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Duality with Linear-Feedback Schemes for the Scalar Gaussian MAC and BC

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Abstract—We show that with perfect feedback and when restricting to linear-feedback schemes, the regions achieved over the two-user scalar Gaussian memoryless MAC and over the two-user scalar Gaussian memoryless BC coincide, if the MAC and the BC have equal channel coefficients and if the same (sum)-power constraint $P$ is imposed on their inputs. Since the achievable region for the MAC is well known (it equals Ozarow’s perfect-feedback capacity region under a sum-power constraint), we can characterize the region that is achievable over the scalar Gaussian BC with linear-feedback schemes.

I. INTRODUCTION

Feedback is known to increase the capacity of multi-user channels such as the multi-access channel (MAC) and the broadcast channel (BC). But for most multi-user channels the exact capacity region is open even with perfect output-feedback. A notable exception is the two-user memoryless Gaussian MAC whose capacity was derived by Ozarow [1]. Ozarow’s capacity-achieving scheme is a linear-feedback scheme, i.e., a scheme where at each time the transmitters send linear combinations of the past feedback signals and of code symbols that only depend on their own message. Kramer extended this scheme to $K \geq 3$ users [2]. Under symmetric power constraints $P$ for all users, Kramer’s scheme achieves the largest sum-rate among all linear-feedback schemes [3], and it achieves the sum-capacity when $P$ is sufficiently large [2]. (It is yet unknown whether the scheme achieves the sum-capacity also when $P$ is small.)

For the memoryless Gaussian BC the capacity region with perfect feedback is unknown even with two receivers. Achievable regions based on linear-feedback schemes have been proposed in [2], [4], [5], [6], [7], [8]. Non-linear feedback schemes have been proposed in [14], [9], [10]. The best known achievable regions are due to linear-feedback schemes.

The linear-feedback schemes in [5], [6], [7] are designed based on control-theoretic considerations. For some setups, e.g., uncorrelated noises of equal variance [7], these schemes achieve the same sum-rate over the Gaussian BC under power constraint $P$ as Ozarow’s scheme achieves over the Gaussian MAC under a sum-power constraint $P$. Thus, there is a duality in terms of sum-rate between the BC-schemes in [5], [6], [7] and Ozarow’s MAC-scheme [1].

In this paper, we prove a duality between arbitrary linear-feedback schemes over the two-user scalar Gaussian MAC and BC, similar to the MIMO (nofeedback) MAC-BC duality in [11], [15]. Specifically, we show that the regions achieved by linear-feedback schemes over the two-user scalar Gaussian MAC under sum-power constraint $P$ and over the two-user scalar Gaussian BC with uncorrelated noises under the same power constraint $P$ coincide, if the channel coefficients of the MAC and the BC are equal. Since the set of achievable regions over the Gaussian MAC using linear-feedback schemes is known—it equals Ozarow’s achievable region under a sum-power constraint—our result allows to characterize the region that is achievable with linear-feedback schemes over the two-user scalar Gaussian BC with uncorrelated noises. We can also identify the optimal linear-feedback schemes over the scalar Gaussian BC and show that for equal noise-variances the control-theoretic schemes in [5], [6], [7] achieve the largest sum-rate among all linear-feedback schemes.

II. GAUSSIAN MAC WITH FEEDBACK

Consider the two-user memoryless scalar Gaussian MAC with perfect output-feedback in Figure 1. At each time $t \in \mathbb{N}$, if $x_{1,t}$ and $x_{2,t}$ denote the real symbols sent by Transmitters 1 and 2, the receiver observes the real channel output

$$Y_t = h_1 x_{1,t} + h_2 x_{2,t} + Z_t, \quad (1)$$

where $h_1$ and $h_2$ are constant non-zero channel coefficients and \{Z_t\} is a sequence of independent and identically distributed (i.i.d.) zero-mean unit-variance\(^1\) Gaussian random variables.

The goal of communication is that Transmitters 1 and 2 convey their independent messages $M_1$ and $M_2$ to the common receiver. The messages $M_1$ and $M_2$ are independent of the noises \{Z_t\} and uniformly distributed over the sets $M_1 \triangleq \{1, \ldots, 2^{nR_1}\}$ and $M_2 \triangleq \{1, \ldots, 2^{nR_2}\}$, where $R_1$ and $R_2$ denote the rates of transmission and $n$ the blocklength.

