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HP-DGFEM FOR SECOND-ORDER ELLIPTIC PROBLEMS IN POLYHEDRA I: STABILITY ON GEOMETRIC MESHES *

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Abstract. We introduce and analyze hp-version discontinuous Galerkin (dG) finite element methods for the numerical approximation of linear second-order elliptic boundary value problems in three dimensional polyhedral domains. In order to resolve possible corner-, edge- and corner-edge singularities, we consider hexahedral meshes that are geometrically and anisotropically refined towards the corresponding neighborhoods. Similarly, the local polynomial degrees are increased $\boldsymbol{s}$-linearly and possibly anisotropically away from singularities. We design interior penalty hp-dG methods and prove that they are well-defined for problems with singular solutions and stable under the proposed hp-refinements, i.e., on σ-geometric anisotropic meshes of mapped hexahedra with anisotropic polynomial degree distributions of μ-bounded variation. We establish (abstract) error bounds that will allow us to prove exponential rates of convergence in the second part of this work.

1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be an open and bounded polyhedron, with Lipschitz boundary $\Gamma = \partial\Omega$ given by a finite union of plane faces[1]. In Ω, we consider the Dirichlet problem for the diffusion-reaction equation

$$\mathsf{L}u \equiv -\nabla \cdot (\boldsymbol{A}\nabla u) + cu = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{1.2}$$

where $\boldsymbol{A} \in \mathsf{L}^\infty(\Omega; \mathbb{R}^{3\times 3}_{\boldsymbol{sym}})$, $c \in L^\infty(\Omega; \mathbb{R})$ are given functions. We assume that $\boldsymbol{A}$ is symmetric and uniformly positive definite:

$$\exists \alpha_0 > 0: \quad \forall \boldsymbol{\zeta} \in \mathbb{R}^3, \text{a.e. } \boldsymbol{x} \in \Omega: \quad \alpha_0^{-1}|\boldsymbol{\zeta}|^2 \geq \boldsymbol{\zeta}^\top \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\zeta} \geq \alpha_0|\boldsymbol{\zeta}|^2, \tag{1.3}$$

and that the reaction coefficient c is nonnegative on Ω, i.e., $c(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega$. Then for every $f \in H^{-1}(\Omega)$, the boundary-value problem (1.1)–(1.2) admits a unique solution $u \in H^1_0(\Omega)$.

This paper is the first in a series of papers in which we study hp-version discontinuous Galerkin (dG) finite element methods for elliptic problems in polyhedral domains. In this part, we shall establish the stability and prove abstract error bounds for interior penalty methods for (1.1)–(1.2) on geometrically refined meshes of mapped hexahedra with anisotropic polynomial degree distributions of bounded variation.

The hp-version of the finite element method (FEM) for elliptic problems was proposed in the mid 80ies by Babuška and his coworkers. They unified the hitherto largely separate developments of fixed-order "h-version FEM" in the sense of Ciarlet, which achieve convergence through reduction of the mesh size h, and the so-called "*spectral* (or p-version) FEM" achieving convergence through increasing the polynomial order p on a fixed mesh. Apart from unifying these two approaches, a key new

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[1]All what follows will also hold if Ω is a finite union of such Lipschitz polyhedra; *finite* union is a restriction—there are Lipschitz polyhedra with infinitely many plane faces.

feature of hp-FEM was the possibility to achieve *exponential convergence rates* in terms of the number N of degrees of freedom. Exponential convergence results for the hp-version of the FEM were shown in one dimension by Babuška and Gui in [11] for the model singular solution $u(x) = x^\alpha - x \in H_0^1(\Omega)$, with $\alpha > 1/2$ and $\Omega = (0,1)$. Specifically, the energy error was shown to be bounded by $\exp(-b\sqrt{N})$ for *any fixed subdivision ratio* $\sigma \in (0,1)$ (in particular, for $\sigma = 1/2$ when geometric element sequences are obtained by successive element bisection) for a constant b depending on the singularity exponent α as well as on σ.

In two dimensions, exponential convergence (i.e., an upper bound of the form $C\exp(-b\sqrt[3]{N})$ for the error of the hp-version FEM in polygons) was obtained by Babuška and Guo in the mid 80ies in a series of landmark papers ([3, 14, 15] and the references therein). Key ingredients in the proof were *geometric mesh refinement* towards the singular support $\mathcal{S}$ (being the set of vertices of the polygon Ω) of the solution and *nonuniform elemental polynomial degrees* which increase $\boldsymbol{s}$-linearly with the elements' distance from $\mathcal{S}$. We mention that the proof of *elliptic regularity in countably normed spaces of the solutions*, which constitutes an essential prerequisite for the exponential convergence proof, has been a major technical achievement.

In the 90ies, steps to extend the analytic regularity and the hp-convergence analysis in [3, 14, 15] to three dimensions were undertaken in [4, 13, 16, 17] and the references therein. While all these works were devoted to conforming finite element methods for second-order elliptic problems, extensions to hp-version mixed methods and conforming methods for higher-order problems in polygons were obtained in [12, 27].

Discontinuous Galerkin methods emerged in the 70ies as stable discretizations of first-order transport-dominated problems (see [20, 21, 26]), and as nonconforming discretizations of second-order elliptic problems (cf. [1, 5, 9, 24, 32]). Later, in the 90ies, dG methods were studied within the hp-version setting for first-order transport and for advection-reaction-diffusion problems in two- and three-dimensional domains (see [18, 19]). Exponential convergence rates were established for *piecewise analytic solutions* excluding, in particular, corner singularities as occurring in polygonal domains. In that context, exponential convergence was established in [33, 34] for diffusion problems and in [31] for the Stokes equations.

In the present paper, we shall consider the hp-dGFEM for the boundary-value problem (1.1)–(1.2) in polyhedra in $\mathbb{R}^3$. Particularly, for solutions with possible *corner and edge singularities* (measured in appropriate weighted Sobolev spaces), we shall prove that the hp-dG discretizations are stable and consistent for suitable combinations of σ-geometric meshes (obtained from mapped hexahedral elements) and anisotropic elemental polynomial degrees (that are $\boldsymbol{s}$-linearly increasing and of μ-bounded variation). The hp-dG approximations are shown to be well-defined and to satisfy the Galerkin orthogonality property. Finally, we derive abstract error estimates for the dG energy error with respect to a suitable discontinuous elemental polynomial interpolation operator.

