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Analytic regularity and polynomial approximation of stochastic, parametric elliptic multiscale PDEs *

V.H. Hoang †and Ch. Schwab ‡

Abstract

A class of second order, elliptic PDEs in divergence form with stochastic and anisotropic conductivity coefficients and n known, separated microscopic length scales ε_i, $i = 1, ..., n$ in a bounded domain $D \subset \mathbb{R}^d$ is considered. Neither stationarity nor ergodicity of these coefficients is assumed. Sufficient conditions are given for the random solution to converge $\mathbb{P}$-a.s, as $\varepsilon_i \to 0$, to a stochastic, elliptic one-scale limit problem in a tensorized domain of dimension $(n+1)d$. It is shown that this stochastic limit problem admits best N-term "polynomial chaos" type approximations which converge at a rate $\sigma > 0$ that is determined by the summability of the random inputs' Karhúnen-Loève expansion. The convergence of the polynomial chaos expansion is shown to hold $\mathbb{P}$-a.s. and uniformly with respect to the scale parameters ε_i. Regularity results for the stochastic, one-scale limiting problem are established. An error bound for the approximation of the random solution at finite, positive values of the scale parameters ε_i is established in the case of two scales, and in the case of $n > 2$ scales convergence is shown, albeit without giving a convergence rate in this case.

1 Problem formulation

1.1 A class of stochastic multiscale elliptic problems

In a bounded Lipschitz domain $D \subset \mathbb{R}^d$ (to which we shall refer as "physical domain"), we consider diffusion problems in D where the diffusion coefficients resp. the permeability is uncertain and exhibits microstructure on one or several microscopic length scales. In what follows, we assume these length scales to be *separated* and *a priori known*. To describe the periodic microstructure, let Y denote the unit cube in $\mathbb{R}^d$ and let $Y_1, Y_2, \ldots, Y_n$ be n copies of Y which we assume to be the ranges of the n fast- or microscopic variables (all our results generalize to the case when the Y_j are nonidentical). To describe the random permeabilities that are admissible in our analysis, we assume given a probability space $(\Omega, \Sigma, \mathbb{P})$, and a random field

$$\Omega \ni \omega \mapsto A(\omega; x, y_1, \ldots, y_n) \in L^\infty(D; C_\#(Y_1 \times \ldots \times Y_n)^{d\times d}_{\rm sym}) \tag{1.1}$$

such that

$$A \in L^\infty(\Omega, d\mathbb{P}; L^\infty(D; C_\#(Y_1 \times \ldots \times Y_n)^{d\times d}_{sym})). \tag{1.2}$$

Throughout, for $0 < p \le \infty$ and a Banach space B, we denote by $L^p(\Omega, d\mathbb{P}; B)$ the Bochner space of strongly $\mathbb{P}$ measurable mappings from (Ω, Σ) to B with the sigma-algebra of Borel sets which are p-summable (resp. $\mathbb{P}$-a.s. bounded in B in case that $p = \infty$). In (1.1) and the following, the notation # indicates that the functions admit Y_i periodic extensions to all of $\mathbb{R}^d$ with respect to each of the variables y_i for $i = 1, \ldots, n$ which locally, i.e. on compact subsets of $\mathbb{R}^d$, belong to the same function spaces on these sets. For notational conciseness, we denote by $\mathbf{Y} = Y_1 \times \ldots \times Y_n$ and by $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbf{Y}$. We will write $C_\#(\mathbf{Y})$ in place of $C_\#(Y_1 \times \ldots \times Y_n)$. Spaces of vector functions with each component function belonging to a Banach space B will be denoted by B^d, and of $d \times d$ matrix functions by $B^{d\times d}$. Integrals over such functions will be understood as vector functions of integrals over all component functions. To ensure well-posedness of our problem, we impose

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Assumption 1.1 *The diffusion matrix A satisfies* (1.2). *In particular, it is uniformly bounded and coercive, i.e. there are positive constants α and β such that for all $\omega \in \Omega$, $x \in D$ and every $\boldsymbol{y} \in \mathbf{Y}$ it holds*

$$\forall \xi \in \mathbb{R}^d : \qquad \alpha|\xi|^2 \le \xi^\top A(\omega; x, \boldsymbol{y})\xi \le \beta|\xi|^2 .$$

We assume $\mathbb{P}$-*a.s. scale separation*. This means that for a nondimensional scale parameter $\varepsilon > 0$, there are n known, deterministic, positive functions $1 > \varepsilon_1(\varepsilon) \ge \ldots \ge \varepsilon_n(\varepsilon) > 0$ which depend continuously and monotonically on ε, and which describe the n microscopic length scales which the random diffusion coefficient depends on. Without loss of generality, we set $\varepsilon_1 = \varepsilon$. *If the random coefficient* (1.4) *has $n > 1$ fast scales, we say that the coefficient is* $\mathbb{P}$-*a.s. scale separated* if, for all $i = 1, \ldots, n-1$ for $\mathbb{P}$-a.s., there holds

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{i+1}}{\varepsilon_i} = 0. \tag{1.3}$$

If $n = 1$, i.e. there is only one fast scale, we denote $\epsilon_1 = \epsilon$ and the condition (1.3) is understood to be void. For A as in (1.1) and satisfying Assumption 1.1, for a given family of scale parameters ε_i satisfying (1.3), we define a family of n-scale, random multiscale diffusion tensors $A^\varepsilon(\omega; x) \in L^\infty(\Omega, d\mathbb{P}; L^\infty(D)^{d\times d}_{sym})$ by

$$A^\varepsilon(\omega; x) := A(\omega; x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}) . \tag{1.4}$$

With $A^\varepsilon(\omega; x)$ defined in this way, for given $f \in H^{-1}(D)$, $0 < \varepsilon < 1$ and for $\omega \in \Omega$, we consider in D the n-scale stochastic Dirichlet problem:

$$-\text{div} A^\varepsilon(\omega; x)\nabla u^\varepsilon = f(x) \quad \text{in } D, \qquad u^\varepsilon|_{\partial D} = 0 . \tag{1.5}$$

For simplicity of exposition, we assume in what follows that the source term $f \in H^{-1}(D)$ is deterministic and independent of ε. At this point we remark that stochastic homogenization problems have been considered before; we mention only [3, 5, 12] and the references there. However, usually only two scales were considered and an ergodic hypothesis was imposed. In this work, neither stationarity nor ergodicity of the random coefficient will be assumed. We begin our analysis by casting problem (1.5) in variational form:

$$\text{find } u^\varepsilon \in H^1_0(D) \text{ such that } \int_D A^\varepsilon(\omega; x)\nabla u^\varepsilon \cdot \nabla\phi dx = \int_D f\phi dx \;\; \forall \phi \in H^1_0(D). \tag{1.6}$$

We equip the space $H^1_0(D)$ with the norm $\|v\|_{H^1_0(D)} = \|\nabla v\|_{L^2(D)}$. Then the random solution u^ε of (1.6) satisfies, for $\mathbb{P} - a.e.\ \omega \in \Omega$,

$$\|u^\varepsilon(\omega; \cdot)\|_{H^1_0(D)} \le \frac{\|f\|_{H^{-1}(D)}}{\alpha} .$$

We assume that the random coefficient A in Assumption 1.1 and in (1.4) is characterized by a sequence $\boldsymbol{z}(\omega) = (z_k(\omega))_{k\ge1}$ of random variables as follows:

$$A(\omega; x, \boldsymbol{y}) = \bar{A}(x, \boldsymbol{y}) + \sum_{k=1}^{\infty} z_k(\omega)\Psi_k(x, \boldsymbol{y}), \quad (\omega, x, \boldsymbol{y}) \in \Omega \times D \times \mathbf{Y} , \tag{1.7}$$

where $\Psi_k(x, \boldsymbol{y}) \in L^\infty(D, \mathbf{Y})^{d\times d}_{\text{sym}}$. Without any assumptions on the normalization of the z_k, Ψ_k, the parametric representation (1.7) is nonunique. We therefore assume that the $z_k(\omega)$ are i.i.d. and $z_k \sim \mathcal{U}(-1, 1)$. We further denote by $\boldsymbol{z}$ the coefficient vector $(z_1, z_2, \ldots) \in U := [-1, 1]^{\mathbb{N}}$ of realizations. For a sequence $\beta = (\beta_k)_{k\ge1} \in \ell^1(\mathbb{N})$, we assume the matrix functions Ψ_k in (1.7) to satisfy

$$\forall k \in \mathbb{N} : \forall \xi \in \mathbb{R}^d,\ x \in D,\ \boldsymbol{y} \in \mathbf{Y} \ : \quad |\xi^\top \Psi_k(x, \boldsymbol{y})\xi| \le \beta_k|\xi|^2 , \tag{1.8}$$

which implies that the series (1.7) converges unconditionally, $\mathbb{P} - a.s.$ We also assume that the mean field in (1.7), i.e. the matrix function $\bar{A} \in L^\infty(D \times \mathbf{Y})^{d\times d}_{sym}$, satisfies

$$\forall \xi \in \mathbb{R}^d,\ x \in D,\ \boldsymbol{y} \in \mathbf{Y} \ : \quad \alpha_0|\xi|^2 \le \xi^\top \bar{A}(x, \boldsymbol{y})\xi \le \beta_0|\xi|^2 . \tag{1.9}$$

To ensure that the random coefficient $A(\omega; x, \boldsymbol{y})$ in (1.7) is well defined and coercive, we assume that in (1.7) the fluctuation expansion of $A - \bar{A}$ is dominated by the mean field $\bar{A}$ in the following sense:

Assumption 1.2 *We assume* (1.8), (1.9) *and that in* (1.7), *the random variables* z_k *are i.i.d. in* $[-1, 1]$. *Moreover, with the ellipticity constant* α_0 *in* (1.9), *we assume that the matrix functions* $\bar{A}$ *and* Ψ_k *satisfy for some* $\kappa > 0$

$$\sum_{k \geq 1} \beta_k \leq \frac{\kappa}{1+\kappa} \alpha_0.$$

Assumption 1.2 implies Assumption 1.1: we may choose

$$\alpha = \alpha_0 - \frac{\kappa}{1+\kappa}\alpha_0 = \frac{1}{1+\kappa}\alpha_0 \ , \qquad \beta = \alpha_0 + \frac{\kappa}{1+\kappa}\alpha_0 \ .$$

From (1.8) and Assumption 1.2, we have

Proposition 1.3 *The following estimate holds*

$$\sum_{k \geq 1} \|\Psi_k\|_{L^\infty(D)^{d\times d}} \leq 2\frac{\kappa}{1+\kappa}\alpha_0 \ .$$

Proof From Assumption 1.2, for each $i = 1, \ldots, d$ and for every $k = 1, 2, \ldots$, we have

$$\|(\Psi_k)_{ii}\|_{L^\infty(D)} \leq \beta_k.$$

Fix two indices $i, j = 1, \ldots, d$, and choose in (1.8) $\xi_i = 1$ and $\xi_j = 1$, and $\xi_l = 0$ for $l \neq i, j$. Then

$$\forall x \in D,\ \boldsymbol{y} \in \mathbf{Y}\ , k \in \mathbb{N}\ : \quad |(\Psi_k(x,\boldsymbol{y}))_{ii} + (\Psi_k(x,\boldsymbol{y}))_{jj} + (\Psi_k(x,\boldsymbol{y}))_{ij} + (\Psi_k(x,\boldsymbol{y}))_{ji}| \leq 2\beta_k \ .$$

From this, we deduce

$$\forall x \in D,\ \forall \boldsymbol{y} \in \mathbf{Y}\ , \forall k \in \mathbb{N}\ : \quad |(\Psi_k(x,\boldsymbol{y}))_{ij}| + |(\Psi_k(x,\boldsymbol{y}))_{ji}| \leq 4\beta_k \ .$$

This implies the assertion. □

1.2 Karhúnen-Loève expansion

We give a particular example of a parametric expansion (1.7), the *Karhúnen-Loève* expansion of a random matrix function $A(\omega; x, \boldsymbol{y})$. We give, in particular, sufficient conditions in order for Assumption 1.2 to hold. We formulate these conditions in terms of the smoothness of the covariance of the matrix function $A(\omega; x, \boldsymbol{y})$, which is given by the fourth order tensor

$$\mathbf{Cov}[A]_{iji'j'}(x, \boldsymbol{y}, x', \boldsymbol{y'}) = \int_\Omega (A_{ij}(\omega; x, \boldsymbol{y}) - \bar{A}_{ij}(x, \boldsymbol{y}))(A_{i'j'}(\omega; x', \boldsymbol{y'}) - \bar{A}_{i'j'}(x', \boldsymbol{y'}))d\mathbb{P}(\omega) \ ,$$

for $i, j, i', j' = 1, \ldots, d$. Then $\mathbf{Cov}[A]_{iji'j'} \in L^\infty((D \times \mathbf{Y}) \times (D \times \mathbf{Y}), \mathbb{R})$, for all i, j, i', j' is the kernel of the (compact and self-adjoint) *covariance operator* $\mathbf{Q}_A : L^2(D \times \mathbf{Y})^{d\times d}_{\text{sym}} \to L^2(D \times \mathbf{Y})^{d\times d}_{\text{sym}}$ defined by

$$(\mathbf{Q}_A \Phi)_{ij}(x, \boldsymbol{y}) := \int_D \int_{\mathbf{Y}} \mathbf{Cov}[A]_{iji'j'}(x, \boldsymbol{y}, x', \boldsymbol{y'}) \Phi_{i'j'}(x', \boldsymbol{y'}) d\boldsymbol{y'} dx' \ .$$

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ denote the eigenvalues of $\mathbf{Q}_A$ and let $\Phi_k \in (L^2(D \times \mathbf{Y}))^{d\times d}$ denote the corresponding eigenvectors. We assume that $\|\Phi_k\|_{L^2(D\times\mathbf{Y})^{d\times d}} = 1$ for all k. Any random field $A \in L^2(\Omega; L^2(D \times \mathbf{Y})^{d\times d}_{\text{sym}})$ can be represented by a Karhúnen-Loève (KL) expansion

$$A(\omega; x, \boldsymbol{y}) = \bar{A}(x, \boldsymbol{y}) + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \Phi_k(x, \boldsymbol{y}) Z_k(\omega), \tag{1.10}$$

where Z_k are pairwise uncorrelated random variables that satisfy

$$Z_k(\omega) = \frac{1}{\sqrt{\lambda_k}} \int_{D\times\mathbf{Y}} (A_{ij} - \bar{A}_{ij})(\Phi_k)_{ij} d\boldsymbol{y} dx \ .$$

By Assumption 1.2, the random coefficients Z_k in (1.10) are uniformly bounded, $\mathbb{P} - a.s.$ for all k. Note also that, due to the normalization Assumption $\|\Phi_k\|_{L^2(D\times\mathbf{Y})^{d\times d}} = 1$ the probability densities of the random variables Z_k are not necessarily supported in $[-1, 1]$. To estimate the eigenvalues λ_k, we will use the following classical result (see, e.g., [16] and the references therein).

Lemma 1.4 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $\mathcal{C}$ be a symmetric, nonegative and compact linear operator from H to H whose eigenpairs are $(\lambda_m, \phi_m)_{m\geq 1}$. If $m \in \mathbb{N}$ and $\mathcal{C}_m$ is an operator of rank at most m, then*

$$\lambda_{m+1} \leq \|\mathcal{C} - \mathcal{C}_m\|_{\mathcal{L}(H,H)}.$$

We then have the following bounds for the eigenvalues λ_k in terms of the regularity of the covariance function of the random diffusion matrix A in (1.1).

Proposition 1.5 *Assume that the random diffusion matrix A in (1.1) satisfies*

$$A \in L^2(\Omega; H^t_{mix}(D \times \mathbf{Y})) \tag{1.11}$$

where, for $t \geq 0$, the space $H^t_{mix}(D\times\mathbf{Y})$ is defined by $H^t_{mix}(D\times\mathbf{Y}) = H^t(D)\otimes H^t_{\#}(Y_1)\otimes\ldots\otimes H^t_{\#}(Y_n)$ with $\otimes$ denoting the tensor product of separable Hilbert spaces and with H^t denoting, for noninteger values of t, the fractional order Sobolev space (e.g [18]).

Then $\mathbf{Cov}[A] \in H^t_{mix}(D\times\mathbf{Y})\otimes H^t_{mix}(D\times\mathbf{Y})$. Moreover, for all $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that for all $k \geq 1$ holds $\lambda_k \leq c(\varepsilon)k^{-t/d+\varepsilon}$.

Proof The proof is adapted from that of Proposition 2.18 in [16]. Let $\mathbf{P}^L$ be the orthogonal projection into $\widehat{\mathbf{V}}^L$ in the $L^2(D\times\mathbf{Y})$ norm. Here $\widehat{\mathbf{V}}^L$ denotes a sparse tensor product space in the sense of [15, 10] of multilevel spaces in D and in Y_i, $i = 1, ..., n$. The rank of the operator $\mathbf{P}^L\mathbf{Q}_A$ is at most $\dim\hat{\mathbf{V}}^L$. Using results on sparse grid interpolation (see, e.g. [17] and the references there) we find that

$$\|\mathbf{Q}_A - \mathbf{P}^L\mathbf{Q}_A\|_{\mathcal{L}(L^2(D\times\mathbf{Y})\otimes L^2(D\times\mathbf{Y}))} \leq cL^{n/2}2^{-Lt}\ .$$

As $\dim\widehat{\mathbf{V}}^L = O(L^n 2^{dL})$, we get with Lemma 1.4 the conclusion by choosing $\mathcal{C} = \mathbf{Q}_A$ and $\mathcal{C}_m = \mathbf{P}^L\mathbf{Q}_A$ by choosing in Lemma 1.4 $m = k = O(L^n 2^{dL})$. □

For the eigenfunctions Φ_k, we have

Proposition 1.6 *Assume that the random coefficient A in (1.1) satisfies (1.11) for some $t > d/2$. Then for every $d/2 < t^* < t$ there is a constant $c > 0$ independent of k such that*

$$\sum_{i,j=1}^{d} \|(\Phi_k)_{ij}\|^2_{L^\infty(D\times\mathbf{Y})} \leq c\lambda_k^{-2t^*/t}\ .$$

Proof The proof of this proposition follows that for Proposition 2.3 of Bieri et al. [4]. We note that

$$\frac{\partial}{\partial x_\alpha}(\Phi_k)_{ij}(x,\boldsymbol{y}) = \frac{1}{\lambda_k}\int_{D\times\mathbf{Y}}\frac{\partial}{\partial x_\alpha}\mathbf{Cov}[A]_{iji'j'}(x,\boldsymbol{y},x',\boldsymbol{y}')\Phi_{i'j'}(x',\boldsymbol{y}')dx'd\boldsymbol{y}' \quad i,j = 1,\ldots,d$$

with summation over repeated indices. Therefore, with the normalization $\|\Phi_k\|_{L^2(D\times\mathbf{Y})^{d\times d}} = 1$,

$$\forall k\in\mathbb{N}: \quad \|(\Phi_k)\|_{H^t_{mix}(D\times\mathbf{Y})^{d\times d}} \leq \frac{1}{\lambda_k}\|\mathbf{Cov}[A]\|_{H^t_{mix}(D\times\mathbf{Y})\otimes L^2(D\times\mathbf{Y})}\ .$$

For $0 < t^* < t$ hold the inclusions $L^2(D\times\mathbf{Y}) = L^2(D)\otimes L^2(Y_1)\otimes\ldots\otimes L^2(Y_n) \supset H^{t^*}_{mix}(D\times\mathbf{Y}) \supset H^t_{mix}(D\times\mathbf{Y})$ which follow by interpolation between L^2 and H^t on D respectively on Y_i and by the fact that the Sobolev norms of mixed highest derivative are cross norms on the tensor products of the respective Hilbert spaces. It follows from the corresponding interpolation inequality (see, e.g., [18, Chap. 1]) that there exists a constant $C(t^*) > 0$ such that

$$\|(\Phi_k)_{ij}\|_{H^{t^*}_{mix}(D\times\mathbf{Y})} \leq C\|(\Phi_k)_{ij}\|^{t^*/t}_{H^t_{mix}(D\times\mathbf{Y})}\|(\Phi_k)_{ij}\|^{1-t^*/t}_{L^2(D\times\mathbf{Y})}\ .$$

Applying Hölder's inequality we then get for all k

$$\begin{aligned}\sum_{i,j=1}^{d}\|(\Phi_k)_{ij}\|^2_{H^{t^*}_{mix}(D\times\mathbf{Y})} &\leq \left(C\sum_{i,j=1}^{d}\|(\Phi_k)_{ij}\|^2_{H^t_{mix}(D\times\mathbf{Y})}\right)^{t^*/t}\left(\sum_{i,j=1}^{d}\|(\Phi_k)_{ij}\|^2_{L^2(D\times\mathbf{Y})}\right)^{1-t^*/t}\\ &\leq C\Big(\frac{c}{\lambda_k^2}\Big)^{t^*/t}\ .\end{aligned}$$

As $t^* > d/2$, we deduce that exists $c > 0$ such that

$$\forall k \in \mathbb{N} \ : \quad \sum_{i,j=1}^{d} \|(\Phi_k)_{ij}\|^2_{L^\infty(D\times\mathbf{Y})} \leq c\lambda_k^{-2t^*/t} \ .$$

The conclusion then follows. □

In the Karhúnen-Loève expansion (1.10), let $\Psi_k = \sqrt{\lambda_k}\Phi_k$. We then find that there exists a constant $c > 0$ (depending on t, t^* and on d) such that for all k

$$\sum_{i,j=1}^{d} \|(\Psi_k)_{ij}\|^2_{L^\infty(D\times\mathbf{Y})} \leq c\lambda_k^{1-2t^*/t}.$$

From Proposition 1.5, we find that

$$\sum_{i,j=1}^{d} \|(\Psi_k)_{ij}\|^2_{L^\infty(D\times Y)} \leq ck^{(-t/d+\varepsilon)(1-2t^*/t)}.$$

For each vector $\xi \in \mathbb{R}^d$, we have

$$|(\Psi_k)_{ij}(x,\boldsymbol{y})\xi_i\xi_j|^2 \leq (\sum_{i,j=1}^{d} \|(\Psi_k)_{ij}\|^2_{L^\infty(D\times\mathbf{Y})})(\sum_{i,j=1}^{d} \xi_i^2\xi_j^2) \leq ck^{(-t/d+\varepsilon)(1-2t^*/t)}|\xi|^4 \ .$$

Therefore we may choose

$$\beta_k = ck^{(-t/d+\varepsilon)(1/2-t^*/t)}. \tag{1.12}$$

When t is sufficiently large, e.g. $(t/d - \varepsilon)(1/2 - t^*/t) > 1$, this implies that $\beta = \{\beta_k\}_{k\geq 1} \in \ell^1(\mathbb{N})$. Assuming that the random variables Z_k in the expansion (1.10) are uniformly bounded, we can and will in what follows assume that they are rescaled so that the support of their laws equals $[-1, 1]$. Assumption 1.2 holds when the constant α_0 is sufficiently large.