The two transmitters observe perfect feedback from the channel outputs. Thus, the time-$t$ channel input at Transmitter $t \in \{1, 2\}$ can depend on all previous channel outputs $Y^{t-1}$ and its message $M_t$:

$$X_{t,t} = f_{t,t}^{(n)}(M_t, Y^{t-1}), \quad t \in \{1, \ldots, n\}, \quad (2)$$

\(^1\)Assuming unit-variance entails no loss in generality because otherwise the receiver can simply scale its outputs appropriately.
for some encoding function $f_{i,t}^{(n)}: M_i \times \mathbb{R}^{t-1} \rightarrow \mathbb{R}$. The channel inputs $\{X_{1,t}\}_{t=1}^n$ and $\{X_{2,t}\}_{t=1}^n$ have to satisfy an expected average sum-power constraint
\begin{equation}
\frac{1}{n} \sum_{t=1}^n (\mathbb{E}[X_{1,t}^2] + \mathbb{E}[X_{2,t}^2]) \leq P, \tag{3}
\end{equation}
where the expectation is over the messages and the realizations of the channel. The receiver produces a guess
$$
(\hat{M}_1^{(n)}, \hat{M}_2^{(n)}) = \Phi^{(n)}(Y^n)
$$
by means of a decoding function $\Phi^{(n)}: \mathbb{R}^n \rightarrow M_1 \times M_2$.

The average probability of error is
$$
P_{e,\text{MAC}}^{(n)} = \Pr[(\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)]. \tag{4}
$$

We say that a rate-pair $(R_1, R_2)$ is achievable over the Gaussian MAC with feedback under a sum-power constraint $P$, if there exists a sequence of encoding and decoding functions $\{(f_{i,t}^{(n)} )_{t=1}^n, \ (f_{i,\hat{M}^{(n)}}^{(n)} )_{t=1}^n, \ (\Phi^{(n)})_{n=1}^\infty\}$ as described above, satisfying (3) and such that the average probability of error $P_{e,\text{MAC}}^{(n)}$ tends to zero as the blocklength $n$ tends to infinity. The closure of the union of all achievable regions is called capacity region.

In the present paper we focus on linear-feedback schemes for the MAC, where the channel inputs can be written as
$$
X_i = V_i + C_i Y, \quad i \in \{1, 2\}, \tag{5}
$$
where $Y \triangleq (Y_1, \ldots, Y_n)^T$ is the channel output vector, $C_1$ and $C_2$ are $n$-by-$n$ strictly lower-triangular matrices and $V_i = \phi_i(M_i)$. The mapping $\phi_i: M_i \rightarrow \mathbb{R}^n$ as well as the decoder mapping $\Phi^{(n)}$ can be arbitrary (also non-linear). The strict-lower-triangularity of the matrices $C_1$ and $C_2$ ensures that the feedback is used in a strictly causal way. (Notice that any nofeedback scheme is of the form (5) with $C_1 = C_2 = 0$.)

The set of all rate-pairs achieved by linear-feedback schemes is called linear-feedback capacity region and is denoted $C_{\text{MAC}}^{\text{lin}}(h_1, h_2; P)$. The largest sum-rate achievable by a linear-feedback scheme is called linear-feedback sum-capacity and is denoted $C_{\text{MAC}}^{\text{lin}}(h_1, h_2; P)$.

Since Ozarow’s capacity-achieving scheme [11] is a linear-feedback scheme, the general feedback capacity region and the linear-feedback capacity region coincide. They are both given by
$$
c_{\text{MAC}}^{\text{lin}}(h_1, h_2; P) = \bigcup_{P_1 + P_2 \leq P} \bigcup_{\rho \in [0,1]} \mathcal{R}_{Oz}^\rho(h_1, h_2; P_1, P_2) \tag{6}
$$
\footnotetext[2]{Notice that also Ozarow’s rate-splitting scheme has the form in (5) because the feedback signals are combined linearly with code-symbols.}

Consider the two-user scalar Gaussian BC with perfect output-feedback in Figure 2. We now have one transmitter and two receivers. At each time $t \in \mathbb{N}$, if $x_t \in \mathbb{R}$ denotes the transmitter’s channel input, Receiver $i \in \{1, 2\}$ observes the real channel output
\begin{equation}
Y_{1,t} = h_1 x_t + Z_{1,t}, \tag{10}
\end{equation}
where $h_1$ and $h_2$ are constant non-zero channel coefficients and $\{Z_{1,t}\}_{t=1}^n$ and $\{Z_{2,t}\}_{t=1}^n$ model the additive noise at Receivers 1 and 2. The noise sequences $\{Z_{1,t}\}_{t=1}^n$ and $\{Z_{2,t}\}_{t=1}^n$ are independent and each consists of i.i.d. centered Gaussian random variables of unit variance.$^3$