The class of hp-dGFEM investigated here contains, in particular, *three-dimensional generalizations of all mesh-degree combinations which were found to be optimal in the univariate case* in [11]. We mention that the stability of mixed hp-dGFEM (based on uniform isotropic, but variable polynomial degrees) for viscous incompressible flow on meshes of this type has been investigated in [28, 29]. In addition, we refer to [35] for hp-dGFEM discretizations of the linear elasticity and Stokes equations in polyhedra, which are related to the results in the present work.

We emphasize that the *hp*-dG subspaces to be introduced in this paper, together with the associated stability and error analysis for solutions in weighted Sobolev spaces (featuring corner and edge singularities), constitute a key ingredient for the proof of exponential convergence. In the second part, [30], we show that the dG energy error converges as $C\exp(-b\sqrt[5]{N})$ when the *hp*-dG discretizations discussed in this article are used for the numerical approximation of (1.1)–(1.2) in polyhedral domains.

The outline of the article is as follows: In Section 2, we recapitulate regularity results in weighted Sobolev spaces for the solution of (1.1)–(1.2) from [7]; see also [23, 22]. In Section 3, we define *hp*-dG finite element spaces on *σ-geometric meshes* of mapped hexahedral elements with possibly anisotropic and *$\boldsymbol{s}$-linearly increasing polynomial degree distributions of μ-bounded variation*. In particular, we give a constructive algorithmic definition of such subspaces in any bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ with a finite number of faces. Furthermore, in Section 4, we establish the stability and prove error estimates for *hp*-interior penalty discontinuous Galerkin discretizations for these *hp*-dG spaces.

Standard notation will be employed throughout the paper. The number of elements in a set A of finite cardinality is denoted by $|A|$. Occasionally, we shall use the notations "$\lesssim$" or "$\simeq$" to mean an inequality or an equivalence containing generic positive multiplicative constants independent of any local mesh sizes and polynomial degrees.

2. Regularity.

Under the assumption that the coefficient functions $\boldsymbol{A}$, c and the source term f in (1.1) are sufficiently smooth in $\overline{\Omega}$, the solution of (1.1)–(1.2) belongs to H^2 away from any corners and edges of Ω. In order to specify the precise regularity in scales of weighted Sobolev spaces, we recall some recent results from [7] (see also [23, 22]).

2.1. Subdomains and Weights.

In the bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ with plane faces, we denote by $\mathcal{C}$ the set of corners $\boldsymbol{c}$, and by $\mathcal{E}$ the set of open[2] edges $\boldsymbol{e}$ of Ω. Then, the singular support is given by

$$\mathcal{S} = \left(\bigcup_{\boldsymbol{c}\in\mathcal{C}} \boldsymbol{c} \right) \cup \left(\bigcup_{\boldsymbol{e}\in\mathcal{E}} \boldsymbol{e} \right) \subset \Gamma. \tag{2.1}$$

For smooth data $\boldsymbol{A}$, c and f in $\overline{\Omega}$, the set $\mathcal{S}$ coincides with the singular support of the solution u of (1.1)–(1.2).

In order to define suitably weighted Sobolev spaces, we split Ω into vicinities of edges $\boldsymbol{e} \in \mathcal{E}$, corners $\boldsymbol{c} \in \mathcal{C}$, or both. To this end, we define, for $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{e} \in \mathcal{E}$ and $\boldsymbol{x} \in \Omega$, the following distance functions:

$$r_{\boldsymbol{c}}(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \boldsymbol{c}), \qquad r_{\boldsymbol{e}}(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \boldsymbol{e}), \qquad \rho_{\boldsymbol{ce}}(\boldsymbol{x}) = \frac{r_{\boldsymbol{e}}(\boldsymbol{x})}{r_{\boldsymbol{c}}(\boldsymbol{x})}. \tag{2.2}$$

We furthermore *assume* that Ω is such that

$$\exists \varepsilon(\Omega) > 0 : \quad \bigcap_{\boldsymbol{c}\in\mathcal{C}} B_\varepsilon(\boldsymbol{c}) = \emptyset, \tag{2.3}$$

where $B_\varepsilon(\boldsymbol{c})$ denotes the open ball in $\mathbb{R}^3$ with center $\boldsymbol{c}$ and radius ε. Note that assumption (2.3) is a separation condition of the vertices of Ω; it is indeed a geometric

[2] In this paper, all geometric objects (except points, but including, e.g., subdomains, faces, edges, elements) are assumed to be open, unless explicitly stated otherwise.

restriction, since it is not satisfied by all Lipschitz polyhedra with straight faces. In addition, for each corner $\boldsymbol{c} \in \mathcal{C}$, we define by

$$\mathcal{E}_{\boldsymbol{c}} = \{\, \boldsymbol{e} \in \mathcal{E} \,:\, \boldsymbol{c} \cap \overline{\boldsymbol{e}} \neq \emptyset \,\}$$

the set of all edges of Ω which meet at $\boldsymbol{c}$. Moreover, for any $\boldsymbol{e} \in \mathcal{E}$, the set of corners of $\boldsymbol{e}$ is given by

$$\mathcal{C}_{\boldsymbol{e}} \equiv \partial \boldsymbol{e} = \{\, \boldsymbol{c} \in \mathcal{C} \,:\, \boldsymbol{c} \cap \overline{\boldsymbol{e}} \neq \emptyset \,\}\,.$$

