Boundary maps and maximal representations of complex hyperbolic lattices in SU \((m, n)\)

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BOUNDARY MAPS AND MAXIMAL REPRESENTATIONS OF COMPLEX HYPERBOLIC LATTICES IN $SU(m,n)$

A thesis submitted to attain the degree of

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presented by

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Abstract

Bounded cohomology can be used to define numerical invariants on the representation variety $\text{Hom}(\Gamma, G)$ of a discrete group $\Gamma$ into a semisimple Lie group $G$. Such invariants, in turn, allow to select some components of $\text{Hom}(\Gamma, G)$ consisting of representations that, often, reflect situations of particular geometric interest: the maximal representations.

In the thesis we study maximal representations when $\Gamma$ is a lattice in $\text{PU}(1, p) = \text{Isom}_0(\mathbb{H}^p_\mathbb{C})$ and $G$ is the Hermitian Lie group $\text{SU}(m, n)$, by analyzing geometric properties of the associated measurable boundary map $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S_{m,n}$. Here $\partial \mathbb{H}^p_\mathbb{C}$ is the visual boundary of the complex hyperbolic space and $S_{m,n}$ is the compact homogeneous $\text{SU}(m, n)$-space consisting of maximal isotropic subspaces of $\mathbb{C}^{m+n}$.

It was conjectured that maximal representations of complex hyperbolic lattices are (essentially) obtained by restriction of a maximal representation of the ambient group $\text{PU}(1, p)$. We prove this conjecture in two cases: when the Zariski closure of the image has no factor of tube type, and when $p$ is big enough with respect to $m$.

The thesis consists of three parts: in the first part we review some background material about Hermitian symmetric spaces, bounded cohomology and maximal representations. This will allow us to define chain geometry preserving maps $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S_{m,n}$ and weakly monotone maps $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S_{m,n}$, singling out the properties of maps that are equivariant with respect to a maximal representation. One chapter of this part (Chapter 5) is devoted to a review and extension of the recent results of Hamlet’s thesis: he classified all maximal representations between simple Hermitian Lie groups, we treat the more general case of classical Hermitian semisimple Lie groups.

In the second part we study maximal representations $\rho : \Gamma \to \text{SU}(m, n)$ with $m < n$ and we analyze chain geometry preserving maps. We are able to show that each such a map agrees almost everywhere with an algebraic map. This allows us to prove the aforementioned conjecture in the case in which no factor in the Zariski closure of the image is of tube type.

In the third part we consider the case $m = n$, and study weakly monotone maps $\phi : \partial \mathbb{H}^p_\mathbb{C} \to S_{m,n}$. In this case our conclusions are much weaker, but we exhibit an explicit, full measure set $R$, on which the restriction of $\phi$ is continuous and strictly equivariant. This allows to prove that all maximal representations are discrete and injective, and hence completely prove the conjecture if the dimension of the complex hyperbolic space is big enough when compared with the real rank of the target group.
Riassunto

Usando la coomologia limitata è possibile definire degli invarianti numerici sulle varietà delle rappresentazioni $\text{Hom}(\Gamma, G)$ di un gruppo discreto $\Gamma$ in un gruppo di Lie semisemplice $G$. A loro volta questi invarianti permettono di selezionare alcune componenti connesse di $\text{Hom}(\Gamma, G)$ formate dalle cosiddette rappresentazioni massimali.

Nella tesi ci concentriamo sulle rappresentazioni massimali nel caso in cui $\Gamma$ sia un reticolo in $\text{PU}(1,p) = \text{Isom}^+(\mathbb{H}^p\mathbb{C})$ e $G$ denoti il gruppo di Lie Hermitiano $\text{SU}(m,n)$. Lo studio è basato sull’analisi della mappa di bordo $\phi: \partial \mathbb{H}^p\mathbb{C} \to S_{m,n}$ naturalmente associata alla rappresentazione. Qui indichiamo con $\partial \mathbb{H}^p\mathbb{C}$ il bordo visuale dello spazio iperbolico complesso e con $S_{m,n}$ lo spazio omogeneo compatto formato dai sottospazi isotropici massimali di $\mathbb{C}^{m+n}$. Si congettura che tutte le rappresentazioni massimali di reticoli iperbolic complessi si ottengano per restrizione di rappresentazioni di $\text{PU}(1,p)$. Dimostriamo la congettura in due casi: quando la chiusura di Zariski dell’immagine non ha fattori tube-type, e quando $p$ è abbastanza grande rispetto a $m$.

Nella prima parte della tesi ricordiamo i concetti fondamentali riguardo a spazi simmetrici Hermitiani, coomologia limitata e rappresentazioni massimali, questo ci permette di definire mappe $\phi: \partial \mathbb{H}^p\mathbb{C} \to S_{m,n}$ che preservano la geometria delle catene e mappe debolmente monotone con immagine $S_{m,m}$. Nell’ultimo capitolo della parte (il Capitolo 5), riassumiamo ed estendiamo parzialmente i recenti risultati della tesi di Hamlet, in cui classifica tutte le rappresentazioni massimali tra gruppi di Lie Hermitiani semplici.

Nella seconda parte del lavoro ci occupiamo del caso $m < n$ e studiamo mappe che preservano la geometria delle catene. Dimostriamo che ogni mappa con questa proprietà coincide quasi ovunque con una mappa algebrica. Quest’ultimo fatto ci permette di dimostrare la summenzionata congettura nel caso in cui nessun fattore della chiusura di Zariski dell’immagine sia di tube-type.

Nella terza parte della tesi consideriamo il caso $m = n$ e studiamo mappe debolmente monotone. In questo caso le nostre conclusioni sono molto più deboli, ma costruiamo esplicitamente un insieme $R$ di misura piena su cui la restrizione della mappa sia continua. Questo ci permette di dimostrare che tutte le rappresentazioni massimali sono discrete e iniettive, e quindi dimostrare la congettura se la dimensione dello spazio iperbolico complesso è abbastanza grande quando confrontata con il rango del gruppo immagine.
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Chapter 1

Introduction

1.1 Historical background

1.1.1 Rigidity for lattices in semisimple Lie groups

Lattices in semisimple Lie groups have remarkable rigidity features: since homomorphisms defined on Lie groups need to preserve much more structure, it would be natural to expect more flexibility for homomorphisms defined on discrete groups, but often this is not true for lattices whose homomorphism, in many cases, are just as rare as those of the ambient Lie group.

The first result in this direction was proven by Selberg [Sel60]. He used geometric techniques to study the homomorphism given by the inclusion $i$ of a cocompact lattice in $\text{SL}_n(\mathbb{R})$, and showed that this cannot be deformed: each other homomorphism sufficiently close to $i$ needs to be conjugate to $i$. A similar result was shown by Calabi [Cal61] for cocompact lattices in $\text{O}(1,n)$, and subsequently Calabi and Vesentini [CV60] used methods based on Hodge theory and Bochner type inequalities to show that also the inclusion of a cocompact lattice in an Hermitian Lie group is an isolated point in the character variety. These result were then generalized by Weil to all cocompact lattices of simple noncompact Lie groups different from $\text{SL}_2(\mathbb{R})$ and by Garlan-Ragunathan [GR70] for nonuniform lattices in rank 1 groups. By now, local rigidity of lattice embeddings in semisimple Lie groups is well understood:

**Theorem 1.1** (see [Rag72, Section VII.5]). Let $\Gamma$ be a lattice in a semisimple Lie group without compact factors and factors locally isomorphic to $\text{SL}_2(\mathbb{R})$ or $\text{SL}_2(\mathbb{C})$. The inclusion of $\Gamma$ in $G$ is locally rigid.

It is worth remarking here that torsionfree lattices in $\text{SL}_2(\mathbb{R})$ are fundamental groups of surfaces and the inclusion of such a lattice is never rigid: up to conjugation the deformation space of such an inclusion, the Teichmüller space, has dimension $6g - 6$. In the case of $\text{SL}_2(\mathbb{C}) = \text{Isom}(\mathbb{H}^3)$ local rigidity still holds for cocompact lattices, but this is not true for nonuniform lattices: via Dehn-Filling it is possible
to construct homomorphisms arbitrarily close to the lattice inclusion that are not even injective (cfr. [BP92, Section E.6]).

A different kind of rigidity phenomena, the so-called strong rigidity was discovered by Mostow [Mos67] for cocompact lattices in rank one groups and generalized by Prasad [Pra73] to the nonuniform setting:

**Theorem 1.2** (Mostow-Prasad rigidity). Let $G$ be a Lie group not isomorphic to $SL_2(\mathbb{R})$, let $\Gamma < G$ be a lattice and $j : \Gamma \to G$ be an injective homomorphism whose image is a lattice in $G$. Then $j$ is conjugate to the lattice embedding.

Mostow’s proof for rank one groups was based on the study of boundary maps: since, by assumption, the image of $\Gamma$ under $j$ is a cocompact lattice, $\Gamma$ and $j(\Gamma)$ are quasi-isometric, hence in particular there exists a continuous boundary map $\phi : \partial H^3 = \partial_\infty \Gamma \to \partial H^3 = \partial_\infty \rho(\Gamma)$. Mostow showed that this map need to be quasiconformal. The analysis of quasiconformal maps allowed him to show that the boundary map, and hence the representation, is conjugate to the identity. It is worth recalling here also Gromov-Thurston’s endgame in their proof of Mostow rigidity for real hyperbolic three manifolds: using the homotopy invariance of simplicial volume and properties of bounded cohomology one deduces that the boundary map sends vertices of regular ideal tetrahedral to vertices of regular ideal tetrahedral. It is easy to deduce from here that the map is conjugate to the identity (see [Thu, Section 6.3] and [BP92, Chapter C] for more details, and cfr.[BBI13] for a purely bounded-cohomological proof of this result).

As opposed to local rigidity, strong rigidity is a global result on the representation variety, but there is no straightforward implication between local and strong rigidity: for example non-uniform lattices in $PSL_2(\mathbb{C})$ are Mostow rigid but not locally rigid. If the group $G$ has higher rank, it is possible to prove much more: Margulis celebrated superrigidity result [Mar75] applies, completely classifying lattice homomorphisms.

**Theorem 1.3** (Margulis superrigidity). Let $G$ be a semisimple Lie group without compact factors and with rank greater or equal to 2, let $\Gamma < G$ be an irreducible lattice, let $H$ be a centerfree simple algebraic group defined over a local field $k$ and let $\rho : \Gamma \to H(k)$ be a Zariski dense, unbounded representation. Then $\rho$ extends uniquely to a representation of $G$.

The strategy of Margulis’ beautiful proof of superrigidity involved constructing measurable $\rho$-equivariant maps $\phi$ between suitably chosen compact homogeneous spaces for $G$ and $H(k)$. One then uses higher rank features, namely the existence of commuting subgroups, to show that such a map coincides almost everywhere with an algebraic map, and deduces from this information that the representation $\rho$ extends to the ambient group.

Corlette, using techniques based on the study of harmonic mappings, managed to extend this result to lattices in the isometry group $Sp(n,1)$ of the quaternionic hyperbolic spaces and in the isometry group, $F_4$, of the hyperbolic plane over the octonions [Cor92]. A new, more direct proof of Margulis superrigidity due to Bader and Furman just appeared [BF13]. We refer to [GP91] for a nice introduction to rigidity properties of lattices.
1.1.2 Representations of lattices of $SU(1, p)$

Since the group $SU(1, p)$ has real rank equal to one, Margulis superrigidity is not known to hold for lattices of this group, and indeed, if $p$ is equal to 2, Livne constructed, in his PhD thesis [Liv81], examples of lattices in $SU(1, 2)$ that surject onto free groups. These groups cannot be superrigid (see [DM93] for more details on the construction of these lattices). Another example of general failure of superrigidity for complex hyperbolic lattices was provided by Mostow: he constructed examples of lattices $\Gamma_1, \Gamma_2$ in $SU(1, 2)$ that surject onto free groups. These groups cannot be superrigid (see [DM93] for more details on the construction of these lattices). In general the study of homomorphisms of complex hyperbolic lattices is to a large extent open. If $p = 1$, the group $SU(1, 1)$ is locally isomorphic to $PSL_2(\mathbb{R})$, we postpone the discussion of the representation theory of fundamental groups of surfaces to the next section and we focus now on the case $p$ greater or equal to 2.

One of the first examples of rigidity properties for lattices in the isometry groups of the complex hyperbolic spaces was discovered by Goldman and Millson [GM87]. The group $SU(1, p)$ has a natural inclusion in $SU(1, q)$ that induces a standard embedding of a lattice $\Gamma$ of $SU(1, p)$ in $SU(1, q)$. Using Hodge theory, Goldman and Millson studied standard embeddings of cocompact lattices and proved a local rigidity result: any representation close to this embedding is conjugate to it, up to a representation of $\Gamma$ in the compact group $SU(q - p)$, that is the centralizer of the image of the standard embedding. In the same paper they suggested a definition of volume for representations of complex hyperbolic lattices in $SU(1, q)$ and conjectured that representations with maximal volume should be all conjugate to the standard embedding.

Goldman and Millson’s volume of a representation $\rho$ is the integral over the quotient $\Gamma \backslash \mathbb{H}_\mathbb{C}^p$ of the $p$-th exterior power of the pullback, under any smooth $\rho$-equivariant map $f : \mathbb{H}_\mathbb{C}^p \rightarrow \mathbb{H}_\mathbb{C}^q$, of the Kähler form $\omega_q$ of $\mathbb{H}_\mathbb{C}^q$. Such volume is bounded, in absolute value, by the volume of the manifold $\Gamma \backslash \mathbb{H}_\mathbb{C}^p$. Corlette, studying properties of harmonic maps, proved that maximal representations of uniform complex hyperbolic lattices with values in $SU(1, q)$ all come from the standard construction [Cor88].

More recently Koziarz and Maubon [KM08a] and Burger and Iozzi [BI08] introduced a new notion of volume, known as generalized Toledo invariant, that attains its maximal values at the same representations as Goldman and Millson’s one. They independently managed to prove that any maximal representation of a lattice in $SU(1, p)$ with values in $SU(1, q)$ admits an equivariant totally geodesic holomorphic embedding $\mathbb{H}_\mathbb{C}^p \rightarrow \mathbb{H}_\mathbb{C}^q$. In the two cases both the definition of volume and the techniques used in the study of maximal representations are different. The invariant introduced by Koziarz and Maubon is the integral on $\Gamma \backslash \mathbb{H}_\mathbb{C}^p$ of $f^*\omega_q \wedge (\omega_p)^{p-1}$, and the study of maximality is done via Higgs bundles and harmonic maps. Burger and Iozzi’s invariant can be defined in terms of bounded cohomology (cfr. Chapter 4) and the study of maximality is done by analyzing good properties of equivariant boundary maps.

It is possible to define a notion of generalized Toledo invariant for representa-
tions of a complex hyperbolic lattice in any Hermitian Lie group, and Koziarz and Maubon managed to prove rigidity for representations of cocompact lattices if the target group is classical of rank 2 [KM08b] (cfr. also [KM10] where a generalized Toledo invariant is defined for homomorphisms of more general Kähler groups and [Spi14] where maximal representations of Kähler groups admitting an equivariant holomorphic map are studied). It is conjectured that this holds in general:

**Conjecture 1.** Let $\Gamma$ be a lattice in $\text{SU}(1,p)$, with $p > 1$, and let $G$ be a Lie group of Hermitian type. There exist maximal representations of $\Gamma$ into $G$ if and only if $G = \text{SU}(m,n)$ with $n \geq pm$. In this case the representation is standard, namely it is conjugate to $\overline{\rho} \times \chi_{\rho}$ where $\overline{\rho}$ is the restriction to $\Gamma$ of a diagonal embedding of $m$ copies of $\text{SU}(1,p)$ in $\text{SU}(m,n)$ and $\chi_{\rho}$ is an homomorphism with values in the compact centralizer of the image of the diagonal embedding.

Klingler recently extended Goldman and Millson’s local rigidity result to a large class of representations: he used nonabelian Hodge theory to prove that all the representations of uniform complex hyperbolic lattices that satisfy a technical algebraic condition are locally rigid [Kli11]. As a particular case his main theorem probably implies that if $\Gamma$ is a cocompact lattice in $\text{SU}(1,p)$ and $\rho : \Gamma \to \text{SU}(m,n)$ is obtained by restricting to $\Gamma$ the diagonal inclusion of $\text{SU}(1,p)$ in $\text{SU}(m,n)$, then $\rho$ is locally rigid.

**1.1.3 Maximal representations of surface groups and boundary maps**

In the case when $p = 1$, the lattice $\Gamma$ is the fundamental group of a surface. In this case the study of maximal representations of $\Gamma$ into an Hermitian Lie group is one of the approaches to what is known as Higher Teichmüller theory: indeed in the case when $G$ is $\text{PSL}_2(\mathbb{R})$ maximal representations of surface groups correspond precisely to points in the Teichmüller space [Gol80, BIW14].

Probably the first result about maximal representations of surface groups in Hermitian Lie groups different from $\text{PSL}_2(\mathbb{R})$ was proven by Toledo: in [Tol89] he proves that a maximal representation $\rho : \Gamma \to \text{SU}(1,q)$ fixes a complex geodesic. His strategy is similar, in spirit, to Gromov-Thurston’s proof of Mostow rigidity: after establishing the existence of a (measurable) $\rho$-equivariant boundary map $\phi$, he proves that the bounded cohomological information on the maximality of the representation implies that the image $\phi(S^1)$ is the boundary of a complex geodesic preserved by the representation. With similar techniques Hernandez [Her91] studied maximal representations $\rho : \Gamma \to \text{SU}(2,q)$ and showed that the image must stabilize a symmetric space associated to the group $\text{SU}(2,2)$.

The symmetric space associated to $\text{SU}(2,2)$ is a maximal subdomain of tube type of $\text{SU}(2,m)$: it is a maximal totally geodesic subspace of the symmetric space $\mathcal{X}_{2,m}$ associated to $\text{SU}(2,m)$ that is biholomorphic to a subdomain of $\mathbb{C}^n$ of the form $\mathbb{R}^n \times iV$ where $V$ in $\mathbb{R}^n$ is a proper open cone. We refer to [FKK+00] for an introduction to Hermitian symmetric spaces of tube type from the point of view of harmonic analysis, and to [BIW07] for new characterizations of these spaces more closely related to bounded cohomology.
Maximal representations of surface groups in Hermitian symmetric spaces have been studied both from the point of view of Higgs bundles [BGPG06] and with the tool of bounded cohomology [BIW10]. Both approaches show that Hernandez’s result holds more generally: maximal representations of surface groups always stabilize a tube-type subdomain [BIW10, BGPG06]. Despite this, a remarkable flexibility result holds for maximal representations: if the image of \( \rho \) is a Hermitian Lie group of tube type, then \( \rho \) can be deformed to a Zariski dense representation [BIW10, KP09].

One approach to the study of maximal representations \( \rho : \Gamma_g \to G \) of surface groups \( \Gamma_g \) is based on the analysis of the properties of some measurable boundary map that is \( \rho \)-equivariant [BIW10, Sections 4-6]. The suitable boundary of the symmetric space \( \mathcal{X}_G \) associated to the group \( G \) in this setting is the Shilov boundary \( S_G \): the unique closed \( G \) orbit in the topological boundary of the bounded domain realization of \( \mathcal{X}_G \). Clerc and Ørsted constructed an extension to the Shilov boundary of the cocycle associated to the Kähler class [CØ03], and one of the characterizations given in [BIW07] of Hermitian symmetric spaces of tube type is that this extension has only a discrete set of values. Using the fact that the pullback in bounded cohomology can be realized via boundary maps [BI02], it is possible to deduce that, since the Poincaré disc is of tube type, the image of a maximal representation has to stabilize a tube type subdomain [BIW09].

The Shilov boundary of an Hermitian symmetric space of tube type has a natural causal structure [BH11] and the boundary map associated to a maximal representation preserves this structure: it sends almost every positively ordered triple of points in \( S^1 \) to a causally ordered triple. Using this property it is possible to deduce that the boundary map associated to a maximal representation of a surface group coincides almost everywhere with a right continuous map, and this implies, for example, that maximal representations of surface groups are discrete and injective [BIW10, Section 4].

In the case of cocompact lattices \( \Gamma_g \), many good properties of maximal representations also come from the fact, proven in [BILW05, BIW], that those representations are Anosov representations in the sense of Labourie [Lab06] and Guichard-Wienhard [GW12]: the flat \( G \)-bundle associated to the representation \( \rho \) has sections on which the geodesic flow lifts to an exponentially contracting (resp. expanding) flow. This allows to prove that the boundary maps associated to maximal representations of fundamental groups of closed surfaces are continuous, that those representations are quasi isometric embeddings, and that the mapping class group acts properly discontinuously on the corresponding components of the representation variety.

We finish this section by mentioning that the requirement of maximality can be weakened: in a recent preprint Ben Simon, Burger, Hartnick, Iozzi and Wienhard studied weakly maximal representations. These are representations for which the pullback of the class corresponding, in bounded cohomology, to the Kähler form of \( \mathcal{X}_G \) is a multiple of the class that corresponds to the Kähler form of \( \mathbb{D} \) in the bounded cohomology of \( \Gamma_g \). Weakly maximal representations share many good properties with maximal representations and form a large and interesting class.
1.2 The results of the thesis

The purpose of the thesis is to approach Conjecture 1 with the aid of bounded cohomology, and to develop techniques to study boundary maps that are associated to maximal representations. As in the case of maximal representations of surface groups there is a big distinction between the case when the target is of tube type and when it is not, and conjecturally there exist maximal representations of complex hyperbolic lattices only if the image is not of tube type (this reflects the fact that SU(1, p) is not of tube type). In particular there are two main parts of the thesis: in Part II we deal with maximal representations in SU(m, n), namely when the target is not of tube type, and in Part III we focus on tube type groups.

The purpose of Part III would be to exclude the existence of maximal representations in tube type groups by showing that there cannot exist boundary maps associated to maximal representations with target SU(m, m). We are unfortunately not able to completely prove this last statement, but our study allows us to prove that maximal representations are discrete and injective, moreover we give four different proofs that there cannot exist maps with properties slightly stronger than the ones that one can deduce on boundary maps associated to maximal representations. These results enlighten some features of the geometric object we are dealing with, and, hopefully, the hypotheses required by some of these proofs can be weakened to encompass also boundary maps associated to maximal representations.

The strategy in Part II is to show that the boundary map associated to a maximal representation needs to be particularly nice, namely algebraic: using the last step of Margulis’ proof of superrigidity this implies that the representation extends to a tight representation of SU(1, p). In order to complete the study we need a good understanding of tight representations of SU(1, p) in SU(m, n) on one side, and of the subgroups of SU(m, n) that can be realized as Zariski closure of the image of a maximal representation on the other. The first result was obtained by Hamlet [Ham11, Ham12], and the second is a generalization of Hamlet’s results that is going to be part of a joint work with him [HP], we include these results in Chapter 5.

Let us now describe more in detail the results of the thesis. Part I contains mostly a review of background material adapted to our purposes. The only chapter that contains really new material is Chapter 5, and the main novelty there is the following:

**Theorem 1.4.** Let \( \rho : G \to H \) be a tight homomorphism between Hermitian Lie groups. Assume that \( G \) has no factor locally isomorphic to SU(1, 1) and that all the simple factors of \( H \) are classical. Then \( \rho \) is holomorphic.

Part II is taken from the Arxiv preprint [Poz14]. There we deal with the groups SU(m, n) and its first result about representations is the proof of superrigidity for Zariski dense maximal representations:

**Theorem 1.5.** Let \( \Gamma \) be a lattice in SU(1, p) with \( p > 1 \). If \( m \) is different from \( n \), then every Zariski dense maximal representation of \( \Gamma \) into PU(m, n) is the restriction of a representation of SU(1, p).
1.2. THE RESULTS OF THE THESIS

This immediately implies the following:

**Corollary 1.6.** Let $\Gamma$ be a lattice in $\text{SU}(1,p)$ with $p > 1$. There are no Zariski dense maximal representations of $\Gamma$ into $\text{SU}(m,n)$, if $1 < m < n$.

Exploiting Hamlet’s results about tight homomorphisms between Hermitian Lie groups and Theorem 1.4, we are able to use the main theorem of Part II to give a structure theorem for all the maximal representations $\rho : \Gamma \to \text{SU}(m,n)$. This gives a complete proof Conjecture 1 with the additional hypothesis that the Zariski closure of the image does not contain any factor of tube type:

**Theorem 1.7.** Let $\rho : \Gamma \to \text{SU}(m,n)$ be a maximal representation. Then the Zariski closure $L = \overline{\rho(\Gamma)}^Z$ splits as the product $\text{SU}(1,p) \times L_t \times K$ where $L_t$ is a Hermitian Lie group of tube type without irreducible factors that are virtually isomorphic to $\text{SU}(1,1)$, and $K$ is a compact subgroup of $\text{SU}(m,n)$.

Moreover there exists an integer $k$ such that the inclusion of $L$ in $\text{SU}(m,n)$ can be realized as

$$\Delta \times i \times \text{Id} : L \to \text{SU}(1,p)^m - k \times \text{SU}(k,k) \times K < \text{SU}(m,n)$$

where $\Delta : \text{SU}(1,p) \to \text{SU}(1,p)^m - k$ is the diagonal embedding and $i : L_t \to \text{SU}(k,k)$ is a tight holomorphic embedding.

It is possible to show that there are no tube-type factors in the Zariski closure of the image of $\rho$ by imposing some non-degeneracy hypothesis on the associated linear representation of $\Gamma$ into $\text{GL}(\mathbb{C}^{m+n})$:

**Corollary 1.8.** Let $\Gamma$ be a lattice in $\text{SU}(1,p)$, with $p > 1$ and let $\rho$ be a maximal representation of $\Gamma$ into $\text{SU}(m,n)$. Assume that the associated linear representation of $\Gamma$ on $\mathbb{C}^{n+m}$ has no invariant subspace on which the restriction of the Hermitian form has signature $(k,k)$ for some $k$. Then

1. $n \geq pm$,

2. $\rho$ is conjugate to $\overline{\sigma} \times \chi_{\rho}$ where $\overline{\sigma}$ is the restriction to $\Gamma$ of the diagonal embedding of $m$ copies of $\text{SU}(1,p)$ in $\text{SU}(m,n)$ and $\chi_{\rho}$ is a representation $\chi_{\rho} : \Gamma \to K$, where $K$ is a compact group.

Since the invariant defining the maximality of a representation is constant on connected components of the representation variety, we get a new proof of Klingler’s result in our specific case, and the generalization of this latter result to non-uniform lattices:

**Corollary 1.9.** Let $\Gamma$ be a lattice in $\text{SU}(1,p)$, with $p > 1$, and let $\rho$ be the restriction to $\Gamma$ of the diagonal embedding of $m$ copies of $\text{SU}(1,p)$ in $\text{SU}(m,n)$. Then $\rho$ is locally rigid.

In the third part of the thesis we deal with maximal representations with values in $\text{SU}(m,m)$. Our main theorem in this setting is the following:
Theorem 1.10. Let $\Gamma$ be a lattice in $\text{SU}(1,p)$. A maximal representation $\rho : \Gamma \to \text{SU}(m,m)$ is discrete and injective.

Combining this result with the classification of tight homomorphisms between Hermitian Lie groups given in Chapter 5 one immediately gets the following:

Corollary 1.11. Let $\Gamma$ be any complex hyperbolic lattice, and let $\rho : \Gamma \to G$ be a maximal representation. Assume furthermore that $G$ is not virtually isomorphic to the product $\text{SO}^*(2p_1) \times \ldots \times \text{SO}^*(2p_k)$. Then the image of $\rho$ is discrete, and the kernel of $\rho$ is contained in the finite center of $\Gamma$.

Moreover Theorem 1.10 allows to prove Conjecture 1 if the dimension of the symmetric space associated to $\text{SU}(m,m)$ is smaller than the cohomological dimension of the lattice $\Gamma$:

Corollary 1.12. Let $\Gamma$ be a lattice in $\text{SU}(1,p)$. There are no maximal representations $\rho : \Gamma \to \text{SU}(m,m)$ if $2m^2 < 2p - 1$.

Combined with Theorem 1.7 this gives a complete proof of Conjecture 1 if the rank of $\text{SU}(m,m)$ is sufficiently small:

Corollary 1.13. Let $\Gamma$ be a lattice in $\text{SU}(1,p)$. Assume that $m^2 < p$, then every maximal representation $\rho : \Gamma \to \text{SU}(m,n)$ is standard.

As already explained, most of the work of the thesis is actually devoted to prove results about boundary maps, that might be of independent interest. In Part II we deal with measurable maps $\phi : \partial \mathbb{H}^p_C \to \mathcal{S}_{m,n}$ where $\mathcal{S}_{m,n}$ is the Shilov boundary of $\text{SU}(m,n)$ (an introduction to these objects will be given in the next section, and we refer to Chapter 2 for precise definitions). Without assuming any equivariance property for the map $\phi$ we are able to prove the following analogue of the fundamental theorem of projective geometry:

Theorem 1.14. Let $p > 1$, $1 < m < n$ and let $\phi : \partial \mathbb{H}^p_C \to \mathcal{S}_{m,n}$ be a measurable map whose essential image is Zariski dense. Assume that, for almost every triple $(x,y,z)$ on a chain, the triple $(\phi(x),\phi(y),\phi(z))$ is a triple of pairwise transversal points contained in the boundary of a maximal tube type subdomain. Then $\phi$ coincides almost everywhere with a rational map.

When $n = m$, the Zariski closure of the image of the representation $\rho$ is already of tube type, and hence the incidence structure given by maximal tube type subdomains is trivial. However the Shilov boundary of a tube-type Hermitian symmetric space carries a natural partial cyclic order [Kan91] and we will see that boundary maps associated to maximal representations with target $\text{SU}(m,m)$ are weakly monotone: they map triples of positively oriented points on a chain to ordered triples. We are not able to conclude such a strong rigidity statement as Theorem 1.14 for weakly monotone boundary maps, but we can exhibit a full measure subset $R \subseteq \partial \mathbb{H}^p_C$ on which the restriction of the map $\phi$ is continuous:

Proposition 1.15. Let $\phi : \partial \mathbb{H}^p_C \to \mathcal{S}_{m,m}$ be a weakly monotone map. There exists a full measure subset $R$ of $\partial \mathbb{H}^p_C$ with the property that $\phi|_R$ is continuous.
1.2. THE RESULTS OF THE THESIS

We finish this introduction by describing more closely the geometric ideas that will be developed in the thesis. For each part we choose the main geometric idea and we explain it in an easy example, for which we only assume the reader to be familiar with the chain geometry in the boundary of $\partial \mathbb{H}^2_C$, that was extensively studied by Cartan [Car32], and is treated in Goldman's book [Gol99]. In the second part of each section we describe how this geometric idea can be generalized to the right setting, and describe more closely what will be done in the text.

1.2.1 Part II

An easy example explained: strictly monotone maps $\phi : \partial \mathbb{H}^2_C \to \partial \mathbb{H}^2_C$

Purpose of this section is to prove a weak version of Theorem 1.14 under the strong assumptions that $m = 1$, $p = n = 2$, and the map $\phi$ is continuous and injective. We summarize in the next proposition the results about the chain geometry of $\partial \mathbb{H}^2_C$ that we will need in this example:

**Proposition 1.16.** Let us fix a point $v_\infty$ in $\partial \mathbb{H}^2_C$:

1. For every pair of points $p, q$, there exists precisely one chain containing them.
2. $\partial \mathbb{H}^2_C \setminus \{v_\infty\} = \text{Heis} = \mathbb{C} \ltimes \mathbb{R}$. We denote by $\pi : \partial \mathbb{H}^2_C \setminus \{v_\infty\} \to \mathbb{C}$ the projection induced by the quotient of Heis by its center $\mathbb{R}$.
3. Chains through $v_\infty$ correspond to vertical lines $L$.
4. Chains that do not contain $v_\infty$ project to Euclidean circles in $\mathbb{C}$.
5. For every Euclidean circle $C$, and every point $z$ projecting to $C$, there exists precisely one chain $T$ containing $z$ and projecting to $C$. In particular two chains projecting on the same circle are parallel: they differ by the addition of an element in $\mathbb{R}$.

In the next proposition we show how one can use these features of the chain geometry to show that the restriction to a chain of a map $\phi : \partial \mathbb{H}^2_C \to \partial \mathbb{H}^2_C$ that maps chains to chains is algebraic.

**Proposition 1.17.** Let $\phi : \partial \mathbb{H}^2_C \to \partial \mathbb{H}^2_C$ be a continuous, injective map, with the property that, for each triple $(x, y, z)$ on a chain, the image $(\phi(x), \phi(y), \phi(z))$ is contained in a chain. Then the restriction of $\phi$ to each chain is algebraic.
Proof. Let us fix a chain \( L \) and a point \( v_\infty \) in \( L \). We can assume, up to composing the map \( \phi \) with an element of \( \text{SU}(1,2) \), that \( \phi(v_\infty) = v_\infty \). In this case \( \phi \) restricts to a map from the Heisenberg group to itself. Moreover the chain \( L \) corresponds to a vertical line.

In order to show that the restriction of \( \phi \) to \( L \), and to each other vertical line, is algebraic, we want to construct an homomorphism \( h : \mathbb{R} \to \mathbb{R} \) which satisfies \( \phi(x+m) = \phi(x) + h(m) \) for every point \( x \in \partial H^2 \setminus \{v_\infty\} \), and every element \( m \in \mathbb{R} \).

Here we consider the action of \( \mathbb{R} \) on the Heisenberg group by translations. Once this is done we can use the fact that continuous homomorphisms from \( \mathbb{R} \) to \( \mathbb{R} \) are linear maps, and in particular they are algebraic.

As a preliminary step, let us denote by \( \alpha : \mathbb{R} \times \text{Heis} \to \mathbb{R} \) the map defined by \( \alpha(m,z) = \phi(z + m) - \phi(z) \). The map \( \alpha \) is well defined: the points \( \phi(z + m) \) and \( \phi(z) \) belong to the same vertical line, since that holds for \( z + m \) and \( z \) and \( \phi \) preserves chains, and hence \( \phi(z + m) \) and \( \phi(z) \) differ by the addition of an element \( \alpha(m,z) \) in \( \mathbb{R} \).

In order to conclude the proof it is enough to prove that the map \( \alpha \) does not depend on the point \( z \). If this is the case it follows from the definition of \( \alpha \) that it is an homomorphism with the desired properties. Let us then fix two points \( z_1, z_2 \) with distinct projections and consider the chain \( T \) containing \( z_1 \) and \( z_2 \). The image \( \phi(T + m) \) is a chain that contains \( \phi(z_1 + m) \) and \( \phi(z_2 + m) \) and projects to the same Euclidean circle as \( \phi(T) \), since this is the case for \( T \) and \( T + m \). In particular \( \phi(T + m) \) is parallel to \( \phi(T) \) and hence can be written as \( \phi(T) + \beta(m) \) for some number \( \beta(m) \) in \( \mathbb{R} \). It is easy to check that \( \beta(m) \) coincides both with \( \alpha(m, z_1) \) and \( \alpha(m, z_2) \), thus concluding the proof.
The objects needed for the generalization

In order to prove Theorem 1.5 we need a suitable generalization of Proposition 1.17: if we want to study maximal representations of a lattice in $SU(1,p)$ with values in $SU(m,n)$ we should replace the domain of definition of the map $\phi$ with $\partial H_p^c$, that is the visual boundary of $\Gamma$, and the target of the map $\phi$ with a suitable homogeneous space for $SU(m,n)$. We will see in Proposition 2.1 that the convenient homogeneous space is the Shilov boundary $S_{m,n}$ that can be defined as the set of maximal isotropic subspaces of $\mathbb{C}^{m+n}$. This space carries an interesting incidence geometry, whose lines will be called $m$-chains. We will prove in Proposition 4.5 that a map $\phi: \partial H_p^c \to S_{m,n}$ that is equivariant with respect to a maximal representation needs to map chains to $m$-chains.

We will study the incidence structure in $S_{m,n}$ whose lines are $m$-chains in Chapter 6: this structure shares many properties with the chain geometry, and the main aim of the chapter is to find the suitable generalizations of the statements of Proposition 1.16. In general, since $SU(m,n)$ has higher rank, there will be transversality issues, and for example the generalization of the Heisenberg model leaves out a Zariski closed subset of $S_{m,m}$ that is not reduced to a point. Also the description of the possible projections of $m$-chains and intersections of different $m$-chains is more complicated, but the fifth statement, the crucial one that makes the proof work, still holds true: for every point there exists precisely one $m$-chain with given projection (see Proposition 6.16).

Also the existence of continuous boundary maps does not come for free. We will recall in Chapter 4 that we can assume, since the representation is maximal, that there exists an equivariant boundary map with values into the Shilov boundary, but since we can only assume it to be measurable, we will have to use Fubini’s Theorem many times, and our proof will get more technical. The requirement that the map $\phi$ is essentially Zariski dense is a replacement of the hypothesis that the map is injective. It is a rather strong requirement that could be probably weakened a lot, but since it is satisfied in the setting of boundary maps equivariant with Zariski dense representations, we did not bother weakening it.

Since two $m$-chains might intersect in a bigger subspace, we will not be able, in general, to deduce that the suitable analogue of the cocycle $\alpha$ is an homomorphism directly, but we will need more involved arguments, checking that we can gain the full information applying the same idea enough many times: this is the subject of Section 7.3. All of this will allow us to prove, at the end of Chapter 7, that also in the general higher rank case, the restriction of the equivariant boundary map to almost every chain is algebraic.

The purpose of Chapter 8 is to deduce that the boundary map itself is rational, and this follows from a generalization of the well known fact that a measurable map $f: \mathbb{R}^2 \to \mathbb{R}$ that is polynomial when restricted to almost all vertical lines and almost all horizontal lines is algebraic, together with the suitable parametrization of a Zariski open subset $\partial H_p^c$ described in Lemma 8.1. This will conclude the proof of Theorem 1.14 that can be restated as
Theorem 1.18. Let $p > 1$, $1 < m < n$ and let $\phi : \partial \mathbb{H}^p \rightarrow S_{m,n}$ be a measurable map whose essential image is Zariski dense. Assume that, for almost every pair $x, y$ in $\partial \mathbb{H}^p$, $\phi$ satisfies $\dim_C(\phi(x), \phi(y)) = 2m$ and that, for almost every triple with $\dim(x, y, z) = 2$, it holds $\dim(\phi(x), \phi(y), \phi(z)) = 2m$. Then $\phi$ coincides almost everywhere with a rational map.

In the last chapter of Part II we prove applications of this result to the study of maximal representations that we announced in this introduction: not only Zariski dense maximal representations in $\text{SU}(m, n)$ are superrigid, but also the standard embedding of the lattice $\Gamma$ in $\text{SU}(m, n)$ is locally rigid, and we are completely able to describe maximal representations that have no tube type factor in the Zariski closure of their image.

Further directions

Some questions related to Theorem 1.14 remain open for future investigations: on one side it is maybe possible to show that the only incidence geometry preserving map $\phi : \partial \mathbb{H}^p \rightarrow S_{m,n}$ comes from a suitable diagonal embedding. It follows from the results of this thesis that this is the case for maps that are equivariant with maximal representations, but in Theorem 1.14 there is no hypothesis of equivariance, and probably the result is true in general.

We suspect that the incidence structure on the Shilov boundary of the other family of Hermitian Lie groups that are not of tube type, that is $\text{SO}^*(4p + 2)$, have features similar to the geometry of $m$-chains. It would be interesting to check whether this is true, and possibly extend the proof of Theorem 1.14 to chain geometry preserving maps $\phi : \partial \mathbb{H}^p \rightarrow S_{\text{SO}^*(2p)}$. Such a theorem would be one of the two missing ingredients for a complete proof of Conjecture 1 for representations to classical domains, the other being the case of representations in $\text{SU}(m, m)$, we will discuss more in depth in the next section.

It would also be interesting to find other realms of applicability of the geometric idea behind the proof of Theorem 1.14, for example in relation with suitable generalizations of the theory of weakly maximal representations to complex hyperbolic lattices. One could try to generalize the definition of weakly maximal representations of surface groups to complex hyperbolic lattices by requiring that the pullback of the Kähler class of $G$ is a multiple of the Kähler class of $\Gamma$, this is true for many interesting representations of the lattice $\Gamma$ (that extend to $\text{SU}(1, p)$), but, for example, the conjecturally non-existing maximal representations in $\text{SU}(m, m)$ would not be weakly maximal with this definition. Moreover the boundary map associated with these representations does not have values in the Shilov boundary.

We conclude this section by remarking that in their proof of Margulis super-rigidity [BF13], Bader and Furman manage to construct directly the extension of the representation $\rho$ by gluing together the information on the homomorphisms of some subgroups acting faithfully on partial flag varieties. Our setting is different, since, being in rank one, there are, unfortunately, no commuting subgroups, but it would be interesting to understand if the language of algebraic gates applies also here to give a more conceptual proof of Theorem 1.5 starting with the sole results of Theorem 7.1.
1.2.2 Part III

The geometric construction

In this section we exemplify the main geometric idea of Part III replacing $S_{m,m}$ with the more familiar space $(S^1)^m$. The torus $(S^1)^m$ is the Shilov boundary of the Hermitian symmetric space $(\mathcal{D})^m$ where $\mathcal{D}$ is the Poincaré disc and it sits in $S_{m,m}$ as a subspace. We say that a triple of points $(x, y, z)$ belonging to $(S^1)^m$ is maximal if for every $i$ the $i$-th projections $(x_i, y_i, z_i)$ is a positively oriented triple in $S^1$. Moreover, whenever two points $a, b$ in $(S^1)^m$ are fixed we call interval with extrema $a$ and $b$ the set $((a, b))$ of points $c$ with $(a, c, b)$ maximal. This set is an $m$-cube $I^m$.

**Proposition 1.19.** Let $\phi : \partial \mathbb{H}^2 \to (S^1)^m$ be a map. Assume that $\phi$ maps positively oriented triples on a chain to maximal triples and for every point $x$ there exists a dense set $Y_x$ of chains through $x$ on which the restriction is continuous in $x$, then $\phi$ is continuous.

*Proof.* Let us fix a point $z \in \partial \mathbb{H}^2$. We want to show that for every neighborhood $V$ of $\phi(z)$ in $(S^1)^m$ there exists an open neighborhood $U$ of $z$ in $\partial \mathbb{H}^2$ with the property that $\phi(U) \subseteq V$.

Let us fix a chain $C$ containing $z$ on which the restriction of $\phi$ is continuous in $z$. Since $\phi$ maps positively oriented triples on a chain to maximal triples, we get that a fundamental neighborhood system for $\phi(z)$ is formed by the interval of the form $((\phi(x), \phi(y)))$ where $x$ and $y$ belong to $C$ and $(x, z, y)$ is positively oriented. In particular it is enough to construct, for every $x, y$ in $C$ with $(x, z, y)$ positively oriented, an open neighborhood $U$ of $z$ with $\phi(U) \subseteq ((\phi(x), \phi(y)))$.

In order to do this let us construct four additional points $a, b, c, d$ in $\partial \mathbb{H}^2$ with the properties that

1. the 6-tuple $(x, a, b, c, d, y)$ is in generic position, meaning that no triple of points are contained in a chain.

2. The 6-tuple $(\phi(x), \phi(a), \phi(b), \phi(c), \phi(d), \phi(y))$ is maximal, meaning that all the projections are positively oriented 6-tuple in $S^1$.

3. There exists a point $s$ in the triangle $\Delta_{x,a,b}$ and a point $t$ in the triangle $\Delta_{c,d,y}$ such that $z$ belongs to the chain between $s$ and $t$. Here, if $e, f, g$ are points in $\partial \mathbb{H}^2$, the triangle $\Delta_{e,f,g}$ is, by definition, the set of points on a chain between $e$ and a point on the chain between $f$ and $g$. 
It is easy to construct the points $a, b, c, d$ using the fact that for each $w \in \partial \mathbb{H}_2^C$ the map $\phi$ is continuous in $w$ and monotone when restricted to a dense subset $Y_w$ of chains through $w$: the point $a$ can be chosen on any chain in $Y_x$ different from $C$, close enough to $x$, similarly we chose $b$ close enough to $a$ in any chain in $Y_a$ that is different from the previous one. Moreover we chose a chain $D$ in $Y_y$ that projects to the interior of the triangle spanned by $x, a, b$ (here the projection is the one described in Proposition 1.17, where the point $z$ is the one that was denoted by $v_\infty$ there). The point $c$ can be chosen on the chain $D$ close enough to $y$ on the left, and $d$ can be constructed in a similar method.

Let us now consider the set $U$ consisting of points that are between the triangle $\Delta_{x,a,b}$ and $\Delta_{c,d,y}$. By our choice of the chain $D$, the point $z$ belongs to $U$. Since the 6-tuple $(x,a,b,c,d,y)$ is in generic position, the set $U$ is open, and it is easy to check that since the 6-tuple $(\phi(x), \phi(a), \phi(b), \phi(c), \phi(d), \phi(y))$ is maximal, every point $u$ in $U$ satisfies $\phi(u) \in (\phi(x), \phi(y))$. This concludes the proof.

Generalizations, applications, and future directions

Since $\text{SU}(m,m)$ is of tube type, the Shilov boundary $S_{m,m}$ carries a partial cyclic order similar to the one described for the torus: the purpose of Chapter 10 is to study this order and describe explicitly the associated intervals.

It turns out that if $\rho: \Gamma \to \text{SU}(m,m)$ is a maximal representation, there exists an associated boundary map $\phi: \partial \mathbb{H}_2^C \to S_{m,m}$ that is weakly monotone: this means that for almost every positively oriented triple $(x, y, z)$ on a chain the image $(\phi(x), \phi(y), \phi(z))$ is maximal. The purpose of Chapter 11 is to show that a map $\phi$ with this property needs to be nicer than what one expects. The good understanding of intervals we achieved in Chapter 10 allows us to prove that for almost every chain there exists a full measure set on which the restriction of $\phi$ is continuous, and we use this fact to define a full measure good set $R$ for the map $\phi$: this consists of those points that belong to the set of continuity of almost every chain they belong to. A geometric construction slightly more involved than the one we described in Proposition 1.19 allows us to show that the restriction of $\phi$ to $R$ is continuous: the details are carried out in Section 11.2 and this proves Proposition 1.15.

