Affine processes from the perspective of path space valued Lévy processes

Author(s):
Gabrielli, Nicoletta

Publication Date:
2014

Permanent Link:
https://doi.org/10.3929/ethz-a-010146407

Rights / License:
In Copyright - Non-Commercial Use Permitted
Affine processes from the perspective of path space valued Lévy processes

A dissertation submitted to

ETH ZURICH

for the degree of

Doctor of Sciences

presented by

Nicoletta Gabrielli

MSc. in Applied Mathematics, “La Sapienza”, Rome
born April 14, 1985
citizen of Italy

accepted on the recommendation of

Prof. Dr. Josef Teichmann examiner
Prof. Dr. Archil Gulisashvili co-examiner

2014
Abstract

Affine processes are continuous-time Markov processes characterized by the property that their Fourier–Laplace transform of the marginal distributions depends in an exponential affine way on the initial state. Due to their analytic tractability and high flexibility to model some specific patterns of the financial markets, the application of affine processes in mathematical finance, has widely increased lately. Such applications range over a broad variety of tasks common in finance like option pricing or term structure models simulation. The aim of this thesis is to obtain new results on affine processes and on their applications to mathematical finance, with a view towards numerics.

In many pricing problems, like pricing of certain exotic options, we are interested in a pathwise approximation of the process, but, for some examples of affine processes, the discrete approximation can be challenging due to the lack of Lipschitz regularity of the vector fields along the boundary of the state space. Not to mention the high–dimensionality of the problem. It is therefore desirable to develop, describe and implement high-order schemes for affine processes. Notice that, even in dimension $d = 1$, this is a delicate question if we are given the process defined by the SDE $dX_t = \sqrt{X_t}dW_t$.

In this thesis we delve into this matter by analyzing a new representation of affine processes as path-space valued Lévy processes. This new representation, building upon well-known relationships between the square root process and Poisson processes with exponential jumps, will lead to a new pathwise perspective on affine processes.
Sommario

Questa tesi si propone di ottenere risultati nel campo dei processi affini che siano utilizzabili in applicazioni, come ad esempio in matematica finanziaria. I processi affini sono una classe di processi di Markov a tempo continuo caratterizzati dal fatto che il logaritmo della trasformata generalizzata di Fourier è dato da una funzione lineare nella variabile di stato. Oltre ad essere modelli facilmente trattabili a livello analitico, i processi affini sono in grado di riprodurre specifiche realtà dei mercati finanziari, come ad esempio code spesse ed elevata curtosi delle distribuzioni dei rendimenti e volatility smiles. Per questo motivo i processi affini sono ora ampiamente utilizzati in matematica finanziaria per prezzaggio di opzioni e modellizzazioni di strutture di tassi a termine.

Quando si affrontano problemi numerici come prezzaggio di opzioni esotiche, risulta indispensabile essere in possesso di metodi di approssimazione che siano in grado di simulare accuratamente le traiettorie dei processi selezionati per modellizzare il prezzo del sottostante. Nel caso dei processi affini, tali approssimazioni si rivelano particolarmente ardue da implementare a causa della mancanza della proprietà di Lipschitz del campo vettoriale in prossimità del bordo dello spazio in cui sono definiti e della elevata dimensionalità del problema. Nel particolare, i metodi maggiormente utilizzati in queste situazioni, come quelli basati su sviluppi di Taylor stocastici e metodi di cubatura, non sono sempre applicabili nella forma standard a causa della degenerazione presente al bordo. Per questo motivo risulta essenziale sviluppare, descrivere ed implementare metodi numerici che siano efficienti per i processi affini. Perfino nel caso unidimensionale la questione si presenta complicata quando consideriamo processi definiti da una equazione differenziale stocastica del tipo $dX_t = \sqrt{X_t}dW_t$.

In questa tesi vogliamo investigare queste problematiche introducendo una rappresentazione dei processi affini in termini di processi di Lévy a valori su traiettorie. Questa rappresentazione, basandosi su ben note corrispondenze tra processi di tipo CIR e processi di Poisson a salti distribuiti in modo esponenziale, ci porterà ad analizzare i processi affini da una diversa prospettiva.
Acknowledgments

First of all, I would like to thank my supervisor Professor Josef Teichmann, who guided my progresses with his excellent support during my doctoral studies. I am grateful for the proposal of a stimulating and interesting subject within the field of affine processes, for his numerous advice and suggestions which considerably improved this thesis.

I am also thankful to Professor Archil Gulisashvili, for kindly accepting to act as the co-examiner of this thesis and providing me with helpful remarks and good inputs.

The financial support of this work was provided by the ETH Zurich and by the SNSF, to which I gratefully acknowledge.

Special thanks are also due to my colleagues in “Gruppe 3” for the good times at ETH. Thanks for the nice work environment, the interesting discussions, the trips in Europe we had together and your support for the SOLA race. Moreover, I would like to thank all my friends for the pleasant time they provided me in Zurich.

I would like to thank my family and Roberto, for their patience, confidence and encouragement over the years leading up to this work.

Finally, I would like to express my deep gratitude to Professor Peter Laurence, who aroused my interest in mathematical finance and gave me the boost to start this great experience in ETH. This thesis is dedicated to him.
# Contents

Abstract i
Sommario iii
Acknowledgments v

Introduction 1

0 Notation 7

1 Affine processes 21
  1.1 Introduction .............................................. 21
  1.2 Definitions and main properties ......................... 22
      1.2.1 Feller property ..................................... 29
      1.2.2 Semimartingale characterization ................... 31
      1.2.3 Infinitely decomposable processes ................ 32
  1.3 Elementary transformations of affine processes .......... 33
      1.3.1 Affine processes in canonical form ................. 33
      1.3.2 From affine processes to linear processes ......... 36
      1.3.3 From $C^*$ to $C^H$ .................................. 38
      1.3.4 Summary of the assumptions ....................... 39
  1.4 Examples .................................................. 39

2 Time–space representation 45
  2.1 Introduction ................................................ 45
## Contents

2.2 Path–space representation of affine processes .......................... 46  
  2.2.1 How to visualize the affine property ............................. 47  
  2.2.2 Examples ......................................................... 51  
  2.2.3 Applications to linear functionals of affine processes .............. 56  

2.3 Lamperti representation .................................................. 59  
  2.3.1 The one–dimensional case ........................................... 60  
  2.3.2 The multi–dimensional case ........................................ 63  
  2.3.3 Existence of random time changes .................................... 68

3 High order numerical schemes for affine processes .......................... 83  
  3.1 Introduction ............................................................. 83  
  3.2 A short survey on weak approximation schemes .......................... 85  
  3.3 Weak approximation schemes for affine processes ....................... 88  
    3.3.1 Preliminary observations ......................................... 89  
    3.3.2 Convergence results .............................................. 94  
    3.3.3 Analysis of the convergence rate .................................. 100  
    3.3.4 Examples ........................................................ 107

4 Kolmogorov equations of affine type ........................................... 113  
  4.1 Introduction ............................................................. 113  
  4.2 Martingale problem and short–time asymptotic formula .................. 115  
  4.3 Markov semigroup on weighted spaces ................................... 119  
    4.3.1 Functions with controlled growth .................................. 119  
    4.3.2 Differentiable functions with controlled growth ................. 121  
    4.3.3 Lévy–type operators on weighted spaces ............................ 123  
  4.4 Regularity results for affine–type operators ............................. 126  
    4.4.1 Results on $C^\infty_{\text{pol}}$ .................................. 129

Conclusion ................................................................. 140

Appendix ................................................................. 140

A On the Skorohod space .................................................... 141  
  A.1 The Skorohod metric .................................................. 141  
  A.2 The uniform topology ................................................ 142  
  A.3 The dual space ........................................................ 143

B Markov processes on Polish spaces ........................................... 145
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.1 Kolmogorov's extension theorem</td>
<td>145</td>
</tr>
<tr>
<td>B.2 Killed Markov processes</td>
<td>146</td>
</tr>
<tr>
<td>C Lévy processes</td>
<td></td>
</tr>
<tr>
<td>C.1 Existence</td>
<td>147</td>
</tr>
<tr>
<td>C.2 Relation with infinitely divisible distributions</td>
<td>147</td>
</tr>
<tr>
<td>C.3 Lévy-Itô decomposition</td>
<td>148</td>
</tr>
<tr>
<td>C.4 Càdlàg version</td>
<td>154</td>
</tr>
<tr>
<td>C.5 Path properties</td>
<td>156</td>
</tr>
<tr>
<td>D Multiparameter random time change</td>
<td></td>
</tr>
<tr>
<td>D.1 Definitions</td>
<td>159</td>
</tr>
<tr>
<td>D.2 Time-change equations</td>
<td>160</td>
</tr>
<tr>
<td>Bibliography</td>
<td>165</td>
</tr>
</tbody>
</table>
Introduction

The need for more realistic and flexible models able to describe the dynamics of financial markets has led to an increased interest from academics and practitioners into the field of Lévy processes. This is due to the fact that they can accurately describe heavy-tailed and skewed distributions typical of asset returns. Moreover, semi-closed form valuation formulas are available for simple contracts, such as plain vanilla options. In the case of exotic options, probabilistic and deterministic numerical methods are more commonly picked. For a self-contained overview of the theoretical and numerical aspects of Lévy processes, such as path approximation and option pricing with Monte Carlo or PDE methods, we refer the interested reader to [CT04]. Although the sample paths of Lévy processes allow for realistic properties such as jumps and tail dependencies, they may be unsatisfactory for modeling features such as a heavy skewed volatility smile or positivity of volatility dynamics. Stochastic volatility models, for instance the Heston model [Hes93], were developed in order to reproduce volatility surfaces with a much larger skew than that typical in the Lévy setting. Square-root diffusion processes are also very popular in mathematical finance due to their property of staying non-negative and having mean-reverting features. Historically, they were introduced by Kawazu and Watanabe as the continuous time limit of Galton-Watson branching processes, see [KW71]. The class of affine processes contains both Lévy processes, continuous state branching processes as well as their multidimensional generalizations such as matrix-valued Wishart processes, see [Bru91]. For practical applications of the theory in pricing options the reader is referred to [DFS03], and for modeling covariance processes to [CFMT09].

In this thesis we focus on affine processes defined on the canonical state space and their applications in option pricing. Suppose that all the information available in the market is described by a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and
that the underlying log-price process is modeled by means of a stochastic process $X = (X_t)_{t \in [0,T]}$ which is a semimartingale. By the fundamental theory of asset pricing, under the assumption that the interest rate is constantly equal to zero, the price at time $t$ of a contract $H$ is given by

$$V_t^H := \mathbb{E} \left[ H \left( (X_t)_{t \in [0,T]} \right) \bigg| \mathcal{F}_t \right],$$

where the expectation is taken under a measure $\mathbb{P}$, under which $e^X$ is a (local) martingale.

If the payoff $H$ depends only on the value of the stock at maturity, we can interpret (*) as the convolution between the payoff function $H$ and the distribution of $X_T$. Then, up to a modification of the payoff function, a variant Parseval’s theorem can be applied in order to write (*) as

$$\mathbb{E} \left[ H(X_T) \right] = \int H(\xi) \mathbb{P}(X_T \in d\xi) = \int (\mathcal{F}H)(u) \mathbb{E} \left[ e^{i(u,X_T)} \right] du,$$

where $\mathcal{F}H$ denotes the Fourier transform of the function $H$.

Although the Fourier transform of affine processes is known analytically up to the solution of a differential equation of Riccati type, the use of Fourier methods has some drawbacks when applied to solving equations of type (*):

1. It works well for European style contracts but is less well applicable for path dependent contracts.

2. When dealing with high dimensional contracts, like basket options, the computation of the integral over a high dimensional domain can be computationally expensive.

3. The method requires the explicit knowledge of the Fourier transform, which is not always available for affine processes since it is known only up to the knowledge of the solution of the nonlinear ODE defined by the Riccati equation.

In this thesis we address these issues by representing affine processes as a realization of path–space valued Markov processes. This perspective gives a good insight on how to deal with path dependent options written on a stock whose dynamics are specified by an affine process. Indeed, suppose we can find a path–space valued Markov process $L$ such that its distribution at time $s = 1$ coincides with the distribution of
the process $X$, seen as a random variable in the path–space. Then it holds

$$\mathbb{E}\left[H\left((X_t)_{t\in[0,T]}\right)\right] = \mathbb{E}\left[H(L_s)\right]_{s=1}.$$

Therefore, under this perspective, the problem of pricing a path dependent option translates into the problem of pricing a European–style option. The theorem which is the cornerstone of the thesis is Theorem 2.5. It provides the existence of the aforementioned path–valued Markov process $(L_s)_{s\geq0}$. Since we can depict the paths of an affine process as realizations of a path–space valued process, we can additionally conclude that

$$\mathbb{E}\left[H(X_t)\right] = \mathbb{E}\left[H(\text{ev}_t(L_s))\right]_{s=1},$$

where $\text{ev}_t$ is the evaluation at time $t$ of the path–valued process $L$. We show that the process $\text{ev}_tL$, for each fixed $t \geq 0$, is a Lévy process. Eventually, in Section 2.2, we derive the existence of a two parameter process $(L_s(t,x))_{s,t\geq0}$ such that

- for fixed $t$, $(L_s(t,x))_{s\geq0}$ is a Lévy process whose distribution at time 1 coincides with the distribution at time $t$ of an affine process $X$ starting from $X_0 = x$,
- for fixed $s$, $(L_s(t,x))_{t\geq0}$ is an affine process $X$ starting from $X_0 = sx$,
- $L(t,x)$ is a process with stationary and independent increments taking values in $D$, the Skorohod space of càdlàg paths.

We call them representing Lévy processes. The theorem extends the results for the genealogy of branching processes presented in [Lam02]. Our formulation is in line with the one for affine process in $R_{\geq0}$ in [Bou09].

At this stage a natural questions arises:

**Question 1:** Is it possible to explicitly characterize the representing Lévy process for a given affine process?

The second part of Chapter 2 focuses on answering this question. Under some assumption on the semimartingale characteristics of the affine process, we can distinguish the representing Lévy process by its path type. In particular we collect in Section 2.2.2 results about their support, path variation and moments. Moreover, we provide in Section 2.2.3 analytic expressions for their characteristic functional, extending the results in [PY82].

The next question we address is the following:
Question 2: Is it possible to provide some “good” approximation for the representing Lévy processes, which can be used in order to evaluate (\(\ast\)) numerically?

Weak approximation plays a crucial role in the numerical calculation of \((\ast)\). Chapter 3 deals with the construction of a Markov chain \(\tilde{X}\), which approximates the trajectories of \(X\) on a time grid. To underline the dependence on the initial point \(x\), we will often write \(X^x\) and \(\tilde{X}^x\). From Chapter 2, we known that, if we fix a partition \(\{t_0 = 0 < t_1 < t_2 < \ldots < t_N = T\}\) of \([0,T]\), the distribution of \((X_{t_0} = x, X_{t_1}, \ldots, X_{t_N})\) coincides with the distribution of \((x, L_1^{(t_1,x)}, \ldots, L_1^{(t_N,x)})\). The approximation presented in Chapter 3 stems from the representation of the finite dimensional distributions of an affine process by means of the representing Lévy processes. In particular, the numerical scheme is based on the approximation of the system of Riccati ODEs defining the Fourier–Laplace transform of \(X\). The main advantage of our scheme is its property of being geometry preserving. For example, in the case of the square root process, usual schemes, based on a short-time expansion for the transition semigroup, require additional conditions to keep the approximated process away from zero. The scheme we propose here keeps the support invariant. We do also provide the analysis for the convergence rates. We suppose that \(\{t_0, \ldots, t_N\}\) is a uniform partition with mesh size \(h > 0\). Then, by definition (see for example [Alf10]), a Markov chain \(\tilde{X}\) is a weak approximation of \(X\) of order \(\nu\) if, for every \(f\) smooth function with compact support, there exists a constant \(C\) such that

\[
\left| \mathbb{E}[f(X^x_T)] - \mathbb{E}[f(\tilde{X}^x_{t_N})] \right| \leq Ch^\nu.
\]

In Theorem 3.12, we show that the sequence \(\tilde{X}\) converges weakly to \(X\) and then, in Theorem 3.13, we show that the proposed scheme converges with order \(\nu \geq 2\). We actually prove the convergence rate on a bigger class, the one of smooth functions with polynomial growth, henceforth denoted by \(C_{\infty}{_{\text{pol}}}\). The choice of \(C_{\infty}{_{\text{pol}}}\) for the analysis of convergence rates is standard, see [Alf10, JKMP05, TT90] and [TK09].

To sum it up, in Chapter 3, we provide a scheme to approximate the trajectories of an affine process. It would be extremely valuable to extend the analysis of the convergence rate to path dependent functionals. The problem now is:

Question 3: Do we observe the same convergence rate when we approximate the quantity \((\ast)\) by replacing \(X\) with the approximation \(\tilde{X}\)?
options, it is possible to augment the state process from $X$ to $(X,Y)$ where $Y$ is an additional process which allows us to reduce to the case studied before. Some examples and additional references are provided in Section 4 in [DG95]. However, for a general path–dependent option $H$, weak convergence of $\overline{X}$ to $X$ is barely enough to conclude convergence of $H(\overline{X})$ to $H(X)$. Actually, even for European–style options, there is the need for a mathematical theory of weak convergence which applies to functions with suitable growth conditions. This is the main focus of Chapter 4. The starting point is the identification of $\mathbb{E}\left[H(X^+_h)\right]$ with the semigroup associated with the affine process $X$ acting on the function $H$. We write

$$P_hH(x) = \mathbb{E}\left[H(X^+_h)\right].$$

The idea is to approximate the value of $P_hH(x)$ with

$$Q_hH(x) = \mathbb{E}\left[H(\overline{X}^+_h)\right],$$

where $\overline{X}_h$ is a random variable such that

$$|P_hf(x) - Q_hf(x)| \leq h^{\nu+1}F(x), \quad h \approx 0,$$

for all $f \in \mathcal{H}$, where $\mathcal{H}$ is a class of functions containing also the target payoff function $H$, and $F$ is a function which depends on the growth of $f$. Then, if the operators $P$ and $Q$ satisfy appropriate stability conditions regarding their action on the class $\mathcal{H}$, we can expand the local error as

$$P_Tf(x) - Q_h^Nf(x) = \sum_{k=1}^{N} Q_h^{N-k}(Q_h - P_h)P_kf(x),$$

where, for all $n = 1, \ldots, N$, $Q_h^n$ is the operator obtained by taking the $n$th–composition of $Q_h$.

Mainly two are properties which are required. The first one is a small time asymptotic of $P_hf$ and $Q_hf$ of some order $O(h^{\nu+1})$ for each $f$ in $\mathcal{H}$. The second one is a regularity property of the semigroup which implies preservation of the local error. The implication of these two requirements is that the function space $\mathcal{H}$ needs to satisfy, at least, the following two conditions

**H1)** for all $f \in \mathcal{H}$, there exists a $\nu \in \mathbb{N}$ such that $(P_h - Q_h)f \simeq O(h^{\nu+1}),$

**H2)** for all $f \in \mathcal{H}$, $P_tf \in \mathcal{H}$ and $Q_tf \in \mathcal{H}$. 

5
While the first property concerns the discretization scheme, the second one relates to the application of the semigroup on $\mathcal{H}$. In Theorem 4.2 we establish a connection between the Kolmogorov equation satisfied by the semigroup $(P_t)_{t \geq 0}$ and the extended generator of the Markov process. This result is well known for Feller processes, but here we allow the class of test functions to be much more general, including functions which are not bounded at infinity. Then, we derive, in Proposition 4.4, a short time asymptotic formula for the transition semigroup. This is a very important step since it gives a good insight on how to construct weak approximation schemes which satisfy $H1)$. The results in Section 4.2 are established under very general assumptions. In order to further deepen the analysis, we focus on the class of smooth functions with growth controlled by a common weight function $F$. When $\mathcal{H}$ is the class of smooth functions with controlled growth, $H2)$ requires that the function $P_t f(x)$, seen as a function of $(t, x) \in (0, T] \times D$ for fixed $f \in \mathcal{H}$, has derivatives of all orders which are again in $\mathcal{H}$. Hence $H2)$ boils down once it holds that

- all the derivatives $\partial^{\alpha}_{(t, x)} P_t f(x)$ exist for any multi–index $\alpha$ running over the time and space variables,

- all the derivatives have the same growth.

Differentiability in time can be handled using the stochastic Taylor expansion derived in the previous section. Differentiability in space of the transition semigroup is deeply related with regularity of the corresponding solution of the Kolmogorov equation derived in Theorem 4.2. For affine processes, the vector field defining the Kolmogorov equation is a second order pseudo differential operator which is degenerate along the boundary of the state space. Hence the analysis of regularity of the solution is involved. Additionally, due to the lack of hypoellipticity property, standard arguments cannot be applied. The way we achieve the result is by making a fruitful use of the time–space shift we introduced in Chapter 2. The idea is to apply the short time asymptotic formula we derived in the previous section, after having done the time–space shift. The main result in Chapter 4 states that the problem of regularity with respect to the space variable can be translated into regularity with respect to time of the representing Lévy processes. Theorem 4.19 summarizes the results on the class $C^\infty_{\text{pol}}$. 
Chapter 0

Notation

Throughout $D$ denotes a closed subset of a $d$-dimensional real vector space $\mathbb{R}^d$ with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $|\cdot|$. The same notation is used also when the scalar product is considered in the space $\mathbb{R}^d + i\mathbb{R}^d$. In this case we mean the extension of $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^d + i\mathbb{R}^d$ without conjugation. With $S_+^d$ we denote the cones of positive semidefinite $d \times d$ matrices.

**Convolution of measures** Given a family of measures $\{\mu_i\}_{i=1,\ldots,n}$ in $D$, the convolution measure $\mu^* := \mu_1 \ast \ldots \ast \mu_n$ is the image under the map

$$T : \quad D^n \quad \rightarrow \quad D$$

$$(x_1, \ldots, x_n) \quad \mapsto \quad x_1 + \ldots + x_n,$$

of the product measure

$$(\mu_1 \otimes \ldots \otimes \mu_n)(B) := \int_{D^n} \mu_1(dx_1) \ldots \mu_n(dx_n) 1_B(x_1, \ldots, x_n).$$

**Infinitely divisible measures** A probability measure $\mu$ on $D$ is called *infinitely divisible* if, for any positive integer $n$, there exists a probability measure $\mu_n$ on $\mathbb{R}^d$ such that $\mu = \mu_n^*$. Infinitely divisible distributions are fully characterized by their characteristic functions by means of the Lévy-Khintchine formula. The Lévy-Khintchine formula defines a one-to-one correspondence between infinitely divisible distributions and triplets $(b, \sigma, \nu)$ with $b \in \mathbb{R}^d$, $\sigma \in S_+^d$ and $\nu$ a Lévy measure, (see Theorem 8.1 in [Sat99]). The drift term $b$ depends on the choice of a truncation function, which is a continuous functions bounded in norm by 1 which equals the identity in a neighborhood of zero. The triplet
(b, σ, ν) is called Lévy triplet or also characteristic triplet. Here we will make use of the first terminology.

**Convolution semigroup** A family of probability measures \((\mu_t)_{t \geq 0}\) is called a convolution semigroup if \(\mu_{s+t} = \mu_s * \mu_t\) for all \(s, t \geq 0\).

**Probability kernel** A probability kernel from \(D\) to itself is a function

\[
p : D \times B(D) \to [0, 1],
\]

such that

1) for any \(B \in B(D)\), \(p(x, B)\) is \(B(D)\) measurable,

2) for any \(x \in D\), \(p(x, B)\) is a probability measure on \(D\).

**Composition of kernels** Given two probability kernels \(p\) and \(q\) from \(D\) to \(D\), their composition is defined by

\[
(p \otimes q)(x, B) = \int p(x, dy)q(y, dz)1_B(y, z), \quad B \in B(D \times D),
\]

and it is a probability kernel from \(D\) to \(D\).

**Product of kernels** Given two probability kernels \(p\) and \(q\) from \(D\) to \(D\), their product is defined by

\[
(p \cdot q)(x, B) = \int p(x, dy)q(y, B) = (p \otimes q)(x, D \times B), \quad B \in B(D),
\]

and it is a probability kernel from \(D\) to \(D\).

Regarding functions spaces, the following notation will used from now on:

- \(m(D)\) is the space of measurable function on \(D\),
- \(m_{b\text{bd}}(D)\) is the space of measurable bounded function on \(D\),
- \(C(D)\) is the space of continuous functions on \(D\),
- \(C_b(D)\) is given by \(C(D) \cap m_{b\text{dd}}(D)\),
- \(C_c(D)\) is given by all the functions in \(C(D)\) with compact support,
- \(C_0(D)\) is given by all the functions in \(C(D)\) vanishing at infinity,
- \(C^\infty(D)\) is given by all the smooth functions,
- \( C_c^{\infty}(D) \) is given by all the functions in \( C^{\infty}(D) \) with compact support.

Given a function, \( f : D \mapsto \mathbb{R} \) the norm \( ||\cdot|| \) denotes the sup norm

\[
||f|| := \sup_{x \in D} |f(x)|.
\]

For ease of notation, we will denote the components of the constant quantities differently from the components of the time dependent vectors. If \( x \in \mathbb{R}^d \), then we write \( x = (x_1, \ldots, x_d) \) where \( x_j \) denotes the \( j \)-th component. On the other hand, if \( (X_t)_{t \geq 0} \) is a \( \mathbb{R}^d \)-valued stochastic process, we select the \( j \)-th component by writing \( (X_t^{(j)})_{t \geq 0} \). Whenever necessary, we use the notation \( X^x \) to underline the initial position of the process \( X \). In this case, we lighten the notation and simply use \( X^{(j)} \) without the dependence on the initial state.

**Markov processes**

We refer to [GS04],[GS74], [CW05] and [RW00] for a general treatment of stochastic processes in the Markovian and non-Markovian setting.

Adjoin the space \( D \) with an additional point \( \Delta \notin D \), called the cemetery state. Denote by \( D_\Delta \) the one-point compactification of \( D \), \( D_\Delta = D \cup \{\Delta\} \). All the functions \( f \) on \( D \) are extended to \( D_\Delta \) by defining \( f(\Delta) = 0 \).

**Markov Semigroup** A semigroup of transition functions is a family of kernels \( (p_t)_{t \geq 0} \) with

\[
p_t : D_\Delta \times \mathcal{B}(D_\Delta) \to [0, 1]
\]

such that

1. \( p_t(x, \cdot) \) is a probability measure on \( \mathcal{B}(D_\Delta) \) for all \( t \geq 0 \) and \( x \in D_\Delta \) such that
   - \( p_t(x, D) \leq 1 \),
   - \( p_t(x, \{\Delta\}) = 1 - p_t(x, D) \),
   - \( p_t(\Delta, \{\Delta\}) = 1 \),

2. \( p_0(x, \cdot) = \delta_x(\cdot) \) for all \( x \in D_\Delta \),

3. the map \( x \to p_t(x, B) \) is \( \mathcal{B}(D_\Delta) \)-measurable for all \( t \geq 0 \) and \( B \in \mathcal{B}(D_\Delta) \),
4. the Chapman-Kolmogorov equations are satisfied

\[ p_{t+s}(x, B) = \int_{D_\Delta} p_s(x, d\xi) p_t(\xi, B), \]

for all \( s, t \geq 0, x \in D_\Delta \) and \( B \in \mathcal{B}(D_\Delta) \).

A semigroup of transition functions \((p_t)_{t \geq 0}\) on \( D \) induces a semigroup \((P_t)_{t \geq 0}\) acting on the space \( \text{mbdd}(D) \) by

\[ P_t f(x) = \int_D f(\xi) p_t(x, d\xi), \quad \text{for } x \in D, t \geq 0. \]

If, for all \((t, x) \in \mathbb{R}_{\geq 0} \times D\), it holds \( p_t(x, D) = 1 \), the process is said to be conservative.

**Markov Process** A time–homogeneous Markov process with state space \((D, \mathcal{B}(D))\) (eventually adjoint with \( \Delta \)) is a family

\[ X = (\Omega, (X_t)_{t \geq 0}, (\mathcal{F}_t^x)_{t \geq 0}, (p_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D_\Delta}), \]

where

- \( \Omega \) is a probability space,
- \((X_t)_{t \geq 0}\) is a stochastic process taking values in \( D_\Delta \),
- \( \mathcal{F}_t^x = \sigma(\{X_s, s \leq t\}) \),
- \((p_t)_{t \geq 0}\) is a semigroup of transition functions on \((D_\Delta, \mathcal{B}(D_\Delta))\),
- \((\mathbb{P}^x)_{x \in D_\Delta}\) is a probability measures on \((\Omega, \mathcal{F}^x)\), with \( \mathcal{F}_t^x = \bigvee_{t \geq 0} \mathcal{F}_t^x \),

satisfying

\[ \mathbb{E}^x \left[ f(X_{t+s}) \middle| \mathcal{F}_t^x \right] = \mathbb{E}^{X_t} \left[ f(X_s) \right], \quad \mathbb{P}^x\text{-a.s. for all } f \in \text{mbdd}(D_\Delta). \quad (0.1) \]

**Canonical version** Given a semigroup of transition functions \((p_t)_{t \geq 0}\) and an initial distribution \( \nu \), Kolmogorov’s extension theorem (see Theorem B.1) gives the
existence of a probability measure on the path–space
\[ \Omega := \{ \omega : \mathbb{R}_{\geq 0} \to D_{\Delta} \text{ such that } \omega(s-) = \Delta \text{ or } \omega(s) = \Delta \implies \omega(t) = \Delta \text{ for all } t \geq s \}. \]

Then, the canonical way to construct a stochastic process corresponding to \((p_t)_{t \geq 0}\) is given by considering the coordinate map on \(\Omega\)

\[ X_t : \Omega \to D_{\Delta} \]
\[ \omega \mapsto X_t(\omega) := \omega(t). \]

The distribution of \(X_t\) is denoted by \(\lambda_t\) and it given by the push forward measure \(\lambda_t = P^x \circ X_t^{-1}\).

The natural filtration generated by a stochastic process \((X_t)_{t \geq 0}\), will be always denoted by \(\mathcal{F}^X\).

Path regularity From Kolmogorov’s existence theorem, we get existence of a Markov process on the path–space of all possible functions from \([0, \infty)\) to \(D\), possibly augmented with \(\Delta\).

A stochastic process \(X\) has càdlàg paths if, for each \(\omega \in \Omega\), the sample paths \(t \mapsto X_t(\omega)\) are right continuous with left limits for all \(t \geq 0\).

Under some conditions, it is possible to prove that the paths of a stochastic process live on the Skorohod space \(\mathcal{D}(D)\) of càdlàg functions. With “living” in specific functions space, we mean that the measure \(P^x\) assigns to \(\mathcal{D}(D_{\Delta})\) outer measure 1 and, therefore, admits a unique restriction as a probability measure on that space. \(\mathcal{C}(D) \subset \mathcal{D}(D)\) denotes the set of continuous paths.

Augmented filtration The notation \(\mathcal{F}^x\) (resp. \((\mathcal{F}^x_t)_{t \geq 0}\)) will be always used to denote the canonical \(\sigma\)-algebra (resp. filtration). For any \(x \in D\), \(\mathcal{F}^x\) is the completion of \(\mathcal{F}^x\) with respect to the probability measure \(P^x\). Analogously the augmentation of \((\mathcal{F}^x_t)_{t \geq 0}\) with respect to \(P^x\) is the \(\sigma\)-algebra generated by
\[(\mathcal{F}_t^x)_{t \geq 0}\] and the sets of \(\mathbb{P}^x\)-measure zero in \(\mathcal{F}^x\). Precisely, let \(\mathcal{N}(\mathcal{F}^x) = \{A \in \mathcal{F}^x \mid \mathbb{P}^x(A) = 0\}\). Then define
\[
\mathcal{F}_t^x = \sigma\left(\{\mathcal{F}_0^x, \mathcal{N}(\mathcal{F}^x)\}\right).
\]

**Usual conditions** A filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) satisfies the usual conditions if

- \(\mathcal{F}\) is \(\mathbb{P}\)-complete,
- \(\mathcal{F}_0\) contains all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\),
- \((\mathcal{F}_t)_{t \geq 0}\) is right continuous, i.e. \(\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s\).

**Feller Process** A time homogeneous Markov process is called a *Feller process* if its transition semigroup \((P_t)_{t \geq 0}\) satisfies

1. \(P_tC_0(D) \subseteq C_0(D)\) for all \(t \geq 0\),
2. \(\lim_{t \to 0} P_tf(x) = f(x)\) uniformly in \(x \in D\) for all \(f \in C_0(D)\).

**Infinitesimal generator** Given a Markov process, suppose that \(f \in \text{mbdd}(D)\) is a function such that
\[
\lim_{t \to 0} \frac{P_tf - f}{t}
\]
exists in the supremum norm. Denote by \(Af\) the limit in (0.2). The set of all the functions \(f \in \text{mbdd}(D)\) such that (0.2) is well defined is denoted by \(\mathcal{D}(A)\). The operator
\[
A : \mathcal{D}(A) \to C_0(D)
\]
is called *infinitesimal generator* of \(X\).

If \(X\) is a Feller process, it holds \(C_0(D) \subseteq \mathcal{D}(A)\). Moreover, for all \(f \in C_0(D)\),
\[
M^f_t := f(X_t) - f(x) - \int_0^t Af(X_s)ds
\]
is a \(\mathbb{P}^x\) true martingale, for all \(x \in D\).

**Extended generator** The *extended generator* \(\mathcal{A}\) of a Markov process \(X\) is defined by
\[
\mathcal{A} := \{(f, h) \in m(D) \times m(D) \mid M^f\text{ is a local martingale}\}\]
where

\[ M^f_t := f(X_t) - f(x) - \int_0^t h(X_s) ds \]

and the function \( s \mapsto h(X_s) \) is integrable \( \mathbb{P}^x \)-a.s. for all \( x \in D \), on \([0, t]\) for all \( t \geq 0 \). The process \( M^f \) is called Dynkin’s local martingale. Given \( f \), the function \( h \) is uniquely determined up to sets of potential zero. Denote all these versions by \( A^f \), if \((f, h) \in \mathcal{A}\).

Let

\[ \mathcal{D}(\mathcal{A}) := \left\{ f \in m(D) \text{ such that } \mathcal{A}f \in m(D) \text{ } \mathbb{P}^x \text{-a.s. for all } x \in D \right\} \]

and \( M^f \) is a \( \mathbb{P}^x \)-local martingale for all \( x \in D \).

The set \( \mathcal{D}(\mathcal{A}) \) is called the domain of the extended generator \( \mathcal{A} \).

### Infinitely decomposable Markov processes

When \( \mathbb{P} \) and \( \mathbb{Q} \) are two measures on the path–space \( (\Omega, \mathcal{F}^x) \), the notation \( \mathbb{P} \ast \mathbb{Q} \) denotes the image of the product measure \( \mathbb{P} \otimes \mathbb{Q} \) by the measurable map

\[ (\Omega \times \Omega, \mathcal{F}^x \otimes \mathcal{F}^x) \to (\Omega, \mathcal{F}^x) \]

\[ (\omega_1, \omega_2) \mapsto \omega_1 + \omega_2. \]

In \( \mathcal{M}(\Omega) \), the space of all the measures in \( \Omega \), we select the subset \( \mathcal{M}'(\Omega) \) containing all the family of measures \( \mathcal{P} := (\mathbb{P}^x)_{x \in D} \) such that \((X, \mathbb{P}^x)\) is a stochastically continuous Markov process with \( \mathbb{P}^x(X_0 = x) = 1 \).

A Markov process \((X, \mathcal{P})\) is infinitely decomposable if, for every \( n \in \mathbb{N} \) there exists a family \( \mathcal{P}^{(n)} \) in \( \mathcal{M}'(\Omega) \) such that every element \( \mathbb{P}^x \in \mathcal{P} \) admits a decomposition

\[ \mathbb{P}^x = \mathcal{P}^{(1)} \ast \ldots \ast \mathcal{P}^{(n)}, \]

where \( \mathcal{P}^{(k)} \in \mathcal{P}^{(n)} \), for each \( k = 1, \ldots, n \) and moreover \( \mathcal{P}^{(k)}(X_0 = x^{(k)}) = 1 \) with \( x = x^{(1)} + \ldots + x^{(n)} \).
Lévy processes and infinite divisibility

If a semigroup of transition functions \((p_t)_{t \geq 0}\) is homogeneous in the space variable, meaning that, for all \(x \in D\) and \(B \in \mathcal{B}(D)\)

\[
p_t(x, B) = p_t(0, B - x),
\]
we can define a convolution semigroup by

\[
\mu_t(B) := p_t(0, B - x).
\]

Starting from a convolution semigroup \((\mu_t)_{t \geq 0}\), Kolmogorov’s extension Theorem (Theorem B.1) gives the existence of a class of Markov processes called Lévy processes.

Due to translation invariance property of the semigroup of transition functions, we do always assume that a Lévy process starts from zero. If necessary, we change the notation and denote \(P_0\) with \(P^{(b, \sigma, \nu)}\).

Lévy process A time homogeneous Markov process \(X\) is a Lévy process if the following three conditions are satisfied:

L1) \(X_0 = 0\) \(P^{(b, \sigma, \nu)}\)-a.s.

L2) \(X\) has independent and stationary increments, i.e. for all \(n \in \mathbb{N}\) and \(0 \leq t_0 < t_1 < \ldots < t_{n+1} < \infty\)

(independence) the random variables \(\{X_{t_{j+1}} - X_{t_j}\}_{j=0,\ldots,n}\) are independent,

(stationarity) the distribution of \(X_{t_{j+1}} - X_{t_j}\) coincides with the distribution of \(X_{(t_{j+1} - t_j)}\).

L3) (stochastic continuity) for each \(a > 0\) and \(s \geq 0\),

\[
\lim_{t \rightarrow s} P(|X_t - X_s| > a) = 0.
\]

The additional regularity assumption, stochastic continuity, allows us to consider the càdlàg version of Lévy processes.

Relation with infinite divisibility The distributions of Lévy processes are infinitely divisible. Given an infinitely divisible measure, there exists a Lévy
process whose one dimensional distributions are specified by that measure.

With $P^{(b,\sigma,\nu)}$ being an infinitely divisible measure we mean that the one-dimensional distributions $\mu_t := P^{(b,\sigma,\nu)} \circ X^{-1}_t$ are infinitely divisible for all $t \in \mathbb{R}_{\geq 0}$.

**Semimartingale theory**

In this section we assume that $X$ is a stochastic process with càdlàg paths.

**Semimartingale** A stochastic process $(X_t)_{t \geq 0}$ is called a semimartingale if it has the following representation

$$X_t = X_0 + M_t + A_t,$$

where $M$ is a local martingale starting from zero and $A$ is an adapted process with bounded variation starting from zero. The above representation is, in general, not unique. If, in addition, $A$ can be chosen to be predictable, then, from Doob-Mayer decomposition, it follows that the representation is unique. In this case the semimartingale is called special.

**Characteristics** The characteristics of a semimartingale can be defined in several equivalent ways, for a comprehensive treatment we refer to [JS87]. Given a semimartingale $X$ and a truncation function $h$, the characteristics of $X$ with respect to $h$ are specified by a triplet $(B(h), A, K)$ such that, for each $u \in \mathbb{R}^d$,

$$e^{\langle u, X_t \rangle} - \int_0^t e^{\langle u, X_s \rangle} d\Xi_s(u), \quad t \geq 0$$

is a (complex-valued) local martingale, where the process $(\Xi_s(u))_{s \geq 0}$ is defined as

$$\Xi_s(u) = \langle B(h)_s, u \rangle + \frac{1}{2} \langle u, A_s u \rangle + \int \left( e^{\langle u, \xi \rangle} - 1 - \langle u, h(\xi) \rangle \right) K_s(d\xi),$$

with

- $B(h) = (B(h)_1, \ldots, B(h)_d)$ a finite variation predictable process,
- $A = (A_{ij})_{i,j=1,\ldots,d}$ predictable process with values in $S^d_+$, of finite variation,
• $K$ a predictable random measure on $\mathbb{R}_{\geq 0} \times \mathbb{R}^d$.

In the whole thesis, we will fix as truncation function $h(x) = x1_{\{|x| \leq 1\}}$. With abuse of notation we will write $B$ instead of $B(h)$ to denote the predictable process associated with the canonical decomposition with this particular choice of truncation function. Observe that, this is the only element on the characteristics which depends on the choice of $h$. However, if $h'$ is another truncation function, we can always write, up to evanescent sets,

$$B(h) - B(h') = \int_{[0,t] \times \mathbb{R}^d} (h(\omega, ds, d\xi) - h'(\omega, ds, d\xi))K(\omega, ds, d\xi). \quad (0.3)$$

In most applications the characteristics $(B, A, K)$ are absolutely continuous, meaning that there exists a triplet $(B, A, K)$ such that

$$B_t = \int_0^t B_s ds,$$
$$A_t = \int_0^t A_s ds,$$
$$K(\omega, [0,t], d\xi) = \int_0^t K_s(\omega, d\xi) ds.$$

The triplet $(B, A, K)$ is called differential characteristics of $X$ and, following the notation in the literature, it will be denoted by $\partial X := (B, A, K)$.

**Markov semimartingale** In this section we follow [CFY05]. Given a conservative Markov process

$$X = (\Omega = D(D), (X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D}),$$

we say that $X$ is a semimartingale if it is a semimartingale in the probability space $(\Omega, \mathcal{F}^x, (\mathcal{F}_t^x)_{t \geq 0}, \mathbb{P}^x)$ for all $x \in D$. In order to deal with explosions and killing, we introduce

$$\tau_\Delta(\omega) := \inf\{t > 0 \mid X_t^{-}(\omega) = \Delta \text{ or } X_t(\omega) = \Delta\}. \quad (0.4)$$

Since $X$ has càdlàg paths, $\tau_\Delta$ is a stopping time. By definition of $D(D_\Delta)$, $X_t 1_{[\tau_\Delta, \infty[} = \Delta$ and, for this reason, the stopping time $\tau_\Delta$ is usually called the lifetime of the process $X$.

A process could explode in finite time to $\Delta$ or it could be sent in $\Delta$ by a jump.
This motivates the following additional notation:

\[
\tau^J_\infty := \begin{cases}
\tau_\Delta & \text{if } \tau_n = \tau_\Delta \text{ for some } n, \\
\infty & \text{if } \tau_n < \tau_\Delta, \text{ for all } n,
\end{cases}
\]  
\( (0.5) \)

\[
\tau_\infty := \begin{cases}
\tau_\Delta & \text{if } \tau_n < \tau_\Delta \text{ for all } n, \\
\infty & \text{if } \tau_n = \tau_\Delta \text{ for some } n,
\end{cases}
\]  
\( (0.6) \)

where

\[
\tau_n(\omega) := \inf \{ t \geq 0 \mid |X_{t^-}(\omega)| \geq n \text{ or } |X_t(\omega)| \geq n \}, \text{ for all } n \geq 1. \quad (0.7)
\]

Note that, we implicitly assumed that \(|\Delta| = \infty\).

In order to use results from semimartingale theory and work with the associated semimartingale characteristics, we need to stop properly the process before it jumps or explodes. Fix a stopping time \(\tau < \tau_\infty\) and define

\[
X^\tau_t := X_t 1_{[0,\tau \wedge \tau^J_\infty]} + X_{\tau \wedge \tau^J_\infty} 1_{[\tau^J_\infty, \infty]}, \quad t \geq 0.
\]  
\( (0.8) \)

We say that \(X\) is a semimartingale, if \(X^\tau\) is a semimartingale for all \(\tau < \tau_\infty\).

**Martingale Problem** Suppose that the following three maps

\[
A : D \to S^d_+, \\
B : D \to \mathbb{R}^d, \\
C : D \to \mathbb{R}_{\geq 0},
\]

are measurable and bounded on every compact subset of \(D\). Consider a transition kernel

\[
K : D \times B(D) \to \mathbb{R}^d
\]

such that the map \(x \mapsto f(\|\xi\|^2 \wedge 1)K(x, d\xi)\) is also bounded on every compact set of \(D\). Consider the operator \(\mathcal{A}\) acting on \(C^2_c(E)\)

\[
\mathcal{A}f(x) = \langle B(x), \nabla f(x) \rangle + \frac{1}{2} \text{Tr}(A(x)D^2 f(x)) - C(x)f(x) \\
+ \int (f(x + \xi) - f(x) - \langle \nabla f(x), h(\xi) \rangle) K(x, d\xi).
\]  
\( (0.9) \)

Under the aforementioned conditions on \((B, A, C, K)\), \(\mathcal{A}\) defines a linear oper-
ator with $\mathcal{AC}_c^2(D) \subseteq \text{mbdd}(D)$.

Let $\mathbb{P}$ a probability measure on $\mathcal{D}(D)$ and $A$ a bounded linear operator as in (0.9). The measure $\mathbb{P}$ is said to be a solution of the martingale problem for $A$ if, for all $f \in C_c^2(D)$ the process

$$M^f_t := f(X_t) - f(X_0) - \int_0^t Af(X_s)ds, \quad t \geq 0,$$

is a $\mathbb{P}$-martingale with respect to $(\mathcal{F}_t^\gamma)_{t \geq 0}$. The martingale problem for $A$ is said to be well posed if, for all $x \in D$, there exists a unique measure on $(\Omega, \mathcal{F}^x)$, denoted by $\mathbb{P}^x$, such that $M^f$ is a $(\mathbb{P}^x, (\mathcal{F}_t^\gamma)_{t \geq 0})$ martingale for all $f \in C_c^2(D)$ and $\mathbb{P}^x \circ X_{t-}^{-1} = \delta_x$.

**Time change**

In this section we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions and $X$ is a stochastic process adapted to it.

**Time change** A stochastic process $(\theta_t)_{t \geq 0}$ with the following properties:

1. it takes values in $[0, \infty]$,
2. its paths are non decreasing and right continuous,
3. for each fixed $t \geq 0$, the random variable $\theta_t$ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$,

is called a random time change. We say that $\theta$ is a finite time change if $\theta_t < \infty$ almost surely for all $t \geq 0$.

Starting from $(X_t)_{t \geq 0}$ and a finite time change $(\theta_t)_{t \geq 0}$, we define a new process $(\tilde{X}_t)_{t \geq 0}$ given by $\tilde{X}_t := X_{\theta_t}$. The filtration defined by $\tilde{\mathcal{F}}_t := \mathcal{F}_{\theta_t}$ is called time–changed filtration. A process $X$ is said to be adapted to the time change $\theta$ if $X$ is constant on $[\theta_{t-}, \theta_t]$ for any $t \in \mathbb{R}_{\geq 0}$.

**Time change semimartingales** Let $X$ be a semimartingale with characteristics $(\mathcal{B}, A, K)$. Corollaire (10.12) in [Jac79] states that the property of being a semimartingales is stable under finite change of time. In particular we have:

**Theorem** (Theoreme (10.16) in [Jac79]) Let $X$ be a semimartingale with the respect to $(\mathcal{F}_t)_{t \geq 0}$ adapted to the time change $\theta$. Then, the time–changed
process $\tilde{X}$ defined by $\tilde{X}_t := X_{\theta_t}$ is a $(\mathcal{F}_t)_{t \geq 0}$ semimartingale on the set $J$, with

$$J := \{ (\omega, t) \mid \theta_t - (\omega) < \infty \} = \bigcup_n [0, C_n], \quad C_n := \inf \{ s > 0 \mid \theta_s > n \}. $$

The following theorem relates the semimartingale characteristics of the original process $X$ with the semimartingale characteristics of the time–changed process $\tilde{X}$. A more general result than the next one can be stated, however for our purposes the following will be enough.