The goal of the communication is that the transmitter conveys Message $M_1$ to Receiver 1 and Message $M_2$ to Receiver 2. The transmitter observes perfect output-feedback from both receivers. Thus, the time-$t$ channel input $X_t$ can depend on all previous channel outputs $Y_{1,t-1}$ and $Y_{2,t-1}$ and the messages $M_1$ and $M_2$:
$$
X_t = g_t^{(n)}(M_1, M_2, Y_{1,t-1}, Y_{2,t-1}), \quad t \in \{1, \ldots, n\}, \tag{11}
$$
for some encoding function $g_t^{(n)}: M_1 \times M_2 \times \mathbb{R}^{t-1} \times \mathbb{R}^{t-1} \rightarrow \mathbb{R}$. We impose an expected average block-power constraint
\begin{equation}
\frac{1}{n} \sum_{t=1}^n \mathbb{E}[X_t^2] \leq P, \tag{12}
\end{equation}
where the expectation is over the messages and the realizations of the channel. Each Receiver $i \in \{1, 2\}$ produces the guess
$$
\hat{M}_i^{(n)} = \phi_i^{(n)}(Y_i^n)
$$
\footnotetext[3]{As for the MAC, assuming $Z_{1,t}$ and $Z_{2,t}$ have unit variance entails no loss in generality.}
for some decoding function \( \phi^{(n)}_i : \mathbb{R}^n \to \mathcal{M}_i \).

The average probability of error is

\[
P_{e,BC}^{(n)} \triangleq \Pr[(\hat{M}_1 \neq M_1) \lor (\hat{M}_2 \neq M_2)] .
\]

A rate-pair \((R_1, R_2)\) is achievable over the Gaussian BC with feedback and power constraint \( P \), if there exists a sequence of encoding and decoding functions \( \{(G_k^{(n)} \mid \quad \alpha_k^{(n)} \mid \quad \phi^{(n)}_1 \mid \quad \phi^{(n)}_2 )\}_{n=1}^\infty \) as described above, satisfying the power constraint (12) and such that the average probability of error \( P_{e,BC}^{(n)} \) tends to zero as \( n \) tends to infinity.

Also here we restrict attention to linear-feedback schemes for the BC where the transmitter’s channel input vector \( X \triangleq (X_1, \ldots, X_n) \) can be written as:

\[
X = W + B_1 Z_1 + B_2 Z_2 ,
\]

where \( Z_i \triangleq (Z_{i1}, \ldots, Z_{in}) \) represents the noise vector at Receiver \( i \), \( B_1 \) and \( B_2 \) are strictly lower-triangular matrices, and \( W \) is an \( n \)-dimensional information-carrying vector

\[
W = \xi(M_1 , M_2) .
\]

The mapping \( \xi : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathbb{R}^n \) and the decoding operations \( \phi^{(n)}_1 \) and \( \phi^{(n)}_2 \) can be arbitrary.

Taking a linear combination of the information-carrying vector \( W \) and the past noise vectors \( Z_1 \) and \( Z_2 \) is equivalent to taking a (different) linear combination of \( W \) and the past outputs \( Y_1 \) and \( Y_2 \). Thus, the strict lower-triangularity of \( B_1 \) and \( B_2 \) ensures that the feedback is used strictly causally.

Linear-feedback capacity and linear-feedback sum-capacity for the BC are defined analogously as for the MAC. We denote them by \( C_{\text{linfb}}^{\tau}(h_1, h_2; P) \) and \( C_{\text{linfb}}^{\tau,\Sigma}(h_1, h_2; P) \).

IV. MAIN RESULTS

We first present multi-letter expressions for \( C_{\text{linfb}}^{\tau}(h_1, h_2; P) \) and \( C_{\text{linfb}}^{\tau,\Sigma}(h_1, h_2; P) \). They are used to prove Theorem 1 ahead.