Proposition 1.15 has applications to the study of maximal representations: in Chapter 12 we use this result to show that maximal representations are discrete and injective, this is stated in the introduction as Theorem 1.10, and, as a direct corollary, allows to prove Conjecture 1 for small values of $m$.

On the other hand one would hope to gain some more regularity of the map $\phi$ that allows to deduce that such a map, and hence also maximal representations in tube type groups, cannot exist. An idea on how approach this problem is to define the map $\phi$ outside $R$ by taking limits along vertical chains. We pursue this direction at the end of Section 11.2.2, but for the moment we are only able to show that the limit along chains is locally constant in a suitable set of chains through $x$ (this is Proposition 11.20).

In the last chapter of the thesis we give four arguments that show that boundary maps with some stronger hypotheses do not exist: in Section 13.1 we deal with
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differentiable boundary maps, in Section 13.2 we deal with continuous maps, in Section 13.3 and Section 13.4 we assume the map to be strictly monotone. There are two reasons why we decided to include these four proofs: on one side the thesis is also about properties of maps between Shilov boundaries and all these statements are interesting in that setting, on the other side each proof is based on a different idea, and the hypotheses of each proof can be slightly weakened, we still hope that it will be possible, in the future, to fill the gap and give a complete proof of Conjecture 1 by strengthening the results obtained in Section 11.2.2 and weakening the hypotheses of some of the proofs of Chapter 13.
Part I

Background material
Chapter 2

Hermitian Symmetric spaces

2.1 The Grassmannian manifold

Both the symmetric space and the Shilov boundary of $\text{SU}(m,n)$ have an explicit description as closed subsets of the Grassmanian of the $m$-planes in $\mathbb{C}^{m+n}$. This will allow us to give an explicit formula for the map $\pi$ introduced in Proposition 1.16 and its generalizations to higher rank. Before going into the details, we recall here a few facts about the rational structure of the Grassmanian manifold and its affine charts that will be needed in the sequel.

In general, if $M$ is a complex algebraic variety, $M$ can be endowed with the structure of a real algebraic variety by choosing a real structure $x \mapsto \bar{x}$ on $\mathbb{C}$, and we will be generally interested in the real Zariski topology whose closed subsets are locally defined by polynomials in the variables $z_i$ and $\bar{z}_i$. A map $f : X \to Y$ between real affine varieties is called regular if it is defined by polynomial equations, and rational if it is defined on a Zariski open subset of $X$ by quotients of polynomials.

If $V$ is a complex vector space of dimension $n+m$, the Grassmannian of $m$-dimensional complex subspaces of $V$ is the quotient of the frame variety, whose points correspond to ordered bases of $m$-dimensional subspaces of $V$, under the natural action of the group $\text{GL}_m(\mathbb{C})$. In particular, once a basis of $V$ is fixed, the frame variety identifies with $M^+ = M^+((m+n) \times m, \mathbb{C})$, the set of matrices with $(m+n)$ rows and $m$ columns that have maximal rank: in fact it is enough to interpret a matrix $X \in M^+$ as the ordered basis of the subspace spanned by its columns. Under this identification the action of the group $\text{GL}_m(\mathbb{C})$ on the frame variety is given by the action of $\text{GL}_m(\mathbb{C})$ on $M^+$ via right multiplication.

We will denote by

$$j : M^+/\text{GL}_m(\mathbb{C}) \to \text{Gr}_m(\mathbb{C}^{n+m})$$

the quotient map, understood with respect to the standard basis: the image under $j$ of a matrix $M \in M^+$ is the subspace of $\mathbb{C}^{m+n}$ spanned by the columns of $M$. The map $j$ is algebraic, and to any $\text{GL}_m(\mathbb{C})$-invariant Zariski closed subset of $M^+$ corresponds a Zariski closed subset of $\text{Gr}_m(V)$. For example the set of points in
Gr$_m(V)$ that are non-transversal to a given $k$-dimensional subspace $W$ of $V$ is Zariski closed since it corresponds to the Zariski closed subset of $M^+$ defined by the vanishing of all the $(m+k)$-minors of the matrix whose columns are given by a basis of $W$ and the columns of $M$.

Let us now fix an $n$-dimensional subspace $W$ of $V$ and consider the Zariski open set $Gr_m(V)^W$ consisting of points $x \in Gr_m(V)$ that are transversal to $W$. For any fixed point $y$ in $Gr_m(V)^W$ there is a natural identification

$$\psi_W : Gr_m(V)^W \rightarrow \text{Lin}(y,W)$$

defined by requiring, for any $t \in y$, that $\psi_W(x)t$ is the unique vector in $W$ such that $t + \psi_W(x)t$ belongs to $x$. The map $\psi_W$ gives an algebraic affine chart of the Grassmannian. The Zariski topology of $Gr_m(V)$ is induced by the Zariski topology on the charts that can in turn be identified with $M(n \times m)$ by choosing bases of $y$ and $W$.

If we fix a real structure $v \rightarrow \pi$ on $V$, we can induce a real structure on the Grassmannian $Gr_m(V)$ and we will be interested in the structure of $Gr_m(V)$ as a real algebraic variety. We also choose real structures on $M^+$ and on $M(n \times m)$ in a compatible way, so that the Zariski closed subsets of $Gr_m(V)$ correspond to the subsets that, in the affine charts, are defined by polynomial equations involving only the coefficients of a matrix and its conjugates.

### 2.2 Models for Symmetric spaces and Shilov boundaries

Let $G$ be a connected semisimple Lie group with finite center and without compact factors and let $K$ be a maximal compact subgroup of $G$. We will denote by $X = G/K$ the associated symmetric space. Throughout this work we will be only interested in Hermitian symmetric spaces, that is in those symmetric spaces that admit a $G$-invariant complex structure $J$. It is a classical fact [Kor00, Theorem III.2.6] that these symmetric spaces have a bounded domain realization, that means that they are biholomorphic to a bounded convex subspace of $\mathbb{C}^n$ on which $G$ acts via biholomorphisms. Hermitian symmetric spaces were classified by Cartan [Car35], and are the symmetric spaces associated to the exceptional Lie groups $E_7(-25)$ and $E_6(-14)$ together with 4 families of classical domains: the ones associated to $SU(p,q)$, of type $I_{p,q}$ in the standard terminology, the ones associated to $SO^+(2p)$, of type $II_p$, the symmetric spaces, $III_p$, of the groups $Sp(2p,\mathbb{R})$, and the symmetric spaces $IV_p$ associated to $SO(2,p)$$^1$.

An Hermitian symmetric space is said to be of tube-type if it is also biholomorphic to a domain of the form $V + i\Omega$ where $V$ is a real vector space and $\Omega \subset V$ is a proper convex open cone. It is well known that the only spaces that are not of tube type are the symmetric space of $E_6(-14)$ and the families $I_{p,q}$ with $q \neq p$ and $II_p$ with $p$ odd. It follows from the classification that any Hermitian symmetric space contains maximal tube-type subdomains, and those are all conjugate under

$^1$In Cartan’s original terminology [Car35] the families $III_p$ and $IV_p$ are exchanged
2.2. MODELS FOR SYMMETRIC SPACES AND SHILOV BOUNDARIES

the $G$-action, are isometrically and holomorphically embedded and have the same rank as the ambient symmetric space.

The $G$-action via biholomorphism on the bounded domain realization of $\mathcal{X}$ extends continuously on the topological boundary $\partial \mathcal{X}$. If the real rank of $G$ is greater than or equal to two, $\partial \mathcal{X}$ is not an homogeneous $G$-space, but contains a unique closed $G$-orbit, the Shilov boundary $\mathcal{S}_G$ of $\mathcal{X}$. If $\mathcal{X}$ is irreducible, the stabilizer of any point $s$ of $\mathcal{S}_G$ is a maximal parabolic subgroup of $G$. Under this respect $\mathcal{S}_G$ is a partial flag variety of $G$, and the action of $G$ on $\mathcal{S}_G$ is algebraic. In the reducible case, if $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$ is the de Rham decomposition in irreducible factors whose isometry group is $G_1$, then $\mathcal{S}_G$ splits as the product $\mathcal{S}_{G_{1_1}} \times \ldots \times \mathcal{S}_{G_{n}}$ as well. Moreover, when $\mathcal{Y}$ is a maximal tube-type subdomain of $\mathcal{X}$, the Shilov boundary of $\mathcal{Y}$ embeds in the Shilov boundary of $\mathcal{X}$.

The diagonal action of $G$ on the pairs of points $(s_1, s_2) \in \mathcal{S}_G^2$ has a unique open orbit corresponding to pairs of opposite parabolic subgroups. Two points in $\mathcal{S}_G$ are transversal if they belong to this open orbit. Whenever a pair $(s_1, s_2)$ of transversal points of $\mathcal{S}_G$ is fixed, there exists a unique maximal tube-type subdomain $\mathcal{Y} = G_T/K_T$ of $\mathcal{X}$ such that $s_1$ belongs to $\mathcal{S}_{G_T}$. In particular this implies that the Shilov boundaries of maximal tube-type subdomains define a rich incidence structure in $\mathcal{S}_G$.

For each family of classical groups both the symmetric space and the Shilov boundary can be described easily as algebraic subsets of some suitable Grassmannian variety, and the bounded domain realization can be obtained by taking a suitable affine chart of the Grassmannian. We now give an explicit description of those models.

The family $I_{m,n} = SU(m,n)$

The group $SU(m,n)$ is the subgroup of $SL(m+n, \mathbb{C})$ that preserves the standard Hermitian form $h$ of signature $(m,n)$. Clearly $SU(m,n)$ acts on the Grassmannian of $m$-dimensional subspaces of $\mathbb{C}^{m+n}$ and preserves the subset $\mathcal{X}_{m,n}$ consisting of subspaces on which $h$ is positive definite:

$$\mathcal{X}_{m,n} = \{ x \in Gr_m(\mathbb{C}^{n+m}) | h|_x > 0 \}.$$

The manifold $\mathcal{X}_{m,n}$ is a model of the symmetric space of $SU(m,n)$: indeed it is an homogeneous $SU(m,n)$-space, and the stabilizer of the point $o = \langle e_1, \ldots, e_m \rangle$ is the group $S(U(m) \times U(n))$ that is a maximal compact subgroup of $SU(m,n)$.

The set $\mathcal{X}_{m,n}$ is contained in the domain of the affine chart $\psi_W$ associated with the subspace $W = \langle e_{m+1}, \ldots, e_{m+n} \rangle$. In particular the map $\psi_W$ allows to realize $\mathcal{X}_{m,n}$ as a bounded domain $X_{m,n}$ in $\mathbb{C}^{m \times n}$:

$$X_{m,n} = \{ X \in M(n \times m, \mathbb{C}) | X^* X < \text{Id} \}.$$

Here and in the sequel, by the expression $X^* X < \text{Id}$, we mean that the Hermitian matrix $\text{Id} - X^* X$ is positive definite. The domain $X_{m,n}$ corresponds under $\psi_W$ to $\mathcal{X}_{m,n}$ since an explicit formula for the composition $\psi_W \circ j$ is given by $\left[ \begin{array}{c} X_1 \\ X_2 \end{array} \right] \rightarrow X_2 X_1^{-1}$.
It is possible to verify that the bounded domain realization \( X_{m,n} \) of the symmetric space associated to \( \text{SU}(m,n) \) we are giving here is what is classically known as the Harish-Chandra realization (cfr. [Kor00, Theorem III.2.6] and [Hel78, D2 page 527]).

It follows from the explicit formula for the chart \( \psi_W \) that the action of \( \text{SU}(m,n) \) on the bounded model \( X_{m,n} \) is by fractional linear transformations:

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot X = (C + DX)(A + BX)^{-1},
\]

in particular it is an action via biholomorphisms of the domain and it extends continuously to the boundary. The unique closed \( \text{SU}(m,n) \)-orbit in the topological boundary of \( X_{m,n} \) is the set

\[
S_{m,n} = \{ X \in M(n \times m, \mathbb{C}) | X^*X = \text{Id} \},
\]

that corresponds, via the affine chart \( \psi_W \), to the set of isotropic subspaces of \((\mathbb{C}^{m+n}, h)\):

\[
S_{m,n} = \{ x \in \text{Gr}_m(\mathbb{C}^{m+n}) | h|_x \equiv 0 \}.
\]

In order to verify this last assertion, recall that an inverse of the composition \( \psi_W \circ j : M^+ \rightarrow M(n \times m, \mathbb{C}) \) is given by the map \( \lambda : X \mapsto \begin{bmatrix} \text{Id} & X \\ 0 & \text{Id} \end{bmatrix} \). It is now clear that the matrix \( X \) satisfies \( X^*X = \text{Id} \) if and only if the subspace of \( \mathbb{C}^{m+n} \) spanned by the columns of \( \lambda(X) \) is isotropic for \( h \).

The family \( II_m : \text{SO}^+(2m) \)

To give an explicit description of some models for the symmetric spaces of type \( II_m \), recall that the group \( \text{SO}^+(2m) \) can be defined as the subgroup of \( \text{GL}_m(\mathbb{H}) \) preserving a nondegenerate, anti-Hermitian form \( \tilde{h} \). All these forms are equivalent, and, if one properly identifies the quaternionic corp \( \mathbb{H} \) with \( \mathbb{C}^2 \), \( \tilde{h} \) can be written as \( ih + js \) where \( h \) is the standard Hermitian form of signature \((m,m)\) and \( s \) is the orthogonal form represented by the matrix \( \begin{bmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{bmatrix} \). This realizes \( \text{SO}^+(2m) \) as the subgroup of \( \text{SU}(m,m) \) that preserves the orthogonal form \( s \), and the symmetric space associated to \( \text{SO}^+(2m) \) is the subspaces \( X_{\text{SO}^+(2m)} \) of \( X_{m,m} \) consisting of subspaces isotropic for \( s \):

\[
X_{\text{SO}^+(2m)} = \{ x \in \text{Gr}_m(\mathbb{C}^{2m}) | h|_x > 0, s|_x \equiv 0 \}.
\]

The intersection with the affine chart \( \psi_W \) of the subspace of \( \text{Gr}_m(\mathbb{C}^{2m}) \) consisting of subspaces that are isotropic for \( s \) is a linear subspace of \( M(m \times m, \mathbb{C}) \), in particular this gives the bounded domain realization for the symmetric space:

\[
X_{\text{SO}^+(2m)} = \{ X \in M(m \times m, \mathbb{C}) | X^*X < \text{Id}, X + X^t = 0 \}.
\]

In order to describe the Shilov boundary one needs to distinguish two cases according to the parity of \( m \). If \( m = 2k \) is even, the Shilov boundary is described as the set of subspaces that are isotropic for both \( h \) and \( s \):

\[
S_{\text{SO}^+(2k)} = \{ x \in \text{Gr}_{2k}(\mathbb{C}^{4k}) | h|_x \equiv 0, s|_x \equiv 0 \}.
\]
If instead \( m = 2k + 1 \) is odd, there exists no \( m \) dimensional subspace that is isotropic for both \( h \) and \( s \), and the Shilov boundary can be equivalently described as the set of \( 2k \) dimensional subspaces that are isotropic for both \( h \) and \( s \), or as the set of \( 2k + 1 \) dimensional subspaces that are isotropic for \( s \) and on which \( h \) has signature \((1, 0)\); given a \( 2k \) dimensional \( h \)-isotropic subspace there exists a unique maximal subspace that is \( s \)-isotropic and on which \( h \) is positive semidefinite, and vice versa, given a \( 2k + 1 \)-dimensional subspace \( x \) that is \( s \)-isotropic and on which \( h \) has signature \((1, 0)\), the radical of \( h \) in \( x \) is a \( 2k \)-dimensional subspace that is isotropic for \( \tilde{h} \).

To summarize we have

\[
S_{\text{SO}^\ast(4k+2)} = \{ y \in \text{Gr}_{2k}(\mathbb{C}^{4k+2}) | h|_y \equiv 0, s|_y \equiv 0 \}
= \{ x \in \text{Gr}_{2k+1}(\mathbb{C}^{4k+2}) | h|_x \text{ has signature } (1, 0), s|x \equiv 0 \}.
\]

The inclusion of the symmetric space of \( \text{SO}^\ast(2m) \) in \( \text{SU}(m, m) \) is holomorphic and isometric.

**The family \( III_m : \text{Sp}(2m, \mathbb{R}) \)**

Let us denote by \( w \) the symplectic form on \( \mathbb{C}^{2m} \) that is represented, with respect to the standard basis by the matrix \( \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix} \), and let us consider the group \( \text{Sp}(2m, \mathbb{C}) \) consisting of endomorphisms of \( \mathbb{C}^{2m} \) preserving \( w \). The group \( \text{Sp}(2m, \mathbb{R}) \) can be realized as the intersection \( \text{Sp}(2m, \mathbb{C}) \cap \text{SU}(m, m) \) (see [BILW05, Section 2.1.2] for more details), and in this way one gets an holomorphic inclusion of the symmetric space \( X_{\text{Sp}(2m, \mathbb{R})} \) in the symmetric space of \( \text{SU}(m, m) \):

\[
X_{\text{Sp}(2m, \mathbb{R})} = \{ x \in \text{Gr}_m(\mathbb{C}^{2m}) | h|_x > 0, w|_x \equiv 0 \}.
\]

Since, again, the intersection with the affine chart \( \psi_W \) of the subset of \( \text{Gr}_m(\mathbb{C}^{2m}) \) consisting of subspaces that are isotropic for \( w \) is a linear subspace of \( M(m \times m, \mathbb{C}) \), we get that also the bounded domain realization for the symmetric space of \( \text{Sp}(2m, \mathbb{R}) \) is the intersection of the bounded domain realization of symmetric space of \( \text{SU}(m, m) \) with an affine subspace:

\[
X_{\text{Sp}(2m, \mathbb{R})} = \{ X \in M(m \times m, \mathbb{C}) | X^t X < \text{Id}, X - X^t = 0 \}.
\]

Similarly the Shilov boundary can be described as the set of subspaces that are isotropic for both \( h \) and \( w \):

\[
S_{\text{Sp}(2m, \mathbb{R})} = \{ x \in \text{Gr}_{2k}(\mathbb{C}^{4k}) | h|_x \equiv 0, w|x \equiv 0 \}.
\]

Once again the inclusion of \( X_{\text{Sp}(2m, \mathbb{R})} \) in \( \text{SU}(m, m) \) is isometric and holomorphic.

**The family \( IV_m : \text{SO}(2, m) \)**

Let us fix on \( \mathbb{R}^{2+m} \) the standard orthogonal form \( t \) of signature \((2, m)\). The group \( \text{SO}(2, m) \) naturally acts on the Grassmanian of two dimensional planes in \( \mathbb{R}^{2+m} \).
preserving the open set consisting of planes on which \( t \) is positive definite. This is a model for the symmetric space of \( \text{SO}(2, m) \).

\[
X_{\text{SO}(2,m)} = \{ x \in \text{Gr}_m(\mathbb{R}^{2+m}) | |t|_x > 0 \}.
\]

Intersecting with the chart associated with the subspace \( V = (e_3, \ldots, e_{m+2}) \) one gets a bounded domain model for this symmetric space:

\[
X_{\text{SO}(2,m)} = \{ X \in M(m \times 2, \mathbb{R}) | X^t X < \text{Id} \}.
\]

The action of \( \text{SO}(2, m) \) is by fractional linear transformations and it is possible to verify that preserves the complex structure given by

\[
J_0 \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} X_2 & -X_1 \end{bmatrix}.
\]

The inclusion of \( \text{SU}(1, p) \) in \( \text{SO}(2, 2p) \) given by identifying \( \mathbb{C}^{p+1} \) with \( \mathbb{R}^{2p+2} \) induces an inclusion of \( \mathbb{H}^{2p+2}_\mathbb{C} \) in \( X_{\text{SO}(2,2p)} \) and it is easy to verify from the expression given above for the complex structure of \( \text{SO}(2, m) \) that the inclusion of the symmetric spaces is holomorphic and isometric.

**The isomorphisms in low degrees**

We recall here, for the reader’s convenience the isomorphisms between Lie algebras of Hermitian type, see [Hel78, Page 519] for more details:

| \( \text{su}(1, 1) \) | \( \text{sp}(2) \) | \( \text{so}(2, 1) \) | \( \text{so}(2, 2) \) | \( \text{su}(1, 1) \oplus \text{su}(1, 1) \) |
| \( \text{so}(2, 2) \) | \( \text{su}(1, 1) \oplus \text{su}(1, 1) \) | \( \text{so}^*(4) \) | \( \text{su}(1, 1) \oplus \text{su}(2) \) |
| \( \text{so}(2, 3) \) | \( \text{so}(2, 3) \) | \( \text{so}^*(6) \) | \( \text{so}^*(8) \) |

The isomorphism \( \text{sp}(4) = \text{so}(2, 3) \) can be proved directly, the computations are carried out, for example, in [Sie43, Paragraph 56], the isomorphism \( \text{so}^*(8) = \text{so}(2, 6) \) can be deduced from the bounded domain realizations (cfr. [Hel78, Chapter X, Example D.2], and [Mor56]).

**2.3 The Kähler form and the Shilov boundary**

Given three points in \( S_G \) there will not, in general, exist a maximal tube-type subdomain \( Y \) of \( X \) whose Shilov boundary contains all the three points. However it is possible to determine when this happens with the aid of the Kähler form. Recall that, since \( X \) is a Hermitian symmetric space, it is possible to define a \( G \)-invariant differential two form via the formula

\[
\omega(X, Y) = g(X, JY)
\]
where \( g \) denotes the \( G \)-invariant Riemannian metric normalized in such a way that its minimal holomorphic sectional curvature is \(-1\), and \( J \) is the complex structure of \( X \). Since \( \omega \) is \( G \)-invariant, it is closed: it is an old observation of Cartan that every \( G \)-invariant differential form on a symmetric space is closed. This implies that \( X \) is a Kähler manifold and \( \omega \) is its Kähler form. Let \( X^{(3)} \) denote the triples of pairwise distinct points in \( X \) and let us consider the function

\[
\beta_X : X^{(3)} \to \mathbb{R}, \quad (x, y, z) \mapsto \frac{1}{\pi} \int_{\Delta(x,y,z)} \omega
\]

where we denote by \( \Delta(x, y, z) \) any smooth geodesic triangle having \((x, y, z)\) as vertices. Since \( \omega \) is closed, Stokes theorem implies that \( \beta_X \) is a well defined continuous \( G \)-invariant cocycle and it is proven in [CØ03] that it extends continuously to the triples of pairwise transversal points in the Shilov boundary.

If a triple \((s_1, s_2, s_3)\) in \( S^3 \) does not consist of pairwise transversal points, the limit of \( \beta_X(x_1^j, x_2^j, x_3^j) \) as \( x_j \) approaches \( s_j \) is not well defined, but Clerc proved that, restricting only to some preferred sequences (the one that converge radially to \( s_j \)), it is possible to get a measurable extension of \( \beta_X \) to the whole Shilov boundary. The obtained extension \( \beta_S : S^3_G \to \mathbb{R} \) is called the Bergmann cocycle\(^2\) and it is a measurable strict cocycle. The maximality of the Bergmann cocycle detects when a triple of points is contained in the Shilov boundary of a tube-type subdomain:

**Proposition 2.1.** 1. \( \beta_S \) is a strict alternating \( G \)-invariant cocycle with values in \([-\text{rk}X, \text{rk}X]\),

2. if \( \beta_S(s_1, s_2, s_3) = \text{rk}X \) then the triple \((s_1, s_2, s_3)\) is contained in the Shilov boundary of a tube-type subdomain.

**Proof.** The first fact was proven in [Cle07], the second can be found in [BIW09, Proposition 5.6]. \( \square \)

We will call a triple \((s_1, s_2, s_3)\) in \( S^3_G \) satisfying \( \beta_S(s_1, s_2, s_3) = \text{rk}(X) \) a maximal triple. In the case where \( G \) is \( \text{SU}(1,p) \), that is a finite cover of the connected component of the identity in \( \text{Isom}(\mathbb{H}^p) \), the maximal tube-type subdomains are complex geodesics of \( \mathbb{H}^p \) and the Bergmann cocycle coincides with Cartan’s angular invariant \( c_p \) [Gol99, Section 7.1.4]. Following Cartan’s notation we will call *chains* the boundaries of the complex geodesics.

We finish this chapter by remarking that the Bergmann cocycle can be used to characterize which domains are of tube type:

**Proposition 2.2** ([BIW07, Theorem 1]). Let \( X \) be an irreducible Hermitian symmetric space. The following are equivalent:

1. \( X \) is not of tube type;

2. the set \( S^{(3)} \) of triples of pairwise transverse points in the Shilov boundary is connected;

3. the Bergmann cocycle attains all values in \([-\text{rk}X, \text{rk}X]\).

\(^2\) We choose the normalization of [Cle07], the normalization chosen in [DT87] is such that \( \beta_{DT} = \pi \cdot \beta_S \), the one of [BIW10] is such that \( \beta_{BIW} = \frac{2}{\pi} \beta_S \).
Chapter 3

Continuous (bounded) cohomology

We introduce now the concepts we will need about continuous and continuous bounded cohomology, standard references are respectively [BW00] and [Mon01]. A quick introduction to the relevant aspects of continuous bounded cohomology can also be found in [BI09].

Throughout the section $G$ will be a locally compact second countable group, every finitely generated group fits in this class when endowed with the discrete topology. The continuous cohomology of $G$ with real coefficients, $H^n_c(G, \mathbb{R})$ is the cohomology of the complex $(C^n_c(G, \mathbb{R})^G, d)$ where

$$C^n_c(G, \mathbb{R}) = \{ f : G^{n+1} \to \mathbb{R} | f \text{ is a continuous function} \},$$

the invariants are taken with respect to the diagonal action, and the differential

$$d^n : C^n_c(G, \mathbb{R}) \to C^{n+1}_c(G, \mathbb{R}),$$

is defined by the expression

$$d^n f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f((g_0, \ldots, \hat{g_i}, \ldots, g_{n+1}).$$

Similarly the continuous bounded cohomology $H^n_{cb}(G, \mathbb{R})$ of $G$ is the cohomology of the subcomplex $(C^n_{cb}(G, \mathbb{R})^G, d)$ of $(C^n_c(G, \mathbb{R})^G, d)$ consisting of bounded functions. The inclusion $i : C^n_{cb}(G, \mathbb{R})^G \to C^n_c(G, \mathbb{R})^G$ induces, in cohomology, the so-called comparison map $c : H^n_{cb}(G, \mathbb{R}) \to H^n_c(G, \mathbb{R})$. The Banach norm on the cochain modules $C^n_{cb}(G, \mathbb{R})$ defined by

$$\| f \|_\infty = \sup_{(g_0, \ldots, g_n) \in G^{n+1}} |f(g_0, \ldots, g_n)|$$

induces a seminorm on $H^n_{cb}(G, \mathbb{R})$ that is usually referred to as the canonical seminorm or Gromov’s norm.

Most of the results about continuous and continuous bounded cohomology are based on the functorial approach to the study of these cohomological theories that
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is classical in the case of continuous cohomology and was developed by Burger and Monod [BM99] in the setting of continuous bounded cohomology. This allows to show that the cohomology of many different complexes realize canonically the given cohomological theory. Since we will only need applications of this machinery that are already present in the literature we will not describe it any further here and we refer instead to [BW00, Mon01] for details on this nice subject.

3.1 Continuous cohomology and differential forms

The most notable application of this approach to continuous cohomology is van Est Theorem [vE53, Dup76] that realizes the continuous cohomology of a semisimple Lie group in terms of $G$-invariant differential forms on the associated symmetric space:

**Theorem 3.1 (van Est).** Let $G$ be a semisimple Lie group without compact factors and let $X$ be the associated symmetric space. Then

$$\Omega^n(X, \mathbb{R})^G \cong H^n_c(G, \mathbb{R}).$$

Under this isomorphism the differential form $\omega$ corresponds to the class of the cocycle $c_\omega$ defined by the formula

$$c_\omega(g_0, \ldots, g_n) = \frac{1}{\pi} \int_{\Delta(g_0 x, \ldots, g_n x)} \omega$$

for any fixed basepoint $x$ in $X$.

3.2 Continuous bounded cohomology and amenable groups

One of the main differences of bounded cohomology with respect to ordinary cohomology is that the bounded cohomology of a group can be computed from a suitable boundary of the group itself. Indeed if $Z$ is an amenable $H$ space then the bounded cohomology of $H$ can be computed isometrically from the complex $(L^\infty_{alt}(Z^*, \mathbb{R})^H, d)$ [BM02]. For example, if $H$ is either $SU(1, p)$ or one of its lattices, the boundary $\partial \mathbb{H}^P_C$ is an amenable $H$-space and hence the complex $(L^\infty_{alt}(\partial \mathbb{H}^P_C)^*, \mathbb{R})^H, d)$ realizes isometrically the bounded cohomology of $H$. Moreover, since the action of $H$ on $\partial \mathbb{H}^P_C$ is doubly ergodic by Howe-Moore Theorem, we get that $L^\infty_{alt}(\partial \mathbb{H}^P_C)^2, \mathbb{R})^H = 0$. This implies that $H^2_b(H, \mathbb{R})$ is isometrically isomorphic to $L^\infty_{alt}(\partial \mathbb{H}^P_C)^3, \mathbb{R})^H$. It is not hard to verify that the image of $k^b_{SU(1, p)}$ under the isomorphism $H^2_b(SU(1, p), \mathbb{R}) \cong L^\infty_{alt}(\partial \mathbb{H}^P_C)^3, \mathbb{R})^{SU(1, p)}$ is Cartan’s angular invariant $c_p$. 
3.3 Continuous (bounded) cohomology in degree two

Let us now focus more specifically on the second continuous cohomology module of a Hermitian Lie group \( G \). By van Est isomorphism the module \( H^2_{cb}(G, \mathbb{R}) \) is isomorphic to the vector space of the \( G \)-invariant differential 2-forms on \( \mathcal{X} \) which are generated, as a real vector space, by the Kähler classes of the irreducible factors of the symmetric space \( \mathcal{X} \). The class corresponding via van Est isomorphism to the Kähler class \( \omega \) of \( \mathcal{X} \) is represented by the cocycle \( c_\omega(g_0, g_1, g_2) = \beta_\mathcal{X}(g_0 x, g_1 x, g_2 x) \) where \( x \in \mathcal{X} \) is any fixed point.

It was proven in [DT87] for the irreducible classical domains and in [CØ03] in the general case that the absolute value of the cocycle \( c_\omega \) is bounded by \( \text{rk}(\mathcal{X}) \), in particular the class \( [c_\omega] \) is in the image of the comparison map \( c : H^2_{cb}(G, \mathbb{R}) \to H^2_{cb}(G, \mathbb{R}) \):

**Proposition 3.2.** Let \( G \) be a connected semisimple Lie group with trivial center and no compact factor. The comparison map

\[ c : H^2_{cb}(G, \mathbb{R}) \to H^2_{cb}(G, \mathbb{R}) \]

is surjective.

By completely different techniques, it was proven in [BM99] that, if \( G \) is a connected semisimple Lie group with finite center and without compact factors, the comparison map \( c \) is injective (hence an isomorphism) in degree 2.

**Proposition 3.3 ([BM99]).** Let \( G \) be a connected semisimple Lie group with trivial center and no compact factor. The comparison map

\[ c : H^2_{cb}(G, \mathbb{R}) \to H^2_{cb}(G, \mathbb{R}) \]

is injective.

We will denote by \( \kappa^b_G \) the bounded Kähler class, that is the class in \( H^2_{cb}(G, \mathbb{R}) \) satisfying \( c(\kappa^b_G) = [c_\omega] \). The Gromov norm of \( \kappa^b_G \) can be computed explicitly:

**Theorem 3.4 ([DT87, CØ03, BIW09]).** Let \( G \) be a Hermitian Lie group with associated symmetric space \( \mathcal{X} \) and let \( \kappa^b_G \) be its bounded Kähler class. If \( \| \cdot \| \) denotes the Gromov norm, then

\[ \| \kappa^b_G \| = \text{rk}(\mathcal{X}). \]

Let now \( M \) be a locally compact second countable topological group, \( G \) a Lie group of Hermitian type, \( \rho : M \to G \) a continuous homomorphism. Precomposing with \( \rho \) at the cochain level induces a pullback map in bounded cohomology \( \rho_\# : H^2_{cb}(G, \mathbb{R}) \to H^2_{cb}(M, \mathbb{R}) \) that is norm non-increasing. The representations \( \rho \) for which the pullback map is norm preserving, namely \( \| \rho^\#(\kappa^b_M) \| = \| \kappa^b_M \| \), are called tight representations. These homomorphisms were first defined in [BIW09], and in the same paper the following structure theorem is proven:
Theorem 3.5 ([BIW09, Theorem 7.1]). Let $L$ be a locally compact second countable group, $G$ a connected algebraic group defined over $\mathbb{R}$ such that $G = G(\mathbb{R})^\circ$ is of Hermitian type. Suppose that $\rho : L \to G$ is a continuous tight homomorphism. Then

1. The Zariski closure $H = \overline{\rho(L)^{\mathbb{Z}}}$ is reductive.

2. The group $H = H(\mathbb{R})^\circ$ almost splits as a product $H_{nc}H_c$ where $H_c$ is compact and $H_{nc}$ is of Hermitian type.

3. If $Y$ is the symmetric space associated to $H_{nc}$, then the inclusion of $Y$ in $X$ is totally geodesic and the Shilov boundary $S_{H_{nc}}$ sits as a subspace of $S_G$. 
Chapter 4

Boundary maps and maximal representations

4.1 Measurable boundary maps

Let us now focus on the only family of Hermitian symmetric spaces of rank one, namely the complex hyperbolic spaces $H^p_C$ that are the symmetric spaces associated to the groups $SU(1,p)$. The bounded domain realization of $H^p_C$ is the unit ball in $C^p$. Moreover, since these symmetric spaces have rank one, the topological boundary of the bounded domain realization is an homogeneous $SU(1,p)$-space and can be naturally identified both with the visual boundary of $H^p_C$ considered as a CAT(0) space, and with the Shilov boundary of $H^p_C$. We will fix on $\partial H^p_C$ the Lebesgue measure class $[\lambda]$ coming from the explicit realization as a $2p - 1$ dimensional sphere.

If $\Gamma$ is a lattice in $SU(1,p)$, then the action of $\Gamma$ on $\partial H^p_C$ preserves the Lebesgue measure class $[\lambda]$. Moreover the space $\partial H^p_C$ is an amenable and doubly ergodic $\Gamma$-space. The first assertion follows from the fact that $\Gamma$ is a lattice in $SU(1,p)$ and $\partial H^p_C$ is an homogeneous space for $SU(1,p)$ with amenable stabilizers [Zim84, Corollary 4.3.7], and the second fact is a consequence of Howe-Moore’s Theorem [Zim84, 2.2.20]. These two properties of the space $\partial H^p_C$ allow to construct equivariant boundary maps with respect to a Zariski dense representation of the group $\Gamma$ in an algebraic group:

**Proposition 4.1.** [BL04, Proposition 7.2] Let $\Gamma$ be a lattice in $SU(1,p)$, $G$ a Lie group of Hermitian type and let $\rho : \Gamma \to G$ be a Zariski dense representation. Then there exists a $\rho$-equivariant measurable map $\phi : \partial H^p_C \to S_G$ such that, for almost every pair of points $x, y$ in $\partial H^p_C$, $\phi(x)$ and $\phi(y)$ are transversal.

Let us now fix a measurable map $\phi : \partial H^p_C \to S_G$, and define the *essential Zariski closure of $\phi$* to be the minimal Zariski closed subset $V$ of $S_G$ such that $\mu(\phi^{-1}(V)) = 1$. Such a set exists since the intersection of finitely many closed subsets of full measure has full measure and $S_G$ is an algebraic variety, in particular
it is Noetherian. We will say that a measurable boundary map $\phi$ is \textit{Zariski dense} if its essential Zariski closure is the whole $S_G$.

**Proposition 4.2.** Let $\rho$ be a Zariski dense representation, then $\phi$ is Zariski dense.

**Proof.** Indeed let us assume by contradiction that the essential Zariski closure of $\phi(\partial \mathbb{H}^p_C)$ is a proper Zariski closed subset $V$ of $S_G$. The set $V$ is $\rho(\Gamma)$-invariant: indeed for every element $\gamma$ in $\Gamma$, we get $\mu(\phi^{-1}(\rho(\gamma)V)) = \mu(\gamma \phi^{-1}(V)) = 1$, hence, in particular, $\rho(\gamma)V = V$ by minimality of $V$.

Let us now recall that the Shilov boundary $S_G$ is an homogeneous space for $G$, and let us fix the preimage $W$ of $V$ under the projection map $G \to G/Q = S_G$. $W$ is a proper Zariski closed subset of $G$, moreover if $g$ is any element in $W$, the Zariski dense subgroup $\rho(\Gamma)$ of $G$ is contained in $Wg^{-1}$ and this gives a contradiction. \qed

In Proposition 4.1, the hypothesis that the representation $\rho : \Gamma \to G$ is Zariski dense is essential: we will exhibit, in Remark 1, examples of unbounded representations $\rho$ that do not admit measurable equivariant maps $\phi : \partial \mathbb{H}^p_C \to S_G$ for some choices of groups $G$. However it follows from Burger, Iozzi and Wienhard's work that we can replace the assumption that the representation $\rho$ is Zariski dense with the one that it is tight:

**Theorem 4.3** ([BIW09, Theorem 8]). Let $\Gamma$ be a lattice in $\text{SU}(1,p)$, let $G$ be a Lie group of Hermitian type, and let $\rho : \Gamma \to G$ be a tight representation. Then there exists a $\rho$-equivariant measurable map $\phi : \partial \mathbb{H}^p_C \to S_G$.

### 4.2 The transfer map and the Toledo invariant

A key feature of bounded cohomology is that, whenever $\Gamma$ is a lattice in $G$, it is possible to construct a left inverse $T^*_b : H^*_b(\Gamma) \to H^*_b(G)$ of the restriction map. Indeed the bounded cohomology of $\Gamma$ can be computed from the complex $(C^*_b(G, \mathbb{R}))^G, d)$ and the transfer map $T^*_b$ can be defined on the cochain level by the formula

$$T^*_b(c(g_0, \ldots, g_k)) = \int_{\Gamma \setminus G} c(gg_0, \ldots, gg_k) d\mu(g)$$

where $\mu$ is the measure on $\Gamma \setminus G$ induced by the Haar measure of $G$ provided it is normalized to have total mass one. It is worth remarking that when we consider instead the continuous cohomology without boundedness assumptions, a transfer map can be defined with the very same formula only for cocompact lattices, but the restriction map is in general not injective if the lattice is not cocompact.

Let us fix a representation $\rho : \Gamma \to G$. Since $H^2_{\text{ch}}(\text{SU}(1,p), \mathbb{R}) = \mathbb{R}^{k^b_{\text{SU}(1,p)}}$, the class $T^*_b \rho^*(\kappa^b_G)$ is a scalar multiple of the Kähler class $k^b_{\text{SU}(1,p)}$. The \textit{generalized Toledo invariant} of the representation $\rho$ is the number $i_\rho$ such that $T^*_b \rho^*(\kappa^b_G) = i_\rho k^b_{\text{SU}(1,p)}$. A consequence of Theorem 3.4, and the fact that the transfer map is norm non-increasing, is that $|i_\rho| \leq \text{rk}(\mathcal{X})$. A representation $\rho$ is called \textit{maximal} if $|i_\rho| = \text{rk}(\mathcal{X})$. Clearly maximal representation are in particular tight representations.
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It is worth remarking that the definition of generalized Toledo invariant, and hence of maximal representation, we are giving here is different from the one that was first introduced by Burger and Iozzi in [BI00]. Indeed the original definition was based on continuous cohomology only and on the fact that the pullback in continuous cohomology factors via the $L^2$-cohomology of the associated locally symmetric space. However it is proven in [BI07, Lemma 5.3] that the invariant that was originally defined in [BI00], $i\rho$ in the notation of that article, and the invariant we defined here, that there was denoted by $t_b(\rho)$, coincide. Since we will not need $L^2$-cohomology in the sequel we will stick to this equivalent definition.

The following lemma will be useful at the very end of the part, in the proof of Corollary 1.9:

**Lemma 4.4.** The generalized Toledo invariant is constant on connected components of the representation variety.

*Proof.* This is proven in [BI08, Page 4].

4.3 A formula for the Toledo invariant

In higher rank it is not anymore true that the Shilov boundary of $G$ is an amenable $G$ space, however $\beta_S$ is a cocycle in the complex $(\mathcal{B}_{alt}(S^*_G, \mathbb{R}), d)$ where $\mathcal{B}_{alt}(S^*_G, \mathbb{R})$ denotes the bounded alternating Borel functions, and there is a natural map $m^*: H^*(\mathcal{B}_{alt}(S^*_G, \mathbb{R})^G, d) \to H^*_b(G, \mathbb{R})$ that has the property that $m^*[\beta_S] = \kappa_G^b$.

It is possible to prove, exploiting once again functoriality properties of bounded cohomology, the following result:

**Proposition 4.5** ([BI09, Theorem 2.41]). Let $H$ be either $SU(1,p)$ or one of its lattices and let $G$ be a Hermitian Lie group. Let $\rho: H \to G$ be a representation, $\beta_S : (S_G)^3 \to \mathbb{R}$ the Bergmann cocycle and $\phi : \partial \mathbb{H}^p_G \to G/Q$ be a measurable $\rho$-equivariant boundary map. Then $\phi^* \beta_S \in L^\infty_b((\partial \mathbb{H}^p_G)^3, \mathbb{R})^H$ corresponds to the class $\rho^* \kappa_G^b$ in $H^2_b(H, \mathbb{R})$. For almost every triple $(x, y, z)$ in $\partial \mathbb{H}^p_G$, the formula

$$i_\rho c_p(x, y, z) = \int_{H \setminus SU(1,p)} \beta_S(\phi(gx), \phi(gy), \phi(gz))d\mu(g)$$

holds.

The measurable function $\phi^* \beta_S$ is well defined because $\beta_S$ is defined everywhere and is a strict cocycle: in general if $\lambda$ denotes the Lebesgue measure on $\partial \mathbb{H}^p_G$ and $\nu$ is the Lebesgue measure on $S_G$, then $\phi_* \lambda$ is not absolutely continuous with respect of $\nu$, and hence considering the pullback under $\phi$ of an element in $L^\infty(S^*_G, \mathbb{R})$ makes no sense. We will now show that, since $\beta_S$ is a strict $G$-invariant cocycle and $SU(1,p)$ acts transitively on pairs of distinct points of $\partial \mathbb{H}^p_G$, the equality of Proposition 4.5 holds for every triple of pairwise distinct points (this is an adaptation in our context of an argument due to Bucher: cfr. the proof of [BBI13, Proposition 3] in case $n = 3$).

**Lemma 4.6.** The equality in Proposition 4.5 holds for every triple $(x, y, z)$ of pairwise distinct points.
Proof. The formula of Proposition 4.5 is an equality between SU(1, p)-invariant strict cocycles: clearly this is true for the left-hand side, moreover the expression on the right-hand side is a strict cocycle since $\beta_S$ is, and is SU(1, p) invariant since $\phi$ is $\rho$-equivariant and $\beta_S$ is $G$-invariant.

Let us now fix a SU(1, p)-invariant full measure set $\mathcal{O} \subseteq (\partial \mathbb{H}^p_C)^3$ on which the equality holds. Since $\mathcal{O}$ is of full measure, an application of Fubini’s Theorem is that for almost every pair $(y_1, y_2) \in (\partial \mathbb{H}^p_C)^2$ the set of points $z \in \partial \mathbb{H}^p_C$ such that $(y_1, y_2, z) \in \mathcal{O}$ is of full measure. Let us fix a pair $(y_1, y_2)$ for which this holds and denote by $W$ the set of points $z$ such that $(y_1, y_2, z) \in \mathcal{O}$. Since the SU(1, p) action on $\partial \mathbb{H}^p_C$ is transitive on pairs of distinct points, for every $i$ there exists an element $g_i$ such that $(x_i, x_{i+1}) = (g_i y_1, g_i y_2)$. Let us now fix a point $x_3$ in the full measure set $g_1 W \cap g_2 W \cap g_3 W$. Since $x_3$ is in $g_i W$, we get that $g_i^{-1} x_3 \in W$, and hence $(x_i, x_{i+1}, x_3) = g_i (y_1, y_2, g_i^{-1} x_3) \in \mathcal{O}$.

In particular, computing the cocycle identity on the 4tuple $(x_0, x_1, x_2, x_3)$, we get that the identity of Proposition 4.5 holds for the triple $(x_0, x_1, x_2)$:

$$i_\rho(c(x_0, x_1, x_2) = i_\rho(c(x_0, x_1, x_3) - i_\rho(c(x_0, x_2, x_3) + i_\rho(c(x_1, x_2, x_3) =$$

$$= \int_{SU(1, p)/\Gamma} \beta_S(\phi(g x_0), \phi(g x_1), \phi(g x_3)) - \beta_S(\phi(g x_0), \phi(g x_2), \phi(g x_3)) +$$

$$+ \beta_S(\phi(g x_1), \phi(g x_2), \phi(g x_3))dg =$$

$$= \int_{SU(1, p)/\Gamma} \beta_S(\phi(g x_0), \phi(g x_1), \phi(g x_2))dg.$$ 

\[ \square \]

Corollary 4.7. Let $\rho : \Gamma \to G$ be a maximal representation and let $\phi : \partial \mathbb{H}^p_C \to S_G$ be a $\rho$-equivariant measurable boundary map. Then for almost every maximal triple $(x, y, z) \in (\partial \mathbb{H}^p_C)^3$, the triple $(\phi(x), \phi(y), \phi(z))$ is contained in the Shilov boundary of a tube-type subdomain and is a maximal triple.

