Split quasicocycles and defect spaces

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SPLIT QUASICOCYCLES AND DEFECT SPACES

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Für d'Jasmin
Füre Urs

Bi erwachet hüt am Morge
Da woni geschter scho bi gsi
Ir Mitti vomne Schrottplatz
Wi dr Esu vorem Bärg
...

(*Hotelsong*)

Abstract

Let Γ be a group. A map $\Gamma \longrightarrow E$, ranging in a Banach Γ-module E, is called a *quasicocycle* if it satisfies the cocycle identity up to bounded error. To a quasicocycle one associates a class in the second bounded cohomology $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$. The author has previously observed (see [47]) that for a free product of groups $\Gamma = A * B$, any given pair of alternating quasicocycles on the factors A,B can be extended to what we call a *split quasicocycle* on the whole group Γ. This thesis is concerned with the study of split quasicocycles and their significance for bounded cohomology.

Under mild assumptions on the factors of $\Gamma = A * B$, we prove that the split quasicocycles yield infinite dimensional subspaces of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$ whenever the coefficient module E is finite-dimensional or of the type $\ell^p(\Gamma)$ for $1 \leq p < \infty$.

In the case $E = \mathbb{R}$ we are dealing with *split quasimorphisms*. Here we show that every cohomology class obtained from our construction has Gromov norm equal to one half of the defect of the homogenous quasimorphism representing it. This property is known to hold for Brooks' counting quasimorphisms and it is an open question whether it holds in general. We identify the induced subspace of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ to be built up from isometrically embedded *defect spaces*. The defect space $\mathcal{D}(H)$ of a group H is introduced as the space of alternating bounded functions $H \longrightarrow \mathbb{R}$, equipped with an certain exotic norm called the *defect norm*.

Since the geometry of the space $\mathrm{H}^2_{\mathrm{b}}$ is poorly understood we are motivated to study the geometry of defect spaces. It turns out that the set of extremal points in the closed unit ball of $\mathcal{D}(H)$ encodes different properties of the group H. For example, we find a subset of extremal points that is naturally homeomorphic to Sikora's space of left-orders on H.

For the case of a free group, we construct split quasimorphisms that vanish on a given subgroup of infinite index. As a result we obtain embeddings of defect spaces into the relative second bounded cohomology. This part is based on joint work with Cristina Pagliantini.

We suggest furthermore a new geometric construction of a quasimorphism on the free group, based on the CAT(0) geometry of a polygonal complex quasi-isometric to the group's Cayley graph. This construction turns out to admit a generalization which also encompasses split quasimorphisms.

Replacing the target space E with a metric group yields a new type of *quasi-representations* whose construction is much clearer than it is the case for the previously known examples due to Kazhdan (see [36]).

Zusammenfassung

Es sei Γ eine Gruppe. Eine Abbildung $\Gamma \longrightarrow E$ in einen Banach Γ-modul E heisst *Quasikozykel*, wenn sie die Kozykelidentität bis auf einen beschränkten Fehler erfüllt. Zu einem Quasikozykel assoziiert man eine Klasse in der zweiten beschränkten Kohomologie $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$. Der Autor hat bereits früher festgestellt (siehe [47]), dass für ein freies Produkt $\Gamma = A * B$ jedes gegebene Paar alternierender Quasikozykel auf den Faktoren A, B zu einem sogenannten *gespaltenen Quasikozykel* auf der ganzen Gruppe Γ fortgesetzt werden kann. Diese Arbeit befasst sich mit dem Studium der gespaltenen Quasikozykel und deren Bedeutung für die beschränkte Kohomologie.

Unter schwachen Bedingungen an die Faktoren A, B zeigen wir das gespaltene Quasikozykel unendlich-dimensionale Unterräume von $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$ ergeben, falls der Koeffizientenmodul E endlich-dimensional ist, oder vom Typ $\ell^p(\Gamma)$, $1 \leq p < \infty$.

Im Fall $E = \mathbb{R}$ haben wir es mit *gespaltenen Quasimorphismen* zu tun. Hier zeigen wir, dass für jede der induzierten Kohomologieklassen die Gromov-Norm übereinstimmt mit dem halben Defekt des homogenen Quasimorphismus, der dieselbe Klasse repräsentiert. Diese Eigenschaft haben auch die Zählquasimorphismen von Brooks, und es ist eine offene Frage ob sie allgemein gilt. Wir identifizieren den zughörigen Unterraum von $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ als Summe von isometrisch eingebetteten *Defekträumen*. Den Defektraum $\mathcal{D}(H)$ einer Gruppe H führen wir ein als Raum der alternierenden beschränkten Funktionen $H \longrightarrow \mathbb{R}$, ausgestattet mit einer exotischen Norm, der *Defektnorm*.

Über die Geometrie des Raumes $\mathrm{H}^2_{\mathrm{b}}$ weiss man erst wenig, und dies motiviert uns, die Geometrie der Defekträume zu untersuchen. Es stellt sich heraus, dass die Menge der Extremalpunkte im abgeschlossenen Einheitsball von $\mathcal{D}(H)$ verschiedene Eigenschaften der Gruppe H kodiert. Beispielsweise bestimmen wir eine Teilmenge der Extremalpunkte, die auf natürliche Weise homöomorph zu Sikoras Raum der Linksordnungen auf H ist.

Für den Fall einer freien Gruppe konstruieren wir gespaltene Quasimorphismen, die auf einer gegebenen Untergruppe mit unendlichem Index verschwinden. Daraus erhalten wir Einbettungen von Defekträumen in die relative zweite beschränkte Kohomologie. Dies basiert auf einer Zusammenarbeit mit Cristina Pagliantini.

Weiter präsentieren wir eine neue geometrische Konstruktion eines Quasimorphismus auf der freien Gruppe, basierend auf der CAT(0)-Geometrie eines polygonalen Komplexes, der zum Cayley-Graph der Gruppe quasiisometrisch ist. Es zeigt sich, dass diese Konstruktion eine gemeinsame Verallgemeinerung mit gespaltenen Quasimorphismen besitzt.

Indem wir den Modul E durch eine metrische Gruppe ersetzen, erhalten wir neue *Quasi-Darstellungen*, deren Konstruktion viel klarer ist als es bei den bekannten Beispielen von Kazhdan (siehe [36]) der Fall ist.

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Introduction

Let Γ be a group and let E be a Banach Γ-module, that is, a Banach space endowed with a linear isometric Γ-action. Recall that $f : \Gamma \longrightarrow E$ is called a *cocycle* if the map

$$\partial f : (g, h) \ \mapsto \ f(g) - f(gh) + g.f(h)$$

vanishes identically on $\Gamma \times \Gamma$. If Γ acts trivially on E then cocycles are homomorphisms. If ∂f has bounded range then f is called a *quasicocycle*, or, in the case $E = \mathbb{R}$, a *quasimorphism*. Bounded perturbations of cocycles are obviously quasicocycles, and it turns out that a quasicocycle $f : \Gamma \longrightarrow E$ is of this trivial type, if and only if it is contained in the kernel of the map

$$\mathrm{Q}\mathcal{Z}(\Gamma, E) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$$

which associates to f the class $[\partial f]_{\mathrm{b}}$ in the second bounded cohomology of Γ with coefficients in E.

The standard example is the counting quasimorphism $C_w : \mathbb{F}_2 = \langle a, b\rangle \longrightarrow \mathbb{R}$ of Brooks, defined on the rank two free group (see [14]). This simple construction has a word $w \in \mathbb{F}_2$ as its parameter. To determine the value $C_w(g)$ one counts in g the number of subwords equal to w, and subtracts the corresponding number for the inverse word w^{-1}. As soon as w has length at least two, C_w is non-trivial. Mitsumatsu showed, by varying the parameter w, that the cohomology classes associated to counting quasimorphisms span an infinite dimensional subspace in $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2, \mathbb{R})$ (see [40]). Epstein and Fujiwara defined generalized counting quasimorphisms for word-hyperbolic groups and showed that the second bounded cohomology of such a group has infinite dimension (unless the group is elementary). A further generalization for groups acting properly on Gromov-hyperbolic spaces was given by Fujiwara (see [26]), and Bestvina–Fujiwara showed that it is sufficient to have an action which is weakly proper (see [9]). This last result includes the notable case of the mapping class group acting on the curve complex.

Barge–Ghys defined a quasimorphism of different nature for the fundamental group Γ of a closed negatively curved manifold M (see [4] and also the discussion in [19], Section 2.3.1). The parameter of their quasimorphism $f_\lambda : \Gamma \longrightarrow \mathbb{R}$ is a differential 1-form λ on M. For $g \in \Gamma$, the value $f_\lambda(g)$ is defined to be the integral of λ along the unique geodesic loop representing g. Using this construction in the case of surfaces they characterized metrics of constant negative curvature via the area of ideal triangles ([4], Théorème 3.11).

In all these examples we have an infinite dimensional space $\mathrm{H}^2_{\mathrm{b}}$. On the other hand, the bounded cohomology of amenable groups vanishes, as was shown by Johnson (see [35]). This means that on such a group every quasimorphism is trivial. By the work of Burger–Monod this property holds as well for higher rank lattices ([15], Corollary 1.3), a fact which they used to establish rigidity results for actions of lattices on the circle. Bestvina–Fujiwara combined this with their construction mentioned above, leading to a new proof of a rigidity statement for representations of lattices into mapping class groups ([9], Corollary 13).

The (non-)existence of non-trivial quasimorphisms has an impact also in the theory of *stable commutator length* (*scl*): Bavard proved that the scl function of a group is identically equal to zero, if and only if every quasimorphism on this group is trivial ([6], Corollaire 1). Moreover, in case scl does not vanish, it can be expressed through quasimorphisms by means of Bavard's duality theorem ([6], Théorème de dualité, p.141). A large number of further results connecting quasimorphisms and scl is found in Calegari's monograph on the topic (see [19]).

A further area in which quasimorphisms have been studied is measured group theory, notably in the work of Björklund–Hartnick on central limit theorems for quasimorphisms (see [11]). In symplectic topology quasimorphisms have been present since the work of Entov–Polterovich (see [22]). In a rather curious way quasimorphisms on the integers were used by A'Campo for a new construction of the real number system (see [1]).

Among the quasicocycles with non-trivial target, those ranging in the left-regular representation have found particular attention. For groups Γ acting on Gromov hyperbolic graphs, Monod–Mineyev–Shalom have constructed non-trivial quasicocycles $\Gamma \longrightarrow \ell^p(\Gamma)$ for $1 \leq p < \infty$ (see [39]). These groups satisfy in particular $\mathrm{H}^2_\mathrm{b}(\Gamma, \ell^2(\Gamma)) \neq 0$, a property which by the work of Monod-Shalom is an invariant of measure equivalence, and is thought of as a cohomological characterization of negative curvature of a group (see [43] and [44]). On a more general level, Hamenstädt used boundary theory to construct ℓ^p-valued quasicocycles for groups which admit a weakly proper action on an arbitrary Gromov hyperbolic space (see [31]). A further generalization was obtained by Hull–Osin, who gave a construction of ℓ^p-quasicocycles for groups that contain a hyperbolically embedded subgroup (see [33]).

Targets other than left-regular representations have been studied as well. A recent example is found in the construction of a distinguished bounded cohomology class for a group acting on a CAT(0) cube complex X, due to Chatterji–Fernós–Iozzi (see [20]). Here the coefficients are of a geometric type, they are defined using the halfspace structure of X. A construction with very general coefficients has been carried out by Bestvina–Bromberg–Fujiwara (see [8]). Their quasicocycles are defined on groups Γ that act on a geodesic metric space, where at least one group element needs to show a "rank 1" behavior. Here the coefficients are an arbitrary uniformly convex Banach module E, for example a module of finite dimension. This construction is inspired by Brooks counting quasimorphisms, and it yields an infinite dimensional space $\mathrm{H}^2_\mathrm{b}(\Gamma, E)$.

Let now Γ be a group and let E be a Banach Γ-module. Assume that we have a splitting $\Gamma = A * B$ of our group into a free product. A cocycle $\Gamma \longrightarrow E$ is uniquely determined by its restrictions to the free factors A and B, and in fact, every pair of cocycles

$$f_A : A \longrightarrow E, \quad f_B : B \longrightarrow E$$

extends in a unique way to a cocycle on Γ. This fact is no longer true if we consider quasicocycles. However, given two alternating quasicocycles f_A and f_B there is still a natural way of defining an extension $f : \Gamma \longrightarrow E$:

$$\begin{aligned}&f(a_1 b_1 \dots a_n b_n) := \\ &f_A(a_1) + a_1.f_B(b_1) + a_1 b_1.f_A(a_2) + \cdots + a_1 b_1 a_2 \cdots b_{n-1} a_n.f_B(b_n).\end{aligned}$$

We write $f = f_A * f_B$ for this extension and call it a *split quasicocycle*. This formula appeared in the author's master thesis (see [47], Remark (iii), p.5), and it was discovered independently by Thom (see [55], Lemma 5.1). By what we said above, f is trivial iff the pair (f_A, f_B) belongs to the kernel of the map

$$\begin{aligned}\Phi : \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(A, E) \times \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(B, E) &\longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, E), \\ (f_A, f_B) &\mapsto [\partial(f_A * f_B)]_{\mathrm{b}}.\end{aligned}$$

We refer to the image of Φ as the space of *split classes*, and we prove that under suitable assumptions this space is infinite dimensional:

Theorem 2.3. *Let Γ be a finitely generated group with a splitting $\Gamma = A * B$, such that A contains an element of infinite order. Then for any finite dimensional Banach Γ-module E the split classes form an infinite dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$.*

Under essentially the same assumptions on Γ we have

Theorem 2.5. *Let Γ be a countable group with a splitting $\Gamma = A * B$, such that A contains an element of infinite order. For $1 < p < \infty$ the split classes form an infinite-dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^p(\Gamma))$. If the factor A is amenable then the same holds for $p = 1$.*

These statements apply in particular in the situation $A = B = \mathbb{Z}$, i.e. when Γ is free of rank two. We note that infinite-dimensionality of the spaces $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$ in the above theorems also follows from the work of Hull–Osin in the case of ℓ^p-coefficients (see [33], Corollary 1.7), and from the work of Bestvina–Bromberg–Fujiwara in the finite dimensional case (see [8], Theorem 5.1). Let us now consider the case of the trivial target $E = \mathbb{R}$. Here we obtain split quasimorphisms from a given splitting $\Gamma = A * B$, and we identify the kernel of the corresponding map

$$\Phi : \mathrm{QM}_{\mathrm{alt}}(A) \times \mathrm{QM}_{\mathrm{alt}}(B) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$$

as the subspace $\mathrm{Hom}(A, \mathbb{R}) \times \mathrm{Hom}(B, \mathbb{R})$. More precisely:

Theorem 3.3. *Let $f = f_A * f_B$ be a split quasimorphism with corresponding cohomology class $\omega_f = [\partial f]_{\mathrm{b}}$. We have*

$$\|\omega_f\| = \tfrac{1}{2}\operatorname{def}\widehat{f} = \operatorname{def} f = \max\{\operatorname{def} f_A, \operatorname{def} f_B\}.$$

In particular, f is a minimal defect representative for its class.

Here $\|\cdot\|$ stands for the Gromov norm on the space $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$, and for a quasimorphism f we denote by $\widehat{f}$ its homogenization and by $\operatorname{def} f$ its defect (see subsection 1.1). Among other things this theorem implies that a split quasimorphism $f = f_A * f_B$ is non-trivial as soon as one of the factors f_A,f_B is not a homomorphism. Therefore we get a linear embedding $\ell^\infty_{\mathrm{alt}}(A) \times \ell^\infty_{\mathrm{alt}}(B) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$. The spaces $\ell^\infty_{\mathrm{alt}}$ of alternating bounded functions on the respective groups are Banach, and so is the space $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ when equipped with the Gromov norm. This leads to the question whether the embedding under consideration respects the involved norms in any way. To give a precise answer we introduce the notion of the defect space of a group H. This is the space

$$\mathcal{D}(H) := \{f : H \longrightarrow \mathbb{R} \mid f \text{ is bounded and alternating}\}$$

equipped with the norm

$$\|f\|_{\mathrm{def}} := \operatorname{def} f = \sup_{g,h \in H} |f(gh) - f(g) - f(h)|.$$

Using this definition we obtain

Theorem 3.6. *For a group $\Gamma = A * B$ there is a linear isometric embedding*

$$\mathcal{D}(A) \oplus_\infty \mathcal{D}(B) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$$

which maps the pair (f_A, f_B) to the bounded cohomology class ω_f of the split quasimorphism $f = f_A * f_B$.

Here the symbol $\oplus_\infty$ stands for the max-norm on the sum of two Banach spaces. The previous two theorems imply in particular that for every cohomology class ω_f in the embedded space $\mathcal{D}(A) \oplus \mathcal{D}(B)$ the Gromov norm satisfies

$$\|\omega_f\| = \tfrac{1}{2}\operatorname{def}\widehat{f}.$$

It is an open question whether this equality holds in general for the class of a quasimorphism. Through Bavard's duality theorem a positive answer to this question would provide a link between commutator geometry and the geometry of the space $\mathrm{H}^2_{\mathrm{b}}$ (see the discussion in [19]). Equality was shown for a single example of a counting quasimorphism by Bavard ([6], p.148), and Calegari established it for all counting quasimorphisms and their finite linear combinations (see [18]).

The defect space $\mathcal{D}(H)$ has the same underlying vector space as $\ell^\infty_{\mathrm{alt}}(H)$ and in fact these spaces are norm-equivalent: We have $\|f\|_\infty \leq \|f\|_{\mathrm{def}} \leq 3\|f\|_\infty$ for any bounded alternating function f on H. Since the geometry of the $\mathrm{H}^2_{\mathrm{b}}$ is poorly understood we are motivated to study the geometry of defect spaces; we are also not aware of any previous discussion of these spaces. We observe that $\mathcal{D}(H)$ is a dual Banach space and, as such, it has a non-empty set of extremal points $\mathcal{E}(H)$ in its closed unit ball. This is obvious in the case of a finite group H where the unit ball is a polytope. We show that several properties of the group H are reflected in the set $\mathcal{E}(H)$. Most notably, we identify a certain subset $\mathcal{E}_1(H) \subset \mathcal{E}(H)$ such that to each function $f \in \mathcal{E}_1(H)$ we can associate a total left-invariant order $\leq_f$ on the group H. The assignment $f \mapsto \leq_f$ has a naturally defined inverse. More precisely we have the following result in which $LO(H)$ stands for Sikora's space of left-orders on H (see [50]):

Corollary 7.22. *If we endow the set $\mathcal{E}_1(H)$ with the topology induced from the weak*-topology on $\mathcal{D}(H)$ then the map*

$$\mathcal{E}_1(H) \longrightarrow LO(H), \quad f \mapsto \leq_f$$

is a homeomorphism.

Another result of this type concerns a subset $\mathcal{E}^*_{(-1,1)}(H) \subset \mathcal{E}(H)$ which detects embedings of H into the circle group $\mathbb{T}$:

Theorem 7.32. *For every group H there is a natural correspondence*

$$\{\text{embeddings } H \hookrightarrow \mathbb{T}\} \quad \longleftrightarrow \quad \mathcal{E}^*_{(-1,1)}(H)$$

This correspondence yields infinitely many points in $\mathcal{E}(\mathbb{Z})$ which, as functions $\mathbb{Z} \longrightarrow \mathbb{R}$, only take irrational values. The next result shows that the set of extremal points contains the extremal points for the quotients of our group:

Theorem 7.24. *For a short exact sequence*

$$1 \longrightarrow N \longrightarrow H \longrightarrow Q \longrightarrow 1$$

in which the group Q is 2-torsion free, we have an induced embedding

$$j : \mathcal{D}(N) \oplus_\infty \mathcal{D}(Q) \hookrightarrow \mathcal{D}(H)$$

which maps extremal points to extremal points. That is, we have an embedding

$$\mathcal{E}(N) \times \mathcal{E}(Q) \hookrightarrow \mathcal{E}(H).$$

For the group $\mathbb{Z}$ this result can be used to obtain extremal points which have both rational and irrational values. The next statement yields points in $\mathcal{E}(\mathbb{Z})$ which are purely rational but non-periodic:

Corollary 7.29. *If the countably infinite group H is residually finite 2-torsion free, then its set of extremal points $\mathcal{E}(H)$ contains uncountably many rational-valued functions.*

We show that defect spaces are also found as isometric subspaces of relative bounded cohomology. Let H be a subgroup of a group Γ. We say that $f : \Gamma \longrightarrow \mathbb{R}$ is a relative quasimorphism for the pair (Γ, H) if it is a quasimorphism with $f|_H \equiv 0$. To such a map one has an associated class in the relative second bounded cohomology $\mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$ (see subsection 1.2). Using the construction of split quasimorphisms, we proved the following result in joint work with Cristina Pagliantini (see [45]):

Theorem 8.1. *Let Γ be a free group of finite rank $n \geq 2$, and let $H < \Gamma$ be a subgroup of finite rank. The following are equivalent*

(i) *H has infinite index in Γ,*

(ii) *The space $\mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$ is non-trivial.*

(iii) *There exists a linear isometric embedding*

$$\oplus^n_\infty \mathcal{D}(\mathbb{Z}) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$$

The crucial step in the proof of this statement is the construction of a suitable basis of $\mathbb{F}_n$, namely of a basis that admits split quasimorphisms which vanish on the subgroup H. This is accomplished through the following lemma which may be of independent interest:

Lemma 8.2. *Let Γ be a free group of finite rank $n \geq 2$ and let $H < \Gamma$ be a subgroup of finite rank and infinite index. There exists a basis $\{x_1, \ldots, x_n\}$ of Γ such that for all $g \in \Gamma$ and for all i we have*

$$gHg^{-1} \cap \langle x_i \rangle = \{e\},$$

which is to say that no conjugate of H contains a power of an element of this basis.

For a group Γ and a metric group (G, d), a map $\mu : \Gamma \longrightarrow G$ is called a quasi-representation (or ε-representation) if the expression $d(\mu(gh), \mu(g)\mu(h))$ is uniformly bounded. This notion goes back to Ulam who, in his 1960 collection of mathematical problems, asked whether quasi-representations are always close to actual representations (see [56]). The idea of extending maps on the factors of a free product to the whole group also works in this context. It yields *split quasi-representations* which can take values in an arbitrary group G endowed with a bi-invariant metric, for example in a compact Lie group. We obtain a non-triviality result in this setting, which bounds the distance of a split quasi-representation to an actual representation from below:

Theorem 4.2. *Let $\Gamma = A * B$ and let $G = (G, d)$ be a group without ε-small subgroups. For bounded alternating maps $\mu_A : A \longrightarrow G$, $\mu_B : B \longrightarrow G$ with*

$$\delta := \max\{\|\mu_A\|_\infty, \|\mu_B\|_\infty\} \leq \frac{\varepsilon}{2}$$

the split quasi-representation $\mu = \mu_A * \mu_B : \Gamma \longrightarrow G$ satisfies

$$D(\mu) \geq \delta.$$

Here we use the notation $\|\mu_A\|_\infty = \sup_{a \in A} d(\mu(a), e)$, and $D(\mu)$ stands for smallest possible distance of μ to an actual representation $\Gamma \longrightarrow G$. An application of split quasi-representations is found in the context of Ulam stability. Kazhdan gave quantitative answers to Ulam's question in certain situations where the target is the unitary group of a Hilbert space. Namely, he showed that for amenable groups any unitary quasi-representation of defect ε is contained in the ε-neighborhood of an actual representation ([36], Theorem 1). On the other hand, he used an involved construction in order to prove that surface groups admit quasi-representations into $U(n)$, such that the defect is less than $1/n$, but the distance to any representation is at least $1/10$ ([36], Theorem 2). More recently, Burger–Ozawa–Thom have made significant contributions to Ulam's stability question (see [17]). In their language the results of Kazhdan say that amenable groups are (strongly) Ulam stable, while surface groups are not Ulam stable. They observed that our split quasi-representations, which are constructed in a much clearer way than Kazhdan's quasi-representations of surface groups, can be used to establish that free groups are not Ulam stable ([17], Proposition 3.3). In fact they showed further that no group containing a rank two free group is (strongly) Ulam stable, which leads to the question whether Ulam stability characterizes amenability.

We also suggest a new geometric construction of quasimorphisms on the free group $\mathbb{F}_2$. For this we take the group's Cayley graph $\mathcal{T}$ (the 4-regular tree) and thicken it to a obtain a piecewise Euclidean CAT(0) polygonal complex $\mathcal{P}$ (a quasi-tree). Using the isometric action of $\mathbb{F}_2$ on this complex, together with a natural subdivision of the CAT(0) geodesics into straight segments, we define a parametrized family of what we call *geometric deformation quasimorphisms*

$$f_{t,\alpha,\beta} : \mathbb{F}_2 \longrightarrow \mathbb{R}.$$

Here the parameter $t \geq 0$ describes the thickness of $\mathcal{P}$. In the degenerate case $t = 0$ we have $\mathcal{P} = \mathcal{T}$ and the corresponding quasimorphisms $f_{0,\alpha,\beta}$ are in fact homomorphisms. On the other hand, $f_{t,\alpha,\beta}$ is non-trivial as soon as $t > 0$, i.e. when the tree $\mathcal{T}$ is "deformed". The proof of this fact is based on pictures and a short elementary geometric calculation. These deformation quasimorphisms are different from other constructions (split, counting) in that they have no obvious combinatorial description and assume irrational values. However, we present a generalization of the construction which encompasses also split quasimorphisms.

Finally, we discuss a third new construction of a quasimorphism, and this is independent from the other results in this thesis. We mimic in a free group setting Yoshida's construction of non-trivial classes in the third singular bounded cohomology $\mathrm{H}^3_\mathrm{b}(\Sigma, \mathbb{R})$ of a closed hyperbolic surface Σ (see [57]). In his construction, such classes are obtained using the fact that the mapping torus of a pseudo-Anosov diffeomorphism of Σ admits the structure of a hyperbolic 3-manifold. The underlying cohomological fact for Yoshida's construction can be stated in the following way: For a group Γ and an automorphism φ of Γ, the bounded cohomology of the mapping torus $\Gamma *_\varphi$ embeds into the bounded cohomology of Γ. This is a general mechanism which we apply, in an analogous manner, to mapping tori over automorphisms of $\mathbb{F}_n$ that are hyperbolic in the sense of Gromov. Such automorphisms only exist when $n \geq 3$. We obtain classes in $\mathrm{H}^2_\mathrm{b}(\mathbb{F}_n, \mathbb{R})$, and therefore quasimorphisms. The precise statement reads

Theorem 9.6. *For each hyperbolic automorphism φ of the free group $\mathbb{F}_n$ there is an embedding*

$$\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) \hookrightarrow \mathrm{H}^2_\mathrm{b}(\mathbb{F}_n, \mathbb{R}).$$

This theorem is meaningful only if the hyperbolic automorphism φ is such that the second cohomology $\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R})$ is non-trivial. We show that the existence of such automorphisms follows from work of Clay-Pettet (see [21]). While we do not have an explicit example it should be noted that our construction is fundamentally different from others, as can be seen from the fact that it can only be carried out for free groups of rank at least 3.

1 Preliminaries

1.1 Quasicocycles and bounded cohomology

The theory of bounded group cohomology has its origin in the work of Johnson on Banach algebras (see [35]). In a seminal paper Gromov introduced bounded singular cohomology of spaces and proved, among numerous other deep results, that the bounded cohomology of a simply connected space vanishes (see [28]). Ivanov developed an approach to bounded cohomology of discrete groups by means of relative homological algebra (see [34]), building on earlier work of Brooks. Ivanov improved Gromov's result by establishing an isomorphism between the bounded cohomology of a space and the bounded cohomology of its fundamental group. More recently, Burger and Monod have developed a functorial approach to the continuous bounded cohomology of topological groups (see [15], [16], [41]). In this thesis we mainly use bounded cohomology as a suitable language, while we only invoke a few results of the theory. For more background on the topic we refer the reader to [41] and [42].

Throughout this work we denote by Γ a finitely generated group. A map $f : \Gamma \longrightarrow \mathbb{R}$ is called a *quasimorphism* if there exists $C > 0$ such that

$$|f(gh) - f(g) - f(h)| < C, \qquad \forall\, g, h \in \Gamma.$$

The *defect* of a quasimorphism f is defined to be

$$\operatorname{def} f := \sup_{g,h\in\Gamma} |f(gh) - f(g) - f(h)|.$$

A Banach space E that is equipped with a linear isometric action of the group Γ is called a *Banach* Γ*-module*. Having such a module amounts to having a representation

$$\rho : \Gamma \longrightarrow \mathrm{Iso}_{\mathcal{L}}(E)$$

of Γ into the group of linear isometries of the space E. For a Banach Γ-module E we say that a map $f : \Gamma \longrightarrow E$ is a *quasicocycle* if there exists $C > 0$ such that

$$\|f(gh) - f(g) - g.f(h)\|_E < C, \qquad \forall\, g, h \in \Gamma.$$

The defect of such a map is defined accordingly. We denote by $\mathrm{Q}\mathcal{Z}(\Gamma, E)$ the space of quasicocycles with values in E, and by $\mathrm{QM}(\Gamma) := \mathrm{Q}\mathcal{Z}(\Gamma, \mathbb{R})$ the space of quasimorphisms on Γ.