1.3 Probability space

A key tool in our analysis will be a parametric deterministic representation of the law of the random multiscale solution u^ε. We shall use this representation in order to prove various convergence results of u^ε as $\varepsilon \to 0$. Below, we shall investigate the precise regularity of dependence of this representation of u^ε on the parameter vector $\boldsymbol{z}$. This, in turn, also allows for the proof of sharp bounds on spectral approximations of the parametric solution u^ε. To this end, following [7, 8], we parametrize the law of $u^\varepsilon(\omega; x)$ in terms of countably many "random coordinates" $z_k(\omega)$ in the representation (1.7). We collect the random coordinates $(z_k)_{k\geq 1}$ in a vector $\boldsymbol{z}$ and define the *parametric, deterministic multiscale* coefficient $A^\varepsilon(\boldsymbol{z}; x)$ as follows:

$$A^\varepsilon(\boldsymbol{z}, x) := A\left(\boldsymbol{z}; x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\right) \ . \tag{1.13}$$

We define a probability measure on the parameter space $U = [-1, 1]^{\mathbb{N}}$. To this end we introduce the σ-algebra $\Theta = (\mathcal{B}^1([-1, 1]))^{\mathbb{N}}$ where $\mathcal{B}^1([-1, 1])$ denotes the Borel σ-algebra on the interval $[-1, 1]$. On the measurable space (U, Θ) thus obtained, we define a probability measure by

$$\rho(d\boldsymbol{z}) := \bigotimes_{j\geq 1} \frac{dz_j}{2}. \tag{1.14}$$

For any set of the form $S = \prod_{j=1}^{\infty} S_j$ with $S_j \in \mathcal{B}^1([-1, 1])$, it holds $S \in \Theta$ and

$$\rho(S) = \prod_{j=1}^{\infty} \mathbb{P}\{\omega : \ z_j(\omega) \in S_j\}.$$

1.4 Parametric deterministic multiscale problem

For each $\boldsymbol{z} \in U$, we define the deterministic coefficient matrix $A(\boldsymbol{z}; x, \boldsymbol{y})$ by

$$A(\boldsymbol{z}; x, \boldsymbol{y}) = \bar{A}(x, \boldsymbol{y}) + \sum_{k \geq 1} z_k \Psi_k(x, \boldsymbol{y}), \tag{1.15}$$

where the matrix functions $\bar{A}$ and Ψ_k are those in (1.7). The convergence of the sum on the right hand side is ensured by Proposition 1.3. For the parametric, deterministic coefficient $A^\varepsilon(\boldsymbol{z}, x)$ defined in (1.13), and for given $0 < \varepsilon < 1$, $\boldsymbol{z} \in U$ and $f \in H^{-1}(D)$, we consider the deterministic multiscale problem: for given $f \in H^{-1}(D)$ and $\boldsymbol{z} \in U$, find $u^\varepsilon(\boldsymbol{z}, \cdot) \in H^1_0(D)$ which satisfies

$$-\mathrm{div} A^\varepsilon(\boldsymbol{z}; \cdot) \nabla u^\varepsilon(\boldsymbol{z}; \cdot) = f(x) \;, \quad u^\varepsilon(\boldsymbol{z}; \cdot)|_{\partial D} = 0 \;. \tag{1.16}$$

Again, Assumption 1.1 holds with $\alpha = \alpha_0/(1+\kappa)$ and $\beta = \alpha_0 + \kappa\alpha_0/(1+\kappa)$, so problem (1.16) admits a unique solution which satisfies

$$\sup_{0<\varepsilon<1} \sup_{\boldsymbol{z} \in U} \|u^\varepsilon(\boldsymbol{z}, \cdot)\|_{H^1_0(D)} \leq \frac{\|f\|_{H^{-1}(D)}}{\alpha} \;. \tag{1.17}$$

We first prove that the solution $u^\varepsilon(\boldsymbol{z}, \cdot)$ depends on $\boldsymbol{z}$ continuously.

Proposition 1.7 *Under Assumption 1.1, there exists a constant $c > 0$ which is independent of ε such that*

$$\forall \boldsymbol{z}, \boldsymbol{z}' \in U : \quad \|u^\varepsilon(\boldsymbol{z}; \cdot) - u^\varepsilon(\boldsymbol{z}'; \cdot)\|_{H^1_0(D)} \leq c \|A(\boldsymbol{z}; \cdot, \cdot) - A(\boldsymbol{z}'; \cdot, \cdot)\|_{L^\infty(D, C(\bar{\mathbf{Y}}))}.$$

Proof Define $w^\varepsilon := u^\varepsilon(\boldsymbol{z}; \cdot) - u^\varepsilon(\boldsymbol{z}'; \cdot)$. The function w^ε is a weak solution of the Dirichlet problem

$$-\mathrm{div} A^\varepsilon(\boldsymbol{z}; \cdot) \nabla w^\varepsilon = -\mathrm{div}[A^\varepsilon(\boldsymbol{z}'; \cdot) - A^\varepsilon(\boldsymbol{z}; \cdot)] \nabla u^\varepsilon(\boldsymbol{z}'; \cdot) \;, \quad w^\varepsilon|_{\partial D} = 0 \;.$$

Therefore, it holds for every $\boldsymbol{z}, \boldsymbol{z}' \in U$

$$\forall\, w \in H^1_0(D) : \quad \int_D A^\varepsilon(\boldsymbol{z}, x) \nabla w^\varepsilon(x) \cdot \nabla w(x) dx = \int_D [A^\varepsilon(\boldsymbol{z}'; x) - A^\varepsilon(\boldsymbol{z}; x)] \nabla u^\varepsilon(\boldsymbol{z}'; \cdot) \cdot \nabla w(x) dx \;.$$

From (1.17) and Assumption 1.1, we obtain the conclusion. □.

To study the law of the solution u^ε of (1.5), we need to prove its measurability.

Proposition 1.8 *For every $0 < \varepsilon < 1$, the solution $U \ni \boldsymbol{z} \mapsto u^\varepsilon(\boldsymbol{z}; \cdot)$ of (1.16) is measurable as a map from U to $H^1_0(D)$.*

Proof As $H^1_0(D)$ is separable, it is sufficient to show that u is weakly measurable, i.e. for all $\phi \in H^1_0(D)$, the $H^1_0(D)$ innerproduct $\langle u^\varepsilon(\boldsymbol{z}, \cdot), \phi \rangle$ is measurable as a map from U to $\mathbb{R}$. For $a \in \mathbb{R}$ we denote by $Y_a = \{\boldsymbol{z} \in U \;:\; \langle u^\varepsilon(\boldsymbol{z}, \cdot), \phi \rangle > a\}$. From Proposition 1.7 it follows that if $\langle u^\varepsilon(\boldsymbol{z}), \phi \rangle > a$, then there exists a positive constant r such that if

$$\sup_{x, \boldsymbol{y}} |A_{ij}(\boldsymbol{z}; x, \boldsymbol{y}) - A_{ij}(\boldsymbol{z}'; x, \boldsymbol{y})| < r \quad \text{for all} \quad i, j = 1, \ldots, d$$

then $\langle u^\varepsilon(\boldsymbol{z}'; \cdot), \phi \rangle > a$. Let T_k be the set of $\boldsymbol{z} \in U$ such that for all $\bar{\boldsymbol{z}} = (z_1, z_2, \ldots, z_k, \bar{z}_1, \bar{z}_2, \ldots)$ $\langle u^\varepsilon(\bar{\boldsymbol{z}}, \cdot), \phi \rangle > a$ for all $\bar{z}_j \in [-1, 1]$, $j = 1, 2, \ldots$. For each $\boldsymbol{z} \in U$, from Proposition 1.3, we deduce that for all $i, j = 1, \ldots, d$,

$$\sup_{x, \boldsymbol{y}} |A_{ij}(\boldsymbol{z}; x, \boldsymbol{y}) - A_{ij}(\bar{\boldsymbol{z}}; x, \boldsymbol{y})| < r,$$

for all $\bar{z}_1, \bar{z}_2, \ldots \in [-1, 1]$ when k is sufficiently large. Therefore each vector $\boldsymbol{z} \in Y_a$ belongs to a set T_k for some constant k. Let $R_k \in [-1, 1]^k$ be the set of $t = (t_1, \ldots, t_k)$ such that $(t_1, t_2, \ldots, t_k, \bar{z}_1, \bar{z}_2, \ldots) \in T_k$ for all $\bar{z}_i \in [-1, 1]$. From Proposition 1.7, R_k is an open set, and therefore is the union of a countable set of open rectangles. Thus T_k is a countable union of sets in Θ and is therefore measurable, so is $Y_a = \cup_k T_k$. □

Remark 1.9 *The random solution* $u^\varepsilon(\omega;\cdot)$ *of problem* (1.5) *can be recovered from the parametric, deterministic solution of* (1.16)*, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *via*

$$\Omega \ni \omega \mapsto u^\varepsilon(\omega;\cdot) = u^\varepsilon(\boldsymbol{z};\cdot)|_{\boldsymbol{z}=\boldsymbol{z}(\omega)} \in H_0^1(D)$$

where, for $0 < \varepsilon < 1$*,* $u^\varepsilon(\boldsymbol{z};x)$ *denotes the weak solution of the parametric, deterministic problem* (1.16).

We shall use Remark 1.9 in what follows to homogenize (1.5). We do this by first passing to the $(n+1)$-scale limit in the parametric, deterministic problem (1.16) and then "reinsert" $\boldsymbol{z} = \boldsymbol{z}(\omega)$.

1.5 One-scale stochastic limiting problem

For each realization $\omega \in \Omega$, we study the limit when $\varepsilon \to 0$ of the solution u^ε of the problem (1.5). Multiscale convergence is an appropriate tool for this purpose. It was first introduced for two-scale problems by Nguetseng [14] and elaborated further by Allaire [1]. The definition of $n+1$-scale convergence we give below is due to Allaire and Briane [2]; we use their notion of multiscale convergence to study solutions of the problem (1.5) as $\varepsilon \to 0$.

Definition 1.10 *A bounded sequence* $\{u^\varepsilon\}_\varepsilon$ *in* $L^2(D)$ $n+1$*-scale converges to a function* $u_0 \in L^2(D\times\mathbf{Y})$ *if for all test functions* $\phi \in L^2(D, C_\#(\mathbf{Y}))$ *it holds*

$$\lim_{\varepsilon\to 0}\int_D u^\varepsilon(x)\phi\Big(x,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_n}\Big)dx = \int_D\int_{\mathbf{Y}} u_0(x,\boldsymbol{y})\phi(x,\boldsymbol{y})d\boldsymbol{y}dx\ .$$

Here and throughout, we denote

$$\int_{\mathbf{Y}} \cdot\ d\boldsymbol{y} = \int_{Y_1}\cdots\int_{Y_n} \cdot\ dy_n\ldots dy_1.$$

The use of the preceding definition in homogenization is due to the following theorem from [2].

Theorem 1.11 *Any bounded sequence* $\{u^\varepsilon\}$ *in* $L^2(D)$ *contains an* $n+1$*-scale convergent subsequence.*

For the variational formulation of the limiting problem of (1.16) using $n+1$-scale convergence, we introduce the space

$$\mathbf{V} = \{\boldsymbol{v} = (v_0,\{v_i\}):\ v_0 \in H_0^1(D),\ v_i \in V_i\}$$

where

$$V_1 = L^2(D;H^1_\#(Y_1)/\mathbb{R}),\quad V_i = L^2(D\times Y_1\times\ldots\times Y_{i-1};H^1_\#(Y_i)/\mathbb{R})\},\quad i=2,3,\ldots,n.$$

We equip V_i with the norm

$$|||\boldsymbol{v}||| = \|\nabla v_0\|_{L^2(D)} + \sum_{i=1}^n \|\nabla_{y_i} v_i\|_{L^2(D\times\mathbf{Y}_i)}\ \text{ where }\ \mathbf{Y}_i := Y_1\times\ldots\times Y_i\ .$$

For each $\boldsymbol{v} \in \mathbf{V}$, we denote by

$$\boldsymbol{\nabla v} = \nabla_x v_0 + \sum_{i=1}^n \nabla_{y_i} v_i\ . \tag{1.18}$$

Theorem 1.12 *For every fixed* $\boldsymbol{z} \in U$*, as* $\varepsilon \to 0$ *the solution* $u^\varepsilon(\boldsymbol{z};\cdot)$ *of the parametric, deterministic multiscale problem* (1.16) *converges weakly in* $H_0^1(D)$ *to a function* $u_0(\boldsymbol{z};\cdot)$*; moreover,* $\nabla u^\varepsilon(\boldsymbol{z},\cdot)$ $n+1$*-scale converges to* $\boldsymbol{\nabla u}$ *where* $\boldsymbol{u}(\boldsymbol{z}) = (u_0,u_1,\ldots,u_n) \in \mathbf{V}$ *is the unique solution of the parametric, deterministic elliptic one-scale limiting problem*

$$b(\boldsymbol{z};\boldsymbol{u},\boldsymbol{v}) \quad = \quad \int_D\int_{\mathbf{Y}} A(\boldsymbol{z};x,\boldsymbol{y})\boldsymbol{\nabla u}\cdot\boldsymbol{\nabla v}d\boldsymbol{y}dx = \int_D f v_0 dx,\quad \forall\, \boldsymbol{v} = (v_0,\{v_i\}) \in \mathbf{V}\ . \tag{1.19}$$

Here, the parametric bilinear form $b(\boldsymbol{z};\boldsymbol{u},\boldsymbol{v}) : \mathbf{V}\times\mathbf{V} \to \mathbb{R}$ *is bounded and coercive uniformly for* $\boldsymbol{z} \in U$*: there exist positive constants* c_1 *and* c_2 *which are independent of* $\boldsymbol{z} \in U$ *such that*

$$\forall \boldsymbol{z} \in U \quad \forall \boldsymbol{u} \in \mathbf{V}\ :\quad b(\boldsymbol{z};\boldsymbol{u},\boldsymbol{u}) \geq c_1|||\boldsymbol{u}|||^2\ , \tag{1.20}$$

and

$$\forall \boldsymbol{u},\boldsymbol{v} \in \mathbf{V}\ :\ \sup_{\boldsymbol{z}\in U}|b(\boldsymbol{z};\boldsymbol{u},\boldsymbol{v})| \leq c_2|||\boldsymbol{u}|||\ |||\boldsymbol{v}|||\ . \tag{1.21}$$

For each fixed $\boldsymbol{z} \in U$, this theorem is a consequence of [2]; the coefficients c_1 and c_2 only depend on α and β in Assumption 1.1 and are, therefore, independent of $\boldsymbol{z}$. From this, we obtain

$$\sup_{\boldsymbol{z}\in U} |||u(\boldsymbol{z})||| \leq \frac{\|f\|_{H^{-1}(D)}}{c_1}.$$

Using this a-priori bound, one verifies that the passage to the $n+1$-scale limit can be achieved uniformly with respect to $\boldsymbol{z} \in U$. Theorem 1.12 establishes convergence of the parametric solutions $u^\varepsilon(\boldsymbol{z};\cdot)$ as $\varepsilon \to 0$ to a solution to the high dimensional, parametric and deterministic one-scale problem for each fixed parameter vector $\boldsymbol{z} \in U$. To establish the connection between the solution of this problem and the laws of the random multiscale solutions u^ε of (1.5), we next verify measurability of the solution $\boldsymbol{u}(\boldsymbol{z})$ with respect to $\rho(d\boldsymbol{z})$.

Proposition 1.13 *The solution $\boldsymbol{u}(\boldsymbol{z})$ of* (1.19) *as a map from $(U, \Theta, \rho(d\boldsymbol{z}))$ to $\mathbf{V}$ is measurable.*

Proof For any two vectors $\boldsymbol{z}, \boldsymbol{z}' \in U$, let $\boldsymbol{u}(\boldsymbol{z})$ and $\boldsymbol{u}(\boldsymbol{z}')$ be the solutions of the problems (1.19). Define $\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{z}) - \boldsymbol{u}(\boldsymbol{z}')$. From (1.19), we find

$$\forall \boldsymbol{v} \in \mathbf{V}: \quad \int_D \int_{\mathbf{Y}} A(\boldsymbol{z}; x, \boldsymbol{y}) \boldsymbol{\nabla w} \cdot \boldsymbol{\nabla v} d\boldsymbol{y} dx = \int_D \int_{\mathbf{Y}} (A(\boldsymbol{z}'; x, \boldsymbol{y}) - A(\boldsymbol{z}; x, \boldsymbol{y})) \boldsymbol{\nabla} u(\boldsymbol{z}') \cdot \boldsymbol{\nabla v} d\boldsymbol{y} dx.$$

We choose $\boldsymbol{v} = \boldsymbol{w}$. From (1.20), $|||\boldsymbol{u}(\boldsymbol{z})|||$ is bounded uniformly for all $\boldsymbol{z} \in U$. Therefore, there exists a constant c which does not depend on $\boldsymbol{z}, \boldsymbol{z}' \in U$ such that

$$|||\boldsymbol{u}(\boldsymbol{z}) - \boldsymbol{u}(\boldsymbol{z}')||| \leq c \|A(\boldsymbol{z}; x, \boldsymbol{y}) - A(\boldsymbol{z}'; x, \boldsymbol{y})\|_{L^\infty(D \times \mathbf{Y})}. \tag{1.22}$$

The proof then follows the argument used in the proof of Proposition 1.8. □

We define

$$\underline{\mathbf{V}} = L^2(U, \rho; \mathbf{V}) = \left\{ \boldsymbol{v} = \{(v_0, \{v_i\}) : \ v_0 \in L^2(U, \rho; H_0^1(D)), \ v_i \in L^2(U, \rho; V_i)\} \right\} . \tag{1.23}$$

We note in passing that $L^2(U, \rho; \mathbf{V}) \cong L^2(U, \rho) \otimes \mathbf{V}$ and consider the *variational parametric, deterministic problem*:

$$\text{find } \ \boldsymbol{u} \in \underline{\mathbf{V}} \quad \text{such that} \quad B(\boldsymbol{u}(\boldsymbol{z}; \cdot), \boldsymbol{v}) = F(\boldsymbol{v}), \quad \forall \boldsymbol{v} = (v_0, \{v_i\}) \in \underline{\mathbf{V}}. \tag{1.24}$$

Here, the linear functional $F : \underline{\mathbf{V}} \to \mathbb{R}$ and the variational form $B(\cdot, \cdot) : \underline{\mathbf{V}} \times \underline{\mathbf{V}} \to \mathbb{R}$ are given by

$$F(\boldsymbol{v}) = \int_U \int_D f(x) v_0(\boldsymbol{z}; x) dx d\rho(\boldsymbol{z}) \ , \quad B(\boldsymbol{u}, \boldsymbol{v}) = \int_U b(\boldsymbol{z}; \boldsymbol{u}, \boldsymbol{v}) d\rho(\boldsymbol{z}) \ .$$

Proposition 1.14 *Problem* (1.24) *admits a unique parametric, deterministic solution $\boldsymbol{u}(\boldsymbol{z}; \cdot, \cdot) \in \mathbf{V}$ which belongs to $L^2(U, \rho; \mathbf{V})$. For ρ-a.e. $\boldsymbol{z} \in U$, this solution coincides with the solution $\boldsymbol{u}(\boldsymbol{z}; \cdot, \cdot)$ of the parametric problem* (1.19).

Proof The existence and uniqueness of a solution to (1.24) follow from Lax-Milgram theorem.

For each $\boldsymbol{z} \in U$, the solution $\boldsymbol{u}(\boldsymbol{z}; \cdot, \cdot) \in \mathbf{V}$ of the parametric, deterministic elliptic one-scale problem (1.19) exists, is unique and is uniformly bounded with respect to $\boldsymbol{z} \in U$. As a mapping $U \ni \boldsymbol{z} \mapsto \boldsymbol{u}(\boldsymbol{z}; \cdot) \in \mathbf{V}$, it is measurable. As $d\rho(\boldsymbol{z})$ is a probability measure on U, this implies that the parametric solution $\boldsymbol{u}(\boldsymbol{z}; \cdot)$ of (1.19) coincides with the solution $\boldsymbol{u} \in \underline{\mathbf{V}}$ of (1.24).

□

Remark 1.15 *The random solution $u^\varepsilon(\omega; \cdot)$ of problem* (1.5) *$(n+1)$-scale converges, for $\mathbb{P}$-a.e. $\omega \in \Omega$, towards the weak solution $\boldsymbol{u}(\omega; x, \boldsymbol{y})$ of the random one-scale limiting problem*

$$b(\omega; \boldsymbol{u}(\omega; \cdot); \boldsymbol{v}) \quad = \quad \int_D \int_{\mathbf{Y}} A(\omega; x, \boldsymbol{y}) \boldsymbol{\nabla u} \cdot \boldsymbol{\nabla v} d\boldsymbol{y} dx = \int_D f v_0 dx, \quad \forall \, \boldsymbol{v} = (v_0, \{v_i\}) \in \mathbf{V} \tag{1.25}$$

where, for $\omega \in \Omega$, we define

$$\Omega \ni \omega \mapsto b(\omega; \boldsymbol{v}, \boldsymbol{w}) := b(\boldsymbol{z}; \boldsymbol{v}, \boldsymbol{w})|_{\boldsymbol{z} = \boldsymbol{z}(\omega)} \ , \quad \boldsymbol{u}(\omega; x, \boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{z}; x, \boldsymbol{y})|_{\boldsymbol{z} = \boldsymbol{z}(\omega)} \ .$$

Our aim is to construct approximations of $u^\varepsilon(\omega; x)$ which are, on the one hand, *robust with respect to* ε, and, on the other hand, allow for *discretization of the randomness with convergence rates superior to that of Monte Carlo Methods*. To this end, we shall investigate next a spectral, "polynomial chaos" type approximation of the solution $\boldsymbol{u}(\boldsymbol{z}; x, \boldsymbol{y})$ of the $(n+1)$-scale limiting problem with respect to the parameter vector $\boldsymbol{z} \in U$, and then investigate the rate of convergence as $\varepsilon \to 0$ of $u^\varepsilon(\omega; x)$ to the solution of the limiting problem.

2 Galerkin Approximations in U

2.1 Orthonormal basis of $L^2(U,\rho)$

We start by defining a "generalized polynomial chaos" basis of $L^2(U,\rho(d\boldsymbol{z}))$. Let $(L_n)_{n\geq 0}$ be the univariate Legendre polynomials normalized so that

$$\frac{1}{2}\int_{-1}^{1} |L_n(t)|^2 dt = 1 \ . \tag{2.1}$$

Let $\mathcal{F}$ be the (countable) set of all sequences $\nu = (\nu_j)_{j\geq 1}$ of nonnegative integers such that only a finite number of ν_j are non zero, i.e. $\mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \ \|\nu\|_1 < \infty\}$. For $\nu \in \mathcal{F}$, we define the tensorized Legendre polynomials as

$$L_\nu(\boldsymbol{z}) = \prod_{j\geq 1} L_{\nu_j}(z_j), \quad \nu \in \mathcal{F}.$$

By the completeness of the Legendre polynomials $(L_n(t))_{n\geq 0}$ in $L^2(-1,1)$, the family L_ν forms an orthonormal basis of $L^2(U,\rho)$: each function $\boldsymbol{u} \in \underline{\mathbf{V}}$ can be expanded in the Legendre series

$$\boldsymbol{u} = \sum_{\nu\in\mathcal{F}} \boldsymbol{u}_\nu L_\nu, \quad \boldsymbol{u}_\nu \in \mathbf{V}. \tag{2.2}$$

2.2 Semidiscretization with respect to $\boldsymbol{z}$

For a subset $\Lambda \subset \mathcal{F}$ of finite cardinality, we define the space

$$\underline{\mathbf{V}}_\Lambda = \{\boldsymbol{u}_\Lambda = \sum_{\nu\in\Lambda} \boldsymbol{u}_\nu(x,\boldsymbol{y}) L_\nu(\boldsymbol{z}) : \ \boldsymbol{u}_\nu \in \mathbf{V}\} \subset \underline{\mathbf{V}} \ .$$

We then consider the following Galerkin semidiscretization in z:

$$\text{Find} \ \ \boldsymbol{u}_\Lambda \in \underline{\mathbf{V}}_\Lambda : \quad B(\boldsymbol{u}_\Lambda, \boldsymbol{v}_\Lambda) = F(\boldsymbol{v}_\Lambda) \quad \forall\, \boldsymbol{v}_\Lambda \in \underline{\mathbf{V}}_\Lambda \ . \tag{2.3}$$

Then the following approximation result holds.

