# Advanced computational methods for structural reliability analysis - applications in civil engineering

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# Advanced computational methods for structural reliability analysis – Applications in civil engineering

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**Abstract.** Metamodelling is an important tool for modern applications in structural reliability, especially in the presence of high fidelity computational models. A review of several state-of-the-art metamodel-based applications in structural reliability is given.

## 1 Introduction

Structural reliability analysis aims at assessing the safety of a system in the presence of uncertainty in its components. Given a random vector that represents the input uncertainty $\boldsymbol{X}$ (*e.g.* a joint probability density function PDF $f_{\boldsymbol{X}}$), the safety level is defined through a probability of failure $P_f$. By defining a performance function $g(\boldsymbol{X})$ that is negative in case of system failure, $P_f$ can be defined as follows (*e.g.* [3]):

$$P_f = \mathbb{P}\left(\{g(\boldsymbol{X}) \leq 0\}\right) = \int_{\mathcal{D}_f = \{\boldsymbol{x} \in \mathbb{R}^M : g(\boldsymbol{X}) \leq 0\}} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{1}$$

where $\mathcal{D}_f$ is the failure domain defined by $g(\boldsymbol{x}) \leq 0$. Introducing the failure indicator function $\mathbf{1}_{\mathcal{D}_f}$ as the characteristic function of $g(\boldsymbol{x}) \leq 0$, $P_f$ can be recast as:

$$P_f = \int_{\mathbb{R}^M} \mathbf{1}_{\mathcal{D}_f}(\boldsymbol{x})\, f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \equiv \mathbb{E}\left[\mathbf{1}_{\mathcal{D}_f}(\boldsymbol{X})\right] \tag{2}$$

with associated Monte-Carlo (MC) estimator:

$$\hat{P}_f = \frac{1}{N} \sum_{k=1}^{N} 1_{\mathcal{D}_f}\left(\boldsymbol{X}^{(k)}\right) = \frac{N_f}{N}. \tag{3}$$

Due to the slow convergence rate of MC methods, the accurate computation of the expectation value estimator in Eq. (3) can rapidly become prohibitive with high fidelity computational models commonly used today in industrial applications. More efficient strategies for the computation of probability of failures are then needed.

A number of improvements on Monte-Carlo based simulation have been proposed in the literature to effectively precondition the estimation in Eq. (3) and reduce the associated computational costs (*e.g.* Importance Sampling, Subset Simulation, etc.), but their overall cost (typically $N = 10^{3-4}$) may remain prohibitive.

Recent developments in surrogate modelling, however, have proven that metamodels can be ideal candidates for the accurate estimation of $P_f$ while keeping computational costs manageable.

# 2 Surrogate models

Surrogate models are functional approximations to the full complex computational models that can be built from a comparatively small set of full model evaluations (the experimental design), typically comprising tens to hundreds of samples. Such approximations are much cheaper to evaluate than their full counterpart and in some cases they can even provide additional useful information. Several metamodelling techniques have been successfully applied for the solution of structural reliability problems (*e.g.* Polynomial Chaos Expansions (PCE) and Kriging, see [6]). This contribution will focus on Kriging.

## 2.1 Kriging

Kriging, or Gaussian process metamodelling, is based on the representation of the random output $Y = \mathcal{M}(\boldsymbol{X})$ as the superimposition of a deterministic trend and a realization of a zero-mean stationary Gaussian process (see, *e.g.* [4]):

$$Y\left(\boldsymbol{x},\, \omega\right) = \boldsymbol{f}\left(\boldsymbol{x}\right)^{\mathsf{T}}\, \boldsymbol{a} + Z\left(\boldsymbol{x}, \omega\right) \tag{4}$$

where $\boldsymbol{f}\left(\boldsymbol{x}\right)^{\mathsf{T}}\, \boldsymbol{a}$ represents the deterministic trend and $Z\left(\boldsymbol{x}, \omega\right)$ represents the Gaussian process with variance $\sigma^2$ and covariance function:

$$C_{YY}\left(\boldsymbol{x},\, \boldsymbol{x}'\right) = \sigma^2\, R\left(\boldsymbol{x} - \boldsymbol{x}'\,,\, \boldsymbol{\theta}\right) \tag{5}$$

The autocorrelation function family $R$ is a parameter of the metamodel. The hyperparameters $\boldsymbol{\theta}$ of the Gaussian process can be calculated from a learning set of full model evaluations by maximum likelihood estimation.

Two properties of Kriging are especially useful in the context of structural reliability: it is an interpolant and it provides a built-in estimation of the local epistemic error of the surrogate model, known as the Kriging variance $\sigma^2_{\widehat{Y}}(\boldsymbol{x})$.

# 3 Surrogate models in structural reliability

## 3.1 As surrogates of $g(\boldsymbol{x})$

The most straightforward application of surrogate models in structural reliability is that of providing an inexpensive surrogate of the performance function, hence offsetting the costs of a direct application of the MC estimator in Eq. (3). However, the accuracy of the metamodel, especially in low probability regions of the model space, can be difficult to assess.