Recall that the group cohomology $\mathrm{H}^*(\Gamma, E)$ is computed by the Eilenberg–Maclane bar complex

$$0 \to E \to \mathrm{Map}(\Gamma, E) \xrightarrow{\partial^1} \mathrm{Map}(\Gamma^2, E) \xrightarrow{\partial^2} \mathrm{Map}(\Gamma^3, E) \xrightarrow{\partial^3} \dots$$

with the coboundary operators

$$\partial^k f(g_1,\dots,g_{k+1}) := g_1.f(g_2,\dots,g_{k+1}) \\ + \sum_{i=1}^{k}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{k+1}) + (-1)^{k+1} f(g_1,\dots,g_k).$$

The bounded cohomology $\mathrm{H}^*_{\mathrm{b}}(\Gamma, E)$ is the cohomology of the subcomplex of bounded maps, which we call the *bounded bar complex*:

$$0 \to E \to \ell^\infty(\Gamma, E) \xrightarrow{\partial^1} \ell^\infty(\Gamma^2, E) \xrightarrow{\partial^2} \ell^\infty(\Gamma^3, E) \xrightarrow{\partial^3} \dots$$

We denote the cocycles in these complexes by $\mathcal{Z}^*(\Gamma, E)$ and $\mathcal{Z}^*_{\mathrm{b}}(\Gamma, E)$ respectively. The 1-coboundary of a quasicocycle $f : \Gamma \longrightarrow E$ as introduced above is given by $\partial^1 f(g,h) = f(g) + g.f(h) - f(gh)$, so f is almost a cocycle in the bar complex, and since $\partial^2\partial^1 = 0$ we have that $\partial^1 f$ is a 2-cocycle in the bounded bar complex. We denote by $\omega_f := [\partial^1 f]_{\mathrm{b}}$ the corresponding bounded cohomology class. To say it short, we have a linear map

$$\mathrm{Q}\mathcal{Z}(\Gamma, E) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, E), \quad f \mapsto \omega_f$$

whose image $\mathrm{EH}^2_{\mathrm{b}}(\Gamma, E)$ is equal to the kernel of the comparison map $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E) \longrightarrow \mathrm{H}^2(\Gamma, E)$. It is straightforward to check that a quasicocycle is in the kernel of the above map if and only if it admits a decomposition $f = \varphi + \beta$ into an actual cocycle $\varphi \in \mathcal{Z}^1(\Gamma, E)$ and a bounded perturbation $\beta \in \ell^\infty(\Gamma, E)$. These are called *trivial* quasicocycles, the decomposition is called a *trivialization.* A trivialization is unique if and only if the module E is trivial. In general the components of two trivializations may differ by *inner cocycles*

$$\iota_v : \Gamma \longrightarrow E, \quad g \mapsto g.v - v, \quad v \in E. \tag{1}$$

These are the 1-coboundaries in the bounded bar complex. With this terminology, the space $\mathrm{H}^1_{\mathrm{b}}(\Gamma, E)$ can be described as the quotient of the bounded 1-cocycles modulo the inner cocycles. Under fairly general conditions every bounded 1-cocycle is inner:

Proposition 1.1 ([41], Proposition 6.2.1, Corollary 7.5.11)**.** *For a group Γ and a Banach Γ-module E we have $\mathrm{H}^1_{\mathrm{b}}(\Gamma, E) = 0$ if*

(i) *E is reflexive as a Banach space, or*

(ii) *Γ is amenable and E is a coefficient module.*

We will not recall the definition of a coefficient module here, for our purposes it is sufficient to know that the Γ-modules $\ell^p(\Gamma)$, $1 \leq p \leq \infty$, are coefficient modules ([41], Examples 1.2.3).

The spaces $\mathrm{H}^k_{\mathrm{b}}(\Gamma, E)$ carry a quotient semi-norm coming from the norms of the ℓ^∞-spaces in the bounded bar complex. For $k = 2$ and a separable coefficient module E, this is a proper norm which turns $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$ into a Banach space ([41], Corollary 11.4.2). Calculating this *Gromov norm* for a cohomology class of the form ω_f amounts to finding the infimum of the defects over all the quasicocycles at bounded distance from f:

$$\begin{aligned}\|\omega_f\| &= \inf\left\{\operatorname{def}\overline{f} \mid \overline{f} \in \mathrm{Q}\mathcal{Z}(\Gamma, E) \text{ such that } f - \overline{f} \text{ is bounded}\right\}\\ &= \inf\left\{\operatorname{def}(f + \beta) \mid \beta \in \ell^\infty(\Gamma, E)\right\}.\end{aligned}$$

In the case of trivial coefficients $E = \mathbb{R}$ the Gromov norm is related to the notion of a *homogenous* quasimorphism. This is a quasimorphism $f : \Gamma \longrightarrow \mathbb{R}$ for which

$$f(g^n) = n \cdot f(g), \quad \forall g \in \Gamma \; \forall n \in \mathbb{Z},$$

which is to say that f restricts to a homomorphism on every cyclic subgroup. We write $\mathrm{HQM}(\Gamma)$ for the corresponding subspace of $\mathrm{QM}(\Gamma)$. Homogenous quasimorphisms are invariant under conjugation, and for each quasimorphism f, there is a unique $\widehat{f} \in \mathrm{HQM}(\Gamma)$ at bounded distance from f. This *homogenization* of f is given by

$$\widehat{f}(g) = \lim_{n\to\infty} \frac{f(g^n)}{n},$$

and the assignment $f \mapsto \widehat{f}$ defines a projection $\mathrm{QM}(\Gamma) \longrightarrow \mathrm{HQM}(\Gamma)$. The following result of Bavard provides a lower bound for the Gromov norm of the class of a quasimorphism

Theorem 1.2 ([6], Section 3.6)**.** *For any group* Γ *and any quasimorphism* $f : \Gamma \longrightarrow \mathbb{R}$ *we have*

$$\|\omega_f\| \geq \tfrac{1}{2} \cdot \operatorname{def}\widehat{f}.$$

It is an open question whether equality holds for all quasimorphisms.

There is, in general, no canonical way of determining a quasimorphism $f \in \mathrm{QM}(\Gamma)$ such that $[\partial f]_{\mathrm{b}} = \omega$ for a given class $\omega \in \mathrm{EH}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$. However, for the free group $\mathbb{F}_n$ with a chosen basis we have

Proposition 1.3. *For* $\mathbb{F}_n = \langle x_1, \ldots, x_n\rangle$ *let*

$$\mathrm{HQM}_0(\mathbb{F}_n) = \{f \in \mathrm{HQM}(\mathbb{F}_n) \mid f(x_i) = 0 \textit{ for all } i\}.$$

There is a unique linear map

$$\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_n, \mathbb{R}) \longrightarrow \mathrm{HQM}_0(\mathbb{F}_n)$$

that is right-inverse to the natural map $\mathrm{HQM}_0(\mathbb{F}_n) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_n, \mathbb{R})$*,* $f \mapsto [\partial f]_{\mathrm{b}}$.

Proof. Let $\omega \in \mathrm{H}^2_\mathrm{b}(\mathbb{F}_n, \mathbb{R})$. Since $\mathrm{H}^2(\mathbb{F}_n, \mathbb{R}) = 0$ we have $\mathrm{H}^2_\mathrm{b}(\mathbb{F}_n, \mathbb{R}) = \mathrm{EH}^2_\mathrm{b}(\mathbb{F}_n, \mathbb{R})$ and therefore $\omega = [\partial g]_\mathrm{b}$ for some $g \in \mathrm{QM}(\mathbb{F}_n)$. Define $\varphi \in \mathrm{Hom}(\mathbb{F}_n, \mathbb{R})$ by $\varphi(x_i) = -\widehat{g}(x_i)$ and set $f_\omega := \widehat{g} + \varphi \in \mathrm{HQM}_0(\mathbb{F}_n)$. □

The following result has been known to the experts for some time it seems; a proof can be found in Huber's thesis ([32], Theorem 2.14).

Theorem 1.4. *An epimorphism $\varphi : \Gamma \longrightarrow \Gamma'$ between countable groups induces an isometric embedding*

$$\varphi^* : \mathrm{H}^2_\mathrm{b}(\Gamma', \mathbb{R}) \hookrightarrow \mathrm{H}^2_\mathrm{b}(\Gamma, \mathbb{R}).$$

For an automorphism $\tau \in \mathrm{Aut}(\Gamma)$ there is an induced isomorphism τ^* in bounded cohomology with trivial coefficients, so that for each $k \geq 0$ we have an action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{H}^k_\mathrm{b}(\Gamma, \mathbb{R})$, given by $\tau.\omega = (\tau^*)^{-1}\omega$. For inner automorphisms this action is trivial ([41], Lemma 8.7.2), and therefore we have an induced action of $\mathrm{Out}(\Gamma)$. This action preserves the semi-norm mentioned above, in particular we have a linear isometric action of $\mathrm{Out}(\Gamma)$ on the Banach space $\mathrm{H}^2_\mathrm{b}(\Gamma, \mathbb{R})$. We call this the *natural action* of $\mathrm{Out}(\Gamma)$ on H^2_b.

1.2 Relative quasicocycles and bounded cohomology

The relative version of bounded cohomology for topological spaces was defined by Gromov (see [28]). A systematic treatment of relative bounded cohomology of both spaces and groups was initiated by Park (see [46]), and these notions have played a role in the recent work of Frigerio–Pagliantini (see [25]).

Let H be a subgroup of a group Γ, and let E be a Banach Γ-module. We denote by

$$\mathrm{Q}\mathcal{Z}(\Gamma, H; E) := \{f \in \mathrm{Q}\mathcal{Z}(\Gamma, E) \,|\, f|_H = 0\}$$

the space of *relative quasicocycles* with values in E for the pair (Γ, H). As a special case we have the space $\mathrm{QM}(\Gamma, H) := \mathrm{Q}\mathcal{Z}(\Gamma, H; \mathbb{R})$ of *relative quasimorphisms*. For each $k \geq 1$ we have the restriction map $\ell^\infty(\Gamma^k, E) \longrightarrow \ell^\infty(H^k, E)$. These maps are compatible with the differentials of the bounded bar complex; they define the second morphism in the following short exact sequence of chain complexes:

$$0 \to \ell^\infty(\Gamma^*, H^*; E) \to \ell^\infty(\Gamma^*, E) \to \ell^\infty(H^*, E) \to 0.$$

Here we omitted the differentials ∂^* from the notation. The leftmost of these chain complexes consists of the spaces

$$\ell^\infty(\Gamma^k, H^k; E) := \{f : \Gamma^k \longrightarrow E \,|\, f \text{ is bounded and } f|_{H^k} = 0\}.$$

Its cohomology, denoted by $\mathrm{H}^*_{\mathrm{b}}(\Gamma, H; E)$, is called the *relative bounded cohomology* of the pair (Γ, H) with coefficients in E. More explicitly, we have

$$\mathrm{H}^k_{\mathrm{b}}(\Gamma, H; E) = \frac{\ker \partial^k}{\operatorname{im} \partial^{k-1}} = \frac{\mathcal{Z}^k_{\mathrm{b}}(\Gamma, H; E)}{\mathcal{B}^k_{\mathrm{b}}(\Gamma, H; E)}$$

By a standard argument from homological algebra, the above short exact sequence induces a long exact sequence in cohomology:

$$\ldots \to \mathrm{H}^{k-1}_{\mathrm{b}}(H, E) \to \mathrm{H}^k_{\mathrm{b}}(\Gamma, H; E) \to \mathrm{H}^k_{\mathrm{b}}(\Gamma, E) \to \mathrm{H}^k_{\mathrm{b}}(H, E) \to \ldots$$

For a relative quasicocycle $f \in \mathrm{Q}\mathcal{Z}(\Gamma, H; E)$ we have $\partial f \in \mathcal{Z}^2_{\mathrm{b}}(\Gamma, H; E)$, so that there is the map

$$\mathrm{Q}\mathcal{Z}(\Gamma, H; E) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, H; E), \quad f \mapsto [\partial f]_{\mathrm{b}}.$$

In particular we have the map $\mathrm{QM}(\Gamma, H) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$. The case of real coefficients is most relevant to us. In dimension 1 we have

Proposition 1.5. *For any pair of groups* (Γ, H) *we have* $\mathrm{H}^1_{\mathrm{b}}(\Gamma, H; \mathbb{R}) = 0$.

Proof. We have

$$\mathcal{Z}^1_{\mathrm{b}}(\Gamma, H; \mathbb{R}) \subset \mathcal{Z}^1_{\mathrm{b}}(\Gamma, \mathbb{R}) = \mathrm{Hom}(\Gamma, \mathbb{R}) \cap \ell^\infty(\Gamma, \mathbb{R}) = 0. \qquad \square$$

Note that the spaces $\mathrm{H}^k_{\mathrm{b}}(\Gamma, H; E)$ carry a quotient semi-norm, as in the absolute case. In the situation $k = 2$ and $E = \mathbb{R}$ we again have an actual norm on cohomology:

Proposition 1.6. *Let H be a subgroup of a group Γ.*

(i) *The space* $\mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$ *is a Banach space.*

(ii) *The map*

$$i : \mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$$

in the long exact sequence above is a norm non-increasing embedding.

Proof. The long exact sequence contains the segment

$$\mathrm{H}^1_{\mathrm{b}}(H, \mathbb{R}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R}) \xrightarrow{\;i\;} \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$$

By Proposition 1.1 we have $\mathrm{H}^1_{\mathrm{b}}(H, \mathbb{R}) = 0$, so that i is an embedding. The map $\ell^\infty(\Gamma^2, H^2; \mathbb{R}) \longrightarrow \ell^\infty(\Gamma^2, \mathbb{R})$ on the cochain level is norm non-increasing, and therefore the induced map i in cohomology has the same property. This proves part (ii) of the proposition. Now since i is a norm non-increasing embedding, and since $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ is a Banach space, the semi-norm on $\mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$ is a norm. This means that the latter space is Banach as well. $\square$

Proposition 1.7. *Let* (Γ, H) *be a pair of groups. If* H *has finite index in* Γ *then* $\mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R}) = 0$.

Proof. The natural map $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(H, \mathbb{R})$ is isometrically injective ([41], Proposition 8.6.6; see also Proposition 9.3). Furthermore we have $\mathrm{H}^1_{\mathrm{b}}(H, \mathbb{R}) = 0$ by Proposition 1.1. Using these facts we obtain the statement from the above long exact sequence. □

2 Split quasicocycles

Let Γ be a group and let E be a Banach Γ-module. We write

$$\mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(\Gamma, E) := \{f \in \mathrm{Q}\mathcal{Z}(\Gamma, E) \mid f(g) + g.f(g^{-1}) = 0\}$$

for the space of *alternating* quasicocycles. Assume that we have a splitting $\Gamma = A * B$. Through the embeddings $A, B \hookrightarrow \Gamma$ the space E is equipped with an A- and B-module structure. In order to construct a split quasicocycle we consider $f_A \in \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(A, E)$ and $f_B \in \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(B, E)$. We define a map

$$f_A * f_B : \Gamma \longrightarrow E$$

as follows: For an element $e \neq g \in \Gamma$ we let

$$g = a_1 b_1 a_2 b_2 \cdots a_n b_n,$$

be its normal form in which $a_i \in A$, $b_i \in B$ and only a_1 or b_n are possibly trivial. We set

$$\begin{aligned}&(f_A * f_B)(g) := \\ &f_A(a_1) + a_1.f_B(b_1) + a_1 b_1.f_A(a_2) + \ldots + a_1 b_1 a_2 \cdots b_{n-1} a_n.f_B(b_n).\end{aligned}$$

Furthermore we set $(f_A * f_B)(e) = 0$.

Proposition 2.1. *The map $f = f_A * f_B$ is an alternating quasicocycle on Γ with* $\operatorname{def} f = \max\{\operatorname{def} f_A, \operatorname{def} f_B\}$. *The induced linear map*

$$\mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(A, E) \times \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(B, E) \longrightarrow \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(\Gamma, E), \quad (f_A, f_B) \mapsto f$$

extends the natural isomorphism

$$\mathcal{Z}^1(A, E) \times \mathcal{Z}^1(B, E) \longrightarrow \mathcal{Z}^1(\Gamma, E).$$

Proof. The fact that the map f is alternating follows immediately from the corresponding property of the factor maps. We show that f is indeed a quasicocycle. Let $g, h \in \Gamma$. If g ends with an A-letter and h begins with B-letter or vice versa, then $\partial f(g, h) = 0$ since the normal form of gh equals the concatenation of the normal forms of g and h. If the normal forms are $g = g'a$ and $h = a^{-1}h'$ then

$$\begin{aligned}\partial f(g, h) &= f(g'h') - f(g'a) - g'a.f(a^{-1}h') \\ &= f(g'h') - f(g') - g'.(f(a) + a.f(a^{-1})) - g'.f(h') \\ &= \partial f(g', h')\end{aligned}$$

since the quasicocycle f_A is alternating. The same holds for B-letters. So we may assume that $g = g'a_1$ and $h = a_2h'$ with $a_1a_2 \neq 1$ (or likewise with B-letters). In this case we have

$$\begin{aligned} \partial f(g,h) =& f(g'a_1a_2h') - f(g'a_1) - g'a_1.f(a_2h') \\ =& f(g') + g'.f(a_1a_2) + g'a_1a_2.f(h') \\ & - f(g') - g'.f(a_1) - g'a_1.f(a_2) - g'a_1a_2.f(h') \\ =& g'.(f(a_1a_2) - f(a_1) - a_1.f(a_2)) \\ =& g'.\partial f_A(a_1,a_2), \end{aligned}$$

so that $\|\partial f(g,h)\|_E = \|\partial f_A(a_1,a_2)\|_E \leq \operatorname{def} f_A$. Hence f is a quasicocycle with the defect indicated above. □

Note that the quasicocycle $f_A * f_B$ is an actual cocycle if and only if f_A and f_B are both cocycles. In particular we have $\iota_v^A * \iota_v^B = \iota_v^\Gamma$, where the inner cocycles (see equation (1) in subsection 1.1) are defined on the groups indicated in the superscript.

We refer to bounded cohomology classes of the form ω_f, where f is a split quasicocycle for the group $\Gamma = A * B$, as *split classes.*

2.1 Finite-dimensional coefficients

Let $\Gamma = A * B$ and let E be a Banach Γ-module. We write

$$\mathcal{L} := \ell^\infty_{\text{alt}}(A,E) \times \ell^\infty_{\text{alt}}(B,E) \subseteq \mathrm{Q}\mathcal{Z}_{\text{alt}}(A,E) \times \mathrm{Q}\mathcal{Z}_{\text{alt}}(B,E)$$

for the space of pairs of alternating bounded maps on the factors of Γ. Furthermore we let

$$\mathcal{B} := \{(f_A, f_B) \in \mathcal{L} \mid f_A * f_B \text{ is bounded}\}$$

be the subspace of pairs which yield a bounded split quasicocycle. The main result of this subsection is

Theorem 2.2. *Let $\Gamma = A * B$ and let E be a Banach Γ-module. If the factor A contains an element of infinite order then the dimension of the space $\mathcal{L} \,/\, \mathcal{B}$ is infinite.*

Proof. We fix a non-zero vector $v \in E$, an element $a \in A$ of infinite order and a non-trivial element $b \in B$. For a prime number p and $n \geq 0$ we define words $w_{p,n} \in \Gamma$ by

$$\begin{aligned} w_{p,0} &:= 1 \\ w_{p,n} &:= ba^p ba^{p^2} \cdots ba^{p^n}, \quad n \geq 1 \end{aligned}$$

Claim: For all prime numbers p we can choose a bounded map $f^p_A \in \ell^\infty_{\mathrm{alt}}(A,E)$ such that the split quasicocycles $f^p := f^p_A * 0$ satisfy

$$\begin{aligned} f^p(w_{p,n}) &= n \cdot v && \forall p, n \\ f^p(w_{q,n}) &= 0 && \forall q \neq p \,\forall n. \end{aligned}$$

Proof of the claim. For $g = a^{p^n}$, $n \geq 1$, define

$$f^p_A(g) = (w_{p,n-1}b)^{-1}.v$$

and extend to negative powers a^{-p^i} in the way needed to make f^p_A alternating. For all $g \in A$ which are not of the form $g = a^{\pm p^i}$ set $f^p_A(g) = 0$. Using the construction of split quasicocycles we obtain

$$\begin{aligned} f^p(w_{p,n}) &= f^p(w_{p,n-1}) + (w_{p,n-1}b).f^p_A(a^{p^n}) \\ &= f^p(w_{p,n-1}) + v \\ &= f^p(w_{p,n-2}) + 2v \\ &= \ldots = n \cdot v. \end{aligned}$$

The property $f^p(w_{q,n}) = 0$ holds by construction and thus the claim is established. We finally show that the intersection

$$\mathcal{B} \cap \operatorname{span}\{(f^p_A, 0) \,|\, p \text{ prime}\}$$

of subspaces of $\mathcal{L}$ is trivial, which implies the statement of the lemma. So assume that $f = \sum_j \lambda_j f^{p_j}$ is a bounded quasicocycle. Evaluating at $w_{p_j,n}$ yields the equation $f(w_{p_j,n}) = \lambda_j n \cdot v$, whence $\lambda_j = 0$ for all j. □

Applied to the case of a finite-dimensional module E this result implies

Theorem 2.3. *Let Γ be a finitely generated group with a splitting $\Gamma = A * B$, such that A contains an element of infinite order. Then for any finite dimensional Banach Γ-module E the split classes form an infinite dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, E)$.*

and in particular

Corollary 2.4. *For a non-abelian free group $\mathbb{F}$ the split classes form an infinite dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}, E)$ for any finite-dimensional Banach $\mathbb{F}$-module E.*

Proof of Theorem 2.3. The construction of split quasicocycles yields a map

$$\Psi : \mathcal{L} \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, E).$$

Note that $\mathcal{B} \subseteq \ker \Psi$ since bounded quasicocycles are trivial. We show that the quotient $\ker \Psi \,/\, \mathcal{B}$ has finite dimension. This implies, by Theorem 2.2, the infinite-dimensionality of the space

$$\operatorname{im} \Psi \quad \cong \quad \mathcal{L} \,/\, \ker \Psi \quad \cong \quad (\mathcal{L} \,/\, \mathcal{B}) \,/\, (\ker \Psi \,/\, \mathcal{B}).$$

So let $(f_A, f_B) \in \ker \Psi$. The quasicocycle $f := f_A * f_B$ has a trivialization $f = \varphi + \beta$, where $\varphi \in \mathcal{Z}^1(\Gamma, E)$ and $\beta \in \ell^\infty(\Gamma, E)$. If $f = \varphi' + \beta'$ is another trivialization then the cocycle $\varphi - \varphi' = \beta' - \beta$ is bounded. This means that we can assign to (f_A, f_B) a cocycle which is well defined up to addition of a bounded cocycle. Hence we have a map

$$\ker \Psi \longrightarrow \mathcal{Z}^1(\Gamma, E) \,/\, \mathcal{Z}^1_{\mathrm{b}}(\Gamma, E).$$

The kernel of this map is equal to the space $\mathcal{B}$ and so we have an embedding

$$\ker \Psi \,/\, \mathcal{B} \hookrightarrow \mathcal{Z}^1(\Gamma, E) \,/\, \mathcal{Z}^1_{\mathrm{b}}(\Gamma, E).$$

Since Γ is finitely generated and E is finite-dimensional the space $\mathcal{Z}^1(\Gamma, E)$ is finite dimensional as well. It follows that the quotient $\ker \Psi \,/\, \mathcal{B}$ has finite dimension. □

2.2 ℓ^p-coefficients

For a countable group Γ and $1 \leq p \leq \infty$ we endow $\ell^p(\Gamma)$ with the usual left action. That is, for $\chi \in \ell^p(\Gamma)$ and $g \in \Gamma$ we set $(g.\chi)(h) := \chi(g^{-1}h)$.

Theorem 2.5. *Let Γ be a countable group with a splitting $\Gamma = A * B$, such that A contains an element of infinite order. For $1 < p < \infty$ the split classes form an infinite-dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^p(\Gamma))$. If the factor A is amenable then the same holds for $p = 1$.*

Corollary 2.6. *For a non-abelian free group $\mathbb{F}$ the split classes form an infinite dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}, \ell^p(\mathbb{F}))$ for $1 \leq p < \infty$.*

We first establish the fact

Lemma 2.7. *Let $\Gamma = A * B$ be a splitting, and let $f : \Gamma \longrightarrow E$ be a trivial quasicocycle that is bounded on the free factors A and B. If either A is amenable and E is a coefficient module, or if E is reflexive, then f has a trivialization of the form*

$$f = 0 * \varphi_B + \beta$$

for some $\varphi_B \in \mathcal{Z}^1(B, E)$ and some $\beta \in \ell^\infty(\Gamma, E)$.

Proof. Let $f = \varphi + \beta$ be a trivialization of f. By assumption, the cocycle φ splits as $\varphi = \varphi_A * \varphi_B$ into bounded cocycles on the factors. By Proposition 1.1 we have that $\mathrm{H}^1_\mathrm{b}(A, E) = 0$, so $\varphi_A = \iota^A_v$ for some $v \in E$. Write

$$\begin{aligned} f &= \iota^A_v * \varphi_B + \beta \\ &= (\iota^A_v * \varphi_B - \iota^A_v * \iota^B_v) + (\iota^A_v * \iota^B_v + \beta) \\ &= 0 * (\varphi_B - \iota^B_v) + (\iota^\Gamma_v + \beta), \end{aligned}$$

which is, up to renaming, a trivialization of the desired type. □

Proof of Theorem 2.5. We construct an embedding

$$\ell^p(\Gamma) \hookrightarrow \ell^\infty_{\mathrm{alt}}(A, \ell^p(\Gamma)), \quad \xi \mapsto r_\xi$$

such that the split quasicocycle $r_\xi * 0$ is trivial if and only if $\xi = 0$. We begin with fixing an infinite order element $a \in A$ and a non-trivial element $b \in B$. Let $w_n \in \Gamma$ be the sequence defined by

$$\begin{aligned} w_0 &= 1 \\ w_1 &= ab \\ w_n &= aba^2b \cdots a^{n-1}ba^nb, \quad n \geq 2. \end{aligned}$$

For $\xi \in \ell^p(\Gamma)$ we define the bounded map $r_\xi \in \ell^\infty_{\mathrm{alt}}(A, \ell^p(\Gamma))$ as follows: Set

$$r_\xi(a^n) = w_{n-1}^{-1}.\xi, \quad n \geq 1,$$

and extend to negative powers of a in the way needed to make r_ξ alternating. For all $g \in A$ that are not powers of a set $r_\xi(g) = 0$. Assume that the split quasicocycle $f_\xi := r_\xi * 0$ is trivial. Since ℓ^p-spaces are reflexive for $1 < p < \infty$, and since we assume A to be amenable in case $p = 1$, Lemma 2.7 yields a trivialization

$$f_\xi = 0 * \varphi_B + \beta.$$

We evaluate this equation at w_n, where we write $\zeta := \varphi_B(b) \in \ell^p(\Gamma)$. By construction we have $f_\xi(w_n) = n \cdot \xi$, so

$$n \cdot \xi = w_1 b^{-1}.\zeta + \ldots + w_{n-1}b^{-1}.\zeta + \beta(w_n).$$

This is an equation of functions in $\ell^p(\Gamma)$, which we evaluate further at $g \in \Gamma$ to obtain

$$n \cdot \xi(g) = \zeta(bw_1^{-1}g) + \ldots + \zeta(bw_{n-1}^{-1}g) + \beta(w_n)(g).$$

Using the Hölder inequality we obtain

$$\begin{aligned}
&|\zeta(bw_1^{-1}g)| + \ldots + |\zeta(bw_{n-1}^{-1}g)| \\
&\leq (n-1)^{1-1/p} \left(|\zeta(bw_1^{-1}g)|^p + \ldots + |\zeta(bw_{n-1}^{-1}g)|^p \right)^{1/p} \\
&\leq (n-1)^{1-1/p} \cdot \|\zeta\|_{\ell^p(\Gamma)} \\
&\leq n^{1-1/p} \cdot \|\zeta\|_{\ell^p(\Gamma)}.
\end{aligned}$$

Furthermore we have

$$|\beta(w_n)(g)| \leq \|\beta(w_n)\|_{\ell^p(\Gamma)} \leq \|\beta\|_{\ell^\infty(\Gamma, \ell^p(\Gamma))},$$

and hence

$$n \cdot |\xi(g)| \leq n^{1-1/p} \cdot \|\zeta\|_{\ell^p(\Gamma)} + \|\beta\|_{\ell^\infty(\Gamma, \ell^p(\Gamma))}.$$

Dividing both sides by n and letting n tend to infinity finally yields $\xi = 0$. Thus we have shown that the composition

$$\ell^p(\Gamma) \xrightarrow{\ I\ } \ell^\infty_{\mathrm{alt}}(A, \ell^p(\Gamma)) \times \ell^\infty_{\mathrm{alt}}(B, \ell^p(\Gamma)) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^p(\Gamma))$$

with $I(\xi) = (r_\xi, 0)$ is an embedding. The statement follows. □

Remark. There are certain types of coefficients for which all split classes are trivial. Indeed, for any group Γ one has $\mathrm{H}^*_{\mathrm{b}}(\Gamma, \ell^\infty(\Gamma)) = 0$, and the same holds more generally for relatively injective coefficient modules ([41], Proposition 7.4.1). Furthermore one can check that for $\Gamma = A * B$ the split classes in $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^\infty(\Gamma)/\mathbb{R})$ vanish. This is unfortunate, since this space is isomorphic ([41], Proposition 10.3.2) to the poorly understood space $\mathrm{H}^3_{\mathrm{b}}(\Gamma, \mathbb{R})$, which is known to be infinite dimensional for free groups ([51], Theorem 3).

