clay shales. This allows the layers to have an important flexural slip-folding component, which slightly decouples the incompetent Heiti Formation from the more competent Malm limestone during the deformation.

5.3.2 The brittle North-East unit

The north-eastern and east part of the mountain, between the Bise pass and the Pavis pass, is highly deformed in the brittle regime. The measure of $S_0$ gives a fold axis normal to the surrounding trend, which is a fold axis with a NW-SE trend (Fig. 5.4)! Actually, the area underwent a post-folding brittle deformation, which could explain that the layers can be tilted and rotated on the fault planes. It is also speculated that a major fault with a N-
Figure 5.4: Stereoplot showing the \( \pi \)-poles (poles to \( S_0 \)) of the Heiti and Malm formation of the Pavis brittle north-East unit. The Bingham axial distribution is estimated and allows to estimate the fold axis (eigenvector V3).

S trend trans-cuts the region from the Bise pass to the Pas de la Bosse (near the Cornette de Bise, Fig. A.2) passing through the Bise-Pavis small valley (comb). This hypothetical fault could have had an impact on the strong brittle deformation observed in the eastern units of the Pavis mountain.

5.3.3 The Sémy-Chillon anticline

The large gently asymmetrical Sémy-Chillon anticline is overthrusting the Pavis anticline unit. The width is estimated to 1.5 km. The quality of the outcrops (Pointe des Fires - Floray pass and the base of the southern cliff of the Pavis mountain) allows to give a good estimate of the fold axis (Fig. 5.5).

It has been chosen not to distinguish the Heiti and the Malm formations because they show the same fold geometry. At the southern front of the anticline the layers are almost vertical whereas they plunge at 20 to 40 degrees to the SE in the direction of Bise. The fold axis is almost horizontal and shows the same trend as observed for the case of the Pavis anticline. The bulk axial plane would dip to the SE similar to the Pavis anticline and the
structure seems to have an overturned apparent vergence.

5.3.4 The fault distribution

Three groups of faults can be distinguished, but the lack of data (linear structural elements, shear senses) make an interpretation of the paleostress orientation difficult.

1. The first group corresponds to conjugate faults observed mainly in the competent limestones, which seems to result from the vertical load and collapse of the blocks and cliffs.

2. The second group corresponds to steep faults with a N-S strike, which can have a displacement from 3 to 40 m. The movement trend is horizontal, functioning as strike-slip faults (Fig. 5.6 - left).

3. The third group corresponds to faults with a NW-SE strike (Fig. 5.6 - right).
Figure 5.6: Stereoplot showing the fault planes of the N-S-striking group on the left and the NW-SE-striking group on the right. The coloured points correspond to the linear structural measures on the fault plane of the corresponding colour. The irregular dashed lines indicate dextral faults whereas regular dashes indicate sinistral faults.
Figure 5.7: Geologic map of the investigated area. The Dolomite from the Triassic has been added for the completeness of the map although this age do not crop out on the Pavis and Chillon mountains. The upper left corner is left blank because this part belongs to the Château d’Oche complex, which is not part of the study.
Figure 5.8: Interpretative geologic profile of the map (Fig. 5.8). The blurred part corresponds to the interpretation of the eroded formations. The pinched syncline at the Pavis mountain is highlighted with the highly deformed Couches-rouges. The depth of the Malm limestone at the syncline trough is extrapolated from the field observations: the north-West vertically plunging limb of the Séméy-Chillon anticline, crops out with the Couches-rouges up to 1700 m south-west close to the Darbon village. Three main second-order folds in the Heiti Formation have been observed along the crest belonging to the Séméy-Chillon anticline (Pointe des Fires). They are represented here whereas the Malm limestone seems to have no second-order folds due to its mechanical properties. A thrust fault is expected at the trough of the syncline and it is indicated with a question mark because not observed (such thrust fault could be found south-west in the direction of the Darbon village). It also have been observed that the south-east part of the Chillon-Sémé anticline plunge also vertically at the syncline (bedding measured in the Couches-rouges on the road from Vacheresse to Bise). Finally, the bedding of the Heiti Formation is only illustrative and do not correspond to the reality.
5.4.2 Deformation process

The fieldwork has shown that the region underwent a deformation typical for a nappe transported over kilometers and sliding on a slope (large asymmetric fold with an apparent vergence pointing to the NW) as it observed in the viscous multi-layers folding simulations presented in [Schmalholz and Schmid, 2012]. The competent Malm limestones (and part of the Sommant Formation) are almost not deformed and accommodate the NW-SE shortening locally at a kink-point where the slightly dipping abruptly plunges almost vertically at the synclines (Fig. A.7). In the region of investigation the massive limestones overlay the more “ductile” Heiti Formation that can easier deform developing second-order folds. The two formations are gently decoupled and accommodate the deformation in a different manner.