Theorem 2.1 *For all $\Lambda \subset \mathcal{F}$, problem* (2.3) *admits a unique solution $\boldsymbol{u}_\Lambda \in \underline{\mathbf{V}}_\Lambda$ which satisfies the following error estimate:*

$$\|\boldsymbol{u} - \boldsymbol{u}_\Lambda\|_{\underline{\mathbf{V}}} \leq \Big(\sum_{\nu\in\mathcal{F}\setminus\Lambda} \|\boldsymbol{u}_\nu\|_{\mathbf{V}}^2\Big)^{1/2} \ . \tag{2.4}$$

Proof As $\underline{\mathbf{V}}_\Lambda$ is a Hilbert space, from (1.20) and (1.21) and from the Lax-Milgram lemma, (2.3) admits a unique solution $\boldsymbol{u}_\Lambda \in \underline{\mathbf{V}}_\Lambda$. From Cea's lemma and from the normalization (2.1) with Parseval's equality, we find that

$$\|\boldsymbol{u} - \boldsymbol{u}_\Lambda\|_{\underline{\mathbf{V}}} \leq \inf_{\boldsymbol{v}_\Lambda\in\underline{\mathbf{V}}_\Lambda} \|\boldsymbol{u} - \boldsymbol{v}_\Lambda\|_{\underline{\mathbf{V}}}.$$

Choosing $\boldsymbol{v}_\Lambda = \sum_{\nu\in\Lambda} \boldsymbol{u}_\nu L_\nu$, we arrive at the conclusion. □

3 Best N-term approximations

From Theorem 2.1, with a fixed cardinality N, we infer that an optimal choice of the set Λ is to select Λ corresponding to N terms $\boldsymbol{u}_\nu$ with largest $\mathbf{V}$ norms. Since these norms are not known a priori, we establish in this section an apriori bound for them, and choose the set Λ according to these bounds. In this way, we obtain a constructive approach for the choices of index sets Λ with the prescribed cardinality which might be, however, suboptimal. Nevertheless, we shall prove that the sets obtained in this way will allow for the best N-term convergence rates to be achieved. The key ingredient for obtaining the rate of convergence in terms of the cardinality of Λ is the following observation, due to Stechkin.

Lemma 3.1 *Let $\alpha = (\alpha_\nu)_{\nu\in\mathcal{F}}$ be a sequence in $\ell^p(\mathcal{F})$. Let $q \geq p \geq 0$. If $\Lambda_N \subset \mathcal{F}$ is the set of indices corresponding to a set of N largest $|\alpha_\nu|$, then for every N holds*

$$\|\alpha\|_{\ell^q(\mathcal{F}\setminus\Lambda_N)} = \Big(\sum_{\nu\in\mathcal{F}\setminus\Lambda_N} |\alpha_\nu|^q \Big)^{1/q} \leq \|\alpha\|_{\ell^p(\mathcal{F})} N^{-\sigma}, \text{ where } \sigma = \frac{1}{p} - \frac{1}{q} \geq 0 .$$

The convergence rate of truncated gpc expansions therefore depends on the p-summability of the sequence $(\|\boldsymbol{u}_\nu\|_{\mathbf{V}})_{\nu\in\mathcal{F}}$. We show that the summability of this sequence depends on the summability of the sequence β_k in (1.8).

Assumption 3.2 *There exists $0 < p < 1$ such that in (1.8) the sequence $(\beta_k)_{k\geq 1} \in \ell^p(\mathbb{N})$.*

Remark 3.3 *Assumption 3.2 holds if the constants t and t^* in (1.12) satisfy*

$$p\Big(\frac{t}{d} - \varepsilon\Big)\Big(\frac{1}{2} - \frac{t^*}{t}\Big) > 1.$$

3.1 Complex extension of the parametric deterministic problem

To bound $\|\boldsymbol{u}_\nu\|_{\mathbf{V}}$, we follow [8] and extend the parametric, deterministic limit problem (1.19) to complex values of the parameters $\boldsymbol{z}$. Let M be a positive constant. Let $K < 1$ be a positive constant such that

$$K \sum_{j=1}^\infty \beta_j < \frac{\alpha}{2M}.$$

We choose a constant J_0 such that

$$\sum_{j>J_0} \beta_j < \frac{\alpha K}{6M(1+K)}.$$

Let $E = \{1, 2, \dots, J_0\}$ and $F = \mathbb{N} \setminus E$. We define

$$|\nu_F| = \sum_{j>J_0} |\nu_j|.$$

For each $\nu \in \mathcal{F}$, we define

$$r_m = K \text{ when } m \leq J_0, \text{ and } r_m = \max\{1, \frac{\alpha\nu_m}{M|\nu_F|\beta_m}\} \text{ when } m > J_0, \tag{3.1}$$

where we make the convention that $\frac{|\nu_j|}{|\nu_F|} = 0$ if $|\nu_F| = 0$. For $m \geq 1$, we let the set $\mathcal{U}_m \in \mathbb{C}$ be defined as

$$[-1,1] \subset \mathcal{U}_m := \{\zeta_m \in \mathbb{C} : \ \text{dist}(\zeta_m, [-1,1]) \leq r_m\} \subset \mathbb{C}.$$

We next extend problem (1.19) to the complex parametric domain

$$\mathcal{U} = \bigotimes_{m=1}^\infty \mathcal{U}_m \subset \mathbb{C}^{\mathbb{N}} . \tag{3.2}$$

We define the complex parametric coefficient $A(\boldsymbol{\zeta}; x, \boldsymbol{y})$

$$A(\boldsymbol{\zeta}; x, \boldsymbol{y}) := \bar{A}(x, \boldsymbol{y}) + \sum_{m=1}^\infty \zeta_m \Psi_m(x, \boldsymbol{y}) , \quad \boldsymbol{\zeta} \in \mathcal{U} , \ x \in D , \ \boldsymbol{y} \in \mathbf{Y} .$$

The sum on the right hand side of this definition converges uniformly for $\boldsymbol{\zeta} \in \mathcal{U}$, $x \in D$ and for $\boldsymbol{y} \in \mathbf{Y}$, as we obtain from Proposition 1.3 for every $\boldsymbol{\zeta} \in \mathcal{U}$, $x \in D$, and $\boldsymbol{y} \in \mathbf{Y}$

$$\begin{aligned} |A_{ij}(\boldsymbol{\zeta}; x, \boldsymbol{y})| &\leq |\bar{A}_{ij}(x, \boldsymbol{y})| + \sum_{m=1}^\infty |(\Psi_m)_{ij}(x, \boldsymbol{y})|(1 + r_m) \\ &\leq \text{esssup}_{(x,\boldsymbol{y})\in D\times\mathbf{Y}} |\bar{A}_{ij}(x, \boldsymbol{y})| + \sum_{m=1}^{J_0} \|(\Psi_m)_{ij}\|_{L^\infty(D\times\mathbf{Y})}(1 + K) \\ &+ \sum_{j>J_0} \Big(2 + \frac{\alpha\nu_m}{M|\nu_F|\beta_m}\Big) \|(\Psi_m)_{ij}\|_{L^\infty(D\times\mathbf{Y})}. \end{aligned}$$

From Proposition 1.3, we find that

$$|A_{ij}(\boldsymbol{\zeta};x,\boldsymbol{y})|\le\|\bar{A}_{ij}\|_{L^\infty(D\times\mathbf{Y})}+4\frac{\kappa}{1+\kappa}\alpha_0+\frac{2\alpha}{M};. \tag{3.3}$$

In what follows, for functions taking values in $\mathbb{C}$ we still denote (with slight abuse of notation) by $\mathbf{V}$

$$H^1_0(D)\times\prod_{i=1}^n L^2(D\times Y_1\times\ldots\times Y_{i-1};H^1_\#(Y_i))\ .$$

Consider the complex parametric, one-scale limiting problem: given $\boldsymbol{\zeta}\in\mathcal{U}$, find $\boldsymbol{u}\in\mathbf{V}$ such that

$$b(\boldsymbol{\zeta};\boldsymbol{u},\boldsymbol{v})\ =\ \int_D\int_{\mathbf{Y}}A(\boldsymbol{\zeta};x,\boldsymbol{y})\boldsymbol{\nabla u}\cdot\overline{\boldsymbol{\nabla v}}d\boldsymbol{y}dx=\int_D f\overline{v_0}dx\quad\forall\,\boldsymbol{v}=(v_0,\{v_i\})\in\mathbf{V}\ . \tag{3.4}$$

Proposition 3.4 *Problem* (3.4) *admits a unique solution which is uniformly bounded in* $\mathbf{V}$ *for all* $\boldsymbol{\zeta}\in\mathcal{U}$.

Proof We first show that the matrix function $A(\boldsymbol{\zeta};x,\boldsymbol{y})$ is uniformly bounded and coercive for all $\boldsymbol{\zeta}\in U$, $x\in D$ and $\boldsymbol{y}\in\mathbf{Y}$. To this end, we observe that for every in $\xi\in\mathbb{C}^d$ and every $\boldsymbol{\zeta}\in U$, we have:

$$\begin{aligned}|\xi^HA(\boldsymbol{\zeta};x,\boldsymbol{y})\xi|&\le|\xi^H\bar{A}(x,\boldsymbol{y})\xi|+\sum_{m=1}^\infty|\zeta_m||(\xi^H\Psi_m\xi)|\\&\le\Big(\beta_0+\sum_{m=1}^{J_0}(1+K)\beta_m+\sum_{m>J_0}(2+\frac{\alpha\nu_m}{M|\nu_F|\beta_m})\beta_m\Big)|\xi|^2\\&\le\Big(\beta_0+\frac{2\kappa}{1+\kappa}\alpha_0+\frac{\alpha}{M}\Big)|\xi|^2.\end{aligned}$$

To prove uniform coercivity, we note that for every $\boldsymbol{\zeta}\in\mathcal{U}$ and for every $\xi\in\mathbb{C}^d$

$$\begin{aligned}\Re(\xi^HA(\boldsymbol{\zeta};x,\boldsymbol{y})\xi)&\ge\Re(\xi^H\bar{A}(x,y)\xi)-\sum_{m=1}^\infty|\zeta_m||\xi^H\Psi_m\xi|\\&\ge\Big(\alpha_0-\sum_{m=1}^{J_0}(1+K)\beta_m-\sum_{m>J_0}(2+\frac{\alpha\nu_m}{M|\nu_F|\beta_m})\beta_m\Big)|\xi|^2\\&\ge\Big(\alpha_0-\frac{\kappa}{1+\kappa}\alpha_0-K\sum_{m=1}^{J_0}\beta_m-2\sum_{m>J_0}\beta_m-\sum_{m>J_0}\frac{\alpha\nu_m}{M|\nu_F|}\Big)|\xi|^2\\&\ge(\alpha-\frac{\alpha}{2M}-\frac{\alpha}{3M}-\frac{\alpha}{M})|\xi|^2\\&>\frac{\alpha}{2}|\xi|^2\end{aligned} \tag{3.5}$$

if $M\ge4$ where $\alpha=\alpha_0/(1+\kappa)$. The proposition then follows from the Lax-Milgram Lemma. □

For an index $\nu\in\mathcal{F}$, we denote the support of ν by supp(ν), i.e. the set of j such that $\nu_j\neq0$. We define the domain

$$\mathcal{U}_\nu=\otimes_{j\in\text{supp}(\nu)}\mathcal{U}_j.$$

The following analyticity properties of $\boldsymbol{u}(\boldsymbol{z};\cdot,\cdot)$ hold.

Proposition 3.5 *For* $\nu\in\mathcal{F}$ *and* $\zeta\in\mathcal{U}$ *with fixed* ζ_k *for all the indices* $k\notin\text{supp}(\nu)$, *the map* $\boldsymbol{u}:\mathcal{U}_\nu\to\mathbf{V}$ *is analytic as a* $\mathbf{V}$*-valued function.*

Proof For $m\in\mathbb{N}$, we fix all coordinates ζ_k for $k\neq m$ and partition each vector $\boldsymbol{\zeta}\in\mathbb{C}^{\mathbb{N}}$ as $\boldsymbol{\zeta}=(\zeta^*_m,\zeta_m)$. It is sufficient to show that there exists a function $\boldsymbol{v}\in\mathbf{V}$ such that for all $\boldsymbol{\zeta}\in\mathcal{U}$ holds

$$\lim_{\delta\to0}\left\|\frac{\boldsymbol{u}(\zeta^*_m,\zeta_m+\delta;\cdot,\cdot)-\boldsymbol{u}(\boldsymbol{\zeta};\cdot,\cdot)}{\delta}-\boldsymbol{v}(\boldsymbol{\zeta};\cdot,\cdot)\right\|_{\mathbf{V}}=0,$$

For $\delta>0$, define the difference quotient $\boldsymbol{v}^\delta:=\delta^{-1}\left(\boldsymbol{u}(\zeta^*_m,\zeta_m+\delta;\cdot,\cdot)-\boldsymbol{u}(\boldsymbol{\zeta};\cdot,\cdot)\right)$.

The function $\boldsymbol{v}^\delta$ is a weak solution of the parametric variational problem

$$\int_D\int_{\mathbf{Y}} A(\boldsymbol{\zeta};x,\boldsymbol{y})\boldsymbol{\nabla}\boldsymbol{v}^\delta\cdot\overline{\boldsymbol{\nabla}\boldsymbol{w}}d\boldsymbol{y}dx=-\int_D\int_{\mathbf{Y}}\Psi_m(x,y)\boldsymbol{\nabla}\boldsymbol{u}(\zeta_m^*,\zeta_m+\delta;x,\boldsymbol{y})\cdot\overline{\boldsymbol{\nabla}\boldsymbol{w}}d\boldsymbol{y}dx,\quad\forall\boldsymbol{w}\in\mathbf{V}.$$

Let $\boldsymbol{v}$ denote the solution of the problem

$$\int_D\int_{\mathbf{Y}} A(\boldsymbol{\zeta};x,\boldsymbol{y})\boldsymbol{\nabla}\boldsymbol{v}\cdot\overline{\boldsymbol{\nabla}\boldsymbol{w}}d\boldsymbol{y}dx=-\int_D\int_{\mathbf{Y}}\Psi_m(x,y)\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{\zeta};x,\boldsymbol{y})\cdot\overline{\boldsymbol{\nabla}\boldsymbol{w}}d\boldsymbol{y}dx,\quad\forall\boldsymbol{w}\in\mathbf{V}.$$

We deduce that for every $\boldsymbol{w}\in\mathbf{V}$ and every $\boldsymbol{\zeta}\in\mathcal{U}$ holds

$$\int_D\int_{\mathbf{Y}} A(\boldsymbol{\zeta};x,\boldsymbol{y})\boldsymbol{\nabla}(\boldsymbol{v}^\delta-\boldsymbol{v})\cdot\overline{\boldsymbol{\nabla}\boldsymbol{w}}d\boldsymbol{y}dx=-\int_D\int_{\mathbf{Y}}\Psi_m(x,y)\boldsymbol{\nabla}(\boldsymbol{u}(\zeta_m^*,\zeta_m+\delta;x,\boldsymbol{y})-\boldsymbol{u}(\boldsymbol{\zeta},x,\boldsymbol{y}))\cdot\overline{\boldsymbol{\nabla}\boldsymbol{w}}d\boldsymbol{y}dx\,.$$

From this we obtain

$$|||\boldsymbol{v}^\delta-\boldsymbol{v}|||\le c(m)|||\boldsymbol{u}(\zeta_m^*,\zeta_m+\delta;x,\boldsymbol{y})-\boldsymbol{u}(\boldsymbol{\zeta};x,\boldsymbol{y})|||\,.$$

From (1.22),

$$\lim_{\delta\to0}|||\boldsymbol{v}^\delta-\boldsymbol{v}|||=0\,.$$

Hartogs' theorem implies that for every $\nu\in\mathcal{F}$, $\boldsymbol{u}(\zeta)$ is analytic as a mapping from $\mathcal{U}_\nu$ to $\mathbf{V}$. This completes the proof. □

We next investigate summability of the Legendre coefficients.

3.2 Coefficient estimates

Proposition 3.6 *For every $\nu\in\mathcal{F}$, there holds*

$$\|\boldsymbol{u}_\nu\|_{\mathbf{V}}\le C\Big(\prod_{m\in\mathrm{supp}(\nu)}\frac{2(1+K)}{K}\eta_m^{-\nu_m}\Big),\tag{3.6}$$

where $\eta_m:=r_m+\sqrt{1+r_m^2}$ with r_m as in (3.1).

Proof We proceed as in the proof of Lemma A.3 in Bieri, Andreev and Schwab [4] and Hoang and Schwab [11]. For $\nu\in\mathcal{F}$, the function $\boldsymbol{u}_\nu\in\mathbf{V}$ in (1.23) can be represented as

$$\boldsymbol{u}_\nu=\int_U\boldsymbol{u}(\boldsymbol{z})L_\nu(\boldsymbol{z})\rho(d\boldsymbol{z})\tag{3.7}$$

where the integral is understood as a Bochner integral of $\mathbf{V}$-valued functions. Let $S=\mathrm{supp}(\nu)\subset\mathbb{N}$ and define $\bar S:=\mathbb{N}\setminus S$. We then denote by $\mathcal{U}_S=\otimes_{m\in S}\mathcal{U}_m$ and $\mathcal{U}_{\bar S}=\otimes_{m\in\bar S}\mathcal{U}_m$, and by $\boldsymbol{z}_S=\{z_i,\ i\in S\}$, $\boldsymbol{z}_{\bar S}=\{z_i,\ i\in\bar S\}$ the extraction from $\boldsymbol{z}$, and analogously $\boldsymbol{\zeta}_S$ and $\boldsymbol{\zeta}_{\bar S}$.

Let $\mathcal{E}_m$ be the ellipse in $\mathcal{U}_m$ with foci at ±1 and the sum of the semiaxes being η_m; and $\mathcal{E}_S=\prod_{m\in\mathrm{supp}(\nu)}\mathcal{E}_m$. We can then write (3.7) as

$$\boldsymbol{u}_\nu=\frac{1}{(2\pi i)^{|\nu|_0}}\int_U L_\nu(\boldsymbol{z})\oint_{\mathcal{E}_S}\frac{\boldsymbol{u}(\boldsymbol{\zeta}_S,\boldsymbol{z}_{\bar S})}{(\boldsymbol{\zeta}_S-\boldsymbol{z}_S)^{\mathbf{1}}}d\boldsymbol{\zeta}_Sd\rho(\boldsymbol{z})\,.$$

For each $m\in\mathbb{N}$, let Γ_m be a copy of $[-1,1]$ and $z_m\in\Gamma_m$. We denote by $U_S=\prod_{m\in S}\Gamma_m$ and $U_{\bar S}=\prod_{m\in\bar S}\Gamma_m$. We then have

$$\boldsymbol{u}_\nu=\frac{1}{(2\pi i)^{|\nu|_0}}\int_{U_{\bar S}}\oint_{\mathcal{E}_S}\boldsymbol{u}(\boldsymbol{\zeta}_S,\boldsymbol{z}_{\bar S})\int_{U_S}\frac{L_\nu(\boldsymbol{z})}{(\boldsymbol{\zeta}_S-\boldsymbol{z}_S)^{\mathbf{1}}}d\rho_S(\boldsymbol{z}_S)d\boldsymbol{\zeta}_Sd\rho_{\bar S}(\boldsymbol{z}_{\bar S}).$$

We recall the definitions of the Legendre functions of the second kind:

$$Q_n(\xi)=\frac12\int_{[-1,1]}\frac{L_n(z)}{(\xi-z)}dz,\quad\xi\in\mathbb{C}\backslash[-1,1]\,,n\in\mathbb{N}_0\,.$$

For $\nu \in \mathcal{F}$, we denote by ν_S the restriction of ν to S. We define for $\boldsymbol{\zeta} \in \mathbb{C}^{\mathbb{N}}$

$$\mathcal{Q}_{\nu_S}(\boldsymbol{\zeta}_S) = \prod_{m \in \mathrm{supp}(\nu)} Q_{\nu_m}(\zeta_m).$$

Under the Joukovski transformation $\zeta_m = \frac{1}{2}(w_m + w_m^{-1})$, the Legendre polynomials of the second kind are written as

$$Q_{\nu_m}(\frac{1}{2}(w_m + w_m^{-1})) = \sum_{k=\nu_m+1}^{\infty} \frac{q_{\nu_m k}}{w_m^k}$$

with $|q_{\nu_m k}| \leq \pi$. Therefore

$$|\mathcal{Q}_{\nu_S}(\boldsymbol{\zeta}_S)| \leq \prod_{m \in S} \sum_{k=\nu_m+1}^{\infty} \frac{\pi}{\eta_m^k} = \prod_{m \in S} \pi \frac{\eta_m^{-\nu_m-1}}{1-\eta_m^{-1}}.$$

We then have

$$\begin{aligned}
\|\boldsymbol{u}_\nu\|_{\mathbf{V}} &= \left\| \frac{1}{(2\pi i)^{|\nu|_0}} \int_{U_{\bar{S}}} \oint_{\mathcal{E}_S} \boldsymbol{u}(\boldsymbol{\zeta}_S, \boldsymbol{z}_{\bar{S}}) \mathcal{Q}_{\nu_S}(\boldsymbol{\zeta}_S) d\boldsymbol{\zeta}_S d\rho_{\bar{S}}(\bar{\boldsymbol{z}}_S) \right\|_{\mathbf{V}} \\
&\leq \frac{1}{(2\pi)^{|\nu|_0}} \int_{U_{\bar{S}}} \oint_{\mathcal{E}_S} \|\boldsymbol{u}(\boldsymbol{\zeta}_S, \boldsymbol{z}_{\bar{S}})\|_{\mathbf{V}} \mathcal{Q}_{\nu_S}(\boldsymbol{\zeta}_S) d\boldsymbol{\zeta}_S d\rho_{\bar{S}}(\bar{\boldsymbol{z}}_S) \\
&\leq \frac{1}{(2\pi)^{|\nu|_0}} \|\boldsymbol{u}(\boldsymbol{\zeta})\|_{L^\infty(\mathcal{E}_S \times U_{\bar{S}}, \mathbf{V})} \max_{\mathcal{E}_S} |\mathcal{Q}_{\nu_S}| \prod_{m \in S} \mathrm{Len}(\mathcal{E}_m) \\
&\leq \frac{1}{(2\pi)^{|\nu|_0}} \|\boldsymbol{u}(\boldsymbol{\zeta})\|_{L^\infty(\mathcal{E}_S \times \mathcal{U}_{\bar{S}}, \mathbf{V})} \prod_{m \in S} \pi \frac{\eta_m^{-\nu_m-1}}{1-\eta_m^{-1}} \mathrm{Len}(\mathcal{E}_m) \\
&\leq C \prod_{m \in S} \frac{2(1+K)}{K} \eta_m^{-\nu_m},
\end{aligned}$$

as $\mathrm{Len}(\mathcal{E}_m) \leq 4\eta_m$, $\eta_m \geq 1+K$ and $\boldsymbol{u}(\boldsymbol{\zeta})$ is uniformly bounded in $\mathbf{V}$. □

To show the $\ell^p(\mathcal{F})$ summability of $\|\boldsymbol{u}_\nu\|_{\mathbf{V}}$, we use the following proposition, whose proof can be found in [7].

Proposition 3.7 *For* $0 < p < 1$, $\left(\frac{|\nu|!}{\nu!} b^\nu\right)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$ *iff (i)* $\sum_{m \geq 1} b_m < 1$ *and (ii)* $(b_m) \in \ell^p(\mathbb{N})$.

Proposition 3.8 *For* $0 < p < 1$ *as in Assumption 3.2,* $(\|u_\nu\|_{\mathbf{V}})_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$.