Question. Is it true that for every reflexive $\mathbb{F}_2$-Banach module E the split classes form an infinite dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2, E)$?

3 Split quasimorphisms

3.1 Embedding defect spaces

Recall that a split quasimorphism $f = f_A * f_B$ on $\Gamma = A * B$ is defined by

$$f(a_1 b_1 \cdots a_n b_n) = f_A(a_1) + f_B(b_1) + \cdots + f_A(a_n) + f_B(b_n),$$

where f_A and f_B are alternating quasimorphisms on the factors. We determine the homogenization $\widehat{f}$ and calculate the Gromov norm of the bounded cohomology class ω_f. A non-trivial element of Γ is called *cyclically reduced* if its normal form begins with an A-letter and ends with a B-letter or vice versa. Note that this is not the standard use of the terminology.

Proposition 3.1. *The homogenization of a split quasimorphism $f = f_A * f_B$ on $\Gamma = A * B$ is given by*

$$\widehat{f}(g) = \begin{cases} \widehat{f_A}(g), & \text{if } g \in A \\ \widehat{f_B}(g), & \text{if } g \in B \\ & \\ f(g'), & \text{if } g \notin A \cup B \\ & \text{where } g' \text{ is any cyclically reduced conjugate of } g. \end{cases}$$

Proof. If $g \in A$ then $f(g^n) = f_A(g^n)$, so $\widehat{f}(g) = \widehat{f_A}(g)$ and likewise for $g \in B$. So assume that $g \notin A \cup B$. For a cyclically reduced conjugate g' of g we have $f({g'}^n) = n \cdot f(g')$ by construction of f. By conjugacy invariance of $\widehat{f}$ we obtain $\widehat{f}(g) = \widehat{f}(g') = f(g')$. □

Lemma 3.2. *Let $\widehat{f}$ be the homogenization of $f = f_A * f_B$. We have*

$$\operatorname{def} \widehat{f} \geq 2 \cdot \max\{\operatorname{def} f_A, \operatorname{def} f_B\}.$$

Proof. We may assume that $\operatorname{def} f_A \geq \operatorname{def} f_B$, and further that we can choose $a \in A$ with $a^2 \neq 1$. (Otherwise $\mathrm{QM}_{\mathrm{alt}}(A) = 0$, hence $\operatorname{def} f_A = \operatorname{def} f_B = \operatorname{def} \widehat{f} = 0$ and the statement is obvious). Let $b \in B$ and $a_1, a_2 \in A$ be non-trivial elements. We consider the words

$$\begin{aligned} g &= ab\, a_2 b\, a_1 b^{-1} a^{-1}, \\ h &= a^{-1} b^{-1} a_2 b\, a_1 b\, a. \end{aligned}$$

for which we have

$$\widehat{f}(g) = \widehat{f}(h) = \widehat{f}(a_2 b a_1) = f(a_1 a_2 b) = f_A(a_1 a_2) + f_B(b)$$

by conjugation invariance of $\widehat{f}$ and Proposition 3.1. Furthermore,

$$\begin{aligned}\widehat{f}(gh) &= \widehat{f}(aba_2ba_1b^{-1}a^{-2}b^{-1}a_2ba_1ba)\\ &= f(a^2ba_2ba_1b^{-1}a^{-2}b^{-1}a_2ba_1b)\\ &= 2(f_A(a_1)+f_A(a_2)+f_B(b)),\end{aligned}$$

and hence

$$\widehat{f}(gh)-\widehat{f}(g)-\widehat{f}(h) = 2\cdot(f_A(a_1)+f_A(a_2)-f_A(a_1a_2))$$

which implies

$$\begin{aligned}\operatorname{def}\widehat{f} &\geq 2\cdot\sup\{|f_A(a_1)+f_A(a_2)-f_A(a_1a_2)|\mid a_1,a_2\in A, a_1,a_2\neq 1\}\\ &= 2\cdot\operatorname{def} f_A\\ &= 2\cdot\max\{\operatorname{def} f_A,\operatorname{def} f_B\}.\end{aligned}$$

□

Theorem 3.3. *Let $f = f_A * f_B$ be a split quasimorphism with corresponding cohomology class $\omega_f = [\partial f]_{\mathrm{b}}$. We have*

$$\|\omega_f\| = \tfrac{1}{2}\operatorname{def}\widehat{f} = \operatorname{def} f = \max\{\operatorname{def} f_A,\operatorname{def} f_B\}.$$

In particular, f is a minimal defect representative for its class.

Proof. We have

$$\max\{\operatorname{def} f_A,\operatorname{def} f_B\} \leq \tfrac{1}{2}\operatorname{def}\widehat{f} \leq \|\omega_f\| \leq \operatorname{def} f = \max\{\operatorname{def} f_A,\operatorname{def} f_B\}$$

by Theorem 1.2, Lemma 3.2 and Proposition 2.1. □

Corollary 3.4. *For a split quasimorphism $f = f_A * f_B$ the following are equivalent:*

(i) *f is trivial*

(ii) *f is a homomorphism*

(iii) *f is homogenous*

(iv) *f_A and f_B are homomorphisms*

Corollary 3.5. *For $\Gamma = A * B$ the kernel of the linear map*

$$\mathrm{QM}_{\mathrm{alt}}(A)\times\mathrm{QM}_{\mathrm{alt}}(B)\longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R}),\quad (f_A,f_B)\mapsto\omega_f$$

with $f = f_A * f_B$, is equal to $\mathrm{Hom}(A,\mathbb{R})\times\mathrm{Hom}(B,\mathbb{R})$.

Since there are no bounded real-valued homomorphisms the last statement yields a linear embedding

$$\ell^\infty_{\mathrm{alt}}(A) \oplus \ell^\infty_{\mathrm{alt}}(B) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}).$$

By renorming the spaces $\ell^\infty_{\mathrm{alt}}$ in a suitable way, this embedding can be made isometric. Namely, we define the defect space $\mathcal{D}(\Gamma)$ of a group Γ to be the space of bounded alternating functions on Γ, equipped with the defect $\|\cdot\|_{\mathrm{def}} := \mathrm{def}(\cdot)$ as a norm. Section 7 is devoted to the study of defect spaces. Using this definition together with Theorem 3.3 we obtain

Theorem 3.6. *For a group $\Gamma = A * B$ there is a linear isometric embedding*

$$\mathcal{D}(A) \oplus_\infty \mathcal{D}(B) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$$

which maps the pair (f_A, f_B) to the bounded cohomology class ω_f of the split quasimorphism $f = f_A * f_B$.

Here the notation $\oplus_\infty$ stands for the direct sum equipped with the max-norm. We note that the construction of split quasicocycles has an obvious generalization to the case of a free product with several factors, and so does the above result and most of the results that follow.

Corollary 3.7. *If $\Gamma = A * B$ such that A contains an infinite order element, then there is a linear isometric embedding*

$$\mathcal{D}(\mathbb{Z}) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}).$$

In particular, the Banach space $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ is non-separable.

Proof. Let $\langle a \rangle \cong \mathbb{Z}$ be an infinite cyclic subgroup of A. The space $\mathcal{D}(\langle a \rangle)$ is non-separable by Corollary 7.2, and by Proposition 7.4 and Theorem 3.6 we have the isometric embeddings

$$\mathcal{D}(\langle a \rangle) \hookrightarrow \mathcal{D}(A) \hookrightarrow \mathcal{D}(A) \oplus_\infty \mathcal{D}(B) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}). \qquad \square$$

Corollary 3.8. *If the group Γ admits an epimorphism $\Gamma \longrightarrow \mathbb{F}_2$ then there is a linear isometric embedding*

$$\mathcal{D}(\mathbb{Z}) \oplus_\infty \mathcal{D}(\mathbb{Z}) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}).$$

Proof. Apply Theorem 1.4. $\square$

We refer to the appendix for a self-contained simple proof of the fact that split quasimorphisms induce an linear embedding of the space ℓ^∞ into $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2, \mathbb{R})$.

In what follows we apply the last statement to certain classes of groups that have been shown to (virtually) surject onto free groups.

Corollary 3.9. *If the non-abelian group* Γ *is*

(i) *a subgroup of a right-angled Artin group, or*

(ii) *the fundamental group of a compact special cube complex,*

then there is a linear isometric embedding $\mathcal{D}(\mathbb{Z}) \oplus_\infty \mathcal{D}(\mathbb{Z}) \longhookrightarrow \mathrm{H}^2_\mathrm{b}(\Gamma, \mathbb{R})$.

Proof. The group surjects onto $\mathbb{F}_2$ if it is of type (i) ([3], Corollary 1.6), and every group of type (ii) is also of type (i) ([30], Theorem 1.1). □

Corollary 3.10. *If the group* Γ

(i) *is word-hyperbolic and admits a proper and cocompact action on a CAT(0) cube complex, or*

(ii) *is the fundamental group of a compact hyperbolic 3-manifold,*

then there is a finite index subgroup $\Gamma' < \Gamma$ *that admits a linear isometric embedding* $\mathcal{D}(\mathbb{Z}) \oplus_\infty \mathcal{D}(\mathbb{Z}) \longhookrightarrow \mathrm{H}^2_\mathrm{b}(\Gamma', \mathbb{R})$.

Proof. A group of the type (i) has a finite index subgroup which is of type (ii) of the previous corollary ([2], Theorem 1.1). Moreover, every group of type (ii) is of type (i) ([7], Theorem 5.3). □

By a result of Epstein–Fujiwara non-elementary word-hyperbolic groups have infinite-dimensional H^2_b ([23], Theorem 1.1), thus we can ask whether every such group (virtually) admits an embedded defect space in its second bounded cohomology:

Question. Does every non-elementary word-hyperbolic group Γ have a finite index subgroup Γ' such that the space $\mathrm{H}^2_\mathrm{b}(\Gamma', \mathbb{R})$ contains an isometrically embedded copy of

(i) $\mathcal{D}(\mathbb{Z})$?

(ii) $\mathcal{D}(\mathbb{Z}) \oplus_\infty \mathcal{D}(\mathbb{Z})$?

3.2 Amalgamated products

It is natural to ask whether the construction of split quasicocycles generalizes to a construction for amalgamated products $\Gamma = A *_C B$. Generalized counting quasimorphisms for such groups have been constructed by Fujiwara (see [27]). If one tries to define quasicocycles $f = f_A *_C f_B$ on Γ by using the normal form for amalgams, it turns out that the required compatibility between $f_A \in \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(A, E)$,

$f_B \in \mathrm{Q}\mathcal{Z}_{\mathrm{alt}}(B,E)$ is so strong that the map f actually descends to a free product quotient of Γ, more precisely to the largest natural such quotient:

$$\pi : A *_C B \longrightarrow A/\langle\langle C\rangle\rangle * B/\langle\langle C\rangle\rangle.$$

Here $\langle\langle C\rangle\rangle$ stands for the normal closure of C in A and B respectively. In other words, quasicocycles constructed this way are merely pullbacks of split quasicocycles on free products. In the case of quasimorphisms we know (Theorem 1.4) that the above quotient map induces an isometric embedding in $\mathrm{H}^2_{\mathrm{b}}$, so that we obtain the following generalization of Theorem 3.6:

Theorem 3.11. *For an amalgamated product $\Gamma = A *_C B$ there is a linear isometric embedding*

$$\mathcal{D}(A/\langle\langle C\rangle\rangle) \oplus_\infty \mathcal{D}(B/\langle\langle C\rangle\rangle) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$$

which maps the pair (f_A, f_B) to the bounded cohomology class $\pi^*\omega_f = \omega_{\pi^* f}$ of the pullback of the split quasimorphism $f = f_A * f_B$.

Example. For the splitting $\mathrm{SL}(2,\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ the above quotient is equal to $\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. In this case we have a unique split class, see Proposition 3.15.

A more interesting application concerns surface groups. Let $\Sigma_{m,k}$ be the compact orientable surface of genus m with k boundary components, and let $\Gamma_{m,k} = \pi_1(\Sigma_{m,k})$. These groups (except the abelian ones) belong to both of the classes in Corollary 3.9, so that we already know that their $\mathrm{H}^2_{\mathrm{b}}$ contains a copy of $\mathcal{D}(\mathbb{Z}) \oplus_\infty \mathcal{D}(\mathbb{Z})$. Glueing $\Sigma_{m,1}$ with $\Sigma_{n,1}$ along the boundaries yields the surface $\Sigma_{m+n,0}$. On the level of fundamental groups this corresponds to an amalgamation

$$\Gamma_{m+n,0} = \Gamma_{m,1} *_{\langle\gamma\rangle} \Gamma_{n,1},$$

where γ is a generator for the cyclic fundamental group of the glueing curve. The corresponding free product quotient is

$$\pi : \Gamma_{m+n,0} \longrightarrow \Gamma_{m,0} * \Gamma_{n,0},$$

which is induced by the pinching map $\Sigma_{m+n,0} \longrightarrow \Sigma_{m,0} \vee \Sigma_{n,0}$ that contracts the glueing curve to a point. We thus have the following

Theorem 3.12. *Let Γ_m be the fundamental group of the closed orientable surface of genus m. For $m, n \geq 1$ there is a linear isometric embedding*

$$\mathcal{D}(\Gamma_m) \oplus_\infty \mathcal{D}(\Gamma_n) \hookrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma_{m+n},\mathbb{R}).$$

Using the same argument one can obtain such embeddings more generally for surfaces with non-empty boundary and also for suitable splittings of higher dimensional manifolds along π_1-injective codimension one submanifolds.

3.3 Actions of automorphisms

For a group Γ we have the natural action of $\mathrm{Out}(\Gamma)$ on $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ which we discussed in the Section 2. For the cohomology class ω_f of a quasimorphism f this action is given by $\tau.\omega_f = \omega_{\tau.f}$, where we write $\tau.f := f \circ \tau^{-1}$. For split quasimorphisms and classes there is another description of this action. We fix a splitting $\Gamma = A * B$ and denote by $S \subset \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ the corresponding subspace of split classes, that is, the image of the embedding of $\mathcal{D}(A) \oplus \mathcal{D}(B)$. For $\tau \in \mathrm{Aut}(\Gamma)$ we have the induced splitting $\Gamma = \tau(A) * \tau(B)$. We denote its space of split classes by S^τ. There is a natural isometric isomorphism $S \longrightarrow S^\tau$ which comes from a map on the level of quasimorphisms, namely

$$f = f_A * f_B \quad \mapsto \quad f^\tau := f_{\tau(A)} * f_{\tau(B)}$$

where

$$f_{\tau(A)} = f_A \circ \tau^{-1}|_{\tau(A)},\ f_{\tau(B)} = f_B \circ \tau^{-1}|_{\tau(B)},$$

which induces

$$S \longrightarrow S^\tau, \quad \omega_f \mapsto \omega_{f^\tau}.$$

Now it follows immediately from the construction of split quasimorphisms that $\tau.f = f^\tau$. This is to say that the automorphism τ turns the split quasimorphism f into a split quasimorphism for the splitting induced by τ. In the case of an inner automorphism σ we know further that $\sigma.f$ defines the same cohomology class as f. Hence we have

Proposition 3.13. *The space S^τ only depends on the outer class of an automorphism τ.*

As a consequence we have

Proposition 3.14. *If the group Γ admits a splitting $\Gamma = A * B$ with finite factors A,B then the subspace $S \subset \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ is independent of the choice of a splitting.*

Proof. As a consequence of Kurosh's theorem (see, e.g., [49], Theorem 14) every splitting of Γ is a conjugate $A^g * B^g$ for some $g \in \Gamma$, so the claim follows from the previous proposition. □

The minimal example of a group with a non-trivial split quasimorphism is the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. (Note that the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ has only trivial quasimorphisms, as it is virtually cyclic.) By the previous proposition the space of split classes $S \subset \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ does not depend on the choice of a particular splitting. We have $\mathcal{D}(\mathbb{Z}/2\mathbb{Z}) \oplus \mathcal{D}(\mathbb{Z}/3\mathbb{Z}) \cong \{0\} \oplus \mathbb{R}$, so S is one-dimensional. It is generated by the class of a split quasimorphism

$$\mathfrak{R} : \mathrm{PSL}(2,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

known as the *Rademacher function*, a function appearing in several different areas of mathematics. A description of $\mathfrak{R}$ as a quasimorphism, from which one can easily see that it splits, was given by Barge–Ghys ([5], p. 246). Thus we have

Proposition 3.15. *For* $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ *there is, up to scaling, a unique non-zero split class* $\omega_{\mathfrak{R}} \in \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$*. It is the class associated to the Rademacher function.*

We may go further and define the total subspace $\mathcal{S}_\Gamma \subset \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ of split classes of Γ to be

$$\begin{aligned}\mathcal{S}_\Gamma := {}& \mathrm{span}\{\omega \,|\, \omega \text{ is a split class for some splitting of } \Gamma\} \\ = {}& \mathrm{span}\{\omega_f \,|\, f \text{ is a split quasimorphism for some splitting of } \Gamma\}.\end{aligned}$$

The group $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{S}_\Gamma$ via the linear extension of the assignment $\omega_f \mapsto \omega_{f^\tau}$, $\tau \in \mathrm{Aut}(\Gamma)$. By Proposition 3.13 these actions descend to $\mathrm{Out}(\Gamma)$.

Example. (i) If $\Gamma = A * B$ with finite factors then $\mathcal{S}_\Gamma$ is equal to the finite-dimensional space S associated to the given splitting, or, by Proposition 3.14, to any splitting. Hence we have a finite-dimensional representation $\mathcal{S}_\Gamma$ of the finite group $\mathrm{Out}(\Gamma)$. Compare this to the usual representation on the infinite dimensional space $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$.

(ii) In $\mathbb{F}_2$ any two splittings are related via an automorphism, so that $\mathcal{S}_{\mathbb{F}_2} = \mathrm{span}\{S^\tau \,|\, \tau \in \mathrm{Out}(\mathbb{F}_2)\}$, where S is the space associated to a preferred splitting.

We have no good understanding of the spaces $\mathcal{S}_\Gamma$ in case they have infinite dimension. It would be interesting to know

Questions. (i) How large is the Gromov-norm closure $\overline{\mathcal{S}_{\mathbb{F}_2}} \subset \mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2,\mathbb{R})$?

(ii) Is the action of $\mathrm{Out}(\Gamma)$ on $\mathcal{S}_\Gamma$ by isometries, so that it extends to an isometric action on the Banach space $\overline{\mathcal{S}_\Gamma}$?

(iii) What are the possible stabilizers $\mathrm{Stab}_{\mathrm{Out}(\mathbb{F}_2)}(\omega)$ of classes $\omega \in \mathcal{S}_{\mathbb{F}_2}$?

For the standard action of $\mathrm{Out}(\mathbb{F}_2)$ on $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2,\mathbb{R})$ we are able to give a partial answer to the third of these questions. Namely we show that there exist split quasimorphisms f on $\mathbb{F}_2 = \langle a\rangle * \langle b\rangle$ such that $f, \widehat{f}$ and ω_f have infinite stabilizers. For this purpose we consider the automorphism

$$\tau_n : \begin{cases} a \mapsto a \\ b \mapsto a^n b \end{cases}$$

In the following statements we call a function $f : \mathbb{Z} \longrightarrow \mathbb{R}$ *p-periodic* if $f(k+p) = f(k)$ for some $p \geq 1$ and all $k \in \mathbb{Z}$, and we say f is *periodic* if it is p-periodic for some p.

Theorem 3.16. *Let $f = f_A * f_B$ be a split quasimorphism on $\mathbb{F}_2 = \langle a \rangle * \langle b \rangle$, with bounded factors f_A, f_B. For each $n \in \mathbb{Z}\backslash\{0\}$ the following are equivalent*

(i) $\tau_n.f = f$

(ii) $\tau_n.\widehat{f} = \widehat{f}$

(iii) $\tau_n.\omega_f = \omega_f$

(iv) *The function f_A is $|n|$-periodic and the function f_B is equal to zero.*

Furthermore, if $|n| \leq 2$ these conditions imply that $f = 0$.

Corollary 3.17. *If $f_A \in \mathcal{D}(\langle a \rangle)$ is periodic then for $f = f_A * 0$ the stabilizers $\mathrm{Stab}_{\mathrm{Aut}(\mathbb{F}_2)}(f)$, $\mathrm{Stab}_{\mathrm{Out}(\mathbb{F}_2)}(\widehat{f})$ and $\mathrm{Stab}_{\mathrm{Out}(\mathbb{F}_2)}(\omega_f)$ are infinite.*

Proof. By the theorem these stabilizers contain the automorphism τ_n (or its outer class) which has infinite order in $\mathrm{Aut}(\mathbb{F}_2)$ (or in $\mathrm{Out}(\mathbb{F}_2)$). □

Write $\mathrm{Fix}(\tau) = \{\omega \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \,|\, \tau.\omega = \omega\}$ for the subspace of cohomology classes that are invariant under $\tau \in \mathrm{Aut}(\Gamma)$.

Corollary 3.18. *For $n \neq 0$ the intersection of $\mathrm{Fix}(\tau_n)$ with the space of split classes S is isometrically isomorphic to $\mathcal{D}(\mathbb{Z}/n\mathbb{Z})$. In particular this intersection is trivial for $n \in \{\pm 1, \pm 2\}$.*

Proof. By Proposition 7.3 the quotient map $\langle a \rangle \longrightarrow \langle a \,|\, a^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z}$ induces an isometric embedding $\mathcal{D}(\langle a \,|\, a^n = 1 \rangle) \hookrightarrow \mathcal{D}(\langle a \rangle)$, the image of which consists precisely of the n-periodic functions. □

Proof of Theorem 3.16. The last statement follows since an alternating n-periodic function on $\mathbb{Z}$ is zero when $n \in \{1, 2\}$. The implications (i) ⇒ (ii) ⇒ (iii) are obvious. In order to prove (iii) ⇒ (iv) assume that $\tau_n.\omega_f = \omega_f$, equivalently $\tau_{-n}.\omega_f = \omega_f$, which is equivalent to

$$\widehat{f} \circ \tau_n = \widehat{f} + \varphi \tag{$*$}$$

for some $\varphi \in \mathrm{Hom}(\mathbb{F}_2, \mathbb{R})$. Since $\tau_{-n} = \tau_n^{-1}$ we may assume that $n \geq 1$. We evaluate ($*$) for different group elements, where we make repeated use of Proposition 3.1. We write $f_A(k)$ instead of $f_A(a^k)$ and likewise for B. Since $\widehat{f}(a) = 0$ the equation yields $\varphi(a) = 0$. For $k \neq 0$ and $l \geq 1$ let $g = a^k b^l \in \mathbb{F}_2$. We have $\tau_n(g) = a^{k+n} b (a^n b)^{l-1}$, so that ($*$) evaluated at g reads

$$f_A(k+n) + f_B(1) + (l-1)[f_A(n) + f_B(1)] = f_A(k) + f_B(l) + l\varphi(b),$$

which we rearrange to

$$f_A(k+n) - f_A(n) + l[f_A(n) + f_B(1) - \varphi(b)] = f_A(k) + f_B(l).$$

Since the right hand side is bounded as a function of l the bracket vanishes, and we rearrange again to obtain

$$f_A(k+n) = f_A(k) + [f_A(n) + f_B(l)].$$

Since f_A is bounded this implies that the bracket in this new equation vanishes, and hence that $f_A(k+n) = f_A(k)$ for all $k \neq 0$. We are left with showing that $f_A(n) = 0$ which will imply that f_A is $|n|$-periodic and, since the bracket in the last equation vanishes for all $l \geq 1$, that $f_B = 0$. To do this we evaluate $(*)$ on the commutator $[a,b]$. The right hand side vanishes and we have $\tau_n([a,b]) = a^{1+n}ba^{-1}b^{-1}a^{-n}$, so that the evaluation yields

$$f_A(1+n) - f_A(1) - f_A(n) = 0$$

which implies that $f_A(n) = 0$, since $f_A(1+n) = f_A(1)$. We finally prove the implication (iv) $\Rightarrow$ (i). We have to show that if f_A is n-periodic and $f_B = 0$ then $f = f_A * f_B$ is such that for all $g \in \mathbb{F}_2$ we have $f(\tau_n(g)) = f(g)$. Consider the quotient map $\pi : \mathbb{F}_2 = \langle a \rangle * \langle b \rangle \longrightarrow \langle a \,|\, a^n = 1 \rangle * \langle b \rangle$. We have $\pi \circ \tau_n = \pi$, and this means that every power a^{k_i} in $g = a^{k_1}b^{l_1} \cdots a^{k_n}b^{l_n}$ corresponds to a power $a^{k_i + p\cdot n}$ in the factorization of $\tau_n(g)$ for some $p \in \mathbb{Z}$, and all other powers of a in $\tau_n(g)$ are of the form $a^{\pm n}$. Since f_A is n-periodic we deduce that $f(g) = f_A(k_1) + \cdots + f_A(k_n) = f(\tau_n(g))$. □

Note that the automorphisms τ_n are reducible as they fix the free factor $\langle a \rangle$, which supports the following

Conjecture. *If $\tau \in \mathrm{Out}(\mathbb{F}_2)$ is irreducible then $\tau.S \cap S = \{0\}$, in particular the stabilizer $\mathrm{Stab}_{\mathrm{Out}(\mathbb{F}_2)}(\omega_f)$ of every split class ω_f consists of reducible outer automorphisms.*

3.4 A relation to counting quasimorphisms

The standard example for a non-trivial quasimorphism on a free group $\mathbb{F}_2 = \langle a, b \rangle$ (and the only known example until [47]) is Brooks' counting quasimorphism, which is defined as follows (see [14]): For $w, g \in \mathbb{F}_2$ we denote by $h_w(g)$ the number of occurrences of w as a subword of g, when these elements are expressed as reduced words over the given generators. If either of w, g is trivial we set $h_w(g) = 0$. Here we allow overlaps, so that for example $h_{aba}(ababa) = 2$. The counting quasimorphism associated to the word w is given by $C_w := h_w - h_{w^{-1}} \in \mathrm{QM}_{\mathrm{alt}}(\mathbb{F}_2)$. It is non-trivial whenever $w \notin \{e, a^{\pm 1}, b^{\pm 1}\}$, and in fact, the classes induced by the family $\{C_w\}_{w \in \mathbb{F}_2}$ span an infinite dimensional subspace of $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_2, \mathbb{R})$ (see [40], Proposition 5.1). (There are however non-trivial linear dependencies, see [28], Assertion 5.1).