Proof. Let us fix a positively oriented triple $(x, y, z)$ of points on a chain. We know from Lemma 4.6 that the equality

$$\int_{SU(1, p)/\Gamma} \beta_S(\phi(g x), \phi(g y), \phi(g z))dg = \text{rk}(\mathcal{X})$$

holds: since $\rho$ is maximal, then $i_\rho = \text{rk}(\mathcal{X})$, and since $(x, y, z)$ are on a chain then $c_\rho(x, y, z) = 1$.

Since $\|\beta_S\|_\infty = \text{rk}(\mathcal{X})$, it follows that $\beta_S(\phi(g x), \phi(g y), \phi(g z)) = \text{rk}(\mathcal{X})$ for almost every $g$ in SU(1, p). By Proposition 2.1, this implies that for almost every $g \in \text{SU}(1, p)$, the triple $(\phi(g x), \phi(g y), \phi(g z))$ is contained in the boundary of a tube-type subdomain. Since maximal triples in $\partial \mathbb{H}^p_C$ form an SU(1, p)-orbit, the fact that the result holds for almost every element $g$ implies that the result holds for almost every triple of positively oriented points in a chain. The same argument applies for negatively oriented triples. 

\[ \square \]

If $G$ is not of tube type we will say that a measurable map $\phi : \partial \mathbb{H}^p_C \to S_G$ with this property preserves the chain geometry: indeed it induces an almost everywhere defined morphism $(\phi, \hat{\phi})$ from the geometry $\partial \mathbb{H}^p_C \times \mathcal{C}$ whose points are points in $\partial \mathbb{H}^p_C$ and whose lines are the chains, to the geometry $S_G \times \mathcal{T}$ whose points are points in
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$S_G$ and whose lines are the Shilov boundaries of maximal tube-type subdomains of $S_G$. The morphism $(\phi, \hat{\phi})$ has the property that it preserves the incidence structure almost everywhere. Purpose of Part II is to show that a measurable Zariski dense map $\phi : \partial H^p \to S_{SU(m,n)}$ that preserves the chain geometry coincides almost everywhere with an algebraic map.

If instead $G$ is of tube type the first statement of Corollary 4.7 is empty, but the second implies that the image of a positively oriented triple of points on a chain is a maximal triple. We will say that a map $\phi$ with this property is weakly monotone and we will study extensively weakly monotone maps with values in $S_{m,m}$.

We finish this chapter by using Proposition 4.5 to give examples of unbounded representations $\rho : \Gamma \to G$ where $G$ is an Hermitian Lie group that do not admit measurable equivariant maps $\phi : \partial H^p \to S_G$.

Example 1 (non-existence of boundary maps). Let $G$ be the group $SO(2, 2p)$ and let us consider the representation $\rho : \Gamma \to SO(2, 2p)$ obtained by composition of the inclusion $\iota : SU(1, p) \to SO(2, 2p)$. There exists no measurable map $\phi : \partial H^p \to S_{SO(2,2p)}$ that is $\rho$-equivariant.

Proof. We already remarked in Chapter 2 that the inclusion $\iota$ of $SU(1, p)$ in $SO(2, 2p)$ is equivariant with an holomorphic and isometric map $f_\iota : H^p \to X_{SO(2,2p)}$. In particular $\omega_{SU(1, p)} = f_\iota^*(\omega_{SO(2,2p)})$ and hence $\rho^*\kappa_{SO(2,2p)}^b = \kappa_{H^p}^b$, as a consequence of the functoriality property of bounded cohomology. Assume then by contradiction that there exists an equivariant map $\phi : \partial H^p \to S_{SO(2,2p)}$ that is $\rho$-equivariant. Then, the pullback in bounded cohomology can be realized also via $\phi$, and in particular the function $\phi^*\beta_{SO(2,2p)}$ equals $c_p$ in $L^\infty(\partial H^p, \mathbb{R})$. But this is impossible: since $SO(2, 2p)$ is of tube type, the function $\beta_{SO(2,2p)}$, and hence also $\phi^*\beta_{SO(2,2p)}$ has values in the discrete set $\{-2, 0, 2\}$ but the function $c_p$ takes all the possible values in $[-1, 1]$. $\square$

With similar arguments one can show that the restriction to a lattice of the inclusion of $SU(1, p)$ in $SU(p, p)$ does not admit an equivariant map with values in the Shilov boundary $S_{m,m}$, and that, even if $SO^+(4k+2)$ is a subgroup of $SU(2k+1, 2k+1)$, the Shilov boundary of $SO^+(4k+2)$ is not a subspace of $S_{2k+1,2k+1}$: the first group is not of tube type, where $SU(m, m)$ is.
Chapter 5

Tight homomorphisms between Hermitian Lie groups

In this chapter we study tight homomorphisms \( \rho : G_1 \to G_2 \) between Hermitian Lie groups. These were introduced and studied by Burger Iozzi and Wienhard in [BIW09], and were recently classified by Hamlet in his thesis [Ham14b] with the assumption that \( G_1 \) is simple. We review his work, giving some alternative proofs in some cases and extending his results by removing the simplicity assumption on \( G_1 \).

The strategy of the proof consists of two steps: one first shows that tight homomorphisms are holomorphic, namely that the \( \rho \) equivariant, totally geodesic map between the associated symmetric spaces is holomorphic (cfr. [Ham12]), then applies Ihara’s classification of holomorphic homomorphisms between simple Hermitian Lie groups (that can be found in [Iha67]) together with a criterion of [BIW09] to check which holomorphic embeddings are tight and conclude the classification. The chapter is divided in three sections: in the first we recall basic facts about tight homomorphisms that will be needed for the classification (more details can be found in [BIW09]), in the second we prove that tight homomorphisms are holomorphic (this extends the results of [Ham12]), in the third we recall the key steps of Ihara’s classification of holomorphic homomorphisms and summarize, without proofs, Hamlet’s results of [Ham11].

5.1 Tight homomorphisms

There is a correspondence between Lie group homomorphisms \( \rho : M \to G \), Lie algebra homomorphisms \( d\rho : m \to g \), and totally geodesic maps \( f_\rho : X_M \to X_G \). We will say that an homomorphism \( \rho \) is holomorphic if the associated totally geodesic map is holomorphic, that an homomorphism \( \rho : M \to G \) is positive if \( \rho^* \kappa^b_G \) is a linear combination with positive coefficients of the bounded Kähler classes of the irreducible factors of \( X_M \), and strictly positive if it is a combination with nonzero positive coefficients. Similarly we will say that a Lie algebra homomor-
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A homomorphism $d\rho : m \to g$ is tight if its associated Lie group homomorphism is. When classifying all tight homomorphisms it is useful to talk about the associated Lie algebra homomorphism, in order not to have to deal with the finite centers of the Hermitian Lie groups, this way the statements are neater. With a slight abuse of notation we will say that a group, or a Lie algebra, is of tube type if the associated symmetric space is.

The following proposition summarizes easy properties of tight maps in this context:

**Proposition 5.1.**

1. If $\rho : M \to G$ is holomorphic and isometric then $\rho$ is tight if and only if $M$ and $G$ have the same rank.

2. The diagonal embedding $M \to M^n$ is tight.

3. A map $\rho : M \to G_1 \times \ldots \times G_n$ is tight if and only if all the induced maps $\rho_i : M \to G_i$ are tight and positive or tight and negative.

4. Let $\rho : M \to G$ and $\tau : G \to L$ be homomorphisms. If the composition $\tau \circ \rho$ is tight and $G$ is simple then both $\rho$ and $\tau$ are tight, if $\rho$ is tight and $\tau$ is tight and positive, then $\tau \circ \rho$ is tight.

**Proof.**

(1) If $\rho$ is holomorphic and isometric, and we denote by $\omega_G$ the normalized Kähler form of the symmetric space associated to $G$, then $f_\rho^*(\omega_G) = \omega_M$. By the naturality of van Est isomorphism this implies that $\rho^*(\kappa_G^b) = \kappa_M^b$, hence $\rho$ is tight if and only if $\rk(\chi_G) = \|\kappa_G^b\| = \|\kappa_M^b\| = \rk(\chi_M)$.

(2) It is easy to verify that in this case $f_\rho^*\omega_{M^n} = n\omega_M$.

(3) This follows from the additivity of the rank with respect to products: if we denote by $\kappa_i^b$ the bounded Kähler class of the group $G_i$ we get that $\kappa_G^b = \sum \kappa_i^b$. In particular $\|\rho^*\kappa_G^b\| = \|\kappa_M^b\| \sum \rho_i^*\kappa_i^b = \sum (-1)^{\epsilon_i} \|\rho_i^*\kappa_i^b\| \kappa_M^b$ where $\epsilon_i$ is 0 if $\rho_i$ is positive and 1 if it is negative. This implies that

$$\|\rho^*\kappa_G^b\| = \|\sum \rho_i^*\kappa_i^b\| \leq \sum \|\kappa_i^b\| = \sum \rk(\chi_i).$$

Since now $\|\kappa_G^b\| = \rk(\chi) = \sum \rk(\chi_i)$ there can only be equality if all homomorphisms $\rho_i$ are tight and all the signs $\epsilon_i$ are equal.

(4) Is analogue to (3).

**Remark 5.2.** If the Hermitian Lie group $G$ is not simple, the notion of tightness for an homomorphism $\rho : M \to G$ depends drastically on the choice of the complex structure on each factor. For example the diagonal embedding $d : M \to M^2$ is tight but if one changes the orientation of one of the two factors, and hence picks as complex structure on $M^2$ the endomorphism $(J, -J)$, the homomorphism $d$ is not anymore tight: the Kähler class of $(M^2, (J, -J))$ is $\kappa_{M^2}^b = \kappa_{M \times \{id\}}^b - \kappa_{\{id\} \times M}^b$ and the pullback $d^*\kappa_{M^2}^b$ is zero.

Similarly if we consider the tight map $(j, \overline{j}) : M^2 \to M^2$, where $j$ is the identity and $\overline{j}$ is an antiholomorphic isomorphism then the composition $d \circ (j, \overline{j})$ is tight even if $d$ is not tight.
5.2. TIGHT HOMOMORPHISMS ARE HOLOMORPHIC

We now turn to an example of a tight homomorphism that will be crucial in Part II: as we will prove in Proposition 5.16, this is the only tight homomorphism of \( SU(1, p) \) into \( SU(m, n) \), and actually in any classical Hermitian Lie group.

**Example 5.3 (The standard embedding).** Let us consider the vector space \( \mathbb{C}^{n+m} \) endowed with a Hermitian form \( h \) of signature \( (m, n) \), and let us fix an orthogonal direct sum decomposition

\[
\mathbb{C}^{n+m} = V_1 \oplus \ldots \oplus V_m \oplus W
\]

where \( V_i \) are subspaces of \( V \) such that \( h|_{V_i} \) has signature \( (1, p) \). We will denote by \( j \) the associated embedding

\[
j : SU(1, p)^m \times SU(n - pm) \hookrightarrow SU(m, n).
\]

If \( \Delta \) denotes the diagonal embedding

\[
\Delta : SU(1, p) \to SU(1, p)^m,
\]

the composition \( j \circ \Delta : SU(1, p) \to SU(m, n) \) is tight: indeed the diagonal embedding is tight and the inclusion \( j \) is holomorphic and isometric, this implies that \( j \) is tight and positive since \( SU(1, p)^m \times SU(n - pm) \) and \( SU(m, n) \) have the same rank. We will call the composition \( \rho = j \circ \Delta \) and the restriction \( \rho| \) of \( \rho \) to a lattice \( \Gamma \) in \( SU(1, p) \) a standard representation.

The characterization of Hermitian Lie groups of tube type we recalled in Lemma 2.2, allows to prove that if \( \rho : G \to H \) is a tight homomorphism and \( G \) is of tube type, then the Zariski closure of the image is of tube type:

**Proposition 5.4 ([BIW09, Theorem 9]).** Let \( \rho : G_1 \to G_2 \) be a tight homomorphism, then

1. If \( G_1 \) is of tube type, then there exists a maximal subgroup \( G^T_2 \) of \( G_2 \) of tube type such that \( \rho(G_1) \subseteq G^T_2 \),

2. If \( \ker \rho \) is finite and \( G_2 \) is of tube type, then \( G_1 \) is of tube type.

5.2 Tight homomorphisms are holomorphic

We will now show that every tight inclusion \( L \hookrightarrow G \), where all simple factors of \( G \) are classical, and no factor of \( L \) is virtually \( SU(1, 1) \) is holomorphic. This was proven by Hamlet in [Ham12] with the additional hypothesis that \( L \) is irreducible. We will now generalize his techniques and remove this latter hypothesis. The strategy of the proof is as follows: our first aim is to show that tight representations in \( SU(m, n) \) are holomorphic (cfr. Proposition 5.8). This is done by studying some cases in which \( L \) has low rank explicitly and deducing the general case by tightly embedding some suitable low rank subgroup. Using the results of Section 5.1, it is possible to check that the only case left to analyze is when \( G = SO^*(4p + 2) \) and in this case we use an ad hoc argument based on counting multiplicities.
5.2.1 Representations of low rank groups in $SU(m, n)$

In the study of homomorphisms of real Lie algebras $\rho : l \to su(m, n)$, it is useful to consider the associated linear representation $\rho$ of $l$ on $\mathbb{C}^{m+n}$ that is composition of $\rho$ with the standard representation $su(m, n) \to gl((\mathbb{C}^{m+n})$. This latter representation can be decomposed into irreducible representations, and it can be studied in terms of its weights.

In particular, if $\rho^i : l_i \to gl(V_i)$ are representations, the tensor product representation $\rho^1 \otimes \rho^2 : l_1 \otimes l_2 \to gl(V_1 \otimes V_2)$ is defined by the expression

$$\rho^1 \otimes \rho^2((l_1, l_2))(v_1 \otimes v_2) = \rho^1(l_1) v_1 \otimes v_2 + v_2 \otimes \rho^2(l_2) v_2.$$  

It is well known and easy to check that every irreducible linear representation $\rho : g \to su(m, n)$ is of the form $\rho^1 \otimes \rho^2$ for some irreducible representations $\rho^i : l_i \to gl(V_i)$.

Let now $\rho : g \to su(m, n)$ be an homomorphism, it is possible to chose the decomposition of $\rho$ in irreducible factors in a way that is compatible with the metric:

**Proposition 5.5** (cfr. [Ham12, Theorem 4.3]). Let $\rho : g \to su(s, t)$ be a representation, there exists a $\rho$ invariant orthogonal splitting $\mathbb{C}^{s,t} = \oplus \mathbb{C}^{r_i,s_i}$. In particular the representation $\rho$ can be written as the direct sum $\oplus \rho_i : g \to \oplus su(s_i,t_i)$, where $\rho_i$ is irreducible for each $i$.

**Proof.** Let us fix a decomposition of $\mathbb{C}^{s,t} = \oplus U_i$ in irreducible $\rho(g)$-modules. We denote by $h$ the Hermitian form with respect to which $su(s,t)$ is defined. It is easy to check, using the fact that $\rho(g) \subset su(s,t)$ that the restriction of $h$ to each invariant module is either non-degenerate or zero. If the restriction to the submodule $U_1$ is nondegenerate, it is easy to check that the orthogonal $U_1^\perp$ is a $\rho(g_1)$-module, and hence we can proceed by induction. In the case in which we find a submodule $U_i$ on which the restriction of $h$ is zero we can find an isomorphic $\rho(g)$ subspace $U_j$ with the property that the restriction of $h$ to $U_i \oplus U_j$ has signature $(m,m)$ for some $m$: using the fact that $h$ is non-degenerate one can find a subspace $U_j$ with the property that the restriction of $h$ to $U_i \oplus U_j$ is nonzero and then shows using that $U_i$ is an irreducible $g$ module that the restriction of $h$ to the sum is non-degenerate and that the two $g$-modules $U_i$ and $U_j$ are isomorphic. It is then easy to find a different splitting $U_i \oplus U_j = V_i \oplus V_j$ such that the restriction of $h$ to $V_i$ is nondegenerate. \qed

From now on we will say that an homomorphism $\rho : g \to su(m, n)$ is irreducible if the associated linear representation is. Using that an irreducible homomorphism of $su(1,1)$ in $su(p,q)$ is tight if and only if its highest weight is odd (cfr. [Ham12, Theorem 6.1] and [BIW09, Lemma 9.5]), and by cleverly choosing some relevant subalgebras isomorphic to $su(1,1)$ it is possible to prove the following:

**Proposition 5.6** ([Ham12, Theorems 6.2 to 6.5]). For the low rank algebras we have:

6.2 Each tight irreducible homomorphism of $su(1,1) \oplus su(1,1)$ factors through one of the factors.
6.3 The only tight irreducible homomorphism of $\mathfrak{sp}(4)$ is the standard representation. In particular it is holomorphic.

6.4 Each tight irreducible homomorphism of $\mathfrak{sp}(4) \oplus \mathfrak{su}(1,1)$ factors through one of the factor. In particular is holomorphic in the $\mathfrak{sp}(4)$-factor.

6.5 The only tight irreducible homomorphism of $\mathfrak{su}(1,2)$ is the standard representation. In particular it is holomorphic.

We will need another case for our purposes: the analysis of tight irreducible homomorphisms of $\mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1)$.

**Lemma 5.7.** Let $\rho = \rho^1 \boxtimes \rho^2 : \mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(m,n)$ be an irreducible homomorphism. If $\rho$ is tight, then either $\rho^1$ is trivial or $\rho^1$ is the standard representation and $\rho^2$ is trivial. This implies that every tight homomorphism $\rho : \mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(m,n)$ is either holomorphic or antiholomorphic in the first factor.

**Proof.** In the proof of [Ham12, Theorem 6.5] it is shown that, for every irreducible representation $\rho^1 : \mathfrak{su}(1,2) \to \mathfrak{su}(p,q)$, if $\rho^1$ is not the standard representation nor the trivial one, then the composition of $\rho^1$ with a disc $\mathfrak{d} : \mathfrak{su}(1,1) \to \mathfrak{su}(1,2)$ is a representation of $\mathfrak{su}(1,1)$ that has some even nonzero weight. Let us assume by contradiction that the representation $\rho^1$ is not the standard representation nor trivial and consider the restriction $\rho|_l$ of $\rho$ to the subalgebra $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$ associated to $\mathfrak{d}$. We get that $\rho|_l$ decomposes as $\sum \rho_i \boxtimes \rho_k$ where we denote by $\rho_j$ the irreducible representation of $\mathfrak{su}(1,1)$ of highest weight $j$. Since some $l$ is even and nonzero, some of the irreducible factors of $\rho|_l$ is non-tight: we know from Proposition 5.6 that the irreducible tight representations of $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$ factor through a tight representation of one of the factor, which has odd highest weight.

It follows from Proposition 5.1 that if there exists a non-tight irreducible factor of the restriction of $\rho$ to $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$, then $\rho$ is itself not tight. This gives the desired contradiction and hence implies that $\rho^1$ is either trivial or the standard representation. In case $\rho^1$ is the standard representation, we get, again restricting to the same subalgebra $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1)$ and decomposing $\rho|_l$ into $\sum \rho_i \boxtimes \rho_k$, that in at least one factor $\rho_i$ is not trivial, in particular $k$ must be zero again by the case 6.3 of Proposition 5.6.

The last statement of the lemma, that every tight homomorphism $\rho : \mathfrak{su}(1,2) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(m,n)$ is either holomorphic or antiholomorphic in the first factor, follows from Proposition 5.5: the representation $\rho$ splits as a direct sum of irreducible tight representations and has values in $\mathfrak{su}(n_1,n_1) \oplus \ldots \oplus \mathfrak{su}(n_k,n_k)$. In particular, all the irreducible factors must be tight, and hence either holomorphic or antiholomorphic in $\mathfrak{su}(1,2)$. We conclude, using Proposition 5.1 (3), that they are all positive (hence all holomorphic) or all negative (hence all antiholomorphic). \qed

### 5.2.2 Homomorphisms in $\text{SU}(m,n)$

If an homomorphism $\rho : L \to M$ is holomorphic then it is positive. In particular, in the next proposition we will need the hypothesis that $\rho$ is positive in order to deduce
that \( \rho \) is holomorphic. This is not a big restriction: whenever an homomorphism \( \rho \) is given, it is always possible to get a positive homomorphism up to changing the complex structures on the irreducible factors of \( L \).

**Proposition 5.8.** Let \( L \) be a Hermitian Lie algebra without factors isomorphic to \( \mathfrak{su}(1,1) \). Then every tight and positive homomorphism \( i : L \to \mathfrak{su}(m,n) \) is holomorphic.

**Proof.** Let \( L = I_1 \times \ldots \times I_k \) be the decomposition of \( L \) into irreducible factors. In order to show that \( i \) is holomorphic, it is enough to find, for every irreducible factor \( I_s \), an holomorphically embedded subalgebra \( j_s : \mathfrak{g}_s \to I_s \) such that the composition \( i \circ j_s : \mathfrak{g}_s \to \mathfrak{su}(m,n) \) is holomorphic. Once that is done, it is easy to conclude that the restriction of \( i \) to \( I_s \) is holomorphic, using the irreducibility of \( I_s \) under the adjoint action of \( I_s \) on itself (see [Ham12, Lemma 5.4]). Clearly if the restriction of \( i \) to each irreducible factor is holomorphic, the same is true for the representation \( i \).

Let us now fix a factor \( I_s \). It is easy to check with a case by case argument that there exists a tight injective and holomorphic embedding \( j_s : \mathfrak{g}_s \to I_s \) where \( \mathfrak{g}_s \) is either \( \mathfrak{su}(1,2) \) in case \( I_s = \mathfrak{su}(1,p) \), \( \mathfrak{sp}(4,\mathbb{R}) \) in case \( \text{rk}(L_s) \) is even or \( \mathfrak{sp}(4,\mathbb{R}) \oplus \mathfrak{su}(1,1) \) in case \( \text{rk}(L_s) \) is odd and greater than one (see [Ham12, Theorem 7.1]).

We denote by \( \kappa^b_s \in H^2_{\text{ch}}(L_s, \mathbb{R}) \) the Kähler class of the factor \( I_s \), and fix \( \alpha_s \), such that \( i^* (\kappa^b_s) = \sum \alpha_t \kappa^b_t \). For every \( t \) different from \( s \), we consider the diagonal disc \( d_t : \mathfrak{su}(1,1) \to I_t \). Clearly \( d_t^* \kappa^b_s = ||\kappa^b_s|| \kappa^b_s \). Let us now assume first that \( I_s \) has higher rank. Then we consider the homomorphism \( \phi_s : \mathfrak{sp}(4) \oplus \mathfrak{su}(1,1) \to I \) that is given by \( (d_1, \ldots, j_s, \ldots, d_k) \). The composition \( i \circ \phi_s : \mathfrak{sp}(4) \oplus \mathfrak{su}(1,1) \to \mathfrak{su}(m,n) \) is tight:

\[
(i \circ \phi_s)^*( \kappa^b_{\mathfrak{SU}(m,n)}) = \phi_s^* \left( \sum_{t=1}^k \alpha_t \kappa^b_t \right) = \left( \sum_{t \neq s} \alpha_t ||\kappa^b_t|| \kappa^b_{\mathfrak{SU}(1,1)} \right) + \alpha_s j_s^* \kappa^b_s.
\]

Since \( j_s \) is tight we get \( ||j_s^* \kappa^b_s|| = ||\kappa^b_s|| \) and hence \( ||(i \circ \phi_s)^*( \kappa^b_{\mathfrak{SU}(m,n)})|| = \sum \alpha_t ||\kappa^b_t|| = ||\kappa^b_{\mathfrak{SU}(m,n)}|| \).

Every tight and positive homomorphism \( \phi : \mathfrak{sp}(4) \oplus \mathfrak{su}(1,1) \) is holomorphic in the first factor (see [Ham12, Theorem 6.4]). By the discussion in the first paragraph, this implies that the restriction of \( i \) to \( I_s \) is holomorphic.

The case in which \( I_s \) is isomorphic to \( \mathfrak{su}(1,1) \) follows analogously applying Lemma 5.7 instead of [Ham12, Theorem 6.2].

**Remark 5.9.** It is easy to show that this result implies that every tight positive representation \( \rho : L \to G \) is holomorphic provided that none of the irreducible factors of \( L \) is virtually isomorphic to \( \mathfrak{su}(1,1) \) and \( G \) is classical of tube type. Indeed it is enough to remember that every classical domain of tube type admits a tight and holomorphic embedding in \( \mathfrak{SU}(m,m) \) for some \( m \). Additional arguments are required to deal with the exceptional domains and for the domains associated with \( \text{SO}^*(4p + 2) \).
5.2.3 Homomorphisms in $\text{SO}^*(4p + 2)$

We first show directly that, if the dimension of the complex hyperbolic space is big enough, every tight representation of $\mathfrak{su}(1, n) \oplus \mathfrak{su}(1, 1)$ factors through the second factor. This is an extension of [Ham12, Lemma 7.6], and we will actually need only the refinement of this argument to the case $n = 2, 3$ that we will prove in Lemma 5.11. Even if the result of Lemma 5.10 can be reproven using Theorem 1.4 and Hamlet’s classification of tight holomorphic embeddings that we will describe in Section 5.3, we decided to include it nevertheless because it is a nice argument that shades some light on what is happening, and is, in any case, preliminary to the proof of Lemma 5.11.

**Lemma 5.10.** Every tight homomorphism $\rho : \mathfrak{su}(1, n) \oplus \mathfrak{su}(1, 1) \to \mathfrak{so}^*(4p + 2)$, if $n$ is greater or equal to 4 factors through a representation of $\mathfrak{su}(1, 1)$.

**Proof.** The main idea behind the proof is that there is not enough space in $\mathfrak{so}^*(4p + 2)$ for a tight representation of $\mathfrak{su}(1, 4)$. We will make this idea precise by comparing multiplicities of irreducible representations. Let us now consider the diagram

\[
\begin{array}{c}
t_1 : \mathfrak{su}(1, n) \oplus \mathfrak{su}(1, 1) \xrightarrow{\rho} \mathfrak{so}^*(4p + 2) \xrightarrow{i_{NT}} \mathfrak{su}(2p + 1, 2p + 1) \\
\downarrow d \quad \quad \downarrow i_2 \quad \quad \downarrow \text{NT} i_3 \\
t_2 : \mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1) \xrightarrow{\rho|} \mathfrak{so}^*(4p) \xrightarrow{i|} \mathfrak{su}(2p, 2p).
\end{array}
\]

All arrows in the diagram, apart from the ones marked as NT correspond to tight homomorphisms. The vertical arrow $d$ is the direct sum of the inclusion $d_1 : \mathfrak{su}(1, 1) \to \mathfrak{su}(1, 4)$ and the identity $d_2 : \mathfrak{su}(1, 1) \to \mathfrak{su}(1, 1)$. The horizontal arrows on the right are the standard embedding of $\mathfrak{so}^*(2k) \to \mathfrak{su}(k, k)$ obtained by forgetting the quaternionic structure on $\mathbb{C}^{2k}$ that defines $\mathfrak{so}^*(2k)$: this is tight if $k$ is even and hence $\mathfrak{so}^*(2k)$ is of tube type, and not tight when $k$ is odd. The fact that the image of the restriction $\rho|$ is contained in $\mathfrak{so}^*(4p)$, that corresponds to the maximal tube type subdomain of $\mathfrak{so}^*(4p + 2)$ follows from Proposition 5.4.

The first observation, that is a consequence of the arguments explained in Lemma 5.7, is that the linear representation of $\mathfrak{su}(1, n) \oplus \mathfrak{su}(1, 1)$ on $\mathbb{C}^{4p+2}$ splits as a direct sum of representations factoring either through $\mathfrak{su}(1, n)$ or $\mathfrak{su}(1, 1)$ and all the representations of $\mathfrak{su}(1, n)$ are isomorphic to the standard representation of $\mathfrak{su}(1, n)$ on $\mathbb{C}^{n+1}$: this is because when one restricts another irreducible representation of $\mathfrak{su}(1, n) \oplus \mathfrak{su}(1, 1)$ to $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$ one gets even weights, but the composition of the representations on the second horizontal line is tight.

The second observation is that, when one considers $\mathbb{C}^{2p,2p}$ as a $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$ module (considering the composition of the tight homomorphisms on the second vertical line), there cannot be trivial modules: otherwise the representation would be contained in $\mathfrak{su}(m, m)$ embedded as a subalgebra of $\mathfrak{su}(2p, 2p)$ (here $\mathbb{C}^{2m}$ is the orthogonal, in $\mathbb{C}^{4p}$ of the trivial modules), and since the embedding of $\mathfrak{su}(m, m)$ in $\mathfrak{su}(2p, 2p)$ corresponds to an holomorphic and isometric map between the symmetric spaces, this can never be tight unless $m = 2p$. 
In particular, since for each irreducible su(1, n)-module in the first representation, there is a n − 1 dimensional trivial module appearing in the decomposition into irreducible of the representation t₁, we get that this is impossible if n ≥ 4.

Lemma 5.11. Let ρ : su(1, 2) ⊕ su(1, 1) → so⁺(4p + 2) be a tight homomorphism. Then either ρ factors through an homomorphism of su(1, 1) or it splits as the direct sum of the tight homomorphisms ρ₁ ⊕ ρ₂ : su(2, 1) ⊕ su(1, 1) → so⁺(6) ⊕ so⁺(4(p − 1)).

In each case it is holomorphic in the first factor.

Proof. Using the same argument as in Lemma 5.10, we know that there can be at most two standard representations of su(1, 2) in the decomposition of t₁ into irreducible complex representation. We claim that the only possibility is that there is a single irreducible representation of su(1, 2) that is obtained as composition of the inclusion of su(1, 2) in su(1, 3) and the identification of su(1, 3) with so⁺(6).

Indeed let us recall that so⁺(4p + 2) is the subgroup of gl(2p + 1, H) preserving a nondegenerate antiHermitian form on the 2p + 1 dimensional quaternionic vector space H²ᵖ+1. The inclusion of so⁺(4p + 2) in su(2p + 1, 2p + 1) is obtained by forgetting the quaternionic structure, the Hermitian form on C⁴ᵖ+² is the complex part of the antiHermitian form on H²ᵖ+1. In particular if we fix a t₁ invariant complex subspace, its quaternionic span is a ρ-invariant subspace of H²ᵖ+1. Let us now consider the module associated to a standard representation of su(1, 2) on C³. The dimension of its quaternionic span is either 3 or 2, in case it is 3 we are done, since that means that the representation is contained in so⁺(6), and it is not possible that it is 2, since the algebra so⁺(4) is isomorphic to su(2) × su(1, 1) (cfr. [Hel78, Page 520]) hence there cannot be any injective homomorphism of su(1, 2) in so⁺(4).

It is easy to deduce the general case from the low rank example:

Proposition 5.12. Every tight homomorphism ρ : g → so⁺(4p + 2) is holomorphic, provided g has no factor virtually isomorphic to su(1, 1).

Proof. The strategy is quite similar to the proof of Proposition 5.8. Let us split the Lie algebra g into g = su(1, p₁) ⊕ ... ⊕ su(1, pₖ) ⊕ gh. Where each simple factor of gh is of higher rank. Up to restricting the representation to a maximal tube type subalgebra of gh we can assume that gh is of tube type: indeed since each simple factor of gh has higher rank, no factor in the maximal tube type subalgebra of gh is isomorphic to su(1, 1), and, by [Ham12, Lemma 5.4], it is enough to prove that ρ is holomorphic when restricted to a maximal tube type subalgebra. Moreover we can assume that the representation ρ is injective: otherwise we can reduce to a smaller Lie algebra g', clearly the homomorphism is holomorphic in the factors that belong to the kernel of the representation.

If k = 0, namely there is no rank one factor in the decomposition of g into simple Lie algebras, we are done: indeed in this case, by Proposition 5.4 we know that ρ(g) is contained in so⁺(4p), and the thesis follows by composing ρ with the tight inclusion of so⁺(4p) into su(2p, 2p) and applying Proposition 5.8.

Assume then that there exists a nontrivial factor of type su(1, p) in the simple decomposition, and consider the homomorphism d : su(1, 2) ⊕ su(1, 1) → g obtained
by considering the inclusion of $\mathfrak{su}(1,2)$ in $\mathfrak{su}(1,p)$ and a map of $\mathfrak{su}(1,1)$ in a direct sum of diagonal discs or opposites of diagonal discs in the other factors, the choice of the sign of the diagonal disc in the factor $\mathfrak{g}_i$ corresponds to the sign of the coefficient of $\rho^*(\kappa_{SO^+(4p+1)}^b)$ along $\kappa_{G_i}$. By the choice of signs we get that $\rho \circ d$ is tight, in particular we can apply Lemma 5.10 and conclude that $\rho$ is holomorphic in the factor $\mathfrak{su}(1,p)$. This concludes the proof.

It is worth remarking that it follows from Lemma 5.10 that if $\rho: G \to \mathfrak{so}^*(2p)$ is a tight homomorphism, there exists at most one simple factor of $\gamma$ that has rank one and is not isomorphic to $\mathfrak{su}(1,1)$. This factor can only be isomorphic to $\mathfrak{su}(1,2)$ or to $\mathfrak{su}(1,3)$.

In turn this was the last step of the proof of the following:

**Theorem 5.13.** Let $\rho: G \to H$ be a tight homomorphism. Assume that $G$ has no factor locally isomorphic to $\text{SU}(1,1)$ and all the simple factors of $H$ are classical. Then $\rho$ is holomorphic.

It is worth remarking here that in [HO], Hamlet and Okuda prove the result also for homomorphisms of simple Hermitian Lie groups with values in an exceptional Lie group.

### 5.3 Holomorphic tight homomorphisms

**Satake and Ihara’s classification of holomorphic homomorphisms**

The advantage of being able to restrict to holomorphic maps is due to the fact that Satake and Ihara completely classified holomorphic homomorphisms between Hermitian Lie algebras. To be more precise Satake, in [Sat65], completely classified homomorphisms of Hermitian Lie algebras in $\mathfrak{sp}(2p)$, and gave partial results for homomorphisms in $\mathfrak{su}(m,n)$ and $\mathfrak{so}^*(2p)$, and Ihara completed the classification in [Iha67].

Satake observed that a Lie algebra homomorphism $\rho: \mathfrak{g} \to \mathfrak{g'}$ corresponds to an holomorphic totally geodesic map (is holomorphic in our notations) if and only if $\rho \circ \text{ad}(H_0) = \text{ad}(H'_0) \circ \rho$, where we denote by $H_0$ (resp. $H'_0$) the element in the center of the Lie algebra of the maximal compact subgroup of $G$ (resp. $G'$) that induces the complex structure. Lie algebra homomorphisms with this property are normally referred to as (H1) homomorphisms.

Moreover Satake introduced the stronger notion of (H2) homomorphism: those homomorphisms such that $\rho(H_0) = H'_0$. Clearly condition (H2) implies condition (H1), and Satake was able to reduce the problem of finding all homomorphisms into $\mathfrak{sp}(2p)$ that satisfy (H1) to the problem of finding (H2) homomorphisms into $\mathfrak{sp}(2p)$ and $\mathfrak{su}(p,q)$ (cfr. [Sat65, Proposition 1]). Moreover he classified (H2) homomorphisms into $\mathfrak{sp}(2p)$, $\mathfrak{su}(m,n)$, $\mathfrak{so}^*(2p)$.

In a subsequent paper [Iha67] Ihara completed the picture and finished the classification of holomorphic homomorphisms between Hermitian symmetric spaces. In his study he introduced regular subalgebras of an Hermitian Lie algebra, and showed
that each holomorphic homomorphism \( \rho : g_0 \to g_2 \) between Hermitian Lie algebras can be written as the composition \( i \circ \rho_1 \) where \( i : g_1 \to g_2 \) is the inclusion of a regular subalgebra and \( \rho_1 : g_0 \to g_1 \) is an (H2) homomorphism (cfr. [Iha67, Theorem 2]).

The definition of a regular subalgebra is based on the analysis of nice properties of roots systems of Hermitian Lie Algebras without compact factors. Let \( g \) be a Lie algebra of Hermitian type without compact factors, we fix a Cartan subalgebra \( h \) of \( g \) contained in the maximal compact subalgebra \( t \) of \( g \), and we denote by \( r \) the root system of \( g_C \) relative to \( h_C \). All these objects are uniquely defined up to conjugation. If we now chose an order on the roots so that all the noncompact roots \( \alpha \), with the property that \( \alpha(H_0) = i \) are positive, we get a fundamental root system \( \Pi \) for \( r \) that has the property that in each connected component of \( \Pi \) there exists precisely one noncompact root. The connected components of \( \Pi \) correspond to the simple factors of \( g \), and the fact that there is precisely one noncompact root reflects, on the level of root systems, the fact that the adjoint action of \( t \) on \( p \) is irreducible if \( g \) is simple. See [Iha67, Section 1.4] for more details.

A regular subalgebra of \( g \) is then a subalgebra spanned by a subset \( \Delta \) of the root system \( r \) of \( g \) that is the fundamental root system of an Hermitian Lie algebra, namely it is an independent system of strongly orthogonal roots with the additional property that in each connected component of the associated diagram there exists precisely one noncompact root (cfr. [Iha67, Section 2.3]).

**Remark 5.14.** It is worth remarking that, at the level of Lie algebras, the representation \( d_j \) that are tangent to the homomorphism \( j : SU(1, p) \oplus SU(n - pm) \to SU(m, n) \) constructed in Example 5.3 corresponds to the inclusion of a regular subalgebra \( l = su(1, p)m \) of \( g = su(m, n) \).

### Classification of tight holomorphic homomorphism

Some representation in the family of irreducible (H2) representations induces canonical isomorphisms, this is the case, for example for the skewsymmetric tensor representation \( \rho_2 : su(1, 3) \to so^*(6) \) that gives the identification of the two algebras. In the following proposition we summarize Hamlet’s classification of tight homomorphisms.

**Proposition 5.15** ([Ham11, Section 6]). The only tight irreducible (H2) representations that do not come from canonical identifications are

1. The inclusion \( i : \mathfrak{sp}(2p) \to \mathfrak{su}(p, p) \);
2. The inclusion \( i : \mathfrak{so}^*(2p) \to \mathfrak{su}(p, p) \) when \( p \) is even;
3. The spin representation \( \rho : \mathfrak{so}(p, 2) \to \mathfrak{sp}(2p') \);
4. The spin representation \( \rho : \mathfrak{so}(p, 2) \to \mathfrak{so}^*(2p') \);
5. The spin representation \( \rho : \mathfrak{so}(p, 2) \to \mathfrak{su}(p', p') \).
5.4. THE CASES RELEVANT TO THE THESIS

In the next table we summarize, for the reader’s convenience, all the tight regular subalgebras of Hermitian Lie algebras. A proof can be found in [Ham11, Section 5]:

<table>
<thead>
<tr>
<th>su(m, n)</th>
<th>su(m1, n1) ⊕ ... ⊕ su(mk, nk)</th>
</tr>
</thead>
<tbody>
<tr>
<td>with ∑mi = m, mi ≤ ni</td>
<td></td>
</tr>
<tr>
<td>so*(4p + 2)</td>
<td>⊕i su(li, li) ⊕ ∑i so*(4qi) ⊕ su(m, m + 1) ⊕ so*(4n + 2)</td>
</tr>
<tr>
<td>with ∑li + ∑qi + m + n = p, at least one of m, n = 0</td>
<td></td>
</tr>
<tr>
<td>so*(4p)</td>
<td>⊕i su(li, li) ⊕ ∑i so*(4qi)</td>
</tr>
<tr>
<td>with ∑li + ∑qi = p</td>
<td></td>
</tr>
<tr>
<td>sp(2p)</td>
<td>⊕i su(li, li) ⊕ ∑i sp(2mi)</td>
</tr>
<tr>
<td>with ∑2li + ∑mi = p</td>
<td></td>
</tr>
<tr>
<td>so(2p, 2)</td>
<td>su(1, 1) ⊕ su(1, 1)</td>
</tr>
<tr>
<td></td>
<td>su(2, 2)</td>
</tr>
<tr>
<td></td>
<td>so(2k, 2)</td>
</tr>
<tr>
<td>so(2p + 1, 2)</td>
<td>su(1, 1) ⊕ su(1, 1)</td>
</tr>
<tr>
<td></td>
<td>su(2, 2)</td>
</tr>
<tr>
<td></td>
<td>su(1, 1)</td>
</tr>
<tr>
<td></td>
<td>so(t, 2)</td>
</tr>
</tbody>
</table>

In the table above all the inclusion come from a construction similar to the one described in Example 5.3, apart from the tight regular subalgebras of so(p, 2). For a description of these last subalgebras cfr. [Iha67].

5.4 The cases relevant to the thesis

We finish this chapter by applying the results we recalled so far in the cases that will be relevant in the rest of this work: in order to prove Theorem 1.7 we will need to understand what are all the tight homomorphisms of SU(1, p) and what are all the possible tightly embedded subgroups of SU(m, n).

The standard representations are the only tight homomorphisms of SU(1, p):

**Proposition 5.16.** Let G be an irreducible classical Lie group of Hermitian type. Assume that ρ : SU(1, p) → G is tight, then g = su(m, n) for some n ≥ pm, and ρ is virtually a standard representation.

**Proof.** It follows from Proposition 5.4 that since SU(1, p) is not of tube type, G is not of tube type, and from Lemma 5.10 that G cannot be SO*(4p + 2). In particular G is the group SU(m, n) and, as a consequence of Proposition 5.5 that dρ is a diagonal embedding, in particular ρ is virtually a standard representation. ∎

We will also need to understand what are the possible tightly embedded subgroups L ⊆ SU(m, n). When the inclusion is holomorphic, the classification of [Ham12] applies:

**Proposition 5.17.** Let i : l → su(m, n) be a tight holomorphic homomorphism. Then each factor of l is either isomorphic to su(s, t) or is of tube type. Moreover
if \( I = I_t \times I_{nt} \) where \( I_t \) is the product of all the irreducible factors of tube type, then there exists a regular subalgebra \( \mathfrak{su}(k, k) \oplus \mathfrak{su}(m - k, n - k) \) of \( \mathfrak{su}(m, n) \) such that \( I_t \) is included in \( \mathfrak{su}(k, k) \) and \( I_{nt} \) is included in \( \mathfrak{su}(m - k, n - k) \).

**Proof.** Once more the inclusion \( i : I \rightarrow \mathfrak{su}(m, n) \) splits as the composition of an \((H2)\)-representation and the inclusion of a regular subalgebra. The analysis of [Ham11, Section 5] gives that the only regular tight subalgebras of \( \mathfrak{su}(m, n) \) are direct copies of \( \mathfrak{su}(k_1, k_2) \), that satisfy that the sum of the ranks equals \( m \). Since in Section 6 of the same paper it is proved that there are tight \((H2)\)-representations of a Lie algebra \( g \) in \( \mathfrak{su}(k_1, k_2) \) different from the inclusion only if \( g \) is of tube type, and in that case the image is contained in a copy of \( \mathfrak{su}(k_1, k_1) \), we get the desired result. \( \square \)
Part II

Maximal representations in $\text{SU}(m, n)$
Chapter 6

The chain geometry of $S_{m,n}$

From now on we will restrict our attention to the Hermitian Lie group $SU(m, n)$ consisting of complex matrices that preserve a non-degenerate Hermitian form of signature $(m, n)$. It is a classical result (see for example [PS69, Section 2.10]) that, if $m$ is different from $n$, the group $SU(m, n)$ is a Hermitian Lie group which is not of tube type. We will stick to this setting throughout this part.

The groups $SU(1,p)$, that are finite index covers of the groups $PU(1,p) = \text{Isom}^\circ(\mathbb{H}_p^\mathbb{C})$, form a subfamily of those groups. This implies that all the results in this chapter hold true in particular for the isometries of the complex hyperbolic spaces, and hence we will also prove Proposition 1.16. We will denote, for the sake of brevity, by $S_{m,n}$ the Shilov boundary associated to the group $SU(m, n)$. The same space was denoted in the previous part by $S_{SU(m, n)}$. The purpose of this chapter is to understand some features of the incidence structure of the subsets of $S_{m,n}$ that arise as Shilov boundaries of the maximal tube-type subdomains of the symmetric space associated to $SU(m, n)$: the $m$-chains.

The main tool that we will introduce in our investigations is a projection map $\pi_x$, depending on the choice of a point $x \in S_{m,n}$. The map $\pi_x$ associates to a point $y$ that is transversal to $x$ the uniquely determined $m$-chain that contains both $x$ and $y$ and generalize the map $\pi$ introduced in Proposition 1.16 to higher rank. The knowledge that we will gather in this chapter will allow us to prove, in Chapter 7, a striking rigidity result for these geometries that is a generalization, in our setting, of the fundamental theorem of projective geometry. The projection $\pi_x$ encodes a part of the incidence structure that is easier to study but yet contains enough information to draw conclusions on the rigidity of the geometry. We will call a $(m,k)$-circle a geometric object that arises as projection of an $m$-chain that does not contain the point $x$, these are generalizations of the Euclidean circles we encountered in Proposition 1.16. We will be mostly interested in understanding what is a good description of a $(m,k)$-circle, what is the intersection of a fiber of $\pi_x$ with an $m$-chain, and what is a parametrization of the different lifts of a fixed $(m,k)$-circle. The central results of the chapter are Proposition 6.16 and Proposition 6.18, together form the generalization to higher rank of the last statement of Proposition 1.16.
6.1 The bounded model $S_{m,n}$

Recall from Section 2.2 that the Shilov boundary of $SU(m,n)$ can be realized as the set of maximal isotropic subdomains of $\mathbb{C}^{m+n}$:

$$S_{m,n} = \{ x \in \text{Gr}_m(\mathbb{C}^{n+m}) | h|_x = 0 \}.$$  

Moreover it will also be useful, in the sequel, to consider the realization of $S_{m,m}$ in the boundary of the bounded domain realization of $X_{m,n}$. We denote it by $S_{m,n}$:

$$S_{m,n} = \{ X \in M(n \times m, \mathbb{C}) | X^*X = \text{Id} \}.$$  

As a warm up we prove directly the transitivity of the action of $SU(m,n)$ on $S_{m,n}$:

**Proposition 6.1.** The Shilov boundary $S_{m,n}$ is a real algebraic subvariety of the Grassmannian variety. The action of $SU(m,n)$ on $S_{m,n}$ is algebraic and transitive.