Then, for $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{e} \in \mathcal{E}$ and $\boldsymbol{e}_{\boldsymbol{c}} \in \mathcal{E}_{\boldsymbol{c}}$ and for a sufficiently small $\varepsilon > 0$ to be specified below, we define

$$\begin{aligned}
\omega_{\boldsymbol{c}} &= \{\, \boldsymbol{x} \in \Omega : \quad r_{\boldsymbol{c}}(\boldsymbol{x}) < \varepsilon \ \wedge \ \rho_{\boldsymbol{ce}}(\boldsymbol{x}) > \varepsilon \quad \forall\, \boldsymbol{e} \in \mathcal{E}_{\boldsymbol{c}} \,\}, \\
\omega_{\boldsymbol{e}} &= \{\, \boldsymbol{x} \in \Omega : \quad r_{\boldsymbol{e}}(\boldsymbol{x}) < \varepsilon \ \wedge \ r_{\boldsymbol{c}}(\boldsymbol{x}) > \varepsilon \quad \forall\, \boldsymbol{c} \in \mathcal{C}_{\boldsymbol{e}} \}, \\
\omega_{\boldsymbol{ce_c}} &= \{\, \boldsymbol{x} \in \Omega : \quad r_{\boldsymbol{c}}(\boldsymbol{x}) < \varepsilon \ \wedge \ \rho_{\boldsymbol{ce_c}}(\boldsymbol{x}) < \varepsilon \,\}.
\end{aligned}$$

When clear from the context, we simply write $\omega_{\boldsymbol{ce}}$ in place of $\omega_{\boldsymbol{ce_c}}$. Possibly by reducing ε in (2.3), we may partition the domain Ω into four *disjoint* parts,

$$\overline{\Omega} = \overline{\Omega_0 \,\dot\cup\, \Omega_{\mathcal{C}} \,\dot\cup\, \Omega_{\mathcal{E}} \,\dot\cup\, \Omega_{\mathcal{CE}}}, \tag{2.4}$$

where

$$\Omega_{\mathcal{C}} = \bigcup_{\boldsymbol{c}\in\mathcal{C}} \omega_{\boldsymbol{c}}, \qquad \Omega_{\mathcal{E}} = \bigcup_{\boldsymbol{e}\in\mathcal{E}} \omega_{\boldsymbol{e}}, \qquad \Omega_{\mathcal{CE}} = \bigcup_{\boldsymbol{c}\in\mathcal{C}} \bigcup_{\boldsymbol{e}\in\mathcal{E}_{\boldsymbol{c}}} \omega_{\boldsymbol{ce}}. \tag{2.5}$$

We shall refer to the subdomains $\Omega_{\mathcal{C}}$, $\Omega_{\mathcal{E}}$ and $\Omega_{\mathcal{CE}}$ as *corner*, *edge* and *corner-edge neighborhoods* of Ω, respectively. The remaining, "interior" part of Ω is defined by

$$\Omega_0 := \Omega \setminus \overline{\Omega_{\mathcal{C}} \cup \Omega_{\mathcal{E}} \cup \Omega_{\mathcal{CE}}}. \tag{2.6}$$

Note that

$$\operatorname{dist}(\Omega_0, \mathcal{S}) > \varepsilon/2 > 0. \tag{2.7}$$

2.2. Weighted Sobolev Spaces. To each $\boldsymbol{c} \in \mathcal{C}$ and $\boldsymbol{e} \in \mathcal{E}$ we associate a corner and an edge exponent $\beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}} \in \mathbb{R}$, respectively. We collect these quantities in the multi-exponent

$$\boldsymbol{\beta} = \{\beta_{\boldsymbol{c}} : \boldsymbol{c} \in \mathcal{C}\} \cup \{\beta_{\boldsymbol{e}} : \boldsymbol{e} \in \mathcal{E}\} \in \mathbb{R}^{|\mathcal{C}|+|\mathcal{E}|}. \tag{2.8}$$

Inequalities of the form $\boldsymbol{\beta} < 1$ and expressions like $\boldsymbol{\beta} \pm s$, where $s \in \mathbb{R}$, are to be understood componentwise. For example,

$$\boldsymbol{\beta} + s = \{\beta_{\boldsymbol{c}} + s : \boldsymbol{c} \in \mathcal{C}\} \cup \{\beta_{\boldsymbol{e}} + s : \boldsymbol{e} \in \mathcal{E}\}.$$

A key issue in the stability and error analysis of *hp*-approximations in three dimensions is the anisotropic regularity of the solution u of (1.1)–(1.2) near the edges $\mathcal{E}$ of Ω. In order to describe it, we introduce, for corners $\boldsymbol{c} \in \mathcal{C}$ and edges $\boldsymbol{e} \in \mathcal{E}$, local coordinate systems in $\omega_{\boldsymbol{e}}$ and $\omega_{\boldsymbol{ce}}$ such that $\boldsymbol{e}$ corresponds to the direction $(0, 0, 1)$. Then, we denote quantities that are transversal to $\boldsymbol{e}$ by $(\cdot)^{\perp}$, and quantities parallel to $\boldsymbol{e}$ by $(\cdot)^{\parallel}$. In particular, if $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ is a multi-index corresponding to the three local coordinate directions in a subdomain $\omega_{\boldsymbol{e}}$ or $\omega_{\boldsymbol{ce}}$, then we have $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^\perp, \alpha^\parallel)$, where $\boldsymbol{\alpha}^\perp = (\alpha_1, \alpha_2)$ and $\alpha^\parallel = \alpha_3$. Likewise notation shall be employed below in anisotropic quantities related to a face. For $m \in \mathbb{N}_0$, we define the semi-norm $|\circ|_{M^m_{\boldsymbol{\beta}}(\Omega)}$ by

$$
\begin{aligned}
|u|^2_{M^m_{\boldsymbol{\beta}}(\Omega)} = |u|^2_{H^m(\Omega_0)} &+ \sum_{\boldsymbol{e}\in\mathcal{E}} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^3\\ |\boldsymbol{\alpha}|=m}} \left\| r_{\boldsymbol{e}}^{\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^\perp|} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{e}})} \\
&+ \sum_{\boldsymbol{c}\in\mathcal{C}} \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{N}_0^3\\ |\boldsymbol{\alpha}|=m}} \left(\left\| r_{\boldsymbol{c}}^{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{c}})} + \sum_{\boldsymbol{e}\in\mathcal{E}_{\boldsymbol{c}}} \left\| r_{\boldsymbol{c}}^{\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \rho_{\boldsymbol{ce}}^{\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^\perp|} \mathsf{D}^{\boldsymbol{\alpha}} u \right\|^2_{L^2(\omega_{\boldsymbol{ce}})} \right),
\end{aligned}
\tag{2.9}
$$

and the norm $\|\circ\|_{M^m_\beta(\Omega)}$ by

$$
\|u\|^2_{M^m_{\boldsymbol{\beta}}(\Omega)} = \sum_{k=0}^{m} |u|^2_{M^k_{\boldsymbol{\beta}}(\Omega)} . \tag{2.10}
$$

Here, $|u|^2_{H^m(\Omega_0)}$ is the usual Sobolev semi-norm of order m on Ω_0, and the operator $\mathsf{D}^{\boldsymbol{\alpha}}$ denotes the derivative in the local coordinate directions corresponding to the multi-index $\boldsymbol{\alpha}$. Finally, $M^m_{\boldsymbol{\beta}}(\Omega)$ is the weighted Sobolev space obtained as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{M^m_{\boldsymbol{\beta}}(\Omega)}$.