Proof We have from the previous proposition that

$$\begin{aligned}
\|\boldsymbol{u}_\nu\|_{\mathbf{V}} &\leq C \prod_{m \in S} \frac{2(1+K)}{K} (1+r_m)^{-\nu_m} \\
&\leq C \Big(\prod_{m \in E, \nu_m \neq 0} \frac{2(1+K)}{K} \eta^{\nu_m} \Big) \Big(\prod_{m \in F, \nu_m \neq 0} \frac{2(1+K)}{K} \Big(\frac{M|\nu_F|\beta_m}{\alpha \nu_m} \Big)^{\nu_m} \Big)
\end{aligned}$$

where $\eta = 1/(1+K)$. Let $\mathcal{F}_E = \{\nu \in \mathcal{F} : \quad \mathrm{supp}(\nu) \subset E\}$ and $\mathcal{F}_F = \mathcal{F} \setminus E$. From this, we have

$$\sum_{\nu \in \mathcal{F}} \|\boldsymbol{u}_\nu\|_{\mathbf{V}}^p \leq C A_E A_F,$$

where

$$A_E = \sum_{\nu \in \mathcal{F}_E} \prod_{m \in E, \nu_m \neq 0} \Big(\frac{2(1+K)}{K} \Big)^p \eta^{p\nu_m},$$

and

$$A_F = \sum_{\nu \in \mathcal{F}_F} \prod_{m \in F, \nu_m \neq 0} \Big(\frac{2(1+K)}{K} \Big)^p \Big(\frac{M|\nu|\beta_m}{\alpha \nu_m} \Big)^{p\nu_m}.$$

We now show that both A_E and A_F are finite. For A_E, we have

$$A_E = \Big(1 + \Big(\frac{2(1+K)}{K}\Big)^p \sum_{m\geq 1} \eta^{pm}\Big)^{J_0},$$

which is finite because $\eta < 1$. For A_F, we note that for $\nu_m \neq 0$,

$$\frac{2(1+K)}{K} \leq \Big(\frac{2(1+K)}{K}\Big)^{\nu_m}.$$

Therefore

$$A_F \leq \sum_{\nu\in\mathcal{F}_F} \prod_{m\in F} \Big(\frac{|\nu| d_m}{\nu_m}\Big)^{p\nu_m}, \quad \text{where } \ d_m = \frac{2M(1+K)\beta_m}{K\alpha}$$

and where we made the convention that $0^0 = 1$. We now proceed as in [8]: from the Stirling estimate $n!e^n/(e\sqrt{n}) \leq n^n \leq n!e^n/\sqrt{2\pi n}$, we infer $|\nu|^{|\nu|} \leq |\nu|!e^{|\nu|}$ and obtain

$$\prod_{m\in F} \nu_m^{\nu_m} \geq \frac{\nu! e^{|\nu|}}{\prod_{m\in F} \max\{1, e\sqrt{\nu_m}\}}\ .$$

Hence

$$A_F \leq \sum_{\nu\in\mathcal{F}_F} \Big(\frac{|\nu|!}{\nu!} d^\nu\Big)^p \Big(\prod_{m\in F} \max\{1, e\sqrt{\nu_m}\}\Big)^p \leq \sum_{\nu\in\mathcal{F}_F} \Big(\frac{|\nu|!}{\nu!} \bar{d}^\nu\Big)^p,$$

where $\bar{d}_m = ed_m$ and where we have used the estimate $e\sqrt{n} \leq e^n$. From this, we have

$$\sum_{m\geq 1} \bar{d}_m \leq \sum_{m\in F} \frac{6M(1+K)\beta_m}{K\alpha} < 1.$$

It is also obvious that

$$\|\bar{d}\|_{\ell^p(\mathbb{N})} < \infty.$$

From these estimates and from Proposition 3.7 we obtain the conclusion. □

3.3 Best N-term Approximation Rates

With Lemma 3.1, we have from Proposition 3.8 and Theorem 2.1 the following result:

Theorem 3.9 *If Assumptions 1.1, 1.2 and 3.2 hold for some $0 < p < 1$, there exists a sequence $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathcal{F}$ of index sets with cardinality not exceeding N such that the solutions $\boldsymbol{u}_{\Lambda_N}$ of the Galerkin semidiscretized problems (2.3) satisfy*

$$\|\boldsymbol{u} - \boldsymbol{u}_{\Lambda_N}\|_{\underline{\mathbf{V}}} \leq CN^{-\sigma}, \qquad \sigma = \frac{1}{p} - \frac{1}{2}.$$

4 Regularity

To obtain convergence rates of sparse tensor finite element discretizations for the fully discretized problem of (2.3), we introduce, following [15, 10], regularity spaces $\mathcal{H}_i$ $(i = 1,\ldots,n)$. The space $\mathcal{H}_i$ consists of all the functions $w(x, y_1,\ldots,y_i)$ that are Y_j-periodic in y_j $(j = 1,\ldots,i)$ such that for any vectors $(\alpha_0, \alpha_1,\ldots,\alpha_i) \in (\mathbb{N}_0^d)^{i+1}$ such that $|\alpha_j| \leq 1$ for $0 \leq j \leq i-1$ and $|\alpha_i| \leq 2$ where $|\alpha_i|$ denotes the sum of all the components of α_i,

$$\frac{\partial^{|\alpha_0|+\ldots+|\alpha_i|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i} \in L^2(D \times Y_1 \times \ldots \times Y_i).$$

The space $\mathcal{H}_i$ is equipped with the norm

$$\|w\|_{\mathcal{H}_i} = \sum_{\substack{|\alpha_i|\leq 2\\ |\alpha_j|\leq 1,\ j=0,\ldots,i-1}} \Big\|\frac{\partial^{|\alpha_0|+\ldots+|\alpha_i|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i}\Big\|_{L^2(D\times Y_1\times\ldots\times Y_i)}.$$

We then define the subspace $\boldsymbol{\mathcal{H}}$ of $\mathbf{V}$ as

$$\boldsymbol{\mathcal{H}} = \{(v_0, \{v_i\}) :\ v_0 \in H^2(D),\ v_i \in \mathcal{H}_i,\ i = 1,\ldots,n\}.$$

4.1 Regularity of the parametric, deterministic problem (1.19)

For each index $i = 1, \ldots, n-1$, we denote

$$\boldsymbol{y}_i = (y_1, \ldots, y_i) \quad \text{and} \quad \mathbf{Y}_i = Y_1 \times \ldots \times Y_i \,. \tag{4.1}$$

We define by $\boldsymbol{C}_i^1$, $i = 1, \ldots, n$ the space of functions $w(x, y_1, \ldots, y_i)$ that are continuous in each variables $x, y_1, \ldots, y_i$ and that are Y_j-periodic with respect to y_j, $j = 1, ..., i$. For a vector $(\gamma_0, \ldots, \gamma_i) \in \{0,1\}^{i+1}$ and the index vector $(j_0, j_1, \ldots, j_i) \in \{1, \ldots, d\}^{i+1}$, the strong derivative

$$\frac{\partial^{\gamma_0+\ldots+\gamma_i} w}{\partial x_{j_0}^{\gamma_0} \partial y_{1j_1}^{\gamma_1} \ldots \partial y_{ij_i}^{\gamma_i}}$$

exists for all $(x, y_1, \ldots, y_i) \in \bar{D} \times \bar{Y}_1 \times \ldots \times \bar{Y}_i$ and is continuous. We define the seminorm

$$\|w\|_{\boldsymbol{C}_i^1} = \sum_{\substack{(\gamma_0,\ldots,\gamma_i)\in\{0,1\}^{i+1} \\ (j_0,j_1,\ldots,j_i)\in\{1,\ldots,d\}^{i+1}}} \Big\| \frac{\partial^{\gamma_0+\ldots+\gamma_i} w}{\partial x_{j_0}^{\gamma_0} \partial y_{1j_1}^{\gamma_1} \ldots \partial y_{ij_i}^{\gamma_i}} \Big\|_{L^\infty(\Omega\times\mathbf{Y}_i)} . \tag{4.2}$$

The following homogenization result is, in principle, well known (see, e.g., [2]). As we require its parametric version, and also use its derivation later, we present its proof.

Proposition 4.1 *There exists a symmetric matrix function $A_0(\boldsymbol{z};\cdot) \in L^\infty(D)^{d\times d}_{\text{sym}}$ that is uniformly bounded and coercive for all $\boldsymbol{z} \in U$ such that the limit function $u_0(\boldsymbol{z},\cdot) \in H_0^1(D)$ in Theorem 1.12 is the solution of the problem:*

$$\int_D A_0(\boldsymbol{z};x)\nabla u_0(\boldsymbol{z};x)\cdot\nabla\phi(x)dx = \int_D f(x)\phi(x)dx, \quad \forall\,\phi \in H_0^1(D).$$

Proof With $\boldsymbol{u}(\boldsymbol{z}) = (u_0, u_1, ..., u_n) \in \mathbf{V}$ as in (1.19), we have (with implied summation over the repeated index $l = 1, ..., d$)

$$u_n = w_{nl}\Big(\frac{\partial u_0}{\partial x_l} + \frac{\partial u_1}{\partial y_{1l}} + \ldots + \frac{\partial u_{n-1}}{\partial y_{(n-1)l}}\Big),$$

where the functions $w_{nl} \in L^2(D \times Y_1 \times \ldots \times Y_{n-1}; H^1_\#(Y_n)/\mathbb{R})$ are the unique solutions of the parametric unit-cell problems

$$\int_{Y_n} A(\boldsymbol{z}; x, \boldsymbol{y}_{n-1}, y_n)(e_l + \nabla_{y_n} w_{nl}) \cdot \nabla_{y_n}\phi_n dy_n = 0, \quad \forall\,\phi_n \in H^1_\#(Y_n); \tag{4.3}$$

(here e_l denotes the lth unit vector in $\mathbb{R}^d$). From (1.19), we have

$$\int_D \int_{\mathbf{Y}} A(I + \nabla_{y_n} w_n) \cdot \Big(\nabla_x u_0 + \sum_{k=1}^{n-1} \nabla_{y_k} u_k\Big) \cdot \nabla_{y_{n-1}}\phi_{n-1} d\boldsymbol{y}dx = 0, \tag{4.4}$$

for all $\phi_{n-1} \in L^2(D \times Y_1 \times \ldots \times Y_{n-2}, H^1_\#(Y_{n-1}))$, where w_n denotes the vector $(w_{n1}, \ldots, w_{nd})$ and I is the identity matrix. By recursion, we define the "upscaled" conductivity matrices $A_{n-1}(\boldsymbol{z}; x, \boldsymbol{y}_{n-1})$ as

$$A_{n-1} = \int_{Y_n} A(I + \nabla_{y_n} w_n)dy_n = \int_{Y_n} A(I + \nabla_{y_n} w_n) \cdot (I + \nabla_{y_n} w_n)dy_n. \tag{4.5}$$

We then consider the parametric unit cell problem on scale $n-1$: find $w_{(n-1)l}$ such that

$$\int_{Y_{n-1}} A_{n-1}(e_l + \nabla_{y_{n-1}} w_{(n-1)l}) \cdot \nabla_{y_{n-1}}\phi_{n-1} dy_{n-1} = 0, \quad \forall\,\phi_{n-1} \in H^1_\#(Y_{n-1}).$$

We then have

$$u_{n-1} = w_{(n-1)l}\Big(\frac{\partial u_0}{\partial x_l} + \frac{\partial u_1}{\partial y_{1l}} + \ldots + \frac{\partial u_{n-2}}{\partial y_{(n-2)l}}\Big).$$

With the convention that $A_n = A$, we define recursively for $i = n-2, n-3, \ldots$ the functions $w_{il} \in L^2(D \times \mathbf{Y}_{i-1}; H^1_\#(Y_i)/\mathbb{R})$ as (unique) solutions of the problems

$$\int_{Y_i} A_i(e_l + \nabla_{y_i} w_{il}) \cdot \nabla_{y_i} \phi_i dy_i = 0, \quad \forall \phi_i \in H^1_\#(Y_i).$$

For $i = 1, 2, \ldots, n$, the scale interaction function u_i is then determined as

$$u_i = w_{il}\Big(\frac{\partial u_0}{\partial x_l} + \frac{\partial u_1}{\partial y_{1l}} + \ldots + \frac{\partial u_{i-1}}{\partial y_{(i-1)l}}\Big)$$

and the "upscaled" matrix A_{i-1} is defined in terms of A_i as

$$A_{i-1}(\boldsymbol{z}; x, \boldsymbol{y}_{i-1}) = \int_{Y_i} A_i(\boldsymbol{z}; x, \boldsymbol{y}_{i-1}, y_i)(I + \nabla_{y_i} w_i) \cdot (I + \nabla_{y_i} w_i) dy_i, \quad \boldsymbol{z} \in U,\ x \in D,\ \boldsymbol{y}_{i-1} \in \mathbf{Y}_{i-1} \tag{4.6}$$

where w_i denotes the vector $(w_{i1}, \ldots, w_{id})$. Upon completing the upscaling recursion at $i = 1$ the effective diffusivity matrix $A_0(x)$ is obtained as

$$A_0(\boldsymbol{z}; x) = \int_{Y_1} A_1(\boldsymbol{z}; x, y_1)(I + \nabla_{y_1} w_1) \cdot (I + \nabla_{y_1} w_1) dy_1$$

and the function $u_0(\boldsymbol{z}; \cdot) \in H^1_0(D)$ satisfies the *homogenized, parametric limiting problem*

$$\int_D A_0(\boldsymbol{z}; x) \nabla u_0(\boldsymbol{z}; x) \cdot \nabla \phi(x) dx = \int_D f(x) \phi(x) dx, \tag{4.7}$$

for all $\phi \in H^1_0(D)$.

As the matrix A is symmetric, all matrices A_i $(i = 0, \ldots, n-1)$ are symmetric. Fix $\xi \in \mathbb{R}^d$. Then (with summation over repeated indices)

$$A_{(n-1)kl}\xi_k\xi_l = \int_{Y_n} A_{rs}\Big(\xi_r + \frac{\partial(w_{nk}\xi_k)}{\partial y_{nr}}\Big)\Big(\xi_s + \frac{\partial(w_{nl}\xi_l)}{\partial y_{ns}}\Big) dy_n\ .$$

For the constant α as in Assumption 1.1, and for every $\boldsymbol{z} \in U$, $x \in D$, $\boldsymbol{y}_{n-1} \in \mathbf{Y}_{n-1}$ and every $\xi \in \mathbb{R}^d$

$$A_{(n-1)kl}(\boldsymbol{z}; x, \boldsymbol{y}_{n-1})\xi_k\xi_l \geq \alpha \int_{Y_n} \Big(\xi_r + \frac{\partial(w_{nk}\xi_k)}{\partial y_{nr}}\Big)\Big(\xi_r + \frac{\partial(w_{nl}\xi_l)}{\partial y_{nr}}\Big) dy_n \geq \alpha|\xi|^2\ .$$

Furthermore with summation over repeated indices,

$$\begin{aligned} A_{(n-1)kl}(\boldsymbol{z}; x, \boldsymbol{y}_{n-1})\xi_k\xi_l &\leq \beta\Big(\sum_{r=1}^d \xi_r^2 + \sum_{r=1}^d \int_{Y_n} \frac{\partial(w_{nk}\xi_k)}{\partial y_{nr}} \frac{\partial(w_{nl}\xi_l)}{\partial y_{nr}} dy_n\Big) \\ &\leq \beta\Big(\sum_{r=1}^d \xi_r^2 + \sum_{r=1}^d \Big(\sum_{k=1}^d \xi_k^2\Big)\Big(\sum_{k=1}^d \int_{Y_n} \Big(\frac{\partial w_{nk}}{\partial y_{nr}}\Big)^2 dy_n\Big)\Big). \end{aligned}$$

From (4.3), we deduce that there is a constant $c = c(d)$ which depends only on the dimension d such that

$$\|\nabla_{y_n} w_{nl}\|_{L^2(Y_n)} \leq \frac{c(d)}{\alpha} \sup_{k,l} \|A_{kl}\|_{L^\infty(D\times\mathbf{Y})}.$$

Therefore, there is a constant $c = c(\alpha, d)$ such that

$$A_{(n-1)kl}(\boldsymbol{z}; x, \boldsymbol{y}_{n-1})\xi_k\xi_l \leq \beta c(\alpha, d)(1 + \sup_{k,l} \|A_{kl}\|_{L^\infty(D\times\mathbf{Y})})^2|\xi|^2.$$

Repeating this argument for $A_i(\boldsymbol{z}; x, \boldsymbol{y}_i)$, $i = n-1, \ldots, 1$, we deduce that for all $z \in U$ and $x \in D$,

$$A_{0kl}(z; x)\xi_k\xi_l \geq \alpha|\xi|^2,$$

and

$$A_{0kl}(z;x)\xi_k\xi_l \le \beta c(\alpha,d)^n(1+\sup_{k,l}\|A_{kl}\|_{L^\infty(D\times\mathbf{Y})})^2(1+\sup_{k,l}\|A_{(n-1)kl}\|_{L^\infty(D\times\mathbf{Y}_{n-1})})^2\cdots$$
$$.(1+\sup_{k,l}\|A_{1kl}\|_{L^\infty(D\times Y)})^2|\xi|^2\ .$$

From (4.5), we deduce that

$$\begin{aligned}\sup_{k,l}\|A_{(n-1)kl}\|_{L^\infty(D\times\mathbf{Y}_{n-1})} &\le c(d)\sup_{k,l}\|A_{kl}\|_{L^\infty(D\times\mathbf{Y})}(1+\|\nabla_{y_n}w_n\|_{L^\infty(D\times\mathbf{Y}_{n-1},L^2(Y_n))})\\ &\le c(\alpha,d)(1+\sup_{k,l}\|A_{kl}\|_{L^\infty(D\times\mathbf{Y})})^2,\end{aligned}$$

so

$$1+\sup_{k,l}\|A_{(n-1)kl}\|_{L^\infty(D\times\mathbf{Y}_{n-1})}\le c(\alpha,d)(1+\sup_{k,l}\|A_{kl}\|_{L^\infty(D\times\mathbf{Y})})^2\ .$$

Repeating this argument for $i=2,\dots,n$, we get

$$1+\sup_{k,l}\|A_{(n-i)kl}\|_{L^\infty(D\times\mathbf{Y}_{n-i})}\le c(\alpha,d)^{2^i-1}(1+\sup_{k,l}\|A_{kl}\|_{L^\infty(D\times\mathbf{Y})})^{2^i}.$$

Therefore

$$A_{0kl}(z;x)\xi_k\xi_l\le\beta[c(\alpha,d)(1+\sup_{k,l}\|A_{kl}\|_{L^\infty(D\times\mathbf{Y})}]^{2^{n+1}-2}|\xi|^2. \tag{4.8}$$

□

Proposition 4.2 *Assume that the domain D is convex and $f\in L^2(D)$. Assume further that $A(\boldsymbol{z})\in\boldsymbol{C}^1_n$ and $\|A(\boldsymbol{z})\|_{\boldsymbol{C}^1_n}$ is uniformly bounded for all $\boldsymbol{z}\in U$. Then $\boldsymbol{u}(\boldsymbol{z})\in\boldsymbol{\mathcal{H}}$ and $\|\boldsymbol{u}(\boldsymbol{z})\|_{\boldsymbol{\mathcal{H}}}$ is uniformly bounded for all $\boldsymbol{z}\in U$.*

Proof The functions $u_i(\boldsymbol{z};x,\boldsymbol{y}_i)$ can be expressed in terms of the functions $w_i=(w_{i1},\dots,w_{id})$ as

$$u_i=w_i\cdot(I+\nabla_{y_{i-1}}w_{i-1})\cdots(I+\nabla_{y_1}w_1)\cdot\nabla u_0. \tag{4.9}$$

From (4.3), for almost all $(x,\boldsymbol{y}_{n-1})\in D\times\mathbf{Y}_{n-1}$

$$\int_{Y_n}A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{y_n}w_{nl}(\boldsymbol{z};x,\boldsymbol{y})\cdot\nabla_{y_n}\phi_n dy_n=\int_{Y_n}(\nabla_{y_n}\cdot(A(\boldsymbol{z};x,\boldsymbol{y})e_l)\phi_n dy_n,\quad\forall\,\phi_n\in H^1_\#(Y_n)\ . \tag{4.10}$$

As any function in $\mathcal{D}(\mathbb{R}^d)$ with a sufficiently small support can be extended to a Y_n-periodic function of the same regularity, we see using a partition of unity, that

$$\int_{\mathbb{R}^d}A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{y_n}w_{nl}(\boldsymbol{z};x,\boldsymbol{y})\cdot\nabla_{y_n}\phi_n dy_n=\int_{\mathbb{R}^d}(\nabla_{y_n}\cdot(A(\boldsymbol{z};x,\boldsymbol{y})e_l)\phi_n dy_n,\quad\forall\,\phi_n\in\mathcal{D}(\mathbb{R}^d).$$

We choose a smooth domain D' such that $Y_n\subset D'$ and $\tau\in\mathcal{D}(D')$ such that $\tau(y_n)=1$ when $y_n\in Y_n$. For $\tau(y_n)w_{nl}(\boldsymbol{z};x,\boldsymbol{y}_{n-1},y_n)$ in D', we deduce that

$$\|w_{nl}(\boldsymbol{z};x,\boldsymbol{y})\|_{H^2(Y_n)}\le C(\|\nabla_{y_n}\cdot(A(\boldsymbol{z};x,\boldsymbol{y})e_l)\|_{L^2(Y_n)}+\|w_{nl}\|_{L^2(Y_n)}),$$

where the constant C depends on the $\boldsymbol{C}^1$ norm of $A(\boldsymbol{z};x,\boldsymbol{y}_{n-1},\cdot)$, α, β and τ, and is in particular independent of $\boldsymbol{z}\in U$ (see, e.g., Wloka [19] page 330).