**Proof.** The fact that $S_{m,n}$ is a real algebraic subvariety of the Grassmannian follows from the fact that $S_{m,n}$ is contained in the domain of the affine chart $\text{Lin}(\langle e_1, \ldots, e_m \rangle, \langle e_{m+1}, \ldots, e_{m+n} \rangle)$ and the image $S_{m,n}$ of $S_{m,n}$ in that chart is defined by polynomial equations involving only the coefficients of a matrix and their conjugates. Moreover the real algebraic group $SU(m,n)$ acts algebraically on the real algebraic variety $\text{Gr}_m(\mathbb{C}^{m+n})$, and the restriction of the action to the Zariski closed subset $S_{m,n}$ of $\text{Gr}_m(\mathbb{C}^{m+n})$ is algebraic. The transitivity of $SU(m,n)$ on the set of isotropic $m$-subspaces follows from Witt’s theorem. \qed

It is worth remarking that in [BI04] it is explicitly constructed a complex variety $S_{m,n}$ such that $S_{m,n} = S_{m,n}(\mathbb{R})$.

**Pairs of transversal points**

We now turn to the study of the action of $G = SU(m,n)$ on pairs of transversal points in $S_{m,n}$. The general theory of Hermitian symmetric spaces recalled in Chapter 2 tells us that there exists a unique open $G$-orbit for the diagonal $G$-action on $S_{m,n}^2$ and any pair of points in this orbit is called transversal. Indeed in this specific realization points are transversal if and only if the underlying vectorspaces are transversal, as proven in the following lemma:

**Lemma 6.2.** The action of $SU(m,n)$ is transitive on pairs of transversal isotropic subspaces. In particular a pair $(x, y) \in S_{m,n}^2$ is transversal if and only if the underlying vector subspaces are transversal.

**Proof.** Let $x_\infty = (e_i - e_{i+m} | 1 \leq i \leq m)$ and $x_0 = (e_i + e_{i+m} | 1 \leq i \leq m)$. It is easy to compute that both $x_\infty$ and $x_0$ are isotropic subspaces, moreover the pair $(x_\infty, x_0)$ forms a pair of transversal subspaces. Let us now fix another pair $(x, y)$ of isotropic subspaces that are transversal. Then the linear span $(x, y)$ is a $2m$ dimensional subspace of $\mathbb{C}^{n+m}$ on which the restriction of $h$ is non-degenerate. In
particular we can find a basis \((x_1, \ldots, x_m)\) of \(x\) (resp. \((y_1, \ldots, y_m)\) of \(y\)) such that 
\[ h(x_i, y_j) = 2\delta_{ij}. \]
If \((z_{2m+1}, \ldots, z_{m+n})\) is an orthonormal basis of \((x, y)^\perp\), then the linear operator \(L\) sending \(e_i - e_{i+m}\) to \(x_i\), \(e_i + e_{i+m}\) to \(y_i\) and \(e_j\) to \(z_j\) induces an element in \(U(m, n)\) sending \((x_\infty, x_0)\) to \((x, y)\). In order to get an element in \(SU(m, n)\), it is enough to rescale the matrix representing \(L\) so that its determinant is 1.

From now on we will often identify a point \(x\) in the Shilov boundary with its underlying vector subspace, and we will use the notation \(x \cap y\), that would be more suited for the linear setting, with the meaning that the pair \((x, y)\) is a pair of transversal points. Moreover we will use the notation \(S_{m,n}^{(2)}\) for the set of pairs of transversal points, and we will denote the set of points in \(S_{m,n}\) that are transversal to a given point \(x\) by 
\[ S_{m,n}^x := \{ y \in S_{m,n} | y \cap x \}. \]

We now want to describe the maximal tube-type subdomains of \(X_{m,n}\). It is well known that any such subspace is associated with the group \(SU(m, m)\). Indeed for every linear \(2m\)-dimensional subspace \(V\) of \(\mathbb{C}^{m,n}\) on which the restriction of \(h\) has signature \((m, m)\), the closed subset 
\[ X_V = \{ z \in \text{Gr}_m(V) | h|_z > 0 \} = X_{m,n} \cap \text{Gr}_m(V) \]
is a subspace of \(X_{m,n}\) that is holomorphically and isometrically embedded in \(X_{m,n}\) and is a symmetric space whose isometry group, \(SU(V)\), is isomorphic to \(SU(m, m)\). In particular \(X_V\) is a maximal tube-type subdomain of \(X_{m,n}\) whose Shilov boundary 
\[ S_V = \{ z \in \text{Gr}_m(V) | h|_z = 0 \} = S_{m,n} \cap \text{Gr}_m(V) \]
is naturally a subspace of \(S_{m,n}\). As already anticipated in the introduction, we will use the term \(m\)-chain to denote the subspaces of the form \(S_V\) for some linear subspace \(V\) of \(\mathbb{C}^{m,n}\).

Let us now fix a pair of transversal points \((x, y) \in S_{m,n}^{(2)}\). We have seen in the proof of Lemma 6.2 that the linear span \(\langle x, y \rangle \subset \mathbb{C}^{n+m}\) is a \(2m\)-dimensional subspace \(V_{xy}\) of \(\mathbb{C}^{n+m}\), on which the restriction of \(h\) has signature \((m, m)\). Let \(T_{xy} := S_{V_{xy}}\) be the subset of \(S_{m,n}\) consisting of maximal isotropic subspaces of \(V_{xy}\). Then \(T_{xy}\) is the unique Shilov boundary of a maximal tube-type subdomain that contains the points \(x\) and \(y\). We will call \(T_{xy}\) the \(m\)-chain through \(x\) and \(y\).

In the case \(m = 1\), that is \(X_{m,n} = \mathbb{H}^n_{\mathbb{C}}\), the \(1\)-chains are boundaries of complex geodesics or \(chains\) in Cartan’s terminology. This is the reason why we chose to call the Shilov boundaries of maximal tube-type subdomains \(m\)-chains. To be more consistent with Cartan’s notation, will omit the 1, and simply call \(chains\) the \(1\)-chains.

### 6.2 The Heisenberg model \(H_{m,n}(x)\)

We now want to give another model for an open subset of \(S_{m,n}\) that will be extremely useful to study the chain geometry in \(S_{m,n}\). The new model we are introducing is sometimes referred to as a Siegel domain of genus two and was studied,
for example, by Koranyi and Wolf in [KW65]. In the case $m = 1$ this model is described in [Go99, Chapter 4] but our conventions here will be slightly different.

We will need, once again, a more concrete description in terms of the Grassmannian manifold in order to be able to compute what is needed. The Siegel model can be described with the aid of a conjugate form, in $\text{SL}(m + n, \mathbb{C})$, of $\text{SU}(m, n)$. In particular let $F$ be the linear endomorphism of $\mathbb{C}^{m+n}$ that is represented, with respect to the standard basis, by the matrix

$$F = \begin{bmatrix}
\frac{1}{\sqrt{2}} \text{Id} & 0 & \frac{1}{\sqrt{2}} \text{Id} \\
-\frac{1}{\sqrt{2}} \text{Id} & 0 & \frac{1}{\sqrt{2}} \text{Id} \\
0 & \text{Id} & 0
\end{bmatrix}_{m \times m},
$$

and

$$\text{det} C \det A \det A^{-*} = 1.$$

Indeed an element of $Q$ must stabilize $v_\infty$ and should belong to $\text{SU}(\mathbb{C}^{m+n}, \bar{h})$, hence it must have this form since
6.2. THE HEISENBERG MODEL $\mathcal{H}_{M,N}(X)$

$$\begin{bmatrix}
0 & 0 & \text{Id}_m \\
0 & -\text{Id}_{n-m} & 0 \\
\text{Id}_m & 0 & 0
\end{bmatrix} = \begin{bmatrix}
A^* & 0 & 0 \\
B^* & C^* & D^* \\
E^* & F^* & G^*
\end{bmatrix} \begin{bmatrix}
0 & 0 & \text{Id}_m \\
0 & -\text{Id}_{n-m} & 0 \\
\text{Id}_m & 0 & 0
\end{bmatrix} \begin{bmatrix}
A & B & E \\
0 & D & G \\
0 & 0 & A^*G
\end{bmatrix}$$

We now want to compute the Langland decomposition of $Q$. This means that we want to write $Q = L \ltimes N$ where $L$ is reducible and $N$ is reductive. We already proved that the group $SU(\mathbb{C}^{n+m}, h)$ acts transitively on pairs of transversal subspaces in $S_{m,n}$. This implies that the conjugate group $SU(\mathbb{C}^{m+n}, \overline{h})$ acts transitively on pairs of transversal points of $\overline{S}_{m,n}$. In particular this implies that the group $Q$ acts transitively on the set of maximal isotropic subspaces of $(\mathbb{C}^{n+m}, \overline{h})$ that are transversal to $v_\infty$.

The Levi factor $L$ in the Langland decomposition for $Q$ is not unique, but it depends on the choice of a point $v_0$ that is transversal to $v_\infty$. We chose, for simplicity, $v_0$ to be the maximal isotropic subspace $v_0 = \langle e_{n+i} \rangle \ 1 \leq i \leq m$, which is clearly transversal to $v_\infty$, and we denote by $L$ the stabilizer in $Q$ of $v_0$. It follows from the explicit expression for elements of $Q$ that

$$L = \left\{ \begin{bmatrix}
A & 0 & 0 \\
0 & C & 0 \\
0 & 0 & A^{-*}
\end{bmatrix} \bigg| \begin{array}{l}
A \in \text{GL}_m(\mathbb{C}), \\
C \in \text{U}(n-m) \\
det C \det A \det A^{-*} = 1
\end{array} \right\} = SU(n-m) \times \text{GL}_m(\mathbb{C})$$

$L$ is a reductive subgroup of $Q$ and it is a Levi factor for $Q$ being the intersection $Q \cap Q_0$ of two opposite parabolic subgroups, the stabilizers of two transversal points in $S_{m,n}$. Here $Q_0$ is the stabilizer of $v_0$.

Let us now consider the group

$$N = \left\{ \begin{bmatrix}
\text{Id} & F^* & E \\
0 & \text{Id} & F \\
0 & 0 & \text{Id}
\end{bmatrix} \bigg| E^* + E - F^* F = 0 \right\}$$

$N$ is a two step nilpotent subgroup that can be identified with the generalized Heisenberg group $H_{m,n} = M((n-m) \times m, \mathbb{C}) \ltimes \mathfrak{u}(m)$ where $\mathfrak{u}(m)$ denotes the group of antiHermitian matrices and the semidirect product structure is given by

$$(F_1, E_1) \cdot (F_2, E_2) = \left( F_1 + F_2, E_1 + E_2 + \frac{F_1^* F_2 - F_2^* F_1}{2} \right).$$
The subgroup $u(m)$ is the center of $H_{m,n}$, and the identification of $N$ with $H_{m,n}$ is defined by the formula
\[
\begin{bmatrix}
\text{Id} & F^* & E \\
0 & \text{Id} & F \\
0 & 0 & \text{Id}
\end{bmatrix}
\mapsto
\begin{pmatrix} F, \frac{E-E^*}{2} \end{pmatrix}.
\]
Since any element of $Q$ can be written as a product $nl$ where $n \in N$ and $l \in L$, the decomposition $Q = NL$ is a Langland decomposition for the maximal parabolic subgroup $Q$.

**Lemma 6.3.** The group $N$ acts simply transitively on the set of points in $\overline{S}_{m,n}$ that are transversal to $v_\infty$.

**Proof.** Let $x \in \overline{S}_{m,n}$ represent any isotropic $m$-dimensional subspace of $(\mathbb{C}^{m+n},\overline{h})$ which is transversal to $v_\infty$. Since $x$ is isotropic and transverse to $v_\infty$, it is also transverse to $v_\infty^\perp = (e_1, \ldots, e_n)$; indeed $v_\infty$ is the radical of $v_\infty^\perp$. This implies that $x$ admits a basis of the form $\begin{bmatrix} A \\ B \\ \text{Id} \end{bmatrix}$, where the requirement that $x$ is isotropic implies that $A^* + A - B^*B = 0$. Let us now consider the element $n \in N$ corresponding to the pair $(B, A + A^*B) \in H_{m,n}$. It follows from the very construction that $n \cdot v_0 = x$ and in particular $N$ acts transitively on $\overline{S}_{v_\infty}$. Since $N \cap Q_0 = \text{Id}$ we get that the action is simple. The thesis follows from the observation that $N$ preserves the set $\overline{S}_{v_\infty}$ since it is contained in the stabilizer of $v_\infty$. □

We just gave a way of identifying the abstract Heisenberg group $H_{m,n}$, the nilpotent radical $N$ of the stabilizer of the point $v_\infty$ and the open $N$-orbit of the point $v_0$. From now on we will denote the open $N$-orbit of the point $v_0$ with the symbol $H_{m,n}(v_\infty)$. It is worth remarking that $H_{m,n}(v_\infty)$ consists precisely of the intersection of $\overline{S}_{m,n}$ with the affine chart of the Grassmanian associated with the subspace $v_\infty^\perp$ and the latter chart realizes both $\overline{S}_{v_\infty}$ and $\overline{X}_{m,n}$ as unbounded subsets of $M(n \times m, \mathbb{C})$. We will not need this realization, that is a realization of $\overline{X}_{m,n}$ as a Siegel domain of second kind, but only the identification of $H_{m,n}(v_\infty)$ with the Heisenberg group $H_{m,n}$.

In the proof of Lemma 6.3 we gave a particularly nice section of the projection from an algebraic subset of the frame variety to $H_{m,n}(v_\infty)$: each point of $H_{m,n}(v_\infty)$ admits a basis of the form $\begin{bmatrix} A \\ B \\ \text{Id} \end{bmatrix}$, and with a slight abuse of notation we will often identify a point $x$ in $H_{m,n}(v_\infty)$ with the matrix representing its basis.

Our next goal is to understand the action of $Q$ on $H_{m,n}(v_\infty)$. We know that $Q = LN$, moreover an explicit formula for the action of $N$ on $H_{m,n}(v_\infty)$ is given by
\[
\begin{bmatrix}
\text{Id} & F^* & E \\
0 & \text{Id} & F \\
0 & 0 & \text{Id}
\end{bmatrix}
\begin{bmatrix} X \\ Y \\ \text{Id} \end{bmatrix} =
\begin{bmatrix} X + F^*Y + E \\ Y + F \\ \text{Id} \end{bmatrix}.
\]

Similarly an element $q \in Q$ corresponds to a pair $(C, A) \in S(U(n-m) \times \text{GL}_m(\mathbb{C}))$.
and the action is given by

\[
\begin{bmatrix}
    A & 0 & 0 \\
    0 & C & 0 \\
    0 & 0 & A^{-*}
\end{bmatrix}
\begin{bmatrix}
    X \\
    Y \\
    \text{Id}
\end{bmatrix}
= \begin{bmatrix}
    AX \\
    CY \\
    A^{-*}
\end{bmatrix}
\cong \begin{bmatrix}
    AXA^* \\
    CYA^* \\
    \text{Id}
\end{bmatrix}.
\]

Under the identification of \( H_{m,n}(v_\infty) \) with the \( N \) orbit of \( v_0 \), the action of \( L \) on \( H_{m,n}(v_\infty) \) corresponds to the action of \( L \) on \( N \) by conjugation: if \( l \) is an element of \( L \), for any \( n \) in \( N \), the element \( lnl^{-1} \) is the unique element in \( N \) such that \( lnl^{-1}v_0 = lnv_0 \) and indeed we have

\[
\begin{bmatrix}
    A & 0 & 0 \\
    0 & C & 0 \\
    0 & 0 & A^{-*}
\end{bmatrix}
\begin{bmatrix}
    \text{Id} - X^* \\
    0 \\
    0
\end{bmatrix}
\begin{bmatrix}
    AX \\
    CY \\
    A^{-*}
\end{bmatrix}
\cong \begin{bmatrix}
    AXA^* \\
    CYA^* \\
    \text{Id}
\end{bmatrix}.
\]

### Maximal triples

We now briefly turn to the study of maximal triples in \( S_{m,n} \). In this case the Bergmann cocycle can be explicitly computed, and it is shown in [DT87] that, given three points \( Z_1, Z_2, Z_3 \) in the bounded domain realization \( X_{m,n} \), and denoting by \( Y_i \) the matrix \( Y_i = \text{Id} - Z_i^*Z_{i+1} \), and by \( (y_i^k)_{k=1}^m \) the eigenvalues of \( Y_i \) counted with multiplicity, one gets

\[
\int_{\Delta(Z_1, Z_2, Z_3)} \omega = \pi \beta_X(Z_1, Z_2, Z_3) = -2 \sum_{i,k} \arg(y_i^k).
\]

For all \( i, k \), \( \arg(1 - y_i^k) \in [-\pi/2, \pi/2] \).

Since the Bergmann cocycle extends continuously to triples of pairwise transversal points in the Shilov boundary, we get that the same formula holds for transversal triples in the boundary. Let us now consider the isotropic subspaces

\[
\begin{align*}
    x_\infty &= \langle e_i - e_{m+i} \rangle, 1 \leq i \leq m \\
    x_0 &= \langle e_i + e_{m+i} \rangle, 1 \leq i \leq m \\
    x_1 &= \langle e_i + ie_{m+i} \rangle, 1 \leq i \leq m.
\end{align*}
\]

**Lemma 6.4.** The triple \((x_\infty, x_0, x_1)\) is maximal.

**Proof.** This is a direct computation. The matrix \( Z_1 \) representing \( x_\infty \) in the bounded domain realization is \( Z_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \), similarly \( x_0 = Z_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), and \( x_1 = Z_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). This implies that \( Y_1 = 2\text{Id} \), \( Y_2 = \text{Id} - i\text{Id} \), \( Y_3 = \text{Id} - i\text{Id} \). In particular \( \arg(y_i^k) = -\pi/4 \) if \( k = 2, 3 \) and \( 0 \) if \( k = 1 \). One can now easily compute that \( \beta_{S_{m,n}}(x_\infty, x_0, x_1) = m \) and hence the triple is maximal. \( \square \)

It is well known that the group \( SU(m, n) \) acts transitively on maximal triples. We check this and the fact that if the Bergmann cocycle is maximal then a triple is contained in an \( m \)-chain for the sake of completeness.
Lemma 6.5. Let \((Z_1, Z_2, Z_3)\) be a triple such that \(\beta_{S_{m,n}}(Z_1, Z_2, Z_3) = m\). There exists an \(m\)-chain containing \(Z_i\) for all \(i\).

Proof. Since we know that \(\text{SU}(m, n)\) is transitive on pairs of transversal point it is enough to justify the statement for maximal triples \((Z_1, Z_2, Z_3)\) such that \(Z_1 = x_\infty\) and \(Z_2 = x_0\). The third element \(Z_3\) can be written as \(Z_3 = [W \ Z]^{-1}\) where \(Z\) has \(n - m\) rows and \(m\) columns and \(W\) is a square \(m\)-dimensional matrix; since \(Z_3\) belongs to \(S_{m,n}\), we get \(W^*W + Z^*Z = \text{Id}\), hence the eigenvalues of \(W\) have absolute value smaller or equal to one and have all absolute value equal to one if and only if \(Z = 0\) that is equivalent to the fact that \(x\) belongs to the \(m\)-chain containing \(x_0\) and \(x_\infty\).

It is easy to check that \(Y_1 = 2\text{Id}, Y_2 = \text{Id} - W\) and \(Y_3 = \text{Id} + W^*\).

If \(w_j = \lambda_j e^{it_j}\) are the eigenvalues of \(W\), we get that

\[
\beta(x_\infty, x_0, x) = -2 \sum_{j=1}^{m} \left( \arctg \left( \frac{-\lambda_j \sin t_j}{1 - \lambda_j \cos t_j} \right) + \arctg \left( \frac{-\lambda_j \sin t_j}{1 + \lambda_j \cos t_j} \right) \right).
\]

Computing the derivative in \(t_j\) of the argument of the sum, one checks that the function attains its minimum in \(t_j = \pi/2\) where it has value \(2 \arctg(-\lambda_i)\). Clearly this is always greater that \(-\pi/2\) and is equal to \(-\pi/2\) precisely when \(\lambda_i\) equals one. This implies that, if the triple is maximal, the matrix \(Z\) must be zero, and hence \(x\) must belong to the \(m\)-chain through \(x_\infty\) and \(x_0\). In the case in which \(\lambda_j = 1\) we get that \(\beta(x_\infty, x_0, x)\) is maximal if and only if \(t_j \in [0, \pi]\) for all \(j\).

In order to show that \(\text{SU}(m, n)\) is transitive on maximal triples, it is easier to work in the Heisenberg model where we have easy formulae for the stabilizer of the pair \((v_\infty, v_0)\). Let us hence consider the images under \(F^*\) of the subspaces we are considering. It is easy to check that

\[
F^*x_\infty = v_\infty = (e_1, \ldots, e_m)
\]
\[
F^*x_0 = v_0 = (e_{n+1}, \ldots, e_{n+m})
\]
\[
F^*x_1 = v_1 = (e_1 + ie_{n+1}, \ldots, e_m + ie_{n+m})
\]

and to verify that the image under \(F^*\) of the subspace corresponding to the point \([W]\) is the isotropic subspace that is spanned by the columns of the matrix \([\text{Id} - W)(\text{Id} + W)^{-1} \ 0 \ \text{Id}]^T\).

Lemma 6.6. \(\text{SU}(m, n)\) is transitive on maximal triple.

Proof. Indeed if \(e^{it_j}\) are the eigenvalues of the matrix \(W\), then the eigenvalues of the matrix \((\text{Id} - W)(\text{Id} + W)^{-1}\) are \(\frac{1 - e^{it_j}}{1 + e^{it_j}} = \frac{-1 \sin t_j}{1 + \cos t_j}\). In particular the eigenvalues of \(W\) are imaginary numbers and are negative precisely when the arguments of the eigenvalues of \(W\) are in \([0, \pi]\). We know that the stabilizer \(L\) in \(\text{SU}(\mathbb{C}^{n+m}, \mathbb{H})\) of the pair \((v_0, v_\infty)\) is isomorphic to \(U(n - m) \times \text{GL}_m(\mathbb{C})\) and the image of the space \(v_1\) under the element \((C, A)\) is the subspace spanned by the columns of \([-iA^*A \ 0 \ \text{Id}]^T\). In particular every element \(x\) such that \(\beta(x_\infty, x_0, x)\) is maximal is in the orbit \(L \cdot v_1\), and this concludes the proof.
6.3 The projection $\pi_x$

Let us now consider the projection on the first factor $\pi : H_{m,n} \to M((n-m) \times m, \mathbb{C})$ that is obtained by quotienting the center $u(m)$ of $H_{m,n}$. Purpose of this section is to give a geometric interpretation of the quotient space $M((n-m) \times m, \mathbb{C})$: it corresponds to a parametrization of the space of chains through the point $v_\infty$. In order to make this more precise let us consider the set

$$W_{v_\infty} = \{ V \in \text{Gr}_{2m}(\mathbb{C}^{m+n}) | \ v_\infty < V \ \text{and} \ \overline{n}_V \ \text{has signature} \ (m,m) \}.$$ 

The following lemma gives an explicit identification of $W_{v_\infty}$ with the quotient space $M((n-m) \times m, \mathbb{C}) = H_{m,n}/u(m)$:

**Lemma 6.7.** There exists a bijection between $M((n-m) \times m, \mathbb{C})$ and $W_{v_\infty}$ defined by the formula

$$i : M((n-m) \times m, \mathbb{C}) \to W_{v_\infty}, \quad A \mapsto \begin{bmatrix} A^* & \text{Id} \\ 0 & 0 \end{bmatrix}.$$ 

**Proof.** Let $V$ be a point in $W_{v_\infty}$. Then $V^\perp$ is a $(n-m)$ dimensional subspace of $\mathbb{C}^{m+n}$ that is contained in $v_\infty^\perp$. This implies that $V^\perp$ admits a basis of the form $[A \ B \ 0]^T$ where $A$ has $m$ rows and $n-m$ columns and $B$ is a square $n-m$ dimensional matrix. Since the restriction of $\overline{n}$ on $V$ has signature $(m,m)$, the restriction of $\overline{n}$ to $V^\perp$ is negative definite, in particular the matrix $B$ must be invertible. This implies that, up to changing the basis of $V^\perp$, we can assume that $B = \text{Id}_{n-m}$. This gives the desired bijection. \hfill $\square$

$W_{v_\infty}$ parametrizes the $m$-chains containing the point $v_\infty$. We will call them *vertical* chains: the intersection $T^v_{v_\infty}$ of a vertical chain $T$ with the Heisenberg model $H_{m,n}(v_\infty)$ consists precisely of the fiber of the point associated to $T$ in Lemma 6.7 under the projection on the first factor in the Heisenberg model.

**Lemma 6.8.** Let $T \subset S_{m,n}$ be a vertical chain and $V$ be the associated linear subspace, then

1. for every $x$ in $T^v_{v_\infty}$, then $\pi(x) = i^{-1}(V) = p_T$,
2. $T^v_{v_\infty} = \pi^{-1}_1(p_T),$
3. the center $M$ of $N$ acts simply transitively on $T^v_{v_\infty}$.

**Proof.** (1) Let us denote by $p_T$ the matrix in $M((n-m) \times m, \mathbb{C})$ corresponding to $V$ under the isomorphism of Lemma 6.7. An element $w$ of $H_{m,n}(v_\infty)$ with basis $[X \ Y \ \text{Id}_m]^T$ belongs to the chain $T$ if and only if $Y = p_T$: indeed the requirement that $V^\perp$ is contained in $w^\perp$ restates as

$$0 = [X^* \ Y^* \ \text{Id}_m] \begin{bmatrix} 0 & 0 & \text{Id} \\ 0 & -\text{Id} & 0 \\ \text{Id} & 0 & 0 \end{bmatrix} \begin{bmatrix} p_T^* \\ \text{Id} \\ 0 \end{bmatrix} = p_T^* - Y^*.$$
This implies that for every \( w \in T \), we have \( \pi_1(w) = p_T \).

Viceversa if \( \pi_1(w) = p_T \), then \( w \) is contained in \( V \) and this proves (2).

(3) The fact that \( M \) acts simply transitively on \( T_{v_\infty} \) is now obvious: indeed \( M \) acts on the Heisenberg model by vertical translation stabilizing every vertical chain.

The stabilizer \( Q \) of \( v_\infty \) naturally acts on the space \( W_{v_\infty} \), and it is easy to deduce explicit formulae for the induced action on \( M((n - m) \times m, \mathbb{C}) \cong W_{v_\infty} \):

**Lemma 6.9.** The action of \( Q \) on \( M((n - m) \times m, \mathbb{C}) \) is described by

1. The pair \((C, A) \in SU(n - m) \times GL_m(\mathbb{C}) \cong L \) acts via
   \[(C, A) \cdot X = CXA^*.
   
2. The pair \((Z, W) \in M((n - m) \times m, \mathbb{C}) \rtimes u(m) \cong N \) acts via
   \[(Z, W) \cdot X = X + Z.

In the sequel, where this will not cause confusion, we will identify \( W_{v_\infty} \) with \( M((n - m) \times m, \mathbb{C}) \) considering implicit the map \( i^{-1} \) and we will denote by \( \pi_{v_\infty} \) the projection \( \pi_1 : H_{m,n}(v_\infty) \to M((n - m) \times m, \mathbb{C}) \). This emphasizes the role of the point at infinity \( v_\infty \) and will be useful to avoid confusions in Chapter 7 where we will need to consider at the same time Heisenberg models for two different Shilov boundaries, the image and the target of a measurable boundary map \( \phi \).

It is worth remarking that everything we did so far does not really depend on the choice of the point \( v_\infty \), and a map \( \pi_x : \mathbb{F}_{m,n}^x \to W_x \) can be defined for every point \( x \in \mathbb{F}_{m,n} \). We decided to stick to the point \( v_\infty \), since the formulae in the explicit expressions are easier.

### 6.4 Projections of chains

We now want to understand what are the possible images under the map \( \pi_{v_\infty} \) of other chains. We define the *intersection index* of an \( m \)-chain \( T \) with a point \( x \in \mathbb{F}_{m,n} \) by

\[ i_x(T) = \dim(x \cap V_T) \]

where \( V_T \) is the \( 2m \) dimensional linear subspace of \( \mathbb{C}^{m+n} \) associated to \( T \). Clearly \( 0 \leq i_{v_\infty}(T) \leq m \), and \( i_{v_\infty}(T) = m \) if and only if the chain \( T \) is vertical. In general we will call \( k \)-vertical a chain whose intersection index is \( k \); with this notation vertical chains are \( m \)-vertical. Sometimes we will call *horizontal* the chains that are \( 0 \)-vertical (in particular they intersect \( v_\infty \) just in the zero vector).

In our investigations it will be precious to be able to relate different situations via the action of the group \( \mathcal{G} = SU(\mathbb{C}^{m+n}, S) \), under this respect the following lemma will be fundamental:

**Lemma 6.10.** For every \( k \in \{0, \ldots, m\} \) the group \( \mathcal{G} \) acts transitively on
1. the pairs \((x, T)\) where \(x \in \mathcal{S}_{m,n}\) is a point and \(T\) is an \(m\)-chain with \(i_x(T) = k\),

2. the triples \((x, y, T)\) where \(x \cap y, y \in T\) and \(i_x(T) = k\).

In particular the intersection index is a complete invariant of \(m\)-chains up to the \(Q\)-action.

Proof. We will prove directly the second statement, that clearly implies also the first one. Since \(Q\) acts transitively on the Shilov boundary, we can assume that \(x = v_\infty\), hence it is enough to show that \(Q\) is transitive on \(k\)-vertical chains, the \(m\)-chains with intersection index \(k\) with \(v_\infty\).

Since the intersection index \(i_{v_\infty}(T)\) equals to \(k\), the span \(W = \langle v_\infty, T \rangle\) has dimension \(3m - k\), moreover since \(W\) contains \(V_T\) that is a subspace on which \(h\) has signature \((m, m)\), the restriction of \(h\) to \(W\) has signature \((m, 2m - k)\).

Since \(Q\) is transitive on \(3m - k\) subspaces of \(\mathbb{C}^{m+n}\) containing \(v_\infty\) and on which the restriction of \(h\) has signature \((m, 2m - k)\), we can reduce to the case \(W = \langle e_1, \ldots, e_{2m-k}, e_{n+1}, \ldots, e_{n+m} \rangle\) and, even more, we can assume that \(n = 2m - k\).

Since \(Q\) is transitive on points transversal to \(v_\infty\) (Lemma 6.2), we can assume that the \(m\)-chain \(T\) contains the point \(v_0\).

In this case the orthogonal \(V_T^\perp\) of \(V_T\) is a \(m - k\) dimensional space on which \(\mathbb{I}\) is definite negative. A point \(Z\) in the frame manifold which has the property that \(jZ = V_T^\perp\) is represented by a matrix with \(3m - k\) rows and \(m - k\) columns that has a block structure \([Z_1 \ Z_2 \ Z_3]^{T}\) where \(Z_1\) and \(Z_3\) are matrices with \(m\) rows and \(Z_2\) is a square \((m - k)\)-dimensional matrix.

Since, by our assumption, the point \(v_0\) belongs to \(T\), the vector space \(V_T^\perp\) is contained in \(v_0^\perp\), hence \(Z_1 = 0\). Together with the fact that the restriction of \(\mathbb{I}\) to \(z\) must be negative definite, this implies that we can assume (up to changing the representative for \(V_T^\perp\)) that \(Z\) has the form \(\begin{bmatrix} \text{Id} & 0 \\ 0 & Z_3 \end{bmatrix}\) for some matrix \(Z_3\).

We claim that the hypothesis that the intersection index of \(T\) and \(v_\infty\) is \(k\) is equivalent to the fact that the rank of \(Z_3\) is \(m - k\). Indeed the intersection \(v_\infty \cap V\) is \(v_\infty \cap z^\perp\) hence corresponds, in the basis \(\langle e_1, \ldots, e_m \rangle\), to the kernel of

\[
\begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} \begin{bmatrix} \text{Id} \\ \text{Id} \end{bmatrix} = Z_3.
\]

This implies that the group \(L = \text{stab}(v_\infty, v_0)\) acts transitively on the set of \(m\)-chains through \(v_0\) with intersection index \(k\), since \(GL_m(\mathbb{C}) \subset L\) is transitive on matrices with \(m\) rows, \(m - k\) columns and rank \(m - k\).

As explained at the beginning of the chapter we want to give a parametrization of a generic chain \(T\) and study the restriction of \(\pi_{v_\infty}\) to \(T\). In view of Lemma 6.10, it is enough to understand, for every \(k\), the parametrization and the projection of a single \(k\)-vertical chain. The \(k\)-vertical chain we will deal with is the chain with associated linear subspace

\[
T_k = \langle e_i, e_j + e_{m+j} + e_{n+j}, v_0 \mid 1 \leq i \leq k < j \leq m \rangle.
\]
Lemma 6.11. \( T_k \) is the linear subspace associated to a \( k \)-vertical chain.

Proof. \( T_k \) is a \( 2m \)-dimensional subspace containing \( v_0 \). Moreover \( T_k \) splits as the orthogonal direct sum

\[
T_k = T_k^0 \oplus T_k^1 = \langle e_i, e_{n+i} | 1 \leq i \leq k \rangle \oplus \langle e_j + e_{m+j-k} + e_{n+j}, e_{n+j} \rangle \quad 1 \leq j \leq m \rangle = \langle v_\infty \cap T_k, e_{n+i} | 1 \leq i \leq k \rangle \oplus \langle e_j + e_{m+j-k} + e_{n+j}, e_{n+j} \rangle \quad k+1 \leq j \leq m \rangle.
\]

Since \( v_\infty \cap T_k = \langle e_1, \ldots, e_k \rangle \), we get that \( i_{v_\infty}(T_k) \) is \( k \). Since \( \overline{h}|_{T_k^0} \) has signature \((k, k)\) and \( \overline{h}|_{T_k^1} \) has signature \((m - k, m - k)\), we get that the restriction of \( \overline{h} \) on \( T_k \) has signature \((m, m)\) and this concludes the proof.

\[\square\]

Lemma 6.12. \( \mathcal{H}_{m,n}(v_\infty) \cap T_k \) consists precisely of those subspaces of \( \mathbb{C}^{m+n} \) that admit a basis of the form

\[
\begin{bmatrix}
E^*E & E^* X \\
E & \text{Id} + X
\end{bmatrix}^k_{m-k}
\]

The projection of \( T_k \) is contained in an affine subspace of \( M((n-m) \times m, \mathbb{C}) \) of dimension \( m^2 - km \), and consists of the points of \( M((n-m) \times m, \mathbb{C}) \) that have expression \( \begin{bmatrix} E \text{Id} + X \\ 0 \end{bmatrix} \) with \( E \in M((m-k) \times k, \mathbb{C}) \) and \( X \in U(m-k) \).

Proof. It is enough to check that the orthogonal to \( T_k \) is

\[
T_k^\perp = \langle e_{m+j} + e_{n+j+k}, e_{m+i} | 1 \leq j \leq m - k < l \leq n - m \rangle.
\]

This implies that any \( m \)-dimensional subspace \( z \) of \( T_k \), that is transversal to \( v_\infty \), has a basis of the form

\[
z = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}^k_{m-k}
\]

and the restriction of \( \overline{h} \) to \( z \) is zero if and only if

\[
\begin{cases}
Z_{11}^* + Z_{12} = Z_{21}^* Z_{22} \\
Z_{21}^* + Z_{22} = Z_{21}^* Z_{22} \\
Z_{12}^* + Z_{21} = Z_{22}^* Z_{21}
\end{cases}
\]

Equation (22) restates as \( Z_{22} = \text{Id} + X \) for some \( X \in U(m-k) \): indeed a square matrix \( Z \) satisfies the equation \( Z^* + Z = Z^* Z \), if and only if the equation \((Z - \text{Id})^*(Z - \text{Id}) = Z^* Z - Z + Z^* + \text{Id} = \text{Id} \) holds, which means that \( Z = \text{Id} \) belongs to \( U(m-k) \).

This concludes the proof of the first part of the lemma: the \((m-k) \times k \) matrix \( Z_{21} \) can be chosen arbitrarily; Equation (12) uniquely determines \( Z_{12} \) in function of
6.4. PROJECTIONS OF CHAINS

$Z_{21}$ and $Z_{22}$, and Equation (11) determines the Hermitian part of $Z_{11}$ in function of $Z_{21}$, but is satisfied independently on the antiHermitian part of $Z_{11}$. This proves the first part of the lemma.

The second part is a direct consequence of the parametrization of $T^n_\infty$ we just gave, together with the identification of $W_{v_\infty}$ and $M((n-m) \times m, \mathbb{C})$ given in Lemma 6.7.

We will call a subset of $W_x$ that is projection of a $k$-vertical chain a $(m, k)$-circle. The reason for the name circle is due to the fact that, in the case $(m, n) = (1, 2)$ the projections of horizontal chains are Euclidean circles in $\mathbb{C}$. This fact was first observed and used by Cartan in [Car32]. In the general case it is important to record both the dimension of the $m$-chain that is projected and the dimension of the $U(m-k)$ factor in the projection. This explains the notation. We will call generalized circle any subset of $M((n-m) \times m, \mathbb{C})$ arising as a projection of an $m$-chain. In particular a generalized circle is an $(m, k)$-circle for some $k$.

The ultimate goal of this chapter is to understand the possible lifts of a given $(m, k)$-circle. We begin by analyzing the stabilizers in $SU(\mathbb{C}^{m+n}, \hat{h})$ of some configurations:

**Lemma 6.13.** The stabilizer in $SU(\mathbb{C}^{m+n}, \hat{h})$ of the triple $(v_\infty, v_0, T_k)$ is the subgroup of $S_0$ of $L \cong S(U(n-m) \times GL_m(\mathbb{C}))$ consisting of pairs of the form

$$
\begin{pmatrix}
C_{11} & 0 \\
0 & C_{22}
\end{pmatrix}
\begin{pmatrix}
Y & X \\
0 & C_{11}
\end{pmatrix}
\begin{array}{l}
C_{11} \in U(m-k) \\
C_{22} \in U(n-2m+k) \\
X \in M(k \times (m-k), \mathbb{C}) \\
Y \in GL_k(\mathbb{C}).
\end{array}
$$

**Proof.** We determined in Section 6.2 that the stabilizer $L$ in $SU(\mathbb{C}^{m+n}, \hat{h})$ of the pair $(v_\infty, v_0)$ is isomorphic to $SU(n-m) \times GL_m(\mathbb{C})$. The stabilizer of the triple $(v_\infty, v_0, T_k)$ is clearly contained in $L$ and consists precisely of the elements of $L$ stabilizing $T_k^\perp$.

In the proof of Lemma 6.12 we saw that the subspace $T_k^\perp$ has a basis of the form $\begin{pmatrix}0 & 0 \\ X & -\end{pmatrix}$ where $X$ denotes the $m \times (n-m)$ matrix $\begin{pmatrix}0 & 0 \\ \Id_{m-k} & 0 \end{pmatrix}$.

From the explicit expression of elements in $L$ we get

$$
\begin{pmatrix}
A & C \\
C & A^{-*}
\end{pmatrix}
\begin{pmatrix}
0 \\
\Id_X
\end{pmatrix}
= 
\begin{pmatrix}
0 & C \\
\Id & A^{-*}X
\end{pmatrix}
\cong 
\begin{pmatrix}
0 & C \\
\Id & A^{-*}XC^{-1}
\end{pmatrix}.
$$

In turn the requirement that $A^{-*}XC^{-1} = X$, that is $X = A^*XC$, implies, in the suitable block decomposition for the matrices, that

$$
\begin{pmatrix}
A_{11}^* & A_{21}^* \\
A_{12}^* & A_{22}^*
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
\Id_{m-k} & 0
\end{pmatrix}
\begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}
= 
\begin{pmatrix}
A_{21}C_{11} & A_{21}C_{12} \\
A_{22}C_{11} & A_{22}C_{12}
\end{pmatrix}
.$$
This implies that $A_{12} = C_{11}^{-1}$ and $C_{12} = A_{21} = 0$. Moreover since $C$ is unitary, also $C_{21}$ must be 0, and both $C_{22}$ and $C_{11}$ must be unitary. This concludes the proof.

Let us now denote by $o$ the point $o = \pi_{v_0}(v_0) = 0$ in $W_{v_0}$ and by $C_k$ the $(m, k)$-circle that is the projection of $T_k$. We will denote by $S_1$ the stabilizer in $Q$ of the pair $(o, C_k)$.

**Lemma 6.14.** The stabilizer of the pair $(o, C_k)$ is the group

$$ S_1 = \text{Stab}_Q(o, C_k) = M \rtimes S_0 $$

where, as above, we denote by $M$ the center of the nilpotent radical $N$ of $Q$ and by $S_0$ the stabilizer in $Q$ of the pair $(v_0, T_k)$.

**Proof.** Recall that any element in $Q$ can be uniquely written as a product $nl$ where $n$ is in $N$, and $l$ belongs to $L$, the Levi component of $Q$. Since any element in $S_1$ fixes, by assumption, the point $o = \pi_{v_0}(v_0)$ and since any element in $L$ fixes $o$, if $nl$ is in $S_1$ then $n(o) = o$ that, in turn, implies that $n$ belongs to $M$. Hence $S_1$ is of the form $M \rtimes S$ for some subgroup $S$ of $L$.

Let now $X$ be a point in $W_{v_0} = M((n - m) \times m, \mathbb{C})$. The action of $(C, A) \in L$ on $W_x$ is $X \mapsto CXA^*$ (cfr. Lemma 6.9). We want to show that if $C_k$ is preserved then $(C, A)$ must belong to $S_0$. We have proven in Lemma 6.12 that any point $z \in \pi_{v_0}(T_k)$ can be written as $\begin{bmatrix} E & \text{Id} + Z \\ 0 & 0 \end{bmatrix}$ for some matrices $E \in M((m - k) \times k, \mathbb{C})$ and $Z \in U(m - k)$. Explicit computations give that

$$ \begin{bmatrix} E & \text{Id} + Z \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} E & \text{Id} + Z \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix}. $$

Since $A$ is invertible, $E$ is arbitrary and both $\text{Id}$ and $-\text{Id}$ are in $U(m - k)$, the matrix $C_{21}$ must be zero. Hence $C$ must have the same block form of a genuine element of $S_0$. In particular $C_{11}$ is invertible. Since $C_{11}(EA_{21}^* + (\text{Id} + Z)A_{22}^*)$ must be an element of $\text{Id} + U(m - k)$ for every $E$, we get that $A_{21}^*$ must be zero.

The result now follows from Claim 6.15 below.

**Claim 6.15.** Let $C \in U(l)$ and $A \in \text{GL}_d(\mathbb{C})$ be matrices and let $\mathcal{U}$ denote the set

$$ \mathcal{U} = \{ \text{Id} + X \mid X \in U(l) \} \subset M(l \times l, \mathbb{C}). $$

If $CUA^* = \mathcal{U}$ then $A = C$.

**Proof.** Let us consider the birational map

$$ i : M(l \times l, \mathbb{C}) \to M(l \times l, \mathbb{C}) $$

$$ X \mapsto X^{-1} $$

that is defined on a Zariski open subset $\mathcal{O}$ of $M(l \times l, \mathbb{C})$. 


The image, under the involution \( i \), of \( \mathcal{U} \) is the set
\[
\mathcal{L} = \{ W \mid \text{Id} - W^* - W = 0 \} = \frac{1}{2} \text{Id} + u(l).
\]

Moreover \( i(CXA^*) = A^{-*}i(X)C^{-1} \), hence in order to show that the subgroup preserving \( \mathcal{U} \) consists precisely of the pairs \((A, A)\), it is enough to check that the subgroup of \( U(l) \times GL_l(\mathbb{C}) \) preserving \( \mathcal{L} \) consists precisely of the pairs \((A, A)\) with \( A \in U(l) \).

This last statement amounts to show that the only matrix \( B \in GL_l(\mathbb{C}) \) such that \( \text{Id} - W^*B^* - BW = 0 \) for all \( W \in \mathcal{L} \) is the identity itself. Choosing \( W \) to be \( \frac{1}{2} \text{Id} \) we get that \( B^* + B = 2\text{Id} \) hence in particular \( B = \text{Id} + Z \) with \( Z \in u(l) \). Since moreover \( \mathcal{L} = \{ \frac{1}{2} \text{Id} + M \mid M \in u(l) \} \) we have to show that if \( ZM + M^*Z^* = ZM + MZ = 0 \) for all \( M \) in \( u(l) \) then \( Z \) must be zero, and this can be easily seen, for example by choosing \( M \) to be the matrix that is zero everywhere apart from the \( l \)-th diagonal entry where it is equal to \( i \).

We now have all the ingredients we need to show the following proposition:

**Proposition 6.16.** Let \( x \in \mathfrak{S}_{m,n} \) be a point, \( T \) be a chain with \( i_x(T) = k \), \( t \in T \) be a point, \( y = \pi_x(t) \in W_x \). Then

1. \( T \) is the unique lift of the \((m, k)\)-circle \( \pi_x(T) \) through the point \( t \),
2. for any point \( t_1 \) in \( T_{x,t} = \pi_{x}^{-1}(y) \) there exists a unique \( m \) chain through \( t_1 \) which lifts of \( \pi_x(T) \).

**Proof.** (1) As a consequence of Lemma 6.10, in order to prove the statements, we can assume that the triple \((x, t, T)\) is the triple \((v_\infty, v_0, T_k)\). Let \( T' \) be another \( m \)-chain containing the point \( t \) that lifts the \((m, k)\)-circle \( C_k \), a consequence of Lemma 6.10 is that there exists an element \( g \in L \) such that \((v_\infty, v_0, T_k) = g(v_\infty, v_0, T') \). Moreover, since \( \pi_{v_\infty}(T') = C_k \), we get that \( g \in S_1 \). But we know that \( S_1 \cap L = S_0 \) and this proves that \( T' = T_k \).