Note that for every coefficient function $\lambda : \mathbb{F}_2 \longrightarrow \mathbb{R}$, the infinite linear combination

$$\sum_{w \in \mathbb{F}_2} \lambda(w) C_w$$

converges to a map $f : \mathbb{F}_2 \longrightarrow \mathbb{R}$, in the topology of pointwise convergence in $\mathrm{Map}(\mathbb{F}_2, \mathbb{R})$. This is due to the fact that for each $g \in \mathbb{F}_2$, the set $\{w \in \mathbb{F}_2 \mid C_w(g) \neq 0\}$ is finite. The subspace $\mathrm{QM}(\mathbb{F}_2)$ is not closed in $\mathrm{Map}(\mathbb{F}_2, \mathbb{R})$ and it is not clear when the limit f is itself a quasimorphism. In [28] Grigorchuk pointed out that a sufficient, but not necessary condition is that λ be an ℓ^1-function. In the same article he showed that a suitably chosen family of counting quasimorphisms forms a Schauder basis for the space of homogenous quasimorphisms:

Theorem 3.19 ([28], Theorem 5.7)**.** *There exists a family $W \subset \mathbb{F}_2 = \langle a, b \rangle$ such that for every homogenous quasimorphism $f : \mathbb{F}_2 \longrightarrow \mathbb{R}$ that vanishes on the generators, there is a unique function $\alpha : W \longrightarrow \mathbb{R}$ with*

$$f = \sum_{w \in W} \alpha(w) \widehat{C}_w.$$

The representation of a class in $\mathrm{EH}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$ by a homogenous quasimorphism f is unique up to homomorphisms, so that it is unique when we require f to vanish on a given generating set (Proposition 1.3). The homogenization of a split quasimorphism $f = f_A * f_B$ on $\mathbb{F}_2$, with bounded factors f_A, f_B, has the property that $\widehat{f}(a) = \widehat{f}(b) = 0$ and has thus an associated coefficient function α from Grigorchuk's theorem. However, the computation of the precise values $\alpha(w)$, which is done recursively in the proof, turns out to be impractical even for the simplest choices for f_A, f_B. Our following observation says that split quasimorphisms actually admit a very explicit decomposition into a linear combination of counting quasimorphisms. For $k \geq 1$ we use the abbreviations

$$\begin{aligned} C_{a,k} &:= C_{ba^kb} + C_{ba^kb^{-1}} + C_{b^{-1}a^kb} + C_{b^{-1}a^kb^{-1}} \\ C_{b,k} &:= C_{ab^ka} + C_{ab^ka^{-1}} + C_{a^{-1}b^ka} + C_{a^{-1}b^ka^{-1}}. \end{aligned}$$

Theorem 3.20. *Let $f = f_A * f_B$ be a split quasimorphism on $\mathbb{F}_2 = \langle a \rangle * \langle b \rangle$ with f_A, f_B bounded. Then f is at bounded distance from the quasimorphism*

$$\sum_{k=1}^{\infty} f_A(a^k) C_{a,k} + f_B(b^k) C_{b,k},$$

in particular, the homogenization can be expressed as

$$\widehat{f} = \sum_{k=1}^{\infty} f_A(a^k)\widehat{C}_{a,k} + f_B(b^k)\widehat{C}_{b,k},$$

and furthermore, if both f_A *and* f_B *have finite support then* f *is at bounded distance from a finite linear combination of counting quasimorphisms.*

It is worthwhile to note that none of the words ba^kb, ba^kb^{-1}, etc. are contained in Grigorchuk's set W, which consists of words that are of minimal length in their conjugacy class and that don't have a prefix equal to a suffix.

Proof. Let F be the function given by the infinite sum in the theorem. Let $g = a^{k_1}b^{k_2}\cdots a^{k_{n-1}}b^{k_n} \in \mathbb{F}_2$. The power b^{k_2} is detected by exactly one of the four counting quasimorphisms appearing in C_{b,k_2}, depending on the signs of k_1 and k_3. It is counted with weight $f_B(b^{k_2})$, also if $k_2 < 0$ as both C_{b,k_2} and f_B are alternating. The same is true for all the powers $b^{k_2}, a^{k_3}, \ldots, b^{k_{n-2}}, a^{k_{n-1}}$. On the other hand, the quasimorphisms appearing in F count nothing but these powers, so that

$$F(g) = f(g) - f_A(a^{k_1}) - f_B(b^{k_n}),$$

which proves that the difference $F - f$ is bounded (and that F is a quasimorphism). □

4 Split quasi-representations

Let $G = (G, d)$ be a group endowed with a bi-invariant metric. For a set X we have an induced distance on the set of maps $X \longrightarrow G$ which is given by $d(f_1, f_2) = \sup_{x \in X} d(f_1(x), f_2(x))$. We say that f_1, f_2 are at bounded distance if $d(f_1, f_2) < \infty$, and we say that f is bounded if it is at bounded distance from the constant map $x \mapsto e$, in which case we write $\|f\|_\infty$ for this distance. A map $\mu : \Gamma \longrightarrow G$ is called a *quasi-representation* (or ε-representation or δ-homomorphism) if the maps $\Gamma \times \Gamma \longrightarrow G$,

$$(g, g') \mapsto \mu(gg') \qquad \text{and} \qquad (g, g') \mapsto \mu(g)\mu(g')$$

are at bounded distance. In this case the distance between these maps is denoted by $\operatorname{def} \mu$. Note that quasi-representations with values in $G = (\mathbb{R}, |\cdot|)$ are nothing but quasimorphisms. We write $\operatorname{QRep}(\Gamma, G)$ for the set of quasi-representations $\Gamma \longrightarrow G$ and

$$\operatorname{QRep}_{\mathrm{alt}}(\Gamma, G) = \left\{ \mu \in \operatorname{QRep}(\Gamma, G) \mid \mu(g^{-1}) = \mu(g)^{-1} \right\}$$

for the subset of alternating quasi-representations. For every quasi-representation $\mu : \Gamma \longrightarrow G$ we have the associated quantity

$$D(\mu) := \inf\{d(\mu, \rho) \mid \rho \in \operatorname{Hom}(\Gamma, G)\}$$

which measures the minimal distance to an actual representation.

As a straightforward generalization of split-quasimorphisms we obtain split quasi-representations on $\Gamma = A * B$ as follows: For $\mu_A \in \operatorname{QRep}_{\mathrm{alt}}(A, G)$ and $\mu_B \in \operatorname{QRep}_{\mathrm{alt}}(B, G)$ we define $\mu = \mu_A * \mu_B : \Gamma \longrightarrow G$ by

$$\begin{aligned} &(\mu_A * \mu_B)(a_1 b_1 \cdots a_n b_n) := \\ &\mu_A(a_1)\mu_B(b_1) \cdots \mu_A(a_n)\mu_B(b_n). \end{aligned}$$

Due to the bi-invariance of the metric on G, the proof of Proposition 2.1 applies in this non-commutative setting as well and we obtain

Proposition 4.1. *The map $\mu = \mu_A * \mu_B$ is an alternating quasi-representation with* $\operatorname{def} \mu = \max\{\operatorname{def} \mu_A, \operatorname{def} \mu_B\}$. *The induced map*

$$\operatorname{QRep}_{\mathrm{alt}}(A, G) \times \operatorname{QRep}_{\mathrm{alt}}(B, G) \longrightarrow \operatorname{QRep}_{\mathrm{alt}}(\Gamma, G), \quad (\mu_A, \mu_B) \mapsto \mu$$

extends the natural isomorphism

$$\operatorname{Hom}(A, G) \times \operatorname{Hom}(B, G) \longrightarrow \operatorname{Hom}(\Gamma, G).$$

In order to make a statement about the quantity $D(\mu)$ for a split quasi-representation μ we assume that the target group (G, d) has *no ε-small subgroups*, which means that the open ε-ball around the identity contains no non-trivial subgroup. We obtain the following result

Theorem 4.2. *Let $\Gamma = A * B$ and let $G = (G, d)$ be a group without ε-small subgroups. For bounded alternating maps $\mu_A : A \longrightarrow G$, $\mu_B : B \longrightarrow G$ with*

$$\delta := \max\{\|\mu_A\|_\infty, \|\mu_B\|_\infty\} \leq \frac{\varepsilon}{2}$$

the split quasi-representation $\mu = \mu_A * \mu_B : \Gamma \longrightarrow G$ satisfies

$$D(\mu) \geq \delta.$$

Proof. For $\delta = 0$ the statement is trivial, so we may assume that $\delta > 0$ and that there exists $\varphi \in \mathrm{Hom}(\Gamma, G)$ with $d(\mu, \varphi) < \delta$. For all $a \in A$ we have

$$d(\varphi(a), e) \leq d(\varphi(a), \mu_A(a)) + d(\mu_A(a), e) < \delta + \delta \leq \varepsilon$$

which means that the subgroup $\varphi(A) < G$ is ε-small and hence trivial. The same argument shows that $\varphi(B)$ is trivial, so the homomorphism φ is trivial. This means that the map μ is bounded with $\|\mu\|_\infty < \delta$. Now let $a \in A$ and $b \in B$ be different from the identity and let $g_\pm := ab^{\pm 1}$. By construction we have $\mu(g_\pm)^n = \mu(g_\pm^n)$. This means that the cyclic subgroups $\langle \mu(g_+) \rangle$ and $\langle \mu(g_-) \rangle$ of G are δ-small and hence trivial. In particular we have $\mu(g_\pm) = \mu_A(a)\mu_B(b)^{\pm 1} = e$, which implies $\mu_B(b)^2 = e$. The subgroup $\{e, \mu_B(b)\} < G$ is again δ-small, so that $\mu_B(b) = e$. It follows that $\mu_B \equiv e$, and likewise $\mu_A \equiv e$. Hence $\delta = 0$, a contradiction. □

An application of our split quasi-representations construced above was given by Burger–Ozawa–Thom. They observed that the parameters of the construction (namely the maps on the free factors) can be chosen in a way that yields unitary split quasi-representations with certain additional properties:

Proposition 4.3 ([17], Proposition 3.3)**.** *For all $n \geq 1$ and all $\delta > 0$ there exists a split quasi-representation $\mu : \mathbb{F}_2 \longrightarrow U(n)$ such that*

(i) $\operatorname{def} \mu \leq \delta$ *and* $D(\mu) \geq 2 - \delta/3$

(ii) μ *has dense image.*

The first statement here says in particular that $\mathbb{F}_2$ admits a sequence $\{\mu_i\}$ of finite-dimensional unitary split quasi-representations with $\operatorname{def}(\mu_i)$ converging to zero, such that $D(\mu_i)$ stays bounded away from zero. In the language of [17] this means that our construction of split quasi-representations can be used to establish that $\mathbb{F}_2$ is not Ulam stable. One should should compare this with Kazhdan's argument used his proof of the same property for surface groups, in which he uses a construction that is rather involved and by no means obvious ([36], Theorem 2).

5 Geometric deformation quasimorphisms

In this section we present a quasimorphism on the rank two free group which is obtained through thickening the group's Cayley graph. We make use of the fact that a function f on a group Γ is a quasimorphism iff the Γ-homogenous map $c : \Gamma^2 \longrightarrow \mathbb{R}$, $c(g_0, g_1) := f(g_0^{-1} g_1)$ is a *homogenous* quasicocycle, which means that

$$\sup_{g_0, g_1, g_2 \in \Gamma} |c(g_1, g_2) - c(g_0, g_2) + c(g_0, g_1)| < \infty.$$

For $\alpha, \beta > 0$ and $0 < t < \min\{\alpha, \beta\}$ we consider the following Euclidean polygonal complex $\mathcal{P}$

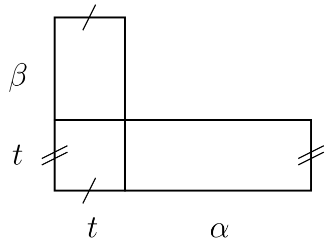


Figure 1: The complex $\mathcal{P}$

This complex consists of a square of side length t, a rectangle of side lengths (α, t) and a rectangle of side lengths (β, t), glued as indicated in the above figure. $\mathcal{P}$ is homotopy equivalent to a wedge of two circles and its universal cover $\mathcal{T}$ is a piecewise Euclidean polygonal tree:

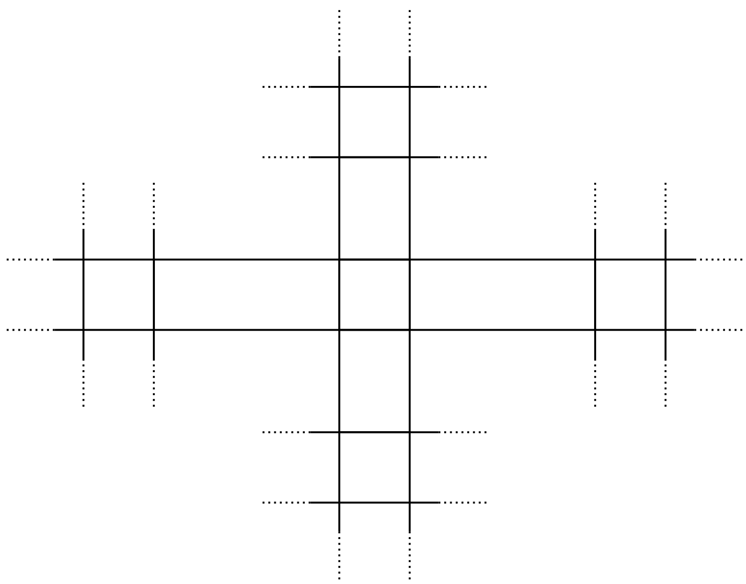


Figure 2: The complex $\mathcal{T} = \widetilde{\mathcal{P}}$

The complex $\mathcal{T}$ carries a natural metric which turns it into a CAT(0) space (see [12], Chapter II.5). In particular, $\mathcal{T}$ is a uniquely geodesic metric space. The geodesics are piecewise linear paths that we will use now to construct a quasimorphism.
Let p_0 be the midpoint of the square in $\mathcal{P}$ and let $p \in \mathcal{T}$ be a point covering p_0. The group $\Gamma := \pi_1(\mathcal{P}, p_0)$ is free of rank two, generated by a horizontal loop a and a vertical loop b. Any non-trivial element $g \in \Gamma$ has a well-defined power factorization of the form $g = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, in which $x_i \in \{a, b\}$, $x_{i+1} \neq x_i$ and $k_i \neq 0$ for all i. The geodesic in $\mathcal{T}$ connecting p with $g.p$ consists of linear segments $s_1, \ldots, s_n$, and there is a correspondence between the segment s_i and the factor $x_i^{k_i}$. This is illustrated in Figure 3.

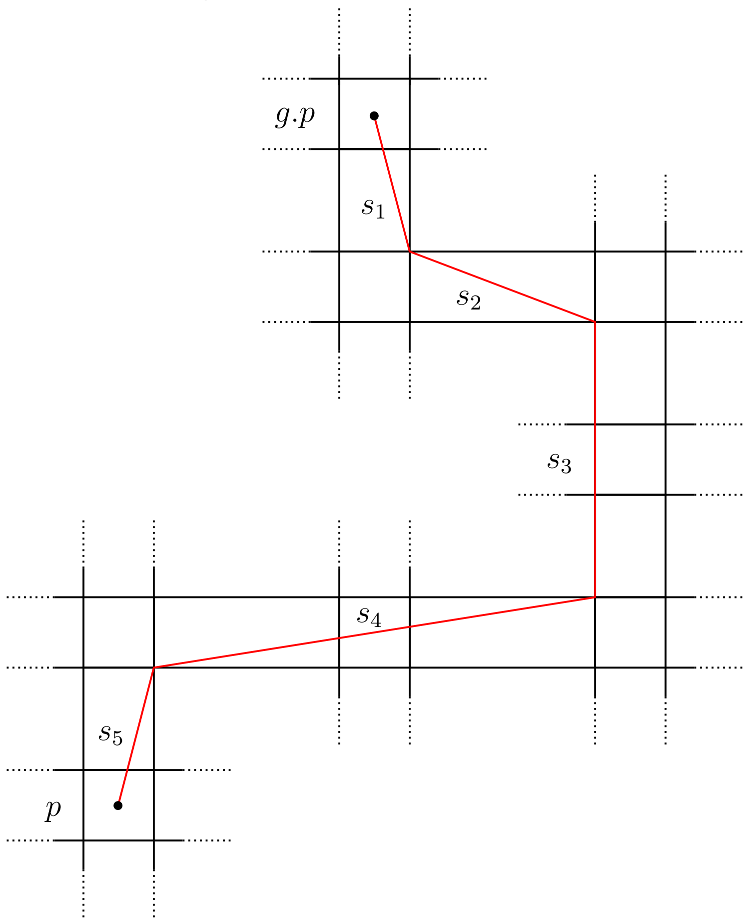


Figure 3: The geodesic $[p, g.p]$ for $g = ba^{-1}b^2a^2b$
and its decomposition into linear segments

At this point we are using the assumption $t < \min\{\alpha, \beta\}$. It guarantees that the geodesic always takes a turn between the power factors of g, so that the number of these factors is equal to the number of segments. We define a map

$f = f_{t,\alpha,\beta} : \Gamma \longrightarrow \mathbb{R}$ as follows. Set $f(e) = 0$. For a non-trivial element $g \in \Gamma$ let

$$g = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \quad \text{and} \quad [p, g.p] = s_n \cup s_{n-1} \cup \cdots \cup s_1$$

as above. We denote by $\ell(s_i)$ the length of the segment s_i and set

$$f(g) = \sum_{i=1}^{n} \operatorname{sgn}(k_i)\, \ell(s_i).$$

In the degenerate case $t = 0$, the complex $\mathcal{T}$ is a 4-regular metric tree which is combinatorially equal to the Cayley graph of Γ (with respect to the generating set $\{a, b\}$). The corresponding map $f_{0,\alpha,\beta}$ is nothing but the homomorphism defined by $a \mapsto \alpha$, $b \mapsto \beta$.

Proposition 5.1. *The map $f_{t,\alpha,\beta}$ is a quasimorphism for all parameters $0 \leq t < \max\{\alpha, \beta\}$.*

Proof. We show that $\sup_{g_0,g_1,g_2} |dc(g_0, g_1, g_2)| < \infty$, where $c(g_0, g_1) = f(g_0^{-1} g_1)$ and $dc(g_0, g_1, g_2) = c(g_1, g_2) - c(g_0, g_2) + c(g_0, g_1)$. Given $g_0, g_1, g_2 \in \Gamma$ we set $p_i := g_i.p$. We look at the geodesic triangle $\Delta \subset \mathcal{T}$ with vertices p_0, p_1, p_2. There are two crucial observations about this triangle. First, note that there is a unique square S in $\mathcal{T}$ which has non-empty intersection with all three sides of Δ. In the case $t = 0$ this square is reduced to a point, the median point of Δ. And second, observe that the triangle is degenerate apart from the linear segments that intersect S, meaning that every point of Δ not in such a segment is contained in two sides. This is due to the fact that the two geodesics going to the vertex p_i coincide once they have taken the first turn after leaving S. All this can be seen in the following picture:

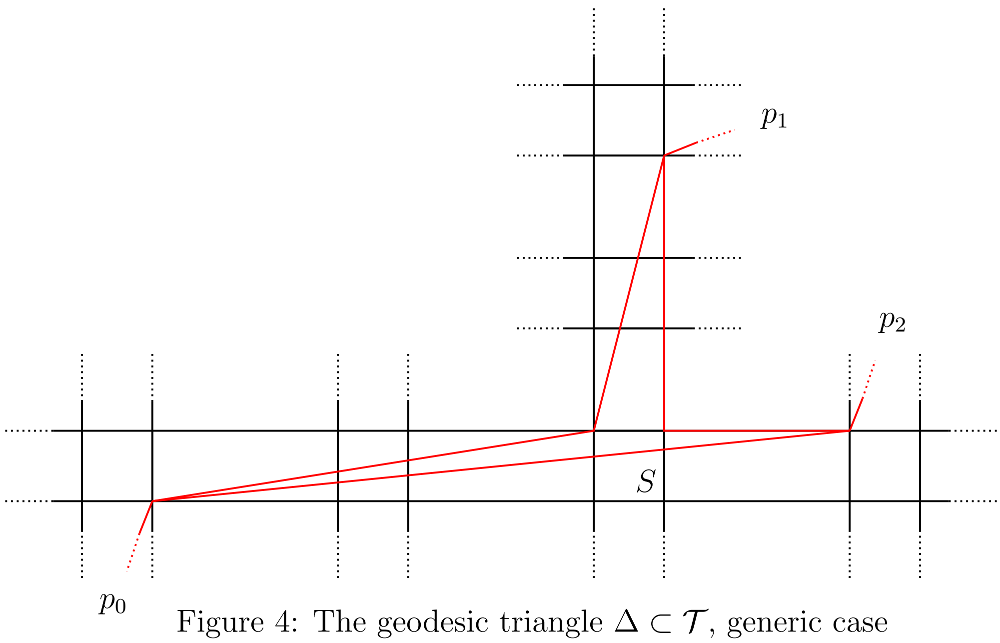


Figure 4: The geodesic triangle $\Delta \subset \mathcal{T}$, generic case

For the computation of $dc(g_0, g_1, g_2)$ we have to add up, with appropriate signs, the lengths of the linear segments of the sides of Δ. Because of cancellation we can ignore the segments in the degenerate part of Δ. Note that the picture shows a triangle that is generic, in the sense that none of its vertices is contained in the median square S. There is one segment passing through S; we break it up into two segments at some point inside S. We are then left with six segments emerging from S:

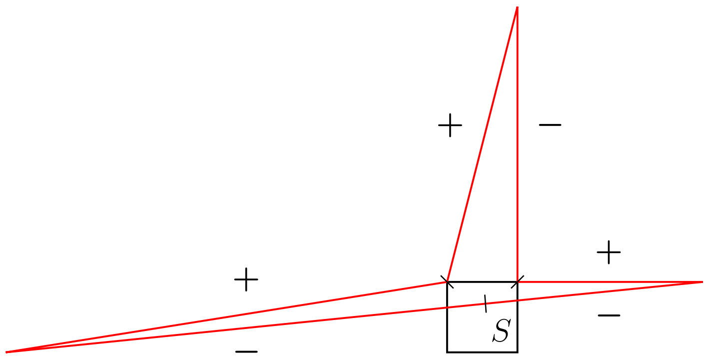


Figure 5: The non-degenerate part of Δ

By construction the lengths of the two segments going from S to a vertex are counted with opposite signs. A possible distribution of the signs is indicated in the picture. Using the triangle inequality we obtain

$$|dc(g_0, g_1, g_2)| \leq 3 \cdot \mathrm{diam}(S).$$

Now assume that Δ is such that at least one of its vertices is contained in S. If there is more than one vertex contained in S then Δ is degenerate and $dc(g_0, g_1, g_2) = 0$. The following picture shows the remaining case where exactly one vertex is inside S:

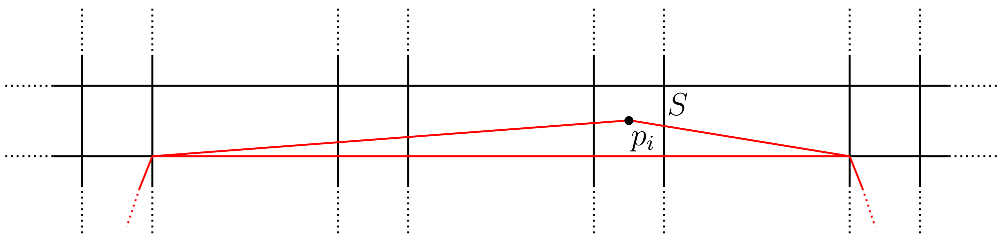


Figure 6: The triangle Δ, exactly one vertex in S

As above we subdivide the segment running through S, so that we get the estimate $|dc(g_0, g_1, g_2)| \leq 2 \cdot \mathrm{diam}(S)$. We thus have shown that

$$\sup_{g_0, g_1, g_2} |dc(g_0, g_1, g_2)| \leq 3 \cdot \mathrm{diam}(S) = 3\sqrt{2t} < \infty. \qquad \square$$

Proposition 5.2. *The quasimorphism $f_{t,\alpha,\beta}$ is non-trivial for all parameters $0 < t < \min\{\alpha, \beta\}$.*

Proof. Let $\widehat{f}$ be the homogenization of $f = f_{t,\alpha,\beta}$, where $0 < t < \min\{\alpha, \beta\}$. We show that $\widehat{f}$ is not a homomorphism, which is equivalent to saying that f is non-trivial. We have $\widehat{f}(a) = \alpha + t$, since for $n > 0$ the geodesic from p to $a^n.p$ consists of a single segment of length $n(\alpha + t)$. Likewise we have $\widehat{f}(b) = \beta + t$. In order to compute $\widehat{f}(ab)$ we consider the geodesic from p to $(ab)^n.p$ for $n > 0$:

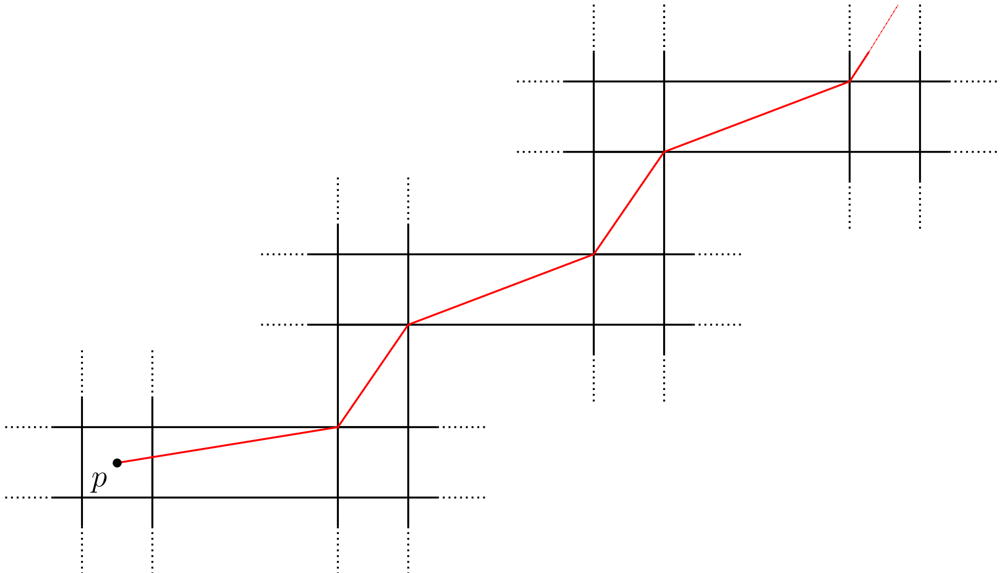


Figure 7: The geodesic $[p, (ab)^n.p]$

If we remove the first two and the last two segments of this geodesic we are left with $n-2$ segments of length $\sqrt{\alpha^2 + t^2}$ and $n-2$ segments of length $\sqrt{\beta^2 + t^2}$. It follows that $\widehat{f}(ab) = \sqrt{\alpha^2 + t^2} + \sqrt{\beta^2 + t^2}$, so that the condition $\widehat{f}(ab) = \widehat{f}(a) + \widehat{f}(b)$ is equivalent to

$$\sqrt{\alpha^2 + t^2} + \sqrt{\beta^2 + t^2} = \alpha + \beta + 2t.$$

This equation does not hold when $t > 0$, and hence, $\widehat{f}$ is not a homomorphism in this case and therefore f is non-trivial for $t > 0$. □

In the remainder of this section we describe a possible generalization of our construction. Let $\Gamma = \langle a, b \rangle$ free of rank two as above and let $X = (X, d_X)$ be a CAT(0) space equipped with a proper and cocompact isometric action of Γ. Choose a basepoint $p \in X$. For $g = x_1^{k_1} \cdots x_n^{k_n} \in \Gamma$ we consider the points

$$q_i := x_i x_{i+1} \cdots x_n.p, \qquad 1 \le i \le n$$

and we set $q_{n+1} := p$. We denote by $\overline{q_i}$ the nearest point projection of q_i onto the geodesic $[p, g.p]$. We then set

$$f(g) = \sum_{i=1}^{n} \operatorname{sgn}(k_i) d_X(\overline{q_i}, \overline{q_{i+1}})$$

to define a function $f : \Gamma \longrightarrow \mathbb{R}$. If we chose for X a polygonal tree $\mathcal{T}$ as above then the points $\overline{q_i}$ are exactly the "turning points" of the geodesic and the resulting function f is the quasimorphism $f_{t,\alpha,\beta}$. In general, the piecewise geodesic path along the points

$$p = \overline{q_{n+1}},\ \overline{q_n},\ \ldots,\ \overline{q_2},\ \overline{q_1} = g.p$$

describes a quasigeodesic from p to $g.p$, which can be seen as a deformed version of the geodesic $[p, g.p]$, and one could call the function f a "geometrically deformed" homomorphism. Unfortunately, we do not know the conditions under which f is a quasimorphism, having verified this property only in the above example. A natural case to consider would be the universal cover a compact hyperbolic surface with boundary, acted upon by the fundamental group.

6 A common generalization of split and deformation quasimorphisms

In this section we introduce a class of quasimorphisms that generalizes both construction that we have discussed so far.