**Theorem (Theorem 8.4 in [BNS10])** Let $X$ be a $\mathbb{R}^d$-valued semimartingale with differential characteristics $\partial X = (B, A, K)$. Suppose that $(\theta_t)_{t \geq 0}$ is a finite, absolutely continuous time change, i.e. it can be written as $\theta_t = \int_0^t \dot{\theta}_s ds$ with non-negative derivative $\dot{\theta}_s$. Then the time–changed process $(\tilde{X}_t)_{t \geq 0} := (X_{\theta_t})_{t \geq 0}$ is a semimartingale relative to the time–changed filtration $(\mathcal{F}_t)_{t \geq 0} := (\mathcal{F}_{\theta_t})_{t \geq 0}$ with differential characteristics $\partial \tilde{X} = (\tilde{B}, \tilde{A}, \tilde{K})$ given by

$$\begin{align*}
\tilde{B}_t &= B_{\theta_t} \dot{\theta}_t, \\
\tilde{A}_t &= A_{\theta_t} \dot{\theta}_t, \\
\tilde{K}_t(G) &= K_{\theta_t}(G) \dot{\theta}_t, \quad \text{for all } G \in \mathcal{B}^d(\mathbb{R}^d).
\end{align*}$$
Chapter 1

Affine processes

1.1 Introduction

History

During the last decades, many alternatives to the Black-Scholes model have been proposed in the literature to overcome its deficiencies. Possible extensions include jumps, stochastic volatility and/or other high dimensional models. Among the most popular ones, we recall the exponential Lévy models, which generalize the Black-Scholes model by introducing jumps. They allow to generate implied volatility smiles and skews similar to the ones observed in the markets. However, in some occasions, independence of increments is a big restriction. Stochastic volatility models give a way to overcome this problem, by introducing an additional source of randomness, which can be interpreted as the instantaneous volatility of the underlying. Replacing the deterministic volatility in Black-Scholes with a CIR model, we get the so called Heston’s model, see [Hes93]. Now jumps can be added in the return component, as in the Bates model (see [Bat96]), and also in the return component, as in the Barndorff-Nielsen and Shephard model (see [BN02]). The class of affine processes includes all the above mentioned examples.

This thesis is devoted to the study of affine processes with state space $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$, henceforth called the canonical state space. It is worth to recall the reader that, affine processes can be defined also on state spaces different from the canonical state space. For example, within the class matrix-valued affine process, we find the class of Wishart processes, introduced by Bru in [Bru91], and now widely used to model multivariate stochastic volatility and correlation, see for example [CFMT09, May12]. Although the importance of such proposed extensions, in this introduction we stick
1. **Affine processes**

to the state space $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ since this the set up we have in mind for the applications presented in the next sections.

**Additional notation**

Henceforth $D$ denotes the subset $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ of $\mathbb{R}^d$.

**Notation 1.1** In order to simplify the notation, we introduce the sets of indices $I$ and $J$ defined as $I = \{1, \ldots, m\}$ and $J = \{m + 1, \ldots, m + n\}$. Moreover, given a set $H \subseteq \{1, \ldots, d\}$, the map $\pi_H$ is the projection of $\mathbb{R}^d$ on the lower dimensional subspace with indices in $H$. In particular

$$
\pi_I : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m_{\geq 0}
$$

$$
x \mapsto \pi_I x := (x_i)_{i \in I}
$$

$$
\pi_J : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n
$$

$$
x \mapsto \pi_J x := (x_j)_{j \in J}.
$$

Due to the geometry of the state space, the function

$$
f_u(x) := e^{(x,u)}
$$

is bounded if and only if

$$
\mathcal{U} := C^m_{\leq 0} \times i \mathbb{R}^n.
$$

Unless differently specified, the notation $\mathbb{E}^x[\cdot]$ indicates that the expectation is taken under the probability measure $\mathbb{P}^x$.

### 1.2 Definitions and main properties

In line with the literature, we introduce the affine processes as a class of time homogeneous Markov processes characterized by two additional properties. The first one being stochastic continuity, the second one a condition which characterizes the Fourier–Laplace transform of the one time marginal distributions.

This introduction of affine processes in taken from [DFS03, CT11] and [KST11].
Definition 1.2 Let
\[(\Omega, (X_t)_{t\geq 0}, (\mathbb{F}_t^x)_{t\geq 0}, (p_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D})\]
be a time homogeneous Markov process. The process \(X\) is said to be an affine process if it satisfies the following properties:

- for every \(t \geq 0\) and \(x \in D\), \(\lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)\) weakly,
- there exist functions \(\varphi : \mathbb{R}_{\geq 0} \times U \to \mathbb{C}\) and \(\Psi : \mathbb{R}_{\geq 0} \times U \to \mathbb{C}^d\) such that
  \[
  \mathbb{E}^x \left[ e^{\langle u, X_t \rangle} \right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\varphi(t, u) + \langle x, \Psi(t, u) \rangle},
  \]
  for all \(x \in D\) and \((t, u) \in \mathbb{R}_{\geq 0} \times U\).

Semiflow property

The affine structure and Markov property give an appealing property for the functions \(\varphi\) and \(\Psi\).

Proposition 1.3 (see Section 3 in [DFS03]) The functions \(\varphi\) and \(\Psi\) satisfy the following semiflow property: for every \(u \in U\) and \(t, s \geq 0\)

\[
\begin{align*}
\varphi(t + s, u) &= \varphi(t, u) + \varphi(s, \Psi(t, u)), & \varphi(0, u) &= 0, \\
\Psi(t + s, u) &= \Psi(s, \Psi(t, u)), & \Psi(0, u) &= u.
\end{align*}
\]

The functions \(\varphi\) and \(\Psi\) can be uniquely chosen so that they are jointly continuous on \(\mathbb{R}_{\geq 0} \times U\).

In the following sections, when we talk about functions \(\varphi\) and \(\Psi\), we will always refer to the unique specification provided by Proposition 1.3 on \(\mathbb{R}_{\geq 0} \times U\).

Regularity and Riccati equations

Regularity is a key feature for an affine process. It gives differentiability of the Fourier–Laplace transform with respect to time.

Definition 1.4 An affine process \(X\) is called regular if, for every \(u \in U\), the derivatives

\[
F(u) := \partial_t \varphi(t, u) \bigg|_{t=0}, \quad R(u) := \partial_t \Psi(t, u) \bigg|_{t=0},
\]

23
exist for all \( u \in U \) and are continuous in
\[
U_m = \left\{ u \in \mathbb{C}^d \mid \sup_{x \in D} \text{Re}(\langle u, x \rangle) \leq m \right\},
\]
for all \( m \geq 1 \).

**Definition 1.5** (see [Kel08]) The couple \((F, R)\) identifies the so called functional characteristics of \( X \).

**Theorem 1.6** (Theorem 6.4 in [CT11]) Every affine process is regular. Moreover, on the set \( \mathbb{R}_{\geq 0} \times U \), the functions \( \varphi \) and \( \Psi \) satisfy the following system of generalized Riccati equations:

\[
\begin{align*}
\partial_t \varphi(t, u) &= F(\Psi(t, u)), \quad \varphi(0, u) = 0, \\
\partial_t \Psi(t, u) &= R(\Psi(t, u)), \quad \Psi(0, u) = u,
\end{align*}
\]

with
\[
F(u) = \langle b, u \rangle + \frac{1}{2} \langle u, au \rangle - c + \int_{D \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) m(d\xi),
\]
\[
R_k(u) = \langle \beta_k, u \rangle + \frac{1}{2} \langle u, \alpha_k u \rangle - \gamma_k + \int_{D \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) M_k(d\xi),
\]
for \( k = 1, \ldots, d \). The set of parameters
\[
(b, \beta, a, \alpha, c, \gamma, m, M)
\]
is specified by
- \( b, \beta_i \in \mathbb{R}^d \) for \( i = 1, \ldots, d \),
- \( a, \alpha_i \in S_+^d \) for \( i = 1, \ldots, d \),
- \( c, \gamma_i \in \mathbb{R}_{\geq 0} \) for \( i = 1, \ldots, d \),
- \( m, M_i \) for \( i = 1, \ldots, d \) are Lévy measures.

This set of parameters is called admissible for \( D = \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n \), if the conditions in Table 1.1 are satisfied with \( I \) and \( J \) defined as
\[
I = \{1, \ldots, m\} \quad \text{and} \quad J = \{m + 1, \ldots, d\}.
\]
1.2. Definitions and main properties

<table>
<thead>
<tr>
<th>diffusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{kl} = 0 ) for ( k \in I ) or ( l \in J ),</td>
</tr>
<tr>
<td>( \alpha_j = 0 ) for all ( j \in J ),</td>
</tr>
<tr>
<td>( (\alpha_{i})_{kl} = 0 ) if ( k \in I \setminus {i} ) or ( l \in I \setminus {i} ),</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>drift</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b \in D ),</td>
</tr>
<tr>
<td>( (\beta_i)_k \geq 0 ) for all ( i \in I ) and ( k \in I \setminus {i} ),</td>
</tr>
<tr>
<td>( (\beta_j)_k = 0 ) for all ( j \in J ), ( k \in I ),</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>killing</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_j = 0 ) for all ( j \in J ),</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>jumps</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{supp} , m \subseteq D ) and ( \int_{D \setminus {0}} \left( (</td>
</tr>
<tr>
<td>( M_j = 0 ) for all ( j \in J ),</td>
</tr>
<tr>
<td>( \text{supp} , M_i \subseteq D ) for all ( i \in I ) and ( \int_{D \setminus {0}} \left( (</td>
</tr>
</tbody>
</table>

**Table 1.1:** Set of conditions for admissible parameters.

The set of admissible parameters fully characterizes an affine process in \( D \).

**Remark 1.7** In some applications, the condition that the process does not explode in finite time will be essential. Clearly, in order to have a conservative process, all the killing rate have to be zero. This condition is however not sufficient. Indeed, explosions could be caused also by the jump measure. Conservativeness is ensured by coupling zero killing rate with the condition

\[
\int \left( |\pi_I \xi| \wedge |\pi_I \xi|^2 \right) M_i(d\xi), \quad \text{for all} \quad i \in I.
\]

**Infinite divisibility property**

If \( X \) is an affine process on the canonical state space, for each \( t \geq 0 \), \( P^x \circ X_t^{-1} \) is infinitely divisible in \( D \). Observe that this property fails for affine processes on a more general state space. The Wishart process, which is an affine process with state space \( S^d_+ \), is an example of affine process whose marginal distributions are not infinitely divisible.
1. Affine processes

**Definition 1.8** (see [DFS03]) A function \( \eta : \mathcal{U} \rightarrow \mathbb{C} \) has the Lévy-Khintchine form on \( D = \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n \) if there exist \( \mathbf{b} \in D, \sigma \in \mathbb{S}^n_+ \) and a Borel measure \( \nu \) on \( D \) satisfying \( \nu(\{0\}) = 0 \) and \( \int_D (|\pi_I \mathbf{x}| + |\pi_J \mathbf{x}|^2) \wedge 1) \nu(d\mathbf{x}) < \infty \) such that for any \( u \in \mathcal{U} \)

\[
\eta(u) = \langle \mathbf{b}, u \rangle + \frac{1}{2} \langle \sigma \pi_J u, \sigma \pi_J u \rangle + \int_{D \setminus \{0\}} e^{\langle u, \xi \rangle} \left( 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu(d\xi),
\]

where \( \pi_I \) and \( \pi_J \) denote the projections of elements of \( \mathbb{R}_0^m \times \mathbb{R}^n \) into \( \mathbb{R}_0^m \) and \( \mathbb{R}^n \) respectively, see Notation 1.1.

**Remark 1.9** In the whole space \( \mathbb{R}^d \), infinitely divisibility is equivalent to the usual representation for \( \eta \) in terms of a Lévy triplet as in Theorem 8.1. in [Sat99]

\[
\eta(u) = \langle \mathbf{b}, u \rangle + \frac{1}{2} \langle u, \sigma u \rangle + \int_{\mathbb{R}^d \setminus \{0\}} e^{\langle u, \xi \rangle} \left( 1 - \langle u, h(\xi) \rangle \right) \nu(d\xi),
\]

with \( u \in i\mathbb{R}^d \), and \((\mathbf{b}, \sigma, \nu)\) a Lévy triplet. From Proposition 11.10 and Theorem 24.7 in [Sat99] we know that a necessary condition to have infinitely divisibility of \( \pi_I X \) is given by

\[
(\sigma_{hk})_{h,k \in I} = 0 \quad \text{and} \quad \pi_I \nu(\mathbb{R}_0^m) < \infty,
\]

or

\[
(\sigma_{hk})_{h,k \in I} = 0 \quad \text{and} \quad \int (1 \wedge |\pi_I \mathbf{x}|) \nu(d\mathbf{x}) < \infty.
\]

Writing \( u = (\pi_I u, \pi_J u) \) we get

\[
\eta(u) = \langle \pi_I b, \pi_I u \rangle + \langle \pi_J b, \pi_J u \rangle + \frac{1}{2} \langle \pi_J u, \sigma \pi_J u \rangle \\
+ \int_{\mathbb{R}^d \setminus \{0\}} e^{\langle u, \xi \rangle} \left( 1 - \langle \pi_I u, \pi_I h(\xi) \rangle - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu(d\xi) \\
= \langle \pi_I b - \int_{D \setminus \{0\}} \pi_I h(\xi) \nu(d\xi), \pi_I u \rangle + \langle \pi_J b, \pi_J u \rangle + \frac{1}{2} \langle \pi_J u, \sigma \pi_J u \rangle \\
+ \int_{\mathbb{R}^d \setminus \{0\}} e^{\langle u, \xi \rangle} \left( 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu(d\xi) \\
= \langle \bar{b}, u \rangle + \frac{1}{2} \langle \pi_J u, \sigma \pi_J u \rangle + \int_{D \setminus \{0\}} e^{\langle u, \xi \rangle} \left( 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu(d\xi)
\]

with \( u \in i\mathbb{R}^d \), and \( \bar{b} = (\pi_I b - \int_{D \setminus \{0\}} \pi_I h(\xi) \nu(d\xi), \pi_J b) \). From Corollary 24.6 in [Sat99], the drift component of the positive part has to satisfy the condition \( \bar{b}_i \geq 0 \).
for all \( i \in I \), i.e.
\[
\pi_I b - \int_{D \setminus \{0\}} \pi_I h(\xi) \nu(d\xi) \geq 0.
\]

To simplify the notation we denoted with \( b \) also the corrected drift. Finally \( \eta \) can be extended in the whole \( U \) defined in (1.2).

**Proposition 1.10 (Proposition 7.4 in [DFS03])** Let \( F \) and \( R \) be defined as in (1.8) and (1.9). The solution of
\[
\begin{align*}
\partial_t \varphi(t, u) &= F(\Psi(t, u)), \quad \varphi(0, u) = 0, \\
\partial_t \Psi(t, u) &= R(\Psi(t, u)), \quad \Psi(0, u) = u,
\end{align*}
\]
consists of functions of type (1.11) up to a constant \( c \).

Due to infinite divisibility of the distributions, for each \( (t, x) \in \mathbb{R}_{\geq 0} \times D \), we can find a triplet \( (b(t, x), \sigma(t, x), \nu(t, x)) \) and a constant \( c(t, x) \geq 0 \), such that
\[
\langle x, \Psi(t, u) \rangle = \langle b(t, x), u \rangle + \frac{1}{2} \langle \pi_{J u}, \sigma(t, x) \pi_{J u} \rangle - c(t, x)
\]
\[
+ \int_D \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_{J u}, \pi_{J h(\xi)} \rangle \right) \nu(t, d\xi).
\]

Due to linearity in \( x \), for all \( i = 1, \ldots, d \),
\[
\Psi_i(t, u) = \langle b_i(t), u \rangle + \frac{1}{2} \langle \pi_{J u}, \sigma_i(t) \pi_{J u} \rangle - c_i(t)
\]
\[
+ \int_{D \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_{J u}, \pi_{J h(\xi)} \rangle \right) \nu_i(t, d\xi),
\]
where the notations are collected and unified in Table 1.2.

<table>
<thead>
<tr>
<th>( \psi_i(t, u), i = 1, \ldots, d )</th>
<th>( \langle x, \psi(t, u) \rangle )</th>
<th>( b(t, x) = B(t)^\top x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_i(t) )</td>
<td>( b(t, x) )</td>
<td>( b(t, x) = B(t)^\top x )</td>
</tr>
<tr>
<td>( \sigma_i(t) )</td>
<td>( \sigma(t, x) )</td>
<td>( \sigma(t, x) = \sum_{i=1}^d x_i \sigma_i(t) )</td>
</tr>
<tr>
<td>( c_i(t) )</td>
<td>( c(t, x) )</td>
<td>( c(t, x) = \langle x, c(t) \rangle )</td>
</tr>
<tr>
<td>( \nu_i(t, \cdot) )</td>
<td>( \nu(t, x, \cdot) )</td>
<td>( \nu(t, x, \cdot) = \langle x, \nu(t, \cdot) \rangle )</td>
</tr>
</tbody>
</table>

**Table 1.2:** The table collects the notation used to express the Lévy triplet of \( \langle x, \Psi(t, u) \rangle \). The matrix \( B(t) \) has as columns the vectors \( b_i(t), i = 1, \ldots, d \).
Càdlàg property

It has been proved in [CT11] that stochastic continuity and affine property are enough to conclude càdlàg property. Therein, it is proved that, if

\[(\Omega, (X_t)_{t \geq 0}, (\mathcal{F}_t^x)_{t \geq 0}, (p_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D})�\]

is a time-homogeneous Markov process defined of a general probability space \(\Omega\) satisfying the properties in Definition 1.2, then, for each \(x \in D\), there exists a \(\mathbb{P}^x\)-version of it which is an affine process with respect to the augmented filtration \((\mathcal{F}_t^x)_{t \geq 0}\) with càdlàg paths.

We summarize here the main results, for the proofs see [CT11].

**Theorem 1.11** Let \((\Omega, (X_t)_{t \geq 0}, (\mathcal{F}_t^x)_{t \geq 0}, (p_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D}) \)

1. For each \(x \in D\), there exists a \(\mathbb{P}^x\)-version, again denoted by \(X\), which is an affine process with respect to the augmented filtration \((\mathcal{F}_t^x)_{t \geq 0}\) and has càdlàg paths in \(D \cup \{\infty\}\).

2. For each \(x \in D\), \(|X_s^-| = \infty\) implies \(|X_t| = \infty\) for all \(t \geq s\) \(\mathbb{P}^x\)-almost surely. This in particular implies that the point at infinity \(\infty\) can be identified with \(\Delta\).

3. Define

\[D(D) := \{\omega \in \Omega \text{ càdlàg such that } |\omega(s^-)| = \infty \text{ or } |\omega(s)| = \infty,\]

implies \(\omega(t) = \Delta\) for all \(t \geq s\),\]

and

\[\mathcal{F} := \bigcap_{x \in D} \mathcal{F}^x \quad \text{and} \quad \mathcal{F}_t := \bigcap_{x \in D} \mathcal{F}_t^x.\]

The process \((\Omega = D(D), (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (p_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D})\) is a Markov process satisfying the affine property and with càdlàg paths.

**Notation 1.12** From now on, we fix \(\Omega = D(D)\) and, whenever necessary, work with the augmented filtration \((\mathcal{F}_t^x)_{t \geq 0}\) with respect of the fixed path-space measure \(\mathbb{P}^x\). To sum it up, we denote by

- \(\mathcal{F}^x\) and \((\mathcal{F}_t^x)_{t \geq 0}\) the \(\mathbb{P}^x\)-augmentations of \(\mathcal{F}\) and \((\mathcal{F}_t)_{t \geq 0}\) respectively,
- \(\mathcal{F}_t = \bigcap_{x \in D} \mathcal{F}_t^x\).
1.2. Definitions and main properties

- \( \mathcal{F} = \bigcap_{x \in D_\Delta} \mathcal{F}^x. \)

The affine transform formula

Until now, we focused on the definition of the affine property for values \( u \) in the set \( \mathcal{U} \subset \mathbb{C}^d \) such that \( f_u(x) \) is a bounded function for all \( x \in D \). We have seen that, for all \( u \in \mathcal{U} \), the functions \( \varphi \) and \( \Psi \) are the solutions of a system of Riccati ODEs. A very interesting problem is to find conditions under which the system of Riccati equations can be extended on a bigger domain \( \mathcal{Y} \subset \mathbb{C}^d \). The following results are taken from [KM11]. Here we simply adapt the notation to our setting.

**Definition 1.13 (Definition 2.7. in [KM11])** Let \( X \) be an affine process with set of admissible parameters \((b, \beta, a, \alpha, 0, 0, m, M)\) and define

\[
\mathcal{Y} = \bigcap_{x \in D} \left\{ y \in \mathbb{R}^d \mid \int_{\{\|\xi\| \geq 1\}} e^{\langle y, \xi \rangle} K(x, d\xi) < \infty \right\}. \tag{1.14}
\]

Denote by

\[
S(\mathcal{Y}) := \left\{ u \in \mathbb{C}^d \mid \Ree(u) \in \mathcal{Y} \right\}.
\]

**Proposition 1.14 (Proposition 2.21 in [KM11])** Let \( X \) be an affine process and suppose that \( \hat{\mathcal{Y}} \neq \emptyset \). Then the functions \( F \) and \( R \) defined in (1.8) and (1.9) admit an analytic extension for all \( u \in S(\hat{\mathcal{Y}}). \)

**Theorem 1.15 (Theorem 2.26 in [KM11])** Let \( X \) be an affine process such that \( \hat{\mathcal{Y}} \neq \emptyset \). Let \( T \geq 0 \) and \( u \in S(\hat{\mathcal{Y}}) \) be fixed and denote by \( y = \Ree(u) \). Suppose that, there exists a couple \((p, q)\) of \( C^1 \) functions mapping \([0, T]\) to \((\mathbb{R}, \mathcal{Y})\) such that

\[
\begin{align*}
\partial_t p(t, y) &= F(q(t, y)), \quad p(0, y) = 0, \tag{1.15} \\
\partial_t q(t, y) &= R(q(t, y)), \quad q(0, y) = y, \tag{1.16}
\end{align*}
\]

for all \( t \in [0, T] \) and that, additionally, \( q(t, y) \in \hat{\mathcal{Y}} \) for all \( t \in [0, T] \). Then, for all \( x \in D \) and \( t \in [0, T] \), \( \mathbb{E}^x \left[ e^{\langle u, X_t \rangle} \right] < \infty \) and

\[
\mathbb{E}^x \left[ e^{\langle u, X_t \rangle} \right] = e^{\varphi(t, u) + \langle x, \Psi(t, u) \rangle}. \tag{1.17}
\]

### 1.2.1 Feller property

**Theorem 1.16 (Theorem 2.7 in [DFS03])** Consider the probability space \((\Omega = \mathcal{D}(D_{\Delta}), \mathcal{F}^t, (\mathcal{F}^t_t)_{t \geq 0})\)
1. Affine processes

and a set \((b, \beta, a, \alpha, c, \gamma, m, M)\) of admissible parameters.

1. There exists a unique Feller semigroup \((P_t)_{t \geq 0}\) with infinitesimal generator \(A\)

\[
A f := \lim_{t \to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}(A) \subset C_0(D).
\]  

(1.18)

of type

\[
A f(x) = \langle B(x), \nabla f(x) \rangle + \frac{1}{2} \text{Tr}(D^2 f(x)A(x)) - C(x)f(x)
+ \int (f(x + \xi) - f(x) - \langle \nabla f(x), h(x) \rangle) K(x,d\xi)
\]  

(1.19)

with

- \(A(x) = a + \sum_{i=1}^{d} \alpha_i x_i\),
- \(B(x) = b + \sum_{i=1}^{d} \beta_i x_i\),
- \(C(x) = c + \sum_{i=1}^{d} \gamma_i x_i\)
- \(K(x,d\xi) = m(d\xi) + \sum_{i=1}^{d} x_i M_i(d\xi)\).

Moreover, for all \((t,u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}\),

\[
P_t f_u(x) = e^{\varphi(t,u) + \langle x, \Psi(t,u) \rangle},
\]

with \(\varphi\) and \(\Psi\) defined as the solution of the ODE (1.7). In particular, \((P_t)_{t \geq 0}\) can be identified with the transition semigroup of an affine process.

2. Let \(X\) be an affine process with state space \(D\) defined on the probability space \((\Omega = \mathcal{D}(D_{\Delta}), \mathcal{F}^2, (\mathcal{F}^2_t)_{t \geq 0})\) Then \(X\) is a Feller process. Let \((A, \mathcal{D}(A))\) be its infinitesimal generator. The class \(\{f_u(x), u \in \mathcal{U}\}\) lies in \(\mathcal{D}(A)\). Moreover, there exists a set of admissible parameters \((b, \beta, a, \alpha, c, \gamma, m, M)\) such that, for all \(u \in \mathcal{U}\) and \(x \in D\), it holds

\[
A f_u(x) = (F(u) + \langle x, R(u) \rangle) f_u(x),
\]

where the functions \(F\) and \(R\) are defined respectively in (1.8) and (1.9).

The domain \(\mathcal{D}(A)\) contains the class \(C^2_c(D)\) and \(A f\) takes the form (1.19) for any \(f \in C^2_c(D)\).
1.2.2 Semimartingale characterization

The following results concerning the semimartingale property of affine processes are taken from [DFS03, Kal06].

**Theorem 1.17** Let $(B, A, C, K)$ be measurable and bounded maps given by

\[
\begin{align*}
A(x) &= a + \sum_{i=1}^{d} \alpha_i x_i, \\
B(x) &= b + \sum_{i=1}^{d} \beta_i x_i, \\
C(x) &= c + \sum_{i=1}^{d} \gamma_i x_i, \\
K(x, d\xi) &= m(d\xi) + \sum_{i=1}^{d} x_i M_i(d\xi),
\end{align*}
\]  

(1.20)

where $b, \beta_i \in \mathbb{R}^d$, $a, \alpha_i \in S^d_+$, $c, \gamma_i \geq 0$ and $m, M_i$ measures on $\mathcal{B}(D)$ for all $i = 1, \ldots, d$.

1. The martingale problem for $A$ defined by

\[
A f(x) = \langle B(x), \nabla f(x) \rangle + \frac{1}{2} \text{Tr}(A(x)D^2 f(x)) - C(x) f(x) + \int f(x + \xi) - f(x) - \langle \nabla f(x), f(\xi) \rangle K(x, d\xi)
\]

is well posed for all $x \in D$ if the restrictions in Table 1.1 are satisfied. Denote by $P^x$ the unique solution for all $x \in D$. The coordinate process $X$ is a time homogeneous Markov process on $(\Omega = D(D_{\Delta}), (\mathcal{F}^x_t)_{t \geq 0}, P^x)$ with state space $D$ satisfying property (1.2).

2. Let $X = (\Omega = D(D_{\Delta}), (\mathcal{F}^x_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P^x)_{x \in D})$ be an affine process with admissible parameters $(b, \beta, a, \alpha, c, \gamma, m, M)$ and $\tau$ a stopping time with $\tau < \tau_{\infty}$ with $\tau_{\infty}$ defined in (0.6). Then $(X^\tau_t)_{t \geq 0}$ defined in (0.8) is a semimartingale with state space $D_{\Delta}$. Its semimartingale characteristics relative to the truncation function $h(x) = x 1_{\{|x| \leq 1\}}$ admit a version of the following form

\[
\begin{align*}
B_t &= \int_0^t B(X^\tau_{s-}) dF_s, \\
A_t &= \int_0^t A(X^\tau_{s-}) dF_s, \\
K(dt, d\xi) &= K(X^\tau_t, d\xi) dF_t,
\end{align*}
\]  

(1.21, 1.22, 1.23)
1. Affine processes

where \( F_t = t 1_{[0, \tau_d]} + \Delta 1_{[\tau_d, \infty]} \) and \((B, A, K)\) are given in (1.20).

1.2.3 Affine processes as infinite decomposable processes

The last characterization we present classifies affine processes as a generalization of continuous state branching processes as introduced in [KW71].

**Theorem 1.18 (Theorem 2.13 in [DFS03])** A Markov process

\[
X = (\Omega = \mathcal{D}(D), (\mathcal{F}_t^x)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D})
\]

is affine if and only if \((\mathbb{P}^x)_{x \in D}\) is infinitely decomposable.

Due to infinitely decomposability, it is possible to show that affine processes are examples of infinitely divisible processes.

**Definition 1.19** A Markov process is called infinitely divisible if all its finite dimensional distributions are infinitely divisible.

Indeed, it is possible to derive a closed form expressions for the Fourier–Laplace transform of the finite dimensional marginals. This is done using the affine property and tower property of the conditional expectation. The analysis of the finite dimensional marginals reveals that any affine process is an infinitely divisible process.

**Proposition 1.20 (see Section 10.3 in [DFS03])** Let \( X \) be an affine process, then all its finite dimensional distributions are infinitely divisible.

**Proof** Let \( \ell = (t_0, \ldots, t_N) \in \mathbb{R}^{N+1}_{\geq 0} \) and denote with \( X(\ell) \) the random vector

\[
X(\ell) := (X_{t_0}, \ldots, X_{t_N}).
\]

For any \( u = (u_0, \ldots, u_N) \in \mathcal{U}^{N+1} \),

\[
\mathbb{E}^x \left[ e^{\sum_{k=0}^N \langle u_k, X_{t_k} \rangle} \right] = e^{\phi^N(\ell, u) + \langle x, \Psi^N(\ell, u) \rangle}
\]

where \( \phi^N(\ell, u) \) and \( \Psi^N(\ell, u) \) are defined by the sequence

\[
\Psi^0(\ell, u) := u_N,
\]

\[
\Psi^{k+1}(\ell, u) := u_{N-k} + \Psi(t_{N-k} - t_{N-(k+1)}, \Psi^k(\ell, u)), \quad k = 0, \ldots, N-1,
\]

\[
\phi^N(\ell, u) := \sum_{k=1}^N \phi(t_{N-k} - t_{N-(k+1)}, \Psi^k(\ell, u)).
\]

32
By construction, for all \( k = 0, \ldots, N \), each \( \Psi^k(t, u) \) has the Lévy-Khintchine decomposition in \( D^{k+1} \). Hence, there exist two infinitely divisible sub-stochastic measures \( \mu_1^\xi \) and \( \mu_2^\xi \) on \( D^{N+1} \) such that, for all \( u_0, \ldots, u_N \in U \)

\[
\int_{D^{N+1}} e^{\sum_{k=0}^N \langle u_k, \xi_k \rangle} \mu_1^\xi (d\xi) = e^{\varphi^N(t, u)} \quad \int_{D^{N+1}} e^{\sum_{k=0}^N \langle u_k, \xi_k \rangle} \mu_2^\xi (d\xi) = e^{\langle x, \Psi^N(t, u) \rangle}
\]

with the notation \( \xi = (\xi_0, \ldots, \xi_N) \in D^{N+1} \). Therefore the convolution \( \lambda_t = \mu_1^\xi * \mu_2^\xi \) is infinitely divisible.

\[\blacksquare\]

### 1.3 Elementary transformations of affine processes

In the set of all continuous functions \( f : U \rightarrow \mathbb{C}^d \), we consider the following subsets

\[
\begin{align*}
\mathcal{C} & := \{ \eta : U \rightarrow \mathbb{C} \text{ of Lévy–Khintchine form (1.11)} \}, \\
\mathcal{C}^* & := \{ \Psi : U \rightarrow \mathbb{C}^d \mid \pi_j \Psi \in \mathcal{C}^m \text{ and } \pi_j \Psi = A \pi_j u, \ A \in \mathbb{R}^{n \times n} \}, \\
\mathcal{C}^H & := \{ \Psi : U \rightarrow \mathbb{C}^d \mid \pi_j \Psi \in \mathcal{C}^m \text{ and } \pi_j \Psi = \pi_j u \} \subset \mathcal{C}^*. 
\end{align*}
\]

#### 1.3.1 Affine processes in canonical form

In this section we want to specify the so-called canonical form for the semimartingale characteristics of \( X \). Here we follow Section 7 in [FM09].

Without loss of generality, we can assume that there exists a \( q \in I \) such that, for all the \( k = q + 1, \ldots, m \), the matrix \( \alpha_k \) has the form

\[
\begin{pmatrix} 
0 & 0 \\
0 & \Sigma 
\end{pmatrix}, \quad \text{with } \Sigma \in S^m_+.
\]

Observe that, due to admissible conditions, the matrix \( a \) is already of this form. For \( k = 1, \ldots, q \) we can assume that \( (\alpha_k)_{k,k} = 1 \).

The next step is to reduce each of the matrices \( \alpha_k, \ k = 1, \ldots, q \) in block diagonal form. In order to do that, we need to get rid of the non zero elements in the \( k \)-th row (and column) of each matrix \( \alpha_k, \ k = 1, \ldots, q \). Let \( \Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be the linear map:

\[
\Lambda : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \\
x \mapsto \Lambda x,
\]
1. **Affine processes**

where

\[
\Lambda = \begin{pmatrix}
-I_m & 0 \\
-(\alpha_1)_{J,1} \cdots - (\alpha_m)_{J,m} & I_n
\end{pmatrix}
= \begin{pmatrix}
I_m \\
-N
\end{pmatrix}.
\]  

(1.27)

Since \( X \) is a killed semimartingale with differential characteristics \( \partial X = (B, A, K) \) given by (1.20) \( \Lambda X \) is again a semimartingale, with same killing rates, but differential characteristics \( \partial \Lambda X = (B^\Lambda, A^\Lambda, K^\Lambda) \). Applying Itô formula we find

- \( B^\Lambda_t = \Lambda B_t + \int_{D \setminus \{0\}} (h(\Lambda \xi) - \Lambda h(\xi)) K(t, d\xi) \),
- \( A^\Lambda_t = \Lambda A_t \Lambda^\top \),
- \( K^\Lambda(X_t, B) = \int_{D \setminus \{0\}} 1_B(\Lambda \xi) K(X_t, d\xi), \quad B \in (D \setminus \{0\}) \),

where \( A^\Lambda_t \) is of block diagonal form. Due to linearity of the map \( \Lambda \), \( \Lambda X \) is still a process with càdlàg paths. However, the drift condition in Table 1.1 could be no more satisfied. We see how to fix it by changing the truncation function. Observe that the kernel \( K^\Lambda \) is given by the push-forward Lévy measures, \( K^\Lambda(X_t, d\xi) = K(X_t, \cdot) \circ \Lambda^{-1}(d\xi) \). Define the new truncation function \( h'(\xi) := \xi 1_{|\Lambda^{-1}\xi| \leq 1} \) and apply (0.3). Then the drift component under the new truncation function is given by

\[ B^\Lambda(h')_t = B^\Lambda(h)_t + \int_{D \setminus \{0\}} (h'(\xi) - h(\xi)) K^\Lambda(X_t, d\xi) \]

\[ = \Lambda B_t + \int_{D \setminus \{0\}} (h(\Lambda \xi) - \Lambda h(\xi)) K(X_t, d\xi) \]

\[ + \int_{D \setminus \{0\}} (h'(\Lambda \xi) - h(\Lambda \xi)) K(X_t, d\xi) \]

\[ = \Lambda B_t + \int_{D \setminus \{0\}} (\Lambda \xi 1_{|\Lambda \xi| \leq 1} - \Lambda \xi 1_{|\xi| \leq 1}) K(X_t, d\xi) \]

\[ + \int_{D \setminus \{0\}} (\Lambda \xi 1_{|\xi| \leq 1} - \Lambda \xi 1_{|\Lambda \xi| \leq 1}) K(X_t, d\xi) \]

\[ = \Lambda B_t = \Lambda \left( b + \sum_{i=1}^d \beta_i X_i^\top \right). \]

Denote by \( B \) the \( d \times d \) matrix obtained by placing each \( \beta_i, \ i = 1, \ldots, d \) as a column, then

\[
\Lambda B = \begin{pmatrix}
-I_m \\
-N
\end{pmatrix} \begin{pmatrix}
*+0 \\
*+0 \\
*+0
\end{pmatrix} = \begin{pmatrix}
*+0 \\
*+0 \\
*+0
\end{pmatrix},
\]
1.3. Elementary transformations of affine processes

and

\[ \Lambda b = \left( \begin{array}{c|c} I_m & 0 \\ \hline N & I_n \end{array} \right) \cdot \left( \begin{array}{c} + \\ \hline \ast \end{array} \right) = \left( \begin{array}{c} + \\ \hline \ast \end{array} \right). \]

**Definition 1.21** Given an affine process on the canonical state space with admissible parameters \((b, \beta, a, \alpha, c, \gamma, m, M)\) and truncation function \(h(\xi) = \xi^1 \{ |\xi| \leq 1 \},\) we say that it is in canonical form if

1. \(A(x) = a + \sum_{i=1}^{d} x_i \alpha_i\) is of block-diagonal form:

\[ A(x) = \left( \begin{array}{c|c} \text{diag}(x_1, \ldots, x_q, 0, \ldots, 0) & 0 \\ \hline 0 & a + \sum_{i=1}^{m} x_i \alpha_i^T \end{array} \right), \]

for some integer \(0 \leq q \leq m\) and \(a, \alpha_i^T, i = 1, \ldots, m\) symmetric positive semidefinite matrices in \(S_n^+\).

2. \(b, \beta_1, \ldots, \beta_d\) already include the correction coming from the integrated jumps.

**Notation 1.22** Let \(B \in \mathbb{R}^{d \times d}\) be the matrix where each column is given by \(\beta_i, i = 1, \ldots, d\). Henceforth, the following notation will be used

\[ B = \left( \begin{array}{c|c} B_I & 0 \\ \hline B_{IJ} & B_J \end{array} \right). \] (1.28)

Hence, up to a linear transformation and change of truncation function, we can assume that each \(R_i, i = 1, \ldots, m\) has the following Lévy-Khintchine form

\[ R_i(u) = \langle \beta_i, u \rangle + \frac{1}{2} \langle u, \alpha_i u \rangle - \gamma_i \]

\[ + \int_{D \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_{J \cup \{i\}} u, \pi_{J \cup \{i\}} h(\xi) \rangle \right) M_i(d\xi), \] (1.29)
where each $\alpha_i \in S^+_d$ is of type

$$\alpha_i = \begin{pmatrix}
0 \\
\vdots \\
0 \\
0 \ldots 0 (\alpha_i)_{ii} 0 \ldots 0 \\
0 \\
\vdots \\
0
\end{pmatrix},$$

with $(\alpha_i)_{ii} \geq 0$ and $\alpha_i^J \in S^+_n$ and $\beta_i, \gamma_i, M_i$ are admissible parameters. At the same time, from admissible conditions, $\pi_j R(u)$ is linear in $\pi_j u$. Precisely, $\pi_j R(u) = B^\top_j \pi_j u$ with $B_j$ defined in (1.28).

### 1.3.2 From affine processes to linear processes

Up to an enlargement of the state space, we can always assume that the Fourier–Laplace transform of $X$ is given by the exponential of a linear function of the state variable. Let $AP(D)$ be the space of affine processes with state space $D$ and denote by $AP^\times(D) \subset AP(D)$ the subset containing processes $X \in AP(D)$ such that, for some $i \in I$ and $c > 0$, $X_t^{(i)} = c$ for all $t \geq 0$, and additionally it holds

$$E^x \left[ e^{(u,X_t)} \right] = e^{(x,\Psi(t,u))}, \quad t \geq 0, x \in D. \quad (1.30)$$

The following holds:

**Proposition 1.23** There exists a bijective map

$$\varphi : AP \left( \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \right) \rightarrow AP^\times \left( \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \right),$$

$$X \mapsto X^\varphi.$$

**Proof** Since $X \in AP \left( \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \right)$, there exist two functions

$$\varphi(t,u) : \mathbb{R}_{\geq 0} \times \mathcal{U} \rightarrow \mathbb{C} \quad \text{and} \quad \Psi(t,u) : \mathbb{R}_{\geq 0} \times \mathcal{U} \rightarrow \mathbb{C}^d$$
such that

\[ E^x \left[ e^{(u, X_t)} \right] = e^{\varphi(t, u) + (x, \Psi(t, u))}, \quad t \geq 0, \, u \in \mathcal{U}. \]  

(1.31)

Fix \( x_0 \in \mathbb{R}_{\geq 0} \) and define

\[
\begin{align*}
  x^\varphi &:= (x_0, x) \in \mathbb{R}^{m+1}_{\geq 0} \times \mathbb{R}^n, \\
  \mathcal{U}^\varphi &:= C_{\leq 0}^{m+1} \times i \mathbb{R}^n, \\
  \Psi^\varphi(t, u_0, u_1, \ldots, u_d) &:= \left( \varphi(t, u_1, \ldots, u_d) + u_0 \right). \tag{1.34}
\end{align*}
\]

Due to regularity in \( t \) of both \( \varphi(t, u) \) and \( \Psi(t, u) \), we conclude that \( \Psi^\varphi(t, \cdot) \) is a regular semiflow. Hence, from Proposition 7.4 in [DFS03], there exists an affine process \( X^\varphi \) with state space \( \mathbb{R}^{m+1}_{\geq 0} \times \mathbb{R}^n \) satisfying

\[
E^{x^\varphi} \left[ e^{(u, X^\varphi_t)} \right] = e^{(x^\varphi, \Psi^\varphi(t, u))}, \quad u \in \mathcal{U}^\varphi.
\]

Moreover, \( \Psi^\varphi(t, u_0, 0, \ldots, 0) = (u_0, 0, \ldots, 0) \) and

\[
E^{x^\varphi} \left[ e^{(u_0, 0, X^\varphi_t)} \right] = e^{x_0 u_0}, \quad \text{for all } t \geq 0.
\]

On the other hand, suppose that \( X \) is an affine process in \( \mathbb{R}^{m+1}_{\geq 0} \times \mathbb{R}^n \) with functional characteristics \( F(u_1, \ldots, u_{d+1}) = 0 \) and \( R_1(0, u_2, \ldots, u_{d+1}) = 0 \). Then, there exists a function \( \Psi \) such that

\[
E^x \left[ e^{(u, X_t)} \right] = e^{(x, \Psi(t, u))}, \quad u \in \mathcal{U}. \tag{1.35}
\]

Observe that, by assumptions,

\[
\Psi_1(t, u) = u_1 + \int_0^t R_1 \left( (\Psi_2(s, u), \ldots, \Psi_{d+1}(s, u)) \right) ds,
\]

\[
\Psi_k(t, u) = u_k + \int_0^t R_k \left( (\Psi_2(s, u), \ldots, \Psi_{d+1}(s, u)) \right) ds, \quad \text{for } k = 2, \ldots, d+1.
\]

Introduce the notation \( u_{[2, d+1]} = (u_2, \ldots, u_{d+1})^\top \in C^m \times i \mathbb{R}^n \). If we define

\[
\varphi^\psi(t, u_{[2, d+1]}) := \Psi_1(t, 0, u_{[2, d+1]}),
\]

\[
\Psi^\psi(t, u_{[2, d+1]}) := \left( \Psi_k(t, 0, u_{[2, d+1]}), \ldots, \Psi_{d+1}(t, 0, u_{[2, d+1]}) \right)^\top,
\]

37

1.3. Elementary transformations of affine processes
and take into account the regularity in \( t \) of \( \Psi(t,u) \), we can conclude that the couple
\((\varphi(t,\cdot),\Psi(t,\cdot))\) forms a regular semiflow (see Definition 7.3 in [DFS03]). Hence,
from Proposition 7.4 in [DFS03], there exists an affine process \( X^{\beta} \) with state space
\( \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \) such that
\[
\mathbb{E}^{x}[e^{\langle u,X_t^\beta \rangle}] = e^{\varphi^{\beta}(t,u)+\langle x,\Psi^\beta(t,u) \rangle}, \quad u \in U^{\beta},
\]
where
\[
x^{\beta} := x_{[2,d+1]} \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n, \quad (1.36)
\]
\[
U^{\beta} := C_{\leq 0} \times i\mathbb{R}^n. \quad (1.37)
\]

We state the next corollary just to fix the notation.

**Corollary 1.24** The function \( \Psi^\varphi \) defined in (1.34) satisfies the semiflow property,
i.e. for all \( s,t \geq 0 \) and \((u_0,u) \in U^\varphi\) it holds
\[
\Psi^\varphi(t+s,u_0,u) = \Psi^\varphi(t,\Psi^\varphi(s,u_0,u)).
\]
The function \( \Psi^\varphi \) is jointly continuous in \( \mathbb{R}_{\geq 0} \times U^\varphi \), differentiable at \( t = 0 \)
and moreover it satisfies
\[
\partial_t \Psi^\varphi(t,u_0,u) = R^\varphi(\Psi^\varphi(t,u_0,u)), \quad \Psi^\varphi(0,u_0,u) = (u_0,u),
\]
with
\[
\partial_t \Psi^\varphi(t,u_0,u)_{t=0} = R^\varphi(u_0,u) = \begin{pmatrix} F(u) \\ R(u) \end{pmatrix}.
\]
Observe that càdlàg property is preserved under the mapping \( \otimes \).

Hereafter, we concentrate on an affine process such that,
\[
\mathbb{E}^x[e^{\langle u,X_t \rangle}] = e^{\langle x,\Psi(t,u) \rangle}, \quad \text{for all} \ (t,x) \in \mathbb{R}_{\geq 0} \times D. \quad (1.38)
\]

**1.3.3 From \( C^* \) to \( C^H \)**

Next, we want to observe that, without loss of generality, we can assume that \( X \)
is an affine process with \( \Psi \in C^H \). The transformation is based on the method of
moving frames and it is taken from [KRST11].
Proposition 1.25 (Theorem 5.1 in [KRST11]) Let $X$ be an affine process. Define the matrix

$$T = \begin{pmatrix} I & 0 \\ 0 & B^\top J \end{pmatrix} \in \mathbb{R}^{d \times d}.$$ 

and the map

$$\mathcal{T} : \text{AP}(D) \rightarrow \text{AP}(D)$$

$$X \mapsto X - T^\top \int_0^X X_s ds.$$ 

The map $\mathcal{T}$ is a bijection between affine processes of type (1.38) with $\Psi(t, \cdot) \in \mathcal{C}^*$ and the class of affine processes of type (1.38) with $\Psi(t, \cdot) \in \mathcal{C}^H$. Moreover, $\mathcal{T}$ can be inverted by

$$\mathcal{T}^{-1}(Z) = Z_t + T^\top \int_0^t e^{T^\top (t-s)} Z_s ds.$$ 

1.3.4 Summary of the assumptions

We collect here the assumptions we are going to use often in the following sections.

$(A^\text{p})$ it holds $(b, a, c, m) = (0, 0, 0, 0)$.

$(A)$ for all $i = 1, \ldots, m$, $\gamma_i = 0$ and $\int (|\pi_I \xi| \wedge |\pi_I \xi|^2) M_i(d\xi) < \infty$.

$(A^H)$ for all $k, j = m + 1, \ldots, d$, $(\beta_j)_k = 0$.

1.4 Examples

Positive orthant $\mathbb{R}^m_{\geq 0}$

Non-negative affine processes with state space $D = \mathbb{R}^m_{\geq 0}$ are also called continuous state branching processes with immigration. When $F = 0$, the process is called branching process without immigration. When $J = \emptyset$, the functional characteristics of an affine process in $\mathbb{R}^m_{\geq 0}$ in the canonical form take the form

$$F(u) = \langle b, u \rangle - c$$

$$+ \int_{\mathbb{R}^m_{\geq 0}} (e^{\langle u, \xi \rangle} - 1) m(d\xi),$$

$$R_k(u) = \langle \beta_k, u \rangle + \frac{1}{2} (\alpha_k)_{kk} u_k^2 - \gamma_k$$

$$+ \int_{\mathbb{R}^m_{\geq 0}} (e^{\langle u, \xi \rangle} - 1 - u_k \xi_1 1_{\{|\xi| \leq 1\}}) M_k(d\xi)$$
1. Affine processes

where

\[
(\alpha_k)_{ij} = 0 \quad \text{if} \ (i, j) \neq (k, k), \\
b \in D, \\
(\beta_k)_i \geq 0 \quad \text{for all} \ k \in I \text{ and} \ i \in I \setminus \{k\}, \\
\text{supp} \ m \subseteq D \quad \text{and} \ \int_{\mathbb{R}_{\geq 0}^m} \left( |\pi_I | \wedge 1 \right) m(d\xi) < \infty, \\
\text{supp} \ M_i \subseteq D \quad \text{for all} \ k \in I \text{ and} \ \\
\int_{\mathbb{R}_{\geq 0}^m} \left( |\pi_I | \wedge 1 \right) M_k(d\xi) < \infty.
\]

Fix \( k = 1, \ldots, m \) and let \( Z^{(k)} \) be an infinitely divisible random variable on \( \mathbb{R}_m^m \geq 0 \) with Lévy triplet \((\beta_k, \alpha_k, M_k)\). Then, for all \( i = 1, \ldots, m, \pi_i Z^{(k)} \) is an infinitely divisible random variable on \( \mathbb{R}_m^m \geq 0 \) with Lévy triplet \((\pi_i \beta_k, \pi_i \alpha_k, \pi_i M_k)\) given by

\[
\pi_i \beta_k = (\beta_k)_i \\
\pi_i \alpha_k = (\alpha_k)_{ii} \\
\pi_i M_k(d\xi) = M_k(d(0, \ldots, \xi_i, \ldots, 0)) =: M_{k,i}(d\xi_i).
\]

Denote by

\[
\eta^{(k)}_i(v) := \log \mathbb{E} \left[ e^{v \pi_i Z^{(k)}} \right], \quad \text{for } v \in i\mathbb{R}.
\]

Due to the restrictions on the admissible parameters

\[
\eta^{(k)}_i(v) := \begin{cases} 
(\beta_k)_i v + \int e^{v \eta - 1 - v \eta 1_{(0,1)}} M_{k,i}(d\eta), & \text{if } i \neq k, \\
(\beta_k)_k v + \frac{1}{2} (\alpha_k)_{kk} v^2 + \int e^{v \eta - 1 - v \eta 1_{(0,1)}} M_{k,i}(d\eta), & \text{if } i = k.
\end{cases}
\]

We use the terminology from [KP08]

**Definition 1.26** Let \( Z \) be a Lévy process in \( \mathbb{R} \) with distribution \( \mathbb{P}^{(b,\sigma,\nu)} \). The process \( Z \) is a subordinator if and only if \( \nu(-\infty, 0) = 0, \int \eta 1_{(0,1)} \nu(d\eta) < \infty, \sigma = 0 \) and \( b - \int \eta 1_{(0,1)} \nu(d\eta) \geq 0 \). If \( \nu(-\infty, 0) = 0 \) and \( Z \) is not a subordinator, then \( Z \) is called spectrally positive Lévy process.