(2) This is a consequence of the first part, together with the transitivity of \( M \) on the vertical chain \( T_{v_0,v_\infty} \). 

We now want to determine what are the lifts of a point \( y \) that are contained in an \( m \)-chain \( T \). We will assume that the point \( x \) is \( v_\infty \): since the group action is transitive on \( \mathfrak{S}_{m,n} \), this is not a real restriction, but is convenient since we gave explicit formulæ only for the stabilizer of \( v_\infty \). Let us fix an \( m \)-chain \( T \) and consider the subgroup
\[
M_T = \text{Stab}_M(T).
\]

Here, as usual, \( M \) denotes the center of \( N \), the nilpotent radical of the stabilizer \( Q \) of \( v_\infty \). Clearly if \( t \) is a lift of a point \( y \in W_{v_\infty} \) that is contained in \( T \), then all the orbit \( M_T \cdot t \) consists of lifts of \( y \) that are contained in \( T \). We want to show that also the other containment holds, namely that the lifts are precisely the \( M_T \) orbit of any point.
We first describe the group $M_T$ in the case in which $T = T_k$, and in this case we denote, for the sake of brevity, by $M_k$ the group $M_{T_k}$. We also define the subgroup of $u(m)$

$$E_k = \{ X \in u(m) | X_{ij} = 0 \text{ if } i > k \text{ or } j > k \} = \left\{ \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} | X_1 \in u(k) \right\},$$

and we consider the identification $\alpha : M \to u(m)$ described in Chapter 6.2.

**Lemma 6.17.** In the notations above, we have $\alpha(M_k) = E_k$.

**Proof.** We already observed that the orthogonal to $T_k$ is

$$T^\perp_k = \langle e_{m+j} + e_{n+j+k}, e_{l+m} \rangle_{1 \leq j \leq m-k < l \leq n-m}.$$ 

Moreover an element of $M$ stabilizes $T_k$ if and only if it stabilizes $T^\perp_k$. If now $m = \begin{bmatrix} \text{id} & 0 & E \\ 0 & \text{id} & 0 \\ 0 & 0 & \text{id} \end{bmatrix}$ is an element of $M$, then the image $m \cdot (e_{m+j} + e_{n+j+k}) = \sum E_{ij} e_{j+k} + e_{m+j} + e_{n+j+k}$ that belongs to $T^\perp_k$ if and only if the $(j+k)$-th column of the matrix $E$ is zero. This implies that the image under $\alpha$ of the subgroup of $M$ that fixes $T^\perp_k$ is contained in $E_k$. Viceversa if $V_k$ is the linear subspace corresponding to $T_k$ it is easy to check that $\alpha^{-1}(E_k)$ belongs to $\text{SU}(V_k)$, in particular it preserves $T_k$. $\Box$

Another invariant of a $k$-vertical chain that will be relevant in the next section is the intersection of the linear subspace $V_T$ underlying $T$ with $v_\infty$. We will denote it by $Z_T$:

$$Z_T = v_\infty \cap V_T.$$ 

In the standard case in which $T = T_k$ we will denote by $Z^k$ the subspace $Z_{T_k}$ which equals to the span of the first $k$ vectors of the standard basis of $\mathbb{C}^m$.

**Proposition 6.18.** Let $T$ be a $k$-vertical chain, then

1. If $g \in Q$ is such that $gT = T_k$, then $M_T = g^{-1}M_kg$.
2. For any point $x \in T$, we have $\pi_{v_\infty}^{-1}(\pi_{v_\infty}(x)) \cap T = M_T x$.
3. If $n \in N$, then $M_{nT} = M_T$.
4. If $a \in \text{GL}(m)$ is such that $a(Z_T) = Z^k$, then $\alpha(M_T) = a^{-1}E_k a^{-*}$.

**Proof.** (1) This follows from the definition of $M_k$ and $M_T$ and the fact that $M$ is normal in $Q$.

(2) Let us first consider the case $T = T_k$. In this case the statement is an easy consequence of the explicit parametrization of the chain $T_k$ we gave in Lemma 6.12: any two points in $T_k$ that have the same projection are in the same $M_k$ orbit. The general case is a consequence of the transitivity of $Q$ on $k$-vertical chains: let $g \in Q$
be such that \(gT = T_k\) and let us denote by \(y\) the point \(gx\). Then we know that 
\[
M_k y = \pi_{v_\infty}^{-1}(\pi_{v_\infty}(y)) \cap T_k.
\]
This implies that 
\[
M_T x = g^{-1}M_k gx = g^{-1}(M_k y) = g^{-1}(\pi_{v_\infty}^{-1}(\pi_{v_\infty}(y)) \cap T_k) = \\
g^{-1}\pi_{v_\infty}^{-1}(\pi_{v_\infty}(y)) \cap g^{-1}T_k = \\
\pi_{v_\infty}^{-1}(\pi_{v_\infty}(x)) \cap T.
\]
Where in the last equality we used that the \(Q\) action on \(H_{m,n}(v_\infty)\) induces an action of \(Q\) on \(W_{v_\infty}\) so that the projection \(\pi_{v_\infty}\) is \(Q\) equivariant.

(3) This is a consequence of the fact that \(M\) is in the center of \(N\): 
\[
M_n T = nM T n^{-1} = M_T.
\]
(4) By (3) we can assume that \(T\) is a chain through the point \(v_0\); indeed there exists always an element \(n \in N\) such that \(nT\) contains \(v_0\), moreover both \(M_n T = M_T\) and \(Z_n T = Z_T\) (the second assertion follows from the fact that any element in \(N\) acts trivially on \(v_\infty\)).

Since \(v_0 \in T\) and we proved in Lemma 6.10 that \(L\) is transitive on \(k\)-vertical chains through \(v_0\), we get that there exists a pair \((C, A) \in U(n-m) \times \text{GL}_m(\mathbb{C})\) such that, denoting by \(g\) the corresponding element in \(L\), we have \(gT = T_k\). It follows from (1) that \(M_T = g^{-1}M_k g\), in particular we have

\[
\begin{bmatrix}
A^{-1} & 0 & 0 \\
0 & C^{-1} & 0 \\
0 & 0 & A^*
\end{bmatrix}
\begin{bmatrix}
\text{Id} & 0 & E \\
0 & \text{Id} & 0 \\
0 & 0 & \text{Id}
\end{bmatrix}
\begin{bmatrix}
A & 0 & 0 \\
0 & C & 0 \\
0 & 0 & A^-
\end{bmatrix}
= 
\begin{bmatrix}
\text{Id} & 0 & A^{-1}EA^- \\
0 & \text{Id} & 0 \\
0 & 0 & \text{Id}
\end{bmatrix}
\]

and hence the subgroup \(\alpha(M_T)\) is the group \(A^{-1}E_k A^-\). Moreover, since \(gT = T_k\) we have in particular that \(gZ_T = Z^k\) and hence \(A(Z_T) = Z^k\) if we consider \(Z_T\) as a subspace of \(v_\infty\).

In order to conclude the proof it is enough to check that for every \(a \in \text{GL}_m(\mathbb{C})\) with \(a(Z_T) = Z^k\) the subgroups \(a^{-1}E_k a^-\) coincide. Indeed it is enough to check that for every element \(a \in \text{GL}_m(\mathbb{C})\) such that \(a(Z^k) = Z^k\) then \(a^{-1}E_k a^- = E_k\). But if \(a\) satisfies this hypothesis, the matrix \(a^-\) has the form \(\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}\). In particular we can compute:

\[
a^{-1}Xa^- = \begin{bmatrix} A_1 & A_2^* \\ 0 & A_3 \end{bmatrix} X_1 \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_1^*X_1A_1 & 0 \\ 0 & 0 \end{bmatrix}
\]

and the latter matrix still belongs to \(E_k\). \(\square\)
Chapter 7

The restriction to a chain is rational

In this chapter we prove that the chain geometry defined in Chapter 6 is rigid in the following sense:

Theorem 7.1. Let $\phi : \partial \mathbb{H}_C^p \to S_{m,n}$ be a measurable, chain geometry preserving, Zariski dense map. For almost every chain $C$ in $\partial \mathbb{H}_C^p$, the restriction $\phi|_C$ coincides almost everywhere with a rational map.

Let us recall that, whenever a point $x \in S_{m,n}$ is fixed, the center $M_x$ of the nilpotent radical $N_x$ of the stabilizer $Q_x$ of $x$ in $\text{SU}(\mathbb{C}^m+n, \mathbb{T})$ acts on the Heisenberg model $\mathcal{H}_{m,n}(x)$ by vertical translations. Moreover, for every $m$-chain $T$ containing the point $x$, the $M_x$ action is simply transitive on the Zariski open subset $T^x$ of $T$. The picture above is true for both $\partial \mathbb{H}_C^p \cong S_{1,p}$ and $S_{m,n}$ where, if $x \in \partial \mathbb{H}_C^p$, the group $M_x$ can be identified with $u(1)$, and, if $\phi : \partial \mathbb{H}_C^p \to S_{m,n}$ is the boundary map, $M_{\phi(x)} \cong u(m)$.

The idea of the proof is the same as in Proposition 1.16: we show that, for almost every point $x \in \partial \mathbb{H}_C^p$, the boundary map is equivariant with respect to a measurable homomorphism $h : M_x \to M_{\phi(x)}$. Since such homomorphism must be algebraic, we get that the restriction of $\phi$ to almost every chain through $x$ must be algebraic.

In order to define the homomorphism $h$ we will prove first that a map $\phi$ satisfying our assumptions induces a measurable map $\phi_x : W_x \to W_{\phi(x)}$. Here $W_x$ can be identified with $\mathbb{C}^{p-1}$ and $W_{\phi(x)}$ can be identified with $M((n-m) \times m, \mathbb{C})$. Both these identifications are not canonical but we fix them once and for all. We will then use the map $\phi_x$ to define a cocycle $\alpha : M_x \times (\partial \mathbb{H}_C^p)^x \to M_{\phi(x)}$ with respect to which $\phi$ is equivariant. We will then show that $\alpha$ is independent on the point $x$, at least when restricted to vertical chains and hence coincides almost everywhere with the desired homomorphism.
CHAPTER 7. THE RESTRICTION TO A CHAIN IS RATIONAL

7.1 First properties of chain preserving maps

Recall from Chapter 4 that a map $\phi$ is Zariski dense if the essential Zariski closure of $\phi(\partial H^p_C)$ is the whole $S_{m,n}$, or, equivalently, if the preimage under $\phi$ of any proper Zariski closed subset of $S_{m,n}$ is not of full measure. Moreover, by definition, a measurable boundary map preserves the chain geometry if the image under $\phi$ of almost every pair of distinct points is a pair of transversal points, and the image of almost every maximal triple $(x_0, x_1, x_2)$ in $(\partial H^p_C)^3$, is a maximal triple $(\phi(x_0), \phi(x_1), \phi(x_2))$, hence in particular it is contained in an $m$-chain.

We will denote by $T_1$ the set of chains in $\partial H^p_C$, and by $T_m$ the set of $m$-chains of $S_{m,n}$. The set $T_1$ is a smooth manifold, indeed an open subset of the Grassmannian $\text{Gr}_2(C^p+1)$. We will endow $T_1$ with its Lebesgue measure class.

The following lemma, an application in this context of Fubini’s theorem, gives the first property of a chain geometry preserving map:

Lemma 7.2. Let $\phi : \partial H^p_C \to S_{m,n}$ be a chain geometry preserving map. For almost every chain $C \in T_1$ there exists an $m$-chain $\hat{\phi}(C) \in T_m$ such that, for almost every point $x$ in $C$, $\phi(x) \in \hat{\phi}(C)$.

Proof. There is a bijection between the set $(\partial H^p_C)^{(3)}$ consisting of triples of distinct points on a chain and the set

$$T_1^{(3)} = \{(C, x, y, z) \mid C \in T_1 \text{ and } (x, y, z) \in C^{(3)}\}.$$

In turn the projection onto the first factor endows the manifold $T_1^{(3)}$ with the structure of a smooth bundle over $T_1$. In particular Fubini’s theorem implies that, for almost every chain $C \in T_1$ and for almost every triple $(x, y, z) \in C^3$, the triple $(\phi(x), \phi(y), \phi(z))$ belongs to the same $m$-chain $\hat{\phi}(C)$. Moreover $\hat{\phi}(C)$ has the desired properties again as a consequence of Fubini theorem.

Lemma 13.2 can be reformulated in the following way

Corollary 7.3. Let $\phi : \partial H^p_C \to S_{m,n}$ be a measurable chain preserving map. Then there exists a measurable map $\hat{\phi} : T_1 \to T_m$ such that, for almost every pair $(x, T) \in \partial H^p_C \times T_1$ with $x \in T$, then $\phi(x) \in \hat{\phi}(T)$.

Proof. The only issue left to check is that the map $\hat{\phi}$ is measurable, but this follows from the fact that the map associating to a pair $(x, y) \in S_{m,n}^{(2)}$ the $m$-chain $T_{xy}$ is algebraic.

Recall that, if $x$ is a point in $\partial H^p_C$, we denote by $W_x$ the set of chains through $x$. We use the identification of $W_x$ as subvariety of $\text{Gr}_2(C^{1-p})$ to endow the space $W_x$ with its Lebesgue measure class.

Corollary 7.4. For almost every $x$ in $\partial H^p_C$, almost every chain in $W_x$ satisfies Lemma 7.2.
Proof. It is again an application of Fubini’s theorem. Let us indeed consider the manifold $T^{(1)} = \{(C,x) | C \in \mathcal{T}, x \in C\}$ the projection on the first two factor $T^{(3)} \to T^{(1)}$ realizes the first manifold as a smooth bundle over the second with fiber $(\mathbb{R} \times \mathbb{R}) \setminus \Delta$. In particular for almost every pair in $T^{(1)}$ the chain satisfies the assumption of Corollary 7.3. Since $T^{(1)}$ is a bundle over $\partial \mathbb{H}^p$ with fiber $W_x$ over $x$, the statement follows applying Fubini again.

We will call a point $x$ generic for the map $\phi$ if it satisfies the hypotheses of Corollary 7.4 and with the property that, for almost every pair $(z,w)$ such that $(x,y,z)$ is maximal, also the triple $(\phi(x),\phi(y),\phi(z))$ is maximal. Let us now fix a point $x$ that is generic for the map $\phi$ and consider the diagram

$$
\begin{array}{ccc}
\mathcal{H}_{1,p}(x) & \xrightarrow{\phi} & \mathcal{H}_{m,n}(\phi(x)) \\
\downarrow \pi_x & \quad & \downarrow \pi_{\phi(x)} \\
W_x & \xrightarrow{\phi_x} & W_{\phi(x)}.
\end{array}
$$

Lemma 7.5. If $x$ is generic for $\phi$, there exists a measurable map $\phi_x$ such that the diagram commutes almost everywhere. Moreover $\phi_x$ induces a measurable map $\hat{\phi}_x$ from the set of circles of $W_x$ to the set of generalized circles of $W_{\phi(x)}$ such that, for almost every chain $T$, we have that $\hat{\phi}(T)$ is a lift of $\hat{\phi}_x(\pi_x(T))$.

Proof. The fact that a map $\phi_x$ exists making the diagram commutative on a full measure set is a direct application of Corollary 7.4.

Since the set of horizontal chains in $\partial \mathbb{H}^p$ is a smooth bundle over the set of Euclidean circles in $W_x \cong \mathbb{C}^{p-1}$, we have that, for almost every Euclidean circle $C$, the map $\hat{\phi}$ is defined on almost every chain $T$ with $\pi_x(T) = C$. Moreover a Fubini-type argument implies that, for almost every circle $C$, the diagram commutes when restricted to the preimage of $C$.

This implies that the projections $\hat{\phi}_x(C) := \pi_{\phi(x)}(\hat{\phi}(T))$ coincide for almost every lift $T_i$ of $C$ if $C$ satisfies the hypotheses of the previous paragraph, and this concludes the proof.

7.2 A measurable cocycle

Recall that if $H,K$ are topological groups and $Y$ is a Borel $H$-space, then a map $\alpha : H \times Y \to K$ is a Borel cocycle if it is a measurable map such that, for every $h_1,h_2$ in $H$ and for almost every $y \in Y$, it holds $\alpha(h_1h_2,y) = \alpha(h_1,h_2 \cdot y)\alpha(h_2,y)$.

Proposition 7.6. Let $\phi$ be a measurable, chain preserving map $\phi : \partial \mathbb{H}^p \to S_{m,n}$. For almost every point $x$ in $\partial \mathbb{H}^p$ there exists a measurable cocycle $\alpha : M_x \times (\partial \mathbb{H}^p)^\# \to M_{\phi(x)}$ such that $\phi$ is $\alpha$-equivariant.

Proof. Let us fix a point $x$ generic for the map $\phi$. For almost every pair $(m,y)$ where $m \in M_x$ and $y \in \mathcal{H}_{1,p}(x)$, we have that the points $\phi(y)$ and $\phi(my)$ are on the same vertical chain in $\mathcal{H}_{m,n}(\phi(x))$. In particular there exists an element
\( \alpha(m, y) \in M_{\phi(x)} \) such that \( \alpha(m, y) \phi(y) = \phi(my) \). We extend \( \alpha \) by defining it to be 0 on pairs that do not satisfy this assumption. The function \( \alpha \) is measurable since \( \phi \) is measurable.

We now have to show that the map \( \alpha \) we just defined is actually a cocycle. In order to do this let us fix the set \( O \) of points \( z \) for which \( \hat{\phi}(T_{xz}) \) is \( m \)-vertical and \( \phi(z) \in \hat{\phi}(T_{xz}) \). \( O \) has full measure as a consequence of Lemma 7.2. Let us now fix two elements \( m_1, m_2 \in M_x \). For every element \( z \) in the full measure set \( O \cap m_2^{-1}O \cap m_1 m_2^{-1}O \), the three points \( \phi(z), \phi(m_2 z), \phi(m_1 m_2 z) \) belong to the same vertical \( m \)-chain, moreover, by definition of \( \alpha \), we have

\[
\alpha(m_1 m_2, z) \phi(z) = \phi(m_1 (m_2 z)) = \alpha(m_1, m_2 z) \phi(m_2 z) = \alpha(m_1, m_2 z) \alpha(m_2, z) \phi(z).
\]

The conclusion follows from the fact that the action of \( M_{\phi(z)} \) on \( H_{m,n}(\phi(x)) \) is simply transitive.

**Proposition 7.7.** Let us fix a point \( x \). Assume that there exists a measurable function \( \beta : M_x \times W_x \to M_{\phi(x)} \) such that for every \( m \in M_x \), for almost every \( T \) in \( W_x \) and for almost every \( z \) in \( T \), the equality \( \alpha(m, z) = \beta(m, T) \) holds. The restriction of the boundary map \( \phi \) to almost every chain through the point \( x \) is rational.

**Proof.** We are assuming that for every \( m \) in \( M_x \) for almost every \( T \) in \( W_x \) and for almost every \( z \) in \( T \), the equality \( \alpha(m, z) = \beta(m, T) \) holds. Fubini’s Theorem then implies that for every \( T \) in a full measure subset \( F \) of \( W_x \), for almost every \( m \) in \( M_x \) and almost every \( z \) in \( T \) the equality \( \alpha(m, z) = \beta(m, T) \) holds. In particular for every vertical chain \( T \) in \( F \) and almost every pair \( (m_1, m_2) \) in \( M_x^2 \) we have \( \beta(m_1, T) \beta(m_2, T) = \beta(m_1 m_2, T) \): it is in fact enough to chose \( m_1 \) and \( m_2 \) so that the equality of \( \alpha(m_i, z) \) and \( \beta(m_i, T) \) holds for almost every \( z \) and compute the cocycle identity for \( \alpha \) in a point \( z \) that works both for \( m_1 \) and \( m_2 \).

It is classical that if \( \pi : G \to J \) satisfies \( \pi(xy) = \pi(x) \pi(y) \) for almost every pair \( (x, y) \) in \( G^2 \), then \( \pi \) coincides almost everywhere with an actual Borel homomorphism (cfr. [Zim84, Theorem B.2]). In particular for every \( T \) in \( F \), we can assume (up to modifying \( \beta|_{T} \) on a zero measure subset) that the restriction of \( \beta \) to \( T \) is a measurable homomorphism \( \beta_T : M_x \to M_{\phi(x)} \) and hence coincides almost everywhere with an algebraic map. Since the action of \( M_x \) and \( M_{\phi(x)} \) on each vertical chain is algebraic and simply transitive, we get that for almost every vertical chain \( T \) the restriction of \( \phi \) to \( T \) is algebraic.

The fact that we let \( \beta \) depend on the vertical chain \( T \) might be surprising, and it is probably possible to prove that the cocycle \( \alpha \) coincides almost everywhere with an homomorphism that does not depend on the vertical chain \( T \). However since it suffices to prove that the restriction of \( \alpha \) to almost every vertical chain coincides almost everywhere with an homomorphism, and since this reduces the technicalities involved, we will restrict to this version.

The remainder of the chapter is devoted to prove, using the chain geometry of \( S_{m,n} \), that the hypothesis of Proposition 7.7 is satisfied whenever \( \phi \) is a Zariski
dense map and \( m < n \). In the following proposition we deal with a preliminary easy case, that requires less technicalities.

**Proposition 7.8.** Let \( \phi : \partial \mathbb{H}_C^p \rightarrow S_{m,n} \) be a measurable, Zariski dense, chain geometry preserving map, and let \( n \geq 2m \). The restriction of \( \phi \) to almost every chain coincides almost everywhere with a rational map.

**Proof.** We want to apply Proposition 7.7 and show that the cocycle \( \alpha : M_x \times \partial \mathbb{H}_C^p \rightarrow M_{\phi(x)} \) only depends on the vertical chain a point belongs to. We consider the set \( F \subseteq W_x \) of chains \( F \) such that

1. \( \hat{\phi}(F) \) is an \( m \)-vertical chain, hence in particular \( \phi_x(F) \) is defined,
2. for almost every circle \( C \) containing the point \( F \in W_x \), for almost every chain \( T \) lifting \( C \) the diagram of Lemma 7.5 commutes almost everywhere.

It follows from the proof of Lemma 7.5 that the set \( F \) is of full measure. Moreover, we get that if \( F \) is an element in \( F \), for almost every point \( z \) in \( F \) and almost every chain \( T \) through \( z \) the diagram of Lemma 7.5 commutes almost everywhere when restricted to \( T \) (this is again an application of Fubini’s Theorem). Using Fubini again, this implies that for almost every point \( w \) in \( \partial \mathbb{H}_C^p \) the diagram of Lemma 7.5 commutes almost everywhere when restricted to the chain \( T_{zw} \).

Let us now fix a chain \( F \in F \) and denote by \( O \) the full measure set of points in \( F \) for which that holds. For every element \( m \in M_x \) we also consider the full measure set \( O_m = O \cap m^{-1}O \). We claim that, given two elements \( z_1, z_2 \in O_m \), the value \( \alpha(m, z_i) \) is the same value \( \beta(m, F) \). In fact let us fix two points \( z_1, z_2 \) in \( O_m \) and let us consider the set \( A_{z_1, z_2, m} \subseteq \partial \mathbb{H}_C^p \) consisting of points \( w \) such that

1. \( \phi(w) \in \hat{\phi}(T_{wz_1}) \cap \hat{\phi}(T_{wz_2}) \)
2. \( \phi(mw) \in \hat{\phi}(T_{mwz_1}) \cap \hat{\phi}(T_{mwz_2}) \)
3. \( \dim(\phi(z_1), \phi(z_2), \phi(w)) = 3m \).

We claim that the set \( A_{z_1, z_2, m} \) is not empty. Indeed, by definition of \( O_m \), the set of points \( w \) satisfying the first two assumption is of full measure. Moreover, since \( n \geq 2m \), the set \( C \) of points in \( S_{m,n} \) such that \( \dim(\phi(z_1), \phi(z_2), \phi(w)) < 3m \) is a proper Zariski closed subset of \( S_{m,n} \). Since the map \( \phi \) is Zariski dense, the preimage
CHAPTER 7. THE RESTRICTION TO A CHAIN IS RATIONAL

of $C$ cannot have full measure, and this implies that $A_{z_1,z_2,m}$ has positive measure, in particular it contains at least one point. The third assumption on the point $w$ implies that the $m$-chain containing $\phi(w)$ and $\phi(z_i)$ is horizontal for $i = 1, 2$.

Let us fix a point $w \in A_{z_1,z_2,m}$ and consider the $m$-chain $\hat{\phi}(mT_{wz_i})$ for $i = 1, 2$. The $m$-chain $\phi(mT_{wz_i})$ is a lift of the $(m,0)$-circle $C_i = \pi_{\phi(x)}(\hat{\phi}(T_{wz_i}))$ that contains both the points $\phi(mz_i)$ and $\phi(mw)$. In particular, since $\alpha(m,z_i)\hat{\phi}(T_{wz_i})$ is a lift of $C_i$ containing $z_i$ we get that $\alpha(m,z_i)\hat{\phi}(T_{wz_i}) = \phi(mT_{wz_i})$. Similarly we get that $\alpha(m,w)\hat{\phi}(T_{wz_i}) = \phi(mT_{wz_i})$. This gives that $\alpha(m,z_i)^{-1}\alpha(m,w) \in \mathcal{E}_{T_{wz_i}}$, but the latter group is the trivial group since we know that the chain $T_{wz_i}$ is 0-vertical. This implies that $\alpha(m,z_1) = \alpha(m,w) = \alpha(m,z_2)$. 

The main difference with the proof of Proposition 1.16 is that here we need an auxiliary point $w$. This is needed to deal with the issue that we only have information almost everywhere. The freedom in the choice of the point $w$ will be crucial in the next section to deal with the case $n < 2m$.

7.3 Possible errors

We now want to understand what information on the difference $\alpha(m,z_1) - \alpha(m,z_2)$ one can get by applying the same argument as in Proposition 7.8 with respect to a generic point $w$ in $\partial \mathbb{H}^2_C$. The main idea is that even if the argument of Proposition 7.7 gives only partial information on the difference $\alpha(m,z_1) - \alpha(m,z_2)$, by running the same argument with many generic points it is possible to reconstruct the full information. As a consequence of Proposition 7.7 it is enough to be able to deduce that $\alpha(z_1,m) = \alpha(z_2,m)$ for pairs $(z_1,z_2)$ such that $(x,z_1,z_2)$ is maximal, and we will stick to this setting.

Let us now consider the maximal triple $(v_\infty,v_0,v_1) \in \mathcal{S}_{m,n}$, where, as before, $v_\infty = (e_1, \ldots, e_m)$, $v_0 = (e_{n+1}, \ldots, e_{n+m})$, and $v_1 = (e_j + i e_{n+j})$ for $1 \leq j \leq m$. For every point $z$ in $\mathcal{S}_{m,n}$, we consider the integer $k(z)$ such that $\langle v_0, v_1, z \rangle = 3m - k(z)$. Both $m$-chains $T_{v_0,z}$ and $T_{v_1,z}$ are $k(z)$-vertical: in fact $\langle v_0, v_\infty, v_1, z \rangle = \langle v_0, v_1, z \rangle = \langle v_\infty, v_0, z \rangle$.

In particular we can define a map

$$
\beta : \mathcal{S}_{m,n} \rightarrow \text{Gr}(v_\infty)^2,
$$

$$
z \rightarrow (\langle v_0, z \rangle \cap v_\infty, \langle v_1, z \rangle \cap v_\infty).
$$

Recall from Section 6.4 that we denote by $Z^k$ the subspace of $\mathbb{C}^m = v_\infty$ spanned by the first $k$ elements of the standard basis, and that $E_k$ denotes the subgroup of $u(m)$ consisting of matrices that can have nonzero values only in the first $k \times k$ block.

Let us now fix a point $z$ in $\mathcal{S}_{m,n}(v_\infty)$ and let $k$ be the integer $k(z)$. We pick elements $g_1, g_2$ in $\text{GL}(m)$ such that $g_i \beta(z) = (\text{e}_1, \ldots, \text{e}_k)$ and we define a subgroup $E(z)$ of $u(m)$ by setting

$$
E(z) = g_1^{-1}E_k g_1^{-*} + g_2^{-1}E_k g_2^{-*}.
$$
Lemma 7.9. Let $x \in \partial \mathbb{H}^m_\mathbb{C}$ be a generic point for the map $\phi$ and $m \in M_x$ be an element.

- For almost every chain $T \in W_x$ and almost every pair of points $z_1, z_2 \in T$, $z_i \in O_m$ and $(\phi(x), \phi(z_1), \phi(z_2))$ is maximal.

- For such a triple, if we conjugate $\phi$ so that $\phi(x) = v_\infty$, $\phi(z_1) = v_0$ and $\phi(z_2) = v_1$ and we fix a point $w \in A_{z_1, z_2, m}$, we get that
  \[
  \alpha(m, z_1) - \alpha(m, z_2) \in E(\phi(w)).
  \]

Proof. Since $x$ is generic for $\phi$, for almost every pair $(z_1, z_2)$ on a vertical chain, it satisfies $(\phi(x), \phi(z_1), \phi(z_2))$ is maximal. Since $\text{SU}(\mathbb{C}^{m+n}, R)$ is transitive on maximal triple, we can assume that $\phi(z_1) = v_0$ and $\phi(z_2) = v_1$.

By definition of $w$, $\phi(w) \in \hat{\phi}(T_{w z_1}) \cap \hat{\phi}(T_{w z_2})$ and similarly $\phi(mw) \in \hat{\phi}(T_{mw z_1}) \cap \hat{\phi}(T_{mw z_2})$. Moreover $\hat{\phi}(T_{mw z_1})$ and $\hat{\phi}(T_{w z_1})$ have the same projection, hence by Proposition 6.18 we get that
\[
\alpha(m, w) - \alpha(m, z_1) \in g_1^{-1}E_k g_1^{-*}.
\]
In the same way one gets that
\[
\alpha(m, w) - \alpha(m, z_2) \in g_2^{-1}E_k g_2^{-*}.
\]
And this concludes the proof. \hfill \square

Throughout the section we will fix the setting provided by Lemma 7.9: we fix $m \in \mathbb{R}$, $z_1, z_2$ in $O_m$ and assume (up to conjugating the map $\phi$) that $\phi(z_1) = v_0$ and $\phi(z_2) = v_1$. The purpose of the rest of the chapter is to show that the intersection
\[
\bigcap_{w \in A_{z_1, z_2, m}} E(\phi(w))
\]
the trivial group. We will informally call errors the subgroups $E(\phi(w))$, since they correspond to the possible defects of the cocycle $\alpha$ to be an actual homomorphism. Our first goal is to prove a criterion to show that the intersection $\bigcap_{w \in I} E(t_i) = \{0\}$ when $t_i$ are points in $S_{m, n}$ and $I$ is a big enough finite set of indices.

In order to understand errors better, we fix a pair of subspaces $(Z_1, Z_2)$ of $\mathbb{C}^m$ and consider the subgroup of $u(m)$ generated as a vector subspace of $u(m)$ by the elementary matrices of the form $t_1 t_2^* - t_2 t_1^*$ where $t_i$ is a vector of $Z_i$. We will denote by $S(Z_1, Z_2)$ this subgroup:
\[
S(Z_1, Z_2) = \langle t_1 t_2^* - t_2 t_1^* | t_i \in Z_i \rangle < u(m).
\]

Lemma 7.10. Let $z$ be a point in $S_{m, n}$. Then
\[
E(z) = S(\beta_1(z), \beta_1(z)) + S(\beta_2(z), \beta_2(z))
\]
where $\beta_i$ denotes the $i$-th component of the map $\beta$. 

Proof. Clearly $E_k$ coincides with $S(Z_k, Z^k)$. Moreover, if $g_i$ is an element such that $g_i\beta_i(z) = Z^k$ then

$$g_i^{-1}S(Z_k, Z^k)g_i^{-*} = S(g_i^{-1}Z_k, g_i^{-1}Z^k) = S(\beta_i(z), \beta_i(z)).$$

The assertion now follows from the definition of the subgroup $E(z)$. \hfill \Box

We will also need the map $\delta$ defined in the following lemma:

**Lemma 7.11.** The map

$$ \delta : \text{Gr}_k(v_{\infty})^2 \rightarrow \text{Gr}(u(m)) $$$$ (Z_1, Z_2) \rightarrow S(Z_1^\perp, Z_2^\perp)$$

is algebraic.

**Proof.** This follows from the fact that both the map that associates to a vector space its orthogonal and the one that associates to a pair of vectors $(t_1, t_2)$ the antiHermitian matrix $t_1t_2^* - t_2t_1^*$ are regular. \hfill \Box

The standard Hermitian form on $\mathbb{C}^m$ induces a positive definite Hermitian form $H$ on $u(m)$ that is defined by the formula

$$H(A, B) = \text{tr}(AB^*).$$

In the next lemma we will introduce a subspace $I(z)$ of $u(m)$ that is easy to work with and has the property that is contained in the orthogonal, with respect to the Hermitian form $H$, to the subgroup $E(z)$ of $u(m)$. The letter $I$ stands for information, namely if two elements $\alpha_1, \alpha_2$ of $u(m)$ differ by an element of $E(z)$, then the orthogonal projections of $\alpha_1$ and $\alpha_2$ on $I(z)$ are equal. In particular if $w$ belongs to $\mathcal{A}_{m, z_1, z_2}$, the group $I(\phi(w))$ measures some of the information on the difference $\alpha(m, z_2) - \alpha(m, z_1)$ we can get applying the argument of Proposition 7.8 to the point $w$.

**Lemma 7.12.** Let $z \in S_{m, n}$ be a point. The subgroup

$$I(z) = \delta \circ \beta(z) = S(Z_1^\perp, Z_2^\perp)$$

is contained in the orthogonal to $E(z)$ in $u(m)$ with respect to $H$.

**Proof.** Recall that $S(Z_1^\perp, Z_2^\perp)$ is generated by matrices of the form $r = r_1r_2^* - r_2r_1^*$ where $r_i$ belongs to the orthogonal to $Z_i$. Similarly a generator of $S(Z_1, Z_1)$ is of the form $a = a_1a_2^* - a_2a_1^*$ for some $a_i$ in $Z_1$. Explicit computations of the trace of the matrix $r^*a$ give

$$H(r, a) = \text{tr}((r_1r_2^*) - r_2r_1^*) = (a_2a_1^* - a_1a_2^*) =$$

$$= -(r_2^*a_1)(a_2^*r_1) + (r_1^*a_1)(a_2^*r_2) + (r_2^*a_2)(a_1^*r_1) - (r_1^*a_2)(a_1^*r_2) = 0$$

since $r_1^*a_i = 0$ for our choice of $r_1$. This proves that $I(z)$ is orthogonal to $S(Z_1, Z_1)$. The proof of the orthogonality to $S(Z_2, Z_2)$ is analogue. \hfill \Box
This allows us to prove the criterium we were looking for:

**Corollary 7.13.** Let \( t_i, i \in I \), be points in \( S_{m,n} \). If \( \langle I(t_i) \mid i \in I \rangle = u(m) \), then \( \bigcap_{i \in I} E(t_i) = \{0\} \).

**Proof.** This is a consequence of Lemma 7.12: since \( I(t_i) \subseteq E(t_i)^\perp \), we have that \( \langle I(t_i) \rangle \subseteq \left( \bigcap_{i \in I} E(t_i) \right)^\perp \) and hence

\[
\bigcap_{i \in I} E(t_i) \subseteq \langle I(t_i) \rangle^\perp = \{0\}.
\]

\[ \square \]

The above lemma will give us an useful criterium to show that \( \bigcap_{i \in I} E(z_i) \) consists generically only the zero matrix if \( |I| > m^2 \). Here by *generically* we mean in an open set in the Zariski topology, namely that the set of tuples \((z_1 \ldots z_s)\) such that \( \cap E(z_i) \) is not reduced to the zero matrix is a proper Zariski closed subset of \((S_{m,n})^s\) if \( s > m^2 \). The first step of the proof of this last result is the surjectivity of the map \( \beta \).

**Surjectivity of \( \beta \)**

We consider the Zariski open subset \( \mathcal{D} \) of \( S_{m,n} \) consisting of points that are transversal to \( v_\infty, v_0 \) and \( v_1 \) and whose intersection with the subspace \( \langle v_\infty, v_0 \rangle \) has minimal dimension. This means that the dimension of the latter intersection is 0 if \( n > 2m \), and \( k = 2m - n \) otherwise. In particular, to avoid trivialities, we will always assume \( 0 < k < m \), and hence \( m < n < 2m \). Otherwise we are back in the case we already treated in Proposition 7.8. We also set \( l = m - k = n - m \).

We will study the restriction to \( \mathcal{D} \) of the rational map \( \beta \):

\[
\beta : \mathcal{D} \subseteq S_{m,n} \rightarrow \text{Gr}_k(v_\infty)^2 \quad z \rightarrow \langle \langle v_0, z \rangle \cap v_\infty, \langle v_1, z \rangle \cap v_\infty \rangle.
\]

The map \( \beta \) factors via the map \(\gamma: \mathcal{D} \rightarrow W_{v_0} \times W_{v_1} \).

**Lemma 7.14.** Parametrizations of \( W_{v_0} \) and \( W_{v_1} \) are given by the maps

\[
M(l \times m, \mathbb{C}) \rightarrow W_{v_0} \quad M(l \times m, \mathbb{C}) \rightarrow W_{v_1}
\]

\[
A_0 \rightarrow \begin{bmatrix} 0 & \text{Id}_l \\ A_0^* \end{bmatrix}^\perp \\
A_1 \rightarrow \begin{bmatrix} A_1^* \\ \text{Id}_l \\ i A_0^* \end{bmatrix}^\perp
\]

A point \( A_i \) in \( W_i \) corresponds to a \( k \)-vertical chain if \( \text{rk}(A_i) = l \).
Two of their underlying vector spaces closed subset in turn this is equivalent to the requirement that the identity, up to changing basis. And this justifies the first assertion, since for every choice of a matrix $X_2$ we get an $m$-chain through $v_1$.

Since the intersection of $V$ with $v_\infty$ is the subspace of $v_\infty$ that is contained in $V$, it corresponds to the kernel of $X_2$ (with respect to the standard basis of $v_\infty$). In particular the requirement that $T$ is $k$-vertical restates as the requirement that $X_2$ has rank $l = m - k$.

Lemma 7.15. Under the parametrizations of Lemma 7.14, the image of $\gamma$ is the closed subset $C$ of $M(l \times m, \mathbb{C}) \times M(l \times m, \mathbb{C})$ defined by the equations

$$C = \{(A_0, A_1)| A_0A_1^* A_1 A_0^* - A_1 A_0^* - A_0 A_1^* = 0\}.$$ 

Proof. Two $m$-chains $T_0, T_1$ intersect in $\mathcal{S}_{m,n}$ if and only if the intersection $V_0 \cap V_1$ of their underlying vector spaces $V_0, V_1$ contains a maximal isotropic subspace. In turn this is equivalent to the requirement that $(V_0 \cap V_1)^\perp$ has signature $(0, l)$. Indeed, since $V_0^\perp$ has signature $(0, l)$ and is contained in $(V_0 \cap V_1)^\perp$, we get that the signature of any subspace of $(V_0 \cap V_1)^\perp$ is $(k_1, l + k_2)$ for some $k_1, k_2$. On the other hand if $V_0 \cap V_1$ contains a maximal isotropic subspace $z$, then $(V_0 \cap V_1)^\perp \subseteq z^\perp$ and the latter space has signature $(0, l)$. In particular the signature of $(V_0 \cap V_1)^\perp$ would be $(0, l)$, and clearly the orthogonal of a subspace of signature $(0, l)$ contains a maximal isotropic subspace.

Since $(V_0 \cap V_1)^\perp = (V_0^\perp, V_1^\perp)$, we are left to check that the requirement that signature of this latter subspace is $(0, l)$ is equivalent to the requirement that $A_1 A_0^*$ belongs to $\text{Id} + \mathcal{U}(l)$. If now we pick a pair $(A_0, A_1) \in M(l \times m, \mathbb{C}) \times M(l \times m, \mathbb{C})$ representing a pair of subspaces $(V_0, V_1) \in W_{v_0} \times W_{v_0}$ we have that the subspace $(V_0 \cap V_1)^\perp$ is spanned by the columns of the matrix

$$\begin{pmatrix} 0 & A_1^* \\ \text{Id} & \text{Id} \\ A_0^* & iA_1^* \end{pmatrix}.$$ 

It is easy to compute the restriction of $\mathcal{U}$ to the given generating system of $(V_0 \cap V_1)^\perp$:
The latter matrix is negative semidefinite and has rank \( l \) if and only if
\[
A_0 A_1^* A_1 A_0^* - A_1 A_0^* - A_0 A_1^* = (A_0 A_1^* - \text{Id})(A_1 A_0^* - \text{Id}) - \text{Id} = 0. \tag{7.1}
\]
In this case the restriction of \( \overline{\theta} \) to \( (V_0 \cap V_1)^\perp \) has signature \((0,l)\).

We now have all the ingredients we need to show that \( \beta \) is surjective.

**Proposition 7.16.** The map \( \beta \) is surjective.

*Proof.* If \( \zeta \) is a map such that \( \beta = \zeta \gamma \), then it is easy to check that the map \( \zeta \) has the following expression, with respect to the coordinates described in Lemma 7.14:
\[
\zeta : W_{v_0} \times W_{v_1} \to \text{Gr}_k(v_\infty)^2 \to (\ker(A_0), \ker(A_1)).
\]

In order to conclude the proof it is enough to show that any pair of \( k \)-dimensional subspaces of \( v_\infty \) can be realized as the kernels of a pair of matrices satisfying Equation 7.1.

We first consider the case in which the subspaces \( V_0, V_1 \) intersect trivially, of course this can only happen if \( k \leq l \). In this case there exists an element \( g \in U(m) \) such that \( g V_0 = V_0 = \langle e_1, \ldots, e_k \rangle \) and that \( V_1 = g V_1 \) is spanned by the columns of the matrix \( \begin{bmatrix} B_{l \times k} \end{bmatrix} \) where \( B \) is a matrix in \( M(k \times k, \mathbb{C}) \). Clearly \( V_1 \) is the kernel of \( A_i \) if and only if \( \overline{V}_i \) is the kernel of \( \overline{A}_i = A_i g^{-1} \) and \( A_i g^{-1} \) satisfies the equation 7.1 if and only if \( A_i \) does. In particular it is enough to exhibit matrices \( A_i \) whose kernel is \( V_i \).

Let us first notice that, for any matrix \( B \in M(k \times k, \mathbb{R}) \) there exists a matrix \( X \in \text{GL}_k(\mathbb{C}) \) such that \( X B \) is a diagonal matrix \( D \) whose elements are only 0 or 1. Let us now consider the matrices
\[
\overline{A}_1 = \begin{bmatrix} 0 & 2\text{Id} & 0 \\ 0 & 0 & 2\text{Id} \end{bmatrix} \quad \text{and} \quad \overline{A}_2 = \begin{bmatrix} X & D & 0 \\ 0 & 0 & \text{Id} \end{bmatrix}.
\]

By construction \( \overline{V}_1 \) is the kernel of \( \overline{A}_1 \) and \( \overline{V}_2 \) is the kernel of \( \overline{A}_2 \), moreover we have that \( \overline{A}_1 \overline{A}_2 \) satisfies Equation 7.1:
\[
\overline{A}_1 \overline{A}_2 = \begin{bmatrix} 0 & 2\text{Id} & 0 \\ 0 & 0 & 2\text{Id} \end{bmatrix} \begin{bmatrix} X^* & 0 \\ D^* & 0 \\ 0 & \text{Id} \end{bmatrix} = \begin{bmatrix} 2D^* & 0 \\ 0 & 2\text{Id} \end{bmatrix} \in \text{Id} + U(m).
\]

This implies that there is a point \( z \in S_{m,n} \) with \( \beta(z) = (\overline{V}_0, \overline{V}_1) \).

The general case, in which the intersection of \( V_i \) is not trivial, is analogous: we can assume, up to the \( U(m) \) action that \( V_0 \cap V_1 = \langle e_1, \ldots, e_s \rangle \) and we can restrict to the orthogonal to \( V_0 \cap V_1 \) with respect to the standard Hermitian form \( \lambda \).

An important corollary of the last proposition is the following

**Corollary 7.17.** For every proper subspace \( L \) of \( \mathfrak{u}(m) \) the set \( C(L) = \{ z \in S_{m,n} \mid I(z) \subseteq L \} \) is a proper Zariski closed subset of \( D \subseteq S_{m,n} \).
Proof. The subspace $C(L)$ is Zariski closed since the subspaces of $u(m)$ that are contained in $L$ form a Zariski closed subset of the Grassmanian $\Gr(u(m))$ of the vector subspaces of $u(m)$, moreover the subspace $I(z)$ is obtained as the composition $I(z) = \delta \circ \beta$ of two regular maps (cfr. Lemma 7.11).

In order to verify that $C(L)$ is a proper subset, unless $L = u(m)$, it is enough to verify that the subspaces of the form $S(Z^1_j, Z^2_j)$ with $Z_j$ transversal subspaces span the whole $u(m)$. Once this is proven, the result follows from the the surjectivity of $\beta$: since $\beta$ is surjective, the preimage of a proper Zariski closed subset is a proper Zariski closed subset. In turn the fact that the span

$$\langle z_1 z^*_2 - z_2 z^*_1 | z_1, z_2 \in C^m \text{ linearly independent} \rangle$$

is the whole $u(m)$ follows from the fact that every matrix of the form $iz_1 z^*_2$ is in the span, since such a matrix can be obtained as the difference $z_1(z_2 - iz_1)^* - (z_2 - iz_1)z^*_1 - (z_1 z^*_2 - z_2 z^*_1)$. □

### 7.4 Proof of Theorem 7.1

Let us now go back to the setting of Theorem 7.1: we fix a measurable, chain geometry preserving, Zariski dense map $\phi : \partial \mathbb{H}_C^m \to S_{m,n}$, a generic point $x \in \partial \mathbb{H}_C^m$ such that for almost every chain $t \in W_x$, for almost every point $y \in t$, $\phi(y) \in \hat{\phi}(t)$. We want to show that the measurable cocycle $\alpha : \partial \mathbb{H}_C^m \times \{1\} \to u(m)$ coincides on almost every vertical chain with a measurable homomorphism.