6.1 The construction

Let $\Gamma = A * B$. Let $\{f_A^{b_1,b_2}\} \subset \mathrm{QM}_{\mathrm{alt}}(A)$ be a family of alternating quasimorphisms, one for each pair $(b_1, b_2) \in B \times B$. Likewise let $\{f_B^{a_1,a_2}\} \subset \mathrm{QM}_{\mathrm{alt}}(B)$. We require these families to satisfy the following conditions:

(i) $f_B^{a_1,a_2} = f_B^{a_2^{-1},a_1^{-1}}$ and $f_A^{b_1,b_2} = f_A^{b_2^{-1},b_1^{-1}}$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

(ii) $\sup\limits_{a_1,a_2 \in A} \|f_B^{a_1,a_2} - f_B^{1,1}\|_\infty < \infty$ and $\sup\limits_{b_1,b_2 \in B} \|f_A^{b_1,b_2} - f_A^{1,1}\|_\infty < \infty$

The second condition is equivalent to the boundedness of the family $\{f_A^{b_1,b_2}\}$ as a subset of the space $(\mathrm{Map}(A, \mathbb{R}), \|\cdot\|_\infty)$, and likewise for the f_B-family. We define a map $f := \{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\} : \Gamma \longrightarrow \mathbb{R}$. For $g \in \Gamma$, $g = a_1 b_1 \cdots a_n b_n$, set

$$f(g) := f_A^{1,b_1}(a_1) + f_B^{a_1,a_2}(b_1) + f_A^{b_1,b_2}(a_2) + \ldots + f_A^{b_{n-1},b_n}(a_n) + f_B^{a_n,1}(b_n).$$

We also write $f(b_1, a, b_2)$ for $f_A^{b_1,b_2}(a)$ and $f(a_1, b, a_2)$ for $f_B^{a_1,a_2}(b)$. With this notation the definition of f reads

$$\begin{aligned} f(g) =& f(1, a_1, b_1) + f(a_1, b_1, a_2) + f(b_1, a_2, b_2) + \ldots \\ &+ f(b_{n-1}, a_n, b_n) + f(a_n, b_n, 1). \end{aligned}$$

Proposition 6.1. *If the families $\{f_A^{b_1,b_2}\}, \{f_B^{a_1,a_2}\}$ satisfy the above two conditions then the map f is an alternating quasimorphism on Γ.*

We first establish some estimates involving the following quantities:

$$\begin{aligned} S &:= \max\{ \sup \|f_B^{a_1,a_2} - f_B^{1,1}\|_\infty,\ \sup \|f_A^{b_1,b_2} - f_A^{1,1}\|_\infty \} \\ D &:= \max\{\operatorname{def} f_A^{1,1},\ \operatorname{def} f_B^{1,1}\} \end{aligned}$$

Lemma 6.2. (i) *For all $b_1, b_2, b_1', b_2' \in B$ and all $a \in A$ we have*

$$|f(b_1, a, b_2) - f(b_1', a, b_2')| \leq 2S,$$

and likewise with A- and B-letters exchanged.

(ii) *For all* $b_1, b_2, b_1', b_2', b_1'', b_2'' \in B$ *and all* $a_1, a_2 \in A$ *we have*

$$|f(b_1, a_1, b_2) + f(b_1', a_2, b_2') - f(b_1'', a_1a_2, b_2'')| \leq 3S + D,$$

and likewise with A*- and* B*-letters exchanged.*

Proof. (i) This is immediate from the definition of S and the triangle inequality.

(ii) By definition of S and D we have

$$\begin{aligned}&|f(b_1, a_1, b_2) + f(b_1', a_2, b_2') - f(b_1'', a_1a_2, b_2'')|\\ &\leq 3S + |f(1, a_1, 1) + f(1, a_2, 1) - f(1, a_1a_2, 1)| \leq 3S + D.\end{aligned}$$ □

Proof of Proposition 6.1. Let $g, h \in \Gamma$.

1st case: There is no cancellation in the product gh, i.e. g, h have factorizations $g = xa$, $h = bx'$ for some $x, x' \in \Gamma$, $a \in A$, $b \in B$. This means that $x = x_1 \cdots x_n$ ends with a B-letter and $x' = x_1' \cdots x_m'$ begins with an A-letter. We have

$$\begin{aligned}f(gh) &= f(1, x_1, x_2) + f(x_1, x_2, x_3) + \cdots + f(x_{n-1}, x_n, a)\\ &\quad + f(x_n, a, b) + f(a, b, x_1') + f(b, x_1', x_2') + \cdots + f(x_{m-1}', x_m', 1)\\ &= f(g) + f(h) + f(x_n, a, b) - f(x_n, a, 1) + f(a, b, x_1') - f(1, b, x_1'),\end{aligned}$$

and hence $|\partial f(g, h)| \leq 4S$ by Lemma 6.2.(i). Note that we get the same estimate if $x = 1$ (or $x' = 1$), in this case the x_n's are replaced by 1 in the last line of the above equation (or the x_1''s are replaced by 1).

2nd case: There is cancellation in gh, i.e. g, h have factorizations of the form $g = xa_1y$, $h = y^{-1}a_2x'$, where $a_1a_2 \neq 1$ (or likewise with B-letters instead of A-letters). Let y have the power factorization $y_1 \cdots y_k$ We have

$$\begin{aligned}f(g) + f(h) =& f(1, x_1, x_2) + \dots\\ &+ f(x_{n-1}, x_n, a_1) + f(x_n, a_1, y_1) + f(a_1, y_1, y_2)\\ &+ f(y_2^{-1}, y_1^{-1}, a_2) + f(y_1^{-1}, a_2, x_1') + f(a_2, x_1', x_2')\\ &+ \cdots + f(x_{m-1}', x_m', 1)\end{aligned}$$

Here we used the property $f(y_{i-1}, y_i, y_{i+1}) = -f(y_{i+1}^{-1}, y_i^{-1}, y_{i-1}^{-1})$, due to which the terms of this type cancel in the above sum. We have $gh = xa_1a_2x'$, and so

$$\begin{aligned}
&f(g) + f(h) - f(gh)\\
&= f(x_{n-1}, x_n, a_1) + f(x_n, a_1, y_1) + f(a_1, y_1, y_2)\\
&\quad + f(y_2^{-1}, y_1^{-1}, a_2) + f(y_1^{-1}, a_2, x_1') + f(a_2, x_1', x_2')\\
&\quad - f(x_{n-1}, x_n, a_1a_2) - f(x_n, a_1a_2, x_1') - f(a_1a_2, x_1', x_2')\\
&= f(x_{n-1}, x_n, a_1) - f(x_{n-1}, x_n, a_1a_2)\\
&\quad + f(a_2, x_1', x_2') - f(a_1a_2, x_1', x_2')\\
&\quad + f(a_1, y_1, y_2) + f(y_2^{-1}, y_1^{-1}, a_2)\\
&\quad + f(x_n, a_1, y_1) + f(y_1^{-1}, a_2, x_1') - f(x_n, a_1a_2, x_1')
\end{aligned}$$

We can estimate each line of this sum using Lemma 6.2, it follows that

$$|\partial f(g, h)| \leq 2S + 2S + 2S + (3S + D) = 9S + D.$$

As usual we can reverse the role of the A- and B-letters in these computations. Altogether we have shown that f is a quasimorphism with $\operatorname{def} f \leq 9S + D$. The fact that f is alternating follows from condition (i) above and the fact that we started with families of alternating quasimorphisms. □

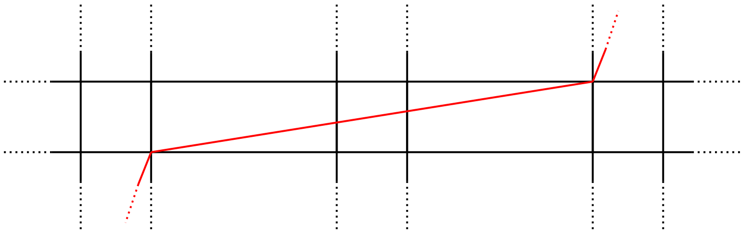
6.2 Examples

Let $f_A \in \mathrm{QM}_{\mathrm{alt}}(A)$, $f_B \in \mathrm{QM}_{\mathrm{alt}}(B)$ and set $f_A^{b_1,b_2} := f_A$ and $f_B^{a_1,a_2} := f_B$ for all a_1, a_2, b_1, b_2. In this case the quasimorphism $\{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\}$ is nothing but the split quasimorphism $f_A * f_B$.

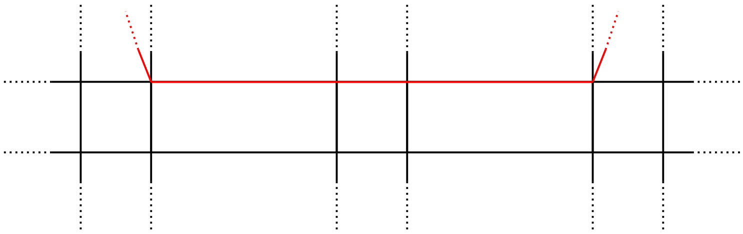
A less obvious example is given by the geometric deformation quasimorphisms $f_{t,\alpha,\beta} \in \mathrm{QM}_{\mathrm{alt}}(\Gamma)$, $\Gamma = \langle a \rangle * \langle b \rangle \cong \mathbb{F}_2$, defined in Section 5. Recall that such a quasimorphism f is given by

$$f(g) = \sum_{i=1}^{n} \operatorname{sgn}(k_i)\ell(s_i)$$

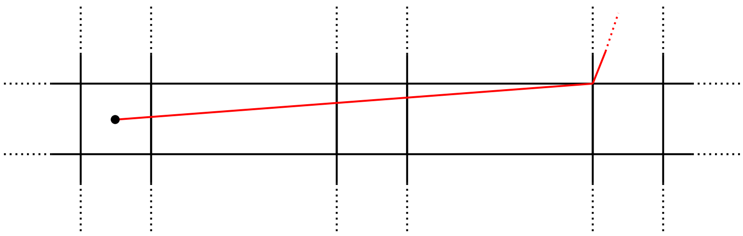
for $g = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, $x_i \in \{a, b\}$, where s_i is the segment corresponding to $x_i^{k_i}$ in the geodesic $[g, g.p]$ in the complex $\mathcal{T}$. Here the length of the segment s_i only depends on the number k_i and the signs of k_{i-1} and k_{i+1}. This can be seen in the following illustrations that show the different possible segments for $b^m a^2 b^n$:



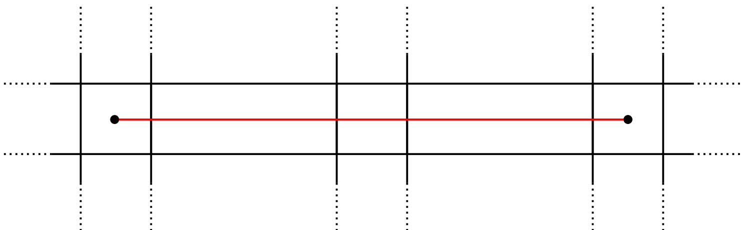
$m, n > 0$



$m < 0, n > 0$



$m = 0, n > 0$



$m = n = 0$

By symmetry of the construction the remaining possibilities for the signs of m, n yield one of the above segments. We have $f = \{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\}$ where the families $\{f_A^{b_1,b_2}\}$, $\{f_B^{a_1,a_2}\}$ are as follows: For $k \neq 0$ the value of $f_A^{b^m,b^n}(a^k) = f(b^m, a^k, b^n)$ is given by the oriented length of the geodesic segment of the geodesic $[p, (b^m a^k b^n).p]$ corresponding to the factor a^k. From the triangle inequality it follows that

$$|f(b^m, a^k, b^n) - f(1, a^k, 1)|$$

is uniformly bounded in m, n. Furthermore the construction is such that

$$f(b^m, a^k, b^n) = f(b^{-n}, a^k, b^{-m})$$

for all m, n, k. This means that the two conditions (i) and (ii) are satisfied.

6.3 Computing the Gromov norm

Here we compute the Gromov norm for certain quasimorphisms of the type constructed in 6.1, notably for geometric deformation quasimorphisms. In fact we verify that, under some assumptions, equality holds in Theorem 1.2 for the the quasimorphism $f = \{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\}$. We show this by adapting the proof of Lemma 3.2.

Lemma 6.3. *Let $f = \{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\}$ as above. The homogenization $\widehat{f}$ is given by $\widehat{f}(g) = \widehat{f_A^{1,1}}(g)$ if $g \in A$, by $\widehat{f}(g) = \widehat{f_B^{1,1}}(g)$ if $g \in B$, and by*

$$\begin{aligned}\widehat{f}(g) =& f(x_n, x_1, x_2) + f(x_1, x_2, x_3) \\ &+ \cdots + f(x_{n-2}, x_{n-1}, x_n) + f(x_{n-1}, x_n, x_1)\end{aligned}$$

if $g \notin A \cup B$. Here $x = x_1 x_2 \cdots x_n$ is any cyclically reduced conjugate of g.

Proof. The case $g \in A$ is immediate since $f(g^n) = f(1, g^n, 1) = f_A^{1,1}(g^n)$, and likewise for $g \in B$. Let $g \notin A \cup B$ and pick any cyclically reduced conjugate x of g as in the statement. For $k > 0$ we compute

$$\begin{aligned}&f(x^k) \\ &= k \cdot [f(x_n, x_1, x_2) + f(x_1, x_2, x_3) \\ &\qquad + \cdots + f(x_{n-2}, x_{n-1}, x_n) + f(x_{n-1}, x_n, x_1)] \\ &+ f(1, x_1, x_2) - f(x_n, x_1, x_2) + f(x_{n-1}, x_n, 1) - f(x_{n-1}, x_n, x_1).\end{aligned}$$

Since $\widehat{f}(g) = \widehat{f}(x)$ the claim follows. □

Let f be a quasimorphism on a group Γ and let X be a subset of $\Gamma \times \Gamma$. We say that $\operatorname{def} f$ *is realized on* X if $\operatorname{def} f = \sup_{(g,h) \in X} |\partial f(g, h)|$. For a free product $\Gamma = A * B$ we consider the sets $S_A, S_B \subset \Gamma \times \Gamma$ given by

$$\begin{aligned}S_A &:= \big\{(x a_1 y, y^{-1} a_2 x') \,|a_1, a_2 \in A,\, a_1 a_2 \neq 1,\, \ell(x), \ell(x'), \ell(y) \geq 2\big\} \\ S_B &:= \big\{(x b_1 y, y^{-1} b_2 x') \,|b_1, b_2 \in B,\, b_1 b_2 \neq 1,\, \ell(x), \ell(x'), \ell(y) \geq 2\big\}\end{aligned}$$

Here the length $\ell(x)$ of $x \in \Gamma$ is the number of power factors needed to write x as a reduced product of A- and B-letters (e.g. $\ell(ab^{-2}a^{-1}b^6) = 4$). We let $S := S_A \cup S_B$.

Lemma 6.4. *Let $f = \{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\}$ as above. If $\operatorname{def} f$ is realized on S then $\operatorname{def} \widehat{f} \geq 2 \operatorname{def} f$.*

Proof. Let $(g,h) \in S$. We may assume that $(g,h) \in S_A$, i.e. $g = xa_1y$, $h = y^{-1}a_2x'$ with $\ell(x), \ell(x'), \ell(y) \geq 2$ and $a_1a_2 \neq 1$. We want to find $\widetilde{g}, \widetilde{h} \in \Gamma$ such that $\partial\widehat{f}(\widetilde{g},\widetilde{h}) = 2\cdot\partial f(g,h)$. We may assume that there is no cancellation in the product $x'x$, since we may replace x' by $x'a$ or $x'b$ without changing the quantity $\partial f(g,h)$. (See case 2 above). Choose $t \in \Gamma$ non-trivial such that there is no cancellation in yty. Define

$$\begin{aligned}\widetilde{g} &:= y^{-1}a_2x'xa_1yt\\ \widetilde{h} &:= t^{-1}y^{-1}a_2x'xa_1y.\end{aligned}$$

We have

$$\begin{aligned}\widehat{f}(\widetilde{g}) + \widehat{f}(\widetilde{h}) = 2[&f(y_2^{-1}, y_1^{-1}, a_2) + f(y_1^{-1}, a_2, x_1') + f(a_2, x_1', x_2')\\ &+ \cdots + f(x_{n-1}, x_n, a_1) + f(x_n, a_1, y_1) + f(a_1, y_1, y_2)]\end{aligned}$$

Note that all the terms involving factors of t cancel in this sum. A cyclically reduced conjugate of $\widetilde{g}\widetilde{h}$ is given by $a_1a_2x'xa_1a_2x'x = (a_1a_2x'x)^2$. We compute

$$\begin{aligned}\widehat{f}(\widetilde{g}\widetilde{h}) &= 2\,\widehat{f}(a_1a_2x'x)\\ &= 2[f(x_n, a_1a_2, x_1') + f(a_1a_2, x_1', x_2') + \cdots + f(x_{n-1}, x_n, a_1a_2)]\end{aligned}$$

Using the computation in the proof of Proposition 6.1, 2nd case, we see that $\partial\widehat{f}(\widetilde{g},\widetilde{h}) = 2\cdot\partial f(g,h)$. The claim follows. □

Theorem 6.5. *Let $f = \{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\}$ as above, with associated cohomology class ω_f. If $\operatorname{def} f$ is realized on S then*

$$\|\omega_f\| = \operatorname{def} f = \tfrac{1}{2}\operatorname{def}\widehat{f}$$

In particular, equality holds in Theorem 1.2 for the quasimorphism f.

Proof. By Theorem 1.2 and Lemma 6.4 we have

$$\operatorname{def} f \geq \|\omega_f\| \geq \tfrac{1}{2}\operatorname{def}\widehat{f} \geq \operatorname{def} f,$$

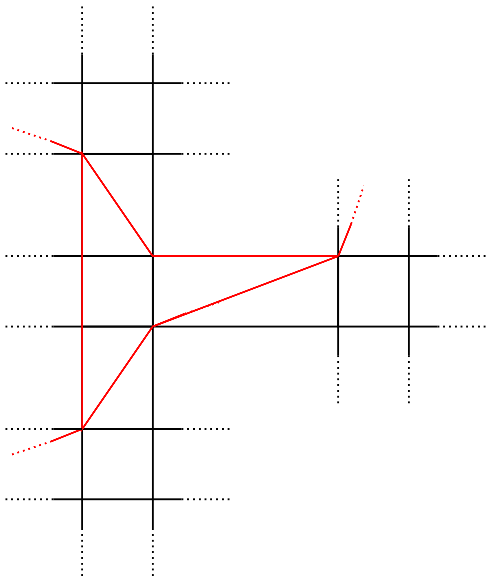
the claim follows. □

It is easy to see that for a split quasimorphism f the defect is realized on S, so Theorem 6.5 applies in this case (and is therefore a generalization of Theorem 3.3).

For the rest of this section we consider geometric deformation quasimorphisms $f_{t,\alpha,\beta}$ on $\Gamma = \mathbb{Z} * \mathbb{Z}$. Again, we denote by $S \subset \Gamma \times \Gamma$ the subset defined above.

Lemma 6.6. *For all $\alpha, \beta > 0$ and $0 < t < \max\{\alpha,\beta\}$ the defect of $f_{t,\alpha,\beta}$ is realized on S.*

Proof. We assume, without loss of generality, that $\alpha \geq \beta$. Using elementary geometry, it is not hard to see that the defect of the homogenous quasicocycle c associated to $f_{t,\alpha,\beta}$ is realized by a triangle with the following core:



One computes

$$\operatorname{def} f_{t,\alpha,\beta} = 2\sqrt{\beta^2 + t^2} - 2\beta - t + \sqrt{\alpha^2 + t^2} - \alpha.$$

This defect is realized for example by $(g, h) = (babab, b^{-1}a^{-1}ba^{-1}b^{-1})$, a pair which belongs to the set S. □

The following statement is now immediate:

Theorem 6.7. *For all geometric deformation quasimorphisms $f_{t,\alpha,\beta}$ with $\alpha, \beta > 0$ and $0 < t < \max\{\alpha, \beta\}$ equality holds in Theorem 1.2.*

In case the defect of a quasimorphism $f = \{f_A^{b_1,b_2}\} * \{f_B^{a_1,a_2}\}$ is not realized on S we are not able to compute the Gromov norm of ω_f. The problem is that for a pair (g, h) which realizes the defect, terms of the form $f(1, a, b)$ or $f(b, a, 1)$ could occur in $\partial f(g, h)$. The homogenization $\widehat{f}$ however ignores these terms, so that our argument showing that $\operatorname{def} \widehat{f} \geq 2 \cdot \operatorname{def} f$ does not apply.

7 Defect spaces

7.1 Definition and first properties

Let Γ be a group. The *defect space* of Γ, denoted by $\mathcal{D}(\Gamma)$, is the space of functions $f : \Gamma \longrightarrow \mathbb{R}$ that are bounded and alternating (i.e. $f(g^{-1}) = -f(g)$ for all $g \in \Gamma$), equipped with the norm

$$\|f\|_{\mathrm{def}} = \operatorname{def} f = \sup_{g,h\in\Gamma} |\partial f(g,h)|,$$

where $\partial f(g,h) = f(g) + f(h) - f(gh)$. This is indeed a norm: If $\|f\|_{\mathrm{def}} = 0$ then f is a bounded homomorphism into $\mathbb{R}$ and hence equal to zero. Homogenity and the triangle inequality are immediate. Thus we have two norms on the space of (alternating) bounded functions on Γ, the ℓ^∞-norm and the defect norm. These norms turn out to be equivalent. This follows essentially from the following statement, in which $\operatorname{ord}(g)$ stands for the (possibly infinite) order of a group element $g \in \Gamma$.

Proposition 7.1. *For $f \in \mathcal{D}(\Gamma)$ and $g \in \Gamma$, $g \neq e$, we have the estimate*

$$|f(g)| \leq \left(1 - \frac{2}{\operatorname{ord}(g)}\right) \|f\|_{\mathrm{def}}.$$

Proof. We may assume that $\|f\|_{\mathrm{def}} = 1$. For $n \geq 1$ we have the estimate

$$|f(g^n) - nf(g)| \leq n - 1,$$

which follows from

$$|f(g^n) - nf(g)| = \left| \sum_{i=1}^{n-1} f(g^{i+1}) - f(g^i) - f(g) \right| \leq (n-1)\|f\|_{\mathrm{def}}.$$

If g has finite even order $2k$ then $f(g^k) = 0$, as f is alternating, so the above estimate implies $k|f(g)| \leq k - 1$ which means that $|f(g)| \leq 1 - \frac{1}{k} = 1 - \frac{2}{\mathrm{ord(g)}}$. If g has order $2k+1$ then we have $f(g^k) + f(g^{k+1}) = 0$, so summation of the estimates

$$|f(g^k) - kf(g)| \leq k - 1$$
$$|f(g^{k+1}) - (k+1)f(g)| \leq k$$

yields $(2k+1)|f(g)| \leq 2k - 1$, so $|f(g)| \leq 1 - \frac{2}{2k+1} = 1 - \frac{2}{\mathrm{ord(g)}}$. Finally, if g has infinite order then letting n tend to infinity in $|\frac{f(g^n)}{n} - f(g)| \leq 1 - \frac{1}{n}$ yields $|f(g)| \leq 1$. □

Corollary 7.2. *The defect norm is equivalent to the supremum norm $\|\cdot\|_\infty$, more precisely, for $f \in \mathcal{D}(\Gamma)$ we have*

$$\|f\|_\infty \leq \|f\|_{\text{def}} \leq 3\|f\|_\infty.$$

The space $\mathcal{D}(\Gamma)$ is therefore a Banach space. It is infinite dimensional if and only if Γ has infinitely many elements of order different from 2, and in this case it is non-separable.

Proof. The lower bound is a consequence of the proposition, the upper bound is immediate from the definition of the defect norm. □

The following function $f \in \mathcal{D}(\mathbb{Z})$ shows that the supremum in the definition of the defect norm need not be attained:

$$f(k) = \begin{cases} 0, & k \text{ even} \\ 1 - 1/k, & k > 0 \text{ odd} \\ 1/k - 1, & k < 0 \text{ odd.} \end{cases}$$

Here we have $\|f\|_{\text{def}} = 2$ but $|\partial f(k,l)| < 2$ for all $k, l \in \mathbb{Z}$.

Proposition 7.3. *An epimorphism $\pi : \Gamma \longrightarrow Q$ induces an isometric embedding $\pi^* : \mathcal{D}(Q) \hookrightarrow \mathcal{D}(\Gamma)$, $f \mapsto f \circ \pi$.*

Proof. By surjectivity of π we have

$$\|\pi^* f\|_{\text{def}} = \sup_{g,h\in\Gamma} |\partial f(\pi(g), \pi(h))| = \sup_{g,h\in Q} |\partial f(g,h)| = \|f\|_{\text{def}}. \qquad \square$$

Proposition 7.4. *For a monomorphism $i : H \longrightarrow \Gamma$, the map*

$$s_i : \mathcal{D}(H) \longrightarrow \mathcal{D}(\Gamma), \quad s_i(f)(g) = \begin{cases} f(h), & g = i(h) \\ 0, & g \notin i(H) \end{cases}$$

is an isometric embedding.

Proof. We write $F := s_i(f)$ and we identify H with its image $i(H)$. Let $g, h \in \Gamma$. Of the three elements g, h, gh either none, one or all three belong to H. In the first case we have $\partial F(g,h) = 0$, in the second case we have $|\partial F(g,h)| \leq \|f\|_\infty \leq \|f\|_{\text{def}}$ (by Proposition 7.1 and Corollary 7.2) and in the last case we have $|\partial F(g,h)| = |\partial f(g,h)| \leq \|f\|_{\text{def}}$. So we have $\|F\|_{\text{def}} \leq \|f\|_{\text{def}}$. As i is injective we also have the reverse inequality, i.e. $\|F\|_{\text{def}} = \|f\|_{\text{def}}$. □

In case of a normal subgroup the last statement can be improved. Consider a short exact sequence

$$1 \longrightarrow N \xrightarrow{i} \Gamma \xrightarrow{\pi} Q \longrightarrow 1. \tag{2}$$

We have the following maps between the defect spaces

$$\mathcal{D}(Q) \xrightarrow{\pi^*} \mathcal{D}(\Gamma) \underset{s_i}{\overset{i_*}{\rightleftarrows}} \mathcal{D}(N),$$

where $i_* \circ \pi^* = 0$ and s_i is a section of i_*. This means that the embeddings π^* and s_i are complementary. More precisely, we have

Proposition 7.5. *For the short exact sequence* (2) *we have an isometric embedding*

$$j : \mathcal{D}(N) \oplus_\infty \mathcal{D}(Q) \hookrightarrow \mathcal{D}(\Gamma)$$

which is given by $j = s_i + \pi^*$, *and, more explicitly, by*

$$j(f, f')(g) = \begin{cases} f(n), & g = i(n) \\ f'(\pi(g)), & g \notin i(N). \end{cases}$$

Here the notation $\oplus_\infty$ stands for the max-norm on the direct sum.