REMARK 2.1. *It will be necessary to consider the $M^m_{\boldsymbol{\beta}}$-norms also for subdomains $K \subset \Omega$: we shall denote by $|\circ|_{M^m_{\boldsymbol{\beta}}(K)}$ the semi-norm* (2.9) *with all domains of integration replaced by their intersections with $K \subset \Omega$ and likewise also for $\|\circ\|_{M^m_{\boldsymbol{\beta}}(K)}$.*

2.3. $M^m_{\boldsymbol{\beta}}$-Regularity. The significance of the weighted Sobolev spaces defined above lies in the fact that (1.1) satisfies a shift theorem in $M^m_{\boldsymbol{\beta}}(\Omega)$ for any $m \in \mathbb{N}$.

PROPOSITION 2.2. *There exist bounds $\beta_{\mathcal{E}}, \beta_{\mathcal{C}} > 0$ (depending on Ω, the coefficients in* (1.1)*, and on the types of boundary conditions on $\partial\Omega$) such that, for $\boldsymbol{\beta}$ satisfying*

$$
0 < \beta_{\boldsymbol{e}} < \beta_{\mathcal{E}}, \quad 0 < \beta_{\boldsymbol{c}} < \frac{1}{2} + \beta_{\mathcal{C}}, \qquad \boldsymbol{e} \in \mathcal{E},\ \boldsymbol{c} \in \mathcal{C}, \tag{2.11}
$$

and every $m \in \mathbb{N}$, a solution $u \in H^1(\Omega)$ of (1.1)*–*(1.2) *with $\mathsf{L}u \in M^m_{1-\boldsymbol{\beta}}(\Omega)$ fulfills $u \in M^m_{-1-\boldsymbol{\beta}}(\Omega)$. Furthermore, there holds the regularity estimate*

$$
\|u\|_{M^m_{-1-\boldsymbol{\beta}}(\Omega)} \leq C_m \|\mathsf{L}u\|_{M^m_{1-\boldsymbol{\beta}}(\Omega)} \tag{2.12}
$$

for all $m \in \mathbb{N}_0$, where $C_m > 0$ is a constant independent of u.

We refer, for example, to [7] for a proof of this result. We emphasize that in the present paper, we only require (2.12) for $m = 2$, that is, we only require that the solution u of (1.1)–(1.2) belongs to $M^2_{-1-\boldsymbol{\beta}}(\Omega)$. Shift theorems such as (2.12) are well-known to hold for rather general second-order elliptic systems in polyhedral domains; see [23, 22] and the references there for precise statements and proofs. In addition, we mention that there are equivalences and relations between the above defined M-spaces and other classes of Sobolev spaces used in the context of elliptic regularity theory in polyhedra; see, e.g., [7, Remark 6.12].

3. hp-Extensions in Ω. The purpose of this section is to introduce a class of hp-finite element spaces for the numerical approximation of (1.1)–(1.2) in polyhedra. They will be employed to establish exponential convergence of the hp-dGFEM in [30]. The spaces considered involve families $\mathfrak{M}_\sigma = \{\mathcal{M}^{(\ell)}\}_{\ell \geq 1}$ of *σ-geometric meshes* in the polyhedron Ω, and ***s**-linear polynomial degree distributions* on elements $K \in \mathcal{M}^{(\ell)}$.

The outline of this section is as follows: we start by introducing a basic hexahedral mesh $\mathcal{M}^0$ in Ω and by discussing some general mapping conditions. Subsequently, we present a specific construction based on subdividing a coarse tetrahedral mesh $\mathcal{T}^0$ into a regular mesh $\mathcal{M}^0$ of (trilinearly mapped) hexahedral elements. Then, families of σ-geometric meshes in general Lipschitz polyhedra $\Omega \subset \mathbb{R}^3$ will be considered. They are obtained by sequences of possibly anisotropic element bisections with a prescribed edge ratio $\sigma \in (0,1)$ (in particular, we allow ratios $\sigma \neq 1/2$ known in the one-dimensional case to yield superior error bounds of hp-approximations for singularities, cf. [11, Theorem 3.2]). While at each stage only elements abutting at the singular support of the solution are subdivided, the polynomial degrees in the remaining elements are increased with each σ-subdivision. As both, mesh and polynomial degrees, are changed simultaneously in hp-FEM, we refer to one step of combined σ-subdivision of elements and directional polynomial degree increase as *hp-extension*, following [11].

We introduce, following [8], the concept of *mesh layers*: as in the two-dimensional case, the aim is to keep the *ratio of the element diameter and the (assumed positive) distance to the solution's singular support* bounded from above and below uniformly over all elements of the geometric mesh family. Naturally, with a view towards exponential convergence, the appearance of edges in the singular support require *anisotropic geometric mesh refinements and anisotropic polynomial degree distributions.*

The hp-setup in this paper will be based on families of meshes $\mathcal{M}$ consisting of (possibly anisotropic) disjoint, open, parametric *hexahedral elements* K_i, such that $\overline{\Omega} = \bigcup_i \overline{K}_i$. To allow for mesh refinement, the hexahedral meshes $\mathcal{M}$ necessarily contain irregular nodes.