## 3.2 Adaptive experimental design

Leveraging on the local error estimate $\sigma^2_{\widehat{Y}}(\boldsymbol{x})$ provided by the Kriging metamodel, it is possible to devise experimental design enrichment strategies aimed at minimizing the

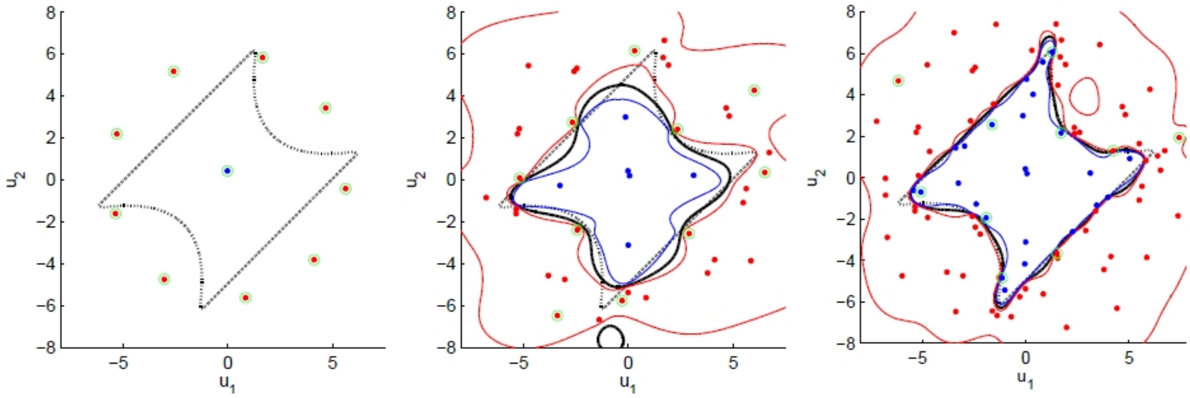


Figure 1: Example of experimental design enrichment based on the learning function in Eq. (6) at three different iterations for the Four Branch Function from [5]. The black line represents the true limit state surface, while the blue and red lines represents the 95% quantiles of the currently estimated failure regions.

metamodelling error near the limit state surface. This allows one to initialize the metamodel with a small experimental design, followed by its iterative refinement in the regions of interest. This can be achieved by defining enrichment criteria that identify the regions where the maximum improvement in the classification power of the metamodel are expected. One example is the learning function from [2]:

$$U(\boldsymbol{x}) = \frac{|\mu_{\widehat{Y}}(\boldsymbol{x})|}{\sigma_{\widehat{Y}}(\boldsymbol{x})} \tag{6}$$

This indicator can be interpreted as *the reliability index of the classification* of each metamodelled sample as being in the safe or failure domain. Lower values of $U(\boldsymbol{x})$ indicate high misclassification probability and closeness of $\boldsymbol{x}$ to the limit state surface. Thus they are ideal candidates for the addition of experimental design points. An example application of adaptive refinement of the failure domain can be found in Figure 1 for a test-case (Four Branch benchmark, see *e.g.* [5]).

## 3.3 Metamodel-based importance sampling

Further improvements can be attained with a metamodel-based extension of the Importance Sampling (IS) Monte-Carlo simulation method. Importance Sampling consists in defining an instrumental probability density function $h(\boldsymbol{X})$ which allows one to recast Eq. (1) as:

$$P_f = \int_{\mathbb{R}^M} \mathbf{1}_{\mathcal{D}_f}(\boldsymbol{x}) \frac{f_{\boldsymbol{X}}(\boldsymbol{x})}{h(\boldsymbol{x})} h(\boldsymbol{x})\, d\boldsymbol{x} = \mathbb{E}_h \left[ \mathbf{1}_{\mathcal{D}_f}(\boldsymbol{X}) \frac{f_{\boldsymbol{X}}(\boldsymbol{X})}{h(\boldsymbol{X})} \right] \tag{7}$$

The choice of $h(\boldsymbol{X})$ significantly affects the accuracy of the Monte-Carlo estimation of $P_f$. An approximation to the optimal instrumental PDF can be built from the Kriging metamodel (see, *e.g.* , [1]), with which Eq. (7) can be rewritten as:

$$P_f = P_{f\varepsilon} \cdot \mathbb{E}_{\tilde{h}} \left[ \frac{\mathbf{1}_{\mathcal{D}_f}(\boldsymbol{x})}{\pi(\boldsymbol{x})} \right] = P_{f\varepsilon} \cdot \alpha_{\text{corr}} \tag{8}$$

where $P_{f\varepsilon}$ is the probability of failure calculated with the inexpensive Kriging surrogate, while $\alpha_{\text{corr}}$ is a correction factor calculated from an additional validation set of full model evaluations sampled from the approximated optimal instrumental PDF $\tilde{h}(\boldsymbol{x})$. Based on the Kriging predictor $\mu_{\widehat{Y}}$ and variance $\sigma^2_{\widehat{Y}}$, the terms in Eq. (8) read:

$$\pi(\boldsymbol{x}) = \Phi\left(-\mu_{\widehat{Y}}(\boldsymbol{x})/\sigma_{\widehat{Y}}(\boldsymbol{x})\right), \quad \tilde{h}(\boldsymbol{x}) = \frac{\pi(\boldsymbol{x}) f_X(\boldsymbol{x})}{P_{f\epsilon}}, \quad P_{f\epsilon} = \mathbb{E}\left[\pi(\boldsymbol{x})\right]. \tag{9}$$

Several application examples of this method show that it can significantly help improving the stability and efficiency existing approaches based on metamodelling or adaptive design alone ([1, 6]).

# 4 Conclusions

Metamodelling has become an established tool in structural reliability applications in the presence of expensive computational models. The research community is actively developing new and more sophisticated approaches than the straightforward surrogation of the limit state function, *e.g.* by exploiting the stochastic properties of Kriging surrogates to create quasi-optimal instrumental densities in Importance Sampling algorithms.

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