Proof. For $(f, f') \in \mathcal{D}(N) \oplus \mathcal{D}(Q)$ we write $F := j(f, f')$. We identify N with its image $i(N)$. We first note that for $g \in N \subset \Gamma$ we have $\pi^*(f')(g) = f'(e_N) = 0$, so $F(g) = f(g)$, which proves the explicit formula for j. Now let $g, h \in \Gamma$. If none of g, h, gh is contained in N then $|\partial F(g,h)| = |\partial f'(\pi(g), \pi(h))| \le \|f'\|_{\text{def}}$. If $g \in N, h \notin N$ then $\pi(h) = \pi(gh)$, so

$$|\partial F(g,h)| = |f(g) + f'(\pi(h)) - f'(\pi(gh))| = |f(g)| \le \|f\|_\infty \le \|f\|_{\text{def}},$$

by Proposition 7.1 and Corollary 7.2. Because of the identities

$$|\partial F(g,h)| = |\partial F(h^{-1}, g^{-1})| = |\partial F(g^{-1}, gh)|$$

the same holds true whenever exactly one of g, h, gh is contained in N. If all three elements are in N then $|\partial F(g,h)| = |\partial f(g,h)| \le \|f\|_{\text{def}}$. It follows that $\|F\|_{\text{def}} \le \max\{\|f\|_{\text{def}}, \|f'\|_{\text{def}}\}$, and the reverse inequality holds since the maps s_i and π^* are isometric. ◻

Lemma 7.6. *For $i = 1, 2$ consider short exact sequences*

$$1 \longrightarrow N_i \longrightarrow \Gamma \longrightarrow Q_i \longrightarrow 1 \tag{3}$$

with $Q_i \neq \{e\}$. We identify N_i with its image in Γ. Assume that $\Gamma = N_1 N_2$. Let V_i be the image of the embedding $j_i : \mathcal{D}(N_i) \oplus_\infty \mathcal{D}(Q_i) \hookrightarrow \mathcal{D}(\Gamma)$. We have

$$V_1 \cap V_2 = \{f \in \mathcal{D}(\Gamma) \,|\, \text{supp}(f) \subset N_1 \cap N_2\}.$$

Proof. Let $f \in V_1 \cap V_2$ and let $g \in \Gamma \setminus N_1$ and $h \in \Gamma \setminus N_2$. Since $N_1 N_2 = \Gamma$ we have

$$\begin{aligned} gN_1 \cap hN_2 &\neq \emptyset \\ gN_1 \cap h^{-1}N_2 &\neq \emptyset \end{aligned}$$

By construction f is constant on non-trivial cosets, so we have $f(g) = f(h)$ but also $f(g) = f(h^{-1}) = -f(h)$ and therefore $f(g) = f(h) = 0$. This proves the "$\subseteq$" part of the claim. The reverse containment is immediate, for f with $\mathrm{supp}(f) \subset N_1 \cap N_2$ we can choose $f_{N_i} = f|_{N_i}$ and $f_{Q_i} = 0$ to obtain $f = j_i(f_{N_i}, f_{Q_i})$ for $i = 1, 2$. □

When applied to direct products this lemma yields the following statement:

Proposition 7.7. *For a product* $\Gamma = \Gamma_1 \times \Gamma_2$ *the induced isometric embeddings*

$$j_1, j_2 : \mathcal{D}(\Gamma_1) \oplus_\infty \mathcal{D}(\Gamma_2) \hookrightarrow \mathcal{D}(\Gamma)$$

are given by

$$j_1(f_1, f_2)(g_1, g_2) = \begin{cases} f_1(g_1), & g_2 = e \\ f_2(g_2), & g_2 \neq e, \end{cases}$$
$$j_2(f_1, f_2)(g_1, g_2) = \begin{cases} f_1(g_1), & g_1 \neq e \\ f_2(g_2), & g_1 = e. \end{cases}$$

and the images of these embeddings intersect trivially in $\mathcal{D}(\Gamma)$.

7.2 Existence of extremal points

Recall that a point of a convex subset C of a real vector space is called *extremal* if it is not a proper convex combination of two points in C. The theorem of Krein-Milman (see for example [58], Theorem 2.3.4) asserts that in a locally convex vector space, any compact convex set is the closed convex hull of its extremal points (and in particular that the set *has* extremal points). This statement applies to the closed unit ball in the isometric dual X^* of a Banach space X, which by the theorem of Banach–Alaoglou (see for example [48], Theorem 3.15) is compact in the weak*-topology. In this subsection we discuss extremal points in the spaces of our interest. For the space $\ell^\infty_{\mathrm{alt}}(\Gamma)$ there is not much to say: This space is the dual of $\ell^1_{\mathrm{alt}}(\Gamma)$ and indeed the set of extremal points in the closed unit ball is easily identified as the set

$$\{f \in \ell^\infty_{\mathrm{alt}}(\Gamma) \mid |f(g)| = 1 \text{ if } g^2 \neq e\}$$

Note that the only information about Γ that is encoded in the space $\ell^\infty_{\mathrm{alt}}(\Gamma)$ is the cardinal

$$\dim \ell^\infty_{\mathrm{alt}}(\Gamma) = |\{g \in \Gamma \mid g^2 \neq e\}|.$$

It turns out that the set of extremal points in the equivalent space $\mathcal{D}(\Gamma)$ has a much richer structure that reflects properties of the group Γ.

In the following we denote by

$$\mathcal{B}(\Gamma) := \{f \in \mathcal{D}(\Gamma) \mid \|f\|_{\text{def}} \leq 1\}$$

the closed unit ball in $\mathcal{D}(\Gamma)$ and by $\mathcal{E}(\Gamma)$ the set of extremal points of $\mathcal{B}(\Gamma)$. In case $\mathcal{D}(\Gamma) = \{0\}$, i.e. when Γ is a 2-torsion group or trivial, we use the convention $\mathcal{E}(\Gamma) := \{0\}$. It is not a priori clear that $\mathcal{E}(\Gamma)$ is non-empty for all groups Γ. This facts follows from

Theorem 7.8. *For a countable discrete group Γ, there exists a Banach space whose isometric dual is $\mathcal{D}(\Gamma)$. More precisely there is an equivalent norm $\|\cdot\|'$ on the space $(\ell^1_{\text{alt}}(\Gamma), \|\cdot\|_{\ell^1})$ such that the isometric dual of $(\ell^1_{\text{alt}}(\Gamma), \|\cdot\|')$ is isomorphic to $\mathcal{D}(\Gamma)$.*

This is a consequence of the following standard lemma from the renorming theory of Banach spaces:

Lemma 7.9 ([24], Lemma 8.8)**.** *Let $(X, \|\cdot\|)$ be a Banach space, and let $\|\cdot\|'$ be an equivalent norm on the isometric dual X^*. Then $\|\cdot\|'$ is a dual norm to some equivalent norm on X if and only if $\|\cdot\|'$ is weak*-lower semicontinuous.*

Proof of Theorem 7.8. For $g \in G$ let $\alpha_g \in \ell^1_{\text{alt}}(G)$ be given by $\alpha_g = \delta_g - \delta_{g^{-1}}$. For $g, h \in G$ we define $\beta_{g,h} := \alpha_g + \alpha_h - \alpha_{gh}$. Now assume that the sequence $\{f_n\} \subset \mathcal{D}(G)$ converges to $f \in \mathcal{D}(G)$ in the weak*-topology. Since $\langle f, \beta_{g,h}\rangle = 2 \cdot \partial f(g, h)$ we deduce that for all $g, h \in G$ we have the convergence $\partial f_n(g, h) \to \partial f(g, h)$. For $\varepsilon > 0$ pick $g_0, h_0 \in G$ such that $\|f\|_{\text{def}} - \varepsilon < |\partial f(g_0, h_0)|$. Now in the inequality $|\partial f_n(g_0, h_0)| \leq \|f_n\|_{\text{def}}$ pass to the limit inferior to obtain

$$\|f\|_{\text{def}} - \varepsilon < |\partial f(g_0, h_0)| \leq \liminf_{n\to\infty} \|f_n\|_{\text{def}},$$

so $\|f\|_{\text{def}} \leq \liminf_{n\to\infty} \|f_n\|_{\text{def}}$. This means that $\|\cdot\|_{\text{def}}$ is weak*-lower semicontinuous. Now the statement follows from Lemma 7.9. □

Corollary 7.10. *The unit ball $\mathcal{B}(\Gamma)$ is the closed convex hull of its set of extremal points $\mathcal{E}(\Gamma)$. In particular we have $\mathcal{E}(\Gamma) \neq \emptyset$ for every group Γ.*

In order to study the geometry of the Banach space $(\ell^1_{\text{alt}}(\Gamma), \|\cdot\|')$ it would be useful if one could express the norm $\|\cdot\|'$ of Theorem 7.8 explicitly, however, we have not succeeded in finding such a description.

We now collect a number of useful facts for later use.

Lemma 7.11. *If $f \in \mathcal{B}(\Gamma)$ satisfies $|\partial f(g, g)| = 1$ for all $g \in \Gamma$ with $g^2 \neq e$, then $f \in \mathcal{E}(\Gamma)$.*

Proof. Assume $f \pm \varepsilon \in \mathcal{B}(\Gamma)$ for some non-zero $\varepsilon \in \mathcal{D}(\Gamma)$. We show that for all $g \in \Gamma$ the relation $\varepsilon(g^2) = 2\varepsilon(g)$ holds. This implies $\varepsilon(g^{2^n}) = 2^n\varepsilon(g)$, hence $\varepsilon = 0$ by boundedness. So let $g \in \Gamma$. If $g^2 = e$ then $\varepsilon(g^2) = 0 = 2\varepsilon(g)$. For $g^2 \neq e$ we may assume that $\partial f(g, g) = 1$, otherwise replace g by g^{-1}. From the inequality

$$1 \geq \partial(f \pm \varepsilon)(g, g) = 1 \pm \partial\varepsilon(g, g)$$

it follows that $\partial\varepsilon(g, g) = 0$, i.e. $\varepsilon(g^2) = 2\varepsilon(g)$. □

For the next statement we consider the subset

$$\mathcal{E}^*(\Gamma) := \{f \in \mathcal{B}(\Gamma) \mid |\partial f(g, h)| = 1 \text{ whenever } g, h, gh \neq e\} \tag{4}$$

As the notation suggests, this set consists of extremal points. These points have the additional property of being stable under restriction to subgroups:

Proposition 7.12. (i) *The set $\mathcal{E}^*(\Gamma)$ is contained in $\mathcal{E}(\Gamma)$.*

(ii) *The set $\mathcal{E}^*(\Gamma)$ is a discrete subset of $\mathcal{D}(\Gamma)$.*

(iii) *If $f \in \mathcal{E}^*(\Gamma)$ then $f|_H \in \mathcal{E}^*(H)$ for every subgroup $H < \Gamma$.*

Proof. The first statement follows at once from Lemma 7.11. In order to prove the second statement let $f, f' \in \mathcal{E}^*(\Gamma)$, $f \neq f'$. We have $\partial f \neq \partial f'$, otherwise $\partial(f - f') = 0$ and hence $f = f'$ by boundedness. Hence there exist $g, h \in \Gamma$ with $\partial f(g, h) \neq \partial f'(g, h)$, which means that $\partial f(g, h) = 1$ and $\partial f'(g, h) = -1$ or vice versa, i.e.

$$\|f - f'\|_{\text{def}} \geq |\partial(f - f')(g, h)| = 2.$$

The last statement is obvious from the definition of $\mathcal{E}^*$. □

Proposition 7.13. *If $f \in \mathcal{E}(\Gamma)$ then for all $g \in \Gamma$ with $g^2 \neq e$ we have*

$$\sup_{h\in\Gamma} |\partial f(g, h)| = \sup_{h\in\Gamma} |\partial f(h, g)| = 1.$$

Proof. Assume that for some $s \in \Gamma$, $s^2 \neq e$, and some $\varepsilon > 0$ we have $\sup_{h\in\Gamma} |\partial f(s, h)| = 1 - \varepsilon$. Let $\alpha_s = \delta_s - \delta_{s^{-1}} \in \mathcal{D}(\Gamma)$. Since $s^2 \neq e$ we have $\alpha_s \neq 0$. We show that the functions

$$f_\pm := f \pm \varepsilon/4 \cdot \alpha_s$$

are contained in $\mathcal{B}(\Gamma)$, so that f is not extremal. Let $g, h \in \Gamma$. If none of g, h, gh is equal to $s^{\pm 1}$ then $|\partial f_\pm(g, h)| = |\partial f(g, h)| \leq 1$. If $g = s$ then

$$|\partial f_\pm(g, h)| \leq |\partial f(g, h)| + \tfrac{1}{4}\varepsilon|\partial\alpha_s(g, h)| \leq 1 - \varepsilon + \tfrac{3}{4}\varepsilon = 1 - \tfrac{1}{4}\varepsilon < 1.$$

Because of the identities

$$|\partial f_\pm(g, h)| = |\partial f_\pm(g^{-1}, gh)| = |\partial f_\pm(h, h^{-1}g^{-1})| = |\partial f_\pm(gh, h^{-1})|$$

the same estimate holds if any of g, h, gh is equal to $s^{\pm 1}$. □

7.3 Extremal points for finite groups

Let Γ be a finite group. In the space $\ell^\infty_{\text{alt}}(\Gamma)$ the closed unit ball

$$\{f \in \ell^\infty_{\text{alt}}(\Gamma) \mid |f(g)| \leq 1 \text{ for all } g \in \Gamma\}$$

can be seen as the unit cube of this space. For the equivalent defect space we have

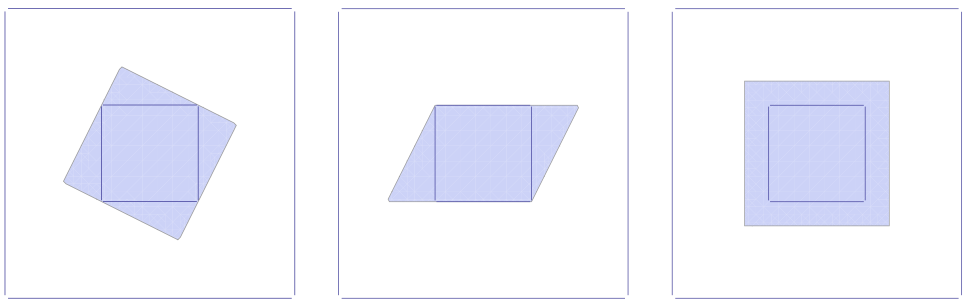
Proposition 7.14. *For a finite group Γ the closed unit ball $\mathcal{B}(\Gamma) \subset \mathcal{D}(\Gamma)$ is a polytope (i.e. a bounded intersection of finitely many closed halfspaces)*

Proof. For $f \in \mathcal{D}(\Gamma)$ the condition $\|f\|_{\text{def}} \leq 1$ means that all of the finitely many linear inequalities

$$f(g) + f(h) - f(gh) \leq 1, \quad g, h \in \Gamma$$
$$f(g) + f(h) - f(gh) \geq -1, \quad g, h \in \Gamma$$

are satisfied simultaneously. Now each of these inequalities has as its set of solutions a closed halfspace in $\mathcal{D}(\Gamma)$. □

The extremal points $\mathcal{E}(\Gamma)$ for a finite group Γ are therefore precisely the vertices of the polytope $\mathcal{B}(\Gamma)$. This polytope depends on the group and not merely on the dimension of the space. This is illustrated by the following pictures, which show the unit balls in the 2-dimensional defect spaces $\mathcal{D}(\mathbb{Z}/5)$, $\mathcal{D}(\mathbb{Z}/6)$ and $\mathcal{D}(\mathbb{Z}/2\times\mathbb{Z}/4)$ (in this order):



Here the shaded regions are the defect-norm unit balls. Furthermore, as an illustration of the inequalities of Corollary 7.2, we see the ℓ^∞-spheres of radius $1/3$ and 1 contained in, respectively containing these unit balls (the frames of the pictures are the spheres of radius 1).

7.4 Left orders and extremal points with maximal sup-norm

Here we identify the subset

$$\mathcal{E}_1(\Gamma) := \{f \in \mathcal{E}(\Gamma) \mid f(\Gamma \setminus \{e\}) = \{\pm 1\}\}$$

of $\mathcal{E}(\Gamma)$. Proposition 7.1 implies that $\mathcal{E}_1(\Gamma) = \emptyset$ whenever Γ has torsion. Indeed, the proposition says that for any $f \in \mathcal{E}(\Gamma)$ and $g \in \Gamma$ of order $1 < n < \infty$ we have $|f(g)| \leq 1 - \frac{2}{n}$, hence $f \notin \mathcal{E}_1(\Gamma)$. It turns out that the obstruction to the existence of points in $\mathcal{E}_1(\Gamma)$ is not just torsion but in fact the lack of a left-order on Γ. We say that a relation $\leq$ on Γ is a *left-order* if it is a total left-invariant order. This means that for all $g, g', g'' \in \Gamma$ we have

(i) $g \leq g' \quad \text{and} \quad g' \leq g \quad \Longrightarrow \quad g = g'$

(ii) $g \leq g' \quad \text{and} \quad g' \leq g'' \quad \Longrightarrow \quad g \leq g''$

(iii) One of $g \leq g'$ and $g' \leq g$ holds.

(iv) $g \leq g' \quad \Longrightarrow \quad hg \leq hg' \quad \text{for all } h \in \Gamma$

As usual we write $g < h$ if we have $g \leq h$ and $g \neq h$. If a relation satisfies the above conditions except perhaps the third one then we call it a partial left-order. We equip the set of partial left-orders $PLO(\Gamma)$ with the topology that is defined by basis sets of the form

$$U_{g_1,g_2,\ldots,g_n} := \{\leq \in PLO(\Gamma) \,|\, g_1 < g_2 < \cdots < g_n\}.$$

The induced topology on the subspace of left-orders $LO(\Gamma)$ was introduced by Sikora in [50]. The following are fundamental results concerning this space:

Theorem 7.15 ([50], Theorem 1.4)**.** *For every group Γ the space $LO(\Gamma)$ is totally disconnected and compact.*

Theorem 7.16 ([37], Theorem 1.3)**.** *For every group Γ the space $LO(\Gamma)$ is either finite or uncountably infinite.*

Let now $\leq$ be a given partial left-order on a group Γ. Since we have $g > e \Longleftrightarrow g^{-1} < e$ we obtain a map $f_{\leq} \in \mathcal{D}(\Gamma)$ by setting

$$f_{\leq}(g) := \begin{cases} 1, & g > e \\ -1, & g < e \\ 0, & \text{else.} \end{cases}$$

We write

$$\mathcal{B}_{\text{int}}(\Gamma) := \{f \in \mathcal{B}(\Gamma) \,|\, f(\Gamma) \subset \{0, \pm 1\}\},$$

this is the set of integral-valued functions that are contained in the unit ball of $\mathcal{D}(\Gamma)$.

Lemma 7.17. *For every $\leq \in PLO(\Gamma)$ we have $f_{\leq} \in \mathcal{B}_{\text{int}}(\Gamma)$.*

Proof. We have to show that $\|f_{\le}\|_{\mathrm{def}} \le 1$. For $g, h \in \Gamma$ write $D = |\partial f_{\le}(g,h)|$. By left-invariance it follows that either none, one or all three of g, h, gh are comparable to e. In the first case we have $D = 0$. In the second case we have $D = 1$. In the third case we have either $f_{\le}(g) = f_{\le}(gh)$ and hence $D = |f_{\le}(h)| = 1$, or we have $f_{\le}(g) = -f_{\le}(gh)$ and we may assume, without loss of generality, $g > e$ and $gh < e$. By left-invariance this means $h < g^{-1} < e$, hence $f_{\le}(h) = -1$ and $D = 1$. □

Now let $f \in \mathcal{B}_{\mathrm{int}}(\Gamma)$. We define a relation $\le_f$ on Γ by

$$g \le_f h \quad :\Longleftrightarrow \quad f(g^{-1}h) = 1$$

for $g \neq h$, and $g \le_f g$ for all $g \in \Gamma$.

Lemma 7.18. *For every $f \in \mathcal{B}_{\mathrm{int}}(\Gamma)$ we have $\le_f \in PLO(\Gamma)$.*

Proof. We show first that the relation is transitive. Let $g \le_f g' \le_f g''$, i.e. $f(g^{-1}g') = f(g'^{-1}g'') = 1$. We have

$$\partial f(g^{-1}g', g'^{-1}g'') = 2 - f(g^{-1}g'')$$

and hence $f(g^{-1}g'') = 1$, so $g \le_f g''$. Now assume that $g \le_f g'$ and $g' \le_f g$. If $g \neq g'$ this means that $f(g^{-1}g') = f(g'^{-1}g) = 1$ which is impossible as f is alternating, so $g = g'$. The fact that $\le_f$ is left-invariant is obvious. □

Lemma 7.19. *The assignments $f \mapsto \le_f$ and $\le \mapsto f_{\le}$ are mutually inverse.*

Proof. Let $f' \in \mathcal{B}_{\mathrm{int}}(\Gamma)$ and let $g \in \Gamma$. We have

$$f_{\le_{f'}}(g) = 1 \Longleftrightarrow g >_{f'} e \Longleftrightarrow f'(g) = 1.$$

Likewise we have $f_{\le_{f'}}(g) = -1 \Longleftrightarrow f'(g) = -1$, and therefore $f_{\le_{f'}} = f'$. On the other hand, for $\preceq \in PLO(\Gamma)$ and $g, h \in \Gamma$ we have

$$g <_{f_{\preceq}} h \Longleftrightarrow f_{\preceq}(g^{-1}h) = 1 \Longleftrightarrow e \prec g^{-1}h \Longleftrightarrow g \prec h$$

and hence $\le_{f_{\preceq}} = \preceq$. □

Lemma 7.20. *The bijection $\mathcal{B}_{\mathrm{int}}(\Gamma) \longrightarrow PLO(\Gamma)$ restricts to a bijection $\mathcal{E}_1(\Gamma) \longrightarrow LO(\Gamma)$.*

Proof. Let $f \in \mathcal{E}_1(\Gamma)$. We have to show that $\le_f$ is a total order. Let $g, h \in \Gamma$. If $g = h$ we have $g \le_f h$. If $g \neq h$ then either $f(g^{-1}h) = 1$ or $f(h^{-1}g) = -f(g^{-1}h) = 1$, hence $g \le_f h$ or $h \le_f g$ □

Theorem 7.21. *If we endow the set* $\mathcal{B}_{\text{int}}(\Gamma)$ *with the topology induced from the weak*-topology on* $\mathcal{D}(\Gamma)$ *then the mutually inverse maps*

$$\mathcal{B}_{\text{int}}(\Gamma) \longrightarrow PLO(\Gamma), \quad f \mapsto \leq_f$$
$$PLO(\Gamma) \longrightarrow \mathcal{B}_{\text{int}}(\Gamma), \quad \leq \ \mapsto f_{\leq}$$

are homeomorphisms.

By Lemma 7.20 we get

Corollary 7.22. *If we endow the set* $\mathcal{E}_1(\Gamma)$ *with the topology induced from the weak*-topology on* $\mathcal{D}(\Gamma)$ *then the mutually inverse maps*

$$\mathcal{E}_1(\Gamma) \longrightarrow LO(\Gamma), \quad f \mapsto \leq_f$$
$$LO(\Gamma) \longrightarrow \mathcal{E}_1(\Gamma), \quad \leq \ \mapsto f_{\leq}$$

are homeomorphisms.

Proof of Theorem 7.21. A subbasis for the topology of the space $PLO(\Gamma)$ is given by the sets

$$U_g = \{\leq \in PLO(\Gamma) \,|\, e < g\}, \qquad g \in \Gamma$$

(see [37], p.1). Denote the first map in the theorem by Φ. We show that for all $g \in \Gamma$ the preimage $\Phi^{-1}(U_g)$ is open in $\mathcal{B}_{\text{int}}(\Gamma)$, which means that the map is continuous. We have

$$\Phi^{-1}(U_g) = \{f \in \mathcal{B}_{\text{int}}(\Gamma) \,|\, e <_f g\} = \{f \in \mathcal{B}_{\text{int}}(\Gamma) \,|\, f(g) = 1\}.$$

Let $\{f_n\}$ be a sequence contained in the complement

$$W := \mathcal{B}_{\text{int}}(\Gamma) \setminus \Phi^{-1}(U_g) = \{f \in \mathcal{B}_{\text{int}}(\Gamma) \,|\, f(g) \in \{0, -1\}\},$$

that weak*-converges to $f \in \mathcal{B}_{\text{int}}(\Gamma)$. By Theorem 7.8 this means that for all $\alpha \in \ell^1_{\text{alt}}(\Gamma)$ we have $\langle \alpha, f_n \rangle \longrightarrow \langle \alpha, f \rangle$. For $\alpha_g := \delta_g - \delta_{g^{-1}}$ we have

$$\langle \alpha_g, f_n \rangle = f_n(g) - f_n(g^{-1}) \in \{0, -2\}$$

for all n, hence $\langle \alpha_g, f \rangle \in \{0, -2\}$ which means that $f(g) \in \{0, -1\}$, i.e. $f \in W$. Therefore W is weak*-closed, which establishes the claim.
We show next that $\mathcal{B}_{\text{int}}(\Gamma)$ is weak*-compact. For this purpose it is sufficient to show that it is a closed subset of the unit ball $\mathcal{B}(\Gamma)$ which is weak*-compact by the theorem of Banach–Alaoglou. For a sequence $\{f_n\}$ in $\mathcal{B}_{\text{int}}(\Gamma)$ with weak*-limit f we have, as above, $\langle \alpha_g, f_n \rangle = 2f_n(g) \in \{0, \pm 2\}$. Hence $\langle \alpha_g, f \rangle \in \{0, \pm 2\}$, which means that $f(g) \in \{0, \pm 1\}$. It follows that f is contained in the set $\mathcal{B}_{\text{int}}(\Gamma)$.
Now the space $\mathcal{B}_{\text{int}}(\Gamma)$ is weak*-compact and the space $PLO(\Gamma)$ is Hausdorff ([50], Proposition 1.3), so the continuous bijection $\Phi : \mathcal{B}_{\text{int}}(\Gamma) \longrightarrow PLO(\Gamma)$ is in fact a homeomorphism. □

Together with Theorem 7.16 we obtain

Corollary 7.23. *The set $\mathcal{E}_1(\Gamma)$ is either finite or uncountably infinite.*

Question. For which $\leq \in PLO(\Gamma)$ is $f_\leq \in \mathcal{E}(\Gamma)$? In other words: What characterizes an extremal partial left-order?

Below, after Theorem 7.25, we will see examples of partial, non-total orders that are extremal.

7.5 Extremal points from quotients

In 7.1 we showed that for a quotient $Q = \Gamma/N$ the space $\mathcal{D}(Q) \oplus_\infty \mathcal{D}(N)$ embeds isometrically into $\mathcal{D}(\Gamma)$. The following result states that this embedding preserves extremal points:

Theorem 7.24. *For a short exact sequence*

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

in which the group Q is 2-torsion free, the induced embedding $j : \mathcal{D}(N) \oplus_\infty \mathcal{D}(Q) \hookrightarrow \mathcal{D}(\Gamma)$ maps extremal points to extremal points. That is, we have an embedding

$$\mathcal{E}(N) \times \mathcal{E}(Q) \hookrightarrow \mathcal{E}(\Gamma).$$

Proof. Let $f_N \in \mathcal{E}(N)$, $f_Q \in \mathcal{E}(Q)$, and let $F = j(f_N, f_Q)$. We have $\|F\|_{\text{def}} = 1$. Assume that for some $E \in \mathcal{D}(\Gamma)$ we have $F \pm E \in \mathcal{B}(\Gamma)$. We have to show that $E = 0$. Since $F|_N = f_N$ and f_N is extremal, we have $E|_N = 0$. In order to show that E vanishes altogether, it is sufficient to show that E descends to the quotient Q, i.e. that $E = e \circ \pi$ for some $e \in \mathcal{D}(Q)$. This is because f_Q is extremal. So we want to prove that $E(g) = E(gn)$ for any given $g \in \Gamma$, $n \in N$. If $g \in N$ we have $E(g) = E(gn) = 0$. Let now $g \notin N$. Since Q is 2-torsion free, we have $\pi(g)^2 \neq e$. By Proposition 7.13, for each $1 > \varepsilon > 0$ there exists $h \in \Gamma$ such that

$$|\partial F(g, h)| = |\partial f_Q(\pi(g), \pi(h))| > 1 - \varepsilon.$$

Since $\pi(n) = e$ this implies

$$|\partial F(gn^k, h)| = |\partial F(gn^{k-1}, nh)| > 1 - \varepsilon$$

for $k \geq 1$. Therefore

$$|\partial E(gn^k, h)| < \varepsilon, \quad |\partial E(gn^{k-1}, nh)| < \varepsilon.$$

Through the triangle inequality these estimates combine to

$$|E(gn^k) - E(gn^{k-1}) + E(h) - E(nh)| < 2\varepsilon, \tag{5}$$

from which we deduce

$$|E(gn^k) - E(g) + k(E(h) - E(nh))| < 2k\varepsilon$$

by telescopic summation. We divide both sides by k and let k tend to infinity. Since E is a bounded function we get

$$|E(h) - E(nh)| < 2\varepsilon.$$

Combining this estimate with the estimate (5) for $k = 1$ we obtain

$$|E(gn) - E(g)| < 4\varepsilon.$$

It follows that $E(gn) = E(n)$. □

If we drop the assumption that Q be 2-torsion free the statement of the theorem is false in general. This can easily be seen in an example like $\mathbb{Z}/6 \longrightarrow \mathbb{Z}/2$. However, we can still produce extremal points for Γ if we start with suitable extremal points for N and Q, as the following statement shows. Recall that we defined the set $\mathcal{E}^*(\Gamma)$ in equation (4) above.