When a spectrally positive Lévy process has bounded variation, then necessarily

\[
Z_t = ct + N_t, \quad t \geq 0,
\]

where \( N \) is a pure jump subordinator and \( c < 0 \) (observe that, if \( c \) were positive, then \( Z \) would be a subordinator).

**Example 1.27 (CIR model)** Consider the one dimensional diffusion \( (X_t)_{t \geq 0} \) given
by the solution of the following SDE:
\[
\begin{aligned}
    \begin{cases}
        dX_t = (b + \beta X_t)dt + \sqrt{\alpha X_t}dW_t, \\
        X_0 = x.
    \end{cases}
\end{aligned}
\]

This is an example of affine process in \( \mathbb{R}_{\geq 0} \). Its Fourier–Laplace transform can be explicitly computed as the solution of the Riccati ODE
\[
\begin{aligned}
    \begin{cases}
        \partial_t \Psi(t, u) = \beta \Psi(t, u) + \frac{1}{2} \alpha \Psi^2(t, u), \\
        \Psi(0, u) = u,
    \end{cases}
\end{aligned}
\]
and
\[
\varphi(t, u) = b \int_0^t \Psi(s, u)ds.
\]

In the following, we will need the explicit form of the Fourier–Laplace transform of this particular example
\[
\begin{aligned}
    \begin{cases}
        dX_t = \sqrt{X_t}dW_t, \\
        X_0 = x.
    \end{cases}
\end{aligned}
\]

In this case, the Riccati ODE is given by
\[
\begin{aligned}
    \begin{cases}
        \partial_t \Psi(t, u) = \frac{1}{2} \Psi^2(t, u), \\
        \Psi(0, u) = u,
    \end{cases}
\end{aligned}
\]
and \( \varphi(t, u) = 0 \). Its solution is given by \( \Psi(t, u) = \frac{u}{1 - \frac{u}{2}} \).

**Example 1.28 (Coupled CIR)** Consider the following two dimensional extension of the CIR model
\[
\begin{aligned}
    \begin{cases}
        dX_t = \sqrt{2X_t}dW_t^1, \\
        dY_t = X_tdt + Y_tdt + \sqrt{2Y_t}dW_t^2, \\
        X_0 = x, \\
        Y_0 = y,
    \end{cases}
\end{aligned}
\]
where \( W = (W^1, W^2) \) is a two dimensional Brownian motion. The process \( (X, Y) \) is an affine process on \( \mathbb{R}^2_{\geq 0} \) with
\[
R(u, v) = (u^2, u + v + v^2)^\top.
\]

**Example 1.29 (CIR with jumps)** Consider the stochastic process
\[
\begin{aligned}
   dZ_t^v &= cZ_t^v dt + \sqrt{Z_t^v}dW_t \\
    &+ \int_{\{\xi > 1\}} \xi \mathcal{J}^Z(dt, d\xi) + \int_{\{\xi \leq 1\}} \xi (\mathcal{J}^Z(dt, d\xi) - \frac{d\xi}{\xi^2} Z_t^v dt) \\
\end{aligned}
\]
\quad (1.41)
with \( c \in \mathbb{R}_{\geq 0} \). We need first this lemma.

**Lemma 1.30** The function \(-u \log(-u)\) identifies a functional characteristics of an affine process \( X \) on \( \mathbb{R}_{\geq 0} \). Moreover, the Fourier–Laplace transform of \( X \) is given by

\[
\mathbb{E}^x \left[ e^{uX_t} \right] = e^{x \Psi(t,u)}, \quad \text{with } \Psi(t,u) = -(-u)^{-t}, \quad t \geq 0, u \in \mathbb{C}_{<0}.
\]

**Proof** Due to the singularity of the functions, we move for the Fourier–Laplace transform to the Laplace transform. It is known that, for \( v \in \mathbb{R}_{\geq 0} \), the following holds

\[
r(v) := v \log(v) = v(1 - \Gamma) + \int_0^\infty \left( e^{-v\xi} - 1 + v\xi 1_{\{\xi \leq 1\}} \right) \frac{d\xi}{\xi^2},
\]

where \( \Gamma \) is the Euler–Mascheroni constant. The solution of

\[
\partial_t \psi(t,v) = -r(\psi(t,v)), \quad \psi(0,v) = v,
\]

is given by

\[
\psi(t,v) = v e^{-t}.
\]

Additionally, for \( v \geq 0 \), since \( e^{-t} \in (0, 1) \) for all \( t > 0 \), \( \psi \) can be seen as the Laplace exponent of an \( \alpha \)-stable subordinator with \( \alpha \in (0, 1) \). Precisely it holds (see [App04], Example 1.3.18)

\[
v^{-t} = \int_0^\infty (1 - e^{-v\xi}) \frac{e^{-t}}{\Gamma(1 - e^{-t})\xi^{1 + e^{-t}}} d\xi.
\]

Translating the notation of affine process on cones to the one used in this thesis, we can conclude that, for all \( u \in \mathbb{C}_{<0} \),

\[
R(u) = r(-u) = -u \log(-u)
\]

\[
= u(\Gamma - 1) + \int_0^\infty \left( e^{u\xi} - 1 - u\xi 1_{\{\xi \leq 1\}} \right) \frac{d\xi}{\xi^2}.
\]

Since the conditions in Table 1.1 are met, \( R \) identifies the functional characteristic of an affine process in \( \mathbb{R}_{\geq 0} \). Moreover it holds

\[
\Psi(t,u) = -\psi(t,-u) = -(-u)^{-t}.
\]
1.4. Examples

Since for \( u \in C_{<0} \)
\[-u \log(u) = u(\Gamma - 1) + \int \left( e^{u\xi} - 1 - u\xi1_{\{\xi \leq 1\}} \right) \frac{d\xi}{\xi^2},\]
for the choice \( c = \Gamma - 1 > 0 \) we get that the process \( Z \) defined in (1.41) is an affine process on \( \mathbb{R}_{\geq 0} \) with
\[ R(u) = \frac{1}{2}u^2 - u \log(u). \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of the semi-flow of Example 1.27 in (a) and the one of Example 1.29 in (b). Plots correspond to \( t = 0, 0.5, 1, 1.5, 2 \).}
\end{figure}

**Heston model**

Consider the two dimensional process

\[
\begin{align*}
    dX_t &= \sqrt{2}X_t dW_t^1, & X_0 &= x, \\
    dY_t &= -X_t dt + \sqrt{2}X_t dW_t^2, & Y_0 &= y,
\end{align*}
\]

where \( W = (W^1, W^2) \) is a two dimensional Brownian motion. The process \( (X,Y) \) is an affine process on \( \mathbb{R}_{\geq 0} \times \mathbb{R} \) with
\[
R(u,v) = \left( -v + u^2 + v^2, 0 \right)^\top.
\]
The exact Fourier–Laplace transform of \((X, Y)\) is given by

\[
E^{(x, y)} \left[ e^{vX + wY} \right] = \exp \left( yw + x\gamma(w) \tan \left( t\gamma(w) + \arctan \frac{v}{\gamma(w)} \right) \right),
\]

where \(\gamma(w) = \sqrt{w(w-1)}\).

The explicit form of its characteristic function was given in the original paper [Hes93]. However different formulations have been introduced in order to obviate some instability observed when working with the original form, see [AMST07] and [dBRFCU10] for example. Much less is known in terms of its triplet and the problem gets particularly complicated if we consider possible multivariate extensions of the model.

**Remark 1.31** In order to add correlation, we consider the linear transformation induced by the matrix

\[
\Lambda = \begin{pmatrix}
1 & 0 \\
\rho & 0
\end{pmatrix},
\]

with \(\rho \in (-1, 1)\).
Chapter 2

Time–space representation of affine processes

2.1 Introduction

This chapter is devoted to the analysis of different representations of affine processes, which can be derived starting from the several characterizations we presented in Chapter 1. The first representation arises from the branching property of affine processes. In analogy with what is done for the genealogy of a branching process (see for example [Lam02]) we depict the paths of an affine process as realizations of a process with independent and stationary increments taking values in the path–space. We will see that, for any fixed $t \geq 0$ and $x \in D$, there exists a Lévy processes such that, in distribution, it holds

$$X^x_t \overset{d}{=} L^{(t,x)}_{|s=1}.$$ 

We call them representing Lévy processes. Additionally, we can lift this construction to path–space. For application purposes, it is very important to have a characterization of the aforementioned processes. Since the class of processes we obtain with this time–space shift is contained in the class of Lévy processes, a full characterization boils down as soon as we identify the Lévy triplet. However, the problem of the identification of the triplet is linked to the solution of the system of nonlinear ODEs defining the Fourier–Laplace transform of the affine processes we take in consideration. Explicit solutions of this type of equations are not always possible. In Section 2.2.2 we collect the cases which are solvable. In Chapter 3 we will focus our attention on the construction of approximation schemes which can be used when the explicit knowledge of the Lévy triplet is not known. The examples provided in Section 2.2.2 will be the building blocks of the numerical approximations presented in Chapter
3. In any case, we provide in Section 2.2.3 analytic expressions for the characteristic functional of the representing Lévy processes, extending the results in [PY82]. The last section collects and unifies other results on time–change representations of affine processes, which are already available in the literature. In 1967, Lamperti introduced a bijection between branching processes, a.k.a. affine processes on $\mathbb{R}_{\geq 0}$ with functional characteristics $(0, R)$, and Lévy processes with Lévy exponent $R$.

In the original paper [Lam67a], the result is stated without a proof. Only recently, two different proofs of the theorem have been presented. Before in [CLUB09], for affine processes in $\mathbb{R}_{\geq 0}$ with functional characteristics $(0, R)$ and, subsequentially, it has been extended in [CPGUB13] including the case of general functional characteristics $(F, R)$. The multidimensional extension can be found in [Kal06]. Therein it is shown that, given a conservative affine process on the canonical state space $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, there exist $d$-independent Lévy processes, on a possibly enlarged probability space, whose time–changed semimartingale characteristics depend in an affine way on the state space. Hence, in comparison with the aforementioned results, the representation in [Kal06], is a weak results because the original affine process and the process constructed as time–changed Lévy processes live in two different probability spaces. In comparison, the construction in [CPGUB13] uses the techniques of multivariate time change as in [Kur80], which allows to reconstruct an affine process (resp. a Lévy process) in a pathwise way from a Lévy process (resp. an affine process). Here we extend this result to the multivariate case first under the assumption of the existence of a vector of random time changes, which is adapted to the filtration generated by the Levy processes. Then, Section 2.3.3 fixes the measurability issues by showing the existence of the solution of the autonomous system of ordinary differential equations which identifies the random time changes.

### 2.2 Path–space representation of affine processes

Let us start with an easy one dimensional example.

**Example 2.1** Let $(X_t)_{t \geq 0}$ be a Feller diffusion

\[
\begin{align*}
\left\{ \begin{array}{l}
dX_t = \sqrt{X_t}dW_t, \\
X_0 = x.
\end{array} \right.
\end{align*}
\]

We know that it is an affine process on $D = \mathbb{R}_{\geq 0}$ with Fourier–Laplace transform given by

\[
\mathbb{E}^x \left[e^{uX_t}\right] = e^{x\Phi(t,u)}, \quad \text{with} \quad \Re(e) < \frac{2}{t},
\]


46
Due to the linear structure, we can consider the initial value $x$ as time parameter and construct a Lévy process $(L_t^x)_{x \geq 0}$ such that the law $X_t$ with initial value $x$ equals the law of $L_t^x$. Indeed, $\Psi(t, u)$ can be interpreted as the Fourier–Laplace transform of a compound Poisson process $(L_t^x)_{x \geq 0}$ with exponentially distributed jumps.

Also in the higher dimensional case, we want to make use of the linearity in $x$ of the logarithm of the Fourier–Laplace transform, to make a time–space shift and consider the space parameter $x$ as time parameter. As announced in the introduction, the construction is made first for fixed parameter $t$. Then, using infinite decomposability of the path–space measures $(P^x)_{x \in D}$, we lift the construction to the path–space.

### 2.2.1 How to visualize the affine property

In this section we assume that $(A^\mathbb{F})$ holds. Recall that Assumption $(A^\mathbb{F})$ implies that $X$ is an affine process in $\mathbb{R}_+^m \times \mathbb{R}^n$ whose Fourier–Laplace transform additionally satisfies

$$E_x \left[ e^{\langle u, X_t \rangle} \right] = e^{\langle x, \Psi(t, u) \rangle},$$

with $\Psi$ having the properties listed in Corollary 1.24.

Under this assumption, the semigroup of transition functions $(p_t)_{t \geq 0}$ does not only satisfy the semigroup property in time, but also a convolution property in space. Indeed, as it is clear from the decomposition $e^{\langle x+y, \Psi(t, u) \rangle} = e^{\langle x, \Psi(t, u) \rangle} e^{\langle y, \Psi(t, u) \rangle}$ it holds:

$$p_t(x+y, \cdot) = p_t(x, \cdot) * p_t(y, \cdot), \quad \text{for all } t \geq 0 \text{ and } x \in D.$$

Since for each fixed $t \geq 0$, $(p_t(sx, \cdot))_{s \geq 0}$ is a convolution semigroup, and we can construct a process with stationary and independent increments as in Section 10 in [Sat99].

**Proposition 2.2** Let $(D(D_\Delta), (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (P^x)_{x \in D_\Delta})$ be an affine process satisfying $(A^\mathbb{F})$. For each fixed $t > 0$ and $x \in D$, there exists a process $(L_s^{(t,x)})_{s \geq 0}$ such that:

1. $L_0^{(t,x)} = 0$,

2. for every $0 \leq s_1 \leq s_2 < \infty$, the increment $L_{s_2}^{(t,x)} - L_{s_1}^{(t,x)}$ is independent of the family $(L_s^{(t,x)})_{s \in [0,s_1]}$ and it is distributed as $X_t^{(s_2-s_1)x}$. 

47
Moreover, for any fixed $t \geq 0$ and $x \in D$, there exists a unique modification $\tilde{L}^{(t,x)}$ of $L^{(t,x)}$ which is a Lévy process with càdlàg paths.

**Proof** The construction is based on the application of Kolmogorov’s extension theorem with the convolution semigroup $p_t(sx, \cdot)_{s \geq 0}$. For the proof of càdlàg property we refer to Appendix C.4. □

**Definition 2.3** Given an affine process $X^x$, we call the Lévy process $(L^{(t,x)}_s)_{s \geq 0}$ constructed in Proposition 2.2 the Lévy process representing $X^x$.

**Remark 2.4** For $D = \mathbb{R}_\geq$, the following decomposition has been considered in [Lam02]. For fixed $t \geq 0$ and $x \in \mathbb{R}_\geq$, the affine property gives the existence of two measures $\nu^0_t$ and $\nu^x_t$ such that $p_t(x, \cdot) = (\nu^0_t * \nu^x_t)(\cdot)$ and moreover

$$\int e^{\langle u, \xi \rangle} \nu^0_t(d\xi) = e^{x\psi(t,u)}, \quad \int e^{\langle u, \xi \rangle} \nu^x_t(d\xi) = e^{x\nu(t,u)}, \quad u \in C_{\leq 0}.$$

This allows to state the previous results also without the assumption $(A^\circ)$ by interpreting $\nu^0_t$ as initial distribution for the process $L^{(t,x)}$. Note that the same arguments holds also in the multidimensional case.

The above construction holds whenever both time $t$ and space $x$ are fixed. Observe that $t$ appears as a parameter for the process, meaning that for all $t \geq 0$ there exists a Lévy process

$$(\mathcal{D}(\mathbb{R}_\geq; D_\Delta), (\mathcal{G}^{(t,x)}_s)_{s \geq 0}, (L^{(t,x)}_s)_{s \geq 0}, P^{(b(t,x),\sigma(t,x),\nu(t,x))}),$$

where

- $\mathcal{D}(\mathbb{R}_\geq; D_\Delta)$ is the space of all the càdlàg paths from $\mathbb{R}_\geq$ in $D$,
- $(\mathcal{G}^{(t,x)}_s)_{s \geq 0}$ is the filtration generated by canonical coordinates,
- $(L^{(t,x)}_s)_{s \geq 0}$ is a stochastic process taking values in $\mathcal{D}(\mathbb{R}_\geq; D_\Delta)$ with stationary and independent increments,
- $P^{(b(t,x),\sigma(t,x),\nu(t,x))}$ is a probability measure on $\mathcal{D}(\mathbb{R}_\geq; D_\Delta)$.

Due to linearity, the construction can be lifted to the path–space by making the following observations.

1: Under the Assumption $(A^\circ)$, there exists an independent copy of $X$, let us call it $X'$, such that $X^{x+y} = X^x + (X')^y$. 

48
2. Path–space representation of affine processes

This implies that the family of path–space valued measures \((P^x)_{x \in D}\) satisfies the following convolution property on \(D(D)\):

\[ P^{x+y} = P^{x} * P^{y} . \]

Additionally, the fact that the Skorohod space \(D(D)\) is a Polish space implies that Kolmogorov’s extension theorem can be applied also with the convolution semigroup \((P^x)_{x \in D}\).

Kolmogorov’s existence theorem applied on the convolutions semigroup \((P^{sx})_{s \geq 0}\) leads to the existence of the following process:

**Theorem 2.5** For each \(x \in D\), there exists a Markov process

\[ \left( D(D)^{\mathbb{R}_{\geq 0}}, (G_s)_{s \geq 0}, (L^s)^{\cdot x}_{s \geq 0}, (\varphi_s(x, \cdot))_{s \geq 0}, P \right) \]

with

- \(D(D)^{\mathbb{R}_{\geq 0}}\) the space of all the paths from \(\mathbb{R}_{\geq 0}\) into the Skorohod space \(D(D)\),
- \((G_s)_{s \geq 0}\) the filtration generated by canonical coordinates,
- \((L^s)^{\cdot x}_{s \geq 0}\) a stochastic process taking values in \(D(D)\) with stationary and independent increments,
- \(\varphi\) a stochastic kernel from \((D, B(D))\) to \((D(D), \mathcal{F})\),
- \(P\) is a probability measure on \(D(D)^{\mathbb{R}_{\geq 0}}\),

such that, in distribution, \(L^1(x, \cdot)\) coincides with \(X^x\).

**Proof** Define the family of Markov kernels \((\varphi_s(x, \cdot))_{s \geq 0}\)

\[ \varphi_s(x, A) : (D, B(D)) \rightarrow (D(D), \mathcal{F}) , \]

by \(\varphi_s(x, A) := P^{sx}(A), A \in \mathcal{F}\). It is a convolution semigroup because, for all \(s, t \geq 0\),

\[ \varphi_{s+t}(x, \cdot) = P^{(s+t)x} = P^{sx} * P^{tx} = \varphi_s(x, \cdot) * \varphi_t(x, \cdot) , \]

and moreover \(\varphi_1 = P^x\). The set

\[ D(D)^{\mathbb{R}_{\geq 0}} := \{ \varpi : \mathbb{R}_{\geq 0} \rightarrow D(D) \} \]
consists of all the paths from $\mathbb{R}_{\geq 0}$ into the Skorohod space $\mathcal{D}(D_\Delta)$. Consider the canonical process defined by the map

$$L^{(s,x)}_s : \mathcal{D}(D_\Delta)^{\mathbb{R}_{\geq 0}} \rightarrow \mathcal{D}(D_\Delta),$$

$$\omega \mapsto \omega_s.$$ for any fixed $x \in D$. Finally let $(G_s)_{s \geq 0}$ be the natural filtration generated by $L$.

The process $(L^{(s,x)}_s)_{s \geq 0}$ satisfies the desired property: under the product measure $P$

$$P(L^{(s,x)}_{s_1} \in dw_1, \ldots, L^{(s,x)}_{s_n} \in dw_n) = \prod_{k=1}^n \varphi_{s_k-s_{k-1}}(x, d\omega_k - \omega_{k-1}),$$

with $0 = s_0 \leq s_1 \leq \ldots \leq s_n < \infty$, $(L^{(s,x)}_s)_{s \geq 0}$ is a Markov process with transition semigroup $(P^{sx})_{s \geq 0}$ and one dimensional distributions $(P^{sx})_{s \geq 0}$. Indeed, for all $B \in \mathcal{F}$ and $s_1, s_2 \in \mathbb{R}_{\geq 0}$,

$$(P \circ (L^{(s,x)}_s - L^{(s,x)}_1)^{-1})(B) = \int_B \varphi_{s_2-s_1}(x, d\omega) = \int_B P^{(s_2-s_1)x}(d\omega) = P^{(s_2-s_1)x}(B).$$

Moreover $L^{(s,x)}_{s_1}, \ldots, L^{(s,x)}_{s_n} - L^{(s,x)}_{s_{n-1}}$ are independent for all $0 = s_0 < \ldots < s_n < \infty$. By construction, given $B_1, \ldots, B_n \in \mathcal{F}$,

$$P(L^{(s,x)}_{s_1} \in B_1, \ldots, L^{(s,x)}_{s_n} - L^{(s,x)}_{s_{n-1}} \in B_n)$$

$$= \int 1_{B_1} \times \ldots \times 1_{B_n} (P \circ (L^{(s,x)}_s - L^{(s,x)}_{s_{n-1}})^{-1})(d\omega)$$

$$= \int 1_{B_1} \times \ldots \times 1_{B_n} \circ \prod_{k=1}^n (L^{(s,x)}_{s_k} - L^{(s,x)}_{s_{k-1}}) P(d\omega)$$

$$= \int 1_{B_1} \ldots \int 1_{B_n} \varphi_{s_1}(x, d\omega_1) \ldots \varphi_{s_n-s_{n-1}}(x, d\omega_n)$$

$$= \prod_{k=1}^n P(L^{s,x}_{s_k} \in B_k).$$

To show that the process $(L^{(s,x)}_s)_{s \geq 0}$ is stochastically continuous, we show that

$$\lim_{s \to t} \varphi_s(x, \cdot) = \varphi_t(x, \cdot)$$

weakly on $\mathcal{D}(D_\Delta)$ for every $t \geq 0$ and $x \in D$. Take a
2.2. Path–space representation of affine processes

function \( f \in C_b(D) \), then we want to show that, for every \( s_1, s_2 \in \mathbb{R}_{\geq 0} \),

\[
\lim_{s_1 \to s_2} \int_{D(D_\Delta)} f(\omega) \varphi_{s_1}(x, d\omega) = \int_{D(D_\Delta)} f(\omega) \varphi_{s_2}(x, d\omega),
\]

or equivalently

\[
\lim_{s_1 \to s_2} \int_{D(D_\Delta)} f(\omega) P^{s_1,x}(d\omega) = \int_{D(D_\Delta)} f(\omega) P^{s_2,x}(d\omega).
\]

Consider functions of the type

\[
f_u(\omega) = e^{(u_1,\omega_1)+\ldots+(u_N,\omega_N)}, \text{ for all } t_1, \ldots, t_N \geq 0 \text{ and } u_1, \ldots, u_N \in \mathcal{U}.
\]

Since

\[
\mathbb{E}^{sx}\left[e^{\sum_{k=0}^n \langle u_k, X_{tk} \rangle}\right] = e^{sx, \Psi^N(t_1, \ldots, t_N, u_1, \ldots, u_N)},
\]

with \( \Psi^N \) defined in Proposition 1.20, easily

\[
\lim_{s_1 \to s_2} \mathbb{E}^{s_1,x}\left[f_u(X)\right] = \lim_{s_1 \to s_2} e^{sx, \Psi^N(t_1, \ldots, t_N, u_1, \ldots, u_N)} = \mathbb{E}^{s_2,x}\left[f_u(X)\right].
\]

2.2.2 Examples

Knowing a set of admissible parameters, the triplet of the Lévy process representing the affine process with the given admissible parameters is known up to the solution of a system of Riccati ODEs. In some cases, like those we present in this section, it is possible to solve the ODE analytically.

**Lemma 2.6** Let \( X \) be an affine process in \( \mathbb{R}^m_{\geq 0} \). Then its representing Lévy process is a subordinator. In particular

1. if

\[
R_i(u) = (\beta_i)_i u_i + \frac{1}{2} (\alpha_i)_i u_i^2,
\]

each component of the representing Lévy process is a compound Poisson process with exponential jumps,

2. if \( d = 1 \) and

\[
R(u) = -u \log(-u), \quad u \in C_{<0}
\]

the representing Lévy process is an \( \alpha \)-stable subordinator with \( \alpha \in (0, 1) \).
Proof For the first fact, observe that the representing Lévy process can only have non negative jumps, if the state space is \( \mathbb{R}^m_{\geq 0} \). For the first example write in vector notation
\[
R(u) = Bu + \frac{1}{2} \text{diag}(u) \text{diag}(\alpha) \text{diag}(u), \quad u \in \mathbb{C}^m_{\leq 0},
\]
with
\[
\alpha = ((\alpha_1)_{1,1}, \ldots, (\alpha_m)_{m,m})^T
\]
and \( \text{diag}(v) \) the \( m \times m \) diagonal matrix with diagonal entries \( v \in \mathbb{R}^m_{\geq 0} \). The Euclidean space \( \mathbb{R}^m_{\geq 0} \) together with the Jordan product given by the componentwise multiplication is a Jordan algebra with corresponding reducible cone \( \mathbb{R}^m_{\geq 0} \). We make use of results in Chapter II in [FK94] to conclude that
\[
\Psi(t, u) = e^{Bt} \left( u^{-1} - \frac{1}{2} \int_0^t e^{-Bs} \alpha^\top ds \right)^{-1},
\]
where \( x^{-1} = (x_1^{-1}, \ldots, x_m^{-1}) \) is the componentwise inverse for \( x \in \mathbb{R}^m_{\geq 0} \). The Lévy-Khintchine decomposition for \( \Psi(t, u) \) is easy when \( B \) is identically zero. In this case, for all \( (t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U} \) such that \( \Re(u) < \frac{\alpha_i}{2t} \), it holds that
\[
\left( u_{i-1} - \frac{(\alpha_i)_{i,i}}{2} t \right)^{-1} = \int_0^t \left( e^{u_i \xi_i} - 1 \right) \nu_i(t, d\xi_i),
\]
where \( \nu_i(t, dx) := \frac{1}{\lambda_i} E_{\lambda_i}(x) dx \), with \( E_{\lambda} \) density of an exponential random variable with parameter \( \lambda \) and \( \lambda_i = \frac{(\alpha_i)_{i,i}}{2} t \). For general case, if \( x \in D \), define
\[
E_{\lambda}(x) = \left( \frac{1}{\lambda_1} E_{\lambda_1}(x_1), \ldots, \frac{1}{\lambda_m} E_{\lambda_m}(x_m) \right)^\top,
\]
with \( \lambda := \frac{1}{2} \int_0^t e^{-Bs} \alpha ds \).

Then, \( \Psi(t, u) \) has the Lévy-Khintchine representation of a pure jump process with integrable Lévy measure \( \nu(t, \cdot) \) given by
\[
\nu(t, d\xi) := e^{Bt} E_\lambda(\xi) d\xi.
\]
Finally, when \( R(u) = -u \log(-u) \) the representation has been already stated in Lemma 1.30. \[ \blacksquare \]

**Proposition 2.7** Suppose that \( X \) additionally satisfies \((\hat{A})\) and \((\hat{A}^H)\). Recall that, under these assumptions, the set of admissible parameters satisfies
2.2. Path–space representation of affine processes

Representing Levy processes for \( X_t = x + \int_0^t \sqrt{2} X_s \, ds \)

Figure 2.1: For the Feller diffusion, the representing Lévy processes are compound Poisson processes with exponential jumps. The plot shows two realization of the Feller diffusion with starting point \( x = 2 \) and \( x = 1 \) (black lines). The paths with different colors are realization of the compound Poisson processes for different time parameters (in the plot the time parameters are set to be \( t_k = k/50, k = 1, \ldots, 50 \)).

- \((b, a, c, m) = (0, 0, 0, 0)\),
- for all \( i = 1, \ldots, m \), \( \gamma_i = 0 \) and \( \int (|\pi_I \xi| \wedge \pi_I \xi|^2) M_i (d\xi) < \infty \),
- for all \( k, j = m + 1, \ldots, d \), \((\beta_j)_k = 0\).

Then, for any \((t, x) \in \mathbb{R}_{\geq 0} \times D\) the representing Lévy process has the following Lévy triplet

\[
\begin{align*}
\mathbf{b}(t, x) &= e^{B^T t} x, \\
\sigma(t, x) &= \int_0^t e^{B^T (t-s)} \alpha^J (x), \\
\nu(t, x, d\xi) &= \langle x, \nu(t, d\xi) \rangle, \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{b}(t, x) &= e^{B^T t} x, \\
\sigma(t, x) &= \int_0^t e^{B^T (t-s)} \alpha^J (x), \\
\nu(t, x, d\xi) &= \langle x, \nu(t, d\xi) \rangle, \\
\end{align*}
\]

\[
\begin{align*}
\mathbf{b}(t, x) &= e^{B^T t} x, \\
\sigma(t, x) &= \int_0^t e^{B^T (t-s)} \alpha^J (x), \\
\nu(t, x, d\xi) &= \langle x, \nu(t, d\xi) \rangle, \\
\end{align*}
\]

53
2. Time–space representation

where \( \nu(t, d\xi) = \int_0^t e^{B_j^T(t-s)} n(s, d\xi) \) and each \( n_i \) for \( i = 1, \ldots, m \) is the Lévy measure

\[
n_i(s, d\eta) := \int_{D\setminus\{0\}} p_s(\xi, d\eta) M_i(d\xi).
\]

**Proof** Under the above set of assumptions on the parameters, for each \( i = 1, \ldots, m \), we can write

\[
R_i(u) = \langle \tilde{\beta}_i, u \rangle + \frac{1}{2} \langle \pi_j u, \alpha^j_i \pi_j u \rangle + \int_{D\setminus\{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_j u, \pi_j \xi \rangle \right) M_i(d\xi),
\]

where \( \tilde{\beta}_i \) are the corrected drift terms. Observe that \( \pi_j \tilde{\beta}_i = \pi_j \beta_i \).

The Riccati equation can be solved by variation of constants. The integral formulation for \( \Psi \) reads

\[
\Psi_i(t, u) = u_i + \int_0^t R_i(\Psi(s, u)) ds
= u_i + \int_0^t \langle \beta_i, \pi_j \Psi(s, u) \rangle ds + \int_0^t \langle \pi_j \tilde{\beta}_i, \pi_j u \rangle ds + \int_0^t \langle \pi_j u, \alpha^j_i \pi_j u \rangle ds + \int_0^t \int_{D\setminus\{0\}} \left( e^{\langle \Psi(s, u), \xi \rangle} - 1 - \langle \pi_j u, \pi_j \xi \rangle \right) M_i(d\xi).
\]

Observe that in the above equation the second order term in \( u \) does not depend anymore on \( \Psi \) but we do still have a non linear term in the jump component.

Write

\[
\int_0^t \int \left( e^{\langle \Psi(s, u), \xi \rangle} - 1 - \langle \pi_j u, \pi_j \xi \rangle \right) M_i(d\xi)
= \int \left( e^{\langle \Psi(s, u), \xi \rangle} - 1 - \int \langle \pi_j u, \pi_j \eta \rangle p_s(\xi, d\eta) \right) M_i(d\xi)
+ \int \left( \int \langle \pi_j u, \pi_j \eta \rangle p_s(\xi, d\eta) - \langle \pi_j u, \pi_j \xi \rangle \right) M_i(d\xi).
\]

Introduce

\[
n_i(s, d\eta) := \int_{D\setminus\{0\}} p_s(\xi, d\eta) M_i(d\xi), \ i = 1, \ldots, m.
\]
For every \( s \geq 0 \) and \( u \in \mathcal{U} \) define \( f(s, u) := (f_1(s, u), \ldots, f_m(s, u)) \) with

\[
\begin{align*}
  f_i(s, u) &= \langle \pi J \beta_i, \pi J u \rangle \\
  &\quad + \langle \pi J u, \alpha_i^J \pi J u \rangle \\
  &\quad + \int_D \left( e^{\langle u, \eta \rangle} - 1 - \langle \pi J u, \pi J h(\xi) \rangle \right) n_i(s, d\xi),
\end{align*}
\]

where \( \beta_i \) is the corrected drift obtained by taking into account the last term in (2.6).

Then, by variation of constant, the solution of

\[
\pi J \Psi(t, u) = \pi J u + \int_0^t \mathcal{B}_J \pi J \Psi(s, u) ds + \int_0^t f(s, u) ds
\]

is given by

\[
\pi J \Psi(t, u) = e^{\mathcal{B}_J^T t} \pi J u + \int_0^t e^{\mathcal{B}_J^T (t-s)} f(s, u) ds.
\]

Writing \( f \) in vector notation we get

\[
\begin{align*}
  f(s, u) &= \mathcal{B}_J^T \pi J u + Q(\pi J u) \\
  &\quad + \int_D \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi J u, \pi J \xi \rangle \right) n(s, d\xi),
\end{align*}
\]

where each \( Q_i(\pi J u) \) is given by \( \frac{1}{2} \langle \pi J u, \alpha_i^J \pi J u \rangle \), for all \( i = 1, \ldots, m \).

Integrating we get

\[
\begin{align*}
  \pi J \Psi(t, u) &= \mathcal{B}_J^T (t) \pi J u + \mathcal{B}_J^T J(t) \pi J u + \int_0^t e^{\mathcal{B}_J^T (t-s)} Q(\pi J u) ds \\
  &\quad + \int_D \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi J u, \pi J \xi \rangle \right) \nu(t, d\xi),
\end{align*}
\]

with

\[
\nu(t, d\xi) = \int_0^t e^{\mathcal{B}_J^T (t-s)} n(s, d\xi).
\]

The result holds by changing again the drift to introduce again the truncation function and taking into consideration that

\[
\mathcal{B}(t) = \left( \begin{array}{c} e^{\mathcal{B}_J t} \\ 0 \end{array} \right) = e^{\mathcal{B}_J t}.
\]
2. Time–space representation

2.2.3 Applications to linear functionals of affine processes

In this section we show how to compute the Fourier–Laplace transform of linear functional of the path–valued representing Lévy processes. As expected, we will show that the affine property is kept also at the level of the path–space. Let \( \mu \) be a Radon measure on \( \mathbb{R}_{\geq 0} \) and suppose it has compact support in \([0, T]\). Define the following function acting on \( D(\mathbb{R}_{\geq 0}, D_\Delta) \):

\[
I_\mu : D(D_\Delta) \to \mathbb{R}^d
I_\mu(\omega) := \int_{\mathbb{R}_{\geq 0}} \omega s \mu(ds).
\]

The following quantity is well defined

\[
\Phi_\mu(x, u) := \mathbb{E}^x \left[ e^{\langle u, I_\mu(X) \rangle} \right], \quad u \in i\mathbb{R}^d
\]

and moreover, from self–decomposability of the family \((P^x)_{x \in D_\Delta}\) it follows that

\[
\Phi_\mu(x, u) = \prod_{k=1}^d \Phi_\mu(x_k e_k, u),
\]

where we decompose \( x = \sum_{k=1}^d x_k e_k \) with \( x_k \in \mathbb{R} \) and \( e_1, \ldots, e_d \) the canonical basis of \( \mathbb{R}^d \). In analogy with the one dimensional case presented in [PY82], we show that the function \( \Phi_\mu(x, u) \) has an exponentially affine structure. The result is not surprising. Indeed, in Proposition 1.20, we have already seen that the logarithm of the Fourier–Laplace transform of the finite dimensional marginals of an affine process is an affine function of the state variable \( x \). The next theorem shows that it holds

\[
\Phi_\mu(x, u) = e^{\varphi_\mu(u) + \langle x, \Psi_\mu(u) \rangle}
\]

and additionally provides an integral formulation for \( \varphi_\mu \) and \( \Psi_\mu \). Before giving the proof of the theorem, we want to spend some additional words on the related existing literature. The result in [PY82] has been extended for affine processes taking values on the cone of positive semidefinite matrices in [KK13]. In none of the two cases the assumption on the positivity of the state space is really used. This last remark has brought us to suppose that it could have been possible to extend the result for affine processes on the canonical state space. Rather than using the martingale approach as in [PY82] and [KK13], the representation here is constructed with an
2.2. Path–space representation of affine processes

Theorem 2.8 Let \( \mu \) be a finite positive Radon measure on \( \mathbb{R}_{\geq 0} \) and \( X \) an affine process on \( D \) with functional characteristics \((F, R)\). Then, for all \( u \in i\mathbb{R}^d \) and \( T \geq 0 \)

\[
\mathbb{E}^x \left[ e^{\langle u, \int_0^T X_s \mu(ds) \rangle} \right] = \exp \left( \varphi_{\mu}(T, u) + \langle x, \Psi_{\mu}(T, u) \rangle \right),
\]

where the exponents satisfy the Riccati equations

\[
\begin{align*}
\Psi_{\mu}(T, u) &= \int_0^T R(\Psi_{\mu}(s, u))ds + uG_T(0), \\
\varphi_{\mu}(T, u) &= \int_0^T F(\Psi_{\mu}(s, u))ds
\end{align*}
\]

with \( G_T(s) := \mu([s, T]) \).

Proof We first transform \( X \) into \( X^\circ \) as done in Section 1.3.2, to calculate with a linear process for the sake of simplicity. Fix \( N \in \mathbb{N} \) and consider a partition \( \{0 = t_0, t_1, \ldots, t_N = T\} \). Integration by parts for a càdlàg paths \( t \mapsto X_t \) yields

\[
\int_0^T X_s^\circ \mu(ds) = xG_T(0) + \int_0^T G_T(t)dX_s^\circ \tag{2.9}
\]

respectively in a discretized version

\[
\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} X_s^\circ \mu(s)ds = xG_T(0) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} G_T(s)dX_s^\circ. \tag{2.10}
\]

Notice that both sides converge in probability to \( \int_0^T X_s^\circ \mu(ds) \), hence it is sufficient to calculate the Fourier–Laplace transforms of approximations of the right hand side. We choose as approximations continuous, piecewise linear approximations \( \tilde{G}_T \) with derivatives constant on \([t_k, t_{k+1}]\), which will converge pointwise (and bounded by a constant) to any generic \( G \) as the partition refines.

We proceed by extending the state space by \( d \) dimensions considering the affine process \((X^\circ, \int_0^t X_s^\circ ds)\), see Theorem 4.10 in [Kel08]: this affine process allows to calculate the Fourier–Laplace transform of linear functionals of integrals of the affine
process. Using tower property, for \( u, v \in \mathcal{U} \) it holds
\[
\mathbb{E}_x \left[ e^{\langle v, X_T^\phi \rangle} + \left(u, \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} g_k X_s^\phi ds \right) \right] = e^{\langle x, \phi_{N+1}(T,v,u) \rangle},
\]
where the function \( q_{N+1} \) is defined iteratively by
\[
q_0(T, v, u) = v \quad \quad q_n(T, v, u) = \Psi(t_{N-n+1} - t_{N-n}, q_n + ug_{N-n}), \quad n = 1, \ldots, N.
\]
Taking \( v = 0 \) we get
\[
\mathbb{E}_x \left[ e^{\left(u, \int_0^T \tilde{G}(s) X_s^\phi ds \right)} \right] = e^{\langle x, \phi_{N+1}(T,u) \rangle}.
\]

Going to the limit leads to the following Riccati equation for the exponents of the Fourier–Laplace transforms
\[
\Psi^\phi_\mu(T, u) = G_T(0)u + \int_0^T \mathcal{R}^\phi(\Psi^\phi_\mu(s, u)) ds.
\]

By convergence in probability, which implies convergence in law, we obtain existence of the desired representation and the integral equation for \( \Psi^\phi_\mu \). Finally, it sufficed to consider the transformation induced by the map \( \phi \) to get the affine form.

**Remark 2.9** Let \( X \) be linear. Notice that the previous theorem calculates the Lévy exponent of the representing Lévy process in the following sense: any \( \mathbb{R}^d \)-valued linear functional \( \ell \) on càdlàg paths \( t \mapsto X_t \) being represented by an integral along a
2.3 Lamperti representation

Radon measure $\mu$ up to time $T$ satisfies the affine transform formula

$$\mathbb{E}\left[ e^{\langle u, \ell(L^{(x)}) \rangle} \right] = \mathbb{E}^x \left[ e^{\langle u, \ell(X) \rangle} \right] = \mathbb{E}^x \left[ e^{\langle u, \int_0^T X_s \mu(ds) \rangle} \right] = \exp \left( \langle x, \Psi_\mu(T, u) \rangle \right),$$

and the integral equation

$$\Psi_\mu(T, u) = G_T(0) u + \int_0^T R(\Psi_\mu(s, u)) ds.$$

It is a non-trivial fact that this equation actually has a solution, which is shown by the previous theorem.

2.3 Lamperti–type representation for affine processes

The starting point of this section is the following question proposed in [Kal06]:

**Question:** It is known that an affine process is characterized by a set of admissible parameters $(b, \beta, a, \alpha, c, \gamma, m, M)$ which, in turn, forms a set of $d+1$ Lévy–Khintchine triplets. Is it possible to express in a pathwise way each affine process in terms of $d+1$ Lévy processes with the corresponding triplets?

Theorem 3.4 in [Kal06] gives the existence of $\mathbb{R}^d$–valued Lévy processes which, properly time changed, generate a semimartingale with differential characteristics coinciding with the ones of a conservative affine process. However in [Kal06] the following problem is left unsolved:

**Question:** Is the process obtained as a time change adapted with respect to the filtration generated by the Lévy process?

In other words, is it possible to write it as a measurable function of the $d+1$ Lévy processes? Measurability, at least in this setting, is a property which does not follow directly from the theory of time–changed processes (see, for example, Theorem 8.2. in [BNS10]).

In the theory of time–change problems (see [Jac79, BNS10]), the main two ingredients are:

- a semimartingale $Z$ with respect to a filtration $(\mathcal{G}_t)_{t \geq 0},$
- a process $(\theta_s)_{s \geq 0}$ which is non decreasing, right continuous process and such that, for each $s$, $\theta_s$ is a stopping time with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}.$

We will see that, the main difficulty of the second question is located in the measurability property for the time–change. In order to clarify better these matters, we
2. Time–space representation

start by considering, in Section 2.3.1, the one–dimensional case. In this case, the solution of the measurability issue can be found in [EK86].

**Additional notation**

Let \( s \in \mathbb{R}_N^{N} \geq 0 \). Whenever we are going to consider \( s \) a time parameter, we emphasize its multidimensionality by writing \( s \). When \( s = (s_1, \ldots, s_N) \) is a multivariate time parameter and \( X \) is a stochastic process in \( \mathbb{R}^N \), we use the notation

\[
\mathbf{X}(s) := (X^{(1)}_{s_1}, \ldots, X^{(N)}_{s_N}) \in \mathbb{R}^N.
\]

Now we want to introduce the notations of multiparameter filtration, stopping times and time–changed filtration. The following definitions are taken from Chapter 2.8 in [EK86] and [Kur80].

Suppose that \( X \) is a càdlàg process defined on a complete probability space \((\Omega, \mathcal{G}, P)\). Define

\[
\mathcal{G}^s := \sigma \left( \{X^{(k)}_{t_k} : t_k \leq s_k, i = 1, \ldots, N\} \right), \quad \text{for } s \in \mathbb{R}_N^{N} \geq 0 \tag{2.11}
\]

and

\[
\mathcal{G}_s = \bigcap_{n \in \mathbb{N}} \mathcal{G}_{s(n)} \vee \sigma(\mathcal{N}), \tag{2.12}
\]

where \( \mathcal{N} \) is the collection of sets in \( \mathcal{G} \) with \( P \)-probability zero and \( s(n) \) is the sequence defined by \( s^{(k)}_k = s_k + 1/n \). A random variable \( \tau = (\tau_1, \ldots, \tau_N) \in \mathbb{R}_N^{N} \) is a \((\mathcal{G}_s)\)-stopping time if

\[
\{\tau \leq s\} := \{\tau_1 \leq s_1, \ldots, \tau_N \leq s_N\} \in \mathcal{G}_s, \quad \text{for all } s \in \mathbb{R}_N^{N}.
\]

If \( \tau \) is a stopping time,

\[
\mathcal{G}_\tau := \{B \in \mathcal{G} : B \cap \{\tau \leq s\} \in \mathcal{G}_s \text{ for all } s \in \mathbb{R}_N^{N}\}.
\]

2.3.1 The one–dimensional case

When \( D = \mathbb{R}_\geq 0 \), it has been proved that there exists one-to-one correspondence between affine processes taking values in \( D \) and Lévy processes, see [CPGUB13]. More precisely, let \( Z^{(1)} = (Z^{(1)}_t)_{t \geq 0} \) be a Lévy process starting from 0 taking values in \( \mathbb{R} \) whose Lévy measure has support \( \mathbb{R}_\geq 0 \). This implies that there exists a function \( R : \mathbb{R} \rightarrow \mathbb{C} \) such that

\[
\mathbb{E} \left[ e^{uZ^{(1)}_s} \right] = e^{sR(u)},
\]
for all \((s, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}\) and \(u \in \mathbb{iR}\). Due to the restrictions on the jump measure, the function \(R\) takes the form

\[
R(u) = \beta u + \frac{1}{2} \alpha^2 u^2 - \gamma + \int_{\mathbb{R}_{\geq 0}} \left( e^{u\xi} - 1 - u\xi \mathbb{1}_{|\xi| \leq 1} \right) M(d\xi),
\]

where \(u \in \mathbb{iR}\), \(\alpha, \beta \in \mathbb{R}\) and \(M\) is a measure on \(\mathbb{R}_{\geq 0}\) which satisfies

\[
\int (1 \wedge |\xi|^2) M(d\xi) < \infty.
\]

Moreover, let \(Z^{(0)}\) be an independent subordinator with

\[
\mathbb{E}\left[ e^{uZ_s^{(0)}} \right] = e^{sF(u)},
\]

for all \((s, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}\) and \(u \in \mathbb{iR}\). Since \(Z^{(0)}\) is a subordinator, there exists a constant \(b \in \mathbb{R}_{\geq 0}\) and a measure \(m\) in \(\mathbb{R}_{\geq 0}\) satisfying

\[
\int (1 \wedge |\xi|) m(d\xi) < \infty,
\]

such that, for all \(u \in \mathbb{iR}\),

\[
F(u) = bu + \int_{\mathbb{R}_{\geq 0}} \left( e^{u\xi} - 1 \right) m(d\xi).
\]

Theorem 2 in [CPGUB13] shows that there exists a solution of the following time-change equation

\[
X_t = x + Z_t^{(0)} + Z_t^{(1)} \int_0^t X_s ds
\]

for all \((t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}\). Moreover, it is proved that the solution is a time homogeneous Markov process, taking values in \(\mathbb{R}\) starting from \(x\), such that the logarithm of the characteristic function of the transition semigroup is given by an affine function of the initial state \(x\). Hence, by definition, is an affine process taking values in \(\mathbb{R}\).