As a consequence of Lemma 7.9, it is enough to show that, for almost every pair $z_1, z_2$ on a vertical chain, the intersection $\bigcap_{w \in C_{z_1, z_2, m}} E(\phi(w)) = \{0\}$. In fact this would imply that the restriction of $\alpha$ to almost every chain essentially does not depend on the choice of the point, hence coincides with a measurable homomorphism.

Let us now show that we can find $m^2$ points $w_1, \ldots, w_{m^2}$ in $\partial \mathbb{H}_C^m$ such that $\bigcap E(\phi(w_i)) = \{0\}$. We work by induction: let us fix $j$ points $w_0, \ldots, w_{j-1}$. If the linear space

$$L_j = \langle I(\phi(w_i)) | i < j \rangle$$

is equal to the whole $u(m)$ we are done. Otherwise it follows from Corollary 7.17 that the subset $C(L_j)$ of $S_{m,n}$ is a proper Zariski closed subset of $D$. Here, as above we denote by $D$ the Zariski open subset of $S_{m,n}$ consisting of points $z$ that are transversal both to $\phi(z_i)$ and to the linear subspace $\langle \phi(z_1), \phi(z_2) \rangle$ and by $C(L_j)$ the subset consisting of those points such that $I(z) \subseteq L_j$.

In particular, since $\phi$ is Zariski dense, its essential image cannot be contained in $C(L_j) \cup (S_{m,n} \setminus D)$ that is a Zariski closed subset of $S_{m,n}$. This implies that we can find a point $w_j$ in the full measure set $A_{z_1, z_2, m}$ such that $\delta \circ \beta(\phi(w))$ is not contained in $L_j$. In turn this implies that $L_{j+1} = \langle I(\phi(w_i)) | i \leq j \rangle$ strictly contains $L_j$, hence has dimension strictly greater. This shows the claim and completes the proof of Theorem 7.1.
Chapter 8

The boundary map is rational

In this chapter we will show that a chain geometry preserving map \( \phi : \partial \mathbb{H}^p \to \mathcal{S}_{m,n} \) whose restriction to almost every chain is rational, coincides almost everywhere with a rational map.

Assume that the chain geometry preserving map \( \phi \) is rational and let us fix a point \( x \). Since the projection \( \pi_\phi(x) : \mathcal{S}_{\phi(x)} \to W_{\phi(x)} \) is regular we get that the map \( \phi_x : W_x \to W_{\phi(x)} \) induced by \( \phi \) is rational as well. The first result of the section is that the converse holds, namely that if there exist enough many points \( s_1, \ldots, s_l \) in \( \partial \mathbb{H}^p \) such that \( \phi_{s_i} \) is rational, then the original map \( \phi \) had to be rational as well.

In what follows we will denote by \( l \) the smallest integer bigger than \( 1 + m/(n - m) \).

**Lemma 8.1.** There exists a Zariski open subset \( \mathcal{O} \subset \mathcal{S}^l_{m,n} \), such that for any \( (x_1, \ldots, x_l) \) in \( \mathcal{O} \), there exists a Zariski open subset \( \mathcal{D}_{x_1, \ldots, x_l} \subset \mathcal{S}_{m,n} \) such that for every \( z \in \mathcal{D}_{x_1, \ldots, x_l} \) we have

\[
\bigcap_{i=1}^l \langle z, x_i \rangle = z.
\]

**Proof.** Let us consider the set \( \mathcal{F} \) of \((l + 1)\)-tuples \((x_1, \ldots, x_l, z)\) in \( \mathcal{S}^{l+1}_{m,n} \) with the property that \( \bigcap_{i=1}^l \langle z, x_i \rangle = z \). This is a Zariski open subset of \( \mathcal{S}^{l+1}_{m,n} \): indeed, since \( z \) is clearly contained in the intersection, the set \( \mathcal{F} \) is defined by the equation \( \dim \bigcap_{i=1}^l \langle z, x_i \rangle \leq m \). In order to conclude the proof it is enough to show that for each \( z \in \mathcal{S}_{m,n} \) the set of tuples \((x_1, \ldots, x_l)\) with the property that \((x_1, \ldots, x_l, z) \in \mathcal{F}\) is non empty: this implies that the set \( \mathcal{F} \) is a non empty Zariski open subset, and in particular there must exist a Zariski open subset of \( \mathcal{S}^l_{m,n} \) consisting of \( l \)-tuples \((x_1, \ldots, x_l)\) satisfying the hypothesis of the lemma.

Let us then fix a point \( z \in \mathcal{S}_{m,n} \). We denote by \( \mathcal{A}^k_z \) the set of \( k \)-tuples \((x_1, \ldots, x_k)\) such that \( \dim \bigcap_{i=1}^k \langle z, x_i \rangle = \max\{2m - (n - m)(k - 1), m\} \). In order to conclude the proof it is enough to exhibit, for every \( k \)-tupla \( x \) in \( \mathcal{A}^k_z \) a non empty subset \( \mathcal{B} \) of \( \mathcal{S}_{m,n} \) such that \((x, b) \in \mathcal{A}^{k+1}_z \) for each \( b \) in \( \mathcal{B} \). If we denote by \( V_k \) the subspace \( \bigcap_{i=1}^k \langle z, x_i \rangle \), that has, by our assumption on the tuple \( x \), dimension

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We consider the set

\[ \text{Proof.} \]

gives a birational isomorphism.

The set \( \mathcal{B} \) is not empty since both transversality conditions are non-empty, Zariski open conditions, and for this choice we get

\[
\dim \bigcap_{i=1}^{k+1} \langle z, x_i \rangle = \dim(\mathcal{V}_k \cap \langle z, x_{k+1} \rangle) = \dim \mathcal{V}_k + \dim \langle z, x_{k+1} \rangle - \dim \mathcal{V}_k, x_{k+1} = 2m - (n - m)(k - 1) + 2m - \min\{m + n, 2m - (n - m)(k - 1) + m\} = \max\{2m - (n - m)k, m\}. \]

If \( n \geq 2m \), the set \( \mathcal{S}_{m,n}^{(2)} \) is contained in \( \mathcal{O} \) and for \( x_1, x_2 \) transversal the set \( \mathcal{D}_{x_1, x_2} \) consists of the points \( z \) that are transversal to \( x_1, x_2 \) and \( \langle x_1, x_2 \rangle \). In general we will assume (up to restricting \( \mathcal{D}_{x_1, ..., x_l} \) to a smaller Zariski open subset) that each \( z \) in \( \mathcal{D}_{x_1, ..., x_l} \) is transversal to \( x_i \) for each \( i \).

**Lemma 8.2.** Let \((x_1, \ldots, x_l)\) be an \( l \)-tuple of pairwise transversal points in the set \( \mathcal{O} \) defined in Lemma 8.1. There exist a quasiprojective subset \( \mathcal{C}_{x_1, \ldots, x_l} \) of \( W_{x_1} \times \ldots \times W_{x_l} \) such that the map \( \beta_{x_1, \ldots, x_l} = \pi_{x_1} \times \ldots \pi_{x_l} : \mathcal{D}_{x_1, \ldots, x_l} \to W_{x_1} \times \ldots \times W_{x_l} \) gives a birational isomorphism.

**Proof.** We consider the set \( \mathcal{C}_{x_1, \ldots, x_l} \) consisting of tuples \((t_1, \ldots, t_l)\) with the property that the associated linear subspaces intersect in an \( m \)-dimensional subspace on which the restriction of \( h \) is zero, and that \( t_j - \pi_j(x_i) \) has maximal rank for every \( i, j \). With this choice \( \mathcal{C}_{x_1, \ldots, x_l} \) is quasiprojective since the condition that the intersection has dimension at least \( m \) and that the restriction of \( h \) to the intersection is degenerate are closed condition (defined by polynomial), the condition that the intersection has dimension at most \( m \) is an open condition. The set \( \mathcal{C}_{x_1, \ldots, x_l} \) is the subset of \( \mathcal{C}_{x_1, \ldots, x_l}^{'} \) that is the image of \( \beta_{x_1, \ldots, x_l} \).

The fact that the map \( \beta_{x_1, \ldots, x_l} \) gives a birational isomorphism follows from the fact that a regular inverse to \( \beta_{x_1, \ldots, x_l} \) is given by the map that associates to an \( l \)-tuple of points their unique intersection.

We now have all the ingredients we need to prove the following

**Proposition 8.3.** Let us assume that for almost every point \( x \in \partial \mathbb{H}_m^d \) the map \( \phi_x \) coincides almost everywhere with a rational map. The same is true for \( \phi \).

**Proof.** Let us fix \( l \) points \( t_1, \ldots, t_l \) which are generic in the sense of Lemma 7.2, which satisfy that \( \phi_{t_i} \) coincides almost everywhere with a rational map, and with the additional property that the \( l \)-tupla \( (\phi(t_1), \ldots, \phi(t_l)) \) belongs to the set \( \mathcal{O} \). We can find such points since the map \( \phi \) is Zariski dense and the set \( \mathcal{O} \) is Zariski open.
Let us now consider the diagram

\[
\partial \mathbb{H}^p_C \setminus \{ t_1, \ldots, t_l \} \xrightarrow{\phi} \mathcal{D}_{\phi(t_1), \ldots, \phi(t_l)} \subseteq \mathcal{S}_{m,n}
\]

\[
\pi_{t_1} \times \ldots \times \pi_{t_l} \downarrow
\]

\[
W_{t_1} \times \ldots \times W_{t_n} \xrightarrow{\phi_{t_1} \times \ldots \times \phi_{t_l}} \mathcal{C}_{\phi(t_1), \ldots, \phi(t_l)}
\]

A consequence of Lemma 7.5 and of the definition of the isomorphisms \( \beta_{t_1, \ldots, t_l} \) is that the diagram commutes almost everywhere. In particular, since the isomorphisms \( \beta_{\phi(t_1), \ldots, \phi(t_l)} \) is birational, and \( \pi_{t_1} \times \ldots \times \pi_{t_l} \) is rational, we get that \( \phi \) coincides almost everywhere with a rational map. \( \square \)

Let us now fix a point \( x \) in \( \partial \mathbb{H}^p_C \), and identify the space \( W_x \) with \( \mathbb{C}^{p-1} \). We want to study the map \( \phi_x : \mathbb{C}^{p-1} \to W_{\phi(x)} \). It follows from Lemma 6.12 restricted to the case \( m = 1 \) that the projections of chains in \( \partial \mathbb{H}^p_C \) to \( \mathbb{C}^{p-1} \) are Euclidean circles \( C \subseteq \mathbb{C}^{p-1} \) (eventually collapsed to points).

**Lemma 8.4.** If \( x \) is generic in the sense of Lemma 7.2, the restriction of \( \phi_x \) to almost every Euclidean circle \( C \) of \( \mathbb{C}^{p-1} \) is rational.

**Proof.** Every Euclidean circle \( C \subseteq \mathbb{C}^{p-1} \) is a circle in our generalized definition, namely is the projection of some 1-chain of \( \partial \mathbb{H}^p_C \). Indeed we know from Lemma 6.12 that the Euclidean circle \( (1 + e^H, 0, \ldots, 0) \in \mathbb{C}^{p-1} \) is the projection of the chain associated to the linear subspace \( \langle e_1 + e_2, e_{p+1} \rangle \) of \( \mathbb{C}^{p+1} \). Moreover the set of Euclidean circles is a homogeneous space under the group of Euclidean similarities of \( \mathbb{C}^{p-1} \) and the group \( Q = \text{stab}(v_\infty) \) acts on \( \mathbb{C}^{p-1} \) as the group of Euclidean similarities.

It follows from the explicit parametrization of a chain given in Lemma 6.12 that, whenever a point \( t \) in \( \pi_x^{-1}(C) \) is fixed, the lift map \( l : C \to T \) is algebraic, where \( T \) is the unique lift of \( C \) containing \( t \).

In particular, if \( T \) is a chain such that the restriction of \( \phi \) to \( T \) coincides almost everywhere with a rational map, the restriction of \( \phi_x \) to \( C = \pi_x(T) \) coincides almost everywhere with a rational map. We can now use a Fubini based argument to get that, for almost every circle \( C \), the restriction of \( \phi_x \) to \( C \) is rational: for almost every chain \( T \), the restriction to \( T \) coincides almost everywhere with a rational map, and the space of chains that do not contain \( x \) is a full measure subset of the space of chains in \( \partial \mathbb{H}^p_C \) which forms a smooth bundle over the space of Euclidean circles in \( \mathbb{C}^{p-1} \). \( \square \)

An usual Fubini type argument implies the following

**Corollary 8.5.** For almost every complex affine line \( L \subseteq \mathbb{C}^m \), for almost every Euclidean circle \( C \) contained in \( L \), the restriction of \( \phi_x \) to \( C \) is algebraic. The same is true for almost every point \( p \) in \( L \) and almost every circle \( C \) containing \( p \).

In order to conclude the proof we will apply many times the following well known lemma that allows to deduce that a map is rational provided that the restriction to enough many subvarieties is rational. Given a map \( \phi : A \times B \to C \) and given a
point \( a \in A \) we denote by \( a \phi : B \to C \) the map \( a \phi(b) = \phi(a, b) \) in the same way, if \( b \) is a point in \( B \), \( b \phi \) will the note the map \( b \phi(a) = \phi(a, b) \).

**Lemma 8.6 ([Zim84, Theorem 3.4.4]).** Let \( \phi : \mathbb{R}^{n+m} \to \mathbb{R} \) be a measurable function. Let us consider the splitting \( \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \). Assume that for almost every \( a \in \mathbb{R}^n \) the function \( a \phi : \mathbb{R}^m \to \mathbb{R} \) coincides almost everywhere with a rational function and for almost every \( b \in \mathbb{R}^m \) the function \( b \phi : \mathbb{R}^n \to \mathbb{R} \) coincides almost everywhere with a rational function, then \( \phi \) coincides almost everywhere with a rational function.

This easily gives that the restriction of \( \phi_x \) to any complex affine line \( \mathcal{L} \) in \( \mathbb{C}^{p-1} \) coincides almost everywhere with a rational map:

**Lemma 8.7.** For almost every complex affine line \( \mathcal{L} \subset \mathbb{C}^{p-1} \), the restriction \( \phi_x|_\mathcal{L} \) coincides almost everywhere with a rational map.

**Proof.** Let us fix a line \( \mathcal{L} \) satisfying the hypothesis of Corollary 8.5 and denote by \( \phi_\mathcal{L} : \mathbb{C} \to W_{\phi(x)} \) the restriction of \( \phi_x \) to \( \mathcal{L} \), composed with a linear identification of \( \mathcal{L} \) with the complex plane \( \mathbb{C} \). By the second assertion of Corollary 8.5, we can find a point \( p \in \mathbb{C} \) such that for almost every Euclidean circle \( C \) through \( p \) the restriction of \( \phi_\mathcal{L} \) to \( C \) coincides almost everywhere with a rational map. Let us now consider the birational map \( i_p : \mathbb{C} \to \mathbb{C} \) defined by \( i_p(z) = (z - p)^{-1} \), and let us denote by \( \psi_\mathcal{L} \) the composition \( \psi_\mathcal{L} = \phi_\mathcal{L} \circ i_p^{-1} \). Since the image under \( i_p \) of Euclidean circles through the point \( p \) are precisely the affine real lines of \( \mathbb{C} \) that do not contain 0, we get that the restriction of \( \psi_\mathcal{L} \) to almost every affine line coincides almost everywhere with a rational map. In particular a consequence of Lemma 8.6 is that the map \( \psi_\mathcal{L} \) itself coincides almost everywhere with a rational map. Since \( \phi_\mathcal{L} \) coincides almost everywhere with \( \psi_\mathcal{L} \circ i_p \) we get that the same is true for the map \( \phi_\mathcal{L} \) and this concludes the proof.

Applying Lemma 8.6 again we deduce the following proposition:

**Proposition 8.8.** Let \( \phi : \partial \mathbb{H}^p_{\mathbb{C}} \to S_{m,n} \) be a map with the property that for almost every chain \( C \) the restriction of \( \phi \) to \( C \) coincides almost everywhere with a rational map. Then for almost every point \( x \in \partial \mathbb{H}^p_{\mathbb{C}} \) the map \( \phi_x \) coincides almost everywhere with a rational map.

In turn this was the last missing ingredient to prove Theorem 1.14.
Chapter 9

Consequences for maximal representations

The last step of Margulis’ original proof of superrigidity involves showing that if a Zariski dense representation $\rho : \Gamma \to H$ of a lattice $\Gamma$ in the algebraic group $G$ admits an algebraic boundary map, then it extends to a representation of $G$ (cfr. [Mar91] and [Zim84, Lemma 5.1.3]). The same argument applies here to deduce the main theorem of the part that we recall for the reader’s convenience.

**Theorem 1.5.** Let $\Gamma$ be a lattice in $SU(1,p)$ with $p > 1$. If $m$ is different from $n$, then every Zariski dense maximal representation of $\Gamma$ into $PU(m,n)$ is the restriction of a representation of $SU(1,p)$.

**Proof.** Let $\rho : \Gamma \to PU(m,n)$ be a Zariski dense maximal representation and let $\psi : \partial H^p \to S_{m,n}$ be a measurable $\rho$ equivariant boundary map, that exists as a consequence of Proposition 4.1 (the difference between $SU(m,n)$ and $PU(m,n)$ plays no role here, since the action of $SU(m,n)$ on $S_{m,n}$ factors through the projection to the adjoint form of the latter group). The essential image of $\psi$ is a Zariski dense subset of $S_{m,n}$ as a consequence of Proposition 4.2, moreover Proposition 4.5 implies that $\psi$ preserves the chain geometry.

Since we proved that any measurable, Zariski dense, chain preserving boundary map $\psi$ coincides almost everywhere with a rational map (cfr. Theorem 1.14), we get that there exists a $\rho$-equivariant rational map $\phi : \partial H^p \to S_{m,n}$. The $\rho$-equivariance follows from the fact that $\phi$ coincides almost everywhere with $\psi$ that is $\rho$ equivariant. In particular, for every $\gamma$ in $\Gamma$ the set on which $\phi(\gamma x) = \rho(\gamma)\phi(x)$ is a Zariski closed, full measure set, and hence is the whole $\partial H^p$.

Since $\phi$ is $\rho$-equivariant and rational, it is actually regular: indeed the set of regular points for $\phi$ is a non-empty, Zariski open, $\Gamma$-equivariant subset of $\partial H^p$. Since, by Borel density [Zim84, Theorem 3.2.5], the lattice $\Gamma$ is Zariski dense in $SU(1,p)$ and $\partial H^p$ is an homogeneous algebraic $SU(1,p)$ space, the only $\Gamma$-invariant proper Zariski closed subset of $\partial H^p$ is the empty set, and this implies that the set of regular points of $\phi$ is the whole $\partial H^p$.  

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In the sequel it will be useful to deal with complex algebraic groups and complex varieties in order to exploit algebraic results based on Nullstellensatz. This is easily achieved by considering the complexification. We will denote by $G$ the algebraic group $\text{SL}(p + 1, \mathbb{C})$ and by $H$ the group $\text{PSL}(m + n, \mathbb{C})$ endowed with the appropriate real structures so that $\text{SU}(1, p) = G(\mathbb{R})$ and $\text{PU}(m, n) = H(\mathbb{R})$. Since $\partial H^t_\mathbb{C}$ and $S_{m,n}$ are homogeneous spaces that are projective varieties, there exist parabolic subgroups $P < \text{SL}(p + 1, \mathbb{C})$ and $Q < \text{PSL}(m + n, \mathbb{C})$ such that $\partial H^t_\mathbb{C} = (G/P)(\mathbb{R}) = G(\mathbb{R})/(P \cap G(\mathbb{R}))$ and $S_{m,n} = (H/Q)(\mathbb{R})$.

The algebraic $\rho$-equivariant map $\phi : \partial H^t_\mathbb{C} \rightarrow S_{m,n}$ lifts to a map $\overline{\phi} : G(\mathbb{R}) \rightarrow S_{m,n}$ and we can extend the latter map uniquely to an algebraic map $T : G \rightarrow H/Q$. The extended map $T$ is $\rho$-equivariant since $G(\mathbb{R})$ is Zariski dense in $G$: whenever an element $\gamma \in \Gamma$ is fixed, the set $\{ g \in G | T(\gamma g) = \rho(\gamma)T(g) \}$ is Zariski closed and contains $G(\mathbb{R})$.

Let us now focus on the graph of the representation $\rho : \Gamma \rightarrow H$ as a subset $\text{Gr}(\rho)$ of the group $G \times H$. Since $\rho$ is an homomorphism, $\text{Gr}(\rho)$ is a subgroup of $G \times H$, hence its Zariski closure $\text{Gr}(\rho)^Z$ is an algebraic subgroup. The image under the first projection $\pi_1$ of $\overline{\text{Gr}(\rho)^Z}$ is a closed subgroup of $G$: indeed the image of a rational morphism (over an algebraically closed field) contains an open subset of its closure, since in our case $\pi_1$ is a group homomorphism, its image is an open subgroup that is hence also closed. Moreover $\pi_1(\overline{\text{Gr}(\rho)^Z})$ contains $\Gamma$ that is Zariski dense in $G$ by Borel density, hence equals $G$.

We now want to use the existence of the algebraic map $T$ and the fact that $\rho(\Gamma)$ is Zariski dense in $H$ to show that $\overline{\text{Gr}(\rho)^Z}$ is the graph of an homomorphism. In fact it is enough to show that $\overline{\text{Gr}(\rho)^Z} \cap \{ e \} \times H = \{ e, e \}$. Let $(e,f)$ be an element in $\overline{\text{Gr}(\rho)^Z} \cap \{ e \} \times H$. Since $H$ is absolutely simple being an adjoint form of a simple Lie group, and $N = \bigcap_{h \in H} hQh^{-1}$ is a normal subgroup of $H$, it is enough to show that $f \in N$ or, equivalently, that $f$ fixes pointwise $H/Q$.

But, since $T$ is a regular map, and the actions of $G$ on itself and of $H$ on $H/Q$ are algebraic, we get that the stabilizer of the map $T$ under the $G \times H$-action,

$$\text{Stab}_{G \times H}(T) = \{(g,h) | ((g,h) \cdot T)(x) = h^{-1}T(gx) = T(x), \forall x \in G\},$$

is a Zariski closed subgroup of $G \times H$. Moreover $\text{Stab}_{G \times H}(T)$ contains $\text{Gr}(\rho)$ and hence also $\overline{\text{Gr}(\rho)^Z}$. In particular $(e,f)$ belongs to the stabilizer of $T$, hence the element $f$ of $H$ fixes the image of $T$ pointwise. Since the image of $T$ is $\rho(\Gamma)$-invariant, $\rho(\Gamma)$ is Zariski dense and the set of points in $H/Q$ that are fixed by $f$ is a closed subset, $f$ acts trivially on $H/Q$.

We can now prove Theorem 1.7:

**Theorem 1.7.** Let $\rho : \Gamma \rightarrow \text{SU}(m,n)$ be a maximal representation. Then the Zariski closure $L = \overline{\rho(\Gamma)^Z}$ splits as the product $\text{SU}(1,p) \times L_t \times K$ where $L_t$ is a Hermitian Lie group of tube type without irreducible factors that are virtually isomorphic to $\text{SU}(1,1)$, and $K$ is a compact subgroup of $\text{SU}(m,n)$.
Moreover there exists an integer $k$ such that the inclusion of $L$ in $\text{SU}(m,n)$ can be realized as
\[
\Delta \times i \times \text{Id} : L \to \text{SU}(1,p)^{m-k} \times \text{SU}(k,k) \times K < \text{SU}(m,n)
\]
where $\Delta : \text{SU}(1,p) \to \text{SU}(1,p)^{m-k}$ is the diagonal embedding and $i : L \to \text{SU}(k,k)$ is a tight holomorphic embedding.

Proof. Let $\rho : \Gamma \to \text{SU}(m,n)$ be a maximal representation and let $L$ be the Zariski closure of $\rho(\Gamma)$ in $\text{SL}(m+n,\mathbb{C})$. Here, as above, $\text{SU}(m,n) = H(\mathbb{R})$ with respect to a suitable real structure on $H = \text{SL}(m+n,\mathbb{C})$. Since the representation $\rho$ is tight, we get, as a consequence of Theorem 3.5, that $L(\mathbb{R})$ almost splits a product $L_{nc} \times L_c$ where $L_{nc}$ is a semisimple Hermitian Lie group tightly embedded in $\text{SU}(m,n)$ and $K = L_c$ is a compact subgroup of $\text{SU}(m,n)$.

Let us now consider the simple factors $L_1, \ldots, L_k$ of $L_{nc}$, namely $L_{nc}$, being semisimple, almost splits as the product $L_{nc} = L_1 \times \ldots \times L_k$ where $L_i$ are simple Hermitian Lie groups. The first observation is that none of the groups $L_i$ can be virtually isomorphic to $\text{SU}(1,1)$. In that case the composition of the representation $\rho$ with the projection $L_{nc} \to L_i$ would be a maximal representation of a complex hyperbolic lattice with values in a group that is virtually isomorphic to $\text{SU}(1,1)$ and this is ruled out by [BI08]: indeed Burger and Iozzi prove, as the last step in their proof of [BI08, Theorem 2], that there are no maximal representations of complex hyperbolic lattices in $\text{PU}(1,1)$.

This implies that the inclusion $i : L_{nc} \to \text{SU}(m,n)$ fulfills the hypotheses of Proposition 5.8 and in particular it is holomorphic. As a consequence of Hamlet’s classification of tight holomorphic homomorphisms between Hermitian Lie groups, it now follows that each factor of the product decomposition $L_1 \times \ldots \times L_k$ is either of tube-type or is isomorphic to $\text{SU}(m_i,n_i)$, with $m_i < n_i$, and $L_{nc}$ is contained in a subgroup of $\text{SU}(m,n)$ of the form $\text{SU}(k,k) \times \text{SU}(m-k,n-k)$.

Let us now fix a factor $L_i$ which is not of tube-type. Since, by Corollary 1.6, there is no Zariski dense representation of $\Gamma$ in $\text{SU}(m_i,n_i)$ if $1 < m_i < n_i$, we get that $m_i = 1$. Moreover, since the only Zariski dense tight representation of $\text{SU}(1,p)$ in $\text{SU}(1,q)$ is the identity map, we get that $n_i = p$ and the composition of $\rho$ with the projection to $L_i$ is conjugate to the inclusion. Assume now that more then one factor of $L_{nc}$ is not of tube-type, then we would have that the diagonal embedding $i : \text{SU}(1,p) \to \text{SU}(1,p)^\tau$ is Zariski dense and this is clearly a contradiction.

\[\square\]

Corollary 1.8. Let $\Gamma$ be a lattice in $\text{SU}(1,p)$, with $p > 1$ and let $\rho$ be a maximal representation of $\Gamma$ into $\text{SU}(m,n)$. Assume that the associated linear representation of $\Gamma$ on $\mathbb{C}^{n+m}$ has no invariant subspace on which the restriction of the Hermitian form has signature $(k,k)$ for some $k$. Then

1. $n \geq pm$,

2. $\rho$ is conjugate to $\overline{\rho} \times \chi_\rho$ where $\overline{\rho}$ is the restriction to $\Gamma$ of the diagonal embedding of $m$ copies of $\text{SU}(1,p)$ in $\text{SU}(m,n)$ and $\chi_\rho$ is a representation $\chi_\rho : \Gamma \to K$, where $K$ is a compact group.
Proof. We know that the Zariski closure of the representation \( \rho \) is contained in a subgroup of \( \text{SU}(m,n) \) isomorphic to \( \text{SU}(1,p)^t \times \text{SU}(m-t,m-t) \times K \). The product \( M = \text{SU}(1,p)^t \times \text{SU}(m-t,m-t) \) corresponds to a splitting \( \mathbb{C}^{m,n} = V_1 \oplus \ldots \oplus V_t \oplus W \oplus Z \) where the restriction of \( h \) to \( V_i \) is non-degenerate and has signature \( (1,p) \) and the restriction of \( h \) to \( W \) is non-degenerate and has signature \( (m-t,m-t) \). The subspace \( W \) is left invariant by \( M \) hence also by \( K \) (since \( K \) commutes with \( M \) and all the invariant subspaces for \( M \) have different signature).

In particular the linear representation of \( \Gamma \) associated with \( \rho \) leaves invariant a subspace on which \( h \) has signature \( (k,k) \) for some \( k \) greater than 1 unless there are no factors of tube-type in the decomposition of \( L \). This latter case corresponds to standard embeddings.

\( \bullet \)

Corollary 1.9. Let \( \Gamma \) be a lattice in \( \text{SU}(1,p) \), with \( p > 1 \), and let \( \rho \) be the restriction to \( \Gamma \) of the diagonal embedding of \( m \) copies of \( \text{SU}(1,p) \) in \( \text{SU}(m,n) \). Then \( \rho \) is locally rigid.

Proof. Let us denote by \( \rho_0 : \Gamma \to \text{SU}(m,n) \) the standard representation. Since by Lemma 4.4 the generalized Toledo invariant is constant on components of the representation variety, we get that any other representation \( \rho \) in the component of \( \rho_0 \) is maximal. By Theorem 1.7 this implies that \( \rho(\Gamma) \) almost splits as a product \( K \times L_t \times \text{SU}(1,p) \), and is contained in a subgroup of \( \text{SU}(m,n) \) of the form \( \text{SU}(1,p)^t \times \text{SU}(m-t,m-t) \times K \). If the group \( L_t \) is trivial then \( \rho \) is a standard embedding, and is hence conjugate to \( \rho_0 \) up to a character in the compact centralizer of the image of \( \rho_0 \). In particular this would imply that \( \rho_0 \) is locally rigid.

Let us then assume by contradiction that there are representations \( \rho_i \) arbitrarily close to \( \rho_0 \) and with the property that the tube-type factor of the Zariski closure of \( \rho_i(\Gamma) \) is not trivial. Up to modifying the representations \( \rho_i \) we can assume that the compact factor \( K \) in the Zariski closure of \( \rho_i \) is trivial.

By Theorem 1.7 this implies that \( \rho_i(\Gamma) \) is contained in a subgroup of \( \text{SU}(m,n) \) isomorphic to \( \text{SU}(m,pm-1) \), moreover we can assume, up to conjugate the representations \( \rho_i \) in \( \text{SU}(m,n) \), that the Zariski closure of \( \rho_i \) is contained in the same subgroup \( \text{SU}(m,pm-1) \) for every \( i \). Since the representations whose image is contained in the subgroup \( \text{SU}(m,pm-1) \) is a closed subspace of \( \text{Hom}(\Gamma,G)/G \), we get that the image of \( \rho_0 \) is contained in \( \text{SU}(m,pm-1) \) and this is a contradiction, since the image of the diagonal embedding does not leave invariant any subspace on which the restriction of \( h \) has signature \( (m,pm-1) \).
Part III

Maximal representations in tube-type groups
Chapter 10

The partial cyclic ordering on $S_{m,m}$

The purpose of this chapter is to analyze the partial cyclic order induced by the Bergmann cocycle on the Shilov boundary of a tube type subdomain. We will only carry out computations for the group $SU(m, m)$, but this is enough to analyze all classical domains: indeed every classical domain of tube type admits a tight embedding in $SU(m, m)$ and in particular the Shilov boundary of any classical Hermitian Lie group is a closed subspace of $S_{m,m}$. Moreover the partial cyclic order induced by the Bergmann cocycle on $S_G$ is the restriction of the one of $S_{m,m}$.

Maximal triples in $S_{m,m}$ and intervals

Let us fix the point $v_\infty$ in $S_{m,m}$. The Bergmann cocycle induces a partial order on $H_{v_\infty}^{v_\infty}$ defined by requiring that $x < y$ if and only if the triple $(x, y, v_\infty)$ is maximal. The relation $<$ is indeed transitive by the cocycle identity: denoting by $\beta_m$ the Bergmann cocycle on $S_{m,m}$ we have

$$\beta_m(y, z, v_\infty) + \beta_m(x, y, v_\infty) = \beta_m(x, z, v_\infty) + \beta_m(x, y, z).$$

In particular if $x < y < z$ the left hand side is equal to $2m$ and the right hand side can be equal to $2m$ only if both triples $(x, z, v_\infty)$ and $(x, y, z)$ are maximal.

We proved in Section 6.2 that the Heisenberg model $H_{v_\infty}^{v_\infty}$ can be identified with $u(m)$. Throughout this part we will consider the linear identification

$$u(m) \rightarrow \text{Her}(m)$$

$$A \rightarrow iA.$$

With this identification, that we will consider fixed for all the part, it is easy to verify that the order $<$ coincides with the partial order on $\text{Her}(m, \mathbb{C})$ defined by $A < B$ if and only if the difference $B - A$ is positive definite.
Lemma 10.1. Let $A, B$ be Hermitian matrices, and let $a, b$ the corresponding points in $\mathcal{H}_{m,m}^{\infty}$. The triple $(a, b, v_\infty)$ is maximal if and only if the difference $B - A$ is positive definite.

Proof. The content of Lemma 6.4 is that the triple $(v_0, v_1, v_\infty)$ is maximal. Moreover, under the identification $\mathcal{H}_{m,m}^{\infty} \cong \text{Her}(m)$, the space $v_0$ corresponds to the matrix 0, and the space $v_1$ corresponds to $\text{Id}$.

Let now $(x, y, v_\infty)$ be a maximal triple. Since $\text{SU}(m, m)$ is transitive on maximal triples (cfr. Lemma 6.6) there exists an element $g \in \text{SU}(m, m)$ such that $g(v_0, v_1, v_\infty) = (x, y, v_\infty)$. Moreover we know that the stabilizer of $v_\infty$ acts on the Heisenberg model, identified with the space of Hermitian matrices, as $(x, y, v_\infty) \mapsto GXG^* + H$ where $G \in \text{GL}_m(\mathbb{C})$ is any matrix and $H$ is Hermitian. This implies that the difference $y - x$ can be written as $G^*G$ for some $G \in \text{GL}_m(\mathbb{C})$ hence is positive definite.

Viceversa assume that $B - A$ is positive definite, and denote by $a, b \in \mathcal{S}_{m,m}$ the vector spaces represented by $[-iA]$ and $[-iB]$. By assumption there exists a matrix $G$ such that $G^*G = B - A$. The triple $(a, b, v_\infty)$ can be written as $g(v_0, v_1, v_\infty)$ where $g$ is the element $\begin{bmatrix} G & AG^{-1} \\ 0 & G^{-1} \end{bmatrix}$ of $\text{SU}(\mathbb{C}^{2m}, \mathbb{H})$. In particular since $(a, b, v_\infty)$ belongs to the $\text{SU}(\mathbb{C}^{2m}, \mathbb{H})$ orbit of a maximal triple, it is maximal.

Let us now consider the norm on $\mathcal{H}_{m,m}^{\infty}$ defined by $\|X\|^2 = \text{tr}(X^2)$. It is easy to verify that this actually defines a norm on the Hermitian matrices, that, moreover, is compatible with the order in the following sense:

Lemma 10.2. Let $X < Y < Z$, then $\|Y - X\| < \|Z - X\|$.

Proof. Indeed let us denote by $C$ the (positive definite) difference $Z - Y$. We have

$$\text{tr}(Z - X)^2 = \text{tr}(C + Y - X)^2 = \text{tr}(Y - X)^2 + \text{tr}(C)^2 + 2\text{tr}(C(Y - X)).$$

Since both $C$ and $Y - X$ are positive definite, we get that the trace of the product $C(Y - X)$ is greater than zero, for example, it is greater than $\text{tr}(Y - X) \lambda$ where $\lambda$ is the smallest eigenvalue of $C$. This concludes the proof.

Let now $(x, y)$ be a pair of transversal points, we define the interval $((x, y))$ to be the set of points

$$((x, y)) = \{ z \in \mathcal{S}_{m,m} | \beta_m(x, z, y) = m \}.$$

Moreover we will denote by $[x, y]$ the closure of the interval $((x, y))$. By definition if $z \in ((x, y))$, then $(z, z) \subseteq ((x, y)$, and the same clearly holds for closed interval, but, the containment $[x, y] \cup [y, x] \subseteq \mathcal{S}_{m,m}$ is strict if $m$ is strictly greater than 1.

Since $\text{SU}(m, m)$ is transitive on pairs of points, it is enough to have a good understanding of a specific interval, as given in the following lemma:

Lemma 10.3. The interval $((v_0, v_\infty))$ is contained in $\mathcal{H}_{m,m}^{\infty}$ and coincides with the set of positive definite Hermitian matrices. It is a symmetric space of $\text{GL}_m(\mathbb{C})$. 

with stabilizer $U(m)$. Moreover the closure $[[v_0, v_\infty]]$ of the interval intersects the Heisenberg model in the set consisting of Hermitian matrices that are positive semidefinite.

**Proof.** The fact that the interval $((v_0, v_\infty))$ coincides with the positive definite Hermitian matrices is a reformulation of Lemma 10.1.

Moreover the interval is a homogeneous space of the stabilizer of the pair $(v_0, v_\infty)$, since $SU(C^{2m}, \mathbb{C})$ acts transitively on the set of maximal triples, hence the stabilizer of the two extrema of the interval act transitively on the interval. This latter group is isomorphic to $GL_m(\mathbb{C})$ acting on the set of positive definite Hermitian matrices by conjugation. Since $U(m)$ is the stabilizer of the identity matrix in this action, the second statement follows.

Clearly the closure of $((v_0, v_\infty))$ in $\mathcal{H}^{v_\infty}_{m,m}$ is the set of positive semidefinite matrices.

Similarly it is easy to determine the intervals of the form $((x, y))$ where $(x, y, v_\infty)$ is maximal:

**Lemma 10.4.** Assume that $(x, y, v_\infty)$ is a maximal triple. Then the interval $((x, y))$ consists of those Hermitian matrices $z$ such that both $z-x$ and $y-z$ are positive definite.

**Proof.** Let us apply the cocycle relation to the four points $(x, z, y, v_\infty)$. Both $(x, z, y)$ and $(x, y, v_\infty)$ are maximal if and only if both $(x, z, v_\infty)$ and $(v_\infty, z, y)$ are positive, and this concludes the proof.

**Example 10.5.** The reason why we decided to call the sets of the form $((x, y))$ intervals is because, in the case of $S_1 = \partial \mathbb{D}$ the intervals are precisely segments in the circle. An interesting, reducible example, is the case of the torus $T = (S^1)^m$. This is the Shilov boundary of the polydisc $\mathbb{D}^m$, and in this case the Bergmann cocycle is given by the sum of the orientation cocycles of the projections. In particular the intervals of $T$ are $m$-cubes, products of intervals. This example is interesting because $T$ sits as a subspace inside $S_{m,m}$ and it is useful to keep in mind to understand part of the structure of the intervals in $S_{m,m}$ as well.

We say that an $n$-tuple $(x_1, \ldots, x_n)$ of points in $S_{m,m}$ is maximal if for every positively oriented triple of indices $(i, j, k)$, the triple $(x_i, x_j, x_k)$ is maximal.

The cocycle relation implies the following

**Lemma 10.6.** Let $(a, b, c, d)$ be a maximal 4-tuple in $S_{m,m}$. Then the interval $((b, c))$ is contained in the interval $((a, d))$.

**Proof.** Indeed let $t$ be a point in $((b, c))$. Applying the cocycle relation to the 4-tuple $(a, b, t, c)$ we get that $(a, t, c)$ is maximal, then considering the 4-tuple $(a, t, c, d)$ we get that $(a, t, d)$ is maximal.

We will also need to define closed intervals with non-necessarily transversal extrema. In order to do this, let us assume that $((x_i, y_i))$ are nested intervals and $x_i \to x$, $y_i \to y$, we define $[[x, y]] = \cap((x_i, y_i))$. 

Lemma 10.7. 1. The interval is well defined.

2. If \( x, y \) are in \( H^\infty_{m,m} \) and \( y - x \) is positive semidefinite, then

\[
[[x, y]] = \left\{ z \in H^\infty_{m,m} \mid z - x \geq 0, \; y - z \geq 0 \right\}.
\]

3. The set \( [[x, y]] \) is a bounded subset of \( H^z_{m,m} \) for each \( z \) in \( ((y_i, x_i)) \).

4. If \( x \) and \( y \) are not transversal, denoting by \( x \cap y \) the intersection of the linear subspaces, then \( x \cap y \subset z \) for every \( z \in [[x, y]] \).

Proof. Let us assume, up to conjugating with a group element, that \( x = v_0 \) and \( y \) is represented by the matrix \( \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \) for some \( d \). It is easy to verify that, since the intervals are nested, for \( i \) big enough we have that \( x - x_i \) and \( y_i - y \) are positive definite. In particular \( v_\infty \) belongs to some interval \( ((y_i, x_i)) \). The formula of point 2 is then an easy consequence of the parametrization of the intervals we gave in Lemma 10.4. This implies also that the interval is well defined, and that the intersection of the interval with an Heisenberg model based in any point contained in the union \( ((y_i, x_i)) \) is bounded.

It follows from fact 2 that if \( y \) is the matrix \( \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \) then the interval \( [[x, y]] \) consists precisely of matrices of the form \( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \) where \( X \) is an Hermitian \( d \)-dimensional matrix whose eigenvalues \( \lambda_i \) satisfy \( 0 \leq \lambda_i \leq 1 \). This shows that the intersection \( x \cap y = \langle e_{m+d+1}, \ldots, e_{2m} \rangle \) is clearly contained in the linear subspace associated to each element \( z \in [[x, y]] \). Since we can always assume, up to conjugation in \( SU(m, m) \), that \( x = 0 \) and \( y = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \) for some \( d \), this concludes the proof.

Not transversal points

Given a point \( z \in \overline{S}_{m,m} \) we will denote by \( Z_z \) the set of points in \( \overline{S}_{m,m} \) that are not transversal to \( z \). The set \( Z_z \) is the complement of the Heisenberg model \( H^z_{m,m} \) based at \( z \), and is a codimension one algebraic subsets of \( \overline{S}_{m,m} \). Moreover we will denote by base of \( Z_z \) the point \( z \):

Lemma 10.8. The intersection of \( Z_{v_0} \) with \( H^v_{m,m} \) consists of

\[
Z_{v_0} = \{ M \in \text{Her}(m, \mathbb{C}) \mid \det(M) = 0 \}.
\]

Proof. This follows from the very definition of \( Z_{v_0} \).

We will denote by \( v_k \) the subspace \( v_k = \langle e_1, \ldots, e_k, e_{m+k+1}, \ldots e_{2m} \rangle \). Clearly \( v_k \) is a maximal isotropic subspace satisfying \( \dim(v_k \cap v_\infty) = k \), moreover it is easy to describe the intersection of \( Z_{v_k} \) with \( H^v_{m,m} \):

Lemma 10.9. The intersection \( Z_{v_k} \cap H^v_{m,m} \) consists of the matrices of the form \( \begin{bmatrix} M_4 & M_2 \\ M_5 & M_4 \end{bmatrix} \) where \( M_4 \) is a square \( n - k \)-dimensional matrix with \( \det(M_4) = 0 \).
Proof. A point \( w \in \mathcal{H}_{m,m}^{v_\infty} \) that is represented, in the model by the matrix \([M_1, M_2, M_3, M_4]\) is not transversal to \( v_k \) if and only if the determinant of the matrix on the left hand side of the equation is zero:
\[
\det \begin{bmatrix}
iM_1 & iM_2 & \text{Id}_k \\
iM_3 & iM_4 & iM_1 \\
\text{Id}_m & \text{Id}_m & \text{Id}_m - k
\end{bmatrix} = \det \begin{bmatrix}
\text{Id} & iM_2 & iM_1 \\
iM_4 & iM_3 & \text{Id}_k \\
\text{Id}_m & \text{Id}_m - k
\end{bmatrix}.
\]
Clearly the determinant of the latter matrix is zero if and only if \( \det(M_4) = 0 \) and this concludes the proof.

Let us now consider curves \( \gamma : S^1 \to S_{m,m} \). We say that a curve \( \gamma \) is causal if for each positively oriented triple \((x, y, z)\) in \( S^1 \), the image \((\gamma(x), \gamma(y), \gamma(z))\) is maximal. Similarly we say that a map \( \gamma : [0, 1] \to S_{m,m} \) is a causal segment if the image of each monotone triple is maximal. We will often identify a curve (resp. a segment) with its image.

It will be useful in the sequel to understand the intersections of a closed strictly monotone curve with sets of the form \( Z_x \). In order to have the tools for properly doing this, let us define the function \( \lambda : \mathcal{H}_{m,m}^{v_\infty} \to \mathbb{R}^m \) that associates to an Hermitian matrix \( M \) the ordered \( m \)-tuple of its eigenvalues \( \lambda_1 \leq \ldots \leq \lambda_m \).

**Lemma 10.10.** Let \( \gamma : [0, 1] \to \mathcal{H}_{m,m} \) be a causal segment. For every \( i \) the function \( \lambda_i \circ \gamma \) is monotone.

**Proof.** This is known as Weyl’s inequality: if \( B = A + C \) where \( A, B \) are Hermitian matrices and \( C \) is positive definite, then \( \lambda_i(B) \geq \lambda_i(A) + \lambda_1(C) \) (cfr. [Bha97, Theorem III.2.1]).

We finish this chapter by showing some relations between the sets of the form \( Z_x \) and the intervals. This will be useful in Chapter 13:

**Lemma 10.11.** For any pair of transversal points \( x, y \), the Shilov boundary \( S_{m,m} \) splits as the disjoint union
\[
S_{m,m} = ((x, y)) \sqcup \bigcup_{z \in [[y, x]]} Z_z.
\]

**Proof.** Since \( \text{SU}(m,m) \) is transitive on pairs of points, we can assume that \( x = v_0 \) and \( y = v_\infty \). Every point that is not transversal to \( v_\infty \) lies only in the second term of the union, in particular we can restrict to points in \( \mathcal{H}_{m,m}^{v_\infty} \). The Heisenberg model coincides with Hermitian matrices, and an element \( M \in \mathcal{H}_{m,m}^{v_\infty} \) belongs to the interval \((v_0, v_\infty)\) (resp. \([v_\infty, v_0]\)) if and only if it is positive definite (resp. negative semidefinite). Moreover an element \( M \) belongs to \( Z_z \) for some negative semidefinite matrix \( z \) if and only if at least one of the eigenvalues of \( M \) are smaller or equal to \( 0 \), and this concludes the proof.