Theorem 7.25. *For a short exact sequence*

$$1 \longrightarrow N \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1.$$

the induced embedding $j : \mathcal{D}(N) \oplus_\infty \mathcal{D}(Q) \hookrightarrow \mathcal{D}(\Gamma)$ *restricts to embeddings*

$$\mathcal{E}_1(N) \times \mathcal{E}^*(Q) \hookrightarrow \mathcal{E}^*(\Gamma)$$

and

$$\mathcal{E}_1(N) \times \mathcal{E}_1(Q) \hookrightarrow \mathcal{E}_1(\Gamma).$$

Proof. Let $f_N \in \mathcal{E}_1(N)$, $f_Q \in \mathcal{E}^*(Q)$ and $F = j(f_N, f_Q)$. Let $g, h \in \Gamma$ such that $g, h, gh \neq e$. If none of g, h, gh is contained in N then $\pi(g), \pi(h), \pi(gh) \neq e$ so

$$|\partial F(g, h)| = |\partial f_Q(\pi(g), \pi(h))| = 1.$$

Assume that exactly one of g, h, gh is contained in N. Because of the identities at the end of the proof of Proposition 7.13 we may assume that $g \in N$, $h, gh \notin N$. We have

$$|\partial F(g, h)| = |f_N(g) + f_Q(\pi(h)) - f_Q(\pi(gh))| = |f_N(g)| = 1.$$

If all three of g, h, gh are contained in N then

$$|\partial F(g,h)| = |f_N(g,h)| = 1$$

since $f_N \in \mathcal{E}_1(N) \subset \mathcal{E}^*(N)$. By definition of $\mathcal{E}^*$ we have $F \in \mathcal{E}^*(\Gamma)$. If we start with $f_N \in \mathcal{E}_1(N)$, $f_Q \in \mathcal{E}_1(Q)$ then $F(\Gamma \setminus \{e\}) = \{\pm 1\}$ by construction, so $F \in \mathcal{E}_1(\Gamma)$. □

Theorem 7.25 can be used to produce extremal points that have values in $\{0, \pm 1\}$ but are not contained in $\mathcal{E}_1$. The quotient $\mathbb{Z} \longrightarrow \mathbb{Z}/2$ for example yields the extremal point $f \in \mathcal{E}^*(\mathbb{Z})$ given by

$$f(k) = \begin{cases} 0, & k = 0 \text{ or } k \text{ odd} \\ 1, & k > 0 \text{ even} \\ -1, & k < 0 \text{ even.} \end{cases}$$

Note that the existence of the second embedding in the theorem reflects the fact that the class of left-orderable groups is closed under extensions.

The following statement gives a condition under which the respective extremal points induced from two different quotients of a group Γ are disjoint.

Proposition 7.26. *For $i = 1, 2$ let*

$$1 \longrightarrow N_i \longrightarrow \Gamma \longrightarrow Q_i \longrightarrow 1$$

be a short exact sequence with a 2-torsion free non-trivial group Q_i. Write S_i for the image of the induced embedding $j_i : \mathcal{E}(N_i) \times \mathcal{E}(Q_i) \hookrightarrow \mathcal{E}(\Gamma)$. If $N_1 N_2 = \Gamma$ then $S_1 \cap S_2 = \emptyset$.

Proof. By Lemma 7.6 any $f \in S_1 \cap S_2$ is of the form $f = j_1(f_{N_1}, 0)$, but 0 is not contained in $\mathcal{E}(Q_1)$. □

This statement says, for example, that the points in $\mathcal{E}(\mathbb{Z})$ obtained from quotients by $p\mathbb{Z}$, p prime, are all distinct.

7.6 Extremal points from infinite chains of normal subgroups

If we have a finite descending chain of normal subgroups

$$\Gamma = N_0 > N_1 > \cdots > N_k$$

with 2-torsion free quotients Γ/N_i then iterated application of Theorem 7.24 yields an embedding

$$\mathcal{E}(N_0/N_1) \times \mathcal{E}(N_1/N_2) \times \cdots \times \mathcal{E}(N_{k-1}/N_k) \times \mathcal{E}(N_k) \hookrightarrow \mathcal{E}(\Gamma)$$

In this subsection we generalize this fact to the situation of suitable infinite chains. Let

$$\Gamma = N_0 > N_1 > \dots$$

be a descending chain of normal subgroups of Γ such that $N_i \neq N_{i+1}$ and such that $\bigcap_{i=0}^{\infty} N_i = \{e\}$. Given a sequence

$$(f_i)_i = (f_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} \mathcal{B}(N_{i-1}/N_i)$$

we define the map $f \in \mathcal{D}(\Gamma)$ by $f(g) = f_i(gN_{i+1})$, where i is the unique index for which $g \in N_i$ and $g \notin N_{i+1}$. For the following statement we equip the above product of unit balls with the norm $\|(f_i)_i\| := \sup_i \|f_i\|_{\text{def}}$.

Proposition 7.27. *The assignment $(f_i)_i \mapsto f$ describes an isometric embedding*

$$J : \prod_{i=0}^{\infty} \mathcal{B}(N_i/N_{i+1}) \longrightarrow \mathcal{B}(\Gamma).$$

Proof. Let $(f_i)_i$ be a given sequence and $f = J((f_i)_i)$. We may assume that $\|(f_i)_i\| = 1$. We first show that $\|f\| \leq 1$. Let $g, h \in \Gamma$. If one of g, h, gh is trivial then $\partial f(g,h) = 0$, since f is alternating. Otherwise, there exists an index i such that $g, h, gh \notin N_{i+1}$. From Proposition 7.5 we obtain an isometric embedding

$$I : \mathcal{B}(N_0/N_1) \times \mathcal{B}(N_1/N_2) \times \cdots \times \mathcal{B}(N_i/N_{i+1}) \times \mathcal{B}(N_{i+1}) \longrightarrow \mathcal{B}(\Gamma).$$

Now set $\alpha := I(f_0, f_1, \dots, f_i, 0)$. By construction, we have $f(g) = \alpha(g)$, $f(h) = \alpha(h)$, $f(gh) = \alpha(gh)$, so that $|\partial f(g,h)| = |\partial \alpha(g,h)| \leq 1$. Therefore we have $\|f\| \leq 1$. To prove the reverse inequality, let i be a given index and let $g, h \in N_i \setminus N_{i+1}$ such that $gh \in N_i \setminus N_{i+1}$. We have $\partial f(g,h) = \partial f_i(gN_{i+1}, hN_{i+1})$. It follows that $\|f\| \geq \|f_i\|_{\text{def}}$, and hence $\|f\| \geq \sup_i \|f_i\|_{\text{def}} = \|(f_i)_i\| = 1$. □

For a group Γ with a descending chain of normal subgroups as above, we now consider the inverse limit

$$\widehat{\Gamma} := \varprojlim_i \Gamma/N_i.$$

We write K_i for the kernel of the natural projection $p_i : \widehat{\Gamma} \longrightarrow \Gamma/N_i$. We have the descending chain of normal subgroups

$$\widehat{\Gamma} = K_0 > K_1 > \dots .$$

The projection p_{i+1} restricts to a map $K_i \longrightarrow N_i/N_{i+1}$, which induces natural isomorphisms $K_i/K_{i+1} \cong N_i/N_{i+1}$. By Proposition 7.27 we have an isometric embedding

$$\widehat{J} : \prod_{i=0}^{\infty} \mathcal{B}(N_i/N_{i+1}) \hookrightarrow \mathcal{B}(\widehat{\Gamma}).$$

By construction, the embeddings J and $\widehat{J}$ are compatible with the restriction map $r : \mathcal{B}(\widehat{\Gamma}) \longrightarrow \mathcal{B}(\Gamma)$, so that $J = r \circ \widehat{J}$. The following result asserts that these embeddings map extremal points to extremal points.

Theorem 7.28. *Let Γ be a group. For an infinite descending chain of normal subgroups*

$$\Gamma = N_0 > N_1 > \dots$$

with trivial intersection and 2-torsion free quotients Γ/N_i, the associated embedding J restricts to an embedding

$$\prod_{i=0}^{\infty} \mathcal{E}(N_i/N_{i+1}) \hookrightarrow \mathcal{E}(\Gamma).$$

Moreover, the restriction of $\widehat{J}$ yields an embedding

$$\prod_{i=0}^{\infty} \mathcal{E}(N_i/N_{i+1}) \hookrightarrow \mathcal{E}(\widehat{\Gamma})$$

and these two embeddings are compatible with the restriction map $\mathcal{D}(\widehat{\Gamma}) \longrightarrow \mathcal{D}(\Gamma)$.

Proof. We only need to prove the first statement, as the second one follows from the observations made before the theorem. Let $(f_i)_i = (f_i)_{i=0}^{\infty} \in \prod_{i=0}^{\infty} \mathcal{E}(N_i/N_{i+1})$ and let $F = J((f_i)_i)$. Assume that there exists $E \in \mathcal{D}(\Gamma)$ such that $F \pm E \in \mathcal{B}(\Gamma)$. We have to show that $E = 0$. For this purpose we consider the embedding

$$\widetilde{J} : \prod_{i=1}^{\infty} \mathcal{B}(N_i/N_{i+1}) \hookrightarrow \mathcal{B}(N_1),$$

which is obtained by shifting the index by 1, and the embedding

$$j : \mathcal{B}(\Gamma/N_1) \oplus_{\infty} \mathcal{B}(N_1) \hookrightarrow \mathcal{B}(\Gamma)$$

from Proposition 7.5. This allows us to write the map F as $F = j(\, f_0, \widetilde{J}(\,(f_i)_{i=1}^{\infty}\,)\,)$. By the same argument as in the proof of Theorem 7.24 the map E descends to a map on the quotient Γ/N_1; more precisely, we have $E(gn) = E(g)$ for $g \notin N_1$ and $n \in N_1$. Since f_0 is extremal, this means that E vanishes on $\Gamma \setminus N_1$. If E also vanishes on N_1 we are done. If not, we iterate the argument to obtain that E vanishes on $N_1 \setminus N_2$, $N_2 \setminus N_3$, etc. Since the intersection of the N_i is trivial, it follows that $E = 0$. □

Corollary 7.29. *If the countably infinite group Γ is residually finite 2-torsion free, then its set of extremal points $\mathcal{E}(\Gamma)$ contains uncountably many rational-valued functions.*

Proof. Let $g_1, g_2, \ldots$ be an enumeration of the non-trivial elements of Γ. By assumption, we can find for each i a finite index normal subgroup H_i of Γ, such that $g_i \notin H_i$, and such that Γ/H_i is 2-torsion free. Now set $N_0 = \Gamma$ and $N_i = \bigcap_{j \le i} H_j$. The groups N_i are finite index normal in Γ and they form a descending sequence as in the theorem. We may assume that $N_{i+1} \neq N_i$. The group N_i/N_{i+1} is finite, non-trivial and 2-torsion free, so its set of extremal points $\mathcal{E}(N_i/N_{i+1})$ consists of rational-valued functions and it has at least two elements. Therefore the product $\prod_i \mathcal{E}(N_i/N_{i+1})$ contains uncountably many sequences of rational-valued functions. □

As another application, we can use the theorem to produce rational-valued functions in $\mathcal{E}(\mathbb{Z})$ that are not periodic. For an odd natural number $\ell > 2$ we have the chain of subgroups

$$\mathbb{Z} > \ell\mathbb{Z} > \ell^2\mathbb{Z} > \ldots \, .$$

The corresponding inverse limit is the group of ℓ-adic numbers $\mathbb{Z}_\ell$, and we have the embeddings

$$\prod_{i=0}^{\infty} \mathcal{E}(\mathbb{Z}/\ell) \hookrightarrow \mathcal{E}(\mathbb{Z}_\ell), \qquad \prod_{i=0}^{\infty} \mathcal{E}(\mathbb{Z}/\ell) \hookrightarrow \mathcal{E}(\mathbb{Z}).$$

To obtain a more concrete example for $\ell = 3$ we consider the extremal point $a \in \mathcal{E}(\mathbb{Z}/3)$, $a(0) = 0$, $a(1) = 1/3$, $a(2) = -1/3$ and the constant sequence $(a) \in \prod_{i=0}^{\infty} \mathcal{E}(\mathbb{Z}/3)$. The induced extremal point $f \in \mathcal{E}(\mathbb{Z})$ has a simple description: For a positive number $k \in \mathbb{Z}$ let $\lambda_{i_0} \in \{1, 2\}$ be the first non-zero coefficient in the 3-adic expansion $k = \sum_i \lambda_i 3^i$. We then have

$$f(k) = a(\lambda_{i_0}) = \begin{cases} 1/3, & \lambda_{i_0} = 1 \\ -1/3, & \lambda_{i_0} = 2. \end{cases}$$

Note that this extremal point f has minimal possible sup-norm $\|f\|_\infty = 1/3$.

7.7 Extremal points with minimal sup-norm

For a group Γ let $f \in \mathcal{E}(\Gamma)$ with minimal norm $\|f\|_\infty = 1/3$. We claim that for all $g \in \Gamma$ with $g^2 \neq e$ we have $f(g) = \pm 1/3$: Indeed: If this is not the case then there is $g \in \Gamma$ with $|f(g)| = 1/3 - \delta$. We then have $|\partial f(g,h)| \le 1/3 - \delta + 1/3 + 1/3 = 1 - \delta$ for all $h \in \Gamma$, a contradiction to Proposition 7.13. If we assume that we are dealing with a 2-torsion free group then the extremal points under consideration are contained in the set

$$\mathcal{E}_{1/3}(\Gamma) := \{f \in \mathcal{E}(\Gamma) \mid f(\Gamma \setminus \{e\}) = \{\pm 1/3\}\}.$$

For each $f \in \mathcal{E}_{1/3}(\Gamma)$ we have a decomposition $\Gamma = \{e\} \sqcup A_f \sqcup A_f^{-1}$, where $A_f = f^{-1}(\{1/3\})$. On the other hand, given a decomposition $\Gamma = \{e\} \sqcup A \sqcup A^{-1}$, we can define $f_A \in \mathcal{B}(\Gamma)$ by

$$f_A(g) = \begin{cases} 1/3, & g \in A \\ -1/3, & g \in A^{-1} \\ 0, & g = e. \end{cases}$$

In the next statement we use the notation

$$A^2 := \{aa' \mid a, a' \in A\}$$
$$A^{(2)} := \{a^2 \mid a \in A\},$$

Proposition 7.30. *Let $f_A \in \mathcal{B}(\Gamma)$ be the function corresponding to a decomposition as above.*

(i) *If $f_A \in \mathcal{E}_{1/3}(\Gamma)$ then $A^{-1} \subset A^2$.*

(ii) *If $A^{(2)} \subset A^{-1}$ then $f_A \in \mathcal{E}_{1/3}(\Gamma)$.*

Proof. The first statement follows from Proposition 7.13: For each $g \in A^{-1}$ there exists an h with $|f_A(g,h)| = 1$. Since $f_A(g) = -1/3$ this means that $f_A(h) = -1/3$ and $f_A(gh) = 1/3$, so we have $g = gh \cdot h^{-1} \in A^2$ by definition of f_A. The second statement is an immediate consequence of Lemma 7.11. □

We note that the condition $A^{(2)} \subset A^{-1}$ is sufficient but not necessary for f_A to be extremal. An example illustrating this fact is given by the subset

$$A = \{1, 3, 4, 5, 6, 11\} \subset \mathbb{Z}/13$$

whose associated function f_A is extremal even though the condition does not hold. Among the smaller cyclic groups only those of order $3, 9, 11$ have non-empty $\mathcal{E}_{1/3}$, but all these extremal points are as in the statement (ii) of the proposition. We have established a correspondence between functions in $\mathcal{E}_1(\Gamma)$ and left orders on the group Γ. In particular, we have $\mathcal{E}_1(\Gamma) \neq \emptyset$ if and only if the group is left-orderable. We do not have such a criterion for the set $\mathcal{E}_{1/3}$. In particular, we would like to know

Question. How can the groups with non-empty $\mathcal{E}_{1/3}$ be characterized? More precisely, how does one determine whether the function $f_A \in \mathcal{B}(\Gamma)$ associated to a decomposition $\Gamma = \{e\} \sqcup A \sqcup A^{-1}$ is extremal?

7.8 Extremal points from the circle group

We denote by

$$\mathbb{T} = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$$

the circle group and we let $\arg : \mathbb{T} \longrightarrow [0, 2\pi)$ be the usual argument function. We define $\mathfrak{f} : \mathbb{T} \longrightarrow (-1, 1)$ by

$$\mathfrak{f}(\zeta) = \begin{cases} \dfrac{\arg(\zeta)}{\pi} - 1, & \zeta \neq 1 \\ 0, & \zeta = 1. \end{cases}$$

For $\zeta \neq 1$ we have

$$\mathfrak{f}(\zeta^{-1}) = \arg(\zeta^{-1})/\pi - 1 = (2\pi - \arg(\zeta))/\pi - 1 = -\mathfrak{f}(\zeta),$$

so that $\mathfrak{f} \in \mathcal{D}(\mathbb{T})$. Note that $\mathfrak{f}$ is equal to a section of the quotient $\mathbb{R} \longrightarrow \mathbb{T}$, composed with the translation $x \mapsto x - 1$ in $\mathbb{R}$.

Proposition 7.31. *We have* $\mathfrak{f} \in \mathcal{E}^*(\mathbb{T})$.

Proof. Let $\zeta, \eta \in \mathbb{T}$ such that $\zeta, \eta, \zeta\eta \neq 1$. We have $\arg(\zeta\eta) = \arg(\zeta) + \arg(\eta) + d$ for $d \in \{0, -2\pi\}$. Hence

$$\partial\mathfrak{f}(\zeta, \eta) = \frac{\arg(\zeta)}{\pi} - 1 + \frac{\arg(\eta)}{\pi} - 1 - \left(\frac{\arg(\zeta) + \arg(\eta) + d}{\pi} - 1\right) = -1 - \frac{d}{\pi} = \pm 1.$$

□

Let $i : \Gamma \hookrightarrow \mathbb{T}$ be an embedding. By Proposition 7.12 we have $\mathfrak{f} \circ i \in \mathcal{E}^*(\Gamma)$. This extremal point has the additional property that it does not attain the values ± 1. It turns out that every such extremal point arises this way. Write

$$\mathcal{E}^*_{(-1,1)}(\Gamma) = \{f \in \mathcal{E}^*(\Gamma) \mid f(\Gamma) \subset (-1, 1)\}.$$

Theorem 7.32. *For every group* Γ *there is a correspondence*

$$\{\text{embeddings } \Gamma \hookrightarrow \mathbb{T}\} \quad \longleftrightarrow \quad \mathcal{E}^*_{(-1,1)}(\Gamma),$$

where the extremal point associated to an embedding i *is given by* $\mathfrak{f} \circ i$*, and the embedding* i_f *associated to* $f \in \mathcal{E}^*_{(-1,1)}(\Gamma)$ *is given by*

$$g \mapsto \begin{cases} -\exp(i\pi f(g)), & g \neq e \\ 1, & g = e. \end{cases}$$

Corollary 7.33. (i) *For a finitely generated group* Γ *we have* $\mathcal{E}^*_{(-1,1)}(\Gamma) \neq \emptyset$ *if and only if* Γ *is of the form* $\mathbb{Z}^k$, $\mathbb{Z}/n$ *or* $\mathbb{Z}^k \times \mathbb{Z}/n$.

(ii) *For a finite group* Γ *we have* $\mathcal{E}^*(\Gamma) \neq \emptyset$ *if and only if* $\Gamma = \mathbb{Z}/n$.

(iii) *For* $f \in \mathcal{E}^*_{(-1,1)}(\mathbb{Z}^k)$ *the values* $f(g)$ *are irrational for all* $g \neq 0$.

Proof. (i) A finitely generated group Γ admits an embedding into $\mathbb{T}$ if and only if it is of this form.

(ii) For a finite group we have $\mathcal{E}^*_{(-1,1)}(\Gamma) = \mathcal{E}^*(\Gamma)$ by Proposition 7.1.

(iii) If $f(g) \in \mathbb{Q}$ then then $i_f(g) \in \mathbb{T}$ has finite order, so that i_f is not an embedding.

□

Note that by Theorem 7.25 there exist functions in $\mathcal{E}^*(\mathbb{Z}^k) \setminus \mathcal{E}^*_{(-1,1)}(\mathbb{Z}^k)$ with rational values. An example is the function $f \in \mathcal{E}^*(\mathbb{Z})$ that for $k > 0$ is given by

$$f(k) = \begin{cases} 1, & k = 3l \\ 1/3, & k = 3l+1 \\ -1/3, & k = 3l+2. \end{cases}$$

Corollary 7.34. *For* $\Gamma = \mathbb{Z}/n$ *the action of* $\mathrm{Aut}(\Gamma)$ *on* $\mathcal{E}^*(\Gamma)$ *is free and transitive. In particular, we have* $|\mathcal{E}^*(\Gamma)| = |\mathrm{Aut}(\Gamma)| = \phi(n)$.

Proof. Given $f, f' \in \mathcal{E}^*(\Gamma)$ with associated embeddings $i, i' : \Gamma \hookrightarrow \mathbb{T}$, there exists a unique $\varphi \in \mathrm{Aut}(\Gamma)$ such that $i' = i \circ \varphi$, i.e. such that $f' = \mathfrak{f} \circ i' = \mathfrak{f} \circ i \circ \varphi = f \circ \varphi$. □

Proof of Theorem 7.32. We first show that for $f \in \mathcal{E}^*_{(-1,1)}(\Gamma)$ the map i_f given in the statement of the theorem is a monomorphism. Let $g, h \in \Gamma$ be such that $g, h, gh \neq e$. We have

$$\begin{aligned} i_f(gh) &= -\exp(i\pi f(gh)) \\ &= -\exp(i\pi(f(g) + f(h) \pm 1)) \\ &= \quad \exp(i\pi(f(g) + f(h)) \\ &= (-\exp(i\pi f(g))(-\exp(i\pi f(h)) = i_f(g) i_f(h). \end{aligned}$$

Furthermore we have $i_f(e) = 1$ by definition, from which one easily deduces that $i_f(gh) = i_f(g) i_f(h)$ also holds when at least one of g, h, gh is trivial. If $i_f(g) = 1$ for some $g \neq e$, then $f(g) = \pm 1$, which is impossible. Hence i_f is injective. Now let $j : \Gamma \hookrightarrow \mathbb{T}$ be an embedding. For $g \neq e$ we have

$$i_{\mathfrak{f} \circ j}(g) = -\exp(i\pi \mathfrak{f}(j(g))) = -\exp(i\pi(\arg(j(g))/\pi - 1) = \exp(i \arg(j(g)) = j(g).$$

and also $i_{\mathfrak{f}\circ j}(e) = j(e) = 1$, so $i_{\mathfrak{f}\circ j} = j$. Furthermore, for $f \in \mathcal{E}^*_{(-1,1)}(\Gamma)$ and $g \neq e$ we have

$$\begin{aligned}(\mathfrak{f} \circ i_f)(g) &= \mathfrak{f}(-\exp(i\pi f(g))) \\ &= \frac{1}{\pi}\arg(-\exp(i\pi f(g))) - 1 \\ &= \frac{1}{\pi}(\pi f(g) + \pi) - 1 = f(g)\end{aligned}$$

and also $(\mathfrak{f} \circ i_f)(e) = f(e) = 0$, so $\mathfrak{f} \circ i_f = f$. □

7.9 Higher defect spaces

Let Γ be a group and E a Banach Γ-module. We call a map $f : \Gamma^n \longrightarrow E$ *alternating* if the associated homogenous map

$$c_f : \Gamma^{n+1} \longrightarrow \mathbb{R}, \quad c_f(g_0, \ldots, g_n) = f(g_0^{-1}g_1, \ldots, g_0^{-1}g_n)$$

is alternating in the sense that

$$c_f(g_{\sigma(0)}, \ldots, g_{\sigma(n)}) = \operatorname{sign}(\sigma) \cdot c_f(g_0, \ldots, g_n)$$

for every permutation σ of $1, \ldots, n$. We define $\ell^\infty_{\text{alt}}(\Gamma^n, E)$ to be the space of bounded alternating maps $\Gamma^n \longrightarrow E$, and we write $\mathcal{Z}^n_{\text{b,alt}}(\Gamma, E)$ for the subspace of bounded alternating n-cocycles. Note that for $n = 1$ this is consistent with our previous definition of alternating maps on Γ. For each $n \geq 1$ we define the *n-th defect space of* Γ *with values in* E:

$$\mathcal{D}^n(\Gamma, E) := \frac{\ell^\infty_{\text{alt}}(\Gamma^n, E)}{\mathcal{Z}^n_{\text{b,alt}}(\Gamma, E)}.$$

This space carries the norm

$$\|[f]\| := \sup_{(g_0,\ldots,g_n)\in\Gamma^{n+1}} \|\partial f(g_0, \ldots, g_n)\|_E.$$

Since $\mathcal{Z}^1_{\text{b,alt}}(\Gamma, \mathbb{R}) = 0$ we obtain for $n = 1$ and $E = \mathbb{R}$ the usual defect space $\mathcal{D}(\Gamma) = \mathcal{D}^1(\Gamma, \mathbb{R})$ together with the defect norm $\| \cdot \|_{\text{def}}$. In view of the rich structure of the spaces $\mathcal{D}(\Gamma)$ it would be interesting to know what information on the group Γ is encoded in the higher defect spaces, for example in the space $\mathcal{D}^2(\Gamma, \mathbb{R})$.

8 Relative bounded cohomology of free groups

This section is based on joint work with Cristina Pagliantini (see [45]). We prove the following result:

Theorem 8.1. *Let Γ be a free group of finite rank $n \geq 2$, and let $H < \Gamma$ be a subgroup of finite rank. The following are equivalent*

(i) *H has infinite index in Γ,*

(ii) *The space $\mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$ is non-trivial.*

(iii) *There exists a linear isometric embedding*

$$\oplus^n_\infty \mathcal{D}(\mathbb{Z}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, H; \mathbb{R})$$

The key to the proof of Theorem 8.1 is the following

Lemma 8.2. *Let Γ be a free group of finite rank $n \geq 2$ and let $H < \Gamma$ be a subgroup of finite rank and infinite index. There exists a basis $\{x_1, \ldots, x_n\}$ of Γ such that for all $g \in \Gamma$ and for all i we have*

$$gHg^{-1} \cap \langle x_i \rangle = \{1\},$$

which is to say that no conjugate of H contains a power of an element of this basis.

In the proof Lemma 8.2 we use the language of Schreier graphs, for which we fix the notation here. Let Γ be a free group with a chosen basis $\{x_1, \ldots, x_n\}$ and let $H < \Gamma$ be a subgroup. The Schreier graph $\mathcal{G}_H$ of the pair (Γ, H) with respect to this basis has as its vertices the set of left cosets

$$\mathrm{vert}(\mathcal{G}_H) = \{gH \mid g \in \Gamma\},$$

and the edges are given by

$$\mathrm{edges}(\mathcal{G}_H) = \{(gH, x_i gH) \mid g \in \Gamma, 1 \leq i \leq n\}.$$

Note that each edge is oriented and naturally labelled with a generator x_i, and that the graph $\mathcal{G}_H$ is $2n$-regular. Let $gH, g'H \in \Gamma/H$ and $x = x_{i_1}^{\epsilon_1} \cdots x_{i_l}^{\epsilon_l} \in \Gamma$, written as a word in the generators and their inverses. We have $xgH = g'H$ if and only if the edge-path $x_{i_1}^{\epsilon_1}, \ldots, x_{i_l}^{\epsilon_l}$ starting at the vertex gH ends at the vertex $g'H$. For each letter x_i^{-1} with a negative power this path is running in the direction opposite to the orientation of the corresponding edge. In particular we have $xgH = gH$ if and only if the path starting at gH and corresponding to x is a loop. We say that a vertex gH is the basepoint of an x-loop if there is $n \geq 1$ such that $x^n gH = gH$. We write

$$L_x(H) \subset \mathrm{vert}(\mathcal{G}_H)$$

for the set of vertices that are the basepoint of an x-loop. For an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ we have an induced bijection

$$\varphi : \mathrm{vert}(\mathcal{G}_H) \longrightarrow \mathrm{vert}(\mathcal{G}_{\varphi(H)}), \quad gH \mapsto \varphi(g)\varphi(H)$$

and by restricting this map we obtain for each $x \in \Gamma$ a bijection

$$L_x(H) \longrightarrow L_{\varphi(x)}(\varphi(H)).$$

The graph $\mathcal{G}_H$ can be identified with a covering graph of a wedge of n loops, namely with the covering that corresponds to the given subgroup H. Furthermore, $\mathcal{G}_H$ can be seen as the quotient of the Cayley graph $\mathcal{G}_{\{1\}}$ with respect to the natural action of H. We denote by $\mathcal{C}_H$ the core of the graph $\mathcal{G}_H$. This is the subgraph that consists of the edges and vertices that are contained in a loop without backtracking. This means that

$$v \in \mathrm{vert}(\mathcal{C}_H) \quad \Longleftrightarrow \quad v \in L_x(H) \text{ for some cyclically reduced } x \in \Gamma$$

and in fact $\mathrm{vert}(\mathcal{C}_H) = \bigcup L_x(H)$, where the union is over all cyclically reduced elements of Γ. Core graphs were introduced by Stallings in [53], and this is also the reference for the following observations:

Proposition 8.3. (i) *The graph $\mathcal{C}_H$ is finite, if and only if the subgroup H has finite rank,*

(ii) *We have $\mathcal{C}_H = \mathcal{G}_H$, if and only if the subgroup H has finite index in Γ.*

Proof of Lemma 8.2. We fix a basis $\{x_1, \ldots, x_n\}$ of Γ. For $1 \leq i \leq n$ and $k \in \mathbb{Z}$ we define $\varphi_{i,k} \in \mathrm{Aut}(\Gamma)$ by

$$\begin{aligned} \varphi_{i,k}(x_i) &= x_i \\ \varphi_{i,k}(x_j) &= x_j x_i^k, \quad \text{for } i \neq j. \end{aligned}$$

We will show that for a suitable concatenation ψ of such automorphisms we obtain a basis of the desired type by setting $y_i = \psi^{-1}(x_i)$. Let $\mathcal{G}_H$ be the Schreier graph of H with respect to the chosen basis and let $\mathcal{C}_H$ be its core. We define

$$L(H) := \bigcup_{i=1}^{n} L_{x_i}(H).$$

This is the set of all vertices in $\mathcal{G}_H$ at which a loop of a basis element is based. Since each x_i is a cyclically reduced element, we have $L(H) \subset \mathrm{vert}(\mathcal{C}_H)$. In particular, the set $L(H)$ is finite since H has finite rank.