Here we are interested in the multivariate generalization of this result.

**Towards the multivariate Lamperti transform**  The next result is already known in the literature

**Theorem 2.10 (Theorem 3.4 in [Kal06])** Let \(X\) be an affine process with ad-
missible parameter satisfying
\[
\int_{|\xi|\geq 1} |\xi_k| M_i (d\xi) < \infty \quad \text{and} \quad \gamma_i = 0, \quad \text{for} \quad 1 \leq i, k \leq m.
\]

On a possibly enlarged probability space, there exists \(d+1\) independent Lévy processes \(Z^{(k)}\) such that
\[
X_t = x + Z^{(0)}_t + \sum_{k=1}^{d} Z^{(k)} \left( \int_0^t X^{(k)}_s \, ds \right).
\]
\hspace{1cm} (2.13)

This result is weak, in the sense that, without any additional assumption, it is not clear who to conclude that the process \(X\) is adapted with respect to the (properly time–changed) filtration generated by the Lévy processes.

Here we provide a solution of (2.13) for a fixed probability space \((\Omega, \mathcal{G}, \mathbb{P})\) which carries \(Z^{(0)}, \ldots, Z^{(d)}\).

We close this section with a two dimensional example which underlines some of the main difficulties one encounters when dealing with time–change equations of type (2.13).

**Example 2.11** Consider the case \(n = m = 1\). Write \(X = (V, Y)\) with \(V\) taking values in \(\mathbb{R}_{\geq 0}\) and \(Y\) in \(\mathbb{R}\). Then the representation in (2.13) reads
\[
\begin{pmatrix}
  V_t \\
  Y_t
\end{pmatrix}
= \begin{pmatrix}
  v \\
  y
\end{pmatrix}
+ Z^{(0)}(t) + Z^{(1)} \left( \int_0^t V_s \, ds \right) + Z^{(2)} \left( \int_0^t Y_s \, ds \right).
\]

As a first step, we introduce transform the process into another affine process with functional characteristic \(F = 0\). We will see that, up to an enlargement of the state space, there is no loss of generality in assuming that the parameters in the Lévy–Khintchine form of \(F\) are all identically zero. In few words, augment the process \(X\) by considering \(\tilde{X} = (X^0, V, Y)\), where \(X^0_t = 1\) for all \(t \geq 0\). Then we can write
\[
\begin{pmatrix}
  X^0_t \\
  V_t \\
  Y_t
\end{pmatrix}
= \begin{pmatrix}
  1 \\
  v \\
  y
\end{pmatrix}
+ Z^{(0)} \left( \int_0^t X^0_s \, ds \right) + Z^{(1)} \left( \int_0^t V_s \, ds \right) + Z^{(2)} \left( \int_0^t Y_s \, ds \right).
\]

Since \(X^0\) is trivially a process with stays positive, we can shift the indices can write
\[
\tilde{X} = x + \sum_{i=1}^{2} Z^{(i)} \left( \int_0^t \tilde{V}^{(i)}_s \, ds \right) + Z^{(3)} \left( \int_0^t \tilde{V}^{(3)}_s \, ds \right).
\]
\[ \widetilde{X} = (V, Y) \] with \( V \in \mathbb{R}_{\geq 0}^2 \) and \( Y \in \mathbb{R} \)

Back to the two dimensional example, we are mainly interested in the solution of

\[
\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix} + Z^{(1)} \left( \int_0^t V_s ds \right) + Z^{(2)} \left( \int_0^t Y_s ds \right).
\]

Observe that, while the first time–changed is defined by means of the path integral of a positive process, and therefore it is increasing, the main difficulty in interpreting the time change equation in a strong sense is located in the second time–change process. Indeed, in this case, the time–change process loses its monotonicity. However, due to the restrictions on the admissible parameters, the Lévy process \( Z^{(k)} \) with \( k = m + 1, \ldots, d \) are characterized by triplets of type \( (\beta_k, 0, 0) \) and hence they are deterministic processes. In Section 1.3 we introduces a set of pathspace transformation which allows us to work only with affine processes with admissible parameters satisfying the additional property \( (\beta_j)_k = 0 \) for all \( j, k \in J \). This means that the Lévy process \( Z^{(k)} \) with \( k = m + 1, \ldots, d \) are not only deterministic but actually identically equal to zero. In the two dimensional case, we are led to focus on equations of type

\[
\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix} + Z^{(1)} \left( \int_0^t V_s ds \right).
\]

Rewrite componentwise as

\[
\begin{align*}
V_t &= v + Z^{(1)}_1 \left( \int_0^t V_s ds \right) \\
Y_t &= y + Z^{(1)}_2 \left( \int_0^t V_s ds \right).
\end{align*}
\]

Then it is evident that only the equation for \( V \) is a real time–change equation. As soon as we provide a strong solution for the system of time–change equations describing the positive components, we automatically find a solution for the components taking values in \( \mathbb{R}^n \) by simple time–change.

2.3.2 The multi–dimensional case

**The setting** We fix \( Z^{(1)}, \ldots, Z^{(d)} \) \( d \) independent \( \mathbb{R}^d \)-valued Lévy processes, each of them with Lévy triplet \( (\beta_k, \alpha_k, M_k) \), \( k = 1, \ldots, d \), defined on the same probability space \( (\Omega, \mathcal{G}, P) \). Henceforth, we assume that the following restrictions on the Lévy triplets hold:

(H) the family \( (\beta, \alpha, M) \) consisting of the collection of the triplets \( (\beta_k, \alpha_k, M_k) \), \( k = 1, \ldots, d \) satisfies the assumptions \( A^H \) and \( \tilde{A} \)
Observe that a family \((\beta, \alpha, M)\) satisfying \((H)\) uniquely identifies an affine process of Heston type by taking the family \((0, \beta, 0, \alpha, 0, 0, M)\) as set of admissible parameters.

Moreover, we will always work with càdlàg version of each \(Z^{(k)}\). Under this assumption, each \(Z^{(k)}\), \(k = 1, \ldots, d\) can be identified with the coordinate process on the canonical space \((\Omega^{(k)}, \mathcal{G}^{(k)}, \mathbb{P}^{(k)})\), where \(\Omega^{(k)} = \mathcal{D}(\mathbb{R}^d)\). We consider the process \(Z = (Z^{(1)}, \ldots, Z^{(d)}) \in \mathbb{R}^{d^2}\) on the product space

\[
(\Omega, \mathcal{G}, \mathbb{P}) := \left( \prod_{k=1}^{d} \Omega^{(k)}, \bigotimes_{k=1}^{d} \mathcal{G}^{(k)}, \bigotimes_{k=1}^{d} \mathbb{P}^{(k)} \right).
\]

We fix \(x \in D\) and consider the functions

\[
f_i^{(k)}(y) := \langle x + Ny, e_k \rangle, \quad \text{for } k = 1, \ldots, d, \ i = 1, \ldots, d, \ y \in \mathbb{R}^{d^2}, \tag{2.14}
\]

where \(N \in \mathbb{R}^{d \times d^2}\) is the matrix obtained by horizontally concatenating \(d\) times the identity matrix of dimension \(d\). In the next section it will be essential to construct the solution of a system of time–change equations of type

\[
Y_i^{(k)}(t) := Z_i^{(k)} \left( \int_0^t f_i^{(k)}(Y_s) ds \right), \quad k, i = 1, \ldots, d. \tag{2.15}
\]

In the next section we will prove the following result aim of this section is to prove the following result:

**Theorem (see Theorem 2.15)** Let \(Z^{(1)}, \ldots, Z^{(d)}\) be \(d\) independent \(\mathbb{R}^d\)-valued Lévy processes each of the with Lévy triplet \((\beta_k, \alpha_k, M_k)\), \(k = 1, \ldots, d\) defined on the same probability space \((\Omega, \mathcal{G}, \mathbb{P})\). Under the assumption that the triplets satisfy \((H)\), for all \(x \in D\), there exists a solution of the following time–change problem

\[
Y_i^{(k)}(t) := Z_i^{(k)} \left( \int_0^t f_i^{(k)}(Y_s) ds \right), \quad k, i = 1, \ldots, d \tag{2.15}
\]

with

\[
f_i^{(k)}(y) := \langle x + Ny, e_k \rangle, \quad k, i = 1, \ldots, d, \ y \in \mathbb{R}^{d^2} \tag{2.16}
\]

and \(N \in \mathbb{R}^{d \times d^2}\) the matrix obtained by horizontally concatenating \(d\) times the identity matrix of dimension \(d\).

The existence of a solution of (2.15) is essential. Let us see how it works for affine processes with state space in \(\mathbb{R}_{\geq 0} \times \mathbb{R}\).
Example 2.12  The previous result in the particular case when \( m = n = 1 \) give the existence of a solution for the time change equation

\[
Y_i^{(k)}(t) := Z_i^{(k)} \left( \int_0^t f_i^{(k)}(Y_s) \, ds \right), \quad k, i = 1, 2.
\]

Under the assumption \((H)\), \( Y_i^{(2)} = 0 \) for all \( t \geq 0 \) and

\[
Y^{(1)}(t) := Z^{(1)} \left( \int_0^t (x_1 + Y_1^{(1)}(s)) \, ds \right).
\]

Define

\[
X = x + NY \quad \text{with} \quad N = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.
\]

Replacing the definitions of the \( Y_i^{(k)} \), it is clear that \( X \) satisfies

\[
\begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} Z_1^{(1)}(\int_0^t X_1^{(1)}(s) \, ds) + Z_2^{(2)}(\int_0^t X_2^{(1)}(s) \, ds) \\ Z_1^{(1)}(\int_0^t X_1^{(1)}(s) \, ds) + Z_2^{(2)}(\int_0^t X_2^{(2)}(s) \, ds) \end{pmatrix}.
\]

In vector notation, we can write

\[
X = x + \sum_{i=1}^2 Z^{(i)} \left( \int_0^t X_s^{(i)} \, ds \right)
\]

which is indeed the formulation in Theorem 2.10.

The next theorem is a generalization of the above argument to the general multivariate case. The additional problem we still need to address is measurability of the time–change process with respect to the filtration generated by the \( Z^{(i)}, i = 1, \ldots, d \).

In order to do it, we will need the notion of multivariate filtration and multivariate stopping time taken from [EK86].

For all \( \underline{s} = (s_1, \ldots, s_d^2) \in \mathbb{R}_{\geq 0}^{d^2} \), define the \( \sigma \)-algebra

\[
G_{\underline{s}} := \sigma \left( \{ Z_{t_h}^{(h)} \mid t_h \leq s_h, \text{for } h = 1, \ldots, d^2 \} \right), \quad (2.17)
\]

and then complete it by

\[
G_{\underline{s}} = \bigcap_{n \in \mathbb{N}} G_{\underline{s}^{(n)}} \vee \sigma(N), \quad (2.18)
\]
where $\mathcal{N}$ is the collection of sets in $\mathcal{G}$ with $P$-probability zero and $s_n^{(k)} = s_k + 1/n$.

**Definition 2.13** A random variable $\tau = (\tau_1, \ldots, \tau_s) \in \mathbb{R}^{d_2}_{\geq 0}$ is a $(\mathcal{G}_s)$-stopping time if
$$\{\tau \leq s\} := \{\tau_1 \leq s_1, \ldots, \tau_d \leq s_d\} \in \mathcal{G}_s, \text{ for all } s \in \mathbb{R}^{d_2}_{\geq 0}.$$ If $\tau$ is a stopping time,
$$\mathcal{G}_\tau := \{B \in \mathcal{G} | B \cap \{\tau \leq s\} \in \mathcal{G}_s \text{ for all } s \in \mathbb{R}^{d_2}_{\geq 0}\}.$$

Now that we have introduced the necessary notation, we are ready to prove the following result.

**Theorem 2.14** Let $(b, \beta, a, \alpha, c, \gamma, m, M)$ be a set of admissible parameters satisfying the Assumptions $A^w$, $A^H$ and $\breve{A}$.

(i) The time–change equation
$$X_t = x + \sum_{i=1}^{d} Z^{(i)}(\theta_t^{(i)}), \text{ with } \theta_t^{(i)} = \int_0^t X_r^{(i)} dr,$$ (2.19)

admits a unique solution.

(ii) Define
$$\theta_t^x := (\theta_t^{(1)}(1), \ldots, \theta_t^{(1)}(d), \ldots, \theta_t^{(d)}(1), \ldots, \theta_t^{(d)}(d)) \in \mathbb{R}^{d_2}.$$ The random variable $\theta_t^x$ is a $\mathcal{G}_s$ stopping time for all $t \geq 0$. Hence the time–change filtration
$$\mathcal{G}_{\theta_t^x} := \{A | A \cap \{\theta_t^x \leq s\} \in \mathcal{G}_s \text{ for all } s \in \mathbb{R}^{d_2}_{\geq 0}\},$$
is well defined.

(iii) Let $R$ be the function defined as in (1.9). The solution of (2.19) is an affine process with functional characteristics $(0, R)$ with respect to the time–changed filtration $(\mathcal{G}_{\theta_t^x})_{t \geq 0}$.

**Proof** Let $Y \in \mathbb{R}^{d_2}$ be the process obtained by casting the solutions of (2.15) as
$$Y := (Y_1^{(1)}, \ldots, Y_1^{(d)}, Y_2^{(1)}, \ldots, Y_2^{(d)}, \ldots, Y_d^{(1)}, \ldots, Y_d^{(d)}).$$
Consider the matrix
\[
N := \begin{pmatrix}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0
\end{pmatrix} \in \mathbb{R}^{d \times d^2}.
\]

Then
\[
X = x + \sum_{k=1}^{d} Y^{(k)}
\]
is a solution of time-change equation (2.19). Indeed, in vector notation, we can write
\[
X = x + NY.
\]

Then, if \(Z^{(k)}_j\) denotes the \(j\)-th coordinate of the \(k\)-th Lévy process,
\[
Z^{(k)}_j \left( \int_0^t f^{(k)}_j(Y_s) \, ds \right) = Z^{(k)}_j \left( \int_0^t (x + NY_s, e_k) \, ds \right) = Z^{(k)}_j \left( \int_0^t X^{(k)}_s \, ds \right)
\]
and
\[
X_j = x_j + \sum_{k=1}^{d} Y^{(k)}_j
\]
\[
= x_j + \sum_{k=1}^{d} Z^{(k)}_j \left( \int_0^t X^{(k)}_s \, ds \right).
\]

Now we move on the measurability of the time–change process. Observe that Theorem 2.15 implies that the vector
\[
\tau(t) := (\tau^{(1)}_1(t), \ldots, \tau^{(1)}_d(t), \ldots, \tau^{(d)}_1(t), \ldots, \tau^{(d)}_d(t))
\]
where
\[
\tau^{(k)}_i(t) = \int_0^t f^{(k)}_i(Y_s) \, ds
\]
is a \(\mathcal{G}_t\) stopping time for all \(t \geq 0\). This follows from Theorem VI.2.2. in [EK86]. From the affine relationship between \(X\) and \(Y\) we conclude that \(\theta_t\) is a \(\mathcal{G}_t\) stopping time and therefore the time–changed filtration is well defined.

Now, we need to check that \(X\) is a homogeneous Markov process with respect to \((\mathcal{G}^k_{\theta_t})_{t \geq 0}\). Applying Proposition I.6 in [Ber98] at each component \(Z^{(k)}\), \(k =\)
2. Time–space representation

1, \ldots, d we get that \((Z(\theta^x_{t+h}) - Z(\theta^x_t))_{h \geq 0}\) has the same law as \(Z(\theta^x_h)_{h \geq 0}\) and it is independent of \(G_{\theta^x_t}\).

Therefore

\[ X^x_{t+h} = X^x_t + N(Z(\theta^x_{t+h}) - Z(\theta^x_t)) = S_t(Z(\theta^x_{t+h}) - Z(\theta^x_t), X^x_t), \]

with

\[ S_t : (\mathbb{R}^d, \prod_{i=1}^d (\mathbb{G}^{\theta^x_i})) \times (\mathbb{R}^d, \mathbb{G}^{\theta^x_t}) \rightarrow (\mathbb{G}^{\theta^x_t}) \]

\[ (Z, X) \rightarrow X + NZ. \]

Therefore, we conclude that the conditional law of \(X^x_{t+h}\), given \(G_{\theta^x_t}\), is \(X^x_t\) measurable. Markov property translates into

\[ X^x_{t+h} = S_0(Z(\theta^y_{y=x^x_t})), y \mid y = X^x_t. \]

Additionally the time–change process is absolutely continuous with

\[ \frac{d}{dt} \theta_i^{(k)}(t) = X_{t-}^{(k)}, \text{ for all } k, i = 1, \ldots, d. \]

The characteristics of the time–changed semimartingale can be computed using the formulas in Theorem 8.4. in [BNS10] to conclude that the process \((S_0(Z(\theta^x_t), x))_{t \geq 0}\) has characteristics \((\beta(X_-), \alpha(X_-), M(X_-))\), where

\[ \beta(x) = x_1 \beta_1 + \ldots + x_m \beta_m, \]
\[ \alpha(x) = x_1 \alpha_1 + \ldots + x_m \alpha_m, \]
\[ M(x, B) = x_1 M_1(B) + \ldots + x_m M_m(B), \quad B \in \mathcal{B}(D). \]

\[ \blacksquare \]

2.3.3 Existence of random time changes

In this section we focus on the proof of the result we announced previously.

**Theorem 2.15** Let \(Z^{(1)}, \ldots, Z^{(d)}\) be \(d\) independent \(\mathbb{R}^d\)-valued Lévy processes each of the with Lévy triplet \((\beta_k, \alpha_k, M_k), k = 1, \ldots, d\) defined on the same probability space \((\Omega, \mathcal{G}, P)\). Under the assumption that the triplets satisfy \((H)\), for all \(x \in D\),
there exists a solution of the following time–change problem

\[ Y_i^{(k)}(t) := Z_i^{(k)} \left( \int_0^t f_i^{(k)}(Y_s) ds \right), \quad k, i = 1, \ldots, d \]  

(2.15)

with

\[ f_i^{(k)}(y) := \langle x + Ny, e_k \rangle, \quad k, i = 1, \ldots, d, \quad y \in \mathbb{R}^{d^2} \]  

(2.20)

and \( N \in \mathbb{R}^{d \times d^2} \) the matrix obtained by horizontally concatenating \( d \) times the identity matrix of dimension \( d \).

The proof of Theorem 2.15 is done in several steps. We first translate the problem of existence and uniqueness of a solution for (2.15) in the problem of existence and uniqueness of a system of ODE with vector field given, for each \( \omega \in \Omega \) by the paths of Lévy processes. In order to construct the solution, we first separate the continuous part of the process from the jump part and then we do a first approximation which relies in the approximation of the starting vector field with paths which have bounded variation. Then we construct the solution of the approximated problem by properly composing the solutions deriving from the evolution of the continuous part and the jump part taken separately. For this purpose we need to introduce an additional approximation which allows us to replace the jump part of the vector field with a piecewise constant approximation. Finally, we construct the solution of the original problem with a limit argument.

We introduce

\[ \tau_i^{(k)}(t) := \int_0^t f_i^{(k)}(Y_s) ds, \quad \text{for} \quad k, i = 1 \ldots, d. \]  

(2.21)

and define

\[ \tau(t) := (\tau_1^{(1)}(t), \ldots, \tau_d^{(1)}(t), \ldots, \tau_1^{(k)}(t), \ldots, \tau_d^{(d)}(t)). \]  

(2.22)

Existence of a solution of (2.15) is straightforward if we can construct a solution of the following system of ODEs

\[
\begin{aligned}
&\dot{\tau}_i^{(k)}(t) = f_i^{(k)}(Z(\tau(t))), \quad \text{for all} \quad k, i = 1, \ldots, d, \\
&\tau_i^{(k)}(0) = 0,
\end{aligned}
\]  

(2.23)

where

\[ Z(\tau(t)) := (Z_1^{(1)}(\tau_1^{(1)}(t)), \ldots, Z_i^{(k)}(\tau_i^{(k)}(t)), \ldots, Z_d^{(d)}(\tau_d^{(d)}(t))). \]
2. Time–space representation

We start by showing that, existence for a solution of (2.23) can be proved by focusing on the components with \( k = 1, \ldots, m \) and \( i = 1, \ldots, m \).

Lemma 2.16 If
\[
\begin{cases}
\tau^{(k)}_i(t) = f^{(k)}_i(Z(\tau(t))), \\
\tau^{(k)}_i(0) = 0,
\end{cases}
\tag{2.24}
\]
admits a solution for all \( k = 1, \ldots, m \) and \( i = 1, \ldots, m \), then it admits also a solution for all \( k = 1, \ldots, d \) and \( i = 1, \ldots, d \).

Proof For all \( k = 1, \ldots, d \) it holds
\[
f^{(k)}_1(y) = x_k + \sum_{h=1}^{d} y^{(h)}_k = f^{(k)}_2 = \ldots = f^{(k)}_d, \quad \text{for all } y \in \mathbb{R}^{d^2},
\]
and therefore
\[
\tau^{(k)}_1(t) = \ldots = \tau^{(k)}_d(t), \quad \text{for all } t \geq 0.
\]
Denote
\[
\tau^{(k)}(t) := \tau^{(k)}_1(t), \quad \text{for all } t \geq 0 \text{ and } k = 1, \ldots, d.
\]
By definition, for each \( k = 1, \ldots, d \),
\[
\tau^{(k)}(t) = \int_0^t \left( x_k + \sum_{i=1}^{d} Y_k^{(i)}(s) \right) ds
\]
\[
= \int_0^t \left( x_k + \sum_{i=1}^{d} Z_k^{(i)}(\tau^{(i)}(s)) \right) ds
\]
\[
= \int_0^t \left( x_k + \sum_{i=1}^{d} Z_k^{(i)}(\tau^{(i)}(s)) \right) ds.
\]
By Assumption, each \( Z^{(j)} \) for \( j \in J \) is a Lévy process with Lévy triplet \((0, 0, 0)\), hence deterministic. In particular, once we find a unique solution for the system
\[
\tau^{(k)}(t) = \int_0^t \left( x_k + \sum_{i=1}^{m} Z_k^{(i)}(\tau^{(i)}(s)) \right) ds \quad \text{for all } k = 1, \ldots, m,
\]
then, for all \( t \geq 0 \) and \( j = m + 1, \ldots, d \), we can compute
\[
\tau^{(j)}(t) = \int_0^t \left( x_j + \sum_{i=1}^{m} Z_j^{(i)}(\tau^{(i)}(s)) \right) ds.
\]
where, for each $i = 1, \ldots, m \pi J Z^{(i)}$ is a Lévy process in $\mathbb{R}^n$ evaluated at the point $\tau^{(i)}(s)$. Therefore, the left hand side of the last equation does not depend anymore on the right hand side.

Henceforth, $Z^{(1)}, \ldots, Z^{(m)}$ are $m$-independent Lévy processes on $\mathbb{R}^m$, each of them with Lévy triplet $(\beta_i, \alpha_i, M_i)$, $i = 1 \ldots, m$, satisfying

$$
(\alpha_i)_{kl} = 0 \text{ for all } k, l \in I \text{ such that } (k, l) \neq (i, i),
(\beta_i)_k \geq 0 \text{ for all } i \in I \text{ and } k \in I \setminus \{i\}.
$$

From the previous lemma, we known that it suffices to study existence and uniqueness of the solution of the following problem

$$
\begin{aligned}
\dot{\tau}^{(k)}(t) &= x + Z_k(\tau^{(k)}(t)), & k = 1, \ldots, m, \\
\tau^{(k)}(0) &= 0,
\end{aligned}
$$

where

$$
Z : \mathbb{R}^m_{\geq 0} \to \mathbb{R}^n
$$

$$
s \mapsto \sum_{i=1}^m Z^{(i)}(s_i).
$$

In vector notation the previous initial value problem reads

$$
\begin{aligned}
\dot{\tau}(t) &= x + Z(\tau(t)), \\
\tau(0) &= 0.
\end{aligned}
$$

**Remark 2.17** By definition it holds

$$
\tau(0) = x \in \mathbb{R}^m_{\geq 0}.
$$

Due to the restrictions on the parameters, each $Z^{(i)}$, $i = 1, \ldots, m$ is a process with no negative jumps. This implies that, whenever a component $\tau^{(i^*)}_s$ reaches zero for some $i^*$, the corresponding component Lévy process $Z^{(i^*)}$ is stopped in zero. In particular, each trajectory of $x + Z$ stays positive until it is absorbed.

**First approximation of the vector field**

In order to construct a solution for (2.27), we start by decomposing of each process $Z^{(i)}$, $i = 1, \ldots, m$ into into its continuous part and jump part. The Lévy–Itô
decomposition together with the canonical form of the admissible parameters give

\[
Z^{(i)}_t = \beta_i t + \sigma_i B^{(i)}_t + \int_0^t \int_{\{\|\xi\| > 1\}} \mathcal{J}^{(i)}(d\xi, ds)
\]

\[
+ \int_0^t \int_{\{\|\xi\| \leq 1\}} (\mathcal{J}^{(i)}(d\xi, ds) - M_i(d\xi)ds)
\]

where \(\sigma_i = \sqrt{(\alpha_i)_{ii}}\), \(B^{(i)}\) is a process in \(\mathbb{R}^m_{\geq 0}\) which evolves only along the \(i\)-th coordinate as Brownian motion and \(\mathcal{J}^{(i)}\) is the jump measure of the process \(Z^{(i)}\).

Let \(\{r_h\}_{h \in \mathbb{N}}\) be a decreasing sequence in \((0, 1]\) chosen so that

\[
\int_{\{\|\xi\| \leq r_h\}} |\xi|^2 M(d\xi) \leq 2^{-(h+1)}.
\]

Define

\[
\Delta_h := \begin{cases} (r_0, \infty) & \text{if } h = 0, \\ (r_h, r_{h-1}] & \text{if } h \geq 1. \end{cases}
\]

Introduce the approximations

\[
Z^{(i,N)}_t := \beta^{(N)}_i t + \sigma_i B^{(i)}_t + \sum_{h=0}^N \int_0^t \int_{\{\|\xi\| \in \Delta_h\}} \xi \mathcal{J}^{(i)}(d\xi, ds)
\]

where

\[
\beta^{(N)}_i = \beta_i - e_i \int_{\{\|\xi\| \in (r_N, 1]\}} \xi_i M_i(d\xi).
\]

Observe that it is possible to specify a unique sequence \(\{r_h\}_{h \in \mathbb{N}}\) for all \(i = 1, \ldots, m\).

Then, for each \(T \in \mathbb{R}_{\geq 0}\)

\[
\sup_{t \in [0,T]} |Z^{(i,N)}_t(\omega) - Z^{(i)}_t(\omega)| \xrightarrow{N \to \infty} 0
\]

for \(P\) almost every \(\omega\), for all \(i = 1, \ldots, m\).

Now, from the assumption on the set of admissible parameter,

\[
\pi_{f\setminus \{i\}} \beta^{(N)}_i \in \mathbb{R}^{m-1}_{\geq 0} \quad \text{and} \quad (\beta^{(N)}_i)_{i} \in \mathbb{R}
\]

so we can decompose

\[
Z^{(i,N)} = Z^{(\tilde{i},N)} + Z^{(\tilde{i},N)}
\]
where \( \tilde{Z}^{(i,N)} \) and \( \tilde{Z}^{(i,N)} \) are two stochastic processes on \( \mathbb{R}^m \) defined by

\[
\begin{align*}
\tilde{Z}^{(i,N)}_k(t) &:= 0, & \text{for } k \neq i, \\
\tilde{Z}^{(i,N)}_i(t) &:= \sigma_i \mathcal{B}^{(i)}(t) + (\beta^{(N)}_i)_t, \\
\tilde{Z}^{(i,N)}(t) &:= \tilde{\beta}^{(N)}_i t + \sum_{h=0}^{N} \int_{\{\xi\in\Delta_h\}} \xi \mathcal{J}^{(i)}(d\xi, ds),
\end{align*}
\]

where

\[
\tilde{\beta}^{(N)}_i = \beta^{(N)}_i - e_i(\beta^{(N)}_i)_i.
\]

Introduce, for all \( s \in \mathbb{R}^m \geq 0 \),

\[
\tilde{Z}^{(N)}(s) := \sum_{i=1}^{m} \tilde{Z}^{(i,N)}(s_i), \\
\tilde{Z}^{(N)}(s) := \sum_{i=1}^{m} \tilde{Z}^{(i,N)}(s_i).
\]

**Proposition 2.18** Let \( t_0 \in \mathbb{R}^m, \tau_0 \in \mathbb{R}^m, \) and \( x \in D \). Suppose that the following system admits a unique solution

\[
\begin{align*}
\dot{\tau}^{(N)}((t_0, \tau_0, x); t) &= (x + \tilde{Z}^{(N)} + \tilde{Z}^{(N)})(((t_0, \tau_0, x); t)), \\
\tau^{(N)}((t_0, \tau_0, x); t) &= \tau_0.
\end{align*}
\]

then, also

\[
\begin{align*}
\dot{\tau}((t_0, \tau_0, x); t) &= (x + Z)((t_0, \tau_0, x); t)), \\
\tau((t_0, \tau_0, x); t) &= \tau_0.
\end{align*}
\]

with \( Z \) defined in (2.26) admits a unique solution and it holds

\[
\lim_{N \to \infty} \tau^{(N)}((t_0, \tau_0, x); t) = \tau((t_0, \tau_0, x); t)
\]

pointwise in \( \omega \).

**Proof** Observe that, by construction,

\[
Z^{(i,N+1)}_i = Z^{(i,N)}_i - e_i \int_{\{\xi\in\Delta_{N+1}\}} \xi_i M_i(d\xi) + \int_0^t \int_{\{\xi\in\Delta_{N+1}\}} \xi \mathcal{J}^{(i)}(d\xi, ds). (2.30)
\]

Consider the increments

\[
Y^{(i,N)} := Z^{(i,N+1)} - Z^{(i,N)} = \int_0^t \int_{\{\xi\in\Delta_{N+1}\}} \tilde{\mathcal{J}}^{(i)}(ds, d\xi),
\]
where $\overline{J}^{(i)}$ is the compensated jump measure
\[
\overline{J}^{(i)}(ds, d\xi) := J(ds, d\xi) - dsM_i(d\xi).
\]

It is clear that $Y^{(i,N)}$ has independent, centered increments and
\[
E\left[|Y_t^{(i,N)} - Y_s^{(i,N)}|^2\right] = t \int_{\{||\xi|| \in \Delta_{N+1}\}} |\xi|^2 M(d\xi).
\]
Fix $T \in \mathbb{R}_{\geq 0}$ and apply Kolmogorov’s inequality to get
\[
P\left(\sup_{t \in [0, T]} |Y_t^{(i,N)}| \geq \varepsilon\right) \leq \frac{T}{\varepsilon^2} \int_{\{||\xi|| \in \Delta_{N+1}\}} |\xi|^2 M(d\xi) \leq \frac{T}{\varepsilon^2} 2^{-N}.
\]

By Borel–Cantelli lemma $Z^{(i,N)} \to Z^{(i)}$ uniformly on compacts up to $P$ null sets.

For ease of notation, write simply $\overline{\tau}^{N+1}(t)$. Now, moving on the convergence of the solution of the ODE, fix $T \gg 0$ and consider
\[
\sigma^{(N,T)} := \overline{\tau}^{N+1} \wedge T - \overline{\tau}\wedge T.
\]

By definition,
\[
\sigma^{(N,T)}(t) = \int_0^t Z^{(N+1)}(\overline{\tau}^{N+1}(s) \wedge T) ds - \int_0^t Z^{(N)}(\overline{\tau}^{N}(s) \wedge T) ds.
\]
Using the previous uniform estimate for $Y^{(i,N)}$ for all $i = 1, \ldots, m$ we conclude that,
\[
P\left(\sup_{t \in [0, T]} \sigma^{(N,T)}(t) \geq \varepsilon\right) \leq \frac{T^2}{\varepsilon^2} 2^{-N}.
\]

By Borel–Cantelli lemma $\overline{\tau}^{N}(\cdot) \wedge T$ uniformly on compacts up to $P$ null sets. Denote this limit by $\overline{\tau}^\ast$. Now we need to show that this limit coincides with the solution of (2.27). From Remark 2.17 we are allowed to use Fubini–Tonelli theorem to conclude that, for any $t_1 \leq t_2$
\[
\lim_{N \to \infty} \int_{t_1}^{t_2} (x + Z^{(N)})(\overline{\tau}^{N}(s)) ds
\]
\[
= \int_{t_1}^{t_2} \lim_{N \to \infty} (x + Z^{(N)})(\overline{\tau}^{N}(s)) ds.
\]
At the same time, for all $i = 1, \ldots, m$,
\[
Z^{(i,N)}(t^-) = \lim_{s \to t^-} Z^{(i,N)}(s) \leq \lim_{s \to t^+} Z^{(i,N)}(s) = Z^{(i,N)}(t)
\]
therefore, since $\tau^N$ is an increasing sequence in $N$,
\[
\tau^*(t_2) - \tau^*(t_1) = \int_{t_1}^{t_2} (x + Z)(\tau^*(s)^-) ds \\
\leq \int_{t_1}^{t_2} (x + Z)(\tau^*(s)) ds
\]
If we show that the last inequality is actually an equality, then we conclude that necessarily $\tau^* = \tau$. Suppose that the strict inequality holds, then necessarily, there exists a subset where $\tilde{\tau}$ is constant. But, by construction, $\tilde{\tau}$ is strictly increasing until it is absorbed. The equality holds and therefore $\tau^* = \tau$. ■

In the following section, we will consider separately the initial value problems with vector fields
\[
x + \tilde{Z}^{(N)} \quad \text{and} \quad \tilde{Z}^{(N)}.
\]
We first show that it is possible to find a solution for the initial value problem
\[
\begin{align*}
\dot{\tau}((t_0, \tau_0, x); t) &= (x + \tilde{Z}^{(N)})(\tau((t_0, \tau_0, x); t)), \\
\tau((t_0, \tau_0, x); t_0) &= \tau_0,
\end{align*}
\]
and then we show how to construct a solution of the general problem.

**Proposition 2.19** There exists a solution of
\[
\begin{align*}
\dot{\tau}((t_0, \tau_0, x); t) &= (x + \tilde{Z}^{(N)})(\tau((t_0, \tau_0, x); t)), \\
\tau((t_0, \tau_0, x); t_0) &= \tau_0,
\end{align*}
\]
with $\tau_0 \in \mathbb{R}^m_{\geq 0}$.

**Proof** Observe that (2.31) is a decoupled system of $m$ equations of type
\[
\begin{align*}
\dot{\tau}_i((t_0, \tau_0, x); t) &= (x_i + \tilde{Z}^{(i,N)})(\tau_i((t_0, \tau_0, x); t)), \quad i = 1, \ldots, m, \\
\tau_i((t_0, \tau_0, x); t_0) &= \pi_{\{i\}} \tau_0.
\end{align*}
\]
where each $\tilde{Z}^{(i,N)}$ is a Brownian motion with drift. Hence, existence of a solution
of (2.32) follows from Section 6.1 in [EK86].

For the proof of the general result, we will need to approximate \( \tilde{Z}^{(N)} \) with piecewise constant functions. Observe that \( \tilde{Z}^{(N)} \) is a process with finite variation and it is piecewise linear.

### Second approximation of the vector field

In order to simplify the notation, we drop the \( N \) and assume that the process \( \tilde{Z} \) is already of finite variation type. Thanks to Proposition 2.18 this is not a real restriction.

We need to introduce some additional notation and prove some preliminary lemmas. Fix \( M \in \mathbb{N} \) and consider the partition

\[
\mathcal{T}_M := \left\{ \frac{k}{2M}, \quad k \geq 0 \right\}.
\]

Define the following approximations on the partition \( \mathcal{T}_M \):

\[
\uparrow \tilde{Z}^{(i, M)}_t := \sum_{k=0}^{\infty} \tilde{Z}_{k/2M}^{(i)} \mathbb{1}_{[\frac{k}{2M}, \frac{k+1}{2M})}(t),
\]

\[
\downarrow \tilde{Z}^{(i, M)}_t := \sum_{k=0}^{\infty} \tilde{Z}_{(k+1)/2M}^{(i)} \mathbb{1}_{[\frac{k}{2M}, \frac{k+1}{2M})}(t).
\]

Introduce, for \( s \in \mathbb{R}_{\geq 0}^m \), the processes \( \uparrow \tilde{Z}^{(M)}(s) \) and \( \downarrow \tilde{Z}^{(M)}(s) \) obtained by taking the sums of \( \uparrow \tilde{Z}^{(i, M)}_{s_i} \) and \( \downarrow \tilde{Z}^{(i, M)}_{s_i} \) respectively.

Observe that the processes \( \uparrow \tilde{Z}^{(M)}(s) \) and \( \downarrow \tilde{Z}^{(M)}(s) \) have jumps either due to the discretization in time or due to the jumps of the process \( \tilde{Z} \).

**Notation 2.20** Let

\[
\Sigma_M := \bigcup_{i=1}^{m} \{ s \geq 0 \mid \Delta Z_s^{(i)} > 0 \}
\]

and augment the partition \( \mathcal{T}_M \) with \( \Sigma_M \). Denote the family obtained in this way by \( \mathcal{T}^\Sigma_M \).
We will first construct a solution for the equation (2.27) when \( \bar{Z} \) is replaced by \( \uparrow \bar{Z}^{(M)} \).

Hereafter, given \( x, y \in \mathbb{R}^m \), we write \( x \leq y \) if \( x_i \leq y_i \), for all \( i = 1, \ldots, m \).

**The algorithm**

Let \( \bar{Z} \) and \( \uparrow \bar{Z}^{(M)} \) be defined as above.

**Input:** Start by defining the random variables

\[
\overleftarrow{\sigma} := (0, \ldots, 0), \quad (2.33)
\]
\[
\overrightarrow{\sigma}(\omega) := (\sigma_1^{(1,M)}(\omega), \ldots, \sigma_m^{(m,M)}(\omega)), \quad (2.34)
\]

where each \( \sigma_i^{(i,M)}(\omega) \) is the first jump in the path \( t \mapsto \uparrow \bar{Z}_t^{(i,M)}(\omega) \).

**Step 1:** Let \( \bar{z}((t_0, \tau_0, x); t) \) be the solution of the system (2.31) starting from

\[
t_0 = 0, \quad \tau_0 = (0, \ldots, 0) \quad \text{and} \quad x \in \mathbb{R}^m_{\geq 0}.
\]

Consider the solution of (2.31) for all times \( t \) such that

\[
\bar{z}((t_0, \tau_0, x); t) < \overrightarrow{\sigma}. \quad (\dagger)
\]

Let \( T \) be the first time such that \( (\dagger) \) does not hold anymore. Stop the solution \( \bar{z}((t_0, \tau_0, x); \cdot) \) at time \( T \). Observe that the condition \( (\dagger) \) is violated if there exists and index \( i^* \in \{1, \ldots, m\} \) such that

\[
\tau_{i^*}((t_0, \tau_0, x); T) = \sigma_1^{(i^*, M)}.
\]

**Step 2:** Update

\[
\overleftarrow{\sigma} := (0, \ldots, \sigma_1^{(i^*, M)}, \ldots, 0), \quad (2.35)
\]
\[
\overrightarrow{\sigma} := (\sigma_1^{(1,M)}, \ldots, \sigma_2^{(i^*, M)}, \ldots, \sigma_m^{(m,M)}), \quad (2.36)
\]
\[
x := x + \Delta \uparrow \bar{Z}^{(M)}(\overrightarrow{\sigma}), \quad (2.37)
\]

where \( \sigma_2^{(i^*, M)}(\omega) \) is the second jump in the path \( t \mapsto \uparrow \bar{Z}_t^{(i^*, M)}(\omega) \). Notice here that there might be more than one \( i^* \), where the above equality is valid, however, for the sake of convenience we only write one.
Step 3: Let \( \tau((t_1, \tau_1, x_1); t) \) be the solution of the system (2.31) starting from the updated values

\[
t_1 = T, \quad \tau_1 = \tau((t_0, \tau_0, x_0); T) \quad \text{and} \quad x_1 = x \in \mathbb{R}^m_{\geq 0}.
\]

As before, we let \( \tau((t_1, \tau_1, x_1); \cdot) \) evolve until

\[
\tau((t_1, \tau_1, x_1); T) < \sigma
\]

holds. As soon as this condition does not holds anymore, we stop again the solution.

End: Do iteratively Step 2 and Step 3 until (†) holds for \( t \) smaller than the time horizon.

The above algorithm describes the guiding principle for the proof of the next result:

**Theorem 2.21** There exists a solution solution of

\[
\begin{cases}
\ddot{\tau}^{(M)}((0, 0, x); t) = (x + \hat{Z} + \uparrow \tilde{Z}^{(M)})(\tau^{(M)}((0, 0, x); t)), \\
\tau^{(M)}((0, 0, x); t_0) = 0.
\end{cases}
\]

**Proof** We already did all the main steps for the proof of this result. Let \( \mathcal{T}_M \) and \( \Sigma \) be the sets defined in Notation 2.20. Recall that \( \mathcal{T}_M^\Sigma \) is a countable family. Enumerate the elements in \( \mathcal{T}_M^\Sigma \) such that \( \sigma_k \) denotes the \( k \)-th jump of \( \uparrow \tilde{Z}^{(i,M)} \). Fix \( x \in D \) and set

\[
(t_0, \tau_0, x) := (0, 0, x).
\]

and

\[
\sigma := (0, \ldots, 0), \\
\tilde{\sigma} := (\sigma_1^{(1,M)}, \ldots, \sigma_1^{(i,M)}, \ldots, \sigma_1^{(m,M)}),
\]

where \( \sigma_k^{(i,M)} \) denotes the \( k \)-th jump in the path \( t \mapsto \uparrow \tilde{Z}_i^{(i,M)} \) for all \( i = 1, \ldots, m \). By definition \( \uparrow \tilde{Z}_i^{(M)}(\underline{s}) = 0 \) for all \( \underline{s} \leq \tilde{\sigma} \). Proposition 2.19 gives the existence of the solution of (2.31) with this set of input parameters. Denote it by \( \tau((t_0, \tau_0, x); t) \). As soon as the solution \( \tau((t_0, \tau_0, x); t) \) reaches a jump time for \( \uparrow \tilde{Z}^{(M)} \), the vector
field in the equation (2.39) changes. Precisely, denote by

\[ t_1 := \sup \{ t > 0 \mid \tau((t_0, \tau_0, x); t) \leq \sigma \} . \]

Again there might be one or more indices \( i^* \), where equality holds. Collect them in a set \( I^* \subseteq \{ 1, \ldots, m \} \). Update the values

\[
\pi_{i^*} \tilde{\tau} := \pi_{i^*} \sigma, \\
\pi_{i^*} \tilde{\sigma} := \pi_{i^*} \sigma_{++}, \tag{2.40}
\]

where \( \sigma_{++} \) contains the next jumps of \( \uparrow \tilde{Z}^{(i, M)} \) for all \( i \in I^* \) after \( \sigma_i \). Then define

\[
\tau_1 := \tau((t_0, \tau_0, x); t_1) \\
x_1 := x + \Delta \uparrow \tilde{Z}^{(M)}(\tilde{\sigma}).
\]

Now, consider again the solution of (2.31), but this time with parameters \( (t_1, \tau_1, x_1) \). Denote it by \( \tau((t_1, \tau_1, x_1); t) \) and observe that it is well defined until all the coordinates of \( \tau((t_1, \tau_1, x_1); t) \) stay below the next jump times of \( \uparrow \tilde{Z}^{(M)} \). We obtain the solution of (2.31) by pasting a finite amount of solutions obtained in the time subintervals defined by \( \mathcal{T}_M \). Define iteratively, for all \( n \geq 1 \)

\[
t_{n+1} := \sup \{ t > 0 \mid \tau((t_n, \tau_n, x_n); t) \leq \sigma \}, \tag{2.41}
\]

\[
\tau_{n+1} := \tau((t_n, \tau_n, x_n); t_{n+1}), \tag{2.42}
\]

\[
x_{n+1} := x_n + \Delta \uparrow \tilde{Z}^{(M)}(\tilde{\sigma}). \tag{2.43}
\]

where, at each step \( \tilde{\tau} \) and \( \tilde{\sigma} \) are updated using the prescription in (2.40). Continuity follows by construction. \( \blacksquare \)

Now that we have found a solution for the approximated problems, we would like to show convergence to the solution of (2.27).

The following results focus on monotonicity and convergence of (2.39).

**Lemma 2.22** Let \( i = 1, \ldots, m \) and \( M \in \mathbb{N} \) be fixed. Then, for all \( t \geq 0 \) it holds

\[
\uparrow \tilde{Z}^{(i, M)}_1 \leq \tilde{Z}^{(i, M)}_t \leq \downarrow \tilde{Z}^{(i, M)}_1 \text{ almost surely}.
\]

Moreover, for each \( \omega \in \Omega \), the sequences \( \{ \uparrow \tilde{Z}^{(i, M)}_t(\omega) \}_{M \in \mathbb{N}} \) and \( \{ \downarrow \tilde{Z}^{(i, M)}_t(\omega) \}_{M \in \mathbb{N}} \)
are monotone in the sense that, for all \( t \geq 0 \),
\[
\uparrow \tilde{Z}_t^{(i, M+1)}(\omega) \geq \uparrow \tilde{Z}_t^{(i, M)}(\omega)
\]
and
\[
\downarrow \tilde{Z}_t^{(i, M+1)}(\omega) \leq \downarrow \tilde{Z}_t^{(i, M)}(\omega).
\]

**Proof** Since \( Z^{(i)} \) has no negative jumps, and, by assumption \( (\beta_i)_k \geq 0 \) for all \( k \neq i \), the paths of \( \tilde{Z}^{(i, M)} \) are increasing. Therefore,
\[
\tilde{Z}_t^{(i, M)} \geq \tilde{Z}_{k/2^M}^{(i, M)} = \uparrow \tilde{Z}_t^{(i, M)}, \text{ a.s. for all } t \in \left[ \frac{k}{2^M}, \frac{k+1}{2^M} \right).
\]
For the same reason,
\[
\tilde{Z}_t^{(i, M)} \leq \tilde{Z}_{(k+1)/2^M}^{(i, M)} = \downarrow \tilde{Z}_t^{(i, M)}, \text{ a.s. for all } t \in \left[ \frac{k}{2^M}, \frac{k+1}{2^M} \right).
\]
Now, since for every \( M \in \mathbb{N} \) the partition \( \mathcal{T}_{M+1} \) is obtained by halving all the subintervals in the partition \( \mathcal{T}_M \), it clearly holds
\[
\uparrow \tilde{Z}_t^{(i, M+1)}(\omega) = \begin{cases} \uparrow \tilde{Z}_t^{(i, M)}(\omega), & \text{for all } t \in \left[ \frac{2k}{2^{M+1}+1}, \frac{2k+1}{2^{M+1}+1} \right), \\ \tilde{Z}_{(2k+1)/2^{M+1}}^{(i, M)}(\omega), & \text{for all } t \in \left[ \frac{2k+1}{2^{M+1}+1}, \frac{2(k+1)}{2^{M+1}+1} \right). \end{cases}
\]
Using again the increasing property of the paths of \( \tilde{Z}^{(i, M)} \) we conclude that
\[
\uparrow \tilde{Z}_t^{(i, M)} \geq \tilde{Z}_t^{(i, M)}, \text{ a.s.}
\]
because
\[
\tilde{Z}_{(2k+1)/2^{M+1}}^{(i, M)} \geq \tilde{Z}_{2k/2^{M+1}}^{(i, M)} = \tilde{Z}_{k/2^M}^{(i, M)}.
\]
The case with \( \downarrow \tilde{Z}^{(i, M)} \) goes analogously. \( \blacksquare \)

**Proposition 2.23** Let \( M \in \mathbb{N} \) be fixed and denote by \( \tau^{(M)}((0,0,x); t) \) the solution of (2.39) constructed in Theorem 2.21. Then, for all \( t \geq 0 \) and \( x \in \mathbb{R}^n_{\geq 0} \) it holds
\[
\tau^{(M)}((0,0,x); t) \leq \tau^{(M+1)}((0,0,x); t), \text{ almost surely.}
\]
2.3. Lamperti representation

Proof This follows by construction using the monotonicity proved in Lemma 2.22. Indeed, denote by \( \mathcal{T}_M := \{ \sigma_k^{(M)} \}_{k \in \mathbb{N}} \) and \( \mathcal{T}_{M+1} := \{ \sigma_k^{(M+1)} \}_{k \in \mathbb{N}} \) the set of jump times for \( \uparrow \hat{Z}^{(M)} \) and \( \uparrow \hat{Z}^{(M+1)} \) respectively. By construction \( \mathcal{T}_M \subset \mathcal{T}_{M+1} \) in the sense that, for each \( \sigma_k^{(M)} \in \mathcal{T}_M \) there exists \( h \in \mathbb{N} \) such that \( \sigma_k^{(M)} = \sigma_h^{(M+1)} \in \mathcal{T}_{M+1} \). Denote by \( \{ \sigma_{kh}^{(M+1)} \}_{h \in \mathbb{N}} \) the jump times of \( \uparrow \hat{Z}^{(M+1)} \) occurring on the subinterval \([\sigma_k^{(M)}, \sigma_{k+1}^{(M+1)}] \). By construction, there is only one jump inside this interval. Write \( \{ \sigma_{kh}^{(M+1)} \}_{h=1,...,3} \) with \( \sigma_{k_1}^{(M+1)} = \sigma_k^{(M)} \) and \( \sigma_{k_3}^{(M+1)} = \sigma_{k+1}^{(M+1)} \). Then \( \tau^{(M+1)} \) is obtained by pasting a finite number of solutions of initial value problems with piecewise linear vector field. For each \( h = 1, 2, 3 \), \( \uparrow \hat{Z}^{(M+1)}(\sigma_{kh}^{(M+1)}) \geq \uparrow \hat{Z}^{(M)}(\sigma_k^{(M)}) \). Therefore, on each subinterval \([\sigma_k^{(M)}, \sigma_{k+1}^{(M+1)}] \), the solution \( \tau^{(M+1)}((t_k, \tau_k, x_k); t) \) is constructed by pasting a finite number of solutions of type \( \tau((t_{kh}; \tau_{kh}, x_{kh}); t) \) where \( x_{kh} \) is increasing sequence in \( h \). Hence we conclude that, for all \( k \in \mathbb{N} \) and for \( t \in [\sigma_k^{(M)}, \sigma_{k+1}^{(M+1)}] \) it holds

\[
\tau^{(M+1)}((t_k, \tau_k, x_k); t) \geq \tau^{(M)}((t_k, \tau_k, x_k); \sigma_k^{(M)}).
\]

Finally, due to monotonicity, we know that the sequence \( \tau^{(M)} \) admits a solution. With the next result we show that the limit is actually finite and, by monotone convergence, it coincides with the solution of (2.27).