The second term in the disjoint union of the previous lemma is indeed redundant: we will see that it is enough to consider the union of the singular sets based in a causal arc contained in \([y, x]\) instead of in the whole closed interval. We start with a preliminary case, in which we assume that \( x = y \):
Lemma 10.12. Let $\gamma : S^{1} \to S_{m,m}$ be a continuous closed causal curve, then

$$S_{m,m} = \bigcup_{z \in \gamma} Z_{z}.$$  

Proof. Assume by contradiction that the statement is false, and let $x$ be a point that is not contained in $\bigcup_{z \in \gamma} Z_{z}$. This implies, in particular, that for every $z \in \gamma$, $z$ is transverse to $x$. This implies that $\gamma$ is contained in $H_{x}^{\tau}$. This is a contradiction: $\gamma$ defines a nontrivial element in the fundamental group of $S_{m,m}$ and cannot be contained in a simply connected set (cfr. Proposition 13.6).

Lemma 10.13. Let $x, y$ be transversal points, for every choice of a continuous causal segment $\gamma_{y,x}$ having $y$ and $x$ as endpoints we have

$$S_{m,m} = (\{(x, y)\} \sqcup \bigcup_{z \in \gamma_{y,x}} Z_{z}.$$  

Proof. As a consequence of Lemma 10.11 we get that the union is disjoint: indeed if $\gamma_{y,x}$ is a causal segment, then $\gamma_{y,x}$ is contained in the closed interval $[[y, x]]$. In particular it is enough to show that, if $z$ is an element in $S_{m,m}$ that does not belong to the interval $((x, y))$, then $z$ is not transversal to $\gamma_{y,x}(t)$ for some $t$.

The transitivity of the $\text{SU}(m, m)$-action on maximal triples in $S_{m,m}$ implies that we can assume that $x = v_{1}$ and $y = v_{0}$ and chose a point $v_{\infty}$ in $((x, y))$ that is transversal to $z$. In particular $z$ belongs to the Heisenberg model $H_{x}^{v_{\infty}}$. Let us consider the function $\lambda : I \to \mathbb{R}^{m}$ that associates to a point $t$ the ordered $m$-upla of eigenvalues of the matrix $z - \gamma_{y,x}(t)$. The function $\lambda$ is continuous, moreover, since $z$ does not belong to the interval, there exists at least one eigenvalue of $z$ that belongs to $[0, 1]$. This implies that one of the values of $\lambda$ has different signs at the endpoints of $I$, and hence must have a zero. The thesis follows since $\lambda(t)$ as a null eigenvalue if and only if $z$ is not transversal to $\gamma_{y,x}(t)$. \qed
Chapter 11

(Weakly) monotone maps

In this chapter we will study maps $\phi : \partial \mathbb{H} \rightarrow S_{m,m}$ that arise as boundary maps of maximal representations. In particular they are equivariant with respect to the action of a lattice and are weakly monotone, namely they preserve, in a sense that will be made more clear later, the cyclic ordering of points on a chain. As it was explained in the introduction such maps should not exist. In this chapter we show that the weak monotonicity implies some additional regularity properties for the maps under consideration. The main result of the section is Proposition 11.15, where we describe explicitly an equivariant full measure subset of $\partial \mathbb{H}$ on which the restriction of the map is continuous and equivariant. This will be used in Chapter 12 to show that maximal representations are discrete and injective. The purpose of the last section is to prove additional continuity properties of the map $\phi$. For every point $x$ we are able to find a positive measure subset $L_x$ of chains $C$ containing $x$ with the property that the right and left limit in $x$ of the restriction of $\phi$ to $C$ is well defined. We also prove that the values of the limits are locally constant in $L_x$. This last result will be used in the last chapter of the thesis.

Definitions

Let $\rho : \Gamma \rightarrow SU(m,m)$ be an homomorphism. A measurable map $\phi : \partial \mathbb{H} \rightarrow S_{m,m}$ is $\rho$-equivariant if there exists a full measure subset $X$ of $\partial \mathbb{H}$ such that for every $\gamma \in \Gamma$ and every $x \in X$ we have $\phi(\gamma x) = \rho(\gamma) \phi(x)$. A map $\phi$ is strictly $\rho$-equivariant if $X$ is the whole $\partial \mathbb{H}$.

Recall, from Theorem 4.3, that if $\rho : \Gamma \rightarrow SU(m,m)$ is a maximal representation then there exists an associated $\rho$-equivariant boundary map $\phi$ that is transverse in the sense that it maps almost every pair of distinct points to pairs of transversal points.

We will denote by $c_p$ the Cartan cocycle on $(\partial \mathbb{H})^3$, normalized to have values in $[-1,1]$, and by $\beta_m$ the Bergmann cocycle on $(S_{m,m})^3$, normalized to have values in $\{-m,-m+2,\ldots,m\}$. A map $\phi : \partial \mathbb{H} \rightarrow S_{m,m}$ is weakly monotone if for almost every triple $(x,y,z) \in (\partial \mathbb{H})^3$ with $c_p(x,y,z) = 1$ we have $\beta_m(\phi(x),\phi(y),\phi(z)) = m$. 107
Example 11.1. If \( \rho : \Gamma \to \text{SU}(m,m) \) is maximal and \( \phi : \partial \mathbb{H}^p_C \to S_{m,m} \) is \( \rho \) equivariant, then \( \phi \) is weakly monotone. More generally if \( \rho : \Gamma \to G \) is maximal, where \( G \) is a classical Lie group of Hermitian-type, we can compose \( \rho \) with a tight holomorphic embedding \( G \to \text{SU}(m,m) \) and get an associated weakly monotone boundary map \( \phi : \partial \mathbb{H}^p_C \to S_{m,m} \) for some \( m \).

Similarly a map \( \phi : S^1 \to S_{m,m} \) is weakly monotone if for almost every positively oriented triple \((x, y, z)\) the image \((\phi(x), \phi(y), \phi(z))\) is maximal. If \( \phi : \partial \mathbb{H}^p_C \to S_{m,m} \) is weakly monotone, then for almost every chain \( C \subseteq \partial \mathbb{H}^p_C \) the restriction \( \phi|_C \) is weakly monotone. Moreover we say that a map \( \phi : \partial \mathbb{H}^p_C \to S_{m,m} \) is monotone (or sometimes even strictly monotone) if for every triple \((x, y, z)\) satisfying \( c_p(x, y, z) = 1 \), we have \( \beta_m(\phi(x), \phi(y), \phi(z)) = m \).

### 11.1 Weakly monotone maps from the interval

Weakly monotone maps defined on \( S^1 \) have been studied by Burger, Iozzi and Wienhard [BIW10, Section 5] with the tool of the essential graph: they study the support of the measure \( \lambda \otimes \phi_* \lambda \) on \( S^1 \times S_{m,m} \) and deduce from the weak monotonicity rigidity properties for this closed subset. Our approach is slightly different: instead of discussing the essential graph, we find a full measure set on which the restriction of the map has good properties. This approach is easier to generalize to maps defined on \( \partial \mathbb{H}^p_C \), where we only have information along chains.

Let us now fix a monotone map \( \phi \) defined on \( S^1 \), and define the essential set \( X \) for \( \phi \) to be

\[
X = \left\{ x \in S^1 \mid \forall y, z \in S^1, |\beta_m(\phi(x), \phi(y), \phi(z))| = m \right\}.
\]

Fubini’s theorem implies that if \( \phi \) is weakly monotone, then the set \( X \) is of full measure. Moreover the restriction of \( \phi \) to \( X \) has good properties, as exploited in the following lemma:

**Lemma 11.2.** The restriction of \( \phi \) to \( X \) is strictly transversal and strictly monotone.

**Proof.** We first check that \( \phi|_X \) is strictly transversal, namely that for every pair of distinct points \((x, y) \in X^2 \), we have \( \phi(x) \not\parallel \phi(y) \). The set of pairs \((w, z)\) for which both \((\phi(x), \phi(w), \phi(z))\) and \((\phi(y), \phi(z), \phi(w))\) are maximal is of full measure. We can pick such a pair with the additional requirement that the four points \((x, w, y, z)\) are positively oriented. From the cocycle identity we get that both \((\phi(x), \phi(w), \phi(y))\) and \((\phi(x), \phi(y), \phi(z))\) are maximal, and in turn this implies that \( x \) and \( y \) are transversal.

Similarly, whenever we fix three pairwise distinct points \((x_1, x_2, x_3)\) that are positively oriented, we can find a triple \(y_1, y_2, y_3\) such that the six points \((x_1, y_1, x_2, y_2, x_3, y_3)\) are positively oriented and the images of the four subtriples \((y_1, x_2, y_2), (y_1, y_2, x_3), (y_1, x_3, y_3), (y_1, y_3, x_1)\) are maximal.

Applying now the cocycle identity to the four points \((y_1, x_2, y_2, x_3)\), one gets that the triple \((\phi(y_1), \phi(x_2), \phi(x_3))\) is maximal, applying the identity to \((x_1, y_1, x_3, y_3)\),
one gets that the triple \((\phi(x_1), \phi(y_1), \phi(x_3))\) is maximal, in turn, applying the co-cycle identity to the four points \((x_1, y_1, x_2, x_3)\), we get the desired result.

The next step consists of understanding the intersection of \(\phi(X)\) with a subset of \(S_{m,m}\) of the form \(Z_z\) for some \(z \in S_{m,m}\).

**Proposition 11.3.** Let \(\phi : S^1 \to S_{m,m}\) be a weakly monotone map, \(X \subseteq S^1\) its essential set, and \(z \in S_{m,m}\) a point. Then

\[
\sum_{x \in X} \dim(\phi(x) \cap z) \leq m.
\]

**Proof.** Assume first that \(z\) can be written as \(\phi(t)\) for a point \(t \in X\). Then the thesis follows: from Lemma 11.2 we get that the image of each other point \(x\) in \(X\) is transversal to \(z\), moreover \(\dim(\phi(t) \cap z) = m\).

Let us now assume that there exists a point \(x \in X\) such that \(\phi(x)\) and \(z\) are transversal. We can fix a suitable Heisenberg model \(\mathcal{H}_{m,m}^{\phi(x)}\) for \(S_{m,m}\) in which \(Z_z\) consists of the set of Hermitian matrices of determinant 0. It is clear that, for a given \(s \in X\), the dimension \(\dim(\phi(s) \cap z)\) is the multiplicity of zero as an eigenvalue of the matrix \(\phi(s)\). Moreover a consequence of Lemma 10.10 is that the eigenvalues of \(\phi(s)\) are strictly increasing functions of \(s\), and this implies the statement.

In order to deal with the remaining case we need an explicit description of the intersection \(Z_z \cap \mathcal{H}_{m,m}^{v_{\infty}}\) in the case in which the points \(z\) and \(v_{\infty}\) are not transversal. This is given in Lemma 10.9: since \(Z_z \cap \mathcal{H}_{m,m}^{v_{\infty}}\) consists of Hermitian matrices \(M\) with the property that some corner submatrix \(M_4\) of \(M\) has determinant zero, the statement follows from the observation that Lemma 10.10 applies to the eigenvalues of the submatrix \(M_4\) as well. Indeed if an Hermitian matrix \(C\) is positive definite, each corner submatrix of \(C\) is positive definite. In particular if \(M < N\) we get that \(N_4 = M_4 + C_4\) where \(C_4\) is a positive definite matrix, and this implies that all the eigenvalues of \(N_4\) are strictly greater than the corresponding eigenvalues of \(M_4\).

We now have all the tools we need to define a right continuous map \(\phi^+ : S^1 \to S_{m,m}\) that is a slight modification of \(\phi\):

**Proposition 11.4.** Let \(\phi : S^1 \to S_{m,m}\) be a weakly monotone map, and let \(X\) be its essential set. For every \(t \in S^1\) the limit

\[
\phi^+(t) = \lim_{x \to t^+, x \in X} \phi(x)
\]

is well defined. Moreover the function \(\phi^+\) is strictly monotone and, for every point \(x \in X\), the difference \(\phi^+(x) - \phi(x)\) is positive semidefinite.

**Proof.** Let us fix a point \(T \in X\) different from \(t\) and consider an Heisenberg model \(\mathcal{H}_{m,m}^{\phi(T)}\). We know that, for every \(w \in X\) that is different from \(T\), \(\phi(w)\) belongs to \(\mathcal{H}_{m,m}^{\phi(T)}\), in particular we will do computations in that model. If we fix two points \(a, b\) in \(X\) such that the four points \((a, t, b, T)\) are positively oriented, we get that the
image of the restriction of $\phi$ to the intersection of $X$ with the interval in $S^1$ with extremal points $(a, b)$ is contained in the compact interval $[[\phi(a), \phi(b)]] \subseteq \mathcal{H}_{m,m}$.

Let $l$ be a limit point of a converging sequence $(\phi(x_i))$ where $(x_i) \in S^1$ is monotone and converges to $t$ on the right. Since $\phi(X) \cap Z_l$ consists of a finite number of points, we get that, up to shrinking $(a, b)$, for every point $s \in (t, b)$, $l \leq \phi(s)$. Moreover it is easy to check using the fact that $l$ is the limit point of a sequence in $X$, that $\phi(s) - l > 0$ for every $s \in X$. This implies that every sequence in $X$ converging to $t$ on the right has limit $l$, hence the limit is well defined.

The fact that the function $\phi^+$ is strictly monotone is a consequence of its definition and the fact that $X$ has full measure, hence for every pair of points $t_1, t_2 \in S^1$ there exist an element $x \in X$ such that $(t_1, x, t_2)$ is positively oriented. The last statement follows from the fact that the limit of positive definite matrices is positive semi-definite.

Analogously it is possible to define a map $\phi^- : S^1 \to S_{m,m}$ by the formula

$$\phi^-(t) = \lim_{x \to t^-} \phi(x).$$

Clearly the difference $\phi^+(t) - \phi^-(t)$ is positive semi-definite and the monotonicity of the eigenvalues under a monotone map allow to prove that $\phi^+(t)$ equals $\phi^-(t)$ outside, in the worst case, a countable set of points. This proves the following:

**Proposition 11.5.** Let $\phi : I \to S_{m,m}$ be a weakly monotone map, then

1. There is a full measure set $X$ such that $\phi|_X$ is monotone.
2. The restriction $\phi|_X$ is continuous almost everywhere.
3. There exist maps $\phi^\pm : S^1 \to S_{m,m}$ such that $\phi^\pm$ are respectively right and left continuous and $\phi^\pm = \phi$ on a full measure set $X'$.

### 11.2 Maps from $\partial \mathbb{H}^P_C$

We will now apply Proposition 11.5 to deduce some good properties of a measurable, weakly monotone map $\phi : \partial \mathbb{H}^P_C \to S_{m,m}$. We will denote by $C^{(n)}$ the configuration space

$$C^{(n)} = \{(x_1, \ldots, x_n, C) \mid C \subset \partial \mathbb{H}^P_C \text{ chain, } x_i \in C \text{ point} \}.$$ 

**Lemma 11.6.** Let $\phi : \partial \mathbb{H}^P_C \to S_{m,m}$ be a weakly monotone map, the following sets have full measure.

1. The subset of $C^{(1)}$ defined by

$$Y = \left\{ (x, C) \in C^{(1)} \left| \begin{array}{l} X_C \text{ of full measure in } C \\ x \in X_C, \\ \phi|_{X_C} \text{ is continuous in } x, \end{array} \right. \right\}.$$
11.2. MAPS FROM $\partial \mathbb{H}^p_C$

2. The subset of $\partial \mathbb{H}^p_C$ defined by

$$R = \left\{ x \in \partial \mathbb{H}^p_C \mid \forall C \in W_x, (x, C) \in Y \right\}.$$ 

3. If $x$ belongs to $R$, the set

$$Y_x = \{ C \in W_x \mid (x, C) \in Y \}.$$ 

Proof. Fubini theorem implies that for almost every chain $C$ the restriction of $\phi$ to $C$ is weakly monotone, Proposition 11.5 then implies point (1), point (2) follows from (1) applying Fubini’s theorem again, point (3) follows from the very definition of $Y_x$. 

Let now $F \subseteq \partial \mathbb{H}^p_C$ and $A \subseteq S_{m,m}$ be subsets. We say that the image of $F$ under $\phi$ is essentially contained in $A$, and we write $\phi(F) \subseteq \text{ess } A$ if $\phi^{-1}(A) \cap F$ is a full measure subset of $F$. This implies that, if $F$ is an open subset of $\partial \mathbb{H}^p_C$, then the essential graph of $\phi$ restricted to $F$ is contained in $\overline{A}$.

The idea behind what will follow is that the set $R$ consists of some of the points for which we are sure that the value of $\phi$ is the correct one (recall that $\phi$ is just a measurable map, defined only up to equality almost everywhere). Ideally, we want to use the values of the map on the set $R$ to determine the correct values of the map $\phi$ also on the other points.

Lemma 11.7. Let $w \in R$ be a point and let $U$ be an open neighborhood of $w$. If $U$ satisfies $\phi(U) \subseteq \langle (a, b) \rangle$, then $\phi(w) \in \langle (a, b) \rangle$.

Proof. This is a consequence of the fact that, by definition of $R$, the restriction of $\phi$ to almost every chain through $w$ is weakly monotone, and $w$ is essentially a point of continuity for the restriction of $\phi$ to almost every chain containing $x$: if, by contradiction, $\phi(w)$ did not belong to $\langle (a, b) \rangle$ it would be possible to find an element $d$ such that either $\langle (\phi(w), d) \rangle$ or $\langle (d, \phi(w)) \rangle$ has empty intersection with $\langle (a, b) \rangle$ (this can be easily checked using the explicit expression for the intervals we gave in Chapter 10). This is a contradiction since we would get, from the continuity of $\phi$ in $w$ along chains, that for almost every chain $C$ through $w$ there would exist a right or left neighborhood of $w$ in $C$ that is outside of $\langle (a, b) \rangle$ and this would give a positive measure subset of $U$ whose image does not belong to $\langle (a, b) \rangle$.

The set $R$ has some other good properties:

Lemma 11.8. The restriction of $\phi$ to $R$ is strictly $\Gamma$ equivariant meaning that if $x$ and $\gamma \cdot x$ belong to $R$, then $\phi(\gamma \cdot x) = \rho(\gamma)\phi(x)$.

Proof. Let $x$ be a point in $R$. For almost every chain $C$ through $x$ there exists a full measure subset $X$ of $C$ such that $\phi(x)$ is the limit of $\phi(t)$ with $t \in X$. Since $\phi$ is essentially $\rho$ equivariant we can find a chain $C$ that satisfies this assumption and such that $\gamma \cdot C$ satisfies the same for $\gamma \cdot x$. Intersecting the full measure sets obtained we get the desired result.
11.2.1 The geometric construction

We now need some geometric constructions.

1. Whenever two distinct points \(x, y\) in \(\partial \mathbb{H}^n_C\) are fixed we denote by \(T_{xy}\) the unique chain containing \(x\) and \(y\) and by \(\gamma_{x,y}\) the segment of \(T_{xy}\) that is between \(x\) and \(y\):

\[
\gamma_{x,y} = \{z \in \partial \mathbb{H}^n_C; c(x, z, y) = 1\}.
\]

2. If \(x_1, \ldots, x_n\) are fixed, the simplex \(\Delta_{x_1,\ldots,x_n}\) with vertices \(x_1, \ldots, x_n\) is defined inductively to be the set

\[
\Delta_{x_1,\ldots,x_n} = \{z \in \gamma_{x_1,t}; t \in \Delta_{x_2,\ldots,x_n}\};
\]

where \(\Delta_{x_1,x_2} = \gamma_{x_1,x_2}\).

3. Given two subsets \(U, V\) of \(\partial \mathbb{H}^n_C\) the positive hull of \(U\) and \(V\) is

\[
H(U, V) = \{z \in \gamma_{u,v}; u \in U, v \in V\}.
\]

Let now \(x \in \partial \mathbb{H}^n_C\) be a point. An oriented chain segment \(\gamma_{x,y}\) is generic if the essential set \(X\) of the chain \(T_{xy}\) through \(x\) and \(y\) is of full measure in \(T_{xy}\) and both points \(x, y\) belong to \(X\). More generally we say inductively that a simplex \(\Delta_{x_1,\ldots,x_n}\) is generic if \(\Delta_{x_2,\ldots,x_n}\) is generic and for almost every point \(t \in \Delta_{x_2,\ldots,x_n}\) the segment \(\gamma_{x,t}\) is generic. We will call \(x_1\) the tip of the simplex \(\Delta_{x_1,\ldots,x_n}\).

**Lemma 11.9.** For almost every \(n\)-tuple of points \((x_1, \ldots, x_n)\) the simplex \(\Delta_{x_1,\ldots,x_n}\) is generic. For every point \(x_1\) in \(R\), almost every simplex with tip \(x_1\) is generic.

**Proof.** It is easy to deduce from Proposition 11.5 that almost every chain segment is generic: indeed since the map \(\phi\) is weakly monotone, the subset of \(C^{(2)}\) consisting of triples \((x, y, C)\) such that the essential set \(X_C\) of \(C\) has full measure and \(x, y\) belong to \(X_C\) is of full measure in \(C^{(2)}\) and hence, as a consequence of Fubini, also the set of generic segments is of full measure.

Let us now assume that for almost every \((n-1)\)-tuple of points \((x_2, \ldots, x_n)\) the simplex \(\Delta_{x_2,\ldots,x_n}\) is generic, we show that for every point \(x_1\) in the full measure set \(R\), and for almost every generic simplex \(\Delta_{x_2,\ldots,x_n}\), the simplex \(\Delta_{x_1,\ldots,x_n}\) is generic. Indeed since \(x_1\) is in \(R\), \(x_1\) is in the full measure essential set of \(C\) for almost every chain \(C\) in \(W_{x_1}\). This implies, applying Fubini’s theorem, that for almost every simplex \(\Delta_{x_2,\ldots,x_n}\) the projection \(\pi_{x_1}(\Delta_{x_2,\ldots,x_n})\) satisfies that \(\pi_{x_1}(\Delta_{x_2,\ldots,x_n}) \cap Y_{x_1}\) is of full measure in \(\pi_{x_1}(\Delta_{x_2,\ldots,x_n})\). In order to conclude it is enough to check that for almost every generic simplex and almost every lift \(L\) of \(\pi_{x_1}(\Delta_{x_2,\ldots,x_n})\), the simplex \(L\) is generic, and that if \(\pi_{x_1}(\Delta_{x_2,\ldots,x_n}) \cap Y_{x_1}\) is of full measure in \(\pi_{x_1}(\Delta_{x_2,\ldots,x_n})\).
then for almost every lift \( L \) satisfies that for almost every point \( t \in L \), \( t \) belongs to the essential set of \( \gamma_{x_1,t} \).

**Lemma 11.10.** Let us assume that \( \Delta_{x_1,\ldots,x_n} \) is a generic simplex, and that the \( n \)-tuple \((\phi(x_1), \ldots, \phi(x_n))\) is maximal, then

\[
\phi(\Delta_{x_1,\ldots,x_n}) \subseteq (\phi(x_1), \phi(x_n)).
\]

**Proof.** We argue, again, by induction on the number \( n \) of vertices of the simplex. If \( n = 2 \) the simplex is an oriented chain segment \( \gamma = \gamma_{x_1,x_2} \), and the hypothesis of genericity amounts to saying that the restriction of \( \phi|_\gamma \) is weakly monotone and \( x_1, x_2 \) belong to the essential set for \( \gamma \). In particular for almost every point \( t \in \gamma \), \((\phi(x_1), \phi(t), \phi(x_2))\) is maximal, hence \( \phi(t) \) belongs to \((\phi(x_1), \phi(x_2))\).

Let us now assume that the statements holds for every generic simplex with at most \( n-1 \) vertices. In particular for almost every point \( t \) in \( \Delta_{x_2,\ldots,x_n} \) we have \( \phi(t) \in ((\phi(x_2), \phi(x_3)) \cup \ldots \cup ((\phi(x_{n-1}), \phi(x_n))) \). Moreover, since the simplex \( \Delta_{x_2,\ldots,x_n} \) is generic, we have that, for almost every point \( t \) in \( \Delta_{x_2,\ldots,x_n} \), the oriented chain simplex \( \gamma_{x_1,t} \) belongs to \( Y_{x_1} \).

This implies that, for almost every \( z \) in \( \Delta_{x_1,\ldots,x_n} \), we have \( \phi(z) \in ((\phi(x_1), \phi(t)) \cup \ldots \cup ((\phi(x_1), \phi(t))) \) and, by induction, this latter interval is contained in \((\phi(x_1), \phi(x_n))\), for almost every \( t \).

In the following lemma we summarize properties of positive hulls

**Lemma 11.11.** Let \( U,V \subset \partial \mathbb{H}_C^P \) be open subsets, then

1. The positive hull \( H(U,V) \) is an open subset of \( \partial \mathbb{H}_C^P \).
2. If \( \phi \) is weakly monotone and there exists a maximal \( 4 \)-tuple \((a,b,c,d)\) in \( S_{m,m} \)
   with \( \phi(U) \subseteq ((a,b)) \), \( \phi(V) \subseteq ((c,d)) \) then \( \phi(H(U,V)) \subseteq ((a,d)) \).

**Proof.** In order to check that the positive hull \( H(U,V) \) is an open subset of \( \partial \mathbb{H}_C^P \) it is enough to observe that \( H(U,V) \) can be written as the union of interior of the the cones with tip points \( u \in U \) and with base \( V \). Since each of these subsets is open (because chain segments depend continuously on their endpoints), we get that the hull itself is open.

We now want to verify the second assertion. In order to check it, it is enough to check that \( H(U,V) \cap R \) is contained in \((a,b))\). Let \( s \) be a point in \( H(U,V) \cap R \), the open subset \( \pi_s(U) \cap \pi_s(V) \subset W_s \) is not empty (since \( s \) belongs to \( H(U,V) \)) hence in particular the intersection \( O = Y_s \cap \pi_s(U) \cap \pi_s(V) \) has positive measure. Since \( \phi(U) \subseteq ((a,b)) \) and \( \phi(V) \subseteq ((c,d)) \) we can find a chain \( C \) in \( O \) that intersects both \( U \) and \( V \) in points \( u,v \) that are in the essential set \( X_C \) and satisfy \( \phi(u) \in ((a,b)) \), \( \phi(v) \in ((c,d)) \); almost every point in \( U \) and \( V \) satisfy \( \phi(u) \in ((a,b)) \) and \( \phi(v) \in ((c,d)) \), moreover almost every chain \( C \) in \( O \) satisfies that \( X_C \) intersect \( U \) in a full measure set of \( C \cap U \). This implies that \( (\phi(u), \phi(s), \phi(v)) \) is maximal, hence \( \phi(s) \in ((a,d)) \).

We will now give a procedure to construct generic simplices satisfying the hypothesis of Lemma 11.10. We start with an easy preliminary lemma:
Lemma 11.12. Let \( w \) be a point in \( R \) and \( a \) be a point in \( S_{m,m} \). For every \( C \in Y_w \), there exists a point \( y \in C \) such that \( \phi(\gamma_{w,y})^\text{ess} \subseteq (\phi(w),a) \).

Proof. This follows from the fact that the intersection of two intervals of \( S_{m,m} \) contains the intersection of a ball of \( S_{m,m} \) with one of the two intervals. Moreover, by definition of \( Y_w \), the restriction of \( \phi \) to a chain \( C \) in \( Y_w \) is essentially continuous in \( w \).

We can now prove the following proposition, that will allow us to show discreteness and injectivity of maximal representations.

Proposition 11.13. Let \( x,y \) be points of \( R \), and assume that \( \phi(x) \preceq \phi(y) \). For every point \( z \in \gamma_{x,y} \) there exists a neighborhood \( U \) of \( z \) such that \( \phi(U) \subseteq (\phi(x),\phi(y)) \).

Proof. We will construct generic simplices \( \Delta_{x_1,...,x_{2p}} \) and \( \Delta_{y_1,...,y_{2p}} \) such that

1. the 4p-tuple \( (x = x_1,\ldots,x_{2p},y_1,\ldots,y_{2p} = y) \) is maximal,
2. there exists \( s \in \Delta_{x_1,...,x_{2p}}^\circ \) and \( t \in \Delta_{y_1,...,y_{2p}}^\circ \) such that \( z \in \gamma_{st} \).

Assuming that this is done, the positive hull \( U = H(\Delta_{x_1,...,x_{2p}}^\circ,\Delta_{y_1,...,y_{2p}}^\circ) \) is an open subset of \( \partial H_p^C \) containing \( z \) and with the property that \( \phi(s) \in ((\phi(x),\phi(y))) \) for almost every point \( s \in U \); this follows from Lemma 11.10 and Lemma 11.11.

We now proceed to the construction of the desired simplices: in order to do this we consider the Heisenberg model \( H_{1+p}^C \) and we fix the point \( x \). Since \( x = x_1 \) belongs to \( R \) the set \( Y_z \) is of full measure, and for almost every \( 2p-1 \)-tuple \( (x_2,\ldots,x_{2p}) \) the simplex \( \Delta_{x_1,...,x_{2p}} \) is generic. In particular for almost every chain \( T \) in \( Y_z \) and almost every point \( x_2 \) in \( T \) for almost every triple \( (x_3,\ldots,x_{2p}) \) the associated simplex is generic. We can then fix a chain \( T \) and a point \( x_2 \) in \( T \) that is close enough to \( x_1 \) so that \( (x_1,x_2,y) \) is maximal. Inductively we can construct the simplex \( \Delta_{x_1,...,x_{2p}} \) applying the same argument. In order to construct the simplex \( \Delta_{y_1,...,y_{2p}} \) we start backward from \( y = y_{2p} \) and we chose the first chain \( T \) so that \( \pi_z(T) \) meets the open subset \( \pi_z(\Delta_{x_1,...,x_{2p}}^\circ) \); this is always possible since \( \pi_z(x) = \pi_z(y) \) by assumption. The same argument above allows to construct the desired 4p-tuple and finishes the proof.

\[ \Delta(x_1,\ldots,x_{2p}) \quad \Delta(y_1,\ldots,y_{2p}) \]
11.2 Maps from \( \partial H^p \)

11.2.2 Consequences on the map \( \phi \)

The purpose of the section is to understand what are the consequences of Proposition 11.13 on the map \( \phi \). In particular we show that the restriction of \( \phi \) to \( R \) is continuous (Proposition 11.15) and that the set \( R \) is big in a very precise sense (Remark 11.19). Moreover for the points \( z \in \partial \mathbb{H}^p \) outside \( R \) we study what are the possible limits in \( z \) of the values \( \phi(t_n) \) where \( (t_n) \in \mathbb{R}^N \) converges to \( z \) along a chain. We are able to define maps \( \phi^\pm_z \) on a positive measure subset \( L_z \) of \( W_z \) and show some properties of those (Proposition 11.20).

Proposition 11.13 implies that, if a chain \( C \) intersects \( R \) in a full measure set, as soon as there are two points in \( C \cap R \) whose image is transverse, then the restriction of \( \phi \) to \( R \cap C \) is monotone and transverse:

**Proposition 11.14.** Let \( C \subset \partial \mathbb{H}^p \) be a chain. Assume that \( C \cap R \) is of full measure in \( C \) and assume that there exist \( x, y \in C \cap R \) with the property that \( \phi(x) \prec \phi(y) \). Then for every triple of distinct points \( z_1 \) in \( C \cap R \) we have \(|\beta_m(\phi(z_1), \phi(z_2), \phi(z_3))| = m\).

**Proof.** Indeed combining Proposition 11.13 and Lemma 11.7 we get that for every point \( z \in C \cap R \) we have \( \beta_m(\phi(x), \phi(z), \phi(y)) = \pm m \) where the sign respects the orientation of the triple \( (x, z, y) \). In particular we get that \( \phi(z) \) is transversal to both \( \phi(x) \) and \( \phi(y) \). This, combined with the cocycle relation for \( \beta_m \), gives the desired statement.

Another immediate consequence of Proposition 11.13 is the continuity of the restriction of \( \phi \) to \( R \):

**Proposition 11.15.** Let \( \phi : \partial \mathbb{H}^p \to S_{m, m} \) be a weakly monotone map. Then the restriction of \( \phi \) to \( R \) is continuous.

**Proof.** Let \( x \) be a point of \( R \). By definition of \( R \) we can find a chain \( C \) in \( W_x \) such that the restriction of \( \phi \) to \( C \) is weakly monotone and essentially continuous in \( x \). We can also assume that almost every point in \( C \) belongs to \( R \). The continuity in \( x \) implies that the intervals of the form \( ((\phi(a), \phi(b))) \) with \( a, b \) in \( C \) form a fundamental system of neighborhoods of the point \( \phi(z) \). The thesis is then a direct consequence of Proposition 11.13.

Proposition 11.15 holds for each weakly maximal map \( \phi \) without the assumption that it is equivariant with a maximal representation of a lattice in \( SU(1, p) \). We now focus on maps that are actually equivariant with respect to a representation of a lattice in \( SU(1, p) \) to be able to say more.

Given a point \( z \in \partial \mathbb{H}^p \) we define a (a priori possibly empty) subset \( \mathcal{P}_z \) of \( \mathbb{R}^2 \) by requiring

\[
\mathcal{P}_z = \{(x, y) \in \mathbb{R}^2 | z \in \gamma_{x,y}, \phi(x) \cap \phi(y), \gamma_{xy} \cap R \text{ has full measure in } \gamma_{xy}\}.
\]

Moreover, whenever \( z \) is fixed, we denote by \( W_z \) the subset of \( W_z \) consisting of chains \( C \) with the property that \( C \cap R \) has full measure in \( C \).
Lemma 11.16. If the weakly monotone map $\phi$ is associated with a maximal representation, the set $P_z$ is not empty for every point $z$. Moreover the projection $L_z = \pi_z(P_z)$ is open in the full measure set $W_z$.

Proof. We first remark that, since $R$ has full measure and the chains through a given point $z$ foliate $\partial \mathcal{H}_C \setminus \{z\}$, Fubini’s theorem tells us that for almost every pair of points $(x, y)$ such that $z$ belongs to $\gamma_{x,y}$ the intersection $\gamma_{x,y} \cap R$ has full measure in $\gamma_{x,y}$.

Moreover, since the map $\phi$ is transversal, for almost every pair of points $(x, y) \in R^2$ we have $\phi(x) \cap \phi(y)$. Applying Fubini theorem, this implies that for almost every point $z$ the set $P_z$ has full measure. Since the restriction of $\phi$ to $R$ is strictly $\rho$-equivariant, the set of points for which $P_z$ has positive measure is a non-empty, $\Gamma$-invariant subset of $\partial \mathcal{H}_C$. We conclude the proof by observing that Proposition 11.13 implies, in particular, that the set of points $z$ for which $P_z$ has positive measure is open: indeed if $z_0$ satisfies that $P_{z_0}$ has positive measure, we get that the same is true for every point $z$ in the neighborhood $U$ of $z_0$ constructed in the proof of that proposition. This concludes: the action of $\Gamma$ on $\partial \mathcal{H}_C$ is minimal, and this implies that the only proper invariant closed subset of $\partial \mathcal{H}_C$ is the empty set.

Let now $x \in \partial \mathcal{H}_C$ be a point and let us denote by $L_x \subseteq W_x$ the projection of $P_x$ to $W_x$: by definition of $P_x$ each pair in $P_x$ determines a single point in $W_x$. For each chain $C$ in $L_x$ the restriction of $\phi$ to $C \cap R$ is strictly transverse and strictly monotone, hence we can define, applying Proposition 11.3, two maps

$$\phi_x^\pm : L_x \to S_{m,m}, \quad C \mapsto \lim_{c \to x^\pm, c \in C \cap R} \phi(c).$$

The maps $\phi_x^\pm$ are $\rho$-equivariant:

Lemma 11.17. For every $\gamma$ in $\Gamma$, and $x \in \partial \mathcal{H}_C$ it holds

1. $L_{\gamma x} = \gamma L_x$

2. $\phi_{\gamma x}^\pm = \rho(\gamma) \phi_x^\pm \gamma^{-1}$.

Proof. This follows directly from the fact that the restriction of $\phi$ to $R$ is strictly $\Gamma$-equivariant.

Lemma 11.18. Let $x \in \partial \mathcal{H}_C$ be a point.

1. If there exists a chain $C \in L_x$ such that $\phi_x^+(C) = \phi_x^-(C)$, for every other chain $C' \in L_x$ we have $\phi_x^+(C') = \phi_x^-(C')$.

2. If there exists a chain $C \in L_x$ such that $\phi_x^+(C)$ is not transversal to $\phi_x^-(C)$, denoting by $\phi^k(C)$ the common subspace of $\phi^+(C)$ and $\phi^-(C)$, we get that, for every other chain $C' \in L_x$, $\phi_x^k(C) \subseteq \phi_x^k(C')$.

Proof. Let us fix sequences $t_n^+, t_n^- \in C \cap R$ converging to $x$ on the right and on the left. By definition $\phi(t_n^\pm)$ converge to $\phi_x^\pm(C)$. As a consequence of Proposition 11.13 we can find points $t_n^\prime$ in $C'$ such that
11.2. MAPS FROM $\partial \mathbb{H}_C^p$

1. $l_n^+(\text{resp. } l_n^-)$ converges to $x$ on the right (resp. on the left)

2. $\phi(l_n^+) \in ((\phi(t_n^-), \phi(t_n^+)))$

Since $\phi^+(C')$ are the limits of $\phi(l_n^+)$ and the intervals $((\phi(t_n^-), \phi(t_n^+)))$ are nested, we get that $\phi^+(C') \in [[\phi^-(C), \phi^+(C)]] = \cap ((\phi(t_n^-), \phi(t_n^+)))$. The first statement is now obvious, since if $\phi^+(C) = \phi^-(C)$ then the interval $[[\phi^-(C), \phi^+(C)]]$ consists of that single point. The second statement is a consequence of the fact that each point in $[[\phi^-(C), \phi^+(C)]]$ contains $\phi^+(C)$ (cfr. Lemma 10.7).

A first application of Lemma 11.18 is that we can assume that the set $R$ is big in the following sense:

**Remark 11.19.** Up to modifying the map $\phi$ on a measure zero subset, we can assume that the set $R$ contains all the points of continuity for the restriction of $\phi$ to each chain on which $\phi$ is weakly monotone. In particular, for every chain $C$ on which the restriction of $\phi$ is weakly monotone, $C \setminus (R \cap C)$ consists of at most countably many points. This remark will be useful in Section 13.4.

We finish this chapter showing that the maps $\phi^\pm_x$ are locally constant:

**Proposition 11.20.** Let $x \in \partial \mathbb{H}_C^p$ be a point. The limits $\phi^+_x(C)$ and $\phi^-_x(C)$ are well defined for every $C \in L_x$ and are locally constant in $L_x$.

**Proof.** Let $C$ be a chain in $L_x$. As a consequence of the definition of $L_x$, $C \cap R$ has full measure in $C$, moreover for every positively oriented triple $(t_1, t_2, t_3) \in (C \cap R)^3$ we have that the triple $(\phi(t_1), \phi(t_2), \phi(t_3))$ is maximal (cfr. Proposition 11.14). Let us then denote by $(l_n^C)$, $(r_n^C)$ sequences in $C \cap R$ converging to $x$ monotonously on the left and on the right respectively.

By definition of $\phi^+_x(C)$ and of $\phi^-_x(C)$ we have that $\phi^+_x(C) = \lim_n \phi(l_n^C)$ and that $\phi^-_x(C) = \lim_n \phi(r_n^C)$. Moreover the intervals $((\phi(r_n^C), \phi(l_n^C)))$ satisfy $((\phi(r_{n+1}^C), \phi(l_{n+1}^C))) \subseteq ((\phi(r_n^C), \phi(l_n^C)))$ and, if $\phi^+_x(C)$ and $\phi^-_x(C)$ are transversal, then $((\phi^+_x(C), \phi^-_x(C))) = \bigcup ((\phi(r_n^C), \phi(l_n^C)))$ and for every $t \in C \cap R$, $\phi(t) \in ((\phi^+_x(C), \phi^-_x(C)))$.

Let us now consider two different chains $C, D$ in $L_x$. As a consequence of Proposition 11.13 we can chose sequences $(r_n^C)$, $(l_n^C)$ and $(r_n^D)$, $(l_n^D)$ with the properties that the sequences consist of points in $R$, belong to the given chain, converge monotonically to $x$ on the respective side and satisfy that $((\phi(r_n^C), \phi(r_M^C)), \phi(r_n^D))$ and $((\phi(l_n^C), \phi(l_M^C)), \phi(r_n^D))$ are maximal for $M > n$ and $(\phi(l_n^D), \phi(r_M^D), \phi(r_n^D))$ and $(\phi(l_n^D), \phi(l_M^D), \phi(r_n^D))$ for $M > n + 1$. 


We claim that this is the case if there exists an \( n \) such that the intervals \((\phi(l_n^C), \phi(r_n^C))\) and \((\phi(l_n^D), \phi(r_n^D))\) intersect. Indeed it is easy to check that the intervals \((\phi(l_n^C), \phi(r_n^C))\) and \((\phi(l_n^D), \phi(r_n^D))\) are nested, that is the 4-tuple \((\phi(l_n^C), \phi(l_n^D), \phi(r_n^C), \phi(r_n^D))\) is maximal if and only if \((\phi(l_n^C), \phi(l_n^D), \phi(r_n^D))\) is maximal. In turn this implies that the 4-tuple \((\phi(l_n^C), \phi(l_n^D), \phi(r_n^D), a)\) is maximal and hence \((\phi(l_n^C), \phi(l_n^D), a)\) is maximal. In turn this implies that the 4-tuple \((\phi(l_n^C), \phi(l_n^D), \phi(r_n^D), a)\) is maximal that allows to deduce that the triple \((\phi(l_n^C), \phi(l_n^D), \phi(r_n^D))\) is maximal.

This implies that in order to prove the proposition it is enough to prove that, locally in \( \mathcal{W}_x \), the intervals \((\phi(r_n^C), \phi(l_n^C))\) and \((\phi(r_n^D), \phi(l_n^D))\) intersect.

Indeed let \( t \) be a point in \( C \cap R \), we want to find a neighborhood of \( \pi_x(t) \) in \( L_x \) consisting of chains \( D \) with the property that \((\phi(r_n^D), \phi(l_n^D))\) intersects \((\phi(r_n^C), \phi(l_n^C))\). But this follows easily from the fact that for almost every Euclidean circle containing \( \pi_x(t) \) there exists an open interval containing \( t \) that consists of chains \( D \) with this property: indeed for almost every Euclidean circle, the lift \( F \) containing \( t \) belongs to \( Y_1 \).

This implies that the point \( t \) is of continuity for such \( F \), in particular, since \( \phi(t) \) belongs to \((\phi(r_n^C), \phi(l_n^C))\), we can find an open interval \( I \) containing \( t \) in \( F \) with the property that every point \( t_D \) in \( I \cap R \) still belongs to \((\phi(r_n^C), \phi(l_n^C))\). The associated vertical chain \( D \) would satisfy that \((\phi(r_n^D), \phi(l_n^D))\) intersects \((\phi(r_n^C), \phi(l_n^C))\), and this proves that the maps \( \phi_{x}^{\pm} \) are locally constant in \( L_x \). 

\( \square \)
Chapter 12

Application: maximal representations are discrete and injective

The purpose of this chapter is to deduce from Proposition 11.13 that maximal representations of complex hyperbolic lattices in $SU(m,n)$ are always discrete and injective. From the main theorem of Part II we know that given a representation in $SU(m,n)$, unless the image is contained in a tube type subdomain, the representation is discrete and injective, in particular it is enough to restrict to maximal representations contained in $SU(m,m)$.

**Proposition 12.1.** Let $\rho : \Gamma \rightarrow SU(m,m)$ be a maximal representation, then the kernel of $\rho$ is contained in the finite center of $\Gamma$.

**Proof.** Let us assume by contradiction that the kernel $N$ of $\rho$ is infinite. In this case the limit set of $N$ is the whole $\partial \mathbb{H}_C^p$: indeed, since $N$ is not finite, its limitset is a non-empty, closed subspace of $\partial \mathbb{H}_C^p$, that is also $\Gamma$-invariant since $N$ is normal. The minimality of the $\Gamma$ action of $\partial \mathbb{H}_C^p$, that coincides with the limitset of $\Gamma$, implies that $\partial \mathbb{H}_C^p$ coincides also with the limitset of $N$, and in particular that the action of $N$ on the boundary is minimal.

Let now $\phi$ be a weakly monotone, measurable $\rho$-equivariant boundary map, and let $R \subseteq \partial \mathbb{H}_C^p$ be the full measure subset defined in Chapter 11. Let us moreover fix a chain $C$ on which the restriction of $\phi$ is weakly monotone, and four cyclically ordered points $x_1, y_1, x_2, y_2$ that are in $C \cap R$. By monotonicity of $\phi|_C$ we get that the intervals $((\phi(x_1), \phi(y_1)))$ and $((\phi(x_2), \phi(y_2)))$ are disjoint. Moreover, since the images $\phi(x_i)$ and $\phi(y_i)$ are transversal points, we get, as a consequence of Proposition 11.13, that there exist open subsets $U_1, U_2$ such that $\phi(U_i) \subseteq ((\phi(x_i), \phi(y_i)))$.