3.1. Basic Hexahedral Mesh $\mathcal{M}^0$ in Ω. We begin by introducing a regular[3] basic (initial) hexahedral mesh $\mathcal{M}^0$ in $\Omega \subset \mathbb{R}^3$ (also called the patch mesh). We suppose that each hexahedron $Q_{j'} \in \mathcal{M}^0 = \{Q_j\}_{j=1}^J$ is an image of the reference patch $\widetilde{Q} = (-1,1)^3$ under a *diffeomorphic* mapping $G_{j'}$,

$$\forall Q_{j'} \in \mathcal{M}^0 : \quad Q_{j'} = G_{j'}(\widetilde{Q}), \qquad j' = 1, \dots, J. \tag{3.1}$$

We collect the (finitely many) maps G_j in the *patch map vector*

$$\mathfrak{G} = \{G_j : j = 1, ..., J\} . \tag{3.2}$$

We assume the patch maps to be *compatible*, i.e.

$$\forall i \neq j : \quad \text{if} \quad \overline{Q}_j \cap \overline{Q}_i \neq \emptyset, \quad \text{then} \quad \left(G_j \circ G_i^{(-1)}\right)|_{\overline{Q}_j \cap \overline{Q}_i} = \mathrm{id} . \tag{3.3}$$

The hexahedral mesh $\mathcal{M}^0$ obtained in this fashion is shape-regular: there exists a constant $C_{\mathcal{M}^0} \geq 1$ (depending only on $\mathfrak{G}$ and $\mathcal{M}^0$) such that

$$C_{\mathcal{M}^0}^{-1} \leq \|\det(\mathsf{D}G)\|_{L^\infty(\widetilde{Q})} \leq C_{\mathcal{M}^0} \qquad \forall G \in \mathfrak{G}, \tag{3.4}$$

[3] By *regular*, we mean that the intersection of the closure of any two elements is either empty, or an entire face, an entire edge or a vertex of both elements.

as well as

$$\left\|\widetilde{\mathsf{D}}^{\boldsymbol{\alpha}} G\right\|_{L^\infty(\widetilde{Q})} + \left\|\mathsf{D}^{\boldsymbol{\alpha}}(G^{-1})\right\|_{L^\infty(G(\widetilde{Q}))} \le C_{\mathcal{M}^0} \qquad \forall G \in \mathfrak{G}, \quad \forall 1 \le |\boldsymbol{\alpha}| \le 2\,. \tag{3.5}$$

Here, we denote by $\widetilde{\mathsf{D}}^{\boldsymbol{\alpha}}$ partial derivatives with respect to the coordinates in the reference patch $\widetilde{Q}$, and by $\mathsf{D}^{\boldsymbol{\alpha}}$ the derivatives with respect to the physical coordinates on $G(\widetilde{Q})$.

For the approximation of solutions with singular support at the edges $\mathcal{E}$ and vertices $\mathcal{C}$ of Ω, we will require that physical edges and vertices of the polyhedron Ω coincide with edges and vertices of certain hexahedra $Q_{j'} \in \mathcal{M}^0$ in exactly one of several canonical ways.

Assumption 3.1. *For each hexahedron $Q \in \mathcal{M}^0$ in the patch mesh exactly one of the following cases is true: the intersection $\mathcal{S} \cap \overline{Q}$*

1. *is empty;*
2. *contains exactly one corner P of Q, and $P \in \mathcal{C}$;*
3. *contains exactly one corner P of Q, where $P \subset \overline{\boldsymbol{e}}$ for some $\boldsymbol{e} \in \mathcal{E}$, and $P \notin \mathcal{C}$;*
4. *contains exactly one closed edge $\overline{\boldsymbol{e}'}$ of Q, where $\overline{\boldsymbol{e}'} \subset \overline{\boldsymbol{e}}$ for some $\boldsymbol{e} \in \mathcal{E}$; moreover, the intersection $\overline{\boldsymbol{e}'} \cap \mathcal{C}$ contains exactly one point P, where P is a corner of both Q and Ω;*
5. *contains exactly one closed edge $\overline{\boldsymbol{e}'}$ of Q, where $\overline{\boldsymbol{e}'} \subset \overline{\boldsymbol{e}}$ for some $\boldsymbol{e} \in \mathcal{E}$, and $\overline{\boldsymbol{e}'} \cap \mathcal{C} = \emptyset$.*

In the sequel, we shall outline one possible construction of a basic hexahedral mesh satisfying Assumption 3.1 for a polyhedron with plane faces: We start the construction from a regular partition $\mathcal{T}^0$ of Ω into open, disjoint tetrahedra $\{T_i\}_i$, such that $\overline{\Omega} = \bigcup_i \overline{T}_i$. Here, we may suppose that the mesh $\mathcal{T}^0$ is sufficiently fine such that, for any tetrahedron $T \in \mathcal{T}^0$, we have that $\overline{T} \cap \mathcal{S}$ is either

(T1) empty;
(T2) one corner of T;
(T3) or the closure of one entire edge $\boldsymbol{e}'$ of T which is a subset of the closure of a singular edge $\boldsymbol{e}$ of Ω (i.e., $\overline{\boldsymbol{e}'} \subset \overline{\boldsymbol{e}} \in \mathcal{E}$). Furthermore, $\overline{\boldsymbol{e}'}$ contains at most one corner of Ω which, if any, is also a corner of T.

Elements in $\mathcal{T}^0$ are assumed to be affine images of the reference tetrahedron $\widehat{T} = \{\,(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \mathbb{R}^3 \,:\, \widehat{x}_1, \widehat{x}_2, \widehat{x}_3 > 0,\ \widehat{x}_1 + \widehat{x}_2 + \widehat{x}_3 < 1\,\}$. More precisely, for each $T_i \in \mathcal{T}^0$, there is an affine mapping $F_i : \widehat{T} \to T_i$ such that $T_i = F_i(\widehat{T})$.