Now, we freeze all the coordinates $(x,\boldsymbol{y}_{n-1})$ except the jth coordinate of the variable y_k for an index $k=1,\dots,n-1$, and denote by (y^*_{kj},y_{kj}) the vector $\boldsymbol{y}_{n-1}$. For $\delta>0$, let

$$\chi^\delta(\boldsymbol{z};x,\boldsymbol{y})=\frac{w_{nl}(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-w_{nl}(\boldsymbol{z};x,\boldsymbol{y})}{\delta}.$$

For all $\phi_n \in H^1_\#(Y_n)$ we have

$$\int_{Y_n} A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{y_n}\chi^\delta\cdot\nabla_{y_n}\phi_n dy_n = -\int_{Y_n}\frac{A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})}{\delta}e_l\cdot\nabla_{y_n}\phi_n dy_n$$
$$-\int_{Y_n}\frac{A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})}{\delta}\nabla_{y_n}w_{nl}(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta,\cdot)\cdot\nabla_{y_n}\phi_n dy_n\ . \qquad (4.11)$$

Let $\chi(\boldsymbol{z};x,\boldsymbol{y}_{n-1},\cdot)\in H^1_\#(Y_n)/\mathbb{R}$ denote the solution of the problem

$$\int_{Y_n} A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{y_n}\chi\cdot\nabla_{y_n}\phi_n dy_n = -\int_{Y_n}\frac{\partial A(\boldsymbol{z};x,\boldsymbol{y})}{\partial y_{kj}}e_l\cdot\nabla_{y_n}\phi_n dy_n$$
$$-\int_{Y_n}\frac{\partial A(\boldsymbol{z};x,\boldsymbol{y})}{\partial y_{kj}}\nabla_{y_n}w_{nl}(\boldsymbol{z};x,\boldsymbol{y})\cdot\nabla_{y_n}\phi_n dy_n. \qquad (4.12)$$

From these equations, we deduce

$$\int_{Y_n} A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{y_n}(\chi^\delta-\chi)\cdot\nabla_{y_n}\phi_n dy_n$$
$$= -\int_{Y_n}\Big(\frac{A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})}{\delta}-\frac{\partial A(\boldsymbol{z};x,\boldsymbol{y})}{\partial y_{kj}}\Big)e_l\cdot\nabla_{y_n}\phi_n dy_n$$
$$-\int_{Y_n}\Big(\frac{A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})}{\delta}-\frac{\partial A(\boldsymbol{z};x,\boldsymbol{y})}{\partial y_{kj}}\Big)\nabla_{y_n}w_{nl}(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta,\cdot)\cdot\nabla_{y_n}\phi_n dy_n$$
$$-\int_{Y_n}\frac{\partial A(\boldsymbol{z};x,\boldsymbol{y})}{\partial y_{kj}}\nabla_{y_n}(w_{nl}(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-w_{nl}(\boldsymbol{z};x,\boldsymbol{y}))\cdot\nabla_{y_n}\phi_n dy_n\ .$$

From (4.11) (ignoring the constant δ), we have for every $\boldsymbol{z}\in U$ and $x\in D$

$$\|w_{nl}(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-w_{nl}(\boldsymbol{z};x,\boldsymbol{y})\|_{H^2(Y_n)/\mathbb{R}}$$
$$\le C(\|A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})\|_{L^\infty(Y_n)}+\|\nabla_{y_n}(A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y}))\|_{L^\infty(Y_n)})\ . \qquad (4.13)$$

Therefore,

$$\begin{aligned}\|\chi^\delta-\chi\|_{H^2(Y_n)/\mathbb{R}} \le\ & C\Big(\Big\|\nabla_{y_n}\Big(\frac{A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})}{\delta}-\frac{\partial A}{\partial y_{kj}}\Big)\Big\|_{L^\infty(Y_n)}\\ &+\Big\|\frac{A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})}{\delta}-\frac{\partial A}{\partial y_{kj}}\Big\|_{L^\infty(Y_n)}\\ &+\|A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y})\|_{L^\infty(Y_n)}\\ &+\|\nabla_{y_n}(A(\boldsymbol{z};x,y^*_{kj},y_{kj}+\delta)-A(\boldsymbol{z};x,\boldsymbol{y}))\|_{L^\infty(Y_n)}\Big)\end{aligned}$$

which converges to 0 when δ tends to 0 as $A\in\boldsymbol{C}^1_n$. Therefore

$$\chi=\frac{\partial w_{nl}}{\partial y_{kj}}\ \text{in}\ H^2_\#(Y_n)/\mathbb{R}.$$

As χ satisfies (4.12), for each $\boldsymbol{z}\in U$, χ as a map from $D\times\mathbf{Y}_{n-1}$ to $H^2(Y_n)/\mathbb{R}$ is continuous, due to the continuity of the coefficient $A(\boldsymbol{z};x,\boldsymbol{y})$ and due to the continuity of w_{nl} as a map from $D\times\mathbf{Y}_{n-1}$ (from (4.13)).

Performing a similar procedure for the remaining functions $\partial w_{nl}/\partial y_{kj}$ and their derivatives, we find $w_{nl}\in\boldsymbol{C}^1_{n-1}(H^2_\#(Y_n)/\mathbb{R})$. Therefore, from (4.5), $A_{n-1}\in\boldsymbol{C}^1_{n-1}$. In the same fashion, we deduce that $w_{il}\in\boldsymbol{C}^1_{i-1}(H^2_\#(Y_i)/\mathbb{R})$ for all $i=1,\dots,n$ and $l=1,\dots,d$, and $\|w_{il}(\boldsymbol{z})\|_{\boldsymbol{C}^1_{i-1}(H^2(Y_i)/\mathbb{R})}$ is uniformly bounded for all $\boldsymbol{z}$. Therefore for every $\boldsymbol{z}\in U$, $A_0(\boldsymbol{z})\in C^1(\bar{D})^{d\times d}$ and $\|A_0(\boldsymbol{z})\|_{(C^1(\bar{D}))^{d\times d}}$ is uniformly bounded for all $\boldsymbol{z}\in U$.

Next we claim $u_0(\boldsymbol{z},\cdot)\in H^2(D)$ and that its $H^2(D)$ norm is uniformly bounded for all $\boldsymbol{z}\in U$. We have shown that for all vectors $\xi\in\mathbb{R}^d$, and for every $\boldsymbol{z}\in U$

$$\alpha|\xi|^2\le\xi^\top A_0(\boldsymbol{z},x)\xi\le\beta'|\xi|^2,$$

where $\alpha>0$ is the constant in Assumption 1.1 and β' is a positive constant that depends only on α, β, n and d. The entries of $A_0(\boldsymbol{z},x)$ are therefore uniformly bounded by a constant depending on α and β'. As D is convex, Theorem 3.2.1.2 of Grisvard [9] shows that for each $\boldsymbol{z}\in U$, $u_0(\boldsymbol{z})\in H^2(D)$. The proofs of Lemma 3.1.3.2 and of Theorem 3.2.1.2 in [9] show that

$$\|u_0(\boldsymbol{z},\cdot)\|_{H^2(D)}\le c\|f\|_{L^2(D)},$$

where the constant c depends on the $C^1(\bar{D})$ norms of A_0, and the $L^\infty(D)$ norms of the entries of the matrix $A_0^{-1/2}(\boldsymbol{z},x)$ which can be bounded by α and β'. Therefore $\|u_0(\boldsymbol{z},\cdot)\|_{H^2(D)}$ is uniformly bounded for all $\boldsymbol{z}\in U$. As $\|w_{il}(\boldsymbol{z},\boldsymbol{y}_i)\|_{\boldsymbol{C}^1_{i-1}(H^2(Y_i)/\mathbb{R})}$ is uniformly bounded for $\boldsymbol{z}\in U$, we get from (4.9) that $u_i\in\mathcal{H}_i$ and $\|u_i(\boldsymbol{z})\|_{\mathcal{H}_i}$ is uniformly bounded for all $\boldsymbol{z}\in U$. Hence $\|\boldsymbol{u}(\boldsymbol{z})\|_{\boldsymbol{\mathcal{H}}}$ is uniformly bounded for all $\boldsymbol{z}\in U$. □

To establish the measurability of $\boldsymbol{u}$, we make the following assumptions.

Assumption 4.3 *We assume that the matrices Ψ_k in* (1.7) *are in $(\boldsymbol{C}^1_n)^{d\times d}$ such that for all $i,j=1,\ldots,d$*

$$\sum_{k=1}^\infty\|\Psi_k\|_{(\boldsymbol{C}^1_n)^{d\times d}}<\infty.$$

Remark 4.4 *When $\mathbf{Cov}[A]_{iji'j'}\in H^{t+1}(D)\otimes H^{t+1}_{\#}(Y_1)\otimes\ldots\otimes H^{t+1}(Y_n)$ for a sufficiently large constant t, for any vectors $(\gamma_0,\gamma_1,\ldots,\gamma_n)\in\{0,1\}^{n+1}$ and any $(j_0,j_1,\ldots,j_n)\in\{1,\ldots,d\}^{n+1}$,*

$$\frac{\partial^{\gamma_0+\ldots+\gamma_n}\mathbf{Cov}[A]_{iji'j'}(x,\boldsymbol{y},x',\boldsymbol{y}')}{\partial x_{j_0}^{\gamma_0}\partial y_{1j_1}^{\gamma_1}\ldots\partial y_{nj_n}^{\gamma_n}}\in H^t(D)\otimes H^t_{\#}(Y_1)\otimes\ldots\otimes H^t_{\#}(Y_n).$$

We then deduce that

$$\|\Psi_k\|_{(\boldsymbol{C}^1_n)^{d\times d}}\le c(\varepsilon)k^{(-t/d+\varepsilon)(1-2t^*/t)}.\tag{4.14}$$

Assumption 4.3 holds when t is sufficiently large.

Proposition 4.5 *With Assumption 4.3, the function $\boldsymbol{u}$ as a map from U to $\boldsymbol{\mathcal{H}}$ is measurable.*

Proof We first prove that there exists a constant c such that for all $\boldsymbol{z},\boldsymbol{z}'\in U$,

$$\|\boldsymbol{u}(\boldsymbol{z})-\boldsymbol{u}(\boldsymbol{z}')\|_{\boldsymbol{\mathcal{H}}}\le c\|A(\boldsymbol{z};\cdot,\cdot)-A(\boldsymbol{z}',\cdot,\cdot)\|_{(\boldsymbol{C}^1_n)^{d\times d}}.\tag{4.15}$$

From (4.3), we have for every fixed $\boldsymbol{z}\in U$, $x\in D$ and $\boldsymbol{y}_{n-1}\in\mathbf{Y}_{n-1}$

$$\begin{aligned}&\int_{Y_n}A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{y_n}(w_{nl}(\boldsymbol{z};x,\boldsymbol{y})-w_{nl}(\boldsymbol{z}';x,\boldsymbol{y}))\cdot\nabla_{y_n}\phi_n dy_n\\&=\int_{Y_n}\Big[\nabla_{y_n}\cdot((A(\boldsymbol{z};x,\boldsymbol{y})-A(\boldsymbol{z}';x,\boldsymbol{y}))e_l)\\&\qquad-\nabla_{y_n}((A(\boldsymbol{z}';x,\boldsymbol{y})-A(\boldsymbol{z};x,\boldsymbol{y}))\nabla_{y_n}w_{nl}(\boldsymbol{z}';x,\boldsymbol{y}))\Big]\phi_n dy_n,\quad\forall\phi_n\in H^1_{\#}(Y_n).\end{aligned}$$

As $\|w_{nl}(\boldsymbol{z};x,\boldsymbol{y}_{n-1},\cdot)\|_{H^2(Y_n)/\mathbb{R}}$ is uniformly bounded for all $(\boldsymbol{z};x,\boldsymbol{y}_{n-1})\in U\times D\times\mathbf{Y}_{n-1}$, we obtain

$$\|(w_{nl}(\boldsymbol{z};x,\boldsymbol{y}_{n-1},\cdot)-w_{nl}(\boldsymbol{z}';x,\boldsymbol{y}_{n-1},\cdot)\|_{H^2(Y_n)/\mathbb{R}}\le c\|A(\boldsymbol{z};\cdot,\cdot)-A(\boldsymbol{z}';\cdot,\cdot)\|_{(\boldsymbol{C}^1_n)^{d\times d}}.\tag{4.16}$$

Similarly, from (4.12)

$$\begin{aligned}&\int_{Y_n}A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{y_n}(\chi(\boldsymbol{z};\cdot,\cdot)-\chi(\boldsymbol{z}';\cdot,\cdot))\cdot\nabla_{y_n}\phi_n dy_n\\&=\int_{Y_n}\nabla_{y_n}\cdot\Big(\frac{\partial A(\boldsymbol{z};x,\boldsymbol{y})-\partial A(\boldsymbol{z}';x,\boldsymbol{y})}{\partial y_{kj}}e_l\Big)\phi_n dy_n\\&+\int_{Y_n}\nabla_{y_n}\cdot\Big(\frac{\partial A(\boldsymbol{z};x,\boldsymbol{y})-\partial A(\boldsymbol{z}';x,\boldsymbol{y})}{\partial y_{kj}}\nabla_{y_n}w_{nl}(\boldsymbol{z};x,\boldsymbol{y})\Big)\phi_n dy_n\\&+\int_{Y_n}\nabla_{y_n}\cdot\Big((A(\boldsymbol{z};x,\boldsymbol{y})-A(\boldsymbol{z}';x,\boldsymbol{y}))\nabla_{y_n}\chi(\boldsymbol{z}';x,\boldsymbol{y})\Big)\phi_n dy_n\,.\end{aligned}$$

Therefore there exists a constant $c > 0$ such that for all $\boldsymbol{z}, \boldsymbol{z}' \in U$

$$\|\chi(\boldsymbol{z}; x, \boldsymbol{y}_{n-1}, \cdot) - \chi(\boldsymbol{z}; x, \boldsymbol{y}_{n-1}, \cdot)\|_{H^2(Y_n)/\mathbb{R}} \le C\|A(\boldsymbol{z}; \cdot, \cdot) - A(\boldsymbol{z}'; \cdot, \cdot)\|_{(\boldsymbol{C}^1_n)^{d\times d}}.$$

Performing a similar procedure for the derivatives of χ, we deduce that

$$\|w_{nl}(\boldsymbol{z}; x, \boldsymbol{y}) - w_{nl}(\boldsymbol{z}'; x, \boldsymbol{y})\|_{C^1_{n-1}(H^2(Y_n)/\mathbb{R})} \le C\|A(\boldsymbol{z}; x, \boldsymbol{y}) - A(\boldsymbol{z}'; x, \boldsymbol{y})\|_{(\boldsymbol{C}^1_n)^{d\times d}}.$$

From this and (4.5),

$$\|A_{n-1}(\boldsymbol{z}; x, \boldsymbol{y}_{n-1}) - A_{n-1}(\boldsymbol{z}'; x, \boldsymbol{y}_{n-1})\|_{(\boldsymbol{C}^1_{n-1})^{d\times d}} \le C\|A(\boldsymbol{z}; x, \boldsymbol{y}) - A(\boldsymbol{z}'; x, \boldsymbol{y})\|_{(\boldsymbol{C}^1_n)^{d\times d}}.$$

Inductively, we then show that for all $i = 1, \dots, n$ and all $l = 1, \dots, n$,

$$\|w_{il}(\boldsymbol{z}; x, \boldsymbol{y}_{i-1}, \cdot) - w_{il}(\boldsymbol{z}'; x, \boldsymbol{y}_{i-1}, \cdot)\|_{\boldsymbol{C}^1_{i-1}(H^2(Y_i)/\mathbb{R})} \le C\|A(\boldsymbol{z}; x, \boldsymbol{y}) - A(\boldsymbol{z}'; x, \boldsymbol{y})\|_{(\boldsymbol{C}^1_n)^{d\times d}}.$$

Therefore for the homogenized coefficient $A_0(\boldsymbol{z}; x)$ holds

$$\|A_0(\boldsymbol{z}; x) - A_0(\boldsymbol{z}'; x)\|_{C^1(\bar{D})^{d\times d}} \le C\|A(\boldsymbol{z}; x, \boldsymbol{y}) - A(\boldsymbol{z}'; x, \boldsymbol{y})\|_{(\boldsymbol{C}^1_n)^{d\times d}}.$$

From (4.7), we obtain for all $\boldsymbol{z}, \boldsymbol{z}' \in U$ and every $\phi \in H^1_0(D)$ that

$$\int_D A_0(\boldsymbol{z}; x)\nabla(u_0(\boldsymbol{z}; x) - u_0(\boldsymbol{z}'; x)) \cdot \nabla\phi(x)dx = \int_D (A_0(\boldsymbol{z}'; x) - A_0(\boldsymbol{z}; x))\nabla u_0(\boldsymbol{z}'; x) \cdot \nabla\phi(x)dx.$$

From this identity and from the assumed $H^2(D)$ regularity for the Dirichlet problem in D we conclude that there exists a constant $C > 0$ such that for all $\boldsymbol{z}, \boldsymbol{z}' \in U$ it holds

$$\begin{aligned}
\|u_0(\boldsymbol{z}; \cdot) - u_0(\boldsymbol{z}'; \cdot)\|_{H^2(D)} &\le C\|\nabla((A_0(\boldsymbol{z}'; x) - A_0(\boldsymbol{z}; x))\nabla u_0(\boldsymbol{z}'; x))\|_{L^2(D)} \\
&\le C \sup_{x\in D}(|A_0(\boldsymbol{z}; x) - A_0(\boldsymbol{z}'; x)| + |\nabla A_0(\boldsymbol{z}; x) - \nabla A_0(\boldsymbol{z}'; x)|) \\
&\le C\|A(\boldsymbol{z}; \cdot, \cdot) - A(\boldsymbol{z}'; \cdot, \cdot)\|_{(\boldsymbol{C}^1_n)^{d\times d}}. \qquad (4.17)
\end{aligned}$$

From (4.9) and the uniform boundedness of w_i in $\boldsymbol{C}^1_{i-1}(H^2(Y_i))$, we get (4.15). A similar argument as in the proof of Proposition 1.8 shows that $\boldsymbol{u}$ as a map from U to $\boldsymbol{\mathcal{H}}$ is measurable. □

From Proposition 4.5, we deduce that $\boldsymbol{u} \in L^2(U, \rho; \boldsymbol{\mathcal{H}})$, so the coefficients $\boldsymbol{u}_\nu$ in the expansion (2.2) are all in $\boldsymbol{\mathcal{H}}$.

4.2 Regularity of the complex parametric, deterministic problems (3.4)

We show that the solution $\boldsymbol{u}(\boldsymbol{\zeta}, \cdot, \cdot)$ of the problem (3.4) belongs to $\boldsymbol{\mathcal{H}}$ when the complex parameter $\boldsymbol{\zeta}$ is in a subset $\bar{\mathcal{U}}$ of the domain $\mathcal{U}$ defined in (3.2). We choose a constant $\bar{K} < 1$ that satisfies

$$\bar{K}\sum_{j=1}^{\infty}(\beta_j + \|\Psi_j\|_{(\boldsymbol{C}^1_n)^{d\times d}}) < \frac{\alpha}{2M}. \qquad (4.18)$$

We then choose a constant $\bar{J}_0$ so that

$$\sum_{j>\bar{J}_0}(\beta_j + \|\Psi_j\|_{(\boldsymbol{C}^1_n)^{d\times d}}) < \frac{\alpha\bar{K}}{6M(1+\bar{K})}. \qquad (4.19)$$

We then denote $\bar{E} = \{1, 2, \dots, \bar{J}_0\}$, $\bar{F} = \mathbb{N} \setminus \bar{E}$ and set

$$|\nu_{\bar{F}}| = \sum_{j>\bar{J}_0} |\nu_j| \ .$$

For each index $\nu \in \mathcal{F}$, we define

$$\bar{r}_m = \bar{K} \text{ when } m \le \bar{J}_0, \text{ and } \bar{r}_m = \max\{1, \frac{\alpha\nu_m}{M|\nu_{\bar{F}}|(\beta_m + \|\Psi_m\|_{(\boldsymbol{C}^1_n)^{d\times d}})}\} \text{ when } m > \bar{J}_0, \qquad (4.20)$$

where we again adopted the convention that $|\nu_m|/|\nu_{\bar{F}}| = 0$ if $|\nu_{\bar{F}}| = 0$. For $m \geq 1$, we define the set $\bar{\mathcal{U}}_m \subset \mathbb{C}$ as

$$[-1,1] \subset \bar{\mathcal{U}}_m := \{\boldsymbol{\zeta}_m \in \mathbb{C} : \ \mathrm{dist}(\zeta_m, [-1,1]) \leq \bar{r}_m\} \subset \mathbb{C} \,. \tag{4.21}$$

We then consider the complex parametric domain $\bar{\mathcal{U}} \subset \mathcal{U}$ defined as

$$\bar{\mathcal{U}} = \bigotimes_{m=1}^{\infty} \bar{\mathcal{U}}_m \subset \mathbb{C}^{\mathbb{N}}.$$

We consider the problem (3.4) for complex valued parameter vectors $\boldsymbol{\zeta} \in \bar{\mathcal{U}}$. For $\boldsymbol{\zeta} \in \bar{\mathcal{U}}$, we have

$$\begin{aligned}
\|A(\boldsymbol{\zeta}; x, \boldsymbol{y})\|_{(\boldsymbol{C}_n^1)^{d\times d}} &\leq \|\bar{A}(x,\boldsymbol{y})\|_{(\boldsymbol{C}_n^1)^{d\times d}} + \sum_{m=1}^{\infty} \|\Psi_m(x,\boldsymbol{y})\|_{(\boldsymbol{C}_n^1)^{d\times d}}(1+\bar{r}_m) \\
&\leq \|\bar{A}(x,\boldsymbol{y})\|_{(\boldsymbol{C}_n^1)^{d\times d}} + \sum_{m=1}^{\bar{J}_0} \|\Psi_m\|_{(\boldsymbol{C}_n^1)^{d\times d}}(1+\bar{K}) \\
&\quad + \sum_{m>\bar{J}_0} \Big(2 + \frac{\alpha\nu_m}{M|\nu_{\bar{F}}|(\beta_m + \|\Psi_m\|_{(\boldsymbol{C}_n^1)^{d\times d}})}\Big)\|\Psi_m\|_{(\boldsymbol{C}_n^1)^{d\times d}} \\
&\leq \|\bar{A}(x,\boldsymbol{y})\|_{(\boldsymbol{C}_n^1)^{d\times d}} + 2\sum_{m=1}^{\infty} \|\Psi_m\|_{(\boldsymbol{C}_n^1)^{d\times d}} + \frac{\alpha}{M}.
\end{aligned} \tag{4.22}$$

As in (3.3), we have

$$|A_{ij}(\zeta; x, y)| \leq \|\bar{A}_{ij}\|_{L^\infty(D\times Y)} + 4\frac{\kappa}{1+\kappa}\alpha_0 + \frac{2\alpha}{M}. \tag{4.23}$$

Therefore $A(\boldsymbol{\zeta}; x, \boldsymbol{y})$ is uniformly bounded in $(\boldsymbol{C}_n^1)^{d\times d}$ for all $\boldsymbol{\zeta} \in \bar{\mathcal{U}}$. We show next that the solution of the parametric problem is jointly holomorphic with respect to any finite set of parameters. For each index $\nu \in \mathcal{F}$, we define the (finite dimensional) domain

$$\bar{\mathcal{U}}_\nu = \bigotimes_{j\in \mathrm{supp}(\nu)} \bar{\mathcal{U}}_j \,.$$

We have the following analyticity result.

Proposition 4.6 *For $\nu \in \mathcal{F}$ and $\zeta \in \bar{\mathcal{U}}$, fixing ζ_k for $k \notin \mathrm{supp}(\nu)$, under Assumption 4.3, if the domain D is convex then $\boldsymbol{u}$ is analytic as a map from $\bar{\mathcal{U}} \to \boldsymbol{\mathcal{H}}$ when the constant M in (4.18) and (4.19) is sufficiently large.*

Proof Let $w_{nl}(\boldsymbol{\zeta})$ be the solution of problem (4.3) for the complex valued coefficient $A(\boldsymbol{\zeta}; x, \boldsymbol{y})$. We show that $w_{nl}(\boldsymbol{\zeta})$ is holomorphic as a mapping from $\bar{\mathcal{U}}_\nu$ to $\boldsymbol{C}_{n-1}^1(H^2(Y_n)/\mathbb{R})$. To this end, we establish complex differentiability by showing that certain difference quotient have limits.