This is already a contradiction: since the action of $N$ on $\partial \mathbb{H}_C^p$ is minimal, the set $N \cdot U_i$ is equal to the whole $\partial \mathbb{H}_C^p$, but since the restriction of $\phi$ to $R$ is strictly $\Gamma$-equivariant, and $N$ is in the kernel of the representation $\rho$, we get that the image
of every point in $R$ must be sent to a point that belongs to the empty intersection $((\phi(x_1), \phi(y_1))) \cap ((\phi(x_2), \phi(y_2)))$. \hfill \qed

A slightly more sophisticated application of Proposition 11.13 allows to prove that maximal representation are discrete:

**Proposition 12.2.** Let $\rho : \partial \mathbb{H}^p \to \text{SU}(m, m)$ be a maximal representation, then the image of $\rho$ is a discrete subgroup of $\text{SU}(m, m)$.

**Proof.** Let us assume first that the representation $\rho$ is Zariski dense. In this case, assuming, by contradiction, that $\rho$ is not discrete, we get that $\rho$ is dense: indeed if we denote by $G$ the closed subgroup of $\text{SU}(m, m)$ that is the connected component of the identity in the closure of $\rho(\Gamma)$, we get that $G$ is normalized by the Zariski dense subgroup $\rho(\Gamma)$ if $\text{SU}(m, m)$. This implies that it is normal in $\text{SU}(m, m)$ and hence, since $\text{SU}(m, m)$ is simple, it equals the whole ambient group. This implies that there exists an element $\gamma$ in $\Gamma$ such that $\rho(\gamma)$ is an elliptic element: elliptic elements form an open subset of $\text{SU}(m, m)$. Moreover, as a consequence of Selberg’s lemma, we can assume that $\gamma$ is not elliptic: there exists a finite index torsion-free subgroup $\Gamma_0$ of $\Gamma$ and we can apply the same argument to the restriction of $\rho$ to $\Gamma_0$.

Let now $I \subseteq S_{m,m}$ be an interval of the form $((\phi(x), \phi(y)))$ for a pair of points $x, y$ with transverse images, and let $U \subseteq \partial \mathbb{H}^p$ be an open subset of $\partial \mathbb{H}^p$ with the property that $\phi(U)$ is essentially contained in $I$, as above this exists as a consequence of Proposition 11.13. We claim that, for every open interval $I'$ containing $I$, we can find a full measure subset $W$ of $\partial \mathbb{H}^p$ with the property that $\phi(W)$ is contained in $I'$. This gives easily a contradiction since, as in the proof of Proposition 12.1, we could then find two disjoint intervals $I'_1, I'_2$ with the property that a full measure subset of $\partial \mathbb{H}^p$ is mapped to both intervals.

We divide the proof of the claim in two (similar) cases. Assume first that $\gamma$ is an hyperbolic isometry in $\partial \mathbb{H}^p$. Since the action of $\Gamma$ on $\partial \mathbb{H}^p$ is minimal there exists a conjugate $\gamma_1$ of $\gamma$ with the additional property that the repulsive fixed point of $\gamma_1$ is contained in $U$. Moreover the image $\rho(\gamma_1)$ will still be an elliptic element being conjugate to $\rho(\gamma)$. Since $\rho(\gamma_1)$ is elliptic we can find a diverging sequence of integers $n_k$ with the property that $\rho(\gamma_1)^{n_k}(I) \subseteq I'$. This implies that the image under $\phi$ of a full measure subset of $\bigcup_k \gamma_1^n U$ is contained in $I'$, and this proves the claim since every point different from the attractive fixedpoint of $\gamma_1$ belongs to the the aforementioned union.

In the case in which $\gamma$ is a nilpotent element we can assume, again up to conjugating $\gamma$, that the fix point of $\gamma$ belongs to $U$. In this case we get that, for every $n$ big enough, the union of $U$ and $\gamma^n : U$ covers the whole $\partial \mathbb{H}^p$. The result then follows along the same lines.

The general case, when the representation $\rho$ is not Zariski dense, can be reduced to this first case by recalling (cfr. Theorem 3.5) that the Zariski closure $G$ of the representation is a reductive Lie group of Hermitian type. Let $G_1 \times \ldots \times G_n$ be the decomposition into irreducible Lie factors of $G$. As a consequence of 5.1 we know that for every $i$ the composition $\rho_i$ of $\rho$ with the projection to the $i$-th factor is maximal, and there exists an $i$ for which $\rho_i$ is not discrete. This implies that we can find an
elliptic element in the image of $\rho_i$. It is then enough to compose $\rho_i$ with a tight embedding of $G_i$ into $SU(m_i, m_i)$ for some $m_i$ to get a maximal representation $\overline{\rho}$ with the property that there exists an infinite order element $\gamma$ whose image is an elliptic element, and this gives, again, a contradiction. 

As remarked by Anna Wienhard, Propositions 12.1 and 12.2 can be combined to show that Conjecture 1 holds for some small values of $m$:

**Corollary 12.3.** Let $\Gamma \in SU(1, p)$ be a lattice and let $m^2 < p$. Let $\rho : \Gamma \to SU(m, n)$ be a maximal representation, then $n > mp$ and $\rho$ is conjugate to a diagonal embedding.

**Proof.** Indeed if $\Gamma$ is a cocompact lattice in $SU(1, p)$ the cohomological dimension of $\Gamma$ is $2p$, and if it is a nonuniform lattice, its cohomological dimension is $2p - 1$. In particular $\Gamma$ cannot act properly discontinuously on a CAT(0) space whose dimension is smaller than $2p$ (resp. $2p - 1$). Since the symmetric space of $SU(m, m)$ has dimension $2m^2$ we get that there exists no maximal representation of $\Gamma$ into $SU(m, m)$ when $m^2 < p$. This, together with Theorem 1.7, implies the claimed statement. 

Chapter 13

Four proofs of nonexistence of boundary maps satisfying some additional hypotheses

In this last chapter of the thesis we give some evidence supporting the conjecture that weakly monotone maps $\phi : \partial \mathbb{H}_C^p \to S_{m,m}$ do not exist. In particular we give four completely different proofs that maps $\phi : \partial \mathbb{H}_C^p \to S_{m,m}$ satisfying some stronger versions of monotonicity do not exist. This illustrates some of the contrasting features of the chain structure on the boundary of the complex hyperbolic space and the partial cyclic order on $S_{m,m}$ and gives possible directions of future research to show the validity of Conjecture 1.

The first proof, with the strong assumption that the map is differentiable, gets its contradiction from the fact that every direction in $T_x \partial \mathbb{H}_C^p$ apart from the ones contained in the contact plane are tangent to chains, whereas a vector tangent to a monotone curve in $S_{m,m}$ needs to be contained in a proper convex cone.

The second proof has the assumption that the map is continuous and deduces the contradiction from the fact that the boundary of the complex hyperbolic space is simply connected, but any causal curve in $S_{m,m}$ defines a nontrivial element in the fundamental group of $S_{m,m}$.

The third proof requires that the map $\phi$ is strictly monotone, namely it maps each pair of distinct points to a pair of transversal points and each triple of positively oriented points on a chain to a maximal triple. The contradiction is given by the fact that each pair of points in $\partial \mathbb{H}_C^p$ is transverse, but, if $\gamma : S^1 \to S_{m,m}$ is a causal curve, then for every point $x \in S_{m,m}$ there exists at least one point $t$ in $S^1$ such that $x$ is not transversal to $\phi(t)$.

The fourth proof requires that the map $\phi$ is equivariant with respect to a maximal representation of a cocompact lattice in $\text{SU}(1,p)$, and has the additional requirement that the map is strictly monotone. Under these hypotheses we are able to deduce continuity by studying Anosov properties of the associated flat $\text{SU}(m,m)$-bundle over the unit tangent bundle of the compact complex hyperbolic manifolds.
13.1 Proof 1: differentiable maps

Causal structure and its convexity

Kaneyuki [Kan91] proved that, for every Lie group $G$ of Hermitian type, the Shilov boundary $S_G$ admits a unique (up to inversions) $G$-invariant causal structure $C$, that is a $G$-invariant family of pointed open cones $C_x$ in the tangent spaces $T_xS_G$. Here by pointed we mean that $\overline{C_x} \cap -\overline{C_x} = \{0\}$.

In the case $G = SU(m, m)$ we identify $S_{m, m}$ with its Heisenberg model

$$H_{m, m}^\infty = \{X \in M(m \times m, \mathbb{C}) | X^* = X\},$$

where the matrix $X$ represents the isotropic subspace with base $[i \, X]$ (see Section 6.2). Since $H_{m, m}$ is a vector space, we identify the tangent space of $S_{m, m}$ with $H_{m, m}^\infty$ itself. Then the cone $C_{v_0}$ consists of the positive definite Hermitian matrices, this is clearly an open pointed cone, and it is invariant by the action of $d_{v_0}q$ for every element $q$ in $Q_0 = \text{Stab}_G(v_0)$: we know that

$$Q_0 = \left\{ \begin{bmatrix} A & 0 \\ B & A^{-1} \end{bmatrix} | A^*B + B^*A = 0 \right\}$$

and it acts on $H_{m, m}^\infty$ as $X \mapsto AX(BX + A^{-})^{-1}$. In particular $d_{v_0}q$ is the linear map given by the $X \mapsto AXA^*$ that clearly preserves the positive definite matrices.

The maximal compact subgroup $K = S(U(m) \times U(m))$ of $SU(m, m)$ acts transitively on $S_{m, m}$, hence $S_{m, m} = K/M$ where $M = SU(m) \times \mathbb{Z}/2\mathbb{Z}$ is the stabilizer, in $K$, of the basepoint $v_0$. It is easy to check that $(A, B) = \text{tr} AB$ gives an $M$-invariant inner product on $T_{v_0}S_{m, m}$ such that the cone $C_{v_0}$ is self dual with respect of $(\, , )$. This means that

$$C_{v_0} = \{ w \in T_{v_0}S_{m, m} | (v, w) > 0 \text{ for all } v \in \overline{C_{v_0}} \}.$$ 

Moreover the identity is a vector $\text{Id} \in T_{v_0}S_{m, m}$ that is fixed by $M$, and, for all $w \in \overline{C_{v_0}}$, we have $(e, w) > \|w\|$, that is the cone $C_{v_0}$ is uniformly acute.

On the contact structure of $\partial \mathbb{H}^p_C$

The manifold $\partial \mathbb{H}^p_C$ is a real $2p - 1$ dimensional submanifold of $\mathbb{C}^p$, in particular, for every point $x \in \partial \mathbb{H}^p_C$, there is a distinguished $(2p - 2)$-dimensional subspace $Z_x$ of $T_x\partial \mathbb{H}^p_C$ defined by $Z_x = T_x\partial \mathbb{H}^p_C \cap iT_x\partial \mathbb{H}^p_C$. This is a contact structure on $\partial \mathbb{H}^p_C$ that is actually the standard contact structure on the sphere $S^{2p-1}$.

Let us now consider an Heisenberg model $H_{1, p}^{\infty}$ so that $(0, 0)$ is the image of the point $x$. The image, in $H_{1, p}^{\infty}$, of the plane $Z_x$ is the horizontal plane $z = 0$ (here we use $z$ to denote the real coordinate in the Heisenberg group). The contact structure in a generic point is obtained translating this plane with an appropriate element of $Q$ and is always transversal to the vertical lines.

Lemma 13.1. For every vector $v \in T_{v_0}H$, that does not belong to the contact plane $Z_{v_0}$, there is precisely one chain $C$ through $v_0$ with $v$ as tangent vector.
Proof. This is clear in the bounded model: if we fix a vector \( v \) that does not belong to the contact plane, the associated complex projective line intersects the ball model for \( \mathbb{H}_C^P \) in a disc whose boundary is the chain we were looking for.

The first proof

Lemma 13.2. Let \( \gamma : S^1 \to S_{m,m} \) be a weakly monotone curve, and \( t \in S^1 \) be a point of differentiability for \( \gamma \), that belongs to the essential set of \( \gamma \). The vector \( \gamma'(t) \) belongs to the closure of the cone \( C_{\gamma(t)} \).

Proof. Since the curve \( \gamma \) is monotone and \( t \) belongs to the essential set of \( \gamma \), we can find a point \( t_1 \in S^1 \) such that for almost every \( t < t_2 < t_1 \), \( (\phi(t), \phi(t_2), \phi(t_1)) \) is maximal.

Since \( SU(m, m) \) acts transitively on \( S_{m,m} \), and since the cone field \( C \) is \( SU(m, m) \)-invariant, we can assume that \( \gamma(t) = v_0, \gamma(t_1) = v_\infty \). In the Heisenberg model both the cone in \( T_{v_0} S_{m,m} \) and the points such that the triple \( (v_0, z, v_\infty) \) is maximal can be identified with the set of positive definite Hermitian matrices. In particular for almost every point \( t_2 \in S^1 \) between \( t \) and \( t_1 \) we have \( \gamma'(t_2) \in C_{v_0} \). The conclusion follows from the fact that \( \gamma'(t) = \lim_{t_2 \to t} \gamma'(t_2)/(t_2 - t) \).

Proposition 13.3. Let \( \phi \) be a differentiable monotone map \( \phi : \partial \mathbb{H}_C^P \to S_{m,m} \), then \( \phi(\partial \mathbb{H}_C^P) \) is contained in a simple causal curve.

Proof. Let us fix a point \( x \in \partial \mathbb{H}_C^P \). We first show that \( d\phi_x(T_x \partial \mathbb{H}_C^P) \) consists of a single line. Indeed for every vector \( v \) in \( T_x \partial \mathbb{H}_C^P \) with positive last coordinate, there exists a chain \( C \) through \( x \) with tangent vector \( v \) at \( x \). In particular, as a consequence of Lemma 13.2, we get that \( d\phi(v) \) is contained in the cone \( C_{\phi(x)} \).

Let us assume by contradiction that there are two vectors \( v_1, v_2 \) of \( T_x \partial \mathbb{H}_C^P \) are sent to two independent vectors. Using the convexity of \( C_x \) we would be able to find a third vector \( v_3 \) in the upperhalf plane that is sent outside \( C_x \): an acute cone cannot contain an halfspace. This is a contradiction: we know that all the vectors in the upperhalf plane are mapped in \( C_{\phi(x)} \). This implies that the image of \( d\phi_x \) is contained in a single line.

Let us now consider the restriction of \( \phi \) to a given chain \( C \). Since \( \phi \) is differentiable, in particular the restriction of \( \phi \) to the smooth submanifold \( C \) of \( \partial \mathbb{H}_C^P \) is differentiable, hence the image \( \phi(C) \) is a smooth causal curve. Since \( \partial \mathbb{H}_C^P \) is connected the same holds for the image, and, since for every point \( x \) with image in \( C \) we have that \( d\phi_x(T_x \partial \mathbb{H}_C^P) \subseteq T_\phi(x) \), we get that \( C \) coincides with the image of \( \phi \).

As a consequence we get the first proof of nonexistence of maximal representations with nice boundary maps:

Corollary 13.4 (First proof). No maximal representation \( \rho : \Gamma \to S_{m,m} \) admits a differentiable boundary map.

Proof. Indeed, since \( \partial \mathbb{H}_C^P \) is simply connected, there is no continuous map \( \phi : \partial \mathbb{H}_C^P \to S^1 \) that induces a surjective map on fundamental groups, but a boundary
map associated to a maximal representation would have this property, since the restriction to a chain would induce a surjective map on fundamental groups by monotonicity.

\[\text{Remark 13.5.} \text{ Even if the differentiability is a strong assumption, we decided to include this proof because, as opposed to the other arguments we will give in the next sections, it is intrinsically of local nature. Maybe it is possible to find a suitable measurable generalization of this argument by considering weak notions of differentiation (cfr. for example [Pan89] and [CK10]).}\]

\section{Proof 2: continuous maps}

In the second proof we assume the map \( \phi \) to be continuous.

\begin{proposition}[Proof 2, continuous map] There are no continuous weakly monotone boundary maps \( \phi : \partial \mathbb{H}^P \to S_{m,m} \) if \( p > 1 \).
\end{proposition}

\begin{proof}
We will show that, if \( \phi \) is continuous and weakly monotone, there exist chains \( C \) in \( \partial \mathbb{H}^P \) such that the image of the restriction of \( \phi \) to \( C \) is a nontrivial loop in \( \pi_1(S_{m,m}) \). This gives a contradiction with the fact that the visual boundary \( \partial \mathbb{H}^P \) is simply connected.

Let us first consider the curve that is described, in the bounded model, by \( \gamma : S^1 \to S_{m,m} \lambda \mapsto \begin{bmatrix} \lambda \Id & \\ & \Id \end{bmatrix} \).

\( \gamma \) defines a nontrivial element in the fundamental group of \( S_{m,m} \); in fact the determinant function \( \det : U(m) \to S^1 \) maps \( \gamma \) to a nontrivial element in \( Z = \pi_1(S^1) \).

Let us now consider the curve \( \gamma_1 : S^1 \to S_{m,m} \) that is the image of the restriction of \( \phi \) to a chain \( C \) on which \( \phi \) is weakly monotone. We will show that \( \gamma_1 \) is homotopic to \( \gamma \), hence in particular defines, as well, a nontrivial curve in \( \pi_1(S_{m,m}) \). We fix three points \( x_0, x_1, x_2 \) in the essential set for the restriction of \( \phi \) to \( C \), and, up to the action of an element \( g \in SU(m,m) \), we can assume that \( \phi(x_0) = \nu_0, \phi(x_1) = \nu_1, \phi(x_2) = \nu_\infty \). Since \( SU(m,m) \) is connected, \( \gamma_1 \) and \( g \circ \gamma_1 \) define the same element in \( \pi_1(S_{m,m}) \). Up to reparametrizing \( \gamma \) we can assume that \( \gamma(x_1) = \gamma_1(x_1) \).

Moreover both \( \gamma|_{[x_0,x_1]} \) and \( \gamma_1|_{[x_0,x_1]} \) are contained in the interval \( ([\nu_0, \nu_1]) \): since the curve \( \gamma_1 \) is weakly monotone, this holds for a full measure subset of \([x_0, x_1]\) and hence for all the points because \( \phi \) is continuous. Since the interval \( ([\nu_0, \nu_1]) \) is simply connected it follows that the restrictions to \([x_0, x_1]\) of the two curves are homotopic. Since the same is true for the restriction \( \gamma_1|_{[x_1, x_0]} \) that is contained in the simply connected interval \( ([x_1, x_0]) \), the two loops are homotopic. This concludes the proof.
\end{proof}

Using Proposition 11.13 it is possible to strengthen the argument of Proposition 13.6 to the following:
13.3. **Proof 3: Strictly Monotone Maps**

In this section we focus on a strictly monotone map \( \phi : \partial \mathbb{H}^n_\mathbb{C} \to S_{m,m} \). Recall that this means that for every pair of distinct points \( x, y \), the image subspaces \( \phi(x), \phi(y) \) are transversal, and that for every triple \( (x, y, z) \) on a chain the dimension of the linear subspace \( \langle \phi(x), \phi(y), \phi(z) \rangle \) is equal to \( 2m \).

**Proposition 13.9.** There exists no strictly monotone map \( \phi : \partial \mathbb{H}^n_\mathbb{C} \to S_{m,m} \).

**Proof.** Let us assume that such a map \( \phi \) exists and let us fix a chain \( C \). Our first aim is to define a continuous and monotone map \( \psi : S^1 \to S_{m,m} \), that extends \( \phi \). To do this we consider two maps \( \phi^\pm : S^1 \to S_{m,m} \) that are respectively right and left continuous and agree with \( \phi \) outside a countable set \( X \subset C \) (cfr. Proposition 11.5). Let us now denote by \( I \) the interval \([0, 1]\) and by \( K \) the Cantor set inside \( I \), and fix a monotone identification of the connected components \( O_i = (O_i^-, O_i^+) \) of the Cantor set.

Let \( \psi : S^1 \to S_{m,m} \) be a strictly monotone map and let \( \gamma \) be a nontrivial element of the fundamental group of \( \partial \mathbb{H}^n_\mathbb{C} \). Then there exists no chain \( C \) on which the restriction of \( \phi \) is continuous, transversal and such that \( C \cap R \) has full measure in \( C \).

**Proof.** We first remark that, since \( C \cap R \) has full measure in \( C \) and \( \phi |_C \) is transversal, the restriction of \( \phi \) to \( C \cap R \) is monotone (cfr. Proposition 11.14), in particular, using the assumption that the restriction of \( \phi \) to \( C \) is continuous, we get that \( \phi |_C \) gives a nontrivial element in the fundamental group of \( S_{m,m} \).

However, since \( \phi \) is weakly monotone we can apply Proposition 11.13 and find an interval \( I = ((a, b)) \) in \( S_{m,m} \) and an open subset \( U \) in \( \partial \mathbb{H}^n_\mathbb{C} \) such that \( \phi(U \cap R) \subseteq ((a, b)) \). Moreover, using that \( \Gamma \) is a lattice and hence its action on \( \partial \mathbb{H}^n_\mathbb{C} \) is minimal, that the map \( \phi \) is \( \rho \)-equivariant and that the set \( R \) is \( \Gamma \)-invariant (cfr. Lemma 11.8), we can assume that the chain \( C \) is contained in \( U \); otherwise we find another chain, of the form \( \gamma \cdot C \) for some \( \gamma \in \Gamma \) with these properties. This implies, using that \( C \cap R \) has full measure in \( C \), that \( \phi(C) \) is contained in the simply connected subset \( I \) of \( S_{m,m} \): indeed the image of a full measure subset of \( C \) is contained in \( I \), and this implies that \( \phi(C) \) is contained in \( I \) since the restriction of \( \phi \) to \( C \) is continuous and monotone. This is in contradiction with the fact that \( \phi(C) \) is a nontrivial element in the fundamental group of \( S_{m,m} \).

**Remark 13.8.** An idea, inspired by this proof, to approach Conjecture 1 is to try to construct quasiconvex surface subgroups \( \Gamma_g \) of the group \( \Gamma \) on which the restriction of the representation is maximal. At least when \( \Gamma \) is cocompact, maybe this can be done combining ideas from Kahn-Markovich seminal work [KM12] on surface subgroups of lattices in \( \text{Isom}(\mathbb{H}^2) \), that has been generalized to all rank one simple Lie groups by Hamnestäd [Ham14a], and the fact that it is possible to control explicitly the Toledo invariant on the fundamental group of pairs of pants [Str12].
of the complement of $K$ with the set $X$, and a surjection of $K$ onto $S^1$ that is an homeomorphism outside $X$. We first assume that for every point $x_i$ in $X$ the limits $\phi^+(x_i)$ and $\phi^-(x_i)$ are transversal. In this case there exists an element $g \in \text{SU}(m, m)$ with the property that $g \cdot v_0 = \phi^-(x_i)$ and $g \cdot v_1 = \phi^+(x_i)$. We define a new map $\psi : S^1 \rightarrow S_{m,m}$ by prescribing that $\psi|_{\partial O}$ is the continuous monotone map defined by $h : \partial O \rightarrow S_{m,m}$ with $h(t) = g \cdot \left[ \frac{t-O_i^-}{O_i^--O_i^+} \text{Id} \right]^T$ and by setting $\psi(k) = \phi(k)$ for every $k$ in $K$. The obtained map $\psi$ is by construction continuous and monotone.

As a consequence of Lemma 10.12, for every point $z \in \partial H^p_C$ there exists a point $t \in I$ such that $\phi(z)$ is not transversal to $\psi(t)$. We claim that this is in contradiction with the hypotheses of strict transversality of the map $\phi$. Indeed let us fix any point $z$ that does not belong to $C$. Since $\phi$ is strictly transversal we get that the point $t$ must belong to $\partial O_i$ for some $i$, we will denote by $x$ the point in $X$ associated to the region $O_i$. Let us now consider the maps $\phi^\pm_x \circ \pi_x : \overline{W}_x \rightarrow S_{m,m}$ constructed in Section 11.2.2. Since the map $\phi$ is everywhere transversal we get that $L_x = \overline{W}_x = W'_x$. In particular, since we proved in Proposition 11.20 that the maps $\phi^\pm_x$ are locally constant, we get that, since $z$ belongs to the chain $\pi_x(z)$, it belongs in particular to the interval $((\phi^+(x), \phi^-(x)))$ and hence cannot be transversal to the point $t$ that belongs to $((\phi^-(x), \phi^+(x)))$ (cfr. Lemma 10.13).

The same argument applies also when there exist points $x_i$ where the limits $\phi^\pm(x_i)$ are distinct but not transversal: with the same procedure one gets a continuous map $\psi : S^1 \rightarrow S_{m,m}$ with the property that, given a point $t \in S^1$, $\phi(t)$ is transversal to each other point apart, maybe, an open interval $O_t$ containing $t$, and the map $\psi$ is strictly monotone on triples of points whose images are pairwise transversal, and if $t_1$ belongs to $O_t$ and $t_2$ does not, it holds that $\psi(t_1) - \psi(t_2)$ is positive semidefinite in the Heisenberg model $H_{t_2}$. It is easy to check that Lemma 10.12 applies also for the curve $\psi$, and this concludes the proof.

Remark 13.10. In the proof of Proposition 13.9 we only used that the map $\phi$ is strictly transversal and there exists a single chain $C$ in which it is monotone. This kind of arguments is rather promising: once one shows that the limits $\phi^\pm$ we introduced in Section 11.2 are well defined, maybe it is possible to study properties of the preimages, in $\partial H^p_C$ of the sets $Z_t$ where $t$ is in the image of a monotone curve: for example it should be easy to see that these subsets are foliated in chains.

### 13.4 Proof 4: weakly Anosov flows

This proof is an adaptation, to our context, of the main proof of [BILW05] (cfr. also [BIW]). Throughout the subsection we will assume that the lattice $\Gamma < \text{SU}(1, p)$ is cocompact and that the maximal representation $\rho : \Gamma \rightarrow \text{SU}(m, m)$ admits an equivariant boundary map $\phi$ that is strictly monotone. We will use the map $\phi$ to define two maps $\phi^\pm$ that coincide with $\phi$ almost everywhere and, in addition, are right (resp. left) continuous when restricted to any chain $C$. The main part of the section will be devoted to showing that the maps $\phi^\pm$ coincide everywhere and hence their restriction to each chain is continuous:
**Proposition 13.11.** Let $\Gamma$ be a cocompact lattice in $\text{SU}(1,p)$ and let $\rho : \Gamma \to \text{SU}(m,m)$ be a maximal representation. Assume that the equivariant boundary map $\phi : \partial_{\text{H}^p_{\mathbb{C}}} \to S_{m,m}$ is strictly monotone, then $\phi$ is continuous along chains.

Combining this with the second proof we gave in this chapter (cfr. Proposition 13.6) we get a fourth proof of the nonexistence of maximal representations under some strong hypothesis on the boundary map:

**Corollary 13.12 (Fourth proof).** No maximal representations $\rho : \Gamma \to \text{SU}(m,m)$ admits a strictly monotone boundary map.

**Right and left continuous maps**

We will first use the results of Chapter 11 to construct right and left continuous maps $\phi^{\pm}$ that agree almost everywhere with the strictly monotone map $\phi$:

**Lemma 13.13.** Let $\phi : \partial_{\text{H}^p_{\mathbb{C}}} \to S_{m,m}$ be a strictly monotone map. Then there exist maps $\phi^{\pm} : \partial_{\text{H}^p_{\mathbb{C}}} \to S_{m,m}$ that agree with $\phi$ almost everywhere and have the additional property that are left (resp. right) continuous whenever restricted to a chain.

**Proof.** In order to define the value of $\phi^{\pm}$ let us fix a point $x \in \partial_{\text{H}^p_{\mathbb{C}}}$ and let us consider the foliation of $\partial_{\text{H}^p_{\mathbb{C}}} \setminus \{x\}$ in vertical chains. Since the map $\phi$ is strictly monotone, we get that the restriction of $\phi$ to each vertical chain is strictly monotone, in particular we can apply Proposition 11.5 to define the value of $\phi^{\pm}(y)$ for every point $y$ different form $x$. It follows from Proposition 11.5, together with an easy application of Fubini’s theorem, that the maps $\phi^{\pm}$ agree with $\phi$ almost everywhere, by construction that they are left (resp. right) continuous along vertical chains, moreover, again by construction, the maps $\phi^{\pm}$ are strictly monotone when restricted to vertical chains. In order to define the values of $\phi^{\pm}$ in $x$, we fix a vertical chain $C$ and define $\phi^{\pm}(x)$ to be the limits in $x$ of the restriction of $\phi$ to $C$.

Since the map $\phi$ is strictly monotone, it follows from Remark 11.19 that for each chain $C$, each point of continuity of $\phi$ on $C$ belongs to $R$, and the restriction of $\phi$ to $C$ is transversal for each chain $C$, in particular, for every point $x \in \partial_{\text{H}^p_{\mathbb{C}}}$ the set $L_x$ coincides with the set $W_x$. We can use this simple observation to show that the values of $\phi^{\pm}(x)$ do not depend on the choice of $C$; indeed we can apply Lemma 11.18 if the two limits coincide and Proposition 11.20 if the limits are different. The same two facts imply that the maps $\phi^{\pm}$ are left (resp. right) continuous and strictly monotone when restricted to any chain, and this concludes the proof.  

**Norms associated with maximal triples**

Our next goal is to associate to a maximal triple $\tau = (\tau^-, \tau^0, \tau^+)$ in $S_{m,m}^3$ of maximal isotropic subspaces of $(\mathbb{C}^{2m}, h)$ a positive definite Hermitian metric $h_\tau$ that has the property that if $(a_1, a_2, a_3, a_4)$ is a maximal triple then $h_{(a_1, a_2, a_3)}|_{a_1} > h_{(a_1, a_2, a_4)}|_{a_1}$.

Let us then fix a triple $\tau$ of pairwise transversal maximal isotropic subspaces $\tau = (\tau^-, \tau^0, \tau^+)$ and define two maps $T^+_{\tau} : \tau^- \to \tau^+$ and $T^-_{\tau} : \tau^+ \to \tau^-$ by
requiring that, if \( v \) is a vector in \( \tau^\mp \), \( T^\pm(v) \) is the unique vector in \( \tau^\pm \) such that \( iT^\pm(v) + v \) belongs to \( \tau_0 \). The map \( T^\pm \) is well defined precisely when the subspace \( \tau^0 \) is transversal to \( \tau^\pm \): in that case \( \tau^0 + \tau^\pm = \mathbb{C}^{2m} \) and each vector in \( \mathbb{C}^{2m} \), hence in particular each vector \( v \) in \( \tau^\pm \) can be written uniquely as \( v = v^0 + v^\pm \) with \( v^0 \in \tau^0, v^\pm \in \tau^\pm \). In this case one has \( T^\pm(v) = iv^\pm \).

We can define a map \( T_\tau : \mathbb{C}^{2m} = \tau^+ \oplus \tau^- \to \mathbb{C}^{2m} \) by setting \( T_\tau(v,w) = (T^-_\tau(w), -T^+_\tau(v)) \).

**Lemma 13.14.** For every element \( g \in SU(m,m) \), and for every \( s \in \mathbb{C}^{2m} \) we have \( T_{g^{-1}}(s) = gT_\tau s \).

**Proof.** By the very definition of \( T^\pm_\tau \), it follows that, for every \( v \in g\tau^\pm \), it holds \( T^\pm_\tau(gv) = gT^\pm_\tau(v) \): indeed \( v + igT^\pm_\tau(g^{-1}v) = g \cdot (g^{-1}v + iT^\pm_\tau(g^{-1}v)) \in g\tau^0 \), and this latter property uniquely characterize the vector \( T^\pm_\tau(gv) \). This proves the lemma by definition of \( T_\tau \). \( \square \)

The endomorphisms \( T_\tau \) allow us to define Hermitian metrics \( h_\tau \) on \( \mathbb{C}^{2m} \):

\[
h_\tau(v,w) = h(v,T_\tau w).
\]

One deduces from Lemma 13.14 what is the effect of an element \( g \) in \( SU(m,m) \) on the metric \( h_\tau \):

**Lemma 13.15.** For every element \( g \) in \( SU(m,m) \) one gets

\[
h_{g^{-1}}(v,w) = h(g^{-1}v,g^{-1}w).
\]

**Proof.** This is an easy computation:

\[
h_{g^{-1}}(v,w) = h(v, T_\tau g^{-1}w) = h(v, gT_\tau g^{-1}w) = h(g^{-1}v, T_\tau g^{-1}w) = h(g^{-1}v, g^{-1}w).
\]

The metric \( h_\tau \) depend monotonically on the triple \( \tau \) in the sense described in the next lemma:

**Lemma 13.16.** The triple \( \tau \) is maximal if and only if the Hermitian metric \( h_\tau \) is positive definite. If \( (a_1, a_2, a_3, a_4) \) is a maximal triple then \( h_{(a_1,a_2,a_4)}|_{a_1} < h_{(a_1,a_3,a_4)}|_{a_1} \) and \( h_{(a_1,a_3,a_4)}|_{a_1} \) tends to zero, as \( a_3 \) converges to \( a_1 \).

**Proof.** Since the group \( SU(m,m) \) is transitive on pairs of transversal points we can assume that \( (a_1,a_4) = (v_0,v_\infty) \) and that the points \( a_2 \) and \( a_3 \) belong to the Heisenberg model \( \mathcal{H}_{m,m} \). In particular, since \( a_j \) is by assumption transversal to \( v_\infty \) and \( v_0 \) there exist invertible Hermitian matrices \( A_2, A_3 \) such that the subspace \( a_j \) is spanned by the columns of the matrix \( [iA_j 0] \).

It is easy to check that the expression for the map \( T_{(v_0,a_j,v_\infty)} \) with respect to the standard basis is \( [A_j^{-1} A_j] \). This implies that the matrix representing the Hermitian form \( h_{(v_0,a_j,v_\infty)} \) with respect to the standard basis is \( [A_j A_j^{-1}] \) and this latter matrix is positive definite if and only if \( A_j \) is positive definite, which, in turn,
is equivalent to the fact that the triple \((v_0, a_j, v_\infty)\) is maximal. In this setting the 4-tuple \((v_0, a_2, a_3, v_\infty)\) is maximal if and only if \(A_3 - A_2\) is positive definite, that in turn, corresponds to the requirement that \(h_{(a_1, a_2, a_4)} - a_j < h_{(a_1, a_3, a_4)} - a_1\).

The last assertion is obvious given the explicit formulae for the objects. □

The geodesic flow

It is well known that the unit tangent bundle of \(\mathbb{H}^p_C\) is parametrized by the set of maximal triples in \((\partial \mathbb{H}^p_C)^3\):

Lemma 13.17. There exists a continuous bijective map \(u : T^1 \mathbb{H}^p_C \to \mathcal{C}^{(3+)}\). Here we denote by \(\mathcal{C}^{(3+)}\) the set of pairwise distinct triples in \(\partial \mathbb{H}^p_C\) that are contained in a chain and are positively oriented, topologized as a subset of \((\partial \mathbb{H}^p_C)^3\).

Proof. Let indeed \((x, v)\) be a vector in \(T^1 \mathbb{H}^p_C\) and let us denote by \(\gamma\) the real oriented geodesic through \(x\) with tangent \(v\) in \(x\) and by \(D\) the unique complex geodesic in \(\mathbb{H}^p_C\) containing \((x, v)\). Moreover let \(\gamma^\perp\) be the unique real geodesic in \(D\) that is orthogonal to \(\gamma\) in \(x\). We denote by \(u^-(v)\) the point of \(\partial \mathbb{H}^p_C\) that is the limit \(\gamma(-\infty)\), by \(u^+(v)\) the point \(\gamma(\infty)\) and by \(u^0(v)\) the endpoint of \(\gamma^\perp\) that has the property that \((u^-(v), u^0(v), u^+(v))\) is positively oriented. By construction the triple \((u^-(v), u^0(v), u^+(v))\) is a maximal triple in \(\partial \mathbb{H}^p_C\), moreover it is easy to check that the map \(u\) defined this way gives the desired parametrization. □

We will study the geodesic flow \(g_t\) on \(T^1 \mathbb{H}^p_C\). It follows from the construction of \(u\) that both coordinates \(u^+\) and \(u^-\) are constant along the orbits of the flow and \(\lim_{t \to \pm \infty} u^0(g_t(x, v)) = u^\pm(x, v)\). Since the geodesic flow \(g_t\) commutes with the \(\Gamma\) action on \(T^1 \mathbb{H}^p_C\), it induces a flow on \(\Gamma\setminus T^1 \mathbb{H}^p_C\) that we will still denote by \(g_t\).

Let us now consider the flat linear bundle \(E_\rho\) associated with the representation \(\rho\): this can be defined as the quotient of \(T^1 \mathbb{H}^p_C \times \mathbb{C}^{2m}\) with respect to the diagonal action of \(\Gamma\): \(\gamma \cdot (z, e) = (\gamma \cdot z, \rho(\gamma) \cdot e)\). Since the bundle \(E_\rho\) is flat, the flow \(g_t\) on \(T^1 \mathbb{H}^p_C\) lifts to a flow, \(\hat{g}_t\) on \(E_\rho\): the flow defined on \(T^1 \mathbb{H}^p_C \times \mathbb{C}^{2m}\) by the formula \(\hat{g}_t(z, e) = (g_t(z), e)\) is invariant for the \(\Gamma\) action and hence induces a flow \(\hat{g}_t\) on the quotient.

The monotone maps \(\phi^\pm\), together with the parametrization \(u\) can be used to define a measurable \(\Gamma\)-equivariant splitting of \(E_\rho\) that, again, is constant along the flow lines: we consider the subbundles \(E^\pm_\rho\) of \(E_\rho\) that are defined pointwise in \(\mathbb{H}^p_C\) by the formula

\[
E^\pm_\rho(z) = \phi^\pm(u^\pm(z)).
\]

Since the map \(\phi\) is, by assumption, strictly monotone, we get that for every pair of points \((x, y)\) the maximal isotropic subspaces \(\phi^+(x), \phi^-(y)\) are transversal, hence in particular the measurable subbundles \(E^\pm_\rho\) actually induce a splitting of \(E_\rho\).
Moreover, using the maps $\phi^\pm$, we can also define a measurable field of Hermitian metrics on the bundle $E$ that descend to a measurable field of Hermitian metrics on the bundle $E_p$; indeed for every $z \in T^1\mathbb{H}^p_C$ we consider the Hermitian metrics $h^\pm(z)$ on $\mathbb{C}^{2m}$ defined by

$$h^\pm(z) = h_{(\phi^-(u^-(z)), \phi^+(u^0(z)), \phi^+(u^+(z)))}.$$

**Lemma 13.18.** The Hermitian metrics $h^\pm(z)$ induce a measurable field of Hermitian metrics on the bundle $E_p$.

**Proof.** In order to check the $\Gamma$-invariance we need to verify that $h^\pm(z)(v, w) = h^\pm(\gamma \cdot z)(\rho(\gamma)(v), \rho(\gamma)(w))$ for all $\gamma$ in $\Gamma$, $z \in T^1\mathbb{H}^p_C$, $v, w \in \mathbb{C}^{2m}$. This follows the equivariance of $\phi$ and Lemma 13.15: indeed if $u = (u^-, u^0, u^+)$ denotes a triple in $\mathcal{D}^m_C$, it follows from the strict $\Gamma$-equivariance of $\phi$ that $(\phi^-(\gamma u^-), \phi^+(\gamma u^0), \phi^+(\gamma u^+)) = \rho(\gamma) \cdot (\phi^-(u^-), \phi^+(u^0), \phi^+(u^+))$, in particular we have:

$$h^\pm(\gamma \cdot z)(\rho(\gamma)v, \rho(\gamma)w) = h_{(\phi^-(\gamma u^-(z)), \phi^+(\gamma u^0(z)), \phi^+(\gamma u^+(z)))}(\rho(\gamma)v, \rho(\gamma)w) = h_{(\phi^-(u^-(z)), \phi^+(u^0(z)), \phi^+(u^+(z)))}(\rho(\gamma)v, \rho(\gamma)w) = h^\pm(z)(\rho(\gamma)^{-1} \rho(\gamma)v, \rho(\gamma)^{-1} \rho(\gamma)w) = h^\pm(z)(v, w).$$

\[\square\]

**A function on $M = \Gamma \setminus T^1\mathbb{H}^p_C$.**

We will now use all the result we proved in the section to show that the maps $\phi^\pm$ agree everywhere and hence the map $\phi$ itself is continuous. In order to do this we will exploit the fact that the geodesic flow $\tilde{g}_t$ is contracting on the bundle $E_p$. We will first need some uniform bound on the norms $\{h^\pm(z)\}_{z \in M}$:

**Lemma 13.19.** The norms $h^\pm(z)$ are locally bounded, in particular they induce norms on $M = \Gamma \setminus T^1\mathbb{H}^p_C$ that are equivalent to a continuous norm.

**Proof.** Since the norms $h^\pm(z)$ are $\Gamma$ invariant, and the action of $\Gamma$ on the space of maximal triples $\mathcal{C}^{(3+)}$ is cocompact, it is enough to show that for every maximal triple $u$ there exists an open neighborhood $O$ of $u$ in $\mathcal{C}^{(3+)}$ with the property that the norms $\{h(z)\}_{z \in O}$ are uniformly bounded. In order to do this we first remark that, whenever we fix six points $a_i, b_i$ in $S_{m,m}$ with the property that the 6-tuple $(a_1, b_1, a_2, b_2, a_3, b_3)$ is maximal, there exist an uniform bound on the norms $h_{x_1, x_2, x_3}$, where $x_i$ belongs to the interval $((a_i, b_i))$. If then $z = (z_1, z_2, z_3) \in \mathcal{C}^{(3+)}$ is a maximal triple, we can use the strict monotonicity of the map $\phi$ to find a maximal 6-tuple in $S_{m,m}$ with the property that $\phi^\pm(z_i) \in ((a_i, b_i))$, moreover using the argument involved in the proof of Proposition 11.13 three times we get the desired open neighborhood $O$. This concludes the proof. \[\square\]

We will now study the properties of the function

$$f : T^1\mathbb{H}^p_C \rightarrow \mathbb{R}, \quad z \mapsto \|T_{(\phi^-(u^-(z)), \phi^-(u^+(z)), \phi^+(u(z)))}\|h^-(z).$$
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here we denote by $\| \cdot \|_{h^-(z)}$ the operatorial norm with respect to the Hermitian metric $h^-(z)$:

$$\| T_\tau \|_{h^\sigma} = \sup_{v \in \mathbb{C}^2} \frac{h^\sigma(T_\tau(v), T_\tau(v))}{h^\sigma(v, v)}.$$

The function $f$ is well defined: the map $\phi$ is strictly monotone, in particular $\phi^-(u^+(z))$ is transversal to $\phi^-(u^-(z))$ since the points $u^+(z)$ and $u^-(z)$ are distinct. Moreover, if we assume, up to composing with an element in $SU(m, m)$, that $\phi^-(u^-(z)) = v_0 = (e_{m+1}, \ldots, e_{2m}), \phi^+(u^+(z)) = v_\infty = (e_1, \ldots, e_m)$ and $\phi^-(u^+(z)) = [iA]$ for some Hermitian matrix $A$, we get that the endomorphism and $T^\tau(\phi^-(u^-(z)), \phi^-(u^+(z)): \phi^+(u^+(z)))$ has expression, with respect to the standard basis, $[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}]$. In particular $f(z) = 0$ if and only if $\phi^+(u^+(z)) = \phi^+(u^-(z))$.

Lemma 13.20. The function $f$ is $\Gamma$-invariant and uniformly bounded.

Proof. In order to verify the $\Gamma$ invariance let us first consider two maximal triples $\sigma, \tau$ in $S^3_{m,m}$ and an element $g \in SU(m, m)$:

$$\| T_{g\sigma} \|_{h^{g\sigma}} = \sup_{v \in \mathbb{C}^2} \frac{h^\sigma(T_{g\sigma}(v), T_{g\sigma}(v))}{h^{g\sigma}(v, v)} = \sup_{v \in \mathbb{C}^2} \frac{h^\sigma(T_\tau(g^{-1}v), T_\tau(g^{-1}v))}{h^\sigma(g^{-1}v, g^{-1}v)} = \| T_\tau \|_{h^\sigma}.$$

In particular, since $\phi$ is $\rho$-equivariant, it follows that the function $f$ is $\Gamma$-invariant. The fact that the function is uniformly bounded follows from the cocompactness of the $\Gamma$ action, along the same lines of the proof of Lemma 13.19.

Proof of Proposition 13.11. As we already remarked, in order to prove the proposition it is enough to show that the function $f$ is always equal to 0. Let us then assume by contradiction that $f$ has a non-zero value in the point $z \in T^1_H\mathbb{H}_C$. Up to conjugating the map $\phi$ with a suitable element in $SU(m, m)$ we can assume that $\phi^+(u^+(z)) = v_\infty$ and $\phi^-(u^-(z)) = v_0$, and that there exists a non-zero Hermitian matrix $A$ such that $\phi^-(u^+(z)) = [iA]$. Moreover, for every time $t$, the maximal isotropic subspace $\phi^+(u^+(z))$ can be written as $[iB_t]$ for some matrices $B_t$. Since, by assumption, the map $\phi^+$ is right continuous when restricted to the chain determined by the vector $z$, we get that all eigenvalues of $B_t$ converge to infinity monotonously as $t$ tends to $\infty$.

It follows from the explicit computations we gave in the proof of Lemma 13.16 that the metrics $h^+(g_t(z))$ have expression $[B_t^{-1} \ 0 \\ 0 \ B_t]$, in particular the function $f(g_t(z))$ is the squareroot of the maximum eigenvalue of $A^*B_tA$. Since this is at least as big as the smallest eigenvalue of $B_t$, and this latter value diverge, we get that the function $f$ cannot be bounded, and hence we reach a contradiction.

Remark 13.21. It is possible to apply the same idea of this proof also in the non-cocompact case, as soon as one is able to exhibit a complex geodesic in $\mathbb{H}_C$ whose projection to $M = \Gamma \backslash T^1_H\mathbb{H}_C$ does not go deep into the cusp: the same argument we used here allows to prove that the function $f$ also in that case is uniformly bounded in each compact region of $M$, and the contradiction comes.
from the fact that the geodesic flow preserves complex geodesics and extends to a weakly expanding/contracting flow on $E^\pm$. Even more, combining this argument with the monotonicity of $\phi$ it should be possible to show that the boundary map can only have discontinuities in points of $\partial \mathbb{H}_C^p$ corresponding to cusps.
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