Then, in order to obtain a basic hexahedral mesh $\mathcal{M}^0$ as described earlier, we split the reference tetrahedron $\widehat{T}$ into four open hexahedra $\{\widehat{Q}_j\}_{j=1}^4$ of equivalent diameter as follows: every face of $\widehat{T}$ is broken into four quadrilaterals by introducing edges joining the edge midpoints and the center of gravity of the face. An additional vertex is introduced in the middle of the tetrahedron and linked with the centers of the faces. Then, each of the four resulting hexahedra $\widehat{Q}_j \subset \widehat{T}$, $j = 1, 2, 3, 4$, is the image of the reference patch $\widetilde{Q} = (-1, 1)^3$ under a *bijective trilinear* transform $\widetilde{Q} \to \widehat{Q}_j$. The $\widehat{Q}_j$ are then transported by affine maps F_i to hexahedra Q_{ij} in Ω; see Figure 3.1. We abuse notation by indexing these hexahedra by a single index $j' = 1, \ldots, J := 4\left|\mathcal{T}^0\right|$. The union of the J hexahedra $Q_{j'} \subset \Omega$ obtained in this fashion constitutes a basic patch mesh $\mathcal{M}^0$ in Ω. In this particular construction, the patch maps $\{G_j\}_{j=1}^J$ from (3.1) are compositions of trilinear and affine transformations, and hence, diffeomorphic.

According to our assumptions, the bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ admits a partition $\mathcal{T}^0$ into a finite number of simplices. Therefore, the above construction

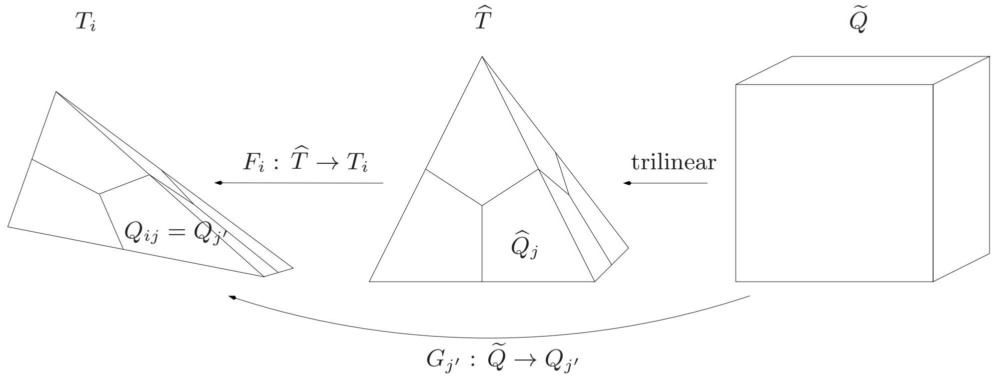

FIG. 3.1. *Trilinear patch mappings from the reference patch* $\widetilde{Q}$ *to the patch mesh* $\mathcal{M}^0$.

gives a mesh of convex, trilinearly mapped hexahedra in any bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$ with plane faces.

LEMMA 3.2. *Assume that the initial tetrahedral mesh* $\mathcal{T}^0$ *in* Ω *satisfies* (T1)–(T3). *Then, the basic hexahedral mesh* $\mathcal{M}^0$ *resulting from the construction above satisfies Assumption 3.1.*

Proof. Let us consider a tetrahedron $T \in \mathcal{T}^0$. The set of the four hexahedra contained in T is denoted by $\mathcal{Q}_T$. We suppose that $\overline{T} \cap \mathcal{S} \neq \emptyset$ (if this intersection were empty, then certainly $\overline{Q} \cap \mathcal{S} = \emptyset$ for all $Q \in \mathcal{Q}_T$, i.e., this is case 1). Recalling (T1)–(T3), the following situations may occur:

(i) $\overline{T} \cap \mathcal{S}$ is a corner of T as well as a corner of Ω: it follows that there is exactly one hexahedron $Q \in \mathcal{Q}_T$ which intersects with $\mathcal{S}$, and this intersection is a corner of Q (because it is a corner of T), i.e., case 2. The closures of the remaining three hexahedra do not intersect with $\mathcal{S}$, i.e., case 1.

(ii) $\overline{T} \cap \mathcal{S}$ is a corner of T, but not a corner of Ω: again, there is exactly one $Q \in \mathcal{Q}_T$ for which $\overline{Q} \cap \mathcal{S} = \overline{T} \cap \Omega$ is a corner of Q. Since this point is not a corner of Ω, it can only be situated on an edge of Ω, i.e., case 3. Furthermore, the closures of the remaining hexahedra in $\mathcal{Q}_T$ do not intersect with $\mathcal{S}$, i.e., case 1.

(iii) $\overline{T} \cap \mathcal{S}$ is an entire closed edge of T containing one corner P of Ω (which by (T3) is also a corner of T): in this case, there exists an edge $\boldsymbol{e} \in \mathcal{E}$ of Ω such that $\overline{T} \cap \mathcal{S} \subset \overline{\boldsymbol{e}}$, and $P = (\overline{T} \cap \mathcal{S}) \cap \partial \boldsymbol{e}$. Then, there is a hexahedron $Q_1 \in \mathcal{Q}_T$ such that P is a corner of Q_1 and $\partial Q_1 \cap \mathcal{S} \subset \overline{\boldsymbol{e}}$ is the closure of an entire edge of Q_1 that contains P. This corresponds to case 4. Furthermore, there is a second hexahedron $Q_2 \in \mathcal{Q}_T$ such that $\partial Q_2 \cap \mathcal{S} \subset \overline{\boldsymbol{e}}$ is an entire closed edge of Q_2 not containing a corner of Ω, i.e., case 5. The closures of the two remaining hexahedra do not have any intersection with $\mathcal{S}$.

(iv) $\overline{T} \cap \mathcal{S}$ is an entire closed edge of T not containing a corner of Ω: there are exactly two hexahedra $Q_1, Q_2 \in \mathcal{Q}_T$ whose boundaries intersect with $\mathcal{S}$. More precisely, $\overline{Q}_1 \cap S$, $\overline{Q}_2 \cap S$ are entire edges of Q_1 and Q_2, respectively, that are subsets of the closure of an edge $\boldsymbol{e} \in \mathcal{E}$ of Ω. The intersections $\overline{\boldsymbol{e}} \cap \overline{Q}_1$ and $\overline{\boldsymbol{e}} \cap \overline{Q}_2$ do not contain any corners of Ω, i.e., case 5. There are no further intersections with $\mathcal{S}$ in this case.