For any m, we fix all coordinates ζ_k for $k \neq m$, and partition $\boldsymbol{\zeta} \in \mathbb{C}^{\mathbb{N}}$ as $\boldsymbol{\zeta} = (\zeta_m^*, \zeta_m)$. Let further $\delta \in \mathbb{C}$ denote the step size of the difference quotients

$$\eta_{mnl}^\delta(\boldsymbol{\zeta}; \cdot, \cdot) := \delta^{-1}\left(w_{nl}(\zeta_m^*, \zeta_m + \delta; \cdot, \cdot) - w_{nl}(\boldsymbol{\zeta}; \cdot, \cdot)\right) \,.$$

The function η_{mnl}^δ satisfies

$$\begin{aligned}
\int_{Y_n} A(\boldsymbol{\zeta}; x, \boldsymbol{y})\nabla_{y_n}\eta_{mnl}^\delta(\boldsymbol{\zeta}; x, y) \cdot \overline{\nabla_{y_n}\phi_n(y_n)}dy_n &= -\int_{Y_n} \Psi_m e_l \cdot \overline{\nabla_{y_n}\phi_n(y_n)}dy_n \\
&\quad - \int_{Y_n} \Psi_m \nabla_{y_n} w_{nl}(\zeta_m^*, \zeta_m + \delta; x, \boldsymbol{y}) \cdot \overline{\nabla_{y_n}\phi_n(y_n)}dy_n, \quad \forall \phi_n \in H_\#^1(Y_n).
\end{aligned}$$

Let $\eta_{mnl}(\boldsymbol{\zeta}; x, \boldsymbol{y})$ denote the solution of the problem

$$\begin{aligned}
\int_{Y_n} A(\boldsymbol{\zeta}; x, \boldsymbol{y})\nabla_{y_n}\eta_{mnl}(\boldsymbol{\zeta}; x, \boldsymbol{y}) \cdot \overline{\nabla_{y_n}\phi_n(y_n)}dy_n &= -\int_{Y_n} \Psi_m e_l \cdot \overline{\nabla_{y_n}\phi_n(y_n)}dy_n \\
&\quad - \int_{Y_n} \Psi_m \nabla_{y_n} w_{nl}(\boldsymbol{\zeta}; x, \boldsymbol{y}) \cdot \overline{\nabla_{y_n}\phi_n(y_n)}dy_n, \quad \forall \phi_n \in H_\#^1(Y_n).
\end{aligned}$$

We then have, for every $\boldsymbol{\zeta} \in U$,

$$\int_{Y_n} A(\boldsymbol{\zeta}; x, \boldsymbol{y}) \nabla_{y_n}(\eta^\delta_{mnl} - \eta_{mnl}) \cdot \overline{\nabla_{y_n}\phi_n} dy_n = -\int_{Y_n} \Psi_n \nabla_{y_n}(w_{nl}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}) - w_{nl}(\boldsymbol{\zeta}; x, \boldsymbol{y})) \cdot \overline{\nabla_{y_n}\phi_n} dy_n \ .$$

Proceeding in the same fashion as in the proof of Proposition 4.2 we obtain

$$\|\eta^\delta_{mnl} - \eta_{mnl}\|_{\boldsymbol{C}^1_{n-1}(H^2(Y_n)/\mathbb{R})} \le c \|\nabla_{y_n} \cdot \Big(\Psi_n \nabla_{y_n}(w_{nl}(\zeta^*_m, \zeta_m + \delta; \cdot, \cdot) - w_{nl}(\boldsymbol{\zeta}; \cdot, \cdot))\Big)\|_{\boldsymbol{C}^1_{n-1}(L^2(Y_n))}$$

which converges to 0 when $\delta \to 0$ as w_{nl} is continuous as a map from $\bar{\mathcal{U}}_m$ to $\boldsymbol{C}^1_{n-1}(H^2(Y_n)/\mathbb{R})$ (which can be shown as in the proof of Proposition 4.5). Therefore w_{nl} is complex differentiable with respect to ζ_m and therefore an analytic function of ζ_m taking values in $\boldsymbol{C}^1_{n-1}(H^2(Y_n)/\mathbb{R})$. From Hartogs' theorem, we conclude that w_{nl} is analytic as a function from $\bar{\mathcal{U}}_\nu$ to $\boldsymbol{C}^1_{n-1}(H^2(Y_n)/\mathbb{R})$. By (4.5), $A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1})$ is an analytic, $\boldsymbol{C}^1_{n-1}$-valued function of $\boldsymbol{\zeta}$ in $\bar{\mathcal{U}}_\nu$.

Next we consider $w_{(n-1)l}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1})$. Again, we verify analyticity by showing complex differentiability via the difference quotients

$$\eta^\delta_{m(n-1)l} = \frac{w_{(n-1)l}(\zeta^*_m, \zeta_m + \delta; \cdot, \cdot) - w_{(n-1)l}(\boldsymbol{\zeta}; \cdot, \cdot)}{\delta}.$$

For these difference quotients, we have for parameter vectors $\boldsymbol{\zeta}$ as above the equation

$$\begin{aligned}
&\int_{\mathbf{Y}_{n-1}} A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}) \nabla_{y_{n-1}} \eta^\delta_{m(n-1)l}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}) \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \\
&= -\int_{\mathbf{Y}_{n-1}} \frac{A_{n-1}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}_{n-1}) - A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1})}{\delta} e_l \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} d\boldsymbol{y}_{n-1} \\
&\quad - \int_{\mathbf{Y}_{n-1}} \frac{A_{n-1}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}_{n-1}) - A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1})}{\delta} \nabla_{y_{n-1}} w_{(n-1)l}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}_{n-1}) \\
&\qquad \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1},
\end{aligned}$$

forall $\phi_{n-1} \in H^1_\#(Y_{n-1})$. We next let $\eta_{m(n-1)l}(\boldsymbol{z}; x, \boldsymbol{y}_{n-2}, y_{n-1}) \in H^1_\#(Y_{n-1})/\mathbb{R}$ satisfy

$$\begin{aligned}
&\int_{Y_{n-1}} A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}) \nabla_{y_{n-1}} \eta_{m(n-1)l}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}) \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \\
&= -\int_{Y_{n-1}} \frac{\partial A_{n-1}}{\partial \zeta_m} e_l \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \\
&\quad - \int_{Y_{n-1}} \frac{\partial A_{n-1}}{\partial \zeta_m} \nabla_{y_{n-1}} w_{(n-1)l}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}) \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \ .
\end{aligned}$$

We deduce

$$\begin{aligned}
&\int_{Y_{n-1}} A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}) \nabla_{y_{n-1}}(\eta^\delta_{m(n-1)l}(\boldsymbol{\zeta}; x, \boldsymbol{y}) - \eta_{m(n-1)l}(\boldsymbol{\zeta}; x, \boldsymbol{y})) \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \\
&= -\int_{Y_{n-1}} \Big(\frac{A_{n-1}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}_{n-1}) - A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1})}{\delta} - \frac{\partial A_{n-1}}{\partial \zeta_m}\Big) e_l \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \\
&\quad - \int_{Y_{n-1}} \Big(\frac{A_{n-1}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}_{n-1}) - A_{n-1}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1})}{\delta} - \\
&\qquad \frac{\partial A_{n-1}}{\partial \zeta_m}\Big) \nabla_{y_{n-1}} w_{(n-1)l}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}) \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \\
&\quad - \int_{Y_{n-1}} \frac{\partial A_{n-1}}{\partial \zeta_m}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}) \nabla_{y_{n-1}} \Big(w_{(n-1)l}(\zeta^*_m, \zeta_m + \delta; x, \boldsymbol{y}_{n-1}) - (w_{(n-1)l}(\boldsymbol{\zeta}; x, \boldsymbol{y}_{n-1}))\Big) \\
&\qquad \cdot \overline{\nabla_{y_{n-1}}\phi_{n-1}(y_{n-1})} dy_{n-1} \ .
\end{aligned}$$

Therefore

$$\|\eta^\delta_{m(n-1)l}(\boldsymbol{\zeta};x,\boldsymbol{y})-\eta_{m(n-1)l}(\boldsymbol{\zeta};x,\boldsymbol{y})\|_{\boldsymbol{C}^1_{n-2}(H^2(Y_{n-1})/\mathbb{R})}\le$$

$$\Bigg\|\nabla_{y_{n-1}}\cdot\Big[\Big(\frac{A_{n-1}(\zeta^*_m,\zeta_m+\delta;x,\boldsymbol{y}_{n-1})-A_{n-1}(\boldsymbol{\zeta};x,\boldsymbol{y}_{n-1})}{\delta}-\frac{\partial A_{n-1}}{\partial\zeta_m}(\boldsymbol{\zeta};x,\boldsymbol{y}_{n-1})\Big)e_l+$$

$$\Big(\frac{A_{n-1}(\zeta^*_m,\zeta_m+\delta;x,\boldsymbol{y}_{n-1})-A_{n-1}(\boldsymbol{\zeta};x,\boldsymbol{y}_{n-1})}{\delta}-\frac{\partial A_{n-1}}{\partial\zeta_m}\Big)\nabla_{y_{n-1}}w_{(n-1)l}(\zeta^*_m,\zeta_m+\delta;x,\boldsymbol{y}_{n-1})$$

$$-\frac{\partial A_{n-1}}{\partial\zeta_m}(\boldsymbol{\zeta};x,\boldsymbol{y}_{n-1})\nabla_{y_{n-1}}\Big(w_{(n-1)l}(\zeta^*_m,\zeta_m+\delta;x,\boldsymbol{y}_{n-1})-(w_{(n-1)l}(\boldsymbol{\zeta};x,\boldsymbol{y}_{n-1}))\Big)\Big]\Bigg\|_{\boldsymbol{C}^1_{n-2}(L^2(Y_{n-1}))},$$

which converges to 0 as A_{n-1} is holomorphic as a mapping from $\bar{\mathcal{U}}_m$ to $\boldsymbol{C}^1_{n-1}$ and $w_{(n-1)l}$ is continuous as a mapping from $\bar{\mathcal{U}}_m$ to $\boldsymbol{C}^1_{n-1}(H^2(Y_{n-1})/\mathbb{R})$.

Similarly, w_{il} is analytic as a map from $\bar{\mathcal{U}}_\nu$ to $\boldsymbol{C}^2_{i-1}(H^2(Y_i))$ for other values of i. To show that u_i is analytic from $\bar{\mathcal{U}}_\nu$ to $\mathcal{H}_i$ and $\boldsymbol{u}$ is analytic from $\bar{\mathcal{U}}_\nu$ to $\boldsymbol{\mathcal{H}}$, it remains to establish the analyticity of u_0 as a map from $\bar{\mathcal{U}}$ to $H^2(D)$ where the domain D is convex. We note that Theorem 3.2.1.2 of Grisvard [9] is not readily applicable to elliptic equations with complex coefficients in a convex domain.

As w_{il} are holomorphic as a map from $\bar{\mathcal{U}}_\nu$ to $\boldsymbol{C}^2_{i-1}(H^2(Y_i))$, the coefficient $A_0(\zeta;x)$ of the complex parametric homogenized equation is analytic. As $\bar{\mathcal{U}}\subset\mathcal{U}$, from (3.5), $\Re(\xi^HA(\zeta;x,\boldsymbol{y})\xi)\ge\alpha|\xi|^2/2$ for all $\zeta\in\bar{\mathcal{U}}$, $x\in D$ and $\boldsymbol{y}\in\mathbf{Y}$. Following the proof of Proposition 4.1, we deduce that

$$\forall\xi\in\mathbb{C}^d,\zeta\in\bar{U},x\in D:\quad\Re(\xi^HA_0(\zeta;x)\xi)\ge\frac{\alpha}{2}|\xi|^2,$$

which implies that for all $\xi\in\mathbb{R}^d$,

$$\xi^\top\Re A_0(\zeta;x)\xi\ge\frac{\alpha}{2}|\xi|^2\,.$$

Further, there is a positive constant β'' that depends only on α, $\sup_{i,j}\|A_{ij}(\zeta;x)\|_{L^\infty(D)}$, d and n such that

$$\xi^\top\Re A_0(\zeta;x)\xi\le\beta''|\xi|^2.$$

From (3.3). β'' can be chosen independently of M (i.e. independently of the complex parametric domain $\bar{\mathcal{U}}$ when M is sufficiently large; here we choose $M\ge4$). Let D_m be a sequence of convex subdomains of D with smooth boundary such that $\mathrm{dist}(\partial D_m,\partial D)\to0$ as in the proof of Theorem 3.2.1.2 in [9]. Consider the Dirichet problems

$$-\nabla\cdot(A_0(\zeta;x)\nabla\phi_m(\zeta;x))=f(x),\quad\phi_m\in H^1_0(D_m),$$

i.e.

$$-\nabla\cdot((\Re A_0(\zeta;x))\nabla\phi_m)=f(x)+i\nabla\cdot((\Im A_0(\zeta;x))\nabla\phi_m)\,.$$

As the boundary of D_m is smooth, $\phi_m\in H^2(D_m)$ ([19], Section 20). Applying the proof of Lemma 3.1.3.2 in [9] for $\Re\phi_m$ and $\Im\phi_m$ respectively, we find that there is a constant c_1 which depends on $\|\Re A_0(z;x)\|_{(C^1(\bar{D}))^{d\times d}}$, the diameter of D, α and β'' such that

$$\|\phi_m\|_{H^2(D_m)}\le c_1\Big(\Big\|\nabla\cdot(A_0(\zeta;x)\nabla\phi_m(\zeta;x))\Big\|_{L^2(D_m)}+\Big\|\nabla\cdot(\Im A_0(\zeta;x)\nabla\phi_m)\Big\|_{L^2(D_m)}\Big)\,.\tag{4.24}$$

We note that

$$\|\nabla\phi_m\|_{L^2(D_m)}\le\frac{2}{\alpha}\|\nabla\cdot(A_0(\zeta;x)\nabla\phi_m(\zeta;x))\|_{L^2(D_m)}\,.$$

Further, from (4.6), $\|A_0(z;x)\|_{(C^1(\bar{D}))^{d\times d}}$ has an upper bound depending on an upper bound of $\|A(\boldsymbol{\zeta};x,\boldsymbol{y})\|_{(\boldsymbol{C}^1_n)^{d\times d}}$, which can be chosen independently of M (from (4.22)). Therefore

$$\|\phi_m\|_{H^2(D_m)}\le c_1\Big(c_2\Big\|\nabla\cdot(A_0(\zeta;x)\nabla\phi_m(\zeta;x))\Big\|_{L^2(D_m)}+c_3\sup_{i,j}\|\Im A_{0ij}(\zeta;x)\|_{L^\infty(D_m)}\|\nabla\nabla\phi_m\|_{L^2(D_m)^{d\times d}}\Big),$$

where the constants c_1 and c_2 are independent of M, and c_3 only depends on the dimension d. Assume that $\sup_{i,j}\|\Im A_{0ij}(\zeta;x)\|_{L^\infty(D)}$ is sufficiently small so that

$$c_3\sup_{i,j}\|\Im A_{0ij}(\zeta;x)\|_{L^\infty(D)}<1/(2c_1),\tag{4.25}$$

we then have

$$\|\phi_m\|_{H^2(D_m)} \le 2c_1c_2 \Big\| \nabla\cdot(A_0(\zeta;x)\nabla\phi_m(\zeta;x))\Big\|_{L^2(D_m)}.$$

Therefore, ϕ_m is uniformly bounded in $H^1_0(D_m)\cap H^2(D_m)$. Arguing as in the proof of Theorem 3.2.1.2 in Grisvard [9], ϕ_m (extended to 0 outside D_m) converges weakly to u_0 in $H^1_0(D)$; and the weak limit in $L^2(D)$ of the second derivatives of ϕ_m (again extended to 0 outside D_m) must be the second derivative of u_0; thus $u_0\in H^1_0(D)\cap H^2(D)$, and

$$\|u_0\|_{H^2(D)} \le c\Big\| \nabla\cdot(A_0(\zeta;x)\nabla u_0(\zeta;x))\Big\|_{L^2(D)}.$$

It remains to show that we can find the a constant M in (4.18) and (4.19) so that (4.25) holds. We note that

$$\begin{aligned}|\Im A_{ij}| &\le \sum_{m=1}^{\infty}\bar r_m\|\Psi_{mij}\|_{L^\infty(D)} \le \sum_{m=1}^{J_0}\bar K 2\beta_m + \sum_{m>J_0}\Big(1+\frac{\alpha\nu_m}{M|\nu_{\bar F}|(\beta_m+\|\psi_m\|_{(\mathbf{C}^1_n)^{d\times d}})}\Big)2\beta_m\\ &\le \frac{\alpha}{M}+\frac{\alpha}{3M}+\frac{2\alpha}{M}\end{aligned}$$

which is small when M is large.

Using the cell problem (4.10) for the complex parametric problem, with $\phi_n=\Im w_{nl}$, taking the imaginary part of both sides, we have

$$\begin{aligned}\int_{Y_n}\Re A(\zeta;x,y)\nabla_{y_n}\Im w_{nl}(\zeta;x,y)\cdot\nabla_{y_n}\Im w_{nl}(\zeta;x,y) = &-\int_{Y_n}\Im A(\zeta;x,y)e_l\cdot\nabla\Im w_{nl}dy_n\\ &-\int_{Y_n}\Im A(\zeta;x,y)\nabla\Re w_{nl}\cdot\nabla\Im w_{nl}dy_n.\end{aligned}$$

Therefore,

$$\|\nabla\Im w_{nl}(\zeta;x,\cdot)\|_{L^\infty(D\times\mathbf{Y}_{n-1},L^2(Y_n))^d} \le c(\alpha,d)\sup_{i,j}\|\Im A_{ij}\|_{L^\infty(D\times\mathbf{Y})}(1+\|\nabla\Re w_{nl}\|_{L^\infty(D\times\mathbf{Y}_{n-1},L^2(Y_n))^d}).$$

We note from (4.10) that

$$\|\nabla\Re w_{nl}\|_{L^\infty(D\times\mathbf{Y}_{n-1},L^2(Y_n))^d}\le c(\alpha,d)(1+\sup_{i,j}\|A_{ij}\|_{L^\infty(D\times\mathbf{Y})}),$$

so

$$\|\nabla\Im w_{nl}(\zeta;x,\cdot)\|_{L^\infty(D\times\mathbf{Y}_{n-1},L^2(Y_n))^d}\le c(\alpha,d)\sup_{i,j}\|\Im A_{ij}\|_{L^\infty(D\times\mathbf{Y})}(1+\sup_{i,j}\|A_{ij}\|_{L^\infty(D\times\mathbf{Y})}).$$

From (4.5), we have

$$\begin{aligned}\sup_{i,j}\|\Im A_{(n-1)ij}\|_{L^\infty(D\times\mathbf{Y}_{n-1})} &\le \sup_{i,j}\|\Im A_{ij}\|_{L^\infty(D\times\mathbf{Y})}+c(d)(\sup_{i,j}\|A_{ij}\|_{L^\infty(D\times\mathbf{Y})}\|\Im\nabla w_n\|_{L^\infty(D\times\mathbf{Y}_{n-1},L^2(Y_n))^{d\times d}}\\ &\qquad+\sup_{i,j}\|\Im A_{ij}\|_{L^\infty(D\times\mathbf{Y})}\|\Re\nabla w_n\|_{L^\infty(D\times\mathbf{Y}_{n-1},L^2(Y_n))^{d\times d}})\\ &\le c(\alpha,d)\sup_{i,j}\|\Im A_{ij}\|_{L^\infty(D\times\mathbf{Y})}(1+\sup_{i,j}\|A_{ij}\|_{L^\infty(D\times\mathbf{Y})})^2.\end{aligned}$$

Repeating this argument we have

$$\begin{aligned}\sup_{i,j}\|\Im A_{0ij}\|_{L^\infty(D)} &\le c(\alpha,d)^n\sup_{i,j}\|\Im A_{ij}\|_{L^\infty(D\times\mathbf{Y})}(1+\sup_{i,j}\|A_{ij}\|_{L^\infty(D\times\mathbf{Y})})^2.(1+\sup_{i,j}\|A_{(n-1)ij}\|_{L^\infty(D\times\mathbf{Y}_{n-1})})^2\\ &\qquad\dots(1+\sup_{i,j}\|A_{1ij}\|_{L^\infty(D\times Y)})^2\\ &\le \sup_{i,j}\|\Im A_{ij}\|_{L^\infty(D\times\mathbf{Y})}[c(\alpha,d)(1+\sup_{i,j}\|A_{ij}\|_{L^\infty(D\times\mathbf{Y})})]^{2^{n+1}-2},\end{aligned}$$

where the last estimate is obtained in a similar fashion as for (4.8). Thus, when the constant M in (4.18) and (4.19) is sufficiently large, the condition (4.25) holds. □

4.3 Summability of $\boldsymbol{u}_\nu$

We now study the summability of the $\boldsymbol{\mathcal{H}}$ norms of $\boldsymbol{u}_\nu$. First, we have the following estimate

Proposition 4.7 *The Legendre coefficients $\boldsymbol{u}_\nu$ in (3.7) of the parametric, deterministic solution $(\boldsymbol{z}; x, \boldsymbol{y})$ satisfy the estimate*

$$\forall \nu \in \mathcal{F}: \quad \|\boldsymbol{u}_\nu\|_{\boldsymbol{\mathcal{H}}} \le C\Big(\prod_{m\in \mathrm{supp}(\nu)} \frac{2(1+\bar{K})}{\bar{K}} \bar{\eta}_m^{\nu_m}\Big),$$

where $\bar{\eta}_m := \bar{r}_m + \sqrt{1+\bar{r}_m^2}$ with $\bar{r}_m$ defined in (4.20).

The proof of this Proposition is identical to that for Proposition 3.6.

To study the summability of the sequence $(\|u_\nu\|_{\boldsymbol{\mathcal{H}}})_{\nu\in\mathcal{F}}$, we make the following

Assumption 4.8 *There is a constant $0 < p < 1$ such that*

$$\sum_{k=1}^{\infty} \|\Psi_k\|^p_{(\boldsymbol{C}^1_n)^{d\times d}} < \infty.$$

Remark 4.9 *Assumption 4.8 holds when in estimate (4.14),*

$$p\Big(\frac{t}{d} - \varepsilon\Big)\Big(1 - 2\frac{t^*}{t}\Big) > 1.$$

We note that if β_k is taken as an upper bound for $\|\mathrm{trace}\,\Psi_k\|_{L^\infty(D\times\mathbf{Y})}$, then Assumption 4.8 implies Assumption 3.2.

Proposition 4.10 *Under Assumption 4.8, $(\|\boldsymbol{u}_\nu\|_{\boldsymbol{\mathcal{H}}})_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$.*

The proof of this Proposition is identical to that of Proposition 3.8 except that we use $\bar{K}$, $\bar{E}$, $\bar{F}$, in places of K, E and F.

Remark 4.11 *All of the above results hold if the domain D is not convex but has a smooth boundary.*

5 Correctors

5.1 Correctors for two scale problems

For two scale problems where the coefficient A does not depend on the slow variable x, an estimate of the solution u^ε in terms of the solution u_0 and the corrector u_1 of the homogenized, high dimensional one-scale problem (4.7) has been established under the provision of sufficient regularity. Specifically, assuming that $u_0 \in C^2(\bar{D})$ and $w_{1l} \in W^{1,\infty}(Y)$ (see e.g. Jikov et al. [12] page 28), we will now prove this result, under slightly weaker regularity requirements for u_0 than what was required in [12]. We give its full proof here to verify the regularity requirements and, more importantly, to show that the error estimate for the two scale parametric problem (1.16) holds uniformly for all $\boldsymbol{z} \in U$. As for two length scales there is only one fast variable, we denote in this case $\boldsymbol{y}$ by y and $\mathbf{Y}$ simply by Y. For two scale problems we denote by $w^l(\boldsymbol{z}; x, y)$ the functions $w_{1l}(\boldsymbol{z}; x, y)$.