Claim I: If $L(H) \neq \emptyset$ then there exists a vertex $v \in L(H)$ such that $v \notin \bigcap_{i=1}^{n} L_{x_i}(H)$.
Proof: Since H has infinite index the graph $\mathcal{G}_H$ has some vertices not contained in $L(H)$, for example any vertex outside the core. Therefore we can choose $w \in \mathrm{vert}(\mathcal{G}_H)$ not in $L(H)$ but adjacent to some $v \in L(H)$. If the edge connecting v with w is labelled x_j then either both or neither of v and w are contained in an x_j-loop. Since w is not in such a loop, v is neither, and therefore $v \notin L_{x_j}(H)$.

Claim II: For all $1 \leq i \leq n$ there exists $k > 0$ such that

(i) $\forall v \in L_{x_i}(H): \quad x_i^k v = v$

(ii) $\forall v \in \mathrm{vert}(\mathcal{C}_H) \setminus L_{x_i}(H): \quad x_i^k v \notin \mathrm{vert}(\mathcal{C}_H)$.

Proof: We define

$$k := N \cdot \mathrm{lcm}\{\ell \,|\, \ell \text{ is the length of a simple } x_i\text{-loop in } \mathcal{G}_H\}$$

for a suitable number $N > 0$. Note that the set of which we are taking the least common multiple is finite, since the graph $\mathcal{G}_H$ contains only finitely many simple loops. If the set is actually empty (i.e. if $L_{x_i}(H) = \emptyset$) then we define the lcm to be 1. The first condition is obviously satisfied for such a k. For v as in the second condition, consider the sequence of vertices $v, x_i v, x_i^2 v, \ldots$. This sequence must eventually leave the core: otherwise it would contain a loop and this loop would contain v, a contradiction since v is not contained in any x_i-loop. Now choosing N sufficiently large we have that $x_i^k v$ is outside the core for all of the finitely many vertices $v \in \mathrm{vert}(\mathcal{C}_H) \setminus L_{x_i}(H)$.

Claim III: Let $1 \leq i \leq n$ and let $k > 0$ be the number from Claim II. For the automorphism $\varphi = \varphi_{i,-k}$ the bijection $\varphi^{-1} = \varphi_{i,k} : \mathrm{vert}(\mathcal{G}_{\varphi(H)}) \longrightarrow \mathrm{vert}(\mathcal{G}_H)$ restricts to a map

$$L(\varphi(H)) \longrightarrow L_{x_i}(H).$$

Proof: Let $v \in L_{x_j}(\varphi(H))$ for some $1 \leq j \leq n$. If $i = j$ then $\varphi^{-1}(v) \in L_{\varphi^{-1}(x_i)}(H) = L_{x_i}(H)$. If $i \neq j$ then $\varphi^{-1}(v) \in L_{\varphi^{-1}(x_j)}(H) = L_{x_j x_i^k}(H)$. Now $x_i^k w$ is outside the core for any vertex w that is not in an x_i-loop, and in particular $x_j x_i^k w$ is outside the core for such a vertex. It follows that $L_{x_j x_i^k}(H) \subset L_{x_i}(H)$, hence $\varphi^{-1}(v) \in L_{x_i}(H)$.

Now Claim I tells us that there is an index $1 \leq i \leq n$ such that $L_{x_i}(H) \subsetneq L(H)$. Let k be the associated number from Claim II and let φ be the automorphism from Claim III. Since the map $L(\varphi(H)) \longrightarrow L_{x_i}(H)$ is injective we have

$$|L(\varphi(H))| \leq |L_{x_i}(H)| < |L(H)|.$$

By replacing the subgroup H with its image $\varphi(H)$ we reduced the number of vertices that are contained in a basic loop. If we iterate this argument we obtain after finitely many steps an automorphism $\psi \in \mathrm{Aut}(\Gamma)$ for which $L(\psi(H)) = \emptyset$. So we have $L_{x_i}(\psi(H)) = \emptyset$ for all i, and this means that

$$\forall i \, \forall g \in \Gamma : \quad g\psi(H)g^{-1} \cap \langle x_i \rangle = \{1\},$$

which amounts to saying

$$\forall i \, \forall g \in \Gamma : \quad gHg^{-1} \cap \langle y_i \rangle = \{1\},$$

where $y_i = \psi^{-1}(x_i)$. □

Lemma 8.4. *Let Γ be a free group of finite rank $n \geq 2$ and let $H < \Gamma$ be a subgroup of finite rank and infinite index. Assume that $\{x_1, \ldots, x_n\}$ is a basis of Γ such that $gHg^{-1} \cap \langle x_i \rangle = \{1\}$ for all i and all $g \in \Gamma$. Then there is a number $m_0 > 0$ such that no element of H contains a power x_i^m with $m > m_0$ as a subword.*

Proof. Assume that, under the assumptions of the lemma, we can find infinitely many powers x_i^m as subwords of cyclically reduced elements of H. In particular, H contains an element of the cyclically reduced form $yx_i^m z$ for some $m > \#\mathrm{vert}(\mathcal{C}_H)$ and some i. (Note that the core $\mathcal{C}_H$ is finite by Proposition 8.3.) The path in the Schreier graph $\mathcal{G}_H$ that starts at the vertex H and is described by this element is then a loop without backtracking based at H. This loop is contained in $\mathcal{C}_H$, in particular the vertices

$$x_i zH, \, x_i^2 zH, \, \ldots, \, x_i^m zH$$

on this loop are contained in $\mathrm{vert}(\mathcal{C}_H)$. By the choice of m, there exist $1 \leq k, l \leq m$, $k \neq l$, such that $x_i^k zH = x_i^l zH$, which means that $x_i^{k-l} \in zHz^{-1}$, a contradiction. □

Combining the last two lemmas immediately yields:

Lemma 8.5. *Let Γ be a free group of finite rank $n \geq 2$ and let $H < \Gamma$ be a subgroup of finite rank and infinite index. There exists a basis $\{x_1, \ldots, x_n\}$ of Γ that has the following property: There is a number $m_0 > 0$ such that no element of H contains a power x_i^m with $m > m_0$ as a subword.*

Proof of Theorem 8.1. The implication (iii)$\Rightarrow$(ii) is obvious and the implication (ii)$\Rightarrow$(i) follows from Proposition 1.7. In order to prove (i)$\Rightarrow$(iii) we let $\{x_1, \dots, x_n\}$ and $m_0 > 0$ be as in Lemma 8.5. Associated to the splitting $\Gamma = \langle x_1 \rangle * \langle x_2 \rangle * \cdots * \langle x_n \rangle$ we have the isometric embedding

$$I : \oplus_\infty^n \mathcal{D}(\mathbb{Z}) \hookrightarrow \mathrm{H}_\mathrm{b}^2(\Gamma, \mathbb{R}).$$

from Theorem 3.6, where we are using the obvious generalization of the theorem to the case of several free factors. By Proposition 7.4 we have an isometric embedding $j : \mathcal{D}(m_0\mathbb{Z}) \hookrightarrow \mathcal{D}(\mathbb{Z})$ from which we obtain the isometric embedding

$$J := \oplus^n j : \oplus_\infty^n \mathcal{D}(m_0\mathbb{Z}) \hookrightarrow \oplus_\infty^n \mathcal{D}(\mathbb{Z}).$$

We claim that the concatenation $I \circ J$ ranges in the image of the embedding

$$i : \mathrm{H}_\mathrm{b}^2(\Gamma, H; \mathbb{R}) \hookrightarrow \mathrm{H}_\mathrm{b}^2(\Gamma, \mathbb{R})$$

given in Proposition 1.6.(ii). So let $\{f_k\} = \{f_k\}_{1 \le k \le n} \in \oplus_\infty^n \mathcal{D}(m_0\mathbb{Z})$. Then $J(\{f_k\}) = \{j(f_k)\}$, and each of the functions $j(f_k) \in \mathcal{D}(\mathbb{Z})$ is supported on the subgroup $m_0\mathbb{Z}$, by construction of the embedding j. Write $f := j(f_1) * \cdots * j(f_n)$ for the split quasimorphism obtained from these functions. This quasimorphism has the property that $f(g) = 0$ if g is an element whose factorization contains no factor of the form $x_i^{k \cdot m_0}$, $k \neq 0$. Since we have chosen m_0 according to Lemma 8.5, we know that in the factorization of each element of H only exponents smaller than m_0 occur. Therefore we have $f|_H = 0$, i.e. $f \in \mathrm{QM}(\Gamma, H)$. In particular we have $\partial f|_{H \times H} = 0$, so the class $(I \circ J)(\{f_k\}) = [\partial f]_\mathrm{b}$ is in fact contained in $\mathrm{H}_\mathrm{b}^2(\Gamma, H; \mathbb{R})$. Therefore we have an embedding

$$\oplus_\infty^n \mathcal{D}(\mathbb{Z}) \cong \oplus_\infty^n \mathcal{D}(m_0\mathbb{Z}) \hookrightarrow \mathrm{H}_\mathrm{b}^2(\Gamma, H; \mathbb{R}).$$

We are left with showing that this embedding is isometric. Let $\{f_k\} \in \oplus_\infty^n \mathcal{D}(m_0\mathbb{Z})$ as before, with associated split quasimorphism f. Let $\omega_f \in \mathrm{H}_\mathrm{b}^2(\Gamma, H; \mathbb{R})$ be the image of $\{f_k\}$ under the embedding in question. We have

$$\operatorname{def} f = \|i(\omega_f)\| \le \|\omega_f\| \le \operatorname{def} f$$

where the equality comes from Theorem 3.3, the first inequality is due to the fact that i is norm non-increasing, and the second inequality is because ∂f is a representative of the relative class ω_f. It follows that

$$\|\omega_f\| = \operatorname{def} f = \max_k \operatorname{def} f_k = \|\{f_k\}\|. \qquad \square$$

9 Quasimorphisms induced from hyperbolic mapping tori

In this section we propose a construction of quasimorphisms in $\mathrm{QM}(\mathbb{F}_n)$ that relies on the fact that mapping tori of suitable automorphisms in $\mathrm{Aut}(\mathbb{F}_n)$ are hyperbolic. The construction, which is independent from the other parts of this thesis, is a translation to the free group setting of a result of Yoshida concerning the bounded cohomology of surfaces (see Theorem 9.2 below). We denote by Σ_f the mapping torus of a diffeomorphism f of a closed orientable surface Σ. The basic form of Yoshida's result is the following:

Theorem 9.1. *For each pseudo-Anosov diffeomorphism f we have an embedding*

$$i_f : \mathrm{H}^3(\Sigma_f, \mathbb{R}) \longhookrightarrow \mathrm{H}^3_{\mathrm{b}}(\Sigma, \mathbb{R})$$

These are the third singular (resp. singular bounded) cohomology spaces of Σ_f (resp. Σ). The space $\mathrm{H}^3(\Sigma_f, \mathbb{R})$ is one-dimensional, generated by the fundamental co-class ν_f of the closed 3-manifold Σ_f. By varying the diffeomorphisms Yoshida proves

Theorem 9.2 ([57], Theorem 1)**.** *Every closed orientable surface Σ of genus at least two admits an infinite family $\{f_n\}$ of pseudo-Anosov diffeomorphisms, such that the classes $i_{f_n}(\nu_{f_n})$ are linearly independent in $\mathrm{H}^3_{\mathrm{b}}(\Sigma, \mathbb{R})$. In particular, this space has infinite dimension.*

In order to sketch a proof of Theorem 9.1 we apply the following fact whose topological version is due to Gromov (see [29], p. 247).

Proposition 9.3 ([41], Proposition 8.6.6)**.** *Let $\Gamma < \overline{\Gamma}$ be groups. If Γ is co-amenable in $\overline{\Gamma}$ then the restriction map $\mathrm{H}^k_{\mathrm{b}}(\overline{\Gamma}, \mathbb{R}) \longrightarrow \mathrm{H}^k_{\mathrm{b}}(\Gamma, \mathbb{R})$ is an isometric embedding for all $k \geq 0$. This holds in particular when there is a short exact sequence*

$$1 \longrightarrow \Gamma \longrightarrow \overline{\Gamma} \longrightarrow A \longrightarrow 1$$

with an amenable group A.

For our purposes the relevant application of this statement is in the case of a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Gamma *_\varphi \longrightarrow \mathbb{Z} \longrightarrow 1 \tag{6}$$

associated to the *mapping torus* $\Gamma *_\varphi$ of an automorphism φ of a group Γ. As the notation suggests, $\Gamma *_\varphi$ is defined to be the HNN extension of Γ over φ. Note that this extension can also be described as the semi-direct product $\Gamma \rtimes_\varphi \mathbb{Z}$. The terminology mapping torus $\Gamma *_{\varphi}$ is due to the fact that the (topological) mapping torus M_f of a homeomorphism $f : X \longrightarrow X$ has fundamental group $\Gamma *_{\pi_1(f)}$, where $\Gamma = \pi_1(X, x_0)$. We will use furthermore the following fundamental result of Gromov–Brooks–Ivanov which allows us to go back and forth between the bounded cohomology of a manifold and the one of its fundamental group:

Theorem 9.4 ([34], Theorem 4.1)**.** *For a connected countable cell complex X the singular bounded cohomology $\mathrm{H}^*_{\mathrm{b}}(X, \mathbb{R})$ is isometrically isomorphic to the bounded group cohomology $\mathrm{H}^*_{\mathrm{b}}(\pi_1(X), \mathbb{R})$.*

Sketch of a proof of Theorem 9.1. Let $f : \Sigma \longrightarrow \Sigma$ be a pseudo-Anosov diffeomorphism of Σ. By a theorem of Thurston the mapping torus M_f can be equipped with a (unique) Riemannian metric that turns it into a closed hyperbolic 3-manifold (see [54]). The fundamental co-class $\nu_f \in \mathrm{H}^3(M_f, \mathbb{R}) \cong \mathbb{R}$ of M_f is bounded (see [29]). More precisely, the canonical representative of ν_f, namely the volume form vol_f, is a bounded 3-cocycle. This yields a canonical embedding $\mathrm{H}^3(M_f, \mathbb{R}) \hookrightarrow \mathrm{H}^3_{\mathrm{b}}(M_f, \mathbb{R})$, determined by the assignment $\nu_f = [\mathrm{vol}_f] \mapsto [\mathrm{vol}_f]_{\mathrm{b}}$. Using the asphericity of M_f and Theorem 9.4 we obtain the group-theoretic counterpart of this embedding, which reads $\mathrm{H}^3(\Gamma *_{\varphi}, \mathbb{R}) \hookrightarrow \mathrm{H}^3_{\mathrm{b}}(\Gamma *_{\varphi}, \mathbb{R})$, where $\Gamma = \pi_1(\Sigma, x_0)$ and $\varphi = \pi_1(f)$. Applying Proposition 9.3 to the corresponding short exact sequence (6) further yields an embedding $\mathrm{H}^3_{\mathrm{b}}(\Gamma *_{\varphi}, \mathbb{R}) \hookrightarrow \mathrm{H}^3_{\mathrm{b}}(\Gamma, \mathbb{R})$. Hence we have the concatenation

$$\mathrm{H}^3(\Gamma *_{\varphi}, \mathbb{R}) \hookrightarrow \mathrm{H}^3_{\mathrm{b}}(\Gamma *_{\varphi}, \mathbb{R}) \hookrightarrow \mathrm{H}^3_{\mathrm{b}}(\Gamma, \mathbb{R}).$$

whose topological counterpart is the desired embedding i_f. □

Our aim is now to apply Proposition 9.3 in a formally similar situation, where a mapping torus $\Gamma *_{\varphi}$ has non-trivial bounded cohomology in some degree k that is larger than the cohomological dimension of Γ. The above strategy will then yield non-trivial classes in $\mathrm{EH}^k_{\mathrm{b}}(\Gamma, \mathbb{R})$. More precisely, we work with mapping tori of hyperbolic automorphisms of free groups. This reinterpretation is in the spirit of a general program in which results in the topological setting of surface groups and mapping class groups are translated into analogous statements in the more combinatorial setting of free groups and $\mathrm{Out}(\mathbb{F}_n)$. It often occurs that in the process of such a translation new difficulties and phenomena arise, and our case is no exception.

Let $\mathbb{F}_n$ be the rank n free group. There are different types of automorphisms of $\mathbb{F}_n$ that can be seen as analogues of pseudo-Anosov diffeomorphisms of a surface. For our purpose the relevant condition is that $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ be *hyperbolic*, which by definition means that there exist numbers $m > 0$ and $\lambda > 1$ such that

$$\lambda |g| \leq \max\{|\varphi^m(g)|, |\varphi^{-m}(g)|\}, \quad \forall\, g \in \Gamma.$$

Here $|\cdot|$ stands for the word length with respect to some generating set. Such automorphisms are characterized by the following theorem, which combines results of Bestvina–Feighn and Brinkmann:

Theorem 9.5 ([13], Theorem 1.2)**.** *For $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ the following are equivalent*

(i) *φ is hyperbolic,*

(ii) *φ is atoroidal, that is, there is no non-trivial conjugacy class $C \subset \mathbb{F}_n$ with $\varphi^m(C) = C$ for an $m \geq 1$,*

(iii) *The mapping torus $\mathbb{F}_n *_\varphi$ does not contain $\mathbb{Z} \times \mathbb{Z}$ as a subgroup,*

(iv) *The mapping torus $\mathbb{F}_n *_\varphi$ is word hyperbolic.*

We note that hyperbolic automorphisms only exist for free groups of rank at least 3. Our observation can now be stated as follows:

Theorem 9.6. *For each hyperbolic automorphism φ of the free group $\mathbb{F}_n$ there is an embedding*

$$\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) \hookrightarrow \mathrm{H}^2_\mathrm{b}(\mathbb{F}_n, \mathbb{R}).$$

Using Proposition 1.3 this yields a construction of quasimorphisms:

Corollary 9.7. *For each hyperbolic automorphism φ of the free group $\mathbb{F}_n$ there is an embedding*

$$\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) \hookrightarrow \mathrm{HQM}_0(\mathbb{F}_n).$$

In order to prove Theorem 9.6 we mimic the proof of Theorem 9.1. There we used the fact that for the hyperbolic 3-manifold group $\Gamma *_\varphi = \pi_1(M_f, \cdot)$ the natural representative of the generator of $\mathrm{H}^3(\Gamma *_\varphi, \mathbb{R})$ is bounded. There is no analogue of this fact for the cohomology $\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R})$. Instead we obtain bounded classes through the following result, which is due to Mineyev:

Theorem 9.8 ([38], Theorem 11)**.** *For a word hyperbolic group Γ the comparison map $\mathrm{H}^k_\mathrm{b}(\Gamma, \mathbb{R}) \longrightarrow \mathrm{H}^k(\Gamma, \mathbb{R})$ is surjective for all $k \geq 2$.*

We note that Mineyev's result applies to the case of non-trivial coefficients as well. The proof given in [38] is carried out by constructing for each $k \geq 2$ a map $r^k : \mathrm{H}^k(\Gamma, \mathbb{R}) \longrightarrow \mathrm{H}^k_\mathrm{b}(\Gamma, \mathbb{R})$ which is right-inverse to the comparison map.

Proof of Theorem 9.6. We obtain the desired embedding as the concatenation

$$\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) \hookrightarrow \mathrm{H}^2_\mathrm{b}(\mathbb{F}_n *_\varphi, \mathbb{R}) \hookrightarrow \mathrm{H}^2_\mathrm{b}(\mathbb{F}_n, \mathbb{R}).$$

Here the first map is Mineyev's right-inverse r^2 to the comparison map and the second map is the restriction map of Proposition 9.3. □

Now in contrast to the manifold group setting, the top dimensional cohomology $\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R})$ may vanish. While the group $\mathbb{F}_n *_\varphi$ has cohomological dimension 2 (it has dimension at least 2 as it is not free, and at most 2 by the Mayer–Vietoris sequence for HNN extensions), this is not necessarily detected by trivial real coefficients:

Proposition 9.9. *For $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ write $\varphi^* = \mathrm{H}^1(\varphi, \mathbb{R})$. Let*

$$d = \dim \ker(\mathrm{id} - \varphi^*)$$

be the dimension of the fixed space of φ^*. Then we have

$$\dim \mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) = d.$$

Corollary 9.10. *For $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ we have $\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) \neq 0$ if and only if the abelianization $\varphi_{\mathrm{ab}} \in \mathrm{GL}(n, \mathbb{Z})$ has an eigenvalue 1.*

Proof. The map $\varphi_{\mathrm{ab}} = \mathrm{H}_1(\varphi, \mathbb{Z})$ has an eigenvalue 1, if and only if

$$\mathrm{H}_1(\varphi, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{H}_1(\varphi, \mathbb{R})$$

has an eigenvalue 1, which is equivalent to the dual map $(\mathrm{H}_1(\varphi, \mathbb{R}))^* = \mathrm{H}^1(\varphi, \mathbb{R}) = \varphi^*$ having an eigenvalue 1, which is to say $d > 0$. □

For the proof of Proposition 9.9 we need

Lemma 9.11. *For $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ there is an exact sequence*

$$\mathrm{H}^1(\mathbb{F}_n *_\varphi, \mathbb{R}) \longrightarrow \mathrm{H}^1(\mathbb{F}_n, \mathbb{R}) \xrightarrow{c} \mathrm{H}^1(\mathbb{F}_n, \mathbb{R}) \xrightarrow{\delta} \mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) \longrightarrow 0$$

where $c = \mathrm{id} - \varphi^*$ with $\varphi^* = \mathrm{H}^1(\varphi, \mathbb{R})$.

Proof. This is a segment of the Mayer–Vietoris sequence associated to the HNN extension $\mathbb{F}_n *_\varphi$ (see [10]). □

Proof of Proposition 9.9. Using the maps δ and c of Lemma 9.11 we get

$$\begin{aligned}
\dim \mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R}) &= \dim \mathrm{im}\, \delta \\
&= \dim \mathrm{H}^1(\mathbb{F}_n, \mathbb{R}) - \dim \ker \delta \\
&= \dim \mathrm{H}^1(\mathbb{F}_n, \mathbb{R}) - \dim \mathrm{im}\, c \\
&= \dim \mathrm{H}^1(\mathbb{F}_n, \mathbb{R}) - (\dim \mathrm{H}^1(\mathbb{F}_n, \mathbb{R}) - \dim \ker c) \\
&= \dim \ker c = d.
\end{aligned}$$

□

In view of Corollary 9.10 the quasimorphism construction of Corollary 9.7 is meaningful only when applied to a hyperbolic automorphism whose abelianization has 1 as an eigenvalue. One important class of hyperbolic automorphisms are the so called Stallings *PV-automorphisms* (see [52]). These automorphisms are characterized by the fact that their abelianization has an eigenvalue of modulus larger than 1 while the other eigenvalues have modulus less than 1. Therefore they cannot serve as an input for our constructions. However there is the following recent result due to Clay–Pettet, that asserts the existence of hyperbolic automorphisms with prescribed abelianization:

Theorem 9.12 ([21], Theorem 6.1)**.** *Let* $n \geq 3$*. For each* $A \in \mathrm{GL}(n, \mathbb{Z})$ *there is a hyperbolic fully irreducible automorphism* $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ *with* $\varphi_{\mathrm{ab}} = A$*.*

We can thus summarize our construction as follows: Let $n \geq 3$. Using Theorem 9.12 pick a hyperbolic $\varphi \in \mathrm{Aut}(\mathbb{F}_n)$ such that the abelianization φ_{ab} has an eigenvalue 1. Then apply Corollary 9.7 to obtain an embedding of the (non-zero) space $\mathrm{H}^2(\mathbb{F}_n *_\varphi, \mathbb{R})$ into $\mathrm{HQM}_0(\mathbb{F}_n)$.

There is no straightforward way of describing the quasimorphisms under consideration explicitly; the reason is that we invoked two existence results (Theorems 9.8 and 9.12) that have rather inexplicit proofs. Compared to the explicit combinatorial definitions of counting and split quasimorphisms, the construction at hand is of a very different nature. This is illustrated by the fact that the construction is only meaningful when applied for a free group of rank at least three.

Appendix: $\mathrm{H}_\mathrm{b}^2(\mathbb{F}_2, \mathbb{R})$ is infinite dimensional, a simple proof

Let $\mathbb{F}_2 = \langle a, b\rangle$ be the free group of rank 2. Given a bounded sequence $s : \mathbb{N} \longrightarrow \mathbb{R}$ we define a map $f_s : \mathbb{F}_2 \longrightarrow \mathbb{R}$ as follows:

Extend the sequence to an alternating map $s : \mathbb{Z} \longrightarrow \mathbb{R}$, i.e. set $s(0) = 0$ and $s(-k) = -s(k)$ for $k < 0$. Define $f_s(e) = 0$ and for $e \neq g \in \mathbb{F}_2$ with normal form $g = a^{k_1} b^{k_2} \cdots a^{k_{n-1}} b^{k_n}$ let

$$f_s(g) := s(k_1) + s(k_2) + \ldots + s(k_n).$$

Theorem A. *The map f_s is a quasimorphism which is trivial if and only if $s = 0$. Hence we have a linear embedding*

$$\ell^\infty \hookrightarrow \mathrm{H}_\mathrm{b}^2(\mathbb{F}_2, \mathbb{R}), \qquad s \mapsto [\partial f_s]_\mathrm{b}$$

and the space $\mathrm{H}_\mathrm{b}^2(\mathbb{F}_2, \mathbb{R})$ is infinite dimensional.

Proof. For $g, h \in \Gamma$ write $\partial f_s(g, h) = f_s(gh) - f_s(g) - f_s(h)$. If (the normal form of) g ends with an a-letter and h begins with b-letter or vice versa, then $\partial f_s(g, h) = 0$ since in this case the normal form of gh is the concatenation of the normal forms of g and h. If the normal forms are $g = g'a^k$ and $h = a^{-k}h'$ then $\partial f_s(g, h) = \partial f(g', h')$, since $s(-k) = -s(k)$, and likewise for b-letters. So we may assume that $g = g'a^k$ and $h = a^l h'$ with $k+l \neq 0$, or likewise with b-letters. In this case we have the normal form $gh = g'a^{k+l}h'$ and so $\partial f_s(g, h) = s(k+l) - s(k) - s(l)$, i.e. $|\partial f_s(g, h)\| \leq 3\|s\|_\infty$. This proves that f_s is a quasimorphism.

Now assume that f_s is trivial, i.e. that $f_s = \varphi + \beta$ where φ is a homomorphism and β is a bounded map. Evaluating this equation at a^n yields $s(n) = n \cdot \varphi(a) + \beta(a^n)$. Since s and β are bounded this means that $\varphi(a) = 0$, and likewise $\varphi(b) = 0$. So $\varphi = 0$, which is to say that $f_s = \beta$ is bounded. For $k \in \mathbb{Z}$, $k \neq 0$, we have $f_s((a^k b^{\pm 1})^n) = n \cdot (s(k) \pm s(1))$. Since f_s is bounded it follows that $s(k) \pm s(1) = 0$, so $s(k) = 0$ and therefore $s = 0$. $\square$

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