To sum up, our assumptions on the initial tetrahedral mesh $\mathcal{T}^0$ imply that only the five cases above may appear. This completes the proof. ☐

REMARK 3.3. *As any Lipschitz polyhedron with plane faces admits a regular triangulation* $\mathcal{T}^0$, *the above construction of a basic hexahedral mesh as well as the*

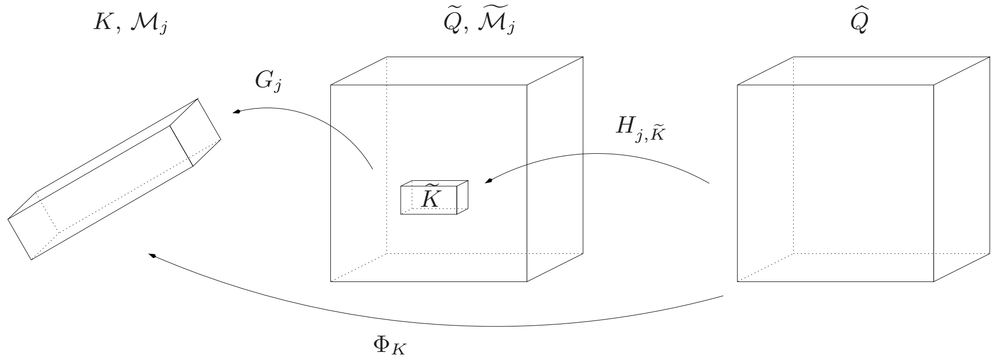

FIG. 3.2. *Element mapping Φ_K from the reference element $\widehat{Q}$ to the physical element K. The reference element $\widehat{Q}$ is first mapped to the cuboid $\widetilde{K}$ of the mesh $\widetilde{\mathcal{M}}_j$ in the reference patch $\widetilde{Q}$ using the dilation-translation $H_{j,\widetilde{K}}$ and then to K via the patch map $G_j : \widetilde{Q} \to Q_j$.*

ensuing one of hp-FE spaces will be possible in any such polyhedron. We emphasize again, however, that partitioning Ω into tetrahedra might not be necessary if a fortiori *a partition $\mathcal{M}^0$ into hexahedra satisfying Assumption 3.1 is available.*

3.2. $(\sigma, \boldsymbol{s})$-Extensions of $(\mathcal{M}^0, \boldsymbol{p}^0)$. For a refinement parameter $\sigma \in (0,1)$, sequences of σ-geometric meshes $\mathcal{M}_\sigma$ in Ω will be obtained by a sequence of *possibly anisotropic σ-subdivisions of those elements $\{Q\}_{Q\in\mathcal{M}^0}$ in the basic hexahedral mesh $\mathcal{M}^0$ which abut at the set $\mathcal{S}$*, combined with simultaneous, possibly anisotropic increase of the elemental polynomial degrees characterized by the *slope parameter* $\boldsymbol{s} > 0$. The resulting spaces of discontinuous, piecewise polynomial functions of (in local coordinates) possibly anisotropic degrees are characterized by the *mesh-degree combinations* $(\mathcal{M}, \boldsymbol{p})$. The combination of simultaneous σ-subdivision and $\boldsymbol{s}$-linear polynomial degree increase of a given mesh-degree combination will be referred to as *$(\sigma, \boldsymbol{s})$-extension*.

In the particular construction described below, all mesh refinements are performed on the *reference patch* $\widetilde{Q} = (-1,1)^3$ (to be distinguished from the *reference element* $\widehat{Q} = (-1,1)^3$). The resulting basic edge, corner and corner-edge geometric refinements in the reference patch will subsequently be mapped into appropriate subsets of the physical domain Ω. To start the construction, consider a patch $Q_j \in \mathcal{M}^0$ for some index j. We build structured geometric meshes $\mathcal{M}_j$ on patch Q_j by mapping several classes of possibly anisotropic reference subdivisions $\widetilde{\mathcal{M}}_j$ defined on the reference patch $\widetilde{Q}$. Then, an element $K \in \mathcal{M}_j$ is the image of an *axiparallel but possibly anisotropic cuboid* $\widetilde{K} \in \widetilde{\mathcal{M}}_j$ under the patch map

$$K = G_j(\widetilde{K}), \qquad \widetilde{K} \in \widetilde{Q} \tag{3.6}$$

from (3.1). For $\widetilde{K} \in \widetilde{\mathcal{M}}_j$, we can further write $\widetilde{K} = H_{j,\widetilde{K}}(\widehat{Q})$, where $H_{j,\widetilde{K}} : \widehat{Q} \to \widetilde{K}$ is a possibly anisotropic dilation combined with a translation from the reference element $\widehat{Q}$ to $\widetilde{K}$. Then, any refinement of the patch Q_j considered below will be given by

$$\mathcal{M}_j = \left\{ K \,:\, K = (G_j \circ H_{j,\widetilde{K}})(\widehat{Q}),\ \widetilde{K} \in \widetilde{\mathcal{M}}_j \right\}, \qquad j = 1, \dots, J;$$

as illustrated in Figure 3.2. The mesh $\mathcal{M}$ in Ω is the union of the J patch refinements:

$$\mathcal{M} = \bigcup_{j=1}^{J} \mathcal{M}_j. \tag{3.7}$$

For the refined meshes $\mathcal{M}$ obtained from $\mathcal{M}^0$ in this fashion, each hexahedral element $K \in \mathcal{M}$ is the image of reference cube $\widehat{Q} = (-1,1)^3$ under an element mapping Φ_K: $K = \Phi_K(\widehat{Q})$ with the following structure:

$$\Phi_K = G_{j(K)} \circ H_K : \widehat{Q} \to K \in \mathcal{M}, \tag{3.8}$$

where $G_j : \widetilde{Q} \to Q_j$, $j = 1, \ldots, J$, is the patch map, and $H_K : \widehat{Q} \to \widetilde{Q}$ is a possibly anisotropic dilation combined with a translation. In particular, Φ_K is analytic from $\overline{\widehat{Q}}$ to $\overline{K}$.