**Proposition 2.24** For all \( t \geq 0 \) and \( x \in \mathbb{R}_{\geq 0}^m \) the sequence \( \tau^{(M)}((0, 0, x); t) \) converges

\[
\lim_{n \to \infty} \tau^{(M)}((0, 0, x); t) = \tau^{(*)}((0, 0, x); t)
\]

and the limit can be identified with the solution of (2.27).

**Proof** Denote by \( \hat{\tau}^{(M)} \) the solution of (2.39) with \( \uparrow \hat{Z}^{(M)} \) replaced by \( \downarrow \hat{Z}^{(M)} \). The same steps done in the proof of Theorem 2.21 prove that \( \hat{\tau}^{(M)} \) is well defined for all \( M \in \mathbb{N} \) and the sequence \( \{ \hat{\tau}^{(M)} \}_{M \in \mathbb{N}} \) is a decreasing sequence converging to a limit \( \hat{\tau}^{(*)} \). Therefore, a similar argument to the one used in the proof of Proposition 2.23 yields

\[
\tau^{(M)}((t_0, \tau_0, x); t) \leq \tau^{(*)}((t_0, \tau_0, x); t) \leq \hat{\tau}^{(M)}((t_0, \tau_0, x); t) \quad \text{almost surely}
\]

for all \( n \in \mathbb{N}, t \geq 0 \) and \( \tau_0 \in \mathbb{R}_{\geq 0}^m \).

At this point, most of the results we need for the proof of Theorem 2.15 have been proved. The final step is to construct the solution of the time change equation (2.15)
using the solution of the system (2.27).

**Proof of Theorem 2.15** Recall that, by definition

\[
\tau^{(k)}_i(t) := \int_0^t f^{(k)}_i(Y_s) ds, \text{ for } k, i = 1, \ldots, d.
\]

Let \( \tau = (\tau^{(1)}, \ldots, \tau^{(m)}) \) be the solution of (2.27) with \( Z := \sum_{i=1}^m \pi_i Z^{(i)} \). Then, for \( k = 1, \ldots, m \) and \( i = 1, \ldots, m \)

\[
Y^{(k)}_i(t) := Z^{(k)}_i(\tau^{(k)}).
\]

Moreover observe that, due to the restrictions in \( H \) the Lévy processes \( Z^{(k)} \) for \( k = m + 1, \ldots, d \) are identically zero and therefore also \( Y^{(k)} \) are identically zero. ■
Chapter 3

High order numerical schemes for affine processes

3.1 Introduction

This chapter focuses on the weak approximation schemes for the trajectories of an affine process. Let $T > 0$ be a fixed time horizon and \{t_0 = 0 < t_1 < \ldots < t_N = T\} a uniform time grid. We want to construct a family of random variables

$$\{X^x_t : t_0 = 0 < t_1 < \ldots < t_N = T\}$$

which approximates $X$ with $X_0 = x$ at the given points.

Recall that, semiflow property for $\Psi$ gives

$$\Psi(t_N, u) = \Psi(t_N - t_{N-1}, \Psi(t_{N-1} - t_{N-2}, \ldots \Psi(t_1, u))), \quad (3.1)$$

and, when the spacing of the partition is assumed to be constantly equal to $h$, we can write (3.1) as

$$\Psi(t_N, u) = \Psi(h, \Psi(h, \ldots \Psi(h, u)))) =: \Psi(h, u)^{\circ N}. \quad (3.2)$$

The previous iterative composition is well defined by closure of the class of functions of Lévy–Khintchine type under the composition operator. In this chapter we propose a numerical scheme which arises by replacing $\Psi(h, u)$ with a function of Lévy–Khintchine type $\varphi(h, u)$ which is “close” to $\Psi(h, u)$ in a sense that we are going to specify in Section 3.3. Inspired by the iterative composition in (3.2), we will construct a sequence $\{y_1, \ldots, y_N\}$ of functions of Lévy–Khintchine form. Starting
3. High order numerical schemes for affine processes

from this sequence, we construct a family of infinitely divisible random variables \( \{X_{t_0}^x, \ldots, X_{t_N}^x\} \) with \( X_{t_0}^x = x \). The convergence of the sequence \( X^x \) to \( X^x \) depends on the choice of the function \( \varphi \). In Theorem 3.5, we see how to construct \( \{y_1, \ldots, y_N\} \) such that the random vector \( (X_{t_0}^x, \ldots, X_{t_N}^x) \) converges weakly to \( (x, X_{t_1}^x, \ldots, X_{t_N}^x) \). Then we introduce the classical setting of weak approximation for stochastic differential equations. In Theorem 3.13, we prove the convergence rate of the proposed scheme on the class of smooth functions with polynomial growth.

**Additional notation**

The following set of functions will be considered:

- \( C_c^\infty(D) \) is the space of \( C^\infty \) functions \( f : D \to \mathbb{R} \) with compact support,
- \( C_{pol}^\infty(D) \) is the spaces of \( C^\infty \) functions \( f : D \to \mathbb{R} \) having all the derivatives with polynomial growth, i.e.

\[
C_{pol}^\infty(D) = \left\{ f \in C^\infty(D), \text{ for all } \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, \eta_\alpha \in \mathbb{N} \mid \text{ for all } x \in D, |\partial^\alpha f(x)| \leq C_\alpha (1 + |x|^{\eta_\alpha}) \right\}.
\]

Observe that, for a multi–index \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \) and \( x \in D \), we write \( x^k = x_1^{k_1} \ldots x_n^{k_n} \). Moreover, with the symbol \( \partial_x^\alpha \) we mean the mixed partial derivative of order \( |\alpha| = \alpha_1 + \ldots + \alpha_d \)

\[
\partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}.
\]

Recall the following class of functions introduced in Section 1.3

- \( \mathcal{C} := \{ \eta : \mathcal{U} \to \mathbb{C} \text{ of Lévy–Khintchine form (1.11)} \} \),
- \( \mathcal{C}^* := \{ \Psi : \mathcal{U} \to \mathbb{C}^d \mid \pi_\ell \Psi \in \mathcal{C}^m \text{ and } \pi_j \Psi = A \pi_j u, A \in \mathbb{R}^{n \times n} \} \),
- \( \mathcal{C}^H := \{ \Psi : \mathcal{U} \to \mathbb{C}^d \mid \pi_\ell \Psi \in \mathcal{C}^m \text{ and } \pi_j \Psi = \pi_j u \} \subset \mathcal{C}^* \).

Additionally, for ease of notation, in this section the notation \( L^\eta \) is used to denote the Lévy process with Lévy exponent \( \eta \), i.e.

\[
\mathbb{E} \left[ f_u(L^\eta_t) \right] = e^{\eta u^{(u)}}, \quad s \geq 0, u \in \mathcal{U}.
\]

Finally, with the symbol \( O \) we denote the big \( O \) notation for the limit behavior of functions.
3.2 A short survey on weak approximation schemes

In mathematical finance, it is essential to have an efficient way to compute quantities of type
\[ \mathbb{E}^x \left[ f(X_t) \right], \tag{3.4} \]
where \( f : D \to \mathbb{R} \) is a payoff function and \( (t, x) \in [0, T] \times D \), with \( T > 0 \) a fixed time horizon. For this purpose, it is common to approximate the trajectories of \( X^x \) with an Euler–Maruyama–type scheme and to compute (3.4) with Monte Carlo methods. Convergence rates for this type of approximation have been studied by many researchers, see for example [TT90, JKMP05]. However this method turns out to be not optimal in numerics when dealing with processes subjected to some domain constraints. Consider, for example, the following example of Cox–Ingersoll–Ross model
\[
\begin{align*}
\begin{cases}
    dX_t &= \sqrt{2X_t}dW_t, \\
    X_0 &= x,
\end{cases}
\end{align*}
\]
where \( W \) is a Brownian motion and \( x \in \mathbb{R}_{\geq 0} \). This is an example of an affine process in \( \mathbb{R}_{\geq 0} \) with
\[
\mathbb{E}^x \left[ e^{uX_t} \right] = e^{x\Psi(t,u)}, \quad t \geq 0, \quad u \in \mathbb{C}_{\leq 0}.
\]
We know that the function \( \Psi \) solves the following ODE
\[
\begin{align*}
\begin{cases}
    \partial_t \Psi(t,u) &= \Psi^2(t,u), \\
    \Psi(0,u) &= u.
\end{cases}
\end{align*}
\tag{3.5}
\]
This is a simple example since the explicit solution of the above ODE is well known, from where we can also get the exact distribution of the one time marginal distributions. In this case, a classical Euler–Maruyama scheme is not even well defined. Indeed, fix \( N \in \mathbb{N} \) and define \( h = T/N \). If we consider the one step approximation
\[
\begin{align*}
\begin{cases}
    \bar{X}_0 := x \in \mathbb{R}_{\geq 0}, \\
    \bar{X}_h := \bar{X}_0 + \int_0^h \sqrt{2\bar{X}_0}dW_s,
\end{cases}
\end{align*}
\]
we see immediately that \( \bar{X}_h \) is a random variable which is negative with positive probability and, therefore, it would not be possible to iterate the construction to define \( \bar{X}_{2h} \). Several methodologies have been introduced in order to overcome this problem. For example, one could impose absorbing conditions in zero, and consider
the following approximation scheme
\[
\begin{align*}
\tilde{X}_0 &:= x \in \mathbb{R}_{\geq 0}, \\
\tilde{X}_{(n+1)h} &:= \tilde{X}_{nh} + \int_{nh}^{(n+1)h} \sqrt{2 \max(0, \tilde{X}_{nh})} dW_s,
\end{align*}
\]

For a review on the possible combinations of reflecting and/or absorbing conditions we refer to [LKDC08].

Geometry preserving schemes can be constructed using splitting schemes. This type of schemes consists in splitting additively the infinitesimal operator in order to obtain subprocesses for which explicit expressions are known or which are easier to approximate by standard methods. A rigorous framework for the weak error analysis of splitting methods has been presented by in [Alf10] and [DT10].

Accordingly to this methodology, the quantity
\[
P_tf(x) = \mathbb{E} \left[ f(\tilde{X}_T) \right], \quad (t, x) \in [0, T] \times D,
\]
if well defined, can be computed using its Taylor expansion for small $t > 0$. In Chapter 4, we will see under which conditions the following limit
\[
\lim_{t \to 0} \frac{P_tf(x) - f(x)}{t},
\]
is well defined and when it is possible to derive the following equation for $(P_t)_{t \geq 0}$
\[
\partial_t P_tf(x) = \mathcal{A} P_tf(x), \quad (t, x) \in [0, T] \times D,
\]
\[
P_0f(x) = f(x), \quad x \in D.
\]

Due to the semigroup property $P_{t_1+t_2} = P_{t_1}P_{t_2}$ for $t_1, t_2 \geq 0$, it is clear that, in order to approximate $f(X_t)$ one only needs to approximate $P_tf$ for $t$ small. Let $T > 0$ and $N \in \mathbb{N}$ be fixed and consider be a stochastic process $\tilde{X}$ with
\[
Q_tf(x) := \mathbb{E} \left[ f(\tilde{X}_t) \right], \quad (t, x) \in [0, T] \times D.
\]

Using telescopic decomposition we can write
\[
\mathbb{E} \left[ f(\tilde{X}_T) \right] - \mathbb{E} \left[ f(X_T) \right] = \sum_{i=1}^{N-1} Q_{\frac{T}{N}}^N \cdots Q_{\frac{T}{N}}^{i+1} (Q_{\frac{T}{N}}^T - P_{\frac{T}{N}}) P_{\frac{T}{N}} f(x).
\]
3.2. A short survey on weak approximation schemes

Suppose that, for \( h \leq \frac{T}{N} \), \( P_h \) and \( Q_h \) coincide on some subspace \( \mathcal{M} \subset m(D) \) up to an order \( \nu \), i.e.

\[
Q_{\frac{T}{N}} P_s f(x) - P_{\frac{T}{N}} P_s f(x) \leq c_f \left( \frac{T}{N} \right)^{\nu + 1} F(x) \quad \text{for all } f \in \mathcal{M},
\]

for some positive function \( F \), a constant \( c_f \) and \( \nu \in \mathbb{N} \). Then, under the assumption that \( Q_{\frac{T}{N} - i} \) preserves the error in \((\ast)\), we obtain a convergence rate of order \( \nu \) from the telescopic sum. In line with the theory of weak approximation schemes for SDE, one needs to check \((\ast)\) for the class \( \mathcal{M} = C^\infty_{\text{pol}} \) introduced at the end of Section 3.1.

In Chapter 4 we will see that, if \( f \in C^\infty_{\text{pol}} \), then, \( P_s f(x) \in C^\infty_{\text{pol}} \) holds for all \( s \geq 0 \) and hence it suffices to check \((\ast)\) for \( s = 0 \). In particular, when \( f \in C^\infty_{\text{pol}} \) and \( s = 0 \), condition \((\ast)\) reads

\[
\left| \mathbb{E} \left[ X_{\frac{T}{N}}^i \right] - \mathbb{E} \left[ X_{\frac{T}{N}}^f \right] \right| \leq c_f \left( \frac{T}{N} \right)^{\nu + 1} (1 + |x|^q),
\]

where \( q \) is a constant which depends on the polynomial growth order of \( f \) and on the index \( \nu \).

For example, consider splitting scheme presented in [Alf10] for the Cox–Ingersoll–Ross model considered before. The approximation is accomplished by the following decomposition of the generator:

\[
\mathcal{A} f(x) = \left( \frac{1}{2} f'(x) + x f''(x) \right) - \frac{1}{2} f'(x) =: \mathcal{A}^1 f(x) + \mathcal{A}^2 f(x).
\]

Let \( \bar{W} \) be a Brownian motion and consider separately the two following equations

\[
\begin{align*}
\begin{cases}
\quad dZ^1_t &= \frac{1}{2} dt + \sqrt{2} Z^1_t d\bar{W}_t, \\
\quad Z^1_0 &= z_1,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
\quad dZ^2_t &= -\frac{1}{2} dt, \\
\quad Z^2_0 &= z_2.
\end{cases}
\end{align*}
\]

Since \( Z^1_t = (\sqrt{z_1} + \bar{W}_t/\sqrt{2})^2 \) and \( Z^2_t = z_2 - \frac{1}{2} t \), the scheme in [Alf10] reads

\[
\tilde{X}_{t_{n+1}} = \left( \sqrt{x - \frac{1}{2} \Delta t_{n+1} + \frac{\bar{W}_{\Delta t_{n+1}}}{\sqrt{2}}} \right)^2 - \frac{1}{2} \Delta t_{n+1},
\]

87
where $\Delta t_{n+1} = t_{n+1} - t_n$. Observe that the scheme is well defined whenever $\bar{X}^x_{t_n} > \frac{1}{2} \Delta t_{n+1}$. In a neighborhood of zero, one has to use a different approximation. Therein, a moment matching method is chosen. Note that, also in a more general case, the main difficulty in discretizing affine processes is located at the boundary of the state space, where the vector fields may lack the Lipschitz property. Additionally observe that, in [Alf10], the condition that the Markov semigroup leaves invariant the function space $C^\infty_{pol}$ is checked using the explicit distribution of the process (see Proposition 2.9 in [Alf05]). In this section we focus on the construction of a numerical schemes which satisfies the short time asymptotic (*) when $f$ is an element of the Fourier modes. Then we see how to extend the result to the class $C^\infty_{pol}$. The proof of the stability of an affine semigroup on the class $C^\infty_{pol}$ is postponed in Section 4.4.1.

3.3 Weak approximation schemes for affine processes

We recall quickly the main notations we fixed in introduction. Henceforth $X$ is the affine process to be approximated and $\bar{X}$ is its approximation. To underline the dependence on the initial point $x$, we will often write $X^x$ and $\bar{X}^x$. In the entire section, the approximation is done on a uniform partition of $[0,T]$ with step size $h = \frac{T}{N}$. The numerical scheme we propose is based on an approximation for the system of Riccati equation. The construction is done as follows:

**Step 1:** We first pick a family $(\varphi_t)_{t \geq 0}$ such that, for each fixed $t \geq 0$, $\varphi_t \in C^H$, and, for each fixed $u \in U$, the function $t \mapsto \varphi_t(u)$ is continuous in $t$ and the right hand derivative $\frac{\partial_t \varphi_t(u)}{t=0^+}$ exists for every $u \in U$ and is continuous at $u = 0$.

**Step 2:** In Proposition 3.2 we observe that $R(u) := \frac{\partial_t \varphi_t(u)}{t=0}$ always identifies a functional characteristic for an affine process.

**Step 3:** For fixed $u \in \bar{U}$ and $t > 0$ we define

$$
\begin{align*}
    y_0(u) &:= u, \\
    y_n(u) &:= \varphi_{t/N} \circ y_{n-1}(u) \in C^H \text{ for all } n = 1, \ldots, N, \text{ with } N \in \mathbb{N}.
\end{align*}
\tag{3.6}
$$

The sequence $\{y_0, y_1, \ldots, y_N\}$ coincides with an $N$ step explicit Euler scheme for the solution at time $t$ of the Riccati equation driven by $R$. We show its convergence and provide some examples.

**Step 4:** Finally we construct $\bar{X}$ by viewing (3.6) in the perspective of Bochner subordination. Using the convergence results obtained in the previous step,
we show weak convergence of $\widehat{X}$ to $X$. Finally we analyze the convergence rate.

### 3.3.1 Preliminary observations

For ease of notation we introduce the following

**Definition 3.1** Let $\circ$ denote the operator of composition in $C^H$. Given two elements $\varphi_1, \varphi_2 \in C^H$, we say that $\varphi_1 \preceq \varphi_2$ if and only if there exists $\varphi_3 \in C^H$ such that $\varphi_2 = \varphi_1 \circ \varphi_3$. A trajectory $(\varphi_t)_{t \geq 0}$ is a subset of $C^H$ which is totally ordered with respect to $\preceq$. A trajectory $(\varphi_t)_{t \geq 0}$ is called regular if $t \mapsto \varphi_t$ is continuous at $t = 0$ and the right hand derivative $\partial_t \varphi_t(u)_{t=0^+}$ exists for every $u \in U$ and is continuous at $u = 0$.

Introduce the following set of functions

$$TC^H := \left\{ R : U \to \mathbb{R}^d \mid \text{if } i \in I, \ R_i \text{ has the Lévy-Khintchine form in } \mathbb{R}^I_{\geq 0} \times \mathbb{R}^J, \right.$$ 

$$\text{if } j \in J, \ R_j(u) = 0 \text{ for all } u \in U \right\},$$

where $I = I \setminus \{i\}$ and $J = J \cup \{i\}$.

For any affine process, up to a transformation, $R_i$ has the Lévy-Khintchine form in $\mathbb{R}^I_{\geq 0} \times \mathbb{R}^J$, for any $i \in I \cup J$. See Section 1.3.3. Here we adapt the proof of Lemma 4.1. in [DFS03] in order to show that, whenever we have a regular trajectory in $C^H$, its derivative at time zero is a function in $TC^H$, up to an additive constant.

**Proposition 3.2** Let $(\varphi_t)_{t \geq 0}$ be a regular trajectory in $C^H$. Then, there exists $\gamma \in \mathbb{R}^d_{\geq 0}$ such that

$$\partial_t \varphi_t|_{t=0^+} - \gamma \in TC^H.$$ 

Viceversa, suppose that $R \in TC^H$. Then the solution of

$$\begin{cases} 
\partial_t \Psi(t, u) = R(\Psi(t, u)), \\
\Psi(0, u) = u,
\end{cases}$$

uniquely defines a regular trajectory in $C^H$.

**Proof** We need to compute the limit

$$\lim_{t \to 0} \frac{\varphi_t(u) - u}{t}.$$
We proceed componentwise, extending results in [Sil68] and using the arguments in the proof of Lemma 4.1. in [DFS03]. By definition, since \((\varphi_t)_{t \geq 0}\) is a trajectory in \(C^H\), for all \(t \geq 0\), it holds
\[
\pi_j \varphi_t(u) = \pi_j u
\]
and hence, for all \(j = m + 1, \ldots, d\), \(R_j(u) = 0\).

For \(i \in I\) the situation is slightly more cumbersome. Indeed, denote by \(\varphi^{(i)}_t\) the \(i\)-th component of \(\varphi_t\). Since \(\varphi^{(i)}_t \in C\), we can write
\[
\frac{\varphi^{(i)}_t(u) - u_t}{t} = \frac{\langle b_t(\xi) - e_i, u \rangle}{t} + \frac{1}{2t} \langle \pi_j u, \sigma_i(\xi) \pi_j u \rangle
\]
\[
+ \int_{D \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_j u, \pi_j \xi (\xi) \rangle \right) \nu_i(t, d\xi) \frac{t}{t}.
\]

Introduce the function
\[
d(\xi) = |\pi_I \xi| + |\pi_J \xi|^2,
\]
and let \(Q\) be the set of points with \(L^2\) norm smaller than 1 in \(D\), i.e.
\[
Q := \{ \xi \in D \mid 0 \leq |\xi| \leq 1 \}.
\]

We split the jump component as
\[
\int_{Q \setminus \{0\}} \frac{\left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_j u, \pi_j \xi \rangle \right)}{d(\xi)} \frac{\nu_i(t, d\xi)}{t} \frac{t}{t}
\]
\[
+ \int_{\{|\xi| > 1\}} \left( e^{\langle u, \xi \rangle} - 1 \right) \frac{\nu_i(t, d\xi)}{t}.
\]

Define
\[
g_u(\xi) := \frac{\left( e^{\langle u, \xi \rangle} - 1 - \langle \pi_j u, \pi_j \xi \rangle \right)}{d(\xi)}.
\]

For \(\xi \in Q \setminus \{0\}\) a Taylor approximation yields:
\[
g_u(\xi) = \int_0^1 \left\langle \pi_I u, \frac{\pi_I \xi}{d(\xi)} \right\rangle \frac{e^{s\langle u, \xi \rangle}}{s} ds + \frac{1}{2} \int_0^1 \left\langle u, \frac{\pi_I \xi \pi_J \xi^T}{2d(\xi)} u \right\rangle \frac{e^{s\langle u, \xi \rangle}(1 - s)}{2s} ds.
\]
The map
\[ \Gamma : Q \rightarrow Q \times \mathbb{R}_{\geq 0}^m \times S_+^m, \]
\[ \xi \mapsto \left( \xi, \frac{\pi_I \xi}{d(\xi)} \left( \pi_J \xi \right)^\top \right), \]
is an homeomorphism which fixes 0. Moreover observe that
\[ \Gamma(Q) = \{ w \in Q \times \mathbb{R}_{\geq 0}^m \times S_+^m \mid |w_2| + \text{Tr}(w_3) = 1 \}. \]
For \( \xi \in Q \setminus \{0\} \), write
\[ g_u(\xi) = \int_0^1 \langle \pi_I u, w_2 \rangle e^{s(u,w_1)} ds + \frac{1}{2} \int_0^1 \langle \pi_J u, w_3 \pi_J u \rangle e^{s(u,w_1)} (1 - s) ds \]
\[ =: \tilde{g}_u(w). \]
The function \( \tilde{g}_u \) can be extended in zero by defining \( \lim_{w \to 0} \tilde{g}_u(w) = 0 \). Moreover, normalize the measure by defining
\[ K_i(t) := \frac{1}{t} \int_D (1 \wedge d(\xi)) \nu_i(t, d\xi), \quad \text{and} \quad m_i(t, \cdot) := \frac{d(\xi) \nu_i(t, \cdot)}{t K_i(t)}. \]
After the change of variables, the first integral in (3.7) becomes
\[ K_i(t) \int_{\Gamma(Q)} \tilde{g}_u(w) (m_i(t, \cdot) \circ \Gamma)(dw). \quad (3.8) \]
Now we want to incorporate the volatility component into this integral by extending the measure in the compactification of \( \Gamma(Q) \). Let \( v \in \mathbb{R}^n \) be such that \( vv^\top = \sigma_i(t) \) and define
\[ \xi^* = \frac{1}{\sqrt{d(v)}} (0, \ldots, 0, v_1, \ldots, v_d)^\top. \]
Clearly, \( d(\xi^*) = 1 \) and
\[ \Gamma(\xi^*) = \left( \xi^*, 0, \frac{\sigma_i(t)}{\text{Tr}(\sigma_i(t))} \right) . \]
Extend the measure \( m_i(t, \cdot) \circ \Gamma \) in the closure of \( \Gamma(Q) \) by defining
\[ m'_i(t, \cdot) \circ \Gamma := m_i(t, \cdot) \circ \Gamma + \frac{\text{Tr}(\sigma_i(t))}{t} K_i(t) \delta_0(\cdot), \]
3. High order numerical schemes for affine processes

with

\[ b = \left( 0, 0, \frac{\sigma_i(t)}{\text{Tr}(\sigma_i(t))} \right). \]

Then we can write

\[ \varphi_t^{(i)}(u) - u_t \]

\[ = \frac{\langle b_i(t) - e_i, u \rangle}{t}
\]

\[ + K_i(t) \int_{\Gamma(Q)} \tilde{g}_u(w)(m'_i(t, \cdot) \circ \Gamma)(dw)
\]

\[ + \int_{D \setminus Q} \left( e^{\langle u, \xi \rangle} - 1 \right) \frac{\nu_i(t, d\xi)}{t}. \]  

(3.9)

The quantity \( K_i(t) \) is nonnegative and, for \( t \) going to 0, it cannot explode. Indeed, if this were the case, the function \( \varphi_t \) would not be differentiable at \( t = 0 \). If

\[ \liminf_{n \to \infty} K_i\left( \frac{1}{n} \right) = 0, \]

in the limit, the jump part does not add any additional mass at the compactification point. If \( \liminf_{n \to \infty} K_i\left( \frac{1}{n} \right) > 0 \), there exists a finite measure \( m^*_i \) on \( \Gamma(Q) \) which is the weak limit, for \( t \) going to 0, of \( m'_i(t, \cdot) \circ \Gamma \). On \( D \setminus Q \), \( \nu_i(t) \) converges to a bounded measure \( m^*_i \) on \( D \setminus Q \) with support contained in the closure of \( D \setminus Q \), namely on \( \{ \xi \in D \mid |\xi| \geq 1 \} \cup \{ \Delta \} \). Working along subsequences

\[ \lim_{n \to \infty} K_i\left( \frac{1}{n} \right) \int_{\Gamma(Q)} \tilde{g}_u(w)(m'_i(n^{-1}, \cdot) \circ \Gamma)(dw)
\]

\[ + \frac{1}{n} \int_{D \setminus Q} \left( e^{\langle u, \xi \rangle} - 1 \right) \nu_i(n^{-1}, d\xi)
\]

\[ =: K_i \int_{\Gamma(Q)} \tilde{g}_u(w)m^*_i,\Gamma(dw) + \int_{D \setminus Q} \left( e^{\langle u, \xi \rangle} - 1 \right) m^*_i(d\xi). \]

Back to (3.9), we observe that, since both the integral terms have a finite limit, the limit for \( t \to 0 \) of (3.9) is finite if and only if also the linear term is finite. Therefore, if \( b_i(t) \neq 0 \), there exists a vector \( x_t \in \mathbb{R}^d \) such that \( \lim_{t \to 0} (b_i(t) - e_i) = x_t \). However we need to analyze the case when \( b_i(t) = 0 \), for all \( t \geq 0 \). By regularity assumptions, it suffices to check that it holds for one fixed \( t \).
Fix $i \in I$, and observe that the measure $\nu_i$ integrates the function
\[
d(\xi) = |\pi_{I \setminus \{i\}} \xi| + |\pi_{J \cup \{i\}} \xi|^2.
\]

Although we use the same notation as before, the definition of $d$ here differs from the previous one since we added an additional integrability along the $i$-th coordinate. Proceeding analogously as before, we split the jump component after moving the integrated small jumps into the integral term
\[
\int_{Q \setminus \{0\}} \frac{(e^{(u, \xi)} - 1 - \left<u_{J \cup \{i\}}, \xi_{J \cup \{i\}}\right>)d(\xi)\nu_i(t, d\xi)}{\nu_i(t, d\xi)t}
+ \int_{|\xi|>1} (e^{(u, \xi)} - 1) \frac{\nu_i(t, d\xi)}{t}.
\]

Since we are going to work with
\[
g_u(\xi) := \frac{(e^{(u, \xi)} - 1 - \left<u_{J \cup \{i\}}, \xi_{J \cup \{i\}}\right>)}{d(\xi)},
\]
we modify the definition of the previously introduced map $\Gamma$, by replacing it with
\[
\Gamma : Q \rightarrow Q \times \mathbb{R}_{\geq 0}^{m-1} \times \mathbb{S}^{n+1}_{+}
\xi \mapsto \left(\xi, \frac{\pi_{I \setminus \{i\}} \xi}{d(\xi)}, \frac{\pi_{J \cup \{i\}} \xi}{d(\xi)}\right).
\]

Hence
\[
\Gamma(Q) = \{w \in Q \times \mathbb{R}_{\geq 0}^{m-1} \times \mathbb{S}^{n+1}_{+} | |w_2| + \text{Tr}(w_3) = 1\}.
\]

As before, we incorporate the volatility component into this integral by extending the measure in the compactification of $\Gamma(Q)$. The construction goes as before, once $v$ is embedded in $\mathbb{R}^{n+1}$. In this case we end up having
\[
\frac{\varphi^{(i)}_t(u) - u_i}{t} = \frac{\int_{|\xi|\leq 1} \xi_i \nu_i(d\xi) - 1}{t}u_i + K_i(t) \int_{\Gamma(Q)} \tilde{g}_u(\xi) m'_i(t, \cdot) \circ \Gamma(dw)
+ \int_{D \setminus Q} (e^{(u, \xi)} - 1) \frac{\nu_i(t, d\xi)}{t}.
\]

Now, we unify the notation in order to compute the limit for $t \rightarrow 0$ of the jump term in both cases. When the drift part is non zero, define $I = I$ and $J = J$, otherwise
3. High order numerical schemes for affine processes

set \( I = I \setminus \{i\} \) and \( J = J \cup \{i\} \). Then the first integral in the above equation reads:

\[
K_i \left[ \int_{Q \setminus \{0\}} g_u(\xi) M^*_i(d\xi) + \left\langle \pi_I u, \int_{\Gamma(Q) \setminus \Gamma(Q)} w_2 m^*_i,\Gamma(dw_2) \right\rangle 
+ \frac{1}{2} \left\langle \pi_J u, \int_{\Gamma(Q) \setminus \Gamma(Q)} w_3 m^*_i,\Gamma(dw_3) \pi_J u \right\rangle \right],
\]

where \( M_i \) is the measure obtained by pushing back the measure \( m^*_i \) by \( \Gamma \). The second integral becomes

\[
\int_{D \setminus Q} \left(e^{(u,\xi)} - 1\right) m^*_i(d\xi) - m^*_i(\{\Delta\}).
\]

The proof of the second part can be found in Proposition 7.4. in [DFS03].

3.3.2 Convergence results

Before presenting the approximation scheme, we need two additional lemmas. The first one being a result on convergence for sequences of type (3.6), the second one being a stochastic representation for it.

**Lemma 3.3** Fix a trajectory \((\varphi_t)_{t \geq 0}\) in \( C^H\) and let \( u \in \mathcal{U} \) and \( t \geq 0 \) be fixed. Let \( N \in \mathbb{N} \) and define recursively \( \{y_n\}_{n=0, \ldots, N} \) as

\[
y_0(u) := u,
\]

\[
y_n(u) := \varphi_{t/n} \circ y_{n-1}(u) \text{ for all } n = 1, \ldots, N,
\]

with \( h = \frac{t}{N} \). Then, the limit \( \lim_{N \to \infty} y_N(u) \) exists and it is given by the value at time \( t \) the solution of system of ODE driven by \( \partial_t \varphi_t(u)_{|t=0} \).

**Proof** Here \( R \) denotes the derivative at zero of the regular trajectory and \( E_R(t, u) \) the exact solution of the Riccati equation driven by \( R \). Define

\[
\eta(h, u) := E_R(h, u) - \varphi_h(u).
\]

In order to control the error up to time \( t \), we introduce the quantity

\[
H(t, N, u) := \sum_{k=0}^{n-1} \eta(h, E_R(kh, u)), \quad u \in \mathcal{U}.
\]
3.3. Weak approximation schemes for affine processes

Split the error as

\[ \varepsilon_{k+1} = (\mathcal{E}_R((k+1)h, u) - y^*_{k+1}(u)) + (y^*_{k+1}(u) - y_{k+1}(u)), \quad (3.11) \]

where \( y^*_{k+1}(u) = \varphi_h(\mathcal{E}_R(kh, u)) \) is the value in \( (k+1)h \) of the approximated problem starting from the exact solution \( \mathcal{E}_R(kh, u) \) at time \( kh \).

Then, due to analyticity for \( u \) in the interior of \( \mathcal{U} \), there exists a constant \( L_\Theta > 0 \) such that

\[ \left| \mathcal{E}_R((k+1), u) - y^*_{k+1}(u) \right| \leq h\tau(h)\rho_\tau(t, u), \]

\[ \left| y^*_{k+1}(u) - y_{k+1}(u) \right| = (1 + hL_\Theta)|\varepsilon_k|, \]

where

\[ \rho_\tau(T, u) := \sup_{s \in [0,T]} |G_\nu(\mathcal{E}_R(s, u))|, \quad (3.12) \]

with

\[ G_\nu(u) = \partial_s^\nu R(\mathcal{E}_R(s, u))|_{s=0}. \]

Therefore it holds

\[ |\varepsilon_{k+1}| \leq h\tau(h)\rho_\tau(t, u) + (1 + hL_\Theta)|\varepsilon_k|. \]

By recursion on \( k \), we find

\[ |\varepsilon_{k+1}| \leq \frac{e^{L_\Theta(k+1)h} - 1}{L_\Theta} \tau(h)\rho_\tau(t, u) \]

and therefore, for all \( k = 0, \ldots, N - 1 \), it holds

\[ |\varepsilon_{k+1}| \leq \frac{e^{L_\Theta t} - 1}{L_\Theta} \tau(h)\rho_\tau(t, u), \]

which clearly implies

\[ |\mathcal{E}_R(t, u) - y_N(u)| \leq \frac{e^{L_\Theta t} - 1}{L_\Theta} \tau(h)\rho_\tau(t, u). \]

\[ \square \]

**Lemma 3.4** For every \( n = 0, \ldots, N \), there exists a distribution \( \hat{p}_n(x, d\xi) \) such that

\[ \int e^{(u, \xi)} \hat{p}_n(x, d\xi) = e^{(x, y_n(u))}, \]
3. High order numerical schemes for affine processes

for all \( x \in D \) and \( u \in \mathcal{U} \). Additionally, this family of measures is defined iteratively as

\[
\hat{p}_0(x, d\xi) = \delta_x(d\xi), \\
\hat{p}_{n+1}(x, d\xi) = \int \hat{p}_n(\xi, d\eta)\hat{\mu}_h(\xi, d\xi), \quad \text{for } n = 0, \ldots, N - 1,
\]

where \( \hat{\mu}_h(x, \cdot) \) is the infinitely divisible measure corresponding with \( \langle x, \varphi_h(u) \rangle \), i.e.

\[
\int e^{(u,\xi)}\hat{\mu}_h(x, d\xi) = e^{(x,\varphi_h(u))}.
\]

Proof For all \( u \in \mathcal{U} \), \( e^{(x,y_N(u))} \) is the Fourier–Laplace transform of an infinitely divisible measure. This is due to the closure of \( C^H \) under \( \circ \). For \( n = 0, \) \( e^{(u,x)} \) is the Fourier–Laplace transform of \( \delta_x \). By induction, suppose that, for every \( x \in D \), there exists a measure \( \hat{p}_n(x, d\xi) \) on \( D \) such that

\[
e^{(x,y_n(u))} = \int_D e^{(u,\xi)}\hat{p}_n(x, d\xi).
\]

In particular, the measure \( \hat{p}_1(x, d\xi) \) is the one corresponding with \( \langle x, \varphi_h(u) \rangle \). Denote it by \( \hat{\mu}_h \). The following holds

\[
e^{(x,y_{n+1}(u))} = e^{(x,\varphi_h(y_n(u)))} \\
= \int \int e^{(\eta, y_n(u))}\hat{\mu}_h(x, d\xi) \\
= \int \int e^{(\eta, u)}\hat{p}_n(\xi, d\eta)\hat{\mu}_h(x, d\xi),
\]

and therefore \( e^{(x,y_{n+1}(u))} \) is the Fourier–Laplace transform of \( (\hat{p}_n \cdot \hat{\mu}_h)(x, \cdot) \). \( \square \)

The next theorem combines Lemmas 3.3 and 3.4 in order to give a preliminary convergence result for the one time marginal distributions of an affine process. We call it a preliminary result since the analysis of the convergence rate will be done only in the next section.

**Theorem 3.5** Fix \( x \in D, T > 0 \) and \( N \in \mathbb{N} \). Let \( \{0 = t_0, \ldots, t_N = T\} \) be a uniform partition with spacing \( h = \frac{T}{N} \). Given a regular trajectory \( (\varphi_t)_{t \geq 0} \in C^H \), consider the Lévy process \( L^{(x, \varphi_h)}(x, \varphi_h) \) such that

\[
\mathbb{E}
\left[
  e^{\left( u, L^{(x, \varphi_h)}_T \right)}
\right] = e^{s(x, \varphi_h(u))}, \quad u \in \mathcal{U}.
\]
Let
\begin{align*}
\{L(\langle \varphi_h, x \rangle, 1), \ldots, L(\langle \varphi_h, x \rangle, N) \},
\end{align*}
be \( N \) independent copies of \( L(\langle x, \varphi_h \rangle) \). The sequence
\begin{align*}
X_0^x &= x,
X_{n+1}^x &= L(\langle \varphi_h, X_n^x \rangle, n), \quad n = 0, \ldots, N - 1,
\end{align*}
defines a Markov chain such that
\begin{enumerate}
\item \( X_t^x \) is infinitely divisible in \( D \), for all \( n = 1, \ldots, N \), with
\begin{align*}
\mathbb{E}\left[ e^{\langle u, X_{t^n}^x \rangle} \right] &= e^{\langle x, y_n(u) \rangle}, \quad u \in \mathcal{U},
\end{align*}
where \( \{y_n\}_{n=0,\ldots,N} \) is defined iteratively as in \((3.6)\).
\item If \( X \) be an affine process with functional characteristic \( R = \partial_t \varphi_t|_{t=0^+} \), then, for all \( f \in \mathcal{C}_b(D) \),
\begin{align*}
\lim_{N \to \infty} \mathbb{E}\left[ f(X_{tN}^x) \right] &= \mathbb{E}^x\left[ f(X_T) \right].
\end{align*}
\end{enumerate}

**Proof** Let \( X_{t^n}^x \) be a random variable with distribution \( \tilde{p}_{t^n}(x, \cdot) \) defined in Lemma 3.4. By construction,
\begin{align*}
\mathbb{E}\left[ e^{\langle u, X_{t^n}^x \rangle} \right] &= e^{\langle x, y_n(u) \rangle},
\end{align*}
and \( X_{t^n}^x \) coincides in distribution with the value at time one of the Lévy process obtained by taking the subordination of \( n \) independent copies of \( L(\langle x, \varphi_h \rangle) \). Moreover, from Lemma 3.3, we know that the sequence \( \{y_n\}_{n=0,\ldots,N} \) coincides with an \( N \)-step explicit Euler scheme of the Cauchy problem
\begin{align*}
\begin{cases}
\partial_t \mathcal{E}_R(t, u) &= R(\mathcal{E}_R(t, u)), \\
\mathcal{E}_R(0, u) &= u,
\end{cases}
\end{align*}
where \( R(u) = \partial_t \varphi_t(u)|_{t=0^+} \). On the other hand, from Proposition 7.4 in [DFS03], there exists an affine process such that
\begin{align*}
\mathbb{E}^x\left[ e^{\langle u, X_t \rangle} \right] &= e^{\langle x, \mathcal{E}_R(t, u) \rangle}, \quad t \geq 0, u \in \mathcal{U}.
\end{align*}
The result now follows applying Lévy’s continuity theorem. Since the Fourier–Laplace transform of \( X_{t_N}^x \) converges pointwise in \( \hat{\mathcal{U}} \) to the Fourier–Laplace of \( X_T^x \),
3. High order numerical schemes for affine processes

Theorem 3.5 can be adapted to construct a sequence $\widehat{X}^{(m)} = (\widehat{X}^{(m)}_{t_1}, \ldots, \widehat{X}^{(m)}_{t_N})$ of random variables such that $\widehat{X}^{(m)}$ converges weakly, as $m \to \infty$, to the random vector $(X_{t_1}, \ldots, X_{t_N})$, for any fixed partition $\{t_1, \ldots, t_N\}$ of $[0, T]$. For ease of notation, we assume that $\{t_1, \ldots, t_N\}$ is a uniform partition with $t_k = kh$, where $h = \frac{T}{N}$ and $k = 1, \ldots, N$.

**Theorem 3.6** Fix $x \in D$, $T > 0$ and $N \in \mathbb{N}$ with $N \geq 2$. Define $h = \frac{T}{N}$ and let $\{t_0 = 0, t_1, \ldots, t_N = T\}$ be a uniform partition of $[0, T]$ with spacing $h$. Let $(\varphi_t)_{t \geq 0}$ and $R$ be defined as in Theorem 3.5. For $M \in \mathbb{N}$, consider the function

$$\Phi_M(h, u) := \varphi^M_M(u). \tag{3.13}$$

Let $u := (u_1, \ldots, u_N) \in \mathcal{U}^N$ and define recursively, for all $n = 2, \ldots, N$,

$$y_M^1(h, u) := \Phi_M(h, u_1),$$

$$y_M^2(h, u) := y_M^1(h, u_1 + \Phi_M(h, u_2)),$$

$$\vdots$$

$$y_M^n(h, u) = y_M^{n-1}(h, u_1, \ldots, u_{N-2}, u_{N-1} + \Phi_M(h, u_N)). \tag{3.14}$$

1. It holds

$$\lim_{M \to \infty} y_M^n(h, u) = \Psi^N(h, u),$$

where $\Psi^N(h, u)$ is defined recursively as

$$\Psi^1(h, u) := \mathcal{E}_R(h, u_1),$$

$$\Psi^2(h, u) := \Psi^1(h, u_1 + \mathcal{E}_R(h, u_2)),$$

$$\vdots$$

$$\Psi^N(h, u) := \Psi^{N-1}(h, u_1, \ldots, u_{N-2}, u_{N-1} + \mathcal{E}_R(h, u_N)),$$

with $\mathcal{E}_R(t, u_k)$ the exact solution of the Riccati ODE driven by $R$ starting from $u_k$, $k = 1, \ldots, N$.

2. There exists a family of infinitely divisible random variables $(\tilde{X}^{(M)}_{t_1}, \ldots, \tilde{X}^{(M)}_{t_N})$ such that

$$\mathbb{E} \left[ e^{\sum_{k=1}^{N} \langle u_k, \tilde{X}^{(M)}_{t_k} \rangle} \right] = e^{\langle x, y_M^n(h, u) \rangle}.$$
3. Let $X$ be an affine process with functional characteristic $R$. Then, for all $f \in C_b(D^N)$,

$$
\lim_{M \to \infty} \mathbb{E} \left[ f(\widehat{X}_{t_1}^{(M)}, \ldots, \widehat{X}_{t_N}^{(M)}) \right] = \mathbb{E}^x \left[ f(X_{t_1}, \ldots, X_{t_N}) \right].
$$

**Proof** We proceed by induction. For $n = 1$, the result holds from Lemma 3.3 by replacing $t$ (resp. $N$) therein with $h$ (resp. $M$). For $n = 2$, write

$$
y_M^2(h, u) := \Phi_M(h, u_1 + \Phi_M(h, u_2)).
$$

The function $\Phi_M$ is defined as the $M$-times composition of the function $\varphi_{h/M}(u)$. Due to regularity of the trajectory, we can take the limit to conclude that

$$
\lim_{M \to \infty} y_M^2(h, u) = \mathcal{E}_R(h, u_1 + \mathcal{E}_R(h, u_2)) = \Psi^2(h, u).
$$

For $n = 3, \ldots, N$, the result holds analogously. Regarding the second point, the identification of $\langle x, y_M^N(t, u) \rangle$ with the logarithm of the Fourier–Laplace transform of the marginals is done by induction. For $n = 1$, there exists an infinitely divisible random variable $\widehat{X}_{t_1}^{(M)}$ such that, for $u_1 \in \mathcal{U}$,

$$
\mathbb{E} \left[ e^{\langle u_1, \widehat{X}_{t_1}^{(M)} \rangle} \right] = e^{\langle x, y_M^1(u_1) \rangle}.
$$

Suppose that, for all $n = 1, \ldots, N-1$, $y_M^n(t, u)$ is such that

$$
\mathbb{E} \left[ e^{\sum_{k=1}^{n} \langle u_k, \widehat{X}_{t_k}^{(M)} \rangle} \right] = e^{\langle x, y_M^n(u) \rangle}.
$$

We show that the result holds for $n + 1$.

$$
\mathbb{E} \left[ e^{\sum_{k=1}^{n+1} \langle u_k, \widehat{X}_{t_k}^{(M)} \rangle} \right] = \mathbb{E} \left[ e^{\sum_{k=1}^{n} \langle u_k, \widehat{X}_{t_k}^{(M)} \rangle + \langle u_{n+1}, \widehat{X}_{t_{n+1}}^{(M)} \rangle} \right]
$$

$$
= \mathbb{E} \left[ e^{\sum_{k=1}^{n} \langle u_k, \widehat{X}_{t_k}^{(M)} \rangle + \langle u_{n+1}, \widehat{X}_{t_{n+1}}^{(M)} \rangle} | \sigma(\widehat{X}_{t_n}^{(M)}) \right]
$$

$$
= \mathbb{E} \left[ e^{\sum_{k=1}^{n-1} \langle u_k, \widehat{X}_{t_k}^{(M)} \rangle + \langle \widehat{X}_{t_n}^{(M)}, u_n + \Phi_M(h, u_{n+1}) \rangle} | \sigma(\widehat{X}_{t_n}^{(M)}) \right]
$$

$$
= e^{\langle x, y_M^n(u_1, \ldots, u_{n-1}, u_n + \Phi_M(h, u_{n+1})) \rangle}
$$

$$
= e^{\langle x, y_M^{n+1}(u) \rangle}.
$$

The last point follows by the identification of $\langle x, \Psi^N(h, u) \rangle$ with the logarithm of the
3. High order numerical schemes for affine processes

joint Fourier–Laplace transform of the marginals distributions of the affine process at \( \{t_1, \ldots, t_N\} \) (apply Proposition 1.20 with \( u_0 = 0 \) or, equivalently, see Corollary 3.3 in [Kal06]).