The numerous trans-cutting strike-slip faults observed at the Pavis mountain and generally in the Prealps [Mosar, 1991] is comparable to the situation of a Piedmont glacier, where the frontal lobe expands onto a lowland forming many fractures along the radial direction. In this situation, the geological nappe, overlaying other geological units, would expand down-slope forming the strike-slip faults to accommodate the radial expansion.
Chapter 6

Final words

On the marker in cell method

The abilities of the MAC-FEM method has been tested successfully to model folding processes and a finer grid has to be used to get results in agreements with the Pure-FEM. The advantage of the MAC-FEM is that a regular mesh can be used independently on how strongly folded the domain is because the properties are carried by the markers and not by the elements any more. It avoids developing a lithology dependent remeshing algorithm which is difficult to make robust for large deformation experiments.

On the fieldwork

The fieldwork has shown that the area of the Pavis mountain underwent a large shortening in the NW-SE direction. The clear rheological contrast between the formations induced different modes of deformation characterized by the formations mechanical properties and the physical conditions (fluid, pressure and temperature) at which the shortening took place. From this work many issues are raised that could be investigated by 3D modelling using visco-plastic rheology: What are the parameters controlling the fold pattern observed at the Pavis mountain namely the asymmetry, the kink-like shape of the Malm limestones, the decoupling between the Heiti and Malm Formations? Which parameters determine the localisation of the kink-point of the Malm limestone fold?
From the knowledge of the regional geology, [Septfontaine, 1984, Septfontaine, 1995] it has been demonstrated that the lithology is also subject to numerous variations and high contrasts in space, both vertically and horizontally (angular unconformity, paleo-cliffs). The following questions remain: what is the influence of high horizontal lithological contrasts in a medium undergoing a large shortening? How the thrust points, detachment points and overturned/recumbent structures are localized by such heterogeneities?¹

Moreover, the observation that the folds are trans-cut by many strike-slip faults also motivates a 3D numerical study of a visco-plastic nappe expanding radially on a lowland like the Piedmont glacier type. It is expected that asymmetric folds with a vergence oriented to the expansion direction form, which are trans-cut radially by many strike-slip faults.

On the linkage process

The mechanical feasibility of the triple-linkage (mainly the fork-linkage) in a single viscous layer has been proved for small and large viscosity ratios. It has been shown that such structures seem to be very stable for a large range of geometric parameters. Moreover, other numerical models tend to show that such structures cannot be an isolated curiosity [Schmid et al., 2008, Fernandez and Kaus, 2013]. There is nevertheless no published work on such patterns in nature and it could be explained by the fact that the linkage process is a quite new concept and, to our knowledge, the possibility of fork structures had never been envisaged before. A plausible candidate would be the Mont-Chasseral in the Jura mountain-belt in Switzerland where a fieldwork with the linkage process in mind should discredit the candidate

¹ The question raised after the observations at the Château-d’Oche mountain (north to the Pavis mountain). The geology of the mountain is still not well understood and resembles a “tectonic melange” (after M. Septfontaine). The structure is like a recumbent fold where at the north face, the Couches-Rouges lays horizontally pinched between the Malm limestones of two different limbs (?), whereas the southern slope of the mountain exhibits the core of the overlaying fold formed by the dolomites of the Triassic. The Malm limestones would be deposed on a very incompetent Triassic bedding (dolomite, evaporite) after a local uplift. How would this rheological contrast have promoted the formation of such recumbent fold and detachment?
The vorticity flow framework has demonstrated its ability to make the comprehension of the folding process and the related fold interplay straightforward. This new point of view, where the fold structures show a characteristic loop-like pattern, turns out to be powerful to understand the observed linkage (simple, triple-linkage and the related locked structures) and merged fold patterns.

Finally, the dominant planar-vorticity wavelength $\lambda_\omega$ has been introduced distinct from the fold amplification dominant wavelength. It has been highlighted that the length based on the vorticity flow could be an important factor controlling the simple-linkage process. As this length determines the vorticity loop periodicity along the shortening direction and that is precisely the loop distribution that controls the linkage process, it is believed that $\lambda_\omega$ could also be the leading parameter for the fold linkage in the general case.

A more detailed study of the linkage process in particular the triple-linkage could afford more informations about the vorticity interaction mechanisms relative to $\lambda_\omega$ (merged and fork pattern). In that purpose, the different region distribution of the phase diagram presented in the triple-linkage could be investigated for a range of lateral separation and mechanical parameters ($R$ and $H$). The graph presenting the linkage transition regarding the lateral separation has also to be completed with different value of $R$ and $H$. The linkage process and vorticity fold pattern could also be investigated in the context of multilayer folding, non-isoviscous and anisotropic layers. A plastic rheology (Von Mises or Drucker-Prager) should finally be introduced to see the effect on the fold hinge curvature in the simple and triple-linkage and to study how the shortening can be accommodated by allowing brittle deformation.