**Definition 1.** Given \( \eta \in \mathbb{N} \) and \( \eta\)-by-\( \eta \) matrices \( D_1 \) and \( D_2 \), let \( Q_1 \) and \( Q_2 \) be the positive square roots of the (positive-definite) \( \eta\)-by-\( \eta \) matrices

\[
M_1 \triangleq (I_n + h_1 D_1)(I_n + h_1 D_1) + h_2^2 D_1^2 D_2
\]

\[
M_2 \triangleq h_2^2 D_1^2 D_2 + (I_n + h_2 D_2)^2(I_n + h_2 D_2)
\]

and let \( \mathcal{R}_{\text{MAC}}(\eta, D_1, D_2, h_1, h_2; P) \) denote the (private-messages) no-feedback capacity (12) of the MIMO MAC

\[
\mathcal{Y}_{\text{MAC}}^{\eta} \triangleq h_1 Q_1^{-1} V_1 + h_2 Q_2^{-1} V_2 + \tilde{Z}
\]

when the \( \eta\)-by-\( \eta \) input vectors \( V_1 \) and \( V_2 \) have to satisfy

\[
\text{tr}(K_{V_1} + K_{V_2}) \leq \max \{0, \eta P - \text{tr}(D_1 D_1^2) - \text{tr}(D_2 D_2^2)\} .
\]

where \( K_{V_i} \) denotes the covariance matrix of \( V_i \) and in (17) \( \tilde{Z} \) is a centered Gaussian vector of identity covariance matrix \( I_\eta \).

**Proposition 1.** The linear-feedback capacity of the scalar Gaussian MAC under sum-power constraint \( P \) satisfies

\[
C_{\text{linfb}}^{\tau,\Sigma}(h_1, h_2; P) = \min \bigg \{ \sup_{P_1, P_2 \geq 0, P_1 + P_2 = P} \frac{1}{2} \log \left( 1 + h_1^2 P_1 + h_2^2 P_2 + 2 \sqrt{h_1^2 h_2^2 P_1 P_2} \rho^* (h_1, h_2; P_1, P_2) \right) \bigg \}
\]

for some decoding function \( \phi^{(n)}_i : \mathbb{R}^n \to \mathcal{M}_i \).

where \( K \) is a centered Gaussian vector of identity covariance matrix \( I \).

**Proposition 2.** Given \( \eta \in \mathbb{N} \) and \( \eta\)-by-\( \eta \) matrices \( B_1 \) and \( B_2 \), let \( S_1 \) and \( S_2 \) be the positive square roots of the (positive-definite) \( \eta\)-by-\( \eta \) matrices

\[
A_1 \triangleq (I_n + h_1 B_1^2)(I_n + h_1 B_1^2)^2 + h_2^2 B_1^2 B_2^2
\]

\[
A_2 \triangleq h_2^2 B_1^2 B_2^2 + (I_n + h_2 B_2^2)(I_n + h_2 B_2^2)^2
\]

and let \( \mathcal{R}_{\text{MAC}}(\eta, B_1, B_2, h_1, h_2; P) \) denote the (private-messages) no-feedback capacity of the MIMO BC (13)

\[
\mathcal{Y}_{\text{MAC}}^{\eta} \triangleq h_1 S_1^{-1} W + \tilde{Z}_i , \quad i \in \{1, 2\},
\]

when the \( \eta\)-by-\( \eta \) input vector \( W \) has to satisfy

\[
\text{tr}(K_W) \leq \max \{0, \eta P - \text{tr}(B_1^2 B_1^2) - \text{tr}(B_2^2 B_2^2)\} .
\]

where \( K_W \) denotes the covariance matrix of \( W \) and where in (24) \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) denote independent centered Gaussian vectors of identity covariance matrix \( I_\eta \).

**Proposition 3.** The linear-feedback capacity region of the Gaussian BC with feedback is:

\[
C_{\text{linfb}}^{\tau}(h_1, h_2; P) = \eta \bigg \{ \sup_{\eta \geq 0, \eta \geq 0} \frac{1}{2} \log \left( 1 + h_1^2 P_1 + h_2^2 P_2 + 2 \sqrt{h_1^2 h_2^2 P_1 P_2} \rho^* (h_1, h_2; P_1, P_2) \right) \bigg \}
\]
where the union is over all positive integers \( \eta \) and strictly lower-triangular \( \eta \)-by-\( \eta \) matrices \( B_1 \) and \( B_2 \).