3.3.3 Analysis of the convergence rate

We start with the classical definition of convergence rate for weak approximation schemes for stochastic differential equations.

**Definition 3.7 (see [Alf10])** A discretization scheme \( \tilde{X}^x \) is a weak \( \nu \)-th order approximation for the process \( X^x \) if, for every \( f \in C^\infty_c(D) \), there exists \( K > 0 \) such that

\[
\left| \mathbb{E}^x[f(X_T)] - \mathbb{E}[f(\tilde{X}^x_T)] \right| \leq K h^\nu, \quad h > 0.
\]

Recall that the approximation proposed in Theorem 3.5 is based on a numerical approximation of a system of ODE of type

\[
\begin{cases}
\partial_t \Psi(t, u) = R(\Psi(t, u)), \\
\Psi(0, u) = u.
\end{cases}
\]

Let \( \Theta(h, u) \) be an approximating operator. You may think about \( \Theta \) as the operator corresponding to a quadrature scheme approximating the problem (3.16) or a splitting operator defined through some exponential splitting. Define iteratively the approximating sequence by

\[
y_0 = u, \\
y_{n+1} = \Theta(h, y_n), \quad \text{for } n = 0, \ldots, N - 1.
\]

Let \( \varepsilon_{n+1} \) be the error defined via

\[
\Psi(t_{n+1}, u) = \Theta(h, \Psi_R(t_n, u)) + \varepsilon_{n+1}, \quad \text{for } n = 0, \ldots, N - 1.
\]

Write

\[
\varepsilon_{n+1} = h \tau_{n+1}(h).
\]

Define

\[
\tau(h) := \max_{n=0, \ldots, N} |\tau_{n+1}(h)|.
\]

We will work under the following assumption
Assumptions 3.8 Let $\Theta(h,u)$ be an approximating operator for the Cauchy problem (3.16). We assume that

1. $(\Theta(h,u))_{h \geq 0}$ is a regular trajectory in $C^H$,
2. $\lim_{h \to 0} \tau(h) = 0$, where $\tau(h)$ is defined in (3.18).

Definition 3.9 The scheme induced by $\Theta$ has order $\nu$ if $\tau(h) = O(h^\nu)$ for $h \to 0$.

Example 3.10 If we take a quadrature scheme, then a one step approximation of (3.16) reads

$$y_{n+1} = y_n + h\Xi(h,y_n;R), \quad \text{for } n = 0, \ldots, N - 1, \quad (3.19)$$

where $\Xi(h,u;R)$ is an increment function (see Chapter II.3 in [HNW11]). Convergence of the scheme is given by

$$\lim_{h \to 0} \Xi(h,u;R) = R(u).$$

In some cases, we are given a setting where it is convenient to split the operator $R$ into two parts as

$$R(u) = R_1(u) + R_2(u).$$

Denote by $\mathcal{E}_{R_1}(t,u)$ and $\mathcal{E}_{R_2}(t,u)$ the exact solution of the evolutionary equations defined separately by $R_1$ and $R_2$ both with initial condition $u$. Some examples of splitting operators $\Theta$ are

$$\Theta^{LT}(h,u) := \mathcal{E}_{R_1}(h,\mathcal{E}_{R_2}(h,u)),$$
$$\Theta^{STR}(h,u) := \mathcal{E}_{R_1}(h/2,\mathcal{E}_{R_2}(h,\mathcal{E}_{R_1}(h/2,u)),$$
$$\Theta^{NV}(h,u) := (\mathcal{E}_{R_1}(h,\mathcal{E}_{R_2}(h,u)) + \mathcal{E}_{R_2}(h,\mathcal{E}_{R_1}(h,u))) / 2.$$

In all three cases, the Assumption 3.8 holds.

When the approximation of $\mathcal{E}_R$ is done via a quadrature or an exponential splitting with $\tau(h) = O(h^\nu)$, the global error depends also on the value $u$. In both cases, we will see that the behavior of the error with respect to $u$ can be controlled by providing an estimate for the following quantity:

$$\rho_\tau(t,u) := \sup_{s \in [0,t]} |G_\nu(\mathcal{E}_R(s,u))|, \quad (3.20)$$

where $G_\nu(u) = \partial^\nu R(\mathcal{E}_R(s,u))|_{s=0}.$

The following lemma will be used in Theorem 3.12.
Lemma 3.11 Let \( R \in TC^H \) such that, for all \( i = 1, \ldots, m \), the Lévy triplet \((\alpha_i, \beta_i, M_i)\) of \( R_i \) satisfies the additional condition
\[
\int_{\{|\xi| \geq 1\}} |\xi| e^{\langle \xi, y \rangle} M_i(d\xi) < \infty, \tag{3.21}
\]
for all \( y \) in a subset \( \mathcal{Y} \subseteq \mathbb{R}^d \). Then, for all \( u \in \mathbb{C}^d \) such that \( \Re(e(u)) \in \mathcal{Y} \), there exists a constant independent of \( u \) such that \( |\partial_u^k R(u)| \leq C \), for all \( k \geq 2 \).

Proof We will use the following inequalities for \( z \in \mathbb{C} \):
\[
|e^z - 1| \leq |z| e^{\Re(z)} \quad \text{and} \quad |e^z - 1 - z| \leq \frac{1}{2} |z|^2 e^{\Re(z)}.
\]

Split the integral
\[
\int |e^{(u, \xi)} - 1 - \langle \pi_{I \cup \{i\}} u, \pi_{I \cup \{i\}} \pi_{I \cup \{i\}} \rangle \rangle M_i(d\xi)
= \int_{\{|\xi| \leq 1\}} |e^{(u, \xi)} - 1 - \langle \pi_{I \cup \{i\}} u, \pi_{I \cup \{i\}} \rangle \rangle M_i(d\xi)
\]
\[
+ \int_{\{|\xi| > 1\}} |e^{(u, \xi)} - 1| M_i(d\xi).
\]

Then estimate
\[
\int_{\{|\xi| \leq 1\}} \langle e^{(u, \xi)} - 1 - \langle \pi_{I \cup \{i\}} u, \pi_{I \cup \{i\}} \rangle \rangle M_i(d\xi)
\leq \frac{1}{2} \int_{\{|\xi| \leq 1\}} |\pi_{I \cup \{i\}} u|^2 |\pi_{I \cup \{i\}} \rangle \rangle^2 e^{\Re(u, \xi)} M_i(d\xi),
\]
\[
\int_{\{|\xi| > 1\}} |e^{(u, \xi)} - 1| M_i(d\xi) \leq \int_{\{|\xi| > 1\}} \langle u, \xi \rangle e^{\Re(u, \xi)} M_i(d\xi).
\]

The first term is integrable because each \( M_i \) is a Lévy measure. Integrability of the second term is given by assumption. Moreover, for \( m \geq 2 \), it holds
\[
\int \partial_u^k (e^{(u, \xi)} - 1 - \langle \pi_{I \cup \{i\}} u, \pi_{I \cup \{i\}} \rangle \rangle M_i(d\xi) \leq \int e^{(u, \xi)} |\xi|^k M_i(d\xi),
\]
which is again finite. Therefore
\[
|\partial_u^k R(u)| \leq \begin{cases} C(1 + |u|^2), & \text{if } k = 0, \\ C(1 + |u|), & \text{if } k = 1, \\ C, & \text{if } k \geq 2. \end{cases}
\]
Theorem 3.12 Let $X$ be an affine process with functional characteristic $R \in TC^H$. Fix an equidistant time discretization with spacing $h = T/N$

$$\{t_0 = 0, \ldots, t_k = kh, \ldots, t_N = T\}.$$

Let $\{y_n\}_{n=0,\ldots,N}$ be an approximation for (3.16) accomplished by an approximating operator $\Theta(h,u)$ satisfying Assumptions 3.8 with global error $\tau(h)$.

1. Define

$$Q_h f(x) := E\left[ f(L^{(x,y_1)}) \right], \quad f \in C_b(D). \quad (3.22)$$

The approximation $\{\bar{X}_t\}_{t_i=0,\ldots,N}$ in Theorem 3.5 satisfies

$$E\left[ f(\bar{X}_t) \right] = Q_h^n f(x), \quad \text{for all } n = 1, \ldots, N,$$

where $Q_h^n$ is given by the composition of (3.22) $n$-times, i.e.

$$Q_h^n f := Q_h \circ \cdots \circ Q_h f, \quad f \in C_b(D).$$

2. For any $u \in \mathcal{U}$, there exists a constant $K_T$ independent of $u$ such that

$$|P_T f_u(x) - Q_h^N f_u(x)| \leq K_T \rho_T(T,u) \tau(h) f_{\Psi(T,\text{Re}(u))}(x),$$

where $\rho_T(T,u)$ has been defined in (3.20).

3. If, additionally, there exists a set $\mathcal{Y} \subseteq \mathbb{R}^d$ such that $E^x\left[e^{\langle y, X_T \rangle}\right] < \infty$ for all $y \in \mathcal{Y}$, then (3.23) holds for all $u \in \mathcal{C}^d$ such that $\text{Re}(u) \in \hat{\mathcal{Y}}$. Moreover, if the approximation is done so that $\tau(h) = O(h^\nu)$ with $\nu \geq 2$, then $\rho_T(T,u)$ can be bounded by a constant which depends only on $\nu$ and $T$. Hence, for all $u \in \mathcal{C}^d$ such that $\text{Re}(u) \in \hat{\mathcal{Y}}$, it holds

$$|P_T f_u(x) - Q_h^N f_u(x)| \leq K'_T h^\nu f_{\Psi(T,\text{Re}(u))}(x).$$

**Proof** We start with the first point. Suppose that, for all $n = 0, \ldots, N-1$, it holds

$$E\left[ f(\bar{X}_t) \right] = Q_h^n f(x), \quad f \in C_b(D).$$

Then

$$Q_h^{n+1} f(x) = Q_h \circ Q_h^n f(x) = E\left[ f(\bar{X}^{(y,\varphi_h)}) \right]_{y = \hat{X}_t} = E\left[ f(\bar{X}_k) \right].$$
3. High order numerical schemes for affine processes

Now we move on point 2. Define, for \( h > 0 \) and \( u \in \mathcal{U} \),

\[ \eta(h, u) = \Psi(h, u) - \varphi_h(u). \]

Then clearly

\[ P_h f_u = e^{\eta(h,u)} Q_h f_u. \]

In order to control the error at time step \( N \) introduce the quantity

\[ H(T, N, u) := \sum_{k=0}^{n-1} \eta(h, \Psi(t_k, u)), \quad u \in \mathcal{U}. \]

Then

\[ P_T f_u = e^{H(T,N,u)} Q^N_h f_u. \quad (3.24) \]

As in the proof of Lemma 3.3 we can show that

\[ |\Psi(T, u) - y_N(u)| \leq \frac{e^{L_\Theta T} - 1}{L_\Theta} \tau(h) \rho_T(T, u). \]

With the above considerations, we conclude that the following holds:

\[ |P_T f_u(x) - Q^N_h f_u(x)| = \left| \left( 1 - e^{H(T,N,u)} \right) P_T f_u(x) \right| \\
\leq \frac{e^{L_\Theta T} - 1}{L_\Theta} \tau(h) \rho_T(T, u) \left( |P_T f_u(x)| + |Q^N_h f_u(x)| \right) \]

To estimate \( |P_T f_u(x)| \) and \( |Q^N_h f_u(x)| \) for \( u \in \mathcal{U} \), note that

\[ |P_T f_u(x)| \leq P_T f_{\mathcal{R}e(u)}(x) = e^{\langle x, \Psi(T, \mathcal{R}e(u)) \rangle} \]

and (3.24) yields

\[ |Q^N_h f_u(x)| = e^{-H(T,N,u)} |P_T f_u(x)| \leq C e^{\langle x, \Psi(T, \mathcal{R}e(u)) \rangle}, \]

for some constant \( C > 0 \). This concludes 2. Now, if \( \mathcal{R}e(u) \in \mathring{\mathcal{Y}} \), there exists \( \varepsilon > 0 \).
3.3. Weak approximation schemes for affine processes

small such that $\Re e(u_k) + \varepsilon \leq y_k$ for $k = 1, \ldots, d$ for all $y \in \mathcal{Y}$. Hence

$$
\int_{\{||\xi|| \geq 1\}} |\xi| e^{\langle u, \xi \rangle} M_i(d\xi) \leq \int_{\{||\xi|| \geq 1\}} \left( e^{\langle \Re e(u) - \varepsilon, \xi \rangle} + e^{\langle \Re e(u) + \varepsilon, \xi \rangle} \right) M_i(d\xi) \\
\leq 2 \int_{\{||\xi|| \geq 1\}} e^{\langle y, \xi \rangle} M_i(d\xi) < \infty,
$$

where the last integrability follows from the Theorem 2.14 in [KM11]. Therefore, if $\nu \geq 2$, from Lemma 3.11, we get the existence of a constant $C$ such that $\rho_\tau(T, u) \leq C$.

The estimate in Theorem 3.12 tells us that, if we start with a $\nu$-th order approximation scheme for the Cauchy problem (3.16), the approximation scheme constructed with the subordination approach, leads to a $\nu$-th order weak scheme, at least when the set of test functions is restricted to the set of the Fourier modes. In the next theorem, we see how to extend the convergence rate to the space of functions with polynomial growth.

**Theorem 3.13** Introduce the sets

$$
\mathcal{Q}^{\text{ext}}(t) := \left\{ u \in \mathbb{C}^d \mid \mathbb{E}^x \left[ e^{\langle \Re e(u), X_t \rangle} \right] < \infty, \text{ for } t \in [0, T] \right\} \tag{3.25}
$$

and

$$
\mathcal{Q}^{\text{ext}}(t_k) := \left\{ u \in \mathbb{C}^d \mid \mathbb{E} \left[ e^{\langle \Re e(u), \tilde{X}_{t_k} \rangle} \right] < \infty, \text{ for } k = 1, \ldots, N\. \right\} \tag{3.26}
$$

Suppose that

$$
\Re e \left( \mathcal{Q}^{\text{ext}}(T) \cap \mathcal{Q}^{\text{ext}}(t_N) \right) \cap \mathbb{R}^d_{\geq 0} \neq \emptyset. \tag{3.27}
$$

If $\tau(h) = \mathcal{O}(h^\nu)$ with $\nu \geq 2$, for all $f \in C^\infty_{\text{pol}}$, there exists a constant $C_f > 0$ which depends on the function $f$ such that

$$
\left| \mathbb{E}^x \left[ f(X_T) \right] - \mathbb{E} \left[ f(\tilde{X}_T^x) \right] \right| \leq C_f h^\nu F(x),
$$

where $F(x) = 1 + \sum_{i=1}^d x_i^\eta$ for some $\eta \in \mathbb{N}$.

**Proof** From Chapter 4 in [KM11] we know that, $\mathcal{Q}^{\text{ext}}(T) \subset \mathcal{Q}^{\text{ext}}(t)$ for all $t \in [0, T]$ and moreover that, for all $u \in \mathcal{Q}^{\text{ext}}(T)$, it holds

$$
\int_{\{||\xi|| \geq 1\}} e^{\langle \Re e(u), \xi \rangle} M_i(d\xi) < \infty, \text{ for all } i = 1, \ldots, d \text{ and } x \in D.
$$
Analogously, since by construction the approximation scheme is built upon \( N \)-times convolution, it holds \( \tilde{Q}^{\text{ext}}(t_{k+1}) \subset \tilde{Q}^{\text{ext}}(t_k) \) for all \( k = 0, \ldots, N - 1 \).

Due to polynomial growth of the function \( f \), there exists a constant \( c \in \mathbb{R} \) such that
\[
g(x) := e^{c|x|} f(x) \in L^1(\mathbb{R}^d).
\]
Planchrel’s theorem can be applied to get
\[
\int f(\xi) e^{-c|\xi|} e^{c|\xi|} (p_t(x, d\xi) - q_t(x, d\xi)) =: \int g(\xi) (\tilde{p}_t(x, d\xi) - \tilde{q}_t(x, d\xi)),
\]
under the condition that the Fourier transform of both \( e^{c|X_T|} \) and \( e^{c|X_{t'}|} \) is well defined. Since \( e^{c|x|} e^{i(v, x)} \leq e^{i(v+c, x)} + e^{i(v-c, x)} \), we need \( \mathbb{E}^x \left[ e^{i(u, X_T)} \right] \) to be finite at least for all \( u \in \mathbb{C}^d \) such that \( |\Re(u)| \leq c \). This translate into existence of an analytic extension of the Fourier transform of \( X_t \) in a complex strip of type \([-c, c]^d \times i\mathbb{R}^d \) for all \( t \geq 0 \). Since \( c \) can be taken arbitrarily small, this is guaranteed by (3.27). The same holds for the approximation sequence.

At the same time,
\[
\mathcal{F}g(v) = \int e^{-i(v, \xi)} g(\xi) d\xi
\]
is well defined for \( v \in \mathbb{R}^d \). Moreover we can write
\[
g(\xi) = \frac{1}{(2\pi)^d} \int e^{i(v, \xi)} \mathcal{F}g(v) dv.
\]
Therefore it holds
\[
\int (\mathcal{F}g(v) e^{i(v, \xi)} e^{c|\xi|} (p_t(x, d\xi) - q_t(x, d\xi))) dv
\]
where in the last step the change of order of the integrals is justified by the integrability we have checked before. Moreover, Lemma 3.11 holds also for \( u \in \mathbb{R}^d \)
3.3. Weak approximation schemes for affine processes

\[ Q^\text{ext}(T) \cap \tilde{Q}^\text{ext}(T) \text{ and therefore} \]

\[
\left| P_T f(x) - Q_h^N f(x) \right| \leq \int |\mathcal{F} g(v) - P_{Tu+c} - Q_h^N f(x) | dv \\
\leq \int |\mathcal{F} g(v)| K_{T,v}^{\nu+1} e^{c(x,y(h,v,|c|))} dv \\
\leq C_f h^{\nu} F(x),
\]

where the constant \( C_f \) depends on the \( L^1 \) norm of the Fourier transform of the dampened function and on the dampening factor \( c \). We finally estimate the function \( F(x) \). Due to the assumption on the Lévy measure, \( X \) is a polynomial process (see [CKT08]). Let \( \alpha \) be the index such that \( |f(x)| \leq C(1 + |x|^\alpha) \). If \( \alpha \) is even, then \( \Pi_\eta(x) := x^{2\eta} \) with \( \eta = \alpha/2 \) is a polynomial. If \( \alpha \) is odd, write \( |x|^{\alpha} \leq 1 + |x|^\eta + 1 =: 1 + \Pi_\eta(x) \), with \( \eta = (\alpha + 1)/2 \). In any case, there exists an \( \eta \geq 0 \) such that

\[
\mathbb{E}^x \left[ |f(X_t)| \right] \leq \mathbb{E}^x \left[ \Pi_\eta(X_t) \right] \leq C F(x)
\]

where \( F(x) := 1 + \sum_{k=1}^d x_k^n \). The same bound holds for polynomial moments of the infinitely divisible random variable \( X_N^x \).

3.3.4 Examples

Example 3.14 (The Heston Model) Let \( (X_t)_{t \geq 0} \) be a Feller diffusion as in Example 2.1 and consider the two dimensional process

\[
\begin{cases}
    dX_t = \sqrt{2X_t} dW^1_t, & X_0 = x, \\
    dY_t = -X_t dt + \sqrt{2X_t} dW^2_t, & Y_0 = y,
\end{cases}
\]

where \( W = (W^1, W^2) \) is a two dimensional Brownian motion. The process \( (X^x, Y^y) \) is an affine process on \( \mathbb{R}_{\geq 0} \times \mathbb{R} \) with

\[
R(u, v) = \left( -v + u^2 + v^2, 0 \right) ^\top.
\]

The exact Fourier–Laplace transform of \( (X, Y) \) is given by

\[
\mathbb{E}^{x,y} \left[ e^{xX_t + yY_t} \right] = \exp \left( yw + x\gamma(w) \tan \left( t\gamma(w) + \arctan \frac{v}{\gamma(w)} \right) \right),
\]

107
where $\gamma(w) = \sqrt{w(w-1)}$.

In higher dimensions, solving this type of ODE is not an easy task. Here we present an approximation scheme which combines both splitting and quadrature methods.

Split

$$R(u, v) = R_1(u) + R_2(v),$$

with

$$R_1(u) := \left( u^2, 0 \right)^\top, \quad \text{and} \quad R_2(v) := \left( -v + v^2, 0 \right)^\top.$$

Then

$$\mathcal{E}_{R_1}(h, u, v) = \left( \frac{u}{1 - hu}, v \right)^\top, \quad \text{and} \quad \mathcal{E}_{R_2}(h, u, v) = (u + h(v + v^2), v)^\top.$$

Define $\varphi_h^{(1)}(u, v) = \mathcal{E}_{STR}^1(h, u, v)$ and $\varphi_h^{(2)}(u, v) = \mathcal{E}_{STR}^2(h, u, v)$, with

$$\mathcal{E}_{STR}^1(h, u, v) = \frac{1}{2} h \left( v^2 + v \right) + u \frac{1}{1 - h \left( \frac{1}{2} h \left( v^2 + v \right) + u \right)} + \frac{1}{2} h \left( v^2 + v \right),$$

$$\mathcal{E}_{STR}^2(h, u, v) = v.$$

Now, we want to identify the subordinated process. Observe that, for all $t \geq 0$, $\mathcal{E}_{STR}^1(h, 0, v) \in \mathcal{C}$ with Lévy triplet $(0, 0, \nu(t, d\xi))$ given by

$$\nu(t, d\xi) = \frac{1}{(2t)^2} \exp \left( -\frac{x}{2t} \right).$$

Let $Z^{R_2}$ be a Lévy process with Lévy exponent $R_2$ (or in other words a Brownian motion with drift). Define

$$(\tilde{X}^x_{0}, \tilde{I}_0, \tilde{Y}^y_0) = (x, 0, y),$$

and, using the same notation of Theorem 3.5, consider, for $n = 0, \ldots, N - 1$,

$$\tilde{X}^x_{tn+1} = L^{\left( \mathcal{E}_{STR}^1(h, 0, \cdot), \tilde{X}^x_{tn}, n \right)} \tilde{X}^x_{tn}, \quad \tilde{I}_{tn+1} = \tilde{I}_{tn} + \frac{h}{2} \left( \tilde{X}^x_{tn} + \tilde{X}^x_{tn+1} \right), \quad \tilde{Y}^y_{tn+1} = \tilde{Y}^y_{tn} + Z^{(R_2, n)}(\tilde{I}_{tn+1}). \quad (3.28)$$
It is not hard to check that
\[ Q_h f_u((x,y)) := \mathbb{E} \left[ f_u(\tilde{X}_h^x, \tilde{Y}_h^y) \right] = e^{x \mathcal{E}^{STR}(h,u,v) + yv}. \]

Since \[ |\mathcal{E}^{STR}(h,u,v) - \mathcal{E}_R(h,u,v)| = O(h^3) \]
the approximation \( (f_u(\tilde{X}^x), f_u(\tilde{Y}^y)) \) defines a global second order scheme for the process \( (f_u(X^x), f_u(Y^y)) \). Moreover, it is known that its characteristic function can be analytically extended around zero (see [dBRFCU10]). Therefore, Theorem 3.13 can be applied.

**Remark 3.15** Observe that, the auxiliary process \( \tilde{I} \) is a trapezoid approximation for the integrated process \( I_t := \int_0^t X_s^x ds \). The approximation scheme proposed in the previous example can be seen as an approximation of the following time change representation
\[ Y_t^y = y + Z(I_t), \quad t \geq 0, \]
where the time change process \( I \) is replaced by its multistep trapezoid approximation. Moreover, that the above scheme can be easily extended to the correlated case as done in Section 1.4.

**Example 3.16 (Coupled CIR)** Affine diffusions in \( \mathbb{R}_{\geq 0}^m \) are identified by a matrix, containing the drift term, and a vector of positive values, identifying the volatility of each component. In analytic terms, the computation of the joint Fourier–Laplace transform is related to the solution of a non linear system of ODE where the dependence is only in the linear term. Here we consider the following two dimensional extension of the CIR model
\[
\begin{align*}
  dX_t &= \sqrt{2}X_t dW^1_t, \\
  dY_t &= X_t dt + Y_t dt + \sqrt{2}Y_t dW^2_t,
\end{align*}
\]
where \( W = (W^1, W^2) \) is a two dimensional Brownian motion. The process \( (X^x, Y^y) \) is an affine process on \( \mathbb{R}_{\geq 0}^2 \) with
\[ R(u,v) = \left( u^2, u + v + v^2 \right)^\top. \]

If we split
\[ R(u,v) = R_1(u,v) + R_2(u,v), \]
with
\[ R_1(u,v) = \left( u^2, v + v^2 \right)^\top \quad \text{and} \quad R_2(u,v) = (0, u)^\top, \]
we can easily integrate to get

\[ \mathcal{E}_{R_1}(t, u, v) = \left( \frac{u - tu}{e^{v(u+v)} - 1} \right), \quad \text{and} \quad \mathcal{E}_{R_2}(t, u, v) = \left( \frac{u}{t(u+v)} \right). \]

If we perform the splitting \( \mathcal{E}^{NV} = (\mathcal{E}_{R_1} \circ \mathcal{E}_{R_2} + \mathcal{E}_{R_2} \circ \mathcal{E}_{R_1})/2 \) we get

\[ \mathcal{E}^{NV}(t, u, v) = \left( \frac{1}{2} \left( -\frac{e^{u(t+u)} - 1}{e^{v(u+v)} - 1} + \frac{tu}{1-tu} + \frac{tu}{e^{v(u+v)} - 1} \right) \right). \]

Now, it is not too hard to identify the Lévy triplet of the subordinated process. Indeed, it holds

\[ \mathcal{E}^{NV}_2(t, u, v) = \int_0^t \left( e^{(c_1(t)u + c_2(t)v)\xi} - 1 \right) \frac{e^{-\xi}}{1 - e^{-t}} d\xi, \]

with \( c_1(t) = t(e^t - 1), \ c_2(t) = e^t - 1 \). Let \( \varphi_\lambda \) be an exponential random variable with mean \( \lambda \). The random variable corresponding to \( \mathcal{E}^{NV}_2(t, \cdot) \) is distributed as \((c_1(t)\varphi_{(1-e^{-t})}, c_2(t)\varphi_{(1-e^{-t})})\) which is \((\varphi_{2t\sinh(t)}, \varphi_{2t\sinh(t)})\). We can identify each

Figure 3.1: The picture shows some approximated paths for the Heston model obtained using the numerical scheme proposed in (3.28). With “Poisson Approximation”, we mean the scheme obtained by replacing the value of the Feller diffusion with the representing Lévy process. Then, \( I \) is approximated by trapezoid scheme and this is used in order to construct the approximation of the trajectories of \( Y^\nu \). The last picture shows the convergence of the Laplace transform of the joint processes \((X, I)\) and \((X, Y)\) at the points \((u, w) = (-1, -2)\). We fixed \( T = \frac{1}{2} \) and \( n = 2, \ldots, 30 \).
3.3. Weak approximation schemes for affine processes

ϕt, with t > 0, with the Fourier–Laplace transform of a compound Poisson process with exponential jumps.

Example 3.17 (CIR with jumps) A possible way to add jumps for affine processes in \( \mathbb{R}^m_0 \) is to add a jump component in the definition of the characteristic functional. Consider the stochastic process

\[
dZ_t^i = cZ_t^i dt + \sqrt{Z_t^i} dW_t + \int_{|\xi| > 1} \xi N(dt, d\xi) + \int_{|\xi| \leq 1} \xi \left( N(dt, d\xi) - \frac{d\xi}{\xi^2} Z_t^i dt \right)
\]

which is an affine process on \( \mathbb{R}_{\geq 0} \) with

\[
R^{C+J}(u) = cu + \frac{1}{2} u^2 + \int_0^\infty \left( e^{u\xi} - 1 - u\xi 1_{|\xi| \leq 1} \right) \frac{d\xi}{\xi^2}.
\]

Split \( R^{C+J} = R^C + R^J \) with

\[
R^C(u) = \frac{u^2}{2}, \quad \text{and} \quad R^J(u) = \Gamma u - u \log(-u).
\]

Then we can identify the exact solutions

\[
\mathcal{E}_R^C(t, u) = \frac{u}{1 - tu/2}
\]

and

\[
\mathcal{E}_R^J(t, u) = -(-u)^{-t}.
\]

We consider a first order scheme. Define \( \varphi_h(u) := \mathcal{E}_R^{LT}(h, u) = \mathcal{E}_R^J(h, \mathcal{E}_R^C(h, u)) \). However, Theorem 3.13 cannot be applied. The main problem is located in the domain of regularity of the Fourier–Laplace transform. Indeed, in this case, \( \mathcal{Q}_t^{ext} \subseteq \mathbb{C}_{<0} \) for all \( t > 0 \).
Chapter 4

Kolmogorov equations of affine type

4.1 Introduction

This chapter is devoted to the study of Markov semigroups acting on function spaces which consist of functions which are not necessarily bounded. More precisely, let $X$ be a Markov process taking values in $D \subseteq \mathbb{R}^d$

$$X = (\Omega, (X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, (p_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D})$$

and $\mathcal{M} \subset m(D)$ a set of measurable functions $f$ such that $\mathbb{E}^x |f(X_t)| < \infty$, for all $t \geq 0$ and $x \in D$. We want to find conditions on $\mathcal{M}$ such that it is possible to identify the transition semigroup $P_t$ acting on $f$ with the solution of the Kolmogorov equation

$$\begin{align*}
\partial_t u(t, x) &= \mathcal{A} u(t, x), \quad (t, x) \in [0, T] \times D, \\
u(0, x) &= f(x), \quad x \in D,
\end{align*}$$

(\ast)

where $\mathcal{A}$ here denotes the (extended) generator of the Markov process $X$.

Starting with the set of functions

$$\mathcal{M} := \left\{ f : D \to \mathbb{R} \mid \text{Borel measurable such that} \right\}
\begin{align*}
P_t|f| < \infty \text{ for all } t \geq 0 \text{ and } M^f_t \text{ is a true martingale},
\end{align*}

$$M^f_t := f(X_t) - f(x) - \int_0^t \mathcal{A} f(X_s)ds,$$
we show that, under the assumption that \( \lim_{t \to 0} P_t f(x) = f(x) \) for all \( f \in \mathcal{M} \), it is possible to conclude that \( P_t f(x) \) coincides with the solution \( u(t, x) \) of the Kolmogorov equation \((*)\).

More precisely, we can conclude that, if \( \mathcal{M} \) is the set of measurable functions \( f \) such that

A1) \( P_t |f| < \infty \),

A2) \( M^f \) is a true martingale,

A3) \( t \mapsto P_t f(x) \) is continuous at \( t = 0 \),

then \( P_t \mathcal{M} \subseteq \mathcal{M} \) and it is possible to derive a short–time asymptotic formula for the function \( t \mapsto P_t f(x) \).

However, in applications, we are given a set of functions \( \mathcal{H} \) and we would like to specify some conditions under which \( P_t \mathcal{H} \subseteq \mathcal{H} \). We first focus on the set \( \mathcal{H} \) of smooth functions with controlled growth such that, for all \( f \in \mathcal{H} \),

B1) \( P_t |f| < \infty \),

B2) \( |A f| \leq K F \), for some constant \( K > 0 \) and a function \( F : D \to \mathbb{R}_{\geq 0} \) such that

\[ \mathbb{E}^t \left[ \sup_{t \in [0,T]} F(X_t) \right] < \infty. \]

We will check that, under these two assumptions, all the three conditions A1), A2) and A3) hold and therefore \( P_t \mathcal{H} \subseteq \mathcal{M} \). Additionally, it is possible to derive a short–time asymptotic formula for \( t \mapsto P_t f \), when \( f \in \mathcal{H} \). However, conditions B1) and B2) are not sufficient to conclude that \( P_t \mathcal{H} \subseteq \mathcal{H} \). We will see that, our main problem hinges on regularity in space of the function \( P_t f(x) \). However, for affine processes, the time–space swap we introduced in Chapter 2 leads to the regularity result we are looking for.

The chapter is organized as follows.

In Section 4.2, we consider functions \( f \) which belong to the domain of the extended generator and provide a framework for the analysis of the Kolmogorov equation.

In Section 4.3, we restrict ourselves to the space \( \mathcal{H} \) of all the functions which are infinitely differentiable and with controlled growth. We analyze under which condi-
tions the function \( u : \mathbb{R}_{\geq 0} \times D \rightarrow \mathbb{R} \) defined by
\[
u(t, x) := P_t f(x)
\]
has derivatives of all orders satisfying the following property
\[
\partial_{(t, x)}^\alpha u(t, x) \in \mathcal{H} \quad \text{for all } (t, x) \in [0, T] \times D.
\]

In this section we will also exploit in details the conditions we obtain when the Markov process \( X \) is a Lévy process.

In the last section, we apply the results derived in Section 4.2 and Section 4.3 to the class of affine processes and the function space \( \mathcal{H} = C_{pol}^\infty \), of smooth functions with polynomial growth.

### 4.2 Martingale problem and short–time asymptotic formula

In this section we fix the problem of giving a comprehensive framework for the transition semigroup of a Markov process acting on a class of functions which is as general as possible. We follow the methodology used in [CKT08]. We start with the definition of the extended generator.

**Definition 4.1 (Definition 7.1. in [ÇJPS80])** Given a Markov process \( X \), the extended generator is an operator \( \mathcal{A} \) with domain \( \mathcal{D}(\mathcal{A}) \) such that, for any \( f \in \mathcal{D}(\mathcal{A}) \)
\[
M^f_t := f(X_t) - f(x) - \int_0^t \mathcal{A} f(X_s) ds
\]
is a local martingale under \( \mathbb{P}^x \) for every \( x \in D \).

Consider a function \( f \in \mathcal{D}(\mathcal{A}) \) such that \( \mathbb{E}^x \left[ |f(X_t)| \right] < \infty \), for all \( (t, x) \in \mathbb{R}_{\geq 0} \times D \) and such that \( (M^f_t)_{t \geq 0} \) is a \( \mathbb{P}^x \) true martingale for every \( x \in D \). Then we can take the expectation on both side of (4.1)
\[
\mathbb{E}^x \left[ f(X_t) - f(x) - \int_0^t \mathcal{A} f(X_s) ds \right] = 0.
\]
4. Kolmogorov equations of affine type

Write
\[ \mathbb{E}^x \left[ f(X_t) \right] = f(x) + \mathbb{E}^x \left[ \int_0^t A f(X_s) ds \right], \] (4.2)
and define \( P_t f(x) := \mathbb{E}^x [ f(X_t) ] \) so that the previous equation reads
\[ P_t f(x) = f(x) + \int_0^t P_s A f(x) ds. \]

From Fubini’s theorem applied to the increments of the martingale \( M^f \) (see Remark 4.1.4) in [CKT08], it follows that \( \mathbb{E}^x \left[ |A f(X_s)| \right] < \infty \), for all \( x \in D \) and \( s \geq 0 \), and therefore we have got an integral equation for \( P_t f \). Hereafter, we restrict ourselves on \( \mathcal{M} \), defined as the space of functions \( f \) such that \( M^f \) is a true martingale:
\[ \mathcal{M} := \left\{ f : D \to \mathbb{R} \mid \text{Borel measurable such that} \right. \]
\[ \left. P_t |f| < \infty \text{ for all } t \geq 0 \text{ and } M^f \text{ is a true martingale} \right\}. \]

However, in order to conclude that \( P_s A f(x) = A P_s f(x) \) for all \( x \in D \) and \( s \geq 0 \), we need some additional regularity in time for the process.

In the next theorem we collect some results from [CKT08].

**Theorem 4.2 (Lemma 2.6. in [CKT08])** Let \( X \) be a time-homogeneous Markov process and \( f \in \mathcal{M} \). Then

1. For any \( s \geq 0 \), \( M^{P_s f} \) is a true martingale.
2. If \( t \to P_t f \) is continuous at \( t = 0 \), \( A P_t f = P_t A f \) for any \( t \geq 0 \).
3. (Feynman–Kac representation) If \( t \to P_t A f(x) \) is continuous at \( t = 0 \), then \( P_t f(x) \) coincides with the solution \( u(t,x) \) of the Kolmogorov equation
\[ \partial_t u(t,x) = A u(t,x), \quad (t,x) \in [0,T] \times D, \]
\[ u(0,x) = f(x), \quad x \in D. \]

**Proof** This is simply an adaptation of the proof in [CKT08]. Since
\[ f(X_t) - f(x) - \int_0^t A f(X_s) ds \]
is a true martingale, all its increments have vanishing expectation. Hence both
4.2. Martingale problem and short–time asymptotic formula

\( f(X_t) \) and \( A f(X_t) \) are integrable for every \( t \geq 0 \). Consider the process

\[
\tilde{M}^P_s f := P_s f(X_t) - P_s f(x) - \int_0^t P_s A f(X_r) \, dr.
\]

Due to the integrability of the increments of \( M^f \), we have that \( \tilde{M}^P_s f \) is well defined for all \( s \) and for all \( t \in [0, T] \). Using Markov property of \( X \) it holds, for \( t_0 \leq t \)

\[
\mathbb{E}^x \left[ \tilde{M}^P_t f - \tilde{M}^P_{t_0} f \mid \mathcal{F}_{t_0} \right] = \mathbb{E}^x \left[ P_s f(X_t) - P_s f(X_{t_0}) - \int_{t_0}^t P_s A f(X_r) \, dr \mid \mathcal{F}_{t_0} \right]
\]

\[
= \mathbb{E}^{X_{t_0}} \left[ P_s f(X_{t-t_0}) - P_s f(X_{t_0}) - \int_{t_0}^t P_s A f(X_{r-t_0}) \, dr \right]
\]

\[
= \left( P_{s+t-t_0} f(y) - P_s f(y) - \int_s^{s+t-t_0} P_r A f(y) \, dy \right) \bigg|_{y = X_{t_0}},
\]

but the last term is identically zero from martingale property of \( M^f \). Taking \( t_0 = 0 \), we get that the map \( r \mapsto P_s A f(X_r) \) is integrable for all \( r \in [0, t] \) for all \( t \) and

\[
\int_0^t P_s |A f(X_r)| \, dr < \infty, \quad \text{for all } t \in [0, T].
\]

By definition of extended generator, together with the fact that \( P_s f \) is integrable, we conclude that \( P_s f \in \mathcal{D}(A) \) and

\[
P_s A f = A P_s f.
\]

Finally, this also implies that \( M^P_s f \) is a true martingale for all \( s \geq 0 \). Kolmogorov’s equation follows by taking the limit of the finite differences and using continuity of \( t \mapsto P_t A f(x) \). Precisely

\[
\partial_t P_t f(x) = \lim_{h \to 0} \frac{P_{t+h} f(x) - P_t f(x)}{h} = \lim_{h \to 0} P_t \left( \frac{P_h f(x) - f(x)}{h} \right)
\]

\[
= \lim_{h \to 0} P_t \frac{1}{h} \int_0^h P_s A f(x) ds
\]

\[
= P_t A f(x) = A P_t f(x),
\]

where in the last line we used commutative property between \( P \) and \( A \). \( \blacksquare \)
4. Kolmogorov equations of affine type

Hence, under continuity assumption, both the formulations

\[ P_t f(x) = f(x) + \int_0^t P_s A f(x) ds \]

and

\[ P_t f(x) = f(x) + \int_0^t A P_s f(x) ds \]

are well defined for all \( t \geq 0 \). In line with the literature, we refer to the first one as Dynkin’s formula and the second one as Kolmogorov equation.

Combining together martingale property and commutative property between the transition semigroup and the extended generator we get

**Corollary 4.3 (Dynkin's formula)** Under the assumptions in Theorem 4.2, for any \( t, s \geq 0 \), it holds

\[ E^x f(X_t) = f(x) + t \int_0^1 E^x [A f(X_{rt})] dr. \] (4.3)

Dynkin’s formula can be iterated by taking the successive powers of the infinitesimal generator. Define iteratively:

\[ A^0 f = f, \]

\[ A^{n+1} f = A(A^n f) \quad \text{for } n \geq 0. \] (4.4)

Then we following holds

**Proposition 4.4 (Iterated Dynkin’s formula)** Let \( X \) be a Markov process and \( \nu \in \mathbb{N} \). For all \( f \in D(A) \) such that, for all \( n = 0, \ldots, \nu \), \( A^{\nu+1} f \in \mathcal{M} \), it holds

\[ E^x f(X_t) = f(x) + \sum_{k=1}^{\nu} \frac{t^k}{k!} A^k f(x) \]

\[ + \frac{t^{\nu+1}}{\nu!} \int_0^1 (1-r)^\nu E^x [A^{\nu+1} f(X_{rt})] dr. \] (4.5)

**Proof** The proof is done by induction on \( \nu \). For \( \nu = 0 \)

\[ E^x f(X_t) = f(x) + t \int_0^1 E^x [A^1 f(X_{st})] ds, \]

coincides with (4.3). Suppose now that the formula holds for \( \nu - 1 \) and we prove it.
for \( \nu \). We can write the right hand side in (4.5) as

\[
f(x) + \sum_{k=1}^{\nu} \frac{t^k}{k!} A^k f(x) + \frac{\nu+1}{\nu!} \int_0^1 (1-s)^\nu E \left[ A^{\nu+1} f(X_{st}) \right] ds
\]

where in the last step we did integration by parts on the integral term.

\[
\text{Corollary 4.5 (Short–time asymptotic formula) Under the same assumptions as in Proposition 4.4, if moreover } t \mapsto P_t A^1 f \text{ is continuous at } t = 0 \text{ the following expansion of the transition semigroup holds:}
\]

\[
P_t f(x) = f(x) + \sum_{k=1}^{\nu} \frac{t^k}{k!} A^k f(x) + \frac{\nu+1}{\nu!} \int_0^t (1-s)^\nu A^{\nu+1} P_s f(x) ds.
\]

4.3 Markov semigroup on weighted spaces

4.3.1 Functions with controlled growth

\[
\text{Definition 4.6 Given a left-continuous, increasing function } \rho : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{> 0} \text{ with } \lim_{u \to \infty} \rho(u) = +\infty, \text{ fix } \eta \in \mathbb{R}_{\geq 0} \text{ and define}
\]

\[
F_\eta(x) : D \to \mathbb{R}_{\geq 0}
\]

\[
x \mapsto \rho(\eta|x|).
\]
4. Kolmogorov equations of affine type

A function $f : D \rightarrow \mathbb{R}$ is a continuous function with growth controlled by $F_\eta$ if there exists a constant $C$ such that

$$|f(x)| \leq CF_\eta(x).$$

The space of all continuous functions with growth controlled by $F$ will be denoted by $C_F$:

$$C_F := \left\{ f \in C \mid \exists C > 0, \eta > 0 |f(x)| \leq CF_\eta(x), \text{ for all } x \in D \right\}. \tag{4.8}$$

**Definition 4.7** Observe that, for each $f \in C_F$, there exists a couple $(C, \eta)$ such that $|f(x)| \leq CF_\eta(x)$. We call $(C, \eta)$ a good couple for $f$.

In the space $C_F$ we introduce the norm

$$||f||_{C_F} := \inf\{C > 0 \mid (C, \eta) \text{ is a good couple for } f \text{ and } |f(x)| \leq CF_\eta(x)\}.$$

**Lemma 4.8** Let $X$ be a time homogeneous Markov process. Given $f \in C_F$ with good couple $(||f||_{C_F}, \eta)$, suppose that, for all $t \in [0, T]$ and $x \in D$, it holds $\mathbb{E}^x\left[F_{\eta}(X_t)\right] < \infty$. Then

$$\lim_{t \to 0^+} \mathbb{E}^x\left[f(X_t)\right] = f(x).$$

**Proof** For any $x \in D$, let $R$ be a constant such that $|x| < R$. We decompose

$$|\mathbb{E}^x[f(X_t)] - f(x)| \leq \mathbb{E}^x\left[|f(X_t) - f(x)|\mathbb{I}_{\{|X_t| \leq R\}}\right] + \mathbb{E}^x\left[|f(X_t)|\mathbb{I}_{\{|X_t| > R\}}\right] + f(x)\mathbb{P}^x(|X_t| > R).$$

The first term can be made arbitrarily small by weak convergence. For $R$ big enough it holds

$$\mathbb{E}^x\left[|f(X_t)|\mathbb{I}_{\{|X_t| > R\}}\right] \leq ||f||_{C_F} \mathbb{E}^x\left[F_{\eta}(X_t)\mathbb{I}_{\{|X_t| > R\}}\right].$$

Moreover,

$$\mathbb{P}^x(|X_t| > R) \leq \frac{1}{\rho(\eta R)} \mathbb{E}^x\left[F_{\eta}(|X_t|)\right].$$

Both terms go to zero as $R$ goes to infinity. \hfill \blacksquare
4.3.2 Differentiable functions with controlled growth

Recall that, given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, with the symbol $\partial^\alpha_x$ we denote the mixed partial derivative of order $|\alpha| = \alpha_1 + \ldots + \alpha_d$

$$\partial^\alpha_x := \frac{\partial |\alpha|}{\partial x_1^\alpha_1 \partial x_2^\alpha_2 \cdots \partial x_d^\alpha_d}.$$

**Definition 4.9** Fix a weight function $F$ and a constant $\eta$. A function $f$ is $k$-times differentiable with growth controlled by $F$ if, for each multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ there exist two constants $C_\alpha > 0$ and $\eta_\alpha > 0$ with $\eta_\alpha \in (0, \eta)$ such that, for all $x \in D$,

$$|\partial^\alpha_x f(x)| \leq C_\alpha F_{\eta_\alpha}(x). \quad (4.9)$$

The space of all the functions which are $k$-times differentiable with growth controlled by $F$ will be denoted by $C^k_F$:

$$C^k_F := \left\{ f \in C^k \mid \text{ for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leq k, \exists C_\alpha > 0, \eta_\alpha > 0 \text{ such that } |f(x)| \leq C_\alpha F_{\eta_\alpha}(x) \text{ for all } x \in D \right\}. \quad (4.10)$$

Therefore, for any $\alpha \in \mathbb{N}^d$, there exists a couple $(C_\alpha, \eta_\alpha)$ satisfying (4.9). Let $\eta_\infty := \max_\alpha \eta_\alpha$. The following norm

$$\|f\|_{C^k_F} := \inf \{ C > 0 \mid |\partial^\alpha f(x)| \leq C F_{\eta_\infty}(x) \text{ for all } \alpha \in \mathbb{R}^d \text{ with } |\alpha| \leq k \}$$

is well defined. This definition extends for smooth functions. We define

$$C^\infty_F := \{ f \in C^\infty \mid \text{ for all } \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, \eta_\alpha > 0 \text{ such that } |\partial^\alpha f(x)| \leq C_\alpha F_{\eta_\alpha}(x) \text{ for all } x \in D \}. \quad (4.11)$$

In line with [Alf10], we call $(C_\alpha, \eta_\alpha)_{\alpha \in \mathbb{N}^d}$ a good sequence. Given $f \in C^\infty_F$ with good sequence $(C_\alpha, \eta_\alpha)$, necessarily there exists $\eta_\infty := \max_\alpha \eta_\alpha$. This implies, in particular, that it is possible to find a constant $\eta_\infty$ such that all the partial derivatives of any orders have growth controlled by $F_{\eta_\infty}$. Therefore the following norm is well defined

$$\|f\|_{C^\infty_F} := \inf \{ C > 0 \mid |\partial^\alpha f(x)| \leq C F_{\eta_\infty}(x) \text{ for all } \alpha \in \mathbb{R}^d \}.$$
Given a function \( f \in C^\infty_F \), we want to exploit conditions under which the process \( M^f \) defined in (4.1) is actually a true martingale. We first add conditions under which, given a weight function \( F \), it holds \( C^\infty_F \subseteq D(\mathcal{A}) \). Then, by definition, \( M^f \) is a local martingale. Obviously, some additional conditions need to be added in order to conclude that the process is a true martingale. We start here with a set of some general conditions which guarantee square integrability of \( M^f \).