Proposition 5.1 *For the parametric two scale problem (1.16), assume that $A(\boldsymbol{z}; x, y) \in L^\infty(U; C^1(\bar{D}; C^1_\#(Y)))$, that the function $u_0(\boldsymbol{z}; x) \in L^\infty(U; H^2(D))$, $w^l(\boldsymbol{z}; x, \boldsymbol{y}) \in L^\infty(U; C^1(\bar{D}; H^2_\#(Y))) \cap L^\infty(U \times \bar{D}; C^1_\#(\bar{Y}))$, and that the domain D has a Lipschitz boundary. Then there exists a constant $c > 0$ such that for every $0 < \varepsilon \le 1$,*

$$\sup_{\boldsymbol{z}\in U} \|u^\varepsilon(\boldsymbol{z}; x) - [u_0(\boldsymbol{z}; x) + \varepsilon u_1(\boldsymbol{z}; x, \frac{x}{\varepsilon})]\|_{H^1(D)} \le c\varepsilon^{1/2} \,. \tag{5.1}$$

Proof For $\boldsymbol{z} \in U$, define

$$u_1^\varepsilon(\boldsymbol{z}; x) = u_0(\boldsymbol{z}; x) + \varepsilon w^l(\boldsymbol{z}; x, \frac{x}{\varepsilon})\frac{\partial u_0(\boldsymbol{z}; x)}{\partial x_l}.$$

We first show that

$$\|\mathrm{div} A^\varepsilon \nabla u_1^\varepsilon - \mathrm{div} A_0 \nabla u_0\|_{H^{-1}(D)} \le c\varepsilon,$$

where c is independent of $\boldsymbol{z}$. We adapt the argument of [12] page 28 for the case where $u_0 \in H^2(D)$ (but not in $C^2(\bar{D})$). We note that

$$
\begin{aligned}
& (A^\varepsilon(\boldsymbol{z};x)\nabla u_1^\varepsilon(\boldsymbol{z};x))_i \\
= \;& \Big(A^\varepsilon_{ij}(\boldsymbol{z};x) + A^\varepsilon_{ik}(\boldsymbol{z};x)\frac{\partial w^j}{\partial y_k}(\boldsymbol{z};x,\frac{x}{\varepsilon})\Big)\frac{\partial u_0}{\partial x_j}(\boldsymbol{z};x) + \varepsilon A^\varepsilon_{ij}(\boldsymbol{z};x)w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial^2 u_0}{\partial x_j \partial x_k}(\boldsymbol{z};x) \\
= \;& A_{0ij}(\boldsymbol{z};x)\frac{\partial u_0}{\partial x_j}(\boldsymbol{z};x) + \Big(A^\varepsilon_{ij}(\boldsymbol{z};x) + A^\varepsilon_{ik}(\boldsymbol{z};x)\frac{\partial w^j}{\partial y_k}(\boldsymbol{z};x,\frac{x}{\varepsilon}) - A_{0ij}(\boldsymbol{z};x)\Big)\frac{\partial u_0}{\partial x_j}(\boldsymbol{z};x) \\
+ \;& \varepsilon A^\varepsilon_{ij}(\boldsymbol{z};x)w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial^2 u_0}{\partial x_j \partial x_k}(\boldsymbol{z};x) \\
= \;& A_{0ij}(\boldsymbol{z};x)\frac{\partial u_0}{\partial x_j}(\boldsymbol{z};x) + g_i^j(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}(\boldsymbol{z};x) + \varepsilon A^\varepsilon_{ij}(\boldsymbol{z};x)w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial^2 u_0}{\partial x_j \partial x_k}(\boldsymbol{z};x),
\end{aligned}
$$

where the functions $g_i^j(\boldsymbol{z};x,y)$ (which are Y-periodic with respect to y) are defined as

$$
g_i^j(\boldsymbol{z};x,y) = A_{ij}(\boldsymbol{z};x,y) + A_{ik}(\boldsymbol{z};x,y)\frac{\partial w^j}{\partial y_k}(\boldsymbol{z};x,y) - A_{0ij}(\boldsymbol{z};x)\;.
$$

By definition, for all $\boldsymbol{z} \in U$ and every $x \in D$ holds

$$
\int_Y g_i^j(\boldsymbol{z};x,y)dy = 0 \quad \text{and} \quad \frac{\partial}{\partial y_i} g_i^j(\boldsymbol{z};x,y) = 0\;.
$$

Therefore, there exist functions $\alpha_{ij}^k(\boldsymbol{z};x,y)$ which are Y-periodic in y such that $\alpha_{ij}^k = -\alpha_{ji}^k$ and

$$
g_i^k(\boldsymbol{z};x,y) = \frac{\partial}{\partial y_j}\alpha_{ij}^k(\boldsymbol{z};x,y). \tag{5.2}
$$

As $w^j \in L^\infty(U;C^1(\bar{D};H^2_\#(Y)))$ and $A \in L^\infty(U;C^1(\bar{D};C^1_\#(\bar{Y}))^{d\times d}_{sym})$, $g_i^j(x,y) \in L^\infty(U;C^1(\bar{D};H^1_\#(Y)))$. The functions α_{ij}^k in (5.2) are constructed as follows (see Jikov et al [12] page 7). We write $\boldsymbol{g}^k = (g_i^k)$ as a Fourier series as

$$
\boldsymbol{g}^k(\boldsymbol{z};x,y) = \sum_{l\in\mathbb{Z}^d, l\neq 0} \boldsymbol{g}_l^k(\boldsymbol{z};x)\exp(\sqrt{-1}l\cdot y)\;.
$$

As $\boldsymbol{g}^k(\boldsymbol{z};x,y) \in L^\infty(U;C^1(\bar{D};H^1_\#(Y)))^d$, we have $\boldsymbol{g}_l^k \in L^\infty(U;C^1(\bar{D}))^d$ and, for all $r = 1,\dots,d$, there exists a constant C_r such that

$$
\sup_{\boldsymbol{z}\in U}\sup_{x\in D}\sum_{l\in\mathbb{Z}^d, l\neq 0} |\boldsymbol{g}_l^k(\boldsymbol{z},x)|^2 l_r^2 \le C_r\;. \tag{5.3}
$$

The functions α_{ij}^k are defined as

$$
\alpha_{ij}^k(\boldsymbol{z};x,y) = \sqrt{-1}\sum_{l\in\mathbb{Z}^d, l\neq 0}\frac{(\boldsymbol{g}_l^k(\boldsymbol{z};x))_j l_i - (\boldsymbol{g}_l^k(\boldsymbol{z};x))_i l_j}{|l|^2}\exp(\sqrt{-1}l\cdot\boldsymbol{y}).
$$

From this definition and (5.3), it is then obvious that for $r,s = 1,\dots,d$

$$
\sup_{\boldsymbol{z}\in U}\sup_{x\in D}\sum_{l\in\mathbb{Z}^d, l\neq 0}\frac{|(\boldsymbol{g}_l^k(\boldsymbol{z};x))_j l_i - (\boldsymbol{g}_l^k(\boldsymbol{z};x))_i l_j|^2}{|l|^4} l_r^2 l_s^2 \le C_{rs}\;.
$$

Therefore $\alpha_{ij}^k(\boldsymbol{z};x,y) \in L^\infty(U;C^1(\bar{D};H^2_\#(Y))$ and, by the embedding theorem, for $d \le 3$ holds $\alpha_{ij}^k(\boldsymbol{z};x,y) \in L^\infty(U;C^1(\bar{D};C_\#(\bar{Y})))$. Next, we observe that

$$
(A^\varepsilon\nabla u_1^\varepsilon(\boldsymbol{z};x) - A_0\nabla u_0(\boldsymbol{z};x))_i = \varepsilon\frac{\partial}{\partial x_j}\Big(\alpha_{ij}^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_k}\Big) + (r_\varepsilon)_i(\boldsymbol{z};x),
$$

where

$$(r_\varepsilon)_i(\boldsymbol{z};x) = -\varepsilon\frac{\partial\alpha^k_{ij}(\boldsymbol{z};x,y)}{\partial x_j}\Big|_{y=x/\varepsilon}\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_k} - \varepsilon\alpha^k_{ij}(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial^2 u_0(\boldsymbol{z};x)}{\partial x_k\partial x_j} + \varepsilon w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})A_{ij}(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial^2 u_0(\boldsymbol{z};x)}{\partial x_k\partial x_j}.$$

As $\alpha^k_{ij}\in L^\infty(U;C^1(\bar D;C(\bar Y)))$, $\|(r_\varepsilon)_i(\boldsymbol{z},\cdot)\|_{L^2(D)}\le c\varepsilon$ for all $\boldsymbol{z}\in U$. As $\alpha^k_{ij}=-\alpha^k_{ji}$,

$$\|\mathrm{div}A^\varepsilon\nabla u_1^\varepsilon(\boldsymbol{z};x) - \mathrm{div}A_0\nabla u_0(\boldsymbol{z};x)\|_{H^{-1}(D)} < c\varepsilon,$$

where the constant c does not depend on $\boldsymbol{z}$. As $\mathrm{div}A_0\nabla u_0 = \mathrm{div}A^\varepsilon\nabla u^\varepsilon$, we find that

$$\|\mathrm{div}A^\varepsilon\nabla u_1^\varepsilon(\boldsymbol{z};x) - \mathrm{div}A^\varepsilon\nabla u^\varepsilon(\boldsymbol{z};x)\|_{H^{-1}(D)} \le c\varepsilon,$$

where the constant c does not depend on z. Let $\tau^\varepsilon\in C_0^\infty(D)$ such that $\tau^\varepsilon=1$ outside an ε neighbourbood of ∂D and such that $\varepsilon|\nabla\tau^\varepsilon(x)|\le c$ for all $\varepsilon>0$. We consider the function

$$w_1^\varepsilon(\boldsymbol{z};x) = u_0(\boldsymbol{z};x) + \varepsilon\tau^\varepsilon(x)w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_k}(\boldsymbol{z};x) = u_1^\varepsilon(x) - \varepsilon(1-\tau^\varepsilon(x))w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_k}.$$

We then get

$$\frac{\partial}{\partial x_j}(u_1^\varepsilon - w_1^\varepsilon)(\boldsymbol{z};x) = -\varepsilon\frac{\partial\tau^\varepsilon}{\partial x_j}(x)w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_k} + (1-\tau^\varepsilon(x))\frac{\partial w^k}{\partial y_j}(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_k}(\boldsymbol{z};x) + \varepsilon(1-\tau^\varepsilon(x))w^k(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial^2 u_0(\boldsymbol{z};x)}{\partial x_k\partial x_j}.$$

For $\varepsilon>0$ sufficiently small, let $D^\varepsilon\subset D$ be an ε neighbourhood of ∂D. As ∂D is Lipschitz, for all functions $\phi\in C^\infty(\overline{D})$ it holds $\|\phi\|^2_{L^2(D^\varepsilon)}\le c\varepsilon^2\|\phi\|^2_{H^1(D)} + c\varepsilon\|\phi\|^2_{L^2(\partial D)}$, so for all $\phi\in H^1(D)$ we have $\|\phi\|_{L^2(D^\varepsilon)}\le c\varepsilon^{1/2}\|\phi\|_{H^1(D)}$. Therefore, since $u_0(\boldsymbol{z};x)\in L^\infty(U;H^2(D))$ and $w^l(\boldsymbol{z};x,\boldsymbol{y})\in L^\infty(U\times\bar D;C^1(\bar Y))$

$$\sup_{\boldsymbol{z}\in U}\|u_1^\varepsilon(\boldsymbol{z};\cdot) - w_1^\varepsilon(\boldsymbol{z};\cdot)\|_{H^1(D)}\le c\varepsilon^{1/2},$$

where the constant c does not depend on ε. Thus,

$$\sup_{\boldsymbol{z}\in U}\|\mathrm{div}(A^\varepsilon(\nabla u_1^\varepsilon(\boldsymbol{z};\cdot) - \nabla w_1^\varepsilon(\boldsymbol{z};\cdot)))\|_{H^{-1}(D)}\le c\varepsilon^{1/2},$$

so

$$\sup_{\boldsymbol{z}\in U}\|\mathrm{div}(A^\varepsilon(\nabla u^\varepsilon(\boldsymbol{z};\cdot) - \nabla w_1^\varepsilon(\boldsymbol{z};\cdot)))\|_{H^{-1}(D)}\le c\varepsilon^{1/2}.$$

From Assumption 1.1, we get

$$\sup_{\boldsymbol{z}\in U}\|u^\varepsilon(\boldsymbol{z};\cdot) - w_1^\varepsilon(\boldsymbol{z};\cdot)\|_{H_0^1(D)}\le c\varepsilon^{1/2}.$$

Hence we have proved that there exists $c>0$ such that for every $0<\varepsilon\le 1$ holds

$$\sup_{\boldsymbol{z}\in U}\|u^\varepsilon(\boldsymbol{z};\cdot) - u_1^\varepsilon(\boldsymbol{z};\cdot)\|_{H^1(D)}\le c\varepsilon^{1/2}.$$

□

We then have the following estimate of the homogenization error for the two scale problems (1.5).

Theorem 5.2 *Assume that $A(\boldsymbol{z};x,y)\in L^\infty(U;C^1(\bar D;C^1_\#(Y))^{d\times d}_{sym})$, that $f\in L^2(D)$ and that the domain D is convex. There exists a constant $c>0$ independent of $\boldsymbol{z}\in U$ such that*

$$\Big\|u^\varepsilon(\boldsymbol{z};\cdot) - [\nabla u_0(\boldsymbol{z};\cdot) + \nabla_y u_1(\boldsymbol{z};x,\frac{x}{\varepsilon})]\Big\|_{L^2(U;H^1(D))}\le c\varepsilon^{1/2}\ .$$

Proof Under the assumptions of the theorem, we have shown that $u_0(\boldsymbol{z};\cdot) \in L^\infty(U;H^2(D))$ and $w^l(\boldsymbol{z};x,\cdot) \in L^\infty(U;C^1(\bar{D},H^2_\#(Y)/\mathbb{R})) \subset L^\infty(U;C^1(\bar{D};C_\#(\bar{Y})))$. To apply Proposition 5.1 we show that $w^l(\boldsymbol{z};x,\cdot) \in L^\infty(U\times\bar{D},C^1_\#(\bar{Y})))$. The functions w^l satisfy

$$\int_Y A(\boldsymbol{z};x,y)\nabla_y w^l(\boldsymbol{z};x,y)\cdot\nabla_y\phi(y)dy = \int_Y \nabla\cdot(A(\boldsymbol{z};x,y)e_l)\phi(y)dy, \quad \forall\phi\in H^1_\#(Y). \tag{5.4}$$

As $d\le 3$, we have the continuous embedding

$$w^l \in L^\infty(U;C^1(\bar{D};H^2(Y))) \subset L^\infty(U;C^1(\bar{D};W^{1,5}(Y)))\ .$$

Therefore, there exists a constant $c>0$ independent of $\boldsymbol{z}\in\mathcal{U}$ and of $x\in D$ such that

$$\forall\boldsymbol{z}\in U\ \forall x\in D: \quad \|w^l(\boldsymbol{z};x,\cdot)\|_{W^{1,5}(\mathbf{Y})/\mathbb{R}} \le c.$$

Let D' be a smooth and bounded domain that contains the unit cube Y and let $\tau\in\mathcal{D}(D')$ be such that $\tau(y)=1$ when $y\in Y$. Applying Theorem 6 of [13, pg. 177], we find that there is a constant $c>0$ (which only depends on the Lipschitz constant of A, α, β and τ) such that

$$\|w^l\|_{W^{2,5}(Y)} \le c(\|\mathrm{div}_y(A(\boldsymbol{z},x,\cdot)e_l)\|_{L^5(Y)} + \|w^l\|_{W^{1,5}(Y)})\ .$$

As $d\le 3$ implies the embedding $W^{2,5}(Y)\subset C^1(\bar{Y})$, it holds $w^l(\boldsymbol{z};x,\cdot)\in L^\infty(U;L^\infty(D;C^1_\#(\bar{Y})))$. Therefore Proposition 5.1 holds.

It remains to show that $\nabla_y u_1(\boldsymbol{z};x,x/\varepsilon)$ as a function from U to $L^2(D)$ is measurable. To this end, we note that for every $\boldsymbol{z},\boldsymbol{z}'\in U$ holds

$$\int_Y A(\boldsymbol{z};x,\boldsymbol{y})\nabla_{\boldsymbol{y}}(w^l(\boldsymbol{z};x,\boldsymbol{y}) - w^l(\boldsymbol{z}';x,\boldsymbol{y}))\cdot\nabla_{\boldsymbol{y}}\phi(\boldsymbol{y})d\boldsymbol{y} = \int_Y \nabla\cdot((A(\boldsymbol{z};x,\boldsymbol{y}) - A(\boldsymbol{z}';x,\boldsymbol{y}))e_l)\phi(\boldsymbol{y})d\boldsymbol{y}$$
$$+\int_Y (A(\boldsymbol{z}';x,\boldsymbol{y}) - A(\boldsymbol{z};x,\boldsymbol{y}))\nabla_y w^l(\boldsymbol{z}';x,\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}}\phi(\boldsymbol{y})d\boldsymbol{y}\ .$$

Thus for all $x\in\bar{D}$ and every $\boldsymbol{z},\boldsymbol{z}'\in U$

$$\|w^l(\boldsymbol{z};x,\cdot) - w^l(\boldsymbol{z}';x,\cdot)\|_{W^{2,5}(Y)/\mathbb{R}} \le c\|A(\boldsymbol{z};\cdot,\cdot) - A(\boldsymbol{z}';\cdot,\cdot)\|_{C^1(\bar{D}\times\bar{Y})^{d\times d}}(1+\|w^l(\boldsymbol{z}')\|_{W^{2,5}(Y)/\mathbb{R}})\ .$$

Therefore,

$$\|w^l(\boldsymbol{z};\cdot,\cdot) - w^l(\boldsymbol{z}';\cdot,\cdot)\|_{L^\infty(\bar{D},C^1(\bar{Y}))} \le c\|A(\boldsymbol{z};\cdot,\cdot) - A(\boldsymbol{z}';\cdot,\cdot)\|_{C^1(\bar{D}\times\bar{Y})^{d\times d}}.$$

From (4.17),

$$\begin{aligned}\|\nabla_y u_1(\boldsymbol{z};x,\frac{x}{\varepsilon}) - \nabla_y u_1(\boldsymbol{z}';x,\frac{x}{\varepsilon})\|_{L^2(D)} &= \|\nabla_y w^l(\boldsymbol{z};x,\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x) - \nabla_y w^l(\boldsymbol{z}';x,\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_l}(\boldsymbol{z}';x)\| \\ &\le \|A(\boldsymbol{z};\cdot,\cdot) - A(\boldsymbol{z}';\cdot,\cdot)\|_{C^1(\bar{D}\times\bar{Y})}\ .\end{aligned}$$

As in the proof of Proposition 1.8, this shows that for every $0<\varepsilon\le 1$, the function $\nabla_y u_1(\boldsymbol{z};x,y)|_{y=x/\varepsilon}$ as a map from U to $L^2(D)$ is strongly measurable. This completes the proof. □

Following [6, Def. 2.16], we define a "folding" or averaging operator

Definition 5.3 *For $\Phi\in L^1(D\times Y)$, we define*

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon t, \Big\{\frac{x}{\varepsilon}\Big\}\Big)dt,$$

where $[x/\varepsilon]$ denotes the "integer" part of x/ε with respect to the period Y and $\{x/\varepsilon\} = x/\varepsilon - [x/\varepsilon]$, where $\Phi(x)=0$ when $x\notin D$.

We shall use the following result from [6, Prop. 2.18]. As we will use it later, we present its proof.

Lemma 5.4 *For $\Phi\in L^1(D\times Y)$,*

$$\int_{D^\varepsilon}\mathcal{U}^\varepsilon(\Phi)(x)dx = \int_{D\times Y}\Phi(x,y)dydx,$$

where D^ε denotes the 2ε neighbourhood of D and where $\Phi(x)=0$ when $x\notin D$.

Proof Let I be a subset of $\mathbb{Z}^d$ such that $D \subset \bigcup_{m\in I} \varepsilon(m+\bar{Y})$. Then

$$\int_{D^\varepsilon}\int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t,\Big\{\frac{x}{\varepsilon}\Big\}\Big)dtdx = \sum_{m\in I}\int_Y\int_{\varepsilon(m+Y)} \Phi\Big(\varepsilon m+\varepsilon t,\Big\{\frac{x}{\varepsilon}\Big\}\Big)dxdt$$
$$= \varepsilon^d \sum_{m\in I}\int_Y\int_Y \Phi(\varepsilon m+\varepsilon t, y)dzdy = \int_{D\times Y}\Phi(x,y)dxdy.$$

□

For the corrector function $u_1(\boldsymbol{z};x,y)$, we have the following result.

Lemma 5.5 *If $u_0 \in L^\infty(U;H^2(D))$ and $w^l \in L^\infty(U;\boldsymbol{C}^1_1)$ for all $l=1,\dots,d$, then there exists a constant c independent of $\boldsymbol{z}$ such that for every $0<\varepsilon\le 1$*

$$\sup_{\boldsymbol{z}\in U}\int_D \Big|\nabla_y u_1(\boldsymbol{z};x,\frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\nabla_y(u_1(\boldsymbol{z};\cdot,\cdot))(x)\Big|^2 dx \le c\varepsilon^2.$$

Proof As

$$u_1(\boldsymbol{z};x,\boldsymbol{y}) = \sum_{l=1}^d \frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)w^l(\boldsymbol{z};x,\boldsymbol{y}),$$

it is sufficient to show that for each $l=1,\dots,d$

$$\int_D\Big|\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)\nabla_y w^l\Big(\boldsymbol{z};x,\frac{x}{\varepsilon}\Big) - \int_Y \frac{\partial u_0}{\partial x_l}\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t\Big)\nabla_y w^l\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t,\frac{x}{\varepsilon}\Big)dt\Big|^2 dx \le c\varepsilon^2.$$

The expression on the left hand side is bounded by

$$\int_D\int_Y\Big|\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)\nabla_y w^l\Big(\boldsymbol{z};x,\frac{x}{\varepsilon}\Big) - \frac{\partial u_0}{\partial x_l}\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t\Big)\nabla_y w^l\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t,\frac{x}{\varepsilon}\Big)\Big|^2 dtdx$$
$$\le 2\int_D\int_Y\Big|\Big(\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x) - \frac{\partial u_0}{\partial x_l}\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t\Big)\Big)\nabla_y w^l\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t,\frac{x}{\varepsilon}\Big)\Big|^2 dtdx +$$
$$2\int_D\int_Y\Big|\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)\Big|^2\Big|\nabla_y w^l\Big(\boldsymbol{z};x,\frac{x}{\varepsilon}\Big) - \nabla_{\boldsymbol{y}} w^l\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t,\frac{x}{\varepsilon}\Big)\Big|^2 dtdx\ .$$

As $w^l \in L^\infty(U;\boldsymbol{C}^1_1)$, there exists a constant c independent of $\boldsymbol{z}\in U$ such that for ε sufficiently small

$$\operatorname*{ess\,sup}_{\boldsymbol{z}\in U}\sup_{t\in Y}\Big|\nabla_y w^l\Big(\boldsymbol{z};x,\frac{x}{\varepsilon}\Big) - \nabla_y w^l\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t,\frac{x}{\varepsilon}\Big)\Big| \le c\varepsilon.$$

Therefore for all $\boldsymbol{z}\in U$

$$\int_D\Big|\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)\nabla_y w^l\Big(\boldsymbol{z};x,\frac{x}{\varepsilon}\Big) - \mathcal{U}^\varepsilon\Big(\frac{\partial u_0}{\partial x_l}(\boldsymbol{z},\cdot)\nabla_y w^l(\boldsymbol{z};\cdot,\cdot)\Big)(x)\Big|^2 dx$$
$$\le c\int_D\int_Y\Big|\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x) - \frac{\partial u_0}{\partial x_l}\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t\Big)\Big|^2 dtdx + c\varepsilon^2\ .$$

Next we claim that for all $\boldsymbol{z}\in U$

$$\int_D\int_Y\Big|\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x) - \frac{\partial u_0}{\partial x_l}\Big(\boldsymbol{z};\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t\Big)\Big|^2 dtdx \le c\varepsilon^2$$

where c is independent of $\boldsymbol{z}$. To prove this, let $\phi(x)$ be a smooth function. Then

$$\begin{aligned}\int_D(\phi(x)-\mathcal{U}^\varepsilon(\phi)(x))^2dx &\le \int_D\int_Y\Big|\phi(x)-\phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]+\varepsilon t\Big)\Big|^2dt\\
&\le \sum_{i=1}^d\int_D\int_Y\Big|\phi\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon t_1,\dots,\varepsilon\Big[\frac{x_{i-1}}{\varepsilon}\Big]+\varepsilon t_{i-1},x_i,\dots,x_d\Big)-\\
&\qquad \phi\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon t_1,\dots,\varepsilon\Big[\frac{x_i}{\varepsilon}\Big]+\varepsilon t_i,x_{i+1},\dots,x_d\Big)\Big|^2dtdx\\
&\le \sum_{i=1}^d\int_D\int_Y\Big|\varepsilon\int_{t_i}^{\{x_i/\varepsilon\}}\frac{\partial\phi}{\partial x_i}\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon t_1,\dots,\varepsilon\Big[\frac{x_i}{\varepsilon}\Big]+\varepsilon\zeta_i,x_{i+1},\dots,x_d)d\zeta_i\Big|^2dtdx\\
&\le \varepsilon^2\sum_{i=1}^d\int_D\int_Y\int_0^1\Big|\frac{\partial\phi}{\partial x_i}\Big(\varepsilon\Big[\frac{x_1}{\varepsilon}\Big]+\varepsilon t_1,\dots,\varepsilon\Big[\frac{x_i}{\varepsilon}\Big]+\varepsilon\zeta_i,x_{i+1},\dots,x_d\Big)\Big|^2d\zeta_idtdx\\
&\le \varepsilon^2\sum_{i=1}^d\int_D\Big|\frac{\partial\phi}{\partial x_i}\Big|^2dx.\end{aligned}$$