Proof: For fixed \( \eta \), \( B_1 \), and \( B_2 \), the rate region
\[
\frac{1}{\eta} R_{BC}(\eta, B_1, B_2, h_1, h_2; P)
\]
achieved by coding over blocks of \( \eta \) channel uses, if the channel inputs in a block are
\[
X = W + B_1 Z_1 + B_2 Z_2,
\]
where \( Z_1 \) and \( Z_2 \) denote the block’s \( \eta \)-by-\( \eta \) noise vectors at Receivers 1 and 2 and \( W \) is an \( \eta \)-by-\( \eta \) input vector that depends on the messages \( (M_1, M_2) \). Receiver \( i \)'s outputs \( Y_i \) in a block are then given by
\[
Y_i = h_i W + h_i B_1 Z_1 + h_i B_2 Z_2 + Z_i, \quad i \in \{1, 2\}.
\]
By coding over the inputs \( W \) of the different blocks, we can achieve the capacity of the MIMO BC in (28), which coincides with the capacity of the MIMO BC in (24).

More details and the converse are omitted for brevity.

Theorem 1. The linear-feedback capacity regions of the scalar Gaussian BC under power constraint \( P \) and of the scalar Gaussian MAC under sum-power constraint \( P \) coincide:
\[
C_{MAC}^{\text{linfb}}(h_1, h_2; P) = C_{BC}^{\text{linfb}}(h_1, h_2; P).
\]

Corollary 1.
\[
C_{MAC,\Sigma}^{\text{linfb}}(h_1, h_2; P) = C_{BC,\Sigma}^{\text{linfb}}(h_1, h_2; P).
\]

Corollary 2. If \( h_1 = h_2 = h \), then
\[
C_{MAC,\Sigma}^{\text{linfb}}(h, h; P) = \frac{1}{2} \log \left( 1 + h^2 P + h^2 P \cdot \rho(h, h; P) \right)
\]
and thus the control-theoretic scheme in [7] achieves the linear-feedback sum-capacity.

Proof: By a symmetry argument. Omitted.

Proof of Theorem 1: We show that
\[
R_{MAC}(\eta, D_1, D_2, h_1, h_2; P) = R_{BC}(\eta, B_1, B_2, h_1, h_2; P),
\]
coincide if
\[
B_i = \bar{D}_i, \quad i \in \{1, 2\},
\]
(33)
where for a matrix \( A \), \( \bar{A} \triangleq E_\eta A E_\eta \) is called its \emph{reversed image} and \( E_\eta \) is the exchange matrix which is 0 everywhere except on the counter-diagonal where it is 1. The theorem then follows by Propositions 1 and 2, and since the mapping in (33) is one-to-one over the set of strictly lower-triangular matrices.

Under (33), the RHSs of the power constraints (18) and (25) coincide. Moreover, under power constraint (18), the MIMO MAC in (17) has the same capacity as the MIMO MAC\footnote{Multiplying \( Y_{MAC} \) by \( E_\eta \) from the left only reverses the order of receiving antennas.}
\[
\hat{Y}_{MAC} \triangleq E_\eta Y_{MAC} = h_1 \hat{Q}_1^{-1} \hat{V}_1 + h_2 \hat{Q}_2^{-1} \hat{V}_2 + \hat{Z},
\]
(34)
where \( \hat{Z} \triangleq E_\eta \hat{Z} \) and where \( \hat{V}_1 \triangleq E_\eta V_1 \) has to satisfy the power constraint (18) when \( V_1 \) replaces \( V_\eta \). Now, Equality (32) follows by the MIMO MAC-BC (nofeedback) duality in [11], [15] and because under (33) the MAC \( Y_{MAC} \) is dual to the BC in (24). In fact, under (33), \( h_1 \hat{Q}_1^{-1} = h_1 S_1^{-1} \) and \( h_2 \hat{Q}_2^{-1} = h_2 S_2^{-1} \).

Remark 1. The optimal MAC scheme is described in [1]. From this we can deduce the optimal MAC-parameters \( V_1 \), \( V_2 \), \( C_1 \), and \( C_2 \) describing the block inputs in (20). (A different set of parameters is required to approach each point on the boundary of the capacity region.) Now, by (32) and comparing (21) and (33), from these parameters we can deduce the optimal BC-parameters \( B_1 \) and \( B_2 \) describing the block inputs in (27). Finally, the results in [13] tell us how to code and decode over the resulting MIMO BC in (28).

Remark 2. Theorem 1 extends to the scalar Gaussian MAC and BC with \( K \geq 3 \) users.

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