**Proposition 4.10** Let \( X \) be a time–homogeneous Markov process with extended generator \( \mathcal{A} \). Suppose that for some \( \eta^* > 0 \)

1. there exists a constant \( K \) such that, for all \( x \in D \),
   
   \[ |A f(x)| \leq K F_{\eta^*}(x), \]

2. for all \( x \in D \), it holds
   
   \[ \mathbb{E}^x \left[ \sup_{t \in [0,T]} F_{\eta^*}^2(X_t) \right] < \infty. \]

Then, for all \( f \in C^\infty_F \) such that \( \eta_{\infty} < \eta^* \), the process \( M^f \) is a true martingale.

**Proof** Since \( f \in D(\mathcal{A}) \) and \( P_t|f| < \infty \), by definition of extended generator, \( M^f \) is a local martingale. Hence, there exists an increasing sequence of stopping times with \( \lim_{n \to \infty} \tau_n = \infty \) \( \mathbb{P}^x \)-a.s. such that \( (M^f_{t \wedge \tau_n})_{t \geq 0} \) are martingales for all \( n \in \mathbb{N} \). Since \( f \in C^\infty_F \), there exist two constants \( C > 0 \) and \( \eta_{\infty} > 0 \) such that \( |f(x)| \leq C F_{\eta_{\infty}}(x) \). Henceforth, \( \tilde{C} \) is a constant which may vary from line to line. For \( t \in [0,T] \), it holds

\[
|M^f_{t \wedge \tau_n}|^2 = |f(X_{t \wedge \tau_n}) - f(x) - \int_0^{t \wedge \tau_n} A f(X_s) ds|^2 \\
\leq \tilde{C} \left( F_{\eta_{\infty}}^2(X_{t \wedge \tau_n}) + F_{\eta_{\infty}}^2(x) + \int_0^t F_{\eta^*}^2(X_{s \wedge \tau_n}) ds \right).
\]

Taking the expectations

\[
\mathbb{E}^x \left[ |M^f_{t \wedge \tau_n}|^2 \right] \leq \tilde{C} \left( \mathbb{E}^x \left[ F_{\eta_{\infty}}^2(X_{t \wedge \tau_n}) \right] + \mathbb{E}^x \left[ F_{\eta_{\infty}}^2(x) \right] + \mathbb{E}^x \left[ \int_0^t F_{\eta^*}^2(X_{s \wedge \tau_n}) ds \right] \right) \\
\leq \tilde{C} \left( 1 + t \mathbb{E}^x \left[ \sup_{s \in [0,t]} F_{\eta^*}^2(X_s) \right] \right).
\]

Using the second assumption, we see that, for all \( x \in D \), there exists a constant \( C_x \)
such that, for all \( n \in \mathbb{N} \) and \( t \in [0, T] \),

\[
\mathbb{E}^x \left[ |M_{t \wedge \tau_n}^f|^2 \right] \leq C_x.
\]

Using Doob’s inequality we conclude that

\[
\mathbb{E}^x \left[ \sup_{t \in [0, T]} |M_{t \wedge \tau_n}^f| \right] \leq \tilde{C} \mathbb{E}^x \left[ |M_{T \wedge \tau_n}^f|^2 \right].
\]

By monotone convergence theorem we conclude that

\[
\mathbb{E}^x \left[ \sup_{t \in [0, T]} |M_{t}^f|^2 \right] < \infty,
\]

from where square integrability of the \( M_{t}^f \) follows. ■

### 4.3.3 Lévy–type operators on weighted spaces

Let \( L \) be a Lévy process with Lévy triplet \((\mathbf{b}, \sigma, \nu)\). Henceforth, the extended generator of a Lévy process is denoted by \( \mathcal{L} \).

For all \( f \in \mathcal{D}(\mathcal{L}) \),

\[
\mathcal{L} f(x) = \langle \mathbf{b}, \nabla f(x) \rangle + \frac{1}{2} \text{Tr}(\sigma D^2 f(x))
\]

\[
+ \int_{\mathcal{D} \setminus \{0\}} (f(x + \xi) - f(x) - \langle h(\xi), \nabla f(x) \rangle) \nu(d\xi),
\]

where \( h(\xi) \) is a fixed truncation function.

**Proposition 4.11** Let \( L \) be a Lévy process on \( \mathbb{R}^d \) with Lévy triplet \((\mathbf{b}, \sigma, \nu)\) and denote by \( \mathcal{L} \) its extended generator. Assume that there exists \( \eta^* \) such that

\[
\int_{\{ |\xi| \geq 1 \}} (|\xi|^2 \wedge F_{\eta^*}(\xi)) \nu(d\xi) < \infty.
\]

For all \( f \in C^\infty_F \) such that \( \eta_\infty < \eta^* \)

(i) \( f \in \mathcal{D}(\mathcal{L}) \) and \( \mathcal{L} f \in C^\infty_F \),
(ii) if $M^f$ is a true martingale, for any fixed $\nu \in \mathbb{N}$, the following expansion holds

$$
\mathbb{E}^y \left[ f(L_s) \right] = f(y) + \sum_{n=1}^{\nu} \frac{s^n}{n!} \mathcal{L}^n f(y) + \frac{s^{\nu+1}}{\nu!} \int_0^1 (1-r)^\nu \mathbb{E}^y \left[ \mathcal{L}^{\nu+1} f(L_{rs}) \right] dr,
$$

where the operators $\mathcal{L}^n, n=1,\ldots,\nu+1$ are defined in (4.4).

**Proof** From Theorem I.4.57 in [JS87], $f(L)$ reads

$$
f(L_s) = f(y) + \int_0^s \langle \nabla f(L_r^-), b \rangle \, dr + \int_0^s \sum_{i=1}^d \partial_{y_i} f(L_r^-) \sum_{j=1}^d \sqrt{\sigma_{ij}} \, dW^j_r,
$$

$$
+ \frac{1}{2} \int_0^s \operatorname{Tr}(D^2 f(L_r^-) \sigma) \, dr
+ \int_0^s \int \left( f(L_r^- + \xi) - f(L_r^-) \right) \left( \mathcal{J}^L(dr, d\xi) - \nu(d\xi) dr \right)
+ \int_0^s \int \left( f(L_r^- + \xi) - f(L_r^-) - \langle h(\xi), \nabla f(L_r^-) \rangle \right) \nu(d\xi) dr.
$$

Since the second and the fourth term on the right hand side are predictable processes of finite variation, $f(L)$ is a special semimartingale. Hence

$$
M^f_s = f(y) + \int_0^s \sum_{i=1}^d \partial_{y_i} f(L_s^-) \sum_{j=1}^d \sqrt{\sigma_{ij}} \, dW^j_r
+ \int_0^s \int \left( f(L_r^- + \xi) - f(L_r^-) \right) \left( \mathcal{J}^L(dr, d\xi) - \nu(d\xi) dr \right),
$$

is a true martingale by assumption and the extended generator $\mathcal{L}$ is given by

$$
\mathcal{L} f(y) = \langle b, \nabla f(y) \rangle + \frac{1}{2} \operatorname{Tr}(\sigma D^2 f(y))
+ \int \left( f(y + \xi) - f(y) - \langle h(\xi), \nabla f(y) \rangle \right) \nu(d\xi).
$$

The following estimate holds

$$
\| \mathcal{L} f \|_{C^\infty} \leq K \| f \|_{C^\infty},
$$

where $K$ is a constant which depends only on the triplet and on the index $\eta^\infty$. Since the growth of the function does not change when the operator $\mathcal{L}$ is applied, martingale property of $M^f$ can be done using Proposition 4.10 with $\eta^* = \eta^\infty$. 

124
By iterating the above estimates, we get
\[ \|L^n f\|_{C^\infty_F} \leq K^n \|f\|_{C^\infty_F}. \] (4.13)

The following result will be used in the following sections.

**Corollary 4.12** Let \( R_\nu f(y, s) \) be the remainder of order \( \nu \) in (4.12),
\[ R_\nu f(y, s) := \frac{s^{\nu+1}}{\nu!} \int_0^1 (1 - r)\nu \mathbb{E}^y \left[ L^{\nu+1} f(L_{rs}) \right] dr. \]

Using the same notations of Proposition 4.11, for \( s \) small, there exists a constant \( C_{\nu, f} \) such that
\[ |R_\nu f(y, s)| \leq s^{\nu+1} C_{\nu, f} F_{\eta^*}(y). \]

**Proof** Since, by assumption,
\[ |\partial^\alpha_x f(x)| \leq \|f\|_{C^\infty_F} F_{\eta^\infty}(x), \text{ for all } \alpha \in \mathbb{N}^d, \]
(4.13) implies
\[ \mathbb{E}^y \left[ |L^{\nu+1} f(L_s)| \right] \leq K^{\nu+1} \|f\|_{C^\infty_F} \mathbb{E}^y \left[ F_{\eta^*}(L_s) \right]. \] (4.14)

Moreover, using Dynkin’s formula,
\[ \mathbb{E}^y \left[ F_{\eta^\infty}(L_s) \right] = F_{\eta^\infty}(y) + \mathbb{E}^y \left[ \int_0^s L F_{\eta^\infty}(L_u) du \right] \leq F_{\eta^\infty}(y) + K \mathbb{E}^y \left[ \int_0^s F_{\eta^\infty}(L_u) du \right]. \]
By Gronwall’s inequality
\[ \mathbb{E}^y \left[ F_{\eta^\infty}(L_s) \right] \leq e^{Ks} F_{\eta^\infty}(y). \] (4.15)

Combining (4.14) and (4.15)
\[ \mathbb{E}^y \left[ |L^{\nu+1} f(L_s)| \right] \leq K^{\nu+1} e^{Ks} \|f\|_{C^\infty_F} F_{\eta^\infty}(y). \] (4.16)
We can estimate
\[
|R^\nu f(y, s)| \leq s^{\nu+1} \frac{1}{\nu!} \int_0^1 (1 - r)\nu E^y \left[ |L^{\nu+1} f(L_sr)\right] dr
\]
\[
\leq s^{\nu+1} K^{\nu+1} ||f||_{C^\infty} F_{\eta\infty}(y) \int_0^1 (1 - r)\nu e^{Kr} dr
\]
\[
\leq s^{\nu+1} C_{\nu,f} F_{\eta\infty}(y)
\]
\[
\leq s^{\nu+1} C_{\nu,f} F_{\eta}(y).
\]

4.4 Regularity results for affine–type operators

In the previous sections we made essentially two main assumptions on the function space \(M\). The first one is continuity at time \(t = 0\) for the transition semigroup \(t \mapsto P_t f(x)\) when \(f \in M\) and the second one is martingale property of the process \(M^f\) for \(f \in M\). We have seen that continuity of the transition semigroup can be achieved once we have enough integrability of the distributions so that a dominate convergence type theorem can be applied. Here we will restrict ourselves on subsets of weighted spaces where martingale property of \(M^f\) holds.

Under these conditions, we found that it is possible derive differentiability in time of the function \(u(t, x)\) by means of an iterated versions of the Dynkin’s lemma. Since we have found a successful way to approach regularity in time, it is desirable to apply results in the previous section also for the analysis of regularity in space.

We recall here some results we have seen in the previous sections.

**Proposition 4.13** For any fixed \(t \geq 0\) and \(x \in D\), the following are equivalent

(i) \(X_t^x\) is the value at time \(t\) of an affine process starting from \(x\) with

\[
E^x \left[ e^{\langle u, X_t \rangle} \right] = e^{\langle x, \Psi(t,u) \rangle}, \quad t \geq 0, u \in U,
\]  

(ii) there exists a Levy process \(L^{(t,x)}\) such that

\[
E^x \left[ e^{\langle u, X_t \rangle} \right] = e^{\langle x, \Psi(t,u) \rangle} = E \left[ e^{\langle L^{(t,x)}_t, u \rangle} \right], \quad u \in U,
\]
(iii) in distribution it holds

\[ X_t \overset{d}{=} L_{x_1}^{(t,e_1)} + \ldots + L_{x_d}^{(t,e_d)}, \]

where \( e_k \) are the canonical coordinates in \( \mathbb{R}^d \)

\[ e_1 := (1, 0, \ldots, 0)^\top, \ldots, e_d := (0, \ldots, 0, 1)^\top. \]

Each \( L^{(t,e_k)} \), \( k = 1, \ldots, d \), is a semimartingale with state space \( D \), by construction. Its semimartingale characteristics, relative to a truncation function \( h \), admit a version of the following form

\[ (s b_k(t), s \sigma_k(t), s \nu_k(t, d\xi)), \]

where \( (b_k(t), \sigma_k(t), \nu_k(t)) \) is a Lévy triplet for each \( k = 1, \ldots, d \).

**Notation 4.14** Let \( L^{(t,e_1)}, \ldots, L^{(t,e_d)} \) be \( d \) independent Lévy processes each of them representing \( X^{e_i} \) for \( i = 1, \ldots, d \). Henceforth, the following notation will be used:

- the extended generator of \( L^{(t,e_i)} \) is denoted by \( \mathcal{L}^{(t,e_i)} \),
- its Markov semigroup is denoted by

\[ \mathbb{E}^y f(L_s^{(t,x)}) = Q^{(t,x)}_s f(y), \quad (4.18) \]

for all \( (t,x), (s,y) \in \mathbb{R}_0^+ \times D \) and \( f \in C_b(D) \).

Unless differently specified, the notation \( Q \) (resp. \( \mathcal{L} \)) denotes the Markov semigroup (resp. the extended generator) of a Lévy process, while the notations \( P \) and \( \mathcal{A} \) are reserved for the same quantities for an affine process.

**Lemma 4.15** Fix \( t \geq 0 \) and \( x \in D \). Suppose that \( \mathbb{E}^x \left[ e^{\langle y, X_t \rangle} \right] < \infty \) for all \( y \in \mathbb{R}^d \).

Let \( F_{\eta^*} \) be a weight functions such that, for some \( y \in \mathbb{R}^d \), it holds

\[ \sup_{x \in D} F_{\eta^*}(x) e^{-\langle y, x \rangle} < \infty. \]

Then, for all \( f \in C_b^\infty \) with \( \eta_{\infty} \leq \eta^* \) it holds \( Q_s^{(t,x)} |f| < \infty \) for any fixed \( (t,x) \in \mathbb{R}_0^+ \times D \) and all \( s \geq 0 \).

**Proof** Let \( q_t(x, \cdot) \) denote the distribution at time 1 of the representing Lévy process \( L^{(t,x)} \). Then, by assumption, it holds \( \int_D e^{\langle y, \xi \rangle} q_t(x, d\xi) < \infty \) for all \( y \in \mathbb{R}^d \). Since
4. **Kolmogorov equations of affine type**

$q_t(x, dξ)$ is infinitely divisible, from Theorem 25.3 in [Sat99], we conclude that $L^{(t,x)}$ is a Lévy process with Lévy measure $ν(t, x, ·)$ satisfying

$$\int_{|ξ|≥1} e^{⟨y, ξ⟩} ν(t, x, dξ) < ∞,$$

for all $y ∈ \mathbb{R}^d$. Moreover, finiteness of the exponential moments is not a time dependent property and therefore $∫_D e^{⟨y, ξ⟩} q_t(sx, dξ) < ∞$ for all $s ≥ 0$ and $y ∈ \mathbb{R}^d$. Hence, if $f$ satisfies the above mentioned growth condition,

$$Q^{(t,x)}_s|f|(z) = ∫ |f(z + ξ)| q_t(sx, dξ) \leq ∫ |f(ξ)| q_t(sx, dξ - z) \leq ∫ F_η^*(ξ) q_t(sx, dξ - z).$$

From spacial homogeneity we conclude that this quantity is finite. ■

**Lemma 4.16** For any $f : D → \mathbb{R}$ such that $Ε^x [ |f(X_t)| ] < ∞$ for all $t ≥ 0$ and $x ∈ D$, the following representation holds

$$u(t, x + hy) = Ε^x [ v^{(t,y)}(h, X_t) ], \quad h > 0, \ y ∈ D.$$

**Proof** For any $x, y ∈ D$ and $t, h > 0$ it holds

$$Ε^{x+hy} [ f(X_t) ] = ∫_D f(ξ) p_t(x + hy, dξ) = ∫_D f(ξ) (p_t(x, ·) * p_t(hy, ·))(dξ) = ∫_D f(ξ) ∫_D p_t(x, dξ - η) p_t(hy, dη) = ∫_D ∫_D f(ξ + η) p_t(hy, dη)p_t(x, dξ) = ∫_D Ε [ f(ξ + L^{(t,y)}_h) ] p_t(x, dξ) = Ε^x [ v^{(t,y)}(h, X_t) ].$$

128
4.4. Regularity results for affine–type operators

4.4.1 Results on \( C^{\infty}_{\text{pol}} \)

In the field of weak approximation of SDE it is essential to have conditions which guarantee that the convergence error obtained for a certain numerical scheme in a small time horizon can be “well propagated” up to a fixed time horizon. Regularity of the Kolmogorov equation with small initial data with polynomial growth allows to control this error (see [Alf10], [TT90] for example).

We first define the following function spaces:

**Definition 4.17** A function \( f \in C^{k}_{\text{pol}} \) if

- \( f \in C^{k} \)
- for all \( \alpha \) multi-index with \( |\alpha| \leq k \), there exist constants \( C_{\alpha} \) and \( \eta_{\alpha} \) such that
  \[ |\partial^{\alpha} f(x)| \leq C_{\alpha}(1 + |x|^{2\eta_{\alpha}}), \quad \text{for all } x \in D. \]

In case the function \( f \) is smooth we can extend the previous definition by taking all the possible derivatives and define

\[ C^{\infty}_{\text{pol}}(D) = \left\{ f \in C^{\infty}(D), \text{ for all } \alpha \in \mathbb{N}^{d} \exists C_{\alpha} > 0, \eta_{\alpha} \in \mathbb{N} \right. \]
\[ \left. \text{such that } |\partial^{\alpha} f(x)| \leq C_{\alpha}(1 + |x|^{2\eta_{\alpha}}) \text{ for all } x \in D \right\}. \]

The first step is to check the basic properties.

**Lemma 4.18** Under the assumption that there exists a \( T > 0 \) such that

\[ \mathbb{E}^{x} \left[ e^{\langle y, X_{T} \rangle} \right] < \infty \text{ for all } y \in \mathbb{R}^{d} \] \hspace{1cm} (4.19)

1. \( t \mapsto P_{t}f \) is continuous at \( t = 0 \) for all \( f \in C^{\infty}_{\text{pol}} \),
2. \( P_{t}C^{\infty}_{\text{pol}} \subseteq C_{\text{pol}} \).

**Proof** We first check that \( P_{t}|f| \) is finite for all \( f \in C^{\infty}_{\text{pol}} \). By assumption, there exist \( C > 0 \) and \( \eta > 0 \) such that \( |f(x)| \leq C(1 + |x|^{2\eta}) =: C\Pi_{2\eta}(x) \), where \( \Pi_{2\eta}(x) \) is a polynomial of order \( 2\eta \). From the estimates in Theorem 2.10 in [CKT08] we know that there exists a \( K > 0 \) such that

\[ \mathbb{E}^{x} \left[ \Pi_{2\eta}(X_{t}) \right] \leq Ce^{Kt}F_{2\eta}(x), \quad t \geq 0, \]
with $F_{2n}(x) := \left(1 + \sum_{i=1}^{d} x_{i}^{2n}\right)$. Since $F_{2n}(x) \leq \Pi_{2n}(x)$, we get integrability of $P_{t}f$. This, together with Lemma 4.8, implies continuity at $t = 0$ of $t \mapsto P_{t}f$. From the previous inequality we do also get the polynomial growth. Now we move on continuity. Since $f \in C_{pol}^\infty$, there exists a $y \in \mathbb{R}^{d}$ such that

$$g(x) := e^{-(y,x)}f(x) \in C_{C}^\infty.$$ 

Let $q_{t}(x, d\xi) := e^{(y,\xi)}p_{t}(x, d\xi)$ and observe that it is a finite measure for all $(t, x) \in [0, T] \times D$. This follows from Lemma 4.2. in [KM11]. Consider the shifted semigroup $\tilde{q}_{t}(x, d\xi) := q_{t}(x, d\xi + x)$. Then we can write

$$\mathbb{E}^{x}\left[f(X_{t})\right] = \int g(\xi)q_{t}(x, d\xi)$$

$$= \int g(\xi + x)\tilde{q}_{t}(x, d\xi).$$

Denote with $\mathcal{F}g$ the Fourier transform of $g$

$$\mathcal{F}g(v) := \int e^{i(v, x)}g(x)dx.$$ 

Since $\mathcal{F}g(v) \in L^{1}(\mathbb{R}^{d})$ there exists a continuous density $G \in L^{1}(\mathbb{R}^{d})$ for $g$. Therefore

$$\lim_{|z| \to 0} P_{t}f(x + z) = \lim_{|z| \to 0} \int g(\xi + x + z)\tilde{q}_{t}(x, d\xi)$$

$$= \int \lim_{|z| \to 0} g(\xi + x + z)\tilde{q}_{t}(x, d\xi)$$

$$= \int g(\xi + x)\tilde{q}_{t}(x, d\xi) = P_{t}f(x),$$

where we used the existence of the density. 

**Theorem 4.19** Let $X$ be an affine process such that, for all $y \in \mathbb{R}^{d}$ and $x \in D$, $\mathbb{E}^{x}\left[e^{(y,X_{T})}\right] < \infty$ for some fixed $T > 0$. Then it holds

(i) $AC_{pol}^\infty \subseteq C_{pol}^{\infty}$,

(ii) for any $f \in C_{pol}^{\infty}$, $P_{t}f$ solves the Kolmogorov’s equation

$$\partial_{t}u(t, x) = Au(t, x),$$

$$u(0, x) = f(x),$$

for $(t, x) \in [0, T] \times D$, 130
(iii) for any \( f \in C^\infty_{pol} \) and \( \nu \in \mathbb{N} \) the following expansion of the transition semigroup holds for \( (t, x) \in [0, T] \times D \):

\[
\mathbb{E}^x \left[ f(X_t) \right] = f(x) + \sum_{k=1}^{\nu} \frac{t^k}{k!} A^k f(x) + R_\nu f(x, t),
\]

where \( R_\nu f(x, t) \) is a remainder of order \( O(t^{\nu+1}) \).

**Proof** Using linearity in \( x \) of the coefficients, we decompose

\[
A f(x) = \sum_{i=1}^{d} x_i A^{(i)} f(x),
\]

where each \( A^{(i)} \) is an operator of Lévy-type. For every \( i = 1, \ldots, m \),

\[
A^{(i)} f(x) = \sum_{k=1}^{d} (\beta_i)_k \partial_{x_k} f(x) + \frac{1}{2} \sum_{k,h=1}^{d} (\alpha_i)_k h \partial_{x_k} \partial_{x_h} f(x) + \int (f(x + \xi) - f(x) - \langle h(\xi), \nabla f(x) \rangle) M_i(d\xi).
\]

From the integrability assumption, it follows that \( \int_{\{ |\xi| \geq 1 \}} e^{\langle y, \xi \rangle} M_i(d\xi) < \infty \), for all \( i = 1, \ldots, m \) and \( y \in \mathbb{R}^d \) (see Theorem 2.14 in [KM11]). Hence

\[
\left| \int_{\{ |\xi| \geq 1 \}} (f(x + \xi) - f(x)) M_i(d\xi) \right| \leq C_1 (1 + |x|^{2m}) \int_{\{ |\xi| \geq 1 \}} |\xi| M_i(d\xi),
\]

\[
\left| \int_{\{ |\xi| \leq 1 \}} (f(x + \xi) - f(x) - \langle \xi, \nabla f(x) \rangle) M_i(d\xi) \right| \leq C_2 (1 + |x|^{2m}) \int_{\{ |\xi| \leq 1 \}} |\xi|^2 M_i(d\xi).
\]

Moreover, since \( f \in C^\infty_{pol} \) there exist two constants \( C \) and \( E \) such that

\[
|f(x)| + \sum_{i=1}^{d} |\partial_{x_i} f(x)| + \sum_{i,j=1}^{d} |\partial^2_{x_ix_j} f(x)| \leq C (1 + |x|^{2E}). \tag{4.20}
\]

Combining the bound on the diffusive part and the jump part, we conclude that there exist two constants \( K \) and \( E \) such that

\[
|A^{(i)} f(x)| \leq K (1 + |x|^{2E}), \quad \text{for all } i = 1, \ldots, d.
\]
4. Kolmogorov equations of affine type

Then
\[ |\mathcal{A}f(x)| \leq K(1 + |x|^{2E})|x| \leq \overline{K}(1 + |x|^{2(E+1)}). \]
This concludes (i).

In order to apply Theorem 4.2, we still need to prove that, for \( f \in C^\infty_{\text{pol}} \),
- process \( M^f \) is a true martingale,
- \( t \mapsto P_tf(x) \) is continuous at \( t = 0 \).

We start with the martingale property. Let \( (C,E) \) be the couple defined in (4.20).
Since \( |\mathcal{A}f| \leq K(1 + |x|^{2(E+1)}) \), in order to apply Proposition 4.10, it remains to check that
\[
\mathbb{E}^x\left[ \sup_{t \in [0,T]} |X_t|^{2(E+1)} \right] < \infty.
\]
By Lemma 2.17 in [CKT08], there exist two constants \( K \) and \( C \) such that
\[
\mathbb{E}^x\left[ \sup_{t \in [0,T]} |X_t|^{2(E+1)} \right] \leq K e^{CT},
\]
if the kernel \( K(x, d\xi) = x_1 M_1(d\xi) + \ldots + x_d M_d(d\xi) \) satisfies
\[
\int_{\{|\xi| \geq 1\}} |\xi|^{2(E+1)} K(X_t, d\xi) \leq (1 + |X_t|^{2(E+1)}).
\] (4.21)
By assumption, for all \( i = 1, \ldots, d, \)
\[
\int_{\{|\xi| \geq 1\}} |\xi|^{2(E+1)} M_i(d\xi) < \infty.
\]
Hence
\[
\int_{\{|\xi| \geq 1\}} |\xi|^{2(E+1)} K(X_t, d\xi) = \int_{\{|\xi| \geq 1\}} |\xi|^{2(E+1)} \sum_{i=1}^d X_t^{(i)} M_i(d\xi) \leq C(1 + |X_t|)
\]
and therefore the (4.21) holds. Now that martingale property has been proved, it remains to check that \( t \mapsto P_tf(x) \) is continuous at \( t = 0 \). This follows from the integral equation
\[
P_tf(x) = f(x) + \int_0^t P_s \mathcal{A}f(x) ds,
\]
paired with the fact that \( \mathcal{A} \) maps \( C^\infty_{\text{pol}} \) into \( C^\infty_{\text{pol}} \). From Theorem 4.2 we conclude that for any \( f \in C^\infty_{\text{pol}} \) and \( P_tf \) solves the Kolmogorov’s equation. Finally (iii) follows as in Proposition 4.4 by considering that \( \mathcal{A} \) maps functions in \( C^\infty_{\text{pol}} \) into functions in...
4.4. Regularity results for affine–type operators

\[ C_{\text{pol}}^{\infty}, \quad k = 1, \ldots, \nu + 1. \]

To show

\[ P_tC_{\text{pol}}^{\infty} \subseteq C_{\text{pol}}^{\infty} \]

we will consider the decomposition

\[ X^{x+he_i, \text{law}} = X^x + \mathcal{X}^{he_i}, \quad h > 0, \ i = 1, \ldots, d, \]

where \( e_1, \ldots, e_d \) are the basis elements in \( \mathbb{R}^d \) and \( \mathcal{X}^{he_i} \) is an independent copy of the process \( X \) starting from \( he_i \).

**Theorem 4.20** Suppose that there exists a \( T > 0 \) such that \( \mathbb{E}^x \left[ e^{(y, X_T)} \right] < \infty \) for all \( y \in \mathbb{R}^d \). Then, all the partial derivatives exist and are continuous. Moreover it holds

\[ \partial_{e_i} u(t, x) = \mathbb{E}^x \left[ \mathcal{L}^{(t, e_i)} f(X_t) \right], \quad t \geq 0, \ x \in D. \quad (4.22) \]

**Proof** Let \( p_t(x, \cdot) \) be the distribution of \( X^x_t \) for \( t \in [0, T] \). From Lemma 4.2 (c) in [KM11], for all \( t \in [0, T] \),

\[ \int_D e^{(y, \xi)} p_t(x, d\xi) < \infty. \]

Recall that \( p_t(x, \cdot) \) are infinitely divisible measures. Denote by \( \nu_t(x, \cdot) \) their Lévy measures. From from Theorem 25.3 in [Sat99], it follows that

\[ \int_{\{|\xi| \geq 1\}} e^{(y, \xi)} \nu_t(x, d\xi) < \infty, \]

for all \( y \in \mathbb{R}^d \) and \( t \in [0, T] \). Due to linearity in \( x \), Proposition 4.11 can be applied to each \( L^{(t, e_i)} \). This allows us to get the following approximate for \( h \) small

\[ u(t, x + he_i) - u(t, x) = \mathbb{E}^x \left[ u^{(t, e_i)}(h, X_t) - u^{(t, e_i)}(0, X_t) \right] \leq h \mathbb{E}^x \left[ L^{(t, e_i)} f(X_t) \right] + O(h^2). \]

By taking the limit of the finite differences, we get existence of the derivatives. Additionally, since \( L^{(t, e_i)} \) maps functions in \( C_{\text{pol}}^{\infty} \) into functions in \( C_{\text{pol}}^{\infty} \), Lemma 4.18 2. leads to continuity of the derivatives.

Higher order partial derivatives can be taken analogously by applying (4.22) several times:
Proposition 4.21 Let $\alpha$ be a multi-index and $f \in C^\infty_{pol}$. Then, under the same assumptions as in Theorem 4.20,

$$\partial^\alpha_x u(t, x) = \mathbb{E}^x \left[ (\mathcal{L}^{(t,e_1)})^{\alpha_1} \cdots (\mathcal{L}^{(t,e_d)})^{\alpha_d} f(X_t) \right], \quad (4.23)$$

and therefore all the derivative exists. Moreover, they are continuous in $x$.

**Proof** The representation of the partial derivatives follows analogously to the case when $|\alpha| = 1$. From (4.23) we can write

$$\partial^\alpha_x u(t, x) = \mathbb{E}^x \left[ (\mathcal{L}^{(t,e_1)})^{\alpha_1} \cdots (\mathcal{L}^{(t,e_d)})^{\alpha_d} f(X_t) \right],$$

and since $P_t C^\infty_{pol} \subseteq C_{pol}$. ■

Proposition 4.22 Let $X^x$ be an affine process satisfying the assumptions in Theorem 4.20. Then given a function $f \in C^\infty_{pol}$, $P_t f(x)$ is again in $C^\infty_{pol}$ for all $t \geq 0$.

The following results is valuable in applications:

**Theorem 4.23** Let $f \in C^\infty_{pol}$. Then, the function $u : \mathbb{R}_{\geq 0} \times D \to \mathbb{R}$ defined by $u(t, x) = \mathbb{E}^x \left[ f(X_t) \right]$ is smooth, with all derivatives satisfying the following property:

$$\text{for all } (t, x) \in [0, T] \times D, \quad |\partial^\alpha_{(t,x)} u(t, x)| \leq K_\alpha(T) \left( 1 + |x|^{2\eta_\alpha(T)} \right), \quad (4.24)$$

where $K_\alpha(T)$ and $\eta_\alpha(T)$ are positive constants depending on the time horizon $T$ and the order of derivative $\alpha$.

**Proof** Let $\alpha$ be a multi-index running over the mixed derivatives with respect to time and space. Split $\alpha = (\alpha_0, \bar{\alpha})$ where $\alpha_0 \in \mathbb{N}$ is the order of derivation in time and $\bar{\alpha} \in \mathbb{N}^d$ is the multi-index for the derivatives in space. Clearly $\alpha_0 = |\alpha| - |\bar{\alpha}|$.

By induction on $\alpha_0$, when $\alpha_0 = 1$,

$$\partial_t \partial^\bar{\alpha}_x u(t, x) = A \partial^\bar{\alpha}_x u(t, x).$$

We need to check that the bound $\partial^\bar{\alpha}_x u(t, x)$ can be taken uniformly for $t \in [0, T]$. From the representation (4.23) and Corollary 4.12,

$$\partial^\bar{\alpha}_x u(t, x) = \mathbb{E}^x \left[ (\mathcal{L}^{(t,e_1)})^{\alpha_1} \cdots (\mathcal{L}^{(t,e_d)})^{\alpha_d} f(X_t) \right]$$

$$\leq \prod_{k=1}^d K(t, x_k)^{\alpha_k} C \left( 1 + |x|^{2\eta_\alpha(T)} \right),$$

where $K(t, x_k)$, $k = 1, \ldots, d$, depends only on the Lévy triplet of the distribution of $X^x_t$. Due to linearity in $x$, for all $t \in [0, T]$ there exists a constant $K$ depending
on the time horizon $T$ such that $K(t, x_k) \leq K(T)x_k$. Therefore

\[ \prod_{k=1}^{d} K(t, x_k)^{\alpha_k} \leq \prod_{k=1}^{d} K(T)^{\alpha_k} x_k^{\alpha_k} \leq K(\alpha)|x|^{|\alpha|}. \]

Using the fact that, for all $u > 0$ and $p, m \in \mathbb{N}$,

\[ u^p(1 + u^{2m}) \leq (1 + u^{2(m+p)}) \]

we get

\[ \partial_{x}^{\alpha}u(t, x) \leq K_{\alpha}(T)(1 + |x|^{2(\eta_{\alpha} + |\alpha|)}), \]

and the result follows by Theorem 4.19 (i). Now suppose that, for all $\alpha_0 = 1, \ldots, |\alpha| - |\eta| - 1$ it holds

\[ |\partial_{t}^{\alpha_0} \partial_{x}^{\alpha}u(t, x)| \leq K_{\alpha}(T)(1 + |x|^{2\eta_{\alpha}(T)}), \text{ for all } t \in [0, T]. \]

Then

\[ \partial_{t}^{\alpha_0 + 1} \partial_{x}^{\alpha}u(t, x) = \partial_{t} \left( \partial_{t}^{\alpha_0} \partial_{x}^{\alpha}u(t, x) \right) = A \partial_{t}^{\alpha_0} \partial_{x}^{\alpha}u(t, x) \]

and, by the induction step, the result follows again as an application of Theorem 4.19.
Conclusion

This thesis was set out to provide applicable results for option pricing when the underlying is driven by an affine processes. Recall that prices of contingent claims are computed as expectations of functionals of the underlying stock, where the expectation is taken under an equivalent martingale measure for the stock. Due to differentiability in time of their Fourier–Laplace transform, affine processes are well suited for Fourier methods. There are no doubts that the Fourier methodology, whenever applicable, is an efficient technique to compute the price of some European style options. The thesis tackles the problem of numerics for affine processes when the Fourier methods turn out to be not a good choice. If we keep aside Fourier methods, there are mainly two classes of numerical methods which can be applied: probabilistic methods, such as Monte Carlo, and/or deterministic methods, which consist on the approximation of the pricing PDE. In both cases, there are some complications, mainly due to the lack of regularity of the vector fields along the boundary of the state space.

On one side, the application of Monte Carlo techniques requires the construction of an approximation of the trajectories of the underlying process. Euler–Maruyama type schemes are the most natural approximation schemes. By means of an easy example (see the beginning of Section 3.1), we highlighted the main problems with this methodology. We stressed that, in our setting

P1) the requirements on the vector field which guarantee convergence are usually not fulfilled,

P2) the geometry of the state space is a priori not preserved.

On the PDE side, we observed that, the identification of the Markov semigroup with the solution of a Kolmogorov P(I)DE needs to be done with care. The reasons being that
the extended generator is a second order integro-differential equation which can be degenerate along the boundary of the state space and

the initial value of the Kolmogorov P(I)DE is a function which is not necessarily bounded.

While in Chapter 3 we focused on the construction of a numerical scheme for path approximations which obviate the the problems listed in P1) and P2), the main focus of Chapter 4 has been the study of conditions under which it is possible to identify a Markov transition semigroup with the solution of a Kolmogorov type equation, with the prescriptions given in D1) and D2). The cornerstone of both chapters is the representation of an affine process as a realization of a path–valued Levy process, presented in Chapter 2.

We started the thesis with an overview on affine processes. In the first part of Chapter 1, we outlined some basic results in this field and fix some common notations. In the second part, we introduced a set of transformations which lead to an easier to handle structure for affine processes. The most relevant one is the one induced by the $\otimes$ operator, presented in Proposition 1.23.

In Chapter 2, we showed how to depict the paths of an affine process as the realizations of a process with independent and stationary increments taking values in the path-space. We called them representing Levy processes. The representation in Proposition 2.2 arises from the branching property of the affine processes and extends the one dimensional findings for the genealogy of a branching process. In Theorem 2.5, we lifted this construction to the path-space. As a product, we gave a proof of affine property of the Fourier–Laplace transform for linear functionals of affine processes. This is done in Theorem 2.8, which is an extension of the one dimensional result in [PY82]. Observe that, this thesis is limited to the existence result for the path–space valued processes. It would be interesting, for future research, to deeper the analysis of this type of processes. Some interesting open issues are:

Q1) Is it possible to prove that the trajectories of the path–space valued representing Lévy processes are càdlàg?

Q2) Can we avail ourselves of these processes for pricing exotic options?

These questions conduct ourselves in merging the theory of affine processes from the path–space valued perspective with the theory of cylindrical Lévy processes in Banach spaces, as in [AR10]. We think that much can be done in this direction and that a positive answer to Q2) would be extremely valuable for applications.
The fact that the transition semigroup of an affine process can be interpret also as a function of the space variable played a pivotal role in our investigation of affine processes. Based on the theory of multivariate time change for Markov processes, in Theorem 2.16, we showed how to identify affine processes as solutions of certain time change equations. Precisely, the theorem shows how to construct the paths of an affine process from the paths of a family of Levy processes properly time–changed. The proof of this theorem relies on the existence of a vector of random time changes, which is adapted to the filtration generated by the Levy processes. Section 2.3.3 is devoted to the proof of its existence.

In only few cases we could explicitly compute the Lévy triplet of the representing Lévy processes. Even though we find very interesting the problem of computing further analytic solutions for the system of Riccati ODEs, in this thesis, we decided to crystallize those processes which are easy to simulate and to build, starting from them, numerical approximations for the trajectories of a general affine process. In Chapter 3, we tackle the problem of the weak approximation for affine processes from the point of view of the representing Levy processes. In Theorem 3.5, we proposed an approximation scheme for the one time marginal distributions of an affine process, which is based on an iterative composition of Levy type transition semigroups. From an analytic perspective, the scheme stems for an explicit Euler scheme for the system of Riccati ODEs identifying the Fourier-Laplace transform of affine processes. In Theorem 3.6 we see how to apply the previous theorem in order to approximate the finite—dimensional distributions of an affine process. In Section 3.3.3. we dealt with the analysis of the convergence rate for the proposed scheme. In Theorem 3.13, we showed how to achieve a convergence rate of order at least 2 on the class of smooth functions with polynomial growth, class of test functions commonly used for the analysis of convergence of weak approximation schemes. The choice of the class of smooth functions with polynomial growth is in agreement with the existing research in the area of high–order weak approximation for stochastic differential equations, see for example [TT90] or [Alf10].

The crucial aid used to get this result is the analysis of the convergence when the set of test functions is given by the class of Fourier modes, proved in Theorem 3.12. Recall that the Riccati ODEs originate from the Kolmogorov PIDE with initial value $f_u(x)$, with $u \in \mathbb{C}^d$ such that $f_u(x)$ is bounded. Hence, the scheme proposed in Chapter 3 falls in the class of high–order schemes based on a short–time asymptotic for the Markov transition semigroup, acting on the class of the Fourier modes.
However, in option pricing, we come across more general function spaces. We are in need of a well posed formulation of the Kolmogorov equation driven by an affine-type operator with an initial value which is not necessarily bounded. Additionally, it order to apply a Feynman–Kac style approach, it is necessary to establish smoothness of the transition semigroup as a function of the state variable. These are the reasons why Chapter 4 is devoted to the study of Markov semigroups acting on function spaces which consist of functions with controlled growth. In Theorem 4.2 we extended the well known connection (in the Feller setting) between the Markov semigroup and the solution of a Kolmogorov equation. In this theorem, we considered functions which belong to the domain of the extended generator and provided a framework for the analysis of the Kolmogorov equation. Additionally, a short-time asymptotic formula for the transition semigroup is provided in Proposition 4.4. In Section 4.3 we focused on the class of smooth functions with controlled growth and, in Section 4.4, we specialized the results obtained in the previous section when the Markov process is an affine process. The concluding result is Theorem 4.23, where we proved an essential property for the affine transition semigroup. Namely, we showed that the class of smooth functions with polynomial growth is left invariant under the action of an affine transition semigroup. This fact is highly relevant in the theory of convergence rates of weak approximation schemes (see, for example, the analysis done in [Alf10]). Despite our results apply for the analysis of the convergence rate of numerical schemes for the trajectories, it would be desirable to have the same type of results also for functions with exponential growth. The main reason why we could not extend the result so far is the lack of an adequate $B_\psi$ formulation of the Markov semigroup for an affine process. For the theory of generalized Feller property and $B_\psi$ formulations, we refer to [DT10]. The fundamental property used in the setting of functions with polynomial growth is the “polynomial property” of an affine process. The polynomial property ensures that the class of polynomials of a certain order is invariant under the action of the affine transition semigroup. We firmly believe that, Theorem 4.23 could be extended to a class of test functions which includes the functions with exponential growth, as soon as a result on an appropriate $B_\psi$ formulation for affine processes will be available in the literature. Unfortunately, by the time we completed this thesis, we could not succeed in finding a reasonable $B_\psi$ formulation.
Appendix A

On the Skorohod space

This survey on the Skorohod space is taken mainly from Section 3.5 in [EK86]. For additional details on the section about the dual space, we refer to [Pes95].

Let \((D, d)\) a complete separable metric space. The Skorohod space \(D(\mathbb{R}_\geq 0; D)\) consists of all functions \(x : \mathbb{R}_\geq 0 \to D\) which are right continuous and with left limits.

A.1 The Skorohod metric

In the space \(D(\mathbb{R}_\geq 0; D)\) it is possible to define a metric \(d_S\) such that \((D(\mathbb{R}_\geq 0; D), d_S)\) is again a complete and separable metric space. Let \(\Lambda\) be the set of all strictly increasing functions \(\lambda\) mapping \(\mathbb{R}_\geq 0\) onto \(\mathbb{R}_\geq 0\). We consider the subset \(\Lambda_L\) of Lipschitz continuous \(\lambda \in \Lambda\) such that

\[ \Gamma(\lambda) := \text{ess sup}_{t \geq 0} |\log \lambda'(t)| < \infty. \]

(A.1)

Given two elements \(x, y \in D(\mathbb{R}_\geq 0; D)\) define

\[ d(x, y, \lambda, s) := \sup_{t \geq 0} \left( x(t \land s), y(\lambda(t) \land s) \right) \wedge 1 \]

(A.2)

and then

\[ d_S(x, y) := \inf_{\lambda \in \Lambda_L} \left( \Gamma(\lambda) \vee \int_0^\infty e^{-s} d(x, y, \lambda, s) ds \right). \]

(A.3)
A. On the Skorohod space

Definition A.1 The metric $d_S$ is called the Skorohod metric and the topology induced on $\mathcal{D}(\mathbb{R}_{\geq 0}; D)$ by this metric is called the Skorohod topology.

In the thesis we used the notion of convergence in the Skorohod metric. In the matter of convergence in Skorohod metric the following holds Proposition A.2 (Proposition 5.2 in [EK86]) Let $\{x_n\}_{n \in \mathbb{N}}$ and $x$ be a elements in $\mathcal{D}(\mathbb{R}_{\geq 0}; D)$. Then $\lim_{n \to \infty} d_S(x_n, x) = 0$ if and only if there exists a sequence of time changes $\{\lambda_n\}_{n \in \mathbb{N}}$ in $\Lambda_L$ such that
1. $\lim_{n \to \infty} \Gamma(\lambda_n) = 0$,
2. $\lim_{n \to \infty} d_S(x_n, x, \lambda_n, s) = 0$ for each continuity point $s$ of $x$.

Moreover it holds

Theorem A.3 (Theorem 5.6 in [EK86]) If $(D, d)$ is a complex separable metric space, than also $(\mathcal{D}(\mathbb{R}_{\geq 0}; D), d_S)$ is a complete separable metric space.

Let $\mathcal{B}(\mathcal{D}(\mathbb{R}_{\geq 0}; D))$ denote the Borel $\sigma$-algebra of $\mathcal{D}(\mathbb{R}_{\geq 0}; D)$. Under the assumptions that $D$ is separable and complete, $\mathcal{B}(\mathcal{D}(\mathbb{R}_{\geq 0}; D))$ coincides with the cylindrical $\sigma$-algebra generated by the coordinate random variables

Proposition A.4 (Proposition 7.1 in [EK86]) Consider the coordinate map

$$\pi_t : \mathcal{D}(\mathbb{R}_{\geq 0}; D) \to D$$

$$x \mapsto x(t).$$

Since $E$ is separable, it holds

$$\sigma(\pi_t ; t \in \mathbb{R}_{\geq 0}) = \mathcal{B}(\mathcal{D}(\mathbb{R}_{\geq 0}; D)).$$

A.2 The uniform topology in the Skorohod space

It is possible to equip $\mathcal{D}(\mathbb{R}_{\geq 0}; D)$ with the uniform metric

$$d_U(x, y) := \sup_{t \geq 0} |x(t) - y(t)|.$$  \hspace{1cm} (A.4)

The topology induced by $d_U(x, y)$ is called the uniform topology. From Section 15 in [Bil09] we known that $(\mathcal{D}(\mathbb{R}_{\geq 0}; D), d_U)$ is not a separable space, which is usually a disadvantageous in probability theory. To simplify the notation, denote by $\mathcal{D}_S$ the space $\mathcal{D}(\mathbb{R}_{\geq 0}; D)$ endowed with the Skorohod topology and by $\mathcal{D}_U$ the same space.
but with the topology induced by $d_U$. Again from Section 15 in [Bil09] it follows that
\[ B(D_S) = B(D_U) \subset B(D_U), \]  
where $B(D_U)$ denotes the set of the balls in $D_U$ and the last inclusion is strict.

## A.3 The dual space

In this section we use again the notation $D_S$ to denote the Skorohod space with the Skorohod topology and $D_U$ the space space with the uniform topology.