In view of this work, it appears clearly that large fold systems do not necessarily grow uniformly in a cylindrical manner. They can stem from the interconnection of smaller embryonic folds growing at the same time and emerging from the dominant perturbations or heterogeneities at the interface between two distinct viscous layers. These embryonic folds grow laterally parallel to their hinge axis and can interact each other leading to a considerable diversity of patterns, which result in neither a fault system nor a regional
orientation stress change. The fold-belt systems should be considered in the light of the fold linkage concept as it has been done for the Zagros fold-and-thrust belt. It could avoid a misinterpretation of the origin of the structures in place and provide a better comprehension of the observed large scale folds.
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For the sake of brevity, I should stop here!
Appendix A

The FEM method applied to the 2D Stokes problem with $Q_2P_1$ elements

A.1 The weak form of the Stokes equations

The Stokes problem for a Newtonian material in $\Omega$ with Dirichlet boundary conditions reads: find $(u, p) \in H^1(\Omega)^3 \times L^2(\Omega)$ verifying (A.1).

\[
\begin{align*}
\left\{ \begin{array}{l}
\eta(u_{i,j} + u_{j,i})_{,j} - p_{,i} - f_i = 0, & \mathbf{x} \in \Omega \\
 u_{k,k} = 0, & \mathbf{x} \in \Omega \\
 u_i = w_i(\mathbf{x}), & \mathbf{x} \in \partial\Omega \\
\end{array} \right. \\
(A.1)
\end{align*}
\]

This are linear equations describing the flow of an incompressible viscous fluid. The weak form of the problem (A.1) is obtained by multiplying the first equation by a test function $v \in V_0$ and $p \in Q$ the second one, integrating over $\Omega$ and then by part (using the Green formula), it comes out (A.2).

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_{\Omega} \eta(u_{i,j} + u_{j,i}) v_{i,j} \, d\Omega - \int_{\Omega} p v_{i,i} \, d\Omega = \int_{\Omega} f_i v_i \, d\Omega \\
\int_{\Omega} q u_{i,i} \, d\Omega = 0 \\
\end{array} \right. \\
(A.2)
\end{align*}
\]

$\forall (v, q) \in V_0 \times Q$. By defining the bilinear forms (A.3),

\[
\begin{align*}
a(u, v) := \int_{\Omega} \eta(u_{i,j} + u_{j,i}) v_{i,j} \, d\Omega, & \quad b(u, q) := \int_{\Omega} q u_{i,i} \, d\Omega, & \quad f(u) := \int_{\Omega} f_i v_i \, d\Omega \\
(A.3)
\end{align*}
\]
the Stokes problem (A.1) can be reformulated as: \( \forall (v, q) \in V_0 \times Q \), find \((u, p) \in V \times Q\) satisfying (A.4)

\[
\begin{align*}
    a(u, v) + b(v, p) &= f(v) \\
    b(u, q) &= 0
\end{align*}
\]  

(A.4)

\( \forall (v, q) \in V_0 \times Q \)

A.2 The \( Q_2P_1 \) elements

Figure A.1: \( Q_2P_1 \) elements with the quadrature points.

The \( Q_2P_1 \) isoparametric elements are used to solve the Stokes problem. In the partition of the initial domain, the \( Q2 \) elements are quadrilateral interconnected with quadratic shape functions interpolating the velocity field. The \( P1 \) elements are triangular and not connected with linear shape functions.
A.3  Fieldwork Tour

Figure A.2: Topographic map of the region of investigation. The reddish area corresponds to the region of investigation. To the upper right corner, the position of the Saint-Laurent mountain has been added. Snapshot from http://www.geoportail.gouv.fr/accueil. The purple lines indicate the walking trails whereas the french-swiss border is indicated by the orange-red bold line.
Figure A.3: On the top: Frontal view of the southern Pavis-cliffs. On the left the bedding is almost vertical whereas to the right it is dipping with less steep angle. The pictures is taken on the road to the Bise pass (here on the right). On the bottom: side view of the Pavis-cliffs where the change of the bedding dip appears clearly. Picture taken beneath the Bise pass.
Figure A.4: On the left: Heiti formation dipping vertically at the southern cliff of the Pavis mountain. On the right: Malm limestones dipping vertically and forming the impressive southern cliffs. Picture taken from the Floray pass. On the bottom: Highly deformed marls of the Couches-Rouges Formation. Picture taken in the syncline of the Pavis mountain.
Figure A.5: The thinly bedded Heiti Formation overlaid by the massive Malm limestones of the Pointe de Bénévant (eastern from the Mont-Chillon). Picture taken from the Pointe des Fires.
Figure A.6: On the top: Picture illustrating the pinched synclines between the Malm limestones of the Pavis cliffs and the one of the Pavis mountain itself. On the bottom: Photograph of the Pavis pass where the Pavis anticline is highlighted on the right. The Malm limestones are clearly dipping almost vertically to the north. The mountain on the left of the graph corresponds to the beginning of the Château-d’Oche. Picture taken from the top of the Aiguilles de Darbon.
Figure A.7: Typical deformation observed in the region. The massive Malm limestones are almost not deformed excepted at some local points where the main accommodation of the shortening occurs.
Bibliography


[federal office for Water and Geology, 2009]


[Fernandez and Kaus, 2013]