The last inequality is derived from Lemma 5.4, freezing the variables $x_{i+1},\dots,x_d$. Fix $\boldsymbol{z}\in U$ and $0<\varepsilon\le 1$ arbitrary, and consider a sequence $\{\phi_n\}_n\subset C^\infty(\bar D)$ which converges to $\partial u_0(\boldsymbol{z};x)/\partial x_l$ in $H^1(D)$. As $n\to\infty$,

$$\begin{aligned}\int_D\Big|\mathcal{U}^\varepsilon(\phi_n)(x)-\mathcal{U}^\varepsilon\Big(\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_l}\Big)(x)\Big|^2dx &\le \int_D\mathcal{U}^\varepsilon\Big(\Big(\phi_n(x)-\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_l}\Big)^2\Big)(x)dx\\
&\le \int_D\Big(\phi_n(x)-\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_l}\Big)^2dx\to 0\ .\end{aligned}$$

Therefore

$$\begin{aligned}&\int_D\Big(\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)-\mathcal{U}^\varepsilon\Big(\frac{\partial u_0(\boldsymbol{z};\cdot)}{\partial x_l}\Big)(x)\Big)^2dx\\
&\le 3\int_D\Big(\frac{\partial u_0}{\partial x_l}-\phi_n\Big)^2dx+3\int_D(\phi_n-\mathcal{U}^\varepsilon(\phi_n))^2dx+3\int_D\Big(\mathcal{U}^\varepsilon(\phi_n)-\mathcal{U}^\varepsilon\Big(\frac{\partial u_0}{\partial x_l}\Big)\Big)^2dx\\
&\le 6\int_D\Big(\frac{\partial u_0}{\partial x_l}-\phi_n\Big)^2dx+3\varepsilon^2\sum_{i=1}^d\int_D\Big|\frac{\partial\phi_n}{\partial x_i}\Big|^2dx.\end{aligned}$$

As $n\to\infty$, we have

$$\int_D\Big(\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)-\mathcal{U}^\varepsilon\Big(\frac{\partial u_0(\boldsymbol{z};\cdot)}{\partial x_l}\Big)(x)\Big)^2\le 3\varepsilon^2\sum_{i=1}^d\int_D\Big|\frac{\partial^2u_0}{\partial x_ix_l}\Big|^2dx.$$

Thus

$$\int_D\Big|\frac{\partial u_0}{\partial x_l}(\boldsymbol{z};x)\nabla_yw^l\Big(\boldsymbol{z};x,\frac{x}{\varepsilon}\Big)-\mathcal{U}^\varepsilon\Big(\frac{\partial u_0(\boldsymbol{z};x)}{\partial x_l}\nabla_yw^l(\boldsymbol{z};x,\boldsymbol{y})\Big)(x)\Big|^2dx\le c\varepsilon^2,$$

for a constant c which does not depend on $\boldsymbol{z}\in U$. □

Lemma 5.6 *With Assumption 4.3, $\mathcal{U}^\varepsilon(\nabla_{\boldsymbol{y}}u_1(\boldsymbol{z};\cdot,\cdot))$ as a map from U to $L^2(D)$ is measurable.*

Proof First we note that $(\mathcal{U}^\varepsilon(\Phi)(x))^2\le\mathcal{U}^\varepsilon(\Phi^2)(x)$ for a.e. x for all functions $\Phi\in L^2(\Omega\times Y)$. Thus

$$\forall\boldsymbol{z}\in U:\quad \|\mathcal{U}^\varepsilon(\nabla_yu_1(\boldsymbol{z};\cdot,\cdot)-\nabla_yu_1(\boldsymbol{z}',\cdot,\cdot))(\cdot)\|_{L^2(D)}\le\|\nabla_yu_1(\boldsymbol{z};\cdot,\cdot)-\nabla_yu_1(\boldsymbol{z}';\cdot,\cdot)\|_{L^2(D\times Y)}.$$

From the proof of Proposition 4.5, there exists $c>0$ independent of $\boldsymbol{z}\in U$ such that

$$\|\mathcal{U}^\varepsilon(\nabla_{\boldsymbol{y}}u_1(\boldsymbol{z};\cdot,\cdot)(\cdot)-\nabla_yu_1(\boldsymbol{z}';\cdot,\cdot)(\cdot))\|_{L^2(D)}\le c\|A(\boldsymbol{z};\cdot,\cdot)-A(\boldsymbol{z}';\cdot,\cdot)\|_{(\boldsymbol{C}^1_2)^{d\times d}}\ .$$

An argument similar to that in the proof of Proposition 1.8 shows that $\mathcal{U}^\varepsilon(\nabla_yu_1)$ as a map from U to $L^2(D)$ is measurable. □

Proposition 5.7 *If $A(\boldsymbol{z};x,y)\in L^\infty(U;\boldsymbol{C}^1_1)$, $f\in L^2(D)$ and if the domain D is convex, then there is a constant $c>0$ such that, for $\varepsilon>0$ sufficiently small, there holds*

$$\Big\|\nabla_y u_1(\boldsymbol{z};x,\frac{x}{\varepsilon})-\mathcal{U}^\varepsilon(\nabla_y u_1(\boldsymbol{z};\cdot,\cdot))(x)\Big\|^2_{L^2(U;L^2(D))}\le c\varepsilon\ .$$

Proof We need to show that $w^l\in L^\infty(U;\boldsymbol{C}^1_1)$. We will do this by analyzing a suitable difference quotient. To define it, we introduce for $x\in\bar D$ and for $\underline{\delta}\in\mathbb{R}^d$, the translation operator

$$\tau_{\underline{\delta}}(x):=x+\underline{\delta}\in\bar D.$$

We then have

$$\begin{aligned}
&\int_Y A(\boldsymbol{z};x,y)\nabla_y\Big(\frac{w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-w^l(\boldsymbol{z};x,y)}{|\underline{\delta}|}\Big)\cdot\nabla_y\phi(y)dy\\
&\quad=\int_Y\nabla_y\cdot\Big(\frac{A(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-A(\boldsymbol{z};x,y)}{|\underline{\delta}|}e_l\Big)\phi(y)dy\\
&\quad-\int_Y\frac{A(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-A(\boldsymbol{z};x,y)}{|\underline{\delta}|}\nabla_y w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)\cdot\nabla_{\boldsymbol{y}}\phi(y)dy,
\end{aligned}\tag{5.5}$$

for all $\phi\in H^1_\#(Y)$. Let $\psi(\boldsymbol{z};x,\cdot)\in H^1_\#(Y)/\mathbb{R}$ denote the solution of

$$\begin{aligned}
\int_Y A(\boldsymbol{z};x,y)\nabla_y\psi(\boldsymbol{z};x,y)\cdot\nabla_y\phi(y)dy&=\int_Y\nabla_y\cdot\Big(\frac{\partial A(\boldsymbol{z};x,y)}{\partial x_i}e_l\Big)\phi(y)dy\\
&-\int_Y\frac{\partial A(\boldsymbol{z};x,y)}{\partial x_i}\nabla_y w^l(\boldsymbol{z};x,y)\cdot\nabla_y\phi(y)dy\quad\forall\phi\in H^1_\#(Y)\ .
\end{aligned}\tag{5.6}$$

We have

$$\begin{aligned}
&\int_Y A(\boldsymbol{z};x,y)\nabla_y\Big(\frac{w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-w^l(\boldsymbol{z};x,y)}{|\underline{\delta}|}-\psi(\boldsymbol{z};x,y)\Big)\cdot\nabla_y\phi(y)dy\\
&=\int_Y\nabla_y\cdot\Big(\Big(\frac{A(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-A(\boldsymbol{z};x,y)}{|\underline{\delta}|}-\frac{\partial A(\boldsymbol{z};x,y)}{\partial x_i}\Big)e_l\Big)\phi(y)dy\\
&-\int_Y\Big(\frac{A(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-A(\boldsymbol{z};x,y)}{|\underline{\delta}|}-\frac{\partial A(\boldsymbol{z},x,y)}{\partial x_i}\Big)\nabla_y w^l(\boldsymbol{z},\tau_{\underline{\delta}}(x),y)\cdot\nabla_y\phi(y)dy\\
&-\int_Y\frac{\partial A}{\partial x_i}(\boldsymbol{z};x,y)\nabla_y(w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-w^l(\boldsymbol{z},x,y))\cdot\nabla_y\phi(y)dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Big\|\frac{w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),\cdot)-w^l(\boldsymbol{z};x,\cdot)}{|\underline{\delta}|}-\psi(\boldsymbol{z};x,\cdot)\Big\|_{W^{2,5}(Y)/\mathbb{R}}\le\\
&c\Big\|\frac{A(\boldsymbol{z};\tau_{\underline{\delta}}(x),\cdot)-A(\boldsymbol{z};x,\cdot)}{|\underline{\delta}|}-\frac{\partial A(\boldsymbol{z};x,\cdot)}{\partial x_i}\Big\|_{C^1(\bar Y))}(1+\|w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),\cdot)\|_{W^{2,5}(Y)/\mathbb{R}})\\
&+c\|w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-w^l(\boldsymbol{z};x,y)\|_{W^{2,5}(\mathbf{Y})/\mathbb{R}}.
\end{aligned}\tag{5.7}$$

From (5.5), we find

$$\|w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-w^l(\boldsymbol{z};x,y)\|_{W^{2,5}(Y)/\mathbb{R}}\le c\|A(\boldsymbol{z};\tau_{\underline{\delta}}(x),y)-A(\boldsymbol{z};x,y)\|_{C^1(\bar D\times\bar Y)}(1+\|w^l(\boldsymbol{z};\tau_{\underline{\delta}}(x),\cdot)\|_{W^{2,5}(Y)}),$$

which converges to 0 when $\delta\to0$. Thus the right hand side of (5.7) converges to 0 as $\delta\to0$. Therefore

$$\frac{\partial w^l(\boldsymbol{z};x,y)}{\partial x_i}=\psi(\boldsymbol{z};x,y)\text{ in }W^{2,5}(Y)/\mathbb{R}\subset C^1(\bar Y).$$

From (5.6), we deduce that $\|\psi\|_{W^{2,5}(Y)/\mathbb{R}}$ is uniformly bounded for all $\boldsymbol{z}\in U$ and is continuous with respect to $x\in D$ due to the continuity of $\nabla_x A$ and w^l in $W^{2,5}(Y)/\mathbb{R}$ so $w^l(\boldsymbol{z};x,y)\in L^\infty(U,C^1(\bar D,W^{2,5}(Y)))\subset L^\infty(U,\boldsymbol{C}^1_1)$. Therefore Lemma 5.5 holds uniformly for all $\boldsymbol{z}\in U$. Lemma 5.6 implies the assertion. □

For the solution $\boldsymbol{u}_{\Lambda_N}$ of the semidiscrete Galerkin approximation (2.3), we denote $\boldsymbol{u}_{\Lambda_N}=(u_{0\Lambda_N},u_{1\Lambda_N})$. We have the following corrector result for the approximation (2.3).

Theorem 5.8 *Assume that $A \in L^\infty(U; (\boldsymbol{C}_1^1)^{d\times d})$, $f \in L^2(D)$ and that D is convex. If Assumptions 1.1, 1.2 and 3.2 hold for some $0 < p < 1$, there exists a constant $c > 0$ (independent of ε and of N) such that for $\varepsilon > 0$ sufficiently small and for N sufficiently large, it holds that*

$$\|u^\varepsilon - [\nabla u_0 + \mathcal{U}^\varepsilon(\nabla_y u_{1\Lambda_N})]\|_{L^2(U;L^2(D))} \leq c(\varepsilon^{1/2} + N^{-\sigma}),$$

where $\sigma = 1/p - 1/2 > 1/2$ and the sets Λ_N are as in Theorem 3.9

Proof From Theorem 5.2 and Proposition 5.7, we get

$$\|u^\varepsilon - [\nabla u_0 + \mathcal{U}^\varepsilon(\nabla_y u_1)]\|_{L^\infty(U;L^2(D))} \leq c\varepsilon^{1/2}.$$

We note that

$$\begin{aligned}
&\int_U \int_D |\mathcal{U}^\varepsilon(\nabla_y u_1(\boldsymbol{z};\cdot,\cdot) - \nabla_y u_{1\Lambda_N}(\boldsymbol{z};\cdot,\cdot))|^2 dx d\rho(\boldsymbol{z}) \\
\leq\ & \int_U \int_D \mathcal{U}^\varepsilon(|\nabla_y u_1(\boldsymbol{z};\cdot,\cdot) - \nabla_y u_{1\Lambda_N}(\boldsymbol{z};\cdot,\cdot)|^2) dx d\rho(\boldsymbol{z}) \\
\leq\ & \int_U \int_D \int_Y |\nabla_y u_1(\boldsymbol{z}; x, y) - \nabla_y u_{1\Lambda_N}(\boldsymbol{z}; x, y)|^2 dy dx d\rho(\boldsymbol{z}) \\
\leq\ & cN^{-2\sigma},
\end{aligned}$$

where the last estimate is deduced from Theorem 3.9. We then get the conclusion. □.

5.2 Correctors for multiple scale problems

For problems with more than two scales, an error estimate analogous to (5.1) appears not to be available. For such problems we will now prove a corrector result; however, we will not give an explicit order of convergence. Moreover, this result does not require any extra regularity beyond the smoothness required for the existence of the $n+1$-scale limit. We start our analysis with the definition of a corrector.

Definition 5.9 *The $n+1$-scale "unfolding" operator $\mathcal{T}_n^\varepsilon : L^1(D) \to L^1(D \times \mathbf{Y})$ is defined by (see also [6]),*

$$\mathcal{T}_n^\varepsilon(\phi)(x, \boldsymbol{y}) = \phi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_2\Big[\frac{y_1}{\varepsilon_2/\varepsilon_1}\Big] + \ldots + \varepsilon_n\Big[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\Big] + \varepsilon_n y_n\Big),$$

where the function ϕ is understood as 0 outside D.

Denoting for $\varepsilon > 0$ sufficiently small by D^ε the 2ε neighbourbood of D, we have

$$\int_D \phi dx = \int_{D^\varepsilon} \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\phi) d\boldsymbol{y} dx \, . \tag{5.8}$$

Fixing $\boldsymbol{z} \in U$, as $\varepsilon \to 0$, we can show that

$$\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(\boldsymbol{z})) \rightharpoonup \boldsymbol{\nabla u} \text{ in } L^2(D \times \mathbf{Y}), \tag{5.9}$$

where $\boldsymbol{u}$ is as defined in Theorem 1.12. Following [6], we next define the "folding" operator $\mathcal{U}_n^\varepsilon$ by

Definition 5.10 *For $\Phi \in L^1(D \times \mathbf{Y})$ (understood to vanish when $x \notin D$) and for $\varepsilon > 0$ sufficiently small, the "folding" operator $\mathcal{U}_n^\varepsilon(\Phi) \in L^1(D)$ is defined as*

$$\begin{aligned}
\mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1} \cdots \int_{Y_n} \Phi\Big(&\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{\varepsilon_1}{\varepsilon_2}\Big\{\frac{x}{\varepsilon_1}\Big\}\Big] + \frac{\varepsilon_2}{\varepsilon_1} t_2, \ldots, \\
&\frac{\varepsilon_n}{\varepsilon_{n-1}}\Big[\frac{\varepsilon_{n-1}}{\varepsilon_n}\Big\{\frac{x}{\varepsilon_{n-1}}\Big\}\Big] + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \Big\{\frac{x}{\varepsilon_n}\Big\}\Big) dt_n \ldots dt_1.
\end{aligned}$$

We have the following measurability result.

Lemma 5.11 *Under Assumption 1.2, for the solution $\boldsymbol{u}(\boldsymbol{z})$ of the parametric, deterministic problem (1.19), the function $\mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}(\boldsymbol{z}))(x)$ (with $\boldsymbol{\nabla u}$ as in (1.18)) as a map from U to $L^2(D)^d$ is measurable.*

Proof For the functions $\Phi \in L^1(D \times \mathbf{Y})$ (which are understood as 0 when $x \notin D$),

$$\int_{D^\varepsilon} \mathcal{U}_n^\varepsilon(\Phi)(x)dx = \int_D \int_{\mathbf{Y}} \Phi d\boldsymbol{y}dx.$$

We note further that for *a.e.* $x \in D$,

$$(\mathcal{U}_n^\varepsilon(\Phi)(x))^2 \leq \mathcal{U}_n^\varepsilon(\Phi^2)(x).$$

From this we obtain, for every $\boldsymbol{z} \in U$ and with $\boldsymbol{\nabla u}$ as in (1.18),

$$\int_D |\mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}(\boldsymbol{z}) - \boldsymbol{\nabla u}(\boldsymbol{z}'))(x)|^2 dx \leq \int_D \mathcal{U}_n^\varepsilon(|\boldsymbol{\nabla u}(\boldsymbol{z}) - \boldsymbol{\nabla u}(\boldsymbol{z}')|^2)(x)dx \leq \int_D \int_{\mathbf{Y}} |\boldsymbol{\nabla u}(\boldsymbol{z}) - \boldsymbol{\nabla u}(\boldsymbol{z}')|^2 d\boldsymbol{y}dx.$$

We then get from (1.22) that there exists a constant c such that

$$\|\mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}(\boldsymbol{z}) - \boldsymbol{\nabla u}(\boldsymbol{z}'))\|_{L^2(D)} \leq c\|A(\boldsymbol{z};\cdot,\cdot) - A(\boldsymbol{z}';\cdot,\cdot)\|_{L^\infty(D\times\mathbf{Y})}.$$

The proof then follows along the lines of the proof of Proposition 1.8. □

We are now in position to prove a corrector result for the best N term approximation. It states that the gpc approximation of the high dimensional limit problem describes $\mathbb{P}$-a.s. all oscillations of the physical solution at small ε.

Theorem 5.12 *Under Assumption 1.2,*

$$\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}_{\Lambda_N})\|_{L^2(U;L^2(D))} = 0\ ,$$

where Λ_N is a subset of $\mathcal{F}$ corresponding to N largest terms of the sequence $(\|\boldsymbol{u}_\nu\|_{\mathbf{V}})_{\nu\in\mathcal{F}}$.

Proof We consider

$$\int_D \int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(A^\varepsilon)(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(\boldsymbol{z}))(x,\boldsymbol{y}) - \boldsymbol{\nabla u}(\boldsymbol{z};x,\boldsymbol{y}))\cdot(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(\boldsymbol{z}))(x,\boldsymbol{y}) - \boldsymbol{\nabla u}(\boldsymbol{z};x,\boldsymbol{y}))d\boldsymbol{y}dx.$$

From (1.5), (1.19), (5.8), (5.9), this expression converges to 0. Therefore the convergence in (5.9) is indeed strong. Fixing $\boldsymbol{z} \in U$, we obtain as $\varepsilon \to 0$

$$\|\nabla u^\varepsilon(\boldsymbol{z}) - \mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}(\boldsymbol{z}))\|_{L^2(D)} = \|\mathcal{U}_n^\varepsilon(\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(\boldsymbol{z})) - \mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}(\boldsymbol{z}))\|_{L^2(D)} \leq \|\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon(\boldsymbol{z})) - \boldsymbol{\nabla u}(\boldsymbol{z})\|_{L^2(D\times\mathbf{Y})} \to 0.$$

As $\|\nabla u^\varepsilon(\boldsymbol{z}) - \mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}(\boldsymbol{z}))\|_{L^2(D)}$ is uniformly bounded for all $z \in U$, we conclude that

$$\lim_{\varepsilon\to 0} \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u})\|_{L^2(U;L^2(D))} = 0.$$

Furthermore, for each $\boldsymbol{z} \in U$

$$\|\mathcal{U}_n^\varepsilon(\boldsymbol{\nabla}(\boldsymbol{u}_{\Lambda_N}(\boldsymbol{z}) - \boldsymbol{u}(\boldsymbol{z})))\|_{L^2(D)} \leq \|\boldsymbol{\nabla}(\boldsymbol{u}_{\Lambda_N}(\boldsymbol{z}) - \boldsymbol{u}(\boldsymbol{z}))\|_{L^2(D\times\mathbf{Y})}.$$

Therefore from Theorem 2.1

$$\lim_{N\to\infty} \|\mathcal{U}_n^\varepsilon(\boldsymbol{\nabla}(\boldsymbol{u}_{\Lambda_N} - \boldsymbol{u}))\|_{L^2(U;L^2(D))} = 0$$

uniformly for all ε. Thus

$$\lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \|\nabla u^\varepsilon - \mathcal{U}_n^\varepsilon(\boldsymbol{\nabla u}_{\Lambda_N})\|_{L^2(U;L^2(D))} = 0\ .$$

□

References

[1] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[2] G. Allaire and M. Briane. Multiscale convergence and reiterated homogenisation. *Proc. Roy. Soc. Edinburgh Sect. A*, 126(2):297–342, 1996.

[3] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1978.

[4] Marcel Bieri, Roman Andreev, and Christoph Schwab. Sparse tensor discretization of elliptic spdes. *SIAM J. Sci. Computing*, 31(6):4281–4304, 2009.

[5] Alain Bourgeat, Andro Mikelić, and Steve Wright. Stochastic two-scale convergence in the mean and applications. *J. Reine Angew. Math.*, 456:19–51, 1994.

[6] D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM J. Math. Anal.*, 40(4):1585–1620, 2008.

[7] Albert Cohen, Ronald DeVore, and Christoph Schwab. Convergence rates of best N-term Galerkin approximations for a class of elliptic spdes. *Found. Comp. Math.*, 10(6):615–646, 2010.

[8] Albert Cohen, Ronald Devore, and Christoph Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's. *Anal. Appl. (Singap.)*, 9(1):11–47, 2011.

[9] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[10] V. H. Hoang and Ch. Schwab. High-dimensional finite elements for elliptic problems with multiple scales. *Multiscale Model. Simul.*, 3(1):168–194, 2004/05.

[11] V. H. Hoang and Ch. Schwab. Sparse Tensor Galerkin discretizations for parametric and random parabolic PDEs I: Analytic regularity and gpc approximation. *Report 2010-11, Seminar for Applied Mathematics, ETH Zürich*, 2010.

[12] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosifyan].

[13] N. V. Krylov. *Lectures on elliptic and parabolic equations in Sobolev spaces*, volume 96 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[14] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[15] Christoph Schwab and Rob Stevenson. Adaptive wavelet algorithms for elliptic PDE's on product domains. *Math. Comp.*, 77(261):71–92 (electronic), 2008.

[16] Christoph Schwab and Radu Alexandru Todor. Karhunen-Loève approximation of random fields by generalized fast multipole methods. *J. Comput. Phys.*, 217(1):100–122, 2006.

[17] Radu-Alexandru Todor. A new approach to energy-based sparse finite-element spaces. *IMA J. Numer. Anal.*, 29(1):72–85, 2009.

[18] Hans Triebel. *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[19] J. Wloka. *Partial differential equations*. Cambridge University Press, Cambridge, 1987. Translated from the German by C. B. Thomas and M. J. Thomas.

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