Let $\mathcal{T}$ denote the family of all the possible collections $\{t_i\}_{i \in I}$ of countable sets of points in $[0,T]$, for a fixed $T > 0$. Given a function $f : [0,T] \to D$ suppose that
\[ ||f||_T := \sup \left\{ \sum_{i \in I} |f(t_i)| : \{t_i\}_{i \in I} \in \mathcal{T} \right\} < \infty \]
Denote by
\[ L^1(\mathcal{T}) := \{ f : [0,T] \to D | ||f||_T < \infty \}. \]

Fix $\mu := (\mu_1, \ldots, \mu_N)$, a family of $N$ Borel measures on $[0,T]$, and $f \in L^1([0,T])$. The define
\[ [\mu, f](x) := \sum_{i=1}^N \int_0^T x^{(i)}(s) d\mu_i(s) + \sum_{t \in [0,T]} \langle \Delta x(t), f(t) \rangle, \]  
where $x^{(i)}$ is the $i$-th coordinate of $x$ and $\Delta x(t) := x(t) - x(t^-)$. Observe that the sum is well defined because, due to the càdlàg property, $x$ can have at most a countable number of jumps. The following holds:

**Theorem A.5 (Theorem 1 in [Pes95])** For every continuous linear functional $\ell$ on the space $D_U$ there exists a unique $\mu$ and $f \in L^1([0,T]; D)$ such that $\ell = [\mu, f]$.

The following theorem relates the continuous linear functionals of $D_U$ with those of $D_S$

**Theorem A.6 (Theorem 2 in [Pes95])** Let $\ell$ a continuous linear functional on $D_U$. Then $\ell : D_S \to \mathbb{R}$ is measurable, i.e. for all $B \in \mathcal{B}(\mathbb{R})$, $\ell^{-1}(B) \in \mathcal{B}(D_S)$.

By means of these two theorems, it is possible to give the following characterization of $\mathcal{B}(D_S)$

**Theorem A.7 (Theorem 3 in [Pes95])** The $\sigma$-algebra $\mathcal{B}(D_S)$ is the one generated by $\ell : D_S \to \mathbb{R}$ with $\ell$ continuous linear functional in $D_U$. 

143
Appendix B

Markov processes on Polish spaces

B.1 Kolmogorov’s extension theorem

Given a semigroup of transition function \((p_t)_{t \geq 0}\) on a Polish space \((D, \mathcal{B}(D))\), and an initial distribution \(\varrho\), Kolmogorov’s extension Theorem gives the existence of a probability space \((\Omega, \mathcal{G}, P)\) where \(\Omega = D^{[0, \infty)}\) is the space of all the functions with values in \(D\) and \(\mathcal{F}\) is the minimal \(\sigma\)-algebra containing all the cylinder sets over \(t_1, \ldots, t_n\) in \([0, \infty)\) of type:

\[
\{ \omega \in \Omega \mid \omega(t_1), \ldots, \omega(t_n) \in B \}, \quad B \in (\mathcal{B}(D))^n.
\]

**Theorem B.1 (Theorem 7.4 in [Kal02])** Let \((D, \mathcal{B}(D))\) be a Polish space with measurable sets specified by the Borel family. Let \((p_t)_{t \geq 0}\) be a semigroup of transition functions on \((D, \mathcal{B}(D))\). Denote by \(\Omega = D^{[0, \infty)}\) and let \(\mathcal{G}_t\) be the minimal \(\sigma\)-algebra containing all the cylinder sets in \(\Omega\) with coordinates up to \(t\), i.e.

\[
\{ \omega \in \Omega \mid (\omega_{t_1}, \ldots, \omega_{t_n}) \in B, \quad t_1, \ldots, t_n \leq t, \quad B \in \mathcal{B}(D)^n \}.
\]

Define \(\mathcal{G} = \bigvee_{t \geq 0} \mathcal{G}_t\). Finally, let \(\varrho\) be a probability measure on \((D, \mathcal{B}(D))\). Then there exists a unique probability measure \(P^\varrho\) on \((\Omega, \mathcal{G})\), such that, for any sequence \(0 = t_0 < t_1 < \ldots < t_n\),

\[
P^\varrho(X_{t_0} \in dx_0, \ldots, X_{t_n} \in dx_n) = \varrho(dx_0) \otimes \bigotimes_{k=1}^n p_{t_k-t_{k-1}}(x_{k-1}, dx_k).
\]
B. Markov processes on Polish spaces

Under $\mathbb{P}^\varrho$, the canonical process, given by the coordinate map on $\Omega$

$$X_t : \Omega \to D$$
$$\omega \mapsto X_t(\omega) := \omega(t),$$

is a Markov process with respect with the filtration $(\mathcal{G}_t)_{t \geq 0}$ with transition probabilities given by $(p_t)_{t \geq 0}$ and initial distribution $\varrho$.

When $\varrho = \delta_x$, the notation $\mathbb{P}^x$ is used rather than $\mathbb{P}^{\delta_x}$.

B.2 Killed Markov processes

Up to now we have assumed that each $p_t$ is a strict probability measure on $D$, meaning that $p_t(x, D) = 1$ for all $t \geq 0$ and $x \in D$. In this case, the Markov processes is said to be conservative. We want to extend the definition in order to allow $p_t(x, D) \leq 1$. In this case the semigroup of transition functions is called sub–Markovian. Intuitively, the condition $p_t(x, D) < 1$ means that the process leaves the state space at a certain random time and we say that the process is killed.

In the thesis we mainly focused on conservative Markov processes. Indeed, it is possible to turn a sub–Markovian semigroup into a conservative semigroup. The idea is to add an additional state at the space by compactification argument. Let $\Delta \not\in D$ and define $D_\Delta = D \cup \{\Delta\}$. Now consider the extension $(\tilde{p}_t)_{t \geq 0}$ of the transition semigroup $(p_t)_{t \geq 0}$ defined by

- $\tilde{p}_t(x, B) = p_t(x, B), \quad B \in \mathcal{B}(D)$,
- $\tilde{p}_t(x, \{\Delta\}) = 1 - p_t(x, D)$,
- $\tilde{p}_t(\Delta, \{\Delta\}) = 1$,
- $\tilde{p}_t(\Delta, D) = 0$.

We modify the canonical probability space accordingly by considering

$$\Omega := \{\omega : \mathbb{R}_{\geq 0} \to D_\Delta \text{ such that } \omega(s^-) = \Delta \text{ or } \omega(s) = \Delta \text{ implies } \omega(t) = \Delta \text{ for all } t \geq s\}.$$

By construction, $p_t(x, D_\Delta) = 1$ for all $t \geq 0$ and $x \in D_\Delta$. From now on, when the process $X$ is defined in $D_\Delta$ we will always denote its semigroup of transition functions with $(p_t)_{t \geq 0}$ although we mean the extended semigroup $(\tilde{p}_t)_{t \geq 0}$. 

146
C.1 Existence

Kolmogorov’s extension theorem applied to semigroup of transition functions which are also spatially homogeneous, gives the canonical construction of Markov processes with stationary and independent increments:

**Theorem C.1 (Proposition 8.5 in [Kal02])** For any convolution semigroup \((\mu_t)_{t \geq 0}\) and initial distribution \(\nu\) there exists a probability measure \(\mathbb{P}^\nu\) on \(\Omega\) such that the canonical process \((X_t)_{t \geq 0}\) is a stochastic process satisfying

1. \(X_0\) has \(\mathbb{P}^\nu\)-distribution \(\nu\),
2. \((X_t)_{t \geq 0}\) has stationary independent increments,
3. the distribution under \(\mathbb{P}^\nu\) of the increments \(X_t - X_s\), with \(t > s\) is given by \(\mu_{t-s}\).

Due to spatial homogeneity, when dealing with a Lévy process, without loss of generality, we will always consider the Markov process associated with initial distribution \(\delta_0\).

C.2 Relation with infinitely divisible distributions

If \(X\) is a Lévy process, then, for any \(t \geq 0\), the random variable \(X_t\) is infinitely divisible. Conversely, from an infinitely divisible distribution \(\mu\), it is possible to construct a convolution semigroup whose corresponding Lévy process admits \(\mu\) as distribution at time 1. Additionally, the Fourier transform of a Lévy process takes
the form:

\[ \mathbb{E}_0 \left[ e^{(u,X_t)} \right] = e^{\eta(u)}, \quad u \in \mathbb{R}^d \]  

(C.1)

\[ \eta(u) = \langle b, u \rangle + \frac{1}{2} \langle u, \sigma u \rangle + \int_{\mathbb{R}^d} \left( e^{\langle u, \xi \rangle} - 1 - \langle h(\xi), u \rangle \right) \nu(d\xi), \]  

(C.2)

where \( b \in \mathbb{R}^d, \sigma \in S^d_+ \) and \( \nu \) is a Lévy measure in \( \mathbb{R}^d \). The Fourier transform (C.1) can be extended in the complex domain and the resulting Fourier–Laplace transform is well defined in

\[ \mathcal{U} := \{ u \in \mathbb{C}^d \mid \eta(\Re(u)) < \infty \}. \]

Under particular circumstances, we will make use of different truncation functions to simplify the notation:

- If \( \int |\xi|^1 \{ |\xi| \leq 1 \} \nu(d\xi) < \infty \), we can take \( h = 0 \) so to get

\[ \eta(u) = \langle b, u \rangle + \frac{1}{2} \langle u, \sigma u \rangle + \int_{\mathbb{R}^d} \left( e^{\langle u, \xi \rangle} - 1 \right) \nu(d\xi), \]

or equivalently we change the drift in order to incorporate the integrated small jumps.

- If \( \int |\xi|^1 \{ |\xi| > 1 \} \nu(d\xi) < \infty \) we can take \( h = 1 \) and write

\[ \eta(u) = \langle b, u \rangle + \frac{1}{2} \langle u, \sigma u \rangle + \int_{\mathbb{R}^d} \left( e^{\langle u, \xi \rangle} - 1 - \langle \xi, u \rangle \right) \nu(d\xi). \]

### C.3 Lévy-Itô decomposition

When dealing with a Lévy process, we make abuse of notation and we denote by \( \mathbb{P}^{(b,\sigma,\nu)} \) the measure \( \mathbb{P}^0 \) associated with the Lévy process with Lévy triplet \( (b,\sigma,\nu) \) starting from 0. In this subsection we follow [Str03].

We first show that the measure \( \mathbb{P}^{(b,\sigma,\nu)} \) can be constructed by superimposition of a pure jump measure \( \mathbb{P}^{(0,0,\nu)} \) on \( \mathcal{D}(\mathbb{R}^d) \) and \( \mathbb{P}^{(b,\sigma,0)} \) on \( C(\mathbb{R}^d) \) so to produce \( \mathbb{P}^{(b,\sigma,\nu)} \) on \( \mathcal{D}(\mathbb{R}^d) \) as the \( \mathbb{P}^{(b,\sigma,0)} \times \mathbb{P}^{(0,0,\nu)} \) distribution of the map

\[ C(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \rightarrow \mathcal{D}(\mathbb{R}^d) \]

\[ (X^{(1)}, X^{(2)}) \rightarrow X^{(1)} + X^{(2)}. \]

The result in Theorem C.2 establishes the identification of the path–space measure \( \mathbb{P}^{(b,\sigma,0)} \) with the Wiener measure on \( C(\mathbb{R}^d) \), properly transformed in order to have
C.3. Lévy-Itô decomposition

Drift $b$ and volatility $\sigma$. Then, we obtain $P^{(0,0,\nu)}$ as a limit of distributions of compound Poisson processes.

On the opposite direction, Theorem C.4 provides a decomposition of the path–space measure $P^{(b,\sigma,\nu)}$ which resemble the Lévy triplet.

**Theorem C.2** Let $P^{(b,\sigma,\nu)}$ be the probability measure defined in Theorem C.1 with initial measure $\delta_0$. Then

1. $P^{(b,\sigma,\nu)}$ admits a unique restriction as a probability measure on $D(\mathbb{R}^d)$ since it assigns to the subspace $D(\mathbb{R}^d) \subset (\mathbb{R}^d)^{(0,\infty)}$ outer measure 1.

2. $P^{(b,\sigma,\nu)}$ is the $P^{(b,\sigma,0)} \times P^{(0,0,\nu)}$ distribution of

   $$(C(\mathbb{R}^d), P^{(b,\sigma,0)}) \times (D(\mathbb{R}^d), P^{(0,0,\nu)}) \rightarrow (D(\mathbb{R}^d), P^{(b,\sigma,\nu)}),$$

   $$(X^{(1)}, X^{(2)}) \mapsto X^{(1)} + X^{(2)}.$$

**Proof** For the construction of $P^{(b,\sigma,0)}$ on $C(\mathbb{R}^d)$ consider first the Wiener measure $P^{(0,1,0)}$ in $C(\mathbb{R})$ as in subsection I.6 in [RW00]. Since $\sigma \in S^d_+$, there exists a matrix $a \in \mathbb{R}^{d \times d}$ such that $\sigma = a a^\top$. We can identify $P^{(b,\sigma,0)}$ with the $(P^{(0,1,0)})^d$-distribution of

$$\tilde{X}_t = a \cdot (X_t^{(1)}, \ldots, X_t^{(d)})^\top + bt,$$

where $\{X^{(k)}\}_{k=1,\ldots,d}$ are $d$-independent copies of $X$ distributed as $P^{(0,1,0)}$.

Now, exclude the trivial case that $\nu$ is zero, since if $\nu(\mathbb{R}^d) = 0$, then

$$P^{(0,0,\nu)}(X_t = 0 \text{ for all } t \geq 0) = 1.$$

If $\nu(\mathbb{R}^d) \in (0, \infty)$ define

$$\tilde{X}_t = X_t + t \int h(d\xi) \nu(d\xi).$$

Then, the $P^{(0,0,\nu)}$ distribution of $\tilde{X}$ is the one of a compound Poisson process with intensity $\nu(\mathbb{R}^d)$ and jump distribution $\frac{\nu}{\nu(\mathbb{R}^d)}$. Indeed, the Lévy exponent of a compound Poisson process with above specified jump time and size distribution is given by

$$\eta(u) = \int \left( e^{\langle u, \xi \rangle} - 1 \right) \nu(d\xi), \quad u \in i\mathbb{R}^d,$$
C. Lévy processes

which can be rewritten as

$$\eta(u) = \left\langle u, \int h(d\xi)\nu(d\xi) \right\rangle + \int \left( e^{\langle u, \xi \rangle} - 1 - \langle u, h(\xi) \rangle \right) \nu(d\xi), \quad u \in \mathbb{R}^d.$$ 

This case includes also the simple Poisson process, when in particular the Lévy measure is given by $\delta_1(d\xi)$.

The construction of $\mathbb{P}^{(0,0,\nu)}$ is done by limit argument. Approximate $\nu$ with

$$\nu_k(d\xi) := \mathbb{1}_{\{2^{-(k+1)} < |\xi| \leq 2^{-k}\}} \nu(d\xi), \quad k \in \mathbb{N},$$

and define the measure $\mathbb{Q} = \prod_{k \in \mathbb{N}} \mathbb{P}^{(0,0,\nu_k)}$ on $\mathcal{D}(\mathbb{R}^d)^\mathbb{N}$. From Theorem 4.1.8. in [Str10], we conclude that each $\mathbb{P}^{(0,0,\nu_k)}$, $k \in \mathbb{N}$, is the canonical measure of a compound Poisson process with intensity

$$\nu(\{2^{-(k+1)} < |\xi| \leq 2^{-k}\})$$

and jump distribution

$$\frac{\nu_k(d\xi)}{\nu(\{2^{-(k+1)} < |\xi| \leq 2^{-k}\}).}$$

Moreover, using again Theorem 4.1.8. in [Str10], $\mathbb{P}^{(0,0,\nu)}$ is the $\mathbb{Q}$ distribution of $\widetilde{X} \in \mathcal{D}(\mathbb{R}^d)$ where $\widetilde{X}$ is, for each $T \in [0, \infty)$, the $\mathbb{Q}$-a.s. limit uniform in $[0, T]$ of the sequence

$$\widetilde{X}_t^{(n)} = \sum_{i=0}^{n} X_t^{(k)}, \text{ with } (X^{(0)}, X^{(1)}, \ldots, X^{(n)}, \ldots) \in \mathcal{D}(\mathbb{R}^d)^\mathbb{N}.$$

This concludes the construction of $\mathbb{P}^{(0,0,\nu)}$ on $\mathcal{D}(\mathbb{R}^d)$.

Now we want to go in the opposite direction. Namely, we are given a Lévy process and we want to decompose it into more elementary processes. Starting with a path $X \in \mathcal{D}(\mathbb{R}^d)$, we remove the jumps so to get a path in $\mathcal{C}(\mathbb{R}^d)$. This is a delicate point since a Lévy process can have both big jumps and infinitely many small jumps, not easily distinguishable from the continuous part. Once this step is done, it is possible to go further and identify $\mathbb{P}^{(b,\sigma,0)}$ with the $\mathbb{P}^{(b,\sigma,\nu)}$-distribution of the path obtain by removing the jumps and $\mathbb{P}^{(0,0,\nu)}$ with the $\mathbb{P}^{(b,\sigma,\nu)}$-distribution of the remaining part.

When working with càdlàg processes, the following notation will be useful:
Definition C.3 A jump function is a map from $[0, \infty) \times \mathcal{D}(\mathbb{R}^d)$ into the space of non-negative Borel measures on $\mathbb{R}^d \setminus \{0\}$ given by
\[
N(t, \Gamma; X) = \sum_{s \leq t} 1_{\Gamma}(X_s - X_{s^-}), \quad \Gamma \subseteq \mathbb{R}^d \setminus \{0\}.
\]
Since the paths of càdlàg processes can have at most a countable number of jumps, the above sum is well defined.

Theorem C.4 (Theorem 2.4.11 in [Str03]) Let $(b, \sigma, \nu)$ be a Lévy triplet and $X$ the corresponding Lévy process. Consider the compensated jump function
\[
\tilde{N}(t, d\xi; X) := N(t, d\xi; X) - t1_{\{||\xi|| \leq 1\}}\nu(d\xi)
\]
and the processes $X^C, X^J$ defined as
\[
X^J_t := \lim_{n \to \infty} \int_{||\xi|| > 2^{-n}} \xi \tilde{N}(t, d\xi; X), \quad (C.3)
\]
\[
X^C_t := X_t - X^J_t. \quad (C.4)
\]
Then
(i) The jump process $X^J$ is well defined and it is an element of $\mathcal{D}(\mathbb{R}^d)$ with $\mathcal{P}^{(b, \sigma, \nu)}$-distribution given by $\mathcal{P}^{(0,0,\nu)}$.
(ii) The component $X^C$ is in $\mathcal{C}(\mathbb{R}^d)$ with $\mathcal{P}^{(b, \sigma, \nu)}$-distribution given by $\mathcal{P}^{(b, \sigma, 0)}$.
(iii) The filtrations generated by $X^C$ and $X^J$ are independent $\mathcal{P}^{(b, \sigma, \nu)}$-almost surely.

For the proof we need two additional results

Lemma C.5 (see Theorem 2.4.11 in [Str03]) Let $\nu$ be a Lévy measure and define
\[
\tilde{N}(t, d\xi; X) := N(t, d\xi; X) - t1_{\{||\xi|| \leq 1\}}\nu(d\xi).
\]
Then, for $\mathcal{P}^{(0,0,\nu)}$-almost every $X \in \mathcal{D}([0, \infty), \mathbb{R}^d)$ the limit
\[
\tilde{X}_t := \lim_{n \to \infty} \int_{||\xi|| > 2^{-n}} \xi \tilde{N}(t, d\xi; X), \quad (C.5)
\]
exists uniformly with respect to $t \in [0, T]$, for each $T \in [0, \infty)$.

Proof We begin with a truncation argument. Define
\[
\tilde{X}_n := \int_{||\xi|| > 2^{-n}} \tilde{N}(\cdot, d\xi; X), \quad n \geq 0. \quad (C.6)
\]
C. Lévy processes

For ease of notation, denote with \( \bar{X}_n(t) \) the value at \( t \) of \( \bar{X}_n \). Consider the increments

\[
\tilde{Y}_n(t) = \bar{X}_{n+1}(t) - \bar{X}_n(t) = \int \xi \mathbb{1}_{\{2^{-(n+1)} < |\xi| \leq 2^{-n}\}} \bar{N}(t, d\xi; X), \quad t \geq 0.
\]

The processes \( \{\tilde{Y}_n\}_{n\geq0} \) have independent, centered increments under \( P^{(0,0,\nu)} \) with

\[
E P^{(0,0,\nu)} \left[ |\tilde{Y}_n(t + s) - \tilde{Y}_n(s)|^2 \right] = t \int |\xi|^2 \mathbb{1}_{\{2^{-(n+1)} < |\xi| \leq 2^{-n}\}} \nu(d\xi).
\]

Fix \( T \in [0, \infty) \), applying Kolmogorov’s inequality on dyadics and then taking the limit we get

\[
P^{(0,0,\nu)} \left( \sup_{t \in [0,T]} |\bar{X}_n(t) - \bar{X}_{n+1}(t)| \geq \varepsilon \right) \leq \frac{T}{\varepsilon^2} \int |\xi|^2 \mathbb{1}_{\{2^{-(n+1)} < |\xi| \leq 2^{-n}\}} \nu(d\xi).
\]

By Borel-Cantelli Lemma the limit is well defined uniformly on compacts up to \( P^{(0,0,\nu)} \)-null sets and it is in \( D(\mathbb{R}^d) \).

This is the last lemma we need for the proof.

Lemma C.6 (see Lemma 2.4.10 in [Str03]) Let \( (b, \sigma, \nu) \) be a Lévy triplet and \( D \in B(\mathbb{R}^d) \backslash \{0\} \) with \( \nu(D) < \infty \). Then \( N(\cdot, D; X) \) under \( P^{(b,\sigma,\nu)} \) has the same distribution of a simple Poisson process with intensity \( \nu(D) \). Define

\[
\bar{X}_t^D := \int_D \xi \bar{N}(t, d\xi; X) - t \int_D h(d\xi) \nu(d\xi), \quad t \geq 0,
\]

and

\[
\nu^D(d\xi) := \mathbb{1}_{\{\xi \in D\}} \nu(d\xi).
\]

The couple \( (\bar{X}_t^D, X - \bar{X}_t^D) \) is \( P^{(b,\sigma,\nu)} \)-distributed as \( P^{(0,0,\nu^D(d\xi))} \times P^{(b,\sigma,\nu - \nu^D(d\xi))} \).

Proof of the Lévy–Itô decomposition The main steps have been proved already. Now it is only a matter of collecting them together. Define

\[
D_n := \{ \xi \in \mathbb{R}^d \mid 2^{-(n+1)} < |\xi| \leq 2^{-n}\}, \quad n \in \mathbb{N}
\]

and consider the process \( \bar{X}_n \) defined as in (C.6). Then the \( P^{(b,\sigma,\nu)} \)-distribution of all \( \bar{X}_n \) given by \( P^{(0,0,\nu_{D_n})} \). Following the proof of Lemma C.5, with \( \tilde{Y}_n = \bar{X}_{n+1}^D \), we
conclude that the limit of $\bar{X}_n$ is well defined and it is a path in $D(\mathbb{R}^d)$. Moreover, $\mathbb{P}^{(0,0,\nu)}$ is the $\mathbb{P}^{(b,\sigma,\nu)}$-distribution of $X^J$. This concludes (i).

Now set $\bar{X}_n^C := X - \bar{X}_n$. By Lemma C.6, $\bar{X}_n^C \in D(\mathbb{R}^d)$ and its $\mathbb{P}^{(b,\sigma,\nu)}$-distribution is given by $\mathbb{P}^{(b,\sigma,\nu-\nu_n)}$, with

$$\nu_n(d\xi) = 1_{\{\xi > 2^{-n}\}} \nu(d\xi).$$

The existence of the limit, together with the identification of the $\mathbb{P}^{(b,\sigma,\nu)}$-distribution, can be handle as in point (i), as soon as we check that the process has continuous paths. Take a sequence $(\varepsilon_n)_{n \geq 0}$ decreasing to 0 such that

$$\int_{\{|\xi| \leq \varepsilon_n\}} |\xi|^2 (\nu - \nu_n)(d\xi) \leq 2^{-n},$$

then

$$\mathbb{P}^{(b,\sigma,\nu)} \left( \exists t \geq 0 \mid |\bar{X}_n^C(t) - \bar{X}_n^C(t^-)| > \varepsilon_n \right) = \mathbb{P}^{(0,0,\nu-\nu_n)} \left( \exists t \geq 0 \mid |\bar{X}_n^C(t) - \bar{X}_n^C(t^-)| > \varepsilon_n \right) = 0,$$

by Borel–Cantelli lemma. Therefore $X^C \in C(\mathbb{R}^d)$. Regarding (iii), since $\mathbb{P}^{(b,\sigma,0)}$ is the Wiener measure, it is independent of the $\sigma$-algebra generated by the jump function $N(t, \cdot, X)$.

\[\blacksquare\]

**Remark C.7** It is common to express Lévy–Itô decomposition in terms of the counting measure associated with $X$, defined by

$$\mathcal{J}((s,t], \Gamma; X) = \#\{ r \in (s,t] \mid X_r - X_{r^-} \in \Gamma \}.$$

In our notation,

$$\mathcal{J}(t, \Gamma; X) = N((0,t], \Gamma; X).$$

Using this notation, the above decomposition reads

$$X_t^J = \lim_{n \to \infty} \int_{\{|\xi| > 2^{-n}\}} \xi \widetilde{N}(t, d\xi; X)$$

$$= \lim_{n \to \infty} \int_{\{|\xi| > 2^{-n}\}} \xi N(t, d\xi; X) - \lim_{n \to \infty} \int_{\{2^{-n} < |\xi| \leq 1\}} \xi t \nu(d\xi)$$

$$= \int_{\{|\xi| > 1\}} \xi N(t, d\xi; X) + \lim_{n \to \infty} \int_{\{2^{-n} < |\xi| \leq 1\}} \xi \widetilde{N}(t, d\xi; X)$$

$$= \int_0^t \int_{\{|\xi| > 1\}} \xi \mathcal{J}(ds, d\xi; X) + \int_0^t \int_{\{0 < |\xi| \leq 1\}} \xi \widetilde{\mathcal{J}}(ds, d\xi; X),$$
where \( \widetilde{J} \) is the compensated random measure defined by
\[
\widetilde{J}(ds, d\xi; X) = J(ds, d\xi; X) - ds\nu(d\xi).
\]
Observe that we used the notation
\[
\lim_{n \to \infty} \int_{\{2^{-n} < |\xi| \leq 1\}} \xi N(t, d\xi; X) = \int_{\{0 < |\xi| \leq 1\}} \xi N(t, d\xi; X).
\]

\section*{C.4 Càdlàg version}

From the Lévy–Itô decomposition, it is clear that sample paths of a Lévy process are càdlàg. In this subsection we want to present another approach to prove the existence of a càdlàg version based on regularization of some martingale related with the original Lévy process. In this subsection suppose that \( X \) is a Lévy process with respect a filtration \( (F_t)_{t \geq 0} \) satisfying the usual conditions. In the Lévy case, we can always obtain a filtration \( (F_t)_{t \geq 0} \) satisfying the usual conditions starting from \( (\mathcal{F}^x_t)_{t \geq 0} \) by taking the \( \mathbb{P}^{(b,\sigma,\nu)} \) completion of \( (\mathcal{F}^x_t)_{t \geq 0} \) and making it right continuous.

The following propositions and theorems are taken from [Pro04], Section I.4. Fix \( u \in \mathcal{U} \) and define the process
\[
M^u_t = \frac{e^{\langle u, X_t \rangle}}{e^{t\eta(u)}}, \quad t \geq 0, \quad u \in \mathcal{U}, \quad (C.7)
\]
where \( \eta \) is the Lévy exponent of \( X \).

**Proposition C.8 (see Theorem I.4.30 in [Pro04])** For any \( u \in \mathcal{U} \) the process \( (M^u_t)_{t \geq 0} \) defined in (C.7) is a martingale with càdlàg paths.

**Proof** Observe that for every \( t \geq 0 \) and \( u \in \mathcal{U} \), \( M^u_t \) is well defined and strictly positive almost surely. For each fixed \( T > 0 \), from tower property, for every \( u \in \mathcal{U} \), it holds
\[
\mathbb{E}^{\mathbb{P}^{(b,\sigma,\nu)}} \left[ \frac{M^u_T}{M^u_t} \big| \mathcal{F}_t \right] = \frac{1}{M^u_t} \mathbb{E}^{\mathbb{P}^{(b,\sigma,\nu)}} \left[ \frac{e^{\langle u, X_T \rangle}}{e^{T\eta(u)}} \big| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}^{(b,\sigma,\nu)}} \left[ e^{\langle u, X_T - X_t \rangle - (T-t)\eta(u)} \big| \mathcal{F}_t \right] = e^{-(T-t)\eta(u)} \mathbb{E}^{\mathbb{P}^{(b,\sigma,\nu)}} \left[ e^{\langle u, X_{T-t} \rangle} \big| \mathcal{F}_t \right] = 1.
\]

Hence we can apply Theorem I.2.9 in [Pro04] to conclude that the functions \( t \mapsto M^u_t(\omega) \) and \( t \mapsto e^{\langle u, X_t(\omega) \rangle} \) are restrictions to \( \mathbb{Q} \cap \mathbb{R}_{\geq 0} \) of càdlàg functions \( \mathbb{P}^{(b,\sigma,\nu)} \)-a.s. on \( \mathbb{R}_+ \).
Now we need to extend càdlàg property for any \( t \geq 0 \). To do that, define the set
\[
\Gamma := \{(\omega, u) \in \Omega \times U \mid t \mapsto M^u_t, t \in Q \cap \mathbb{R}_{\geq 0} \text{ is not the restriction of a càdlàg function}\},
\]
which is \( \mathcal{F} \times \mathcal{B}(\mathcal{U}) \)-measurable. It holds \( \int \mathbb{1}_\Gamma(\omega, u) \mathbb{P}^{(b,\sigma,\nu)}(d\omega) = 0 \) for each \( u \in \mathcal{U} \).

In order to extend it to the whole \( \mathbb{R}_{\geq 0} \) we apply Fubini’s theorem:
\[
\int_\Omega \int_\mathcal{U} \mathbb{1}_\Gamma(\omega, u) d\lambda \mathbb{P}^{(b,\sigma,\nu)}(d\omega) = \int_\mathcal{U} \int_\Omega \mathbb{1}_\Gamma(\omega, u) \mathbb{P}^{(b,\sigma,\nu)}(d\omega) d\lambda = 0.
\]
where \( \lambda \) denotes the Lebesgue measure.

From the càdlàg property of the martingale \( (M^u_t)_{t \geq 0} \) for all \( u \in \mathcal{U} \), we can establish path regularity for the process \( X \).

**Lemma C.9 (see [Pro04])** Let \( \{x_n\}_{n \geq 0} \) be a sequence of real numbers such that \( \{e^{(u,x_n)}\} \) converges, as \( n \to \infty \), for almost all \( u \in \mathcal{U} \). Then \( \{x_n\} \) converges to a finite limit.

We finally can conclude the existence of a càdlàg version for \( X \).

**Theorem C.10 (see Theorem I.4.30 in [Pro04])** Let \( X \) a Lévy process, then there exists a process \( \tilde{X} \) which is a \( \mathbb{P}^{(b,\sigma,\nu)} \)-version of \( X \) with càdlàg paths and again a Lévy process.

**Proof** Define, for all \( t > 0 \),
\[
\tilde{X}_t(\omega) = \begin{cases} 
\lim_{q_k \in Q \cap \mathbb{R}_{\geq 0}, q_k \to t^+} X_{q_k}(\omega), & \omega \in \tilde{\Omega}, \\
0, & \omega \in \Gamma,
\end{cases}
\]
where \( \tilde{\Omega} \) is the projection onto \( \Omega \) of the set \( \{\Omega \times i\mathbb{R}^d\} \setminus \Gamma \). Analogously define \( \tilde{\mathcal{U}} \) as the projection onto \( \mathcal{U} \) of the set \( \{\Omega \times \mathcal{U}\} \setminus \Gamma \). Then \( t \mapsto M^u_t \) is the restriction to \( Q \cap \mathbb{R}_{\geq 0} \) of a càdlàg function for all \( u \in \tilde{\mathcal{U}} \). Hence for every \( \omega \in \tilde{\Omega} \), and all \( u \in \tilde{\mathcal{U}} \), both limits
\[
\lim_{q_k \in Q \cap \mathbb{R}_{\geq 0}, q_k \to t^+} M^u_{q_k}, \quad \lim_{q_k \in Q \cap \mathbb{R}_{\geq 0}, q_k \to t^-} M^u_{q_k},
\]
exist and are finite. From the above lemma we conclude that for every \( \omega \in \tilde{\Omega} \) both
\[
\lim_{q_k \in Q \cap \mathbb{R}_{\geq 0}, q_k \to t^+} X_{q_k}, \quad \lim_{q_k \in Q \cap \mathbb{R}_{\geq 0}, q_k \to t^-} X_{q_k},
\]
exist and are finite.
exist and are finite. Therefore \( \bar{X}_t \) defines a càdlàg function in \( t \). The last step is to show that \( \bar{X} \) is a version for \( X \). By assumption \( \mathcal{F}_t \) is contains all the \( \mathbb{P}^{(b,\sigma,\nu)} \)-null sets of \( \mathcal{F} \). Therefore the process \( \bar{X} \) is \( (\mathcal{F}_t)_{t \geq 0} \)-adapted. The fundamental assumption for finishing the proof is stochastic continuity. By using the fact that \( \lim_{s \to t} X_s = X_t \) in probability and that stochastic continuity implies almost sure convergence along subsequences, we get \( \mathbb{P}^{(b,\sigma,\nu)}(\bar{X}_t = X_t) = 1 \) for all \( t \geq 0 \). Indeed,

\[
\bar{X}_t = \lim_{q_k \to t^+, q_k \in \mathbb{Q} \cap \mathbb{R} \geq 0} X_{q_k}, \quad \mathbb{P}^{(b,\sigma,\nu)} \text{-a.s.}
\]

\[
X_t = \lim_{q_k \to t^+, q_k \in \mathbb{Q} \cap \mathbb{R} \geq 0} X_{q_k}, \quad \mathbb{P}^{(b,\sigma,\nu)} \text{-a.s.}
\]

Finally

\[
\mathbb{E}^{\mathbb{P}^{(b,\sigma,\nu)}} \left[ e^{\langle u, \bar{X}_t \rangle} \right] = \mathbb{E}^{\mathbb{P}^{(b,\sigma,\nu)}} \left[ e^{\langle u, X_t \rangle} \right] = e^{t \eta(u)},
\]

and hence \( \bar{X} \) is a Lévy process with Lévy exponent \( \eta(u) \) with càdlàg paths. ■

### C.5 Path properties

The last topic treated in this appendix deals with the characterization of path regularity of a Lévy process in terms of its triplet. We start with relations between time independent path properties and the Lévy-Khintchine triplet.

Some first observations involve the jump function \( N \) introduced in (C.3).

1. A Lévy process \( X \in \mathcal{C}(\mathbb{R}^d) \) if and only if

\[
N(t, \cdot; X) = 0 \text{ for all } t > 0.
\]

2. A Lévy process \( X \) is a pure jump process if and only if

\[
\int (1 \wedge |\xi|) N(t, d\xi; X) < \infty \text{ for all } t > 0.
\]

Additionally, in this case

- \( X \) admits the representation \( X_t = \int \xi N(t, d\xi; X) \).

- If moreover \( N(t, \mathbb{R}^d; X) < \infty \) for all \( t > 0 \), then its paths increase only by jumps.
Another issue we want to address is continuity of the distribution of a Lévy process in terms of its Lévy triplet. The following are some results in this direction.

**Theorem C.11 (Theorem 27.4 in [Sat99])** Let $X$ be a Lévy process with Lévy triplet $(b, \sigma, \nu)$, and denote by $\mu_t = \mathbb{P}^{(b, \sigma, \nu)} \circ X_t^{-1}$ the following are equivalent:

1. $\mu_t$ has no atoms for all $t > 0$, i.e. $\mu_t(\{x\}) = 0$ for all $x \in \mathbb{R}^d$ and $t > 0$,
2. $\mu_t$ has no atoms for some $t > 0$,
3. $\sigma \neq 0$ or $\nu(\mathbb{R}^d) = \infty$.

Clearly, when $\sigma \neq 0$, since $\mu_t$ contains a convolution factor which is absolutely continuous, we get that $\mu_t$ is actually absolutely continuous for all $t > 0$. Otherwise:

**Theorem C.12 (Theorem 27.16 in [Sat99])** Let $X$ be a Lévy process with Lévy triplet $(b, 0, \nu)$ with $\nu(\mathbb{R}^d) = \infty$ and discrete, i.e. there exists a countable set $C$ such that $\nu(\mathbb{R}^d \setminus C) = 0$. Then $\mu_t$ is either absolutely continuous or continuous singular.

**Remark C.13** In this section we are using notations and definitions from [Sat99]. A measure $\mu$ is continuous singular if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^d$ and there exists a set $B \in \mathcal{B}(\mathbb{R}^d)$ with Lebesgue measure zero such that $\mu(\mathbb{R}^d \setminus B) = 0$. Absolute continuity means that $\mu(B) = 0$ whenever $B \in \mathcal{B}(\mathbb{R}^d)$ is a set with Lebesgue measure zero.

Therefore, being able to identify the Lévy triplet for an infinitely divisible triplet is a good achievement.

Moment condition for a Lévy process is both related with integrability of the Lévy measure and differentiability of its Laplace-Fourier transform:

**Proposition C.14 (Corollary 25.8 in [Sat99])** Let $X$ be a Lévy process and denote by $\mu_t$ the distribution of $X_t$. The existence of moments is not a time dependent property meaning that if $\int |\xi|^k \mu_t(d\xi) < \infty$ for some $t$, then it is also true for all $t > 0$. Let $\nu$ be its Lévy measure. Then $X$ has finite $k$-th moment if and only if $\int |\xi|^k \mathbb{1}_{\{|\xi| > 1\}} \nu(d\xi)$.

From the integrability condition of the Lévy measure we get the following result

**Corollary C.15** A Lévy process has finite $k$-th moment with $k \geq 2$ if and only if the $k$-th moment of the Lévy measure is finite.
On the other hand, marginal moments condition can be handle by means of regularity of the Fourier transform:

**Theorem C.16** Let $X$ be a Lévy process with one dimensional distribution $(\mu_t)_{t \geq 0}$ and denote by $\hat{\mu}_t$ the Fourier transform of $\mu_t$. Then $\int |\xi|^2 \mu_t(d\xi) < \infty$ for all $t > 0$ if $\partial^2_{\xi_i} \hat{\mu}_1(0)$ exists for all $i = 1, \ldots, d$.

**Proof** Since moments conditions is not a time dependent property for a Lévy process, we need only to check that $\partial^2_{\xi_i} \hat{\mu}_1(0)$ implies $\int \xi_i^2 \mu_1(d\xi) < \infty$ which is equivalent to $\int \xi_i^2 \nu(d\xi) < \infty$, where $\nu$ is the Lévy measure of $X$. We can therefore also drop the time index by denoting with $\mu = \mu_1$. Given $s > 0$ and $e_k$ the $k$-th basis element in $\mathbb{R}^d$, $\hat{\mu}(se_k)$ is the Fourier transform of the marginal distribution $\mu_t$. Hence results on one dimensional moment generating functions can be used, see Appendix A in [DFS03] and reference therein.

\[\square\]
Appendix D

Multiparameter random time change

This survey on results on multiparameter random time changes is taken from Section [EK86, Kur80]. For the theory in weak and strong solution of general stochastic equations we refer to [Kur07, Kur13].

D.1 Definitions

We start with the definition of solution for general stochastic equations as in [Kur13].

Definition D.1 Let $D_1$ and $D_2$ be Polish spaces and let $(Y, Z)$ be two random variables in $D_1 \times D_2$. Denote by $P^{(Y,Z)}$ their joint distribution. This is an element in $\mathcal{M}(D_1 \times D_2)$, the set of all measures in the product space $D_1 \times D_2$. Let $Q$ be a given distribution on $D_2$ and $\Gamma$ a set of constraints relating $Y$ and $Z$. Consider the following set

$$\mathcal{M}_Q(D_1 \times D_2) = \{P \in \mathcal{M}(D_1 \times D_2) \mid P(D_1 \times \cdot) = Q\}.$$

Let $\mathcal{M}(\Gamma, Q)(D_1 \times D_2)$ be the subset of $\mathcal{M}_Q(D_1 \times D_2)$ containing all the possible distributions of $(Y, Z)$ under which the constraints in $\Gamma$ are met. We say that, $(Y, Z)$ is a solution for $(\Gamma, Q)$ if $P^{(Y,Z)} \in \mathcal{M}(\Gamma, Q)$.

Definition D.2 A solution $(Y, Z)$ for $(\Gamma, Q)$ is a strong solution if there exists a measurable function $S : D_2 \to D_1$ such that $Y = S(Z)$ almost surely.

In particular, if a strong solution exists on some probability space, then a strong solution exists for any $Z$ with distribution $Q$. Observe that, being a strong solution
means that the measure $P^{(Y,Z)}$ can be fully characterized by the distribution $Q$ and the function $S$.

**Definition D.3** The following definitions are taken from [Kur13].

(Pathwise uniqueness) If $(Y, Z)$ and $(\tilde{Y}, \tilde{Z})$ are two solutions of $(\Gamma, Q)$ on the same probability space $\Omega$, with both $P^{(Y,Z)}$ and $P^{(\tilde{Y},\tilde{Z})}$ elements in $M_{(\Gamma,Q)}(D_1 \times D_2)$, then $Y = \tilde{Y}$ for almost every $\omega \in \Omega$.

(Weak joint uniqueness) There exists at most one measure in $M_{(Q,\Gamma)}(D_1 \times D_2)$.

The following is a generalization of the Yamada and Watanabe theorem on existence and uniqueness of weak and strong solutions.

**Theorem D.4 (Theorem 1.5 in [Kur13])** The following are equivalent:

1. There exists a strong solution of $(\Gamma, Q)$ and weak joint uniqueness holds.
2. There exists at least a measure $P^{(Y,Z)} \in M_{(\Gamma,Q)}$ and pathwise uniqueness holds.

In the thesis, we came across on time–change equations, which are examples of general stochastic equation where the constraints are of type $\Gamma(Y, Z) = 0$. In this case, pathwise uniqueness and weak joint uniqueness are equivalent, see Proposition 2.10 in [Kur07].

**Proposition D.5 (Proposition 2.10 in [Kur07])** Under the assumption that the constraints are of type $\Gamma(Y, Z) = 0$, the following are equivalent

(i) If $(Y, Z)$ and $(\tilde{Y}, \tilde{Z})$ are two solutions of $(\Gamma, Q)$ on the same probability space $\Omega$, with both $P^{(Y,Z)}$ and $P^{(\tilde{Y},\tilde{Z})}$ elements in $M_{(Q,\Gamma)}(D_1 \times D_2)$, then $Y = \tilde{Y}$ for almost every $\omega \in \Omega$.

(ii) Fix $P \in M_{(Q,\Gamma)}(D_1 \times D_2)$ and suppose that $(Y, Z)$ and $(\tilde{Y}, \tilde{Z})$ are two solutions of $(\Gamma, Q)$ on the same probability space $\Omega$, with both $P^{(Y,Z)} = P^{(\tilde{Y},\tilde{Z})} = P$, then $Y = \tilde{Y}$ for almost every $\omega \in \Omega$.

(iii) There exists at most one measure in $M_{(Q,\Gamma)}(D_1 \times D_2)$.

(iv) All the measures $P \in M_{(Q,\Gamma)}(D_1 \times D_2)$ have the same marginal distributions on $D_1$.

### D.2 Time–change equations

Let $D$ be a complete, separable metric space and $\{Z^{(h)}\}_{h=1,...,N}$ a family of stochastic processes with càdlàg paths taking values in $D$ defined on a common probability
D.2. Time–change equations

Let

\[ f_h : D^N \to \mathbb{R}_{\geq 0} \]

be nonnegative measurable functions. In this section we want to study stochastic equations of type

\[ Y_t^{(h)} = Z^{(h)} \left( \int_0^t f_h(Y_s)ds \right). \]  \hspace{1cm} (D.1)

We start by translating the notion of general solutions from the previous setting to the notion of solutions for the time change equation. Observe that, since each \( Z^{(h)} \) has càdlàg paths, if there exists a family \( Y = (Y^{(1)}, \ldots, Y^{(N)}) \) satisfying (D.1), then, necessarily, \( Y \) has càdlàg paths. Henceforth, for ease of notation, we denote by

\[ Z = (Z^{(1)}, \ldots, Z^{(N)}) \quad \text{and} \quad Y = (Y^{(1)}, \ldots, Y^{(N)}). \]

**Definition D.6** A couple \((Y,Z) \in D(\mathbb{R}^N) \times D(\mathbb{R}^N)\) is a solution of (D.1) if their joint distribution \( P(Y,Z) \) is an element of \( M(\Gamma,Q) \), where \( Q \) the distribution of the process \( Z \) seen as an element in \( D(\mathbb{R}^N) \) and

\[ \Gamma : D(\mathbb{R}^N) \times D(\mathbb{R}^N) \to D(\mathbb{R}^N) \]

is defined as

\[ \Gamma(Y,Z) := \gamma(Y,Z) - Y, \]  \hspace{1cm} (D.2)

where

\[ \gamma_h(Y,Z) := Z^{(h)} \left( \int_0^t f_h(Y_s)ds \right), \quad \text{for} \ h = 1, \ldots, N. \]  \hspace{1cm} (D.3)

In case it is possible to find a solution \((Y,Z)\) with both \( Y, Z \) defined on the same probability space \((\Omega, \mathcal{G}, P)\), then, we can consider (D.1) as an equation of type

\[ Y_t^{(h)}(\omega) = Z^{(h)} \left( \int_0^t f_h(Y_s)ds, \omega \right), \quad \omega \in \Omega. \]  \hspace{1cm} (D.4)

**Lemma D.7** (Lemma 2.1 in [EK86]) Suppose that for almost every \( \omega \in \Omega \) a solution of (D.4) exists in the strong sense. Then \( Y \) is a stochastic process in \((\Omega, \mathcal{G}, P)\)

161
Now we want to introduce the notations of multiparameter filtrations, stopping times and time–changed filtrations. The following definitions are taken from Chapter 2.10 in [EK86] and [Kur80].

Suppose that \( Z = (Z^{(1)}, \ldots, Z^{(N)}) \) is a càdlàg process defined on a complete probability space \( (\Omega, \mathcal{G}, P) \). Define

\[
\mathcal{G}^s := \sigma \left( \{ Z_{t_h}^{(i)}, t_h \leq s_h, i = 1, \ldots, N \} \right), \quad \text{for } s \in \mathbb{R}^N_{\geq 0}
\]

and

\[
\mathcal{G}_s := \bigcap_{n \in \mathbb{N}} \mathcal{G}^{\tilde{s}^{(n)}} \vee \sigma(\mathcal{N}),
\]

where \( \mathcal{N} \) is the collection of sets in \( \mathcal{G} \) with \( P \)-probability zero and \( \tilde{s}^{(n)} \) is the sequence defined by \( s^{(k)}_k = s_k + 1/n \).

**Definition D.8** A random variable \( \tau = (\tau_1, \ldots, \tau_N) \in \mathbb{R}^N_{\geq 0} \) is a \( (\mathcal{G}_s) \)-stopping time if

\[
\{ \tau \leq s \} := \{ \tau_1 \leq s_1, \ldots, \tau_N \leq s_N \} \in \mathcal{G}_s, \quad \text{for all } s \in \mathbb{R}^N_{\geq 0}.
\]

If \( \tau \) is a stopping time,

\[
\mathcal{G}_\tau := \{ B \in \mathcal{G} \mid B \cap \{ \tau \leq \tilde{s} \} \in \mathcal{G}_\tilde{s} \text{ for all } \tilde{s} \in \mathbb{R}^N_{\geq 0} \}.
\]

Back to the equation (D.1), define

\[
\tau_t^{(h)} := \int_0^t f_h(Y_s)ds, \quad \text{for } h = 1, \ldots, N.
\]

**Theorem D.9** (Theorem 2.2 in [EK86]) Suppose that for almost every \( \omega \in \Omega \) a solution of \((Y, Z)\) exists and it is (strongly) unique. Then, for all \( t \geq 0 \)

\[
\{ \tau_t \leq \tilde{s} \} \in \mathcal{G}_s.
\]


Bibliography


<table>
<thead>
<tr>
<th>Reference</th>
<th>Author(s)</th>
<th>Title and Details</th>
</tr>
</thead>
</table>