Now we store the element mappings Φ_K defined in (3.8) in the *mapping vector*

$$\boldsymbol{\Phi}(\mathcal{M}) := \{\, \Phi_K \ : \ K \in \mathcal{M} \,\}. \tag{3.9}$$

Keeping in mind the proof of exponential convergence of *hp*-finite element discretizations, we will show in [30] that the anisotropic weights appearing in the definition of $M^m_{\boldsymbol{\beta}}(\Omega)$ yield exponential convergence even with a directionally reduced, *anisotropic* choice of the elementwise polynomial degrees, in addition to geometric mesh refinement towards $\mathcal{S}$. To that end, we associate with each element $K \in \mathcal{M}$ a polynomial degree vector $\boldsymbol{p}_K = (p_{K,1}, p_{K,2}, p_{K,3}) \in \mathbb{N}_0^3$, whose components correspond to the coordinate directions in $\widehat{Q} = \Phi_K^{-1}(K)$. The polynomial degree vector $\boldsymbol{p}_K$ is called *isotropic* if $p_{K,1} = p_{K,2} = p_{K,3} = p_K$. Often, we shall also decompose $\boldsymbol{p}_K = (\boldsymbol{p}_K^\perp, p_K^\parallel)$ into a degree vector $\boldsymbol{p}_K^\perp = (p_{K,1}^\perp, p_{K,2}^\perp)$ and a scalar degree $p_K^\parallel$ perpendicular respectively parallel to an edge $\boldsymbol{e}$. We shall be mainly concerned with the situation where $p_{K,1}^\perp = p_{K,2}^\perp =: p_K^\perp$; in this case we simply write

$$\boldsymbol{p}_K = (p_K^\perp, p_K^\parallel). \tag{3.10}$$

Given a mesh $\mathcal{M}$ of hexahedral elements in Ω, we combine the elemental polynomial degrees $\boldsymbol{p}_K$ into the *polynomial degree vector*

$$\boldsymbol{p}(\mathcal{M}) := \{\, \boldsymbol{p}_K \ : \ K \in \mathcal{M} \,\}. \tag{3.11}$$

We now recall from Lemma 3.2 that there are five possibilities for each patch $Q_j \in \mathcal{M}^0$ to intersect with $\mathcal{S}$. Our error analysis below and in [30] will show that the resolution of singularities along $\mathcal{S}$ in these patches will require four different types of $(\sigma, \boldsymbol{s})$-extensions. To define them, let $\ell \geq 0$ be a refinement level, $\sigma \in (0,1)$ a grading factor and $\boldsymbol{s} > 0$ an order increment parameter.

Then, by ℓ-fold iteration, we generate four basic geometric mesh sequences $\widetilde{\mathcal{R}}_i^{(\ell)}$ on $\widetilde{Q}$, $i = 1,2,3,4$, and associated polynomial degree distributions $\boldsymbol{p}(\widetilde{\mathcal{R}}_i^{(\ell)})$ as in (3.11). For any $i = 1,2,3,4$ and $\ell \geq 0$, the polynomial degree vector $\boldsymbol{p}(\widetilde{\mathcal{R}}_i^{(\ell)})$ will be defined by means of a *co-polynomial* degree vector $\check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_i^{(\ell)})$ with non-negative real entries. More precisely, $\boldsymbol{p}(\widetilde{\mathcal{R}}_i^{(\ell)})$ is defined elementwise by

$$\boldsymbol{p}_K = \lceil \check{\boldsymbol{p}}_K \rceil \qquad \forall K \in \widetilde{\mathcal{R}}_i^{(\ell)}, \tag{3.12}$$

where $\lceil \cdot \rceil$ denotes componentwise rounding to the next greater integer, and $\check{\boldsymbol{p}}_K = (\check{p}_{K,1}, \check{p}_{K,2}, \check{p}_{K,3})$ denotes the elementwise co-polynomial degree vector (with $\check{p}_{K,j} \in \mathbb{R}$,

$\check{p}_{K,j} \geq 1,\ j = 1,2,3)$ on $K \in \widetilde{\mathcal{R}}_i^{(\ell)}$. These quantities together with the geometric mesh refinements shall be described in the sequel. The advantage of employing a co-polynomial degree vector is that they allow for non-integer order increment parameters $\boldsymbol{s}$. For $\ell = 0$, we define $\widetilde{\mathcal{R}}_1^{(0)} = \cdots = \widetilde{\mathcal{R}}_4^{(0)} = \{\widetilde{Q}\}$ and set the elemental co-polynomial degree vectors to $\check{\boldsymbol{p}}_K = (1,1,1)$ for all $K \in \widetilde{\mathcal{R}}_i^{(0)}$. In order to obtain the *hp*-extensions for refinement levels $\ell \geq 1$, we proceed iteratively:

(Ex1) *No element subdivision.* The new mesh-degree combination $(\widetilde{\mathcal{R}}_1^{(\ell)}, \check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_1^{(\ell)}))$ is obtained from $(\widetilde{\mathcal{R}}_1^{(\ell-1)}, \check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_1^{(\ell-1)}))$ by increasing the elemental co-polynomial degree vectors isotropically in each coordinate direction by $\boldsymbol{s}$. That is, on level ℓ the elemental co-polynomial degree vector $\check{\boldsymbol{p}}_{\widetilde{K}}^{(\ell)}$ on $\widetilde{K}$ is updated from the one on level $\ell - 1$ by:

$$\check{\boldsymbol{p}}_{\widetilde{K}}^{(\ell)} := \left(\check{p}_{\widetilde{K},1}^{(\ell-1)} + \boldsymbol{s}, \check{p}_{\widetilde{K},2}^{(\ell-1)} + \boldsymbol{s}, \check{p}_{\widetilde{K},3}^{(\ell-1)} + \boldsymbol{s}\right). \tag{3.13}$$

Extension (Ex1) will be used in elements located in the subdomain Ω_0 where the solution is analytic due to (2.3).

(Ex2) $(\sigma, \boldsymbol{s})$*-Extension towards a corner* $\boldsymbol{c}$. This so-called corner extension is obtained as follows: in the mesh $\widetilde{\mathcal{R}}_2^{(\ell-1)}$, there is a unique hexahedral element Q abutting at corner $\boldsymbol{c}$. The mesh-degree combination $(\widetilde{\mathcal{R}}_2^{(\ell)}, \check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_2^{(\ell)}))$ is obtained from $(\widetilde{\mathcal{R}}_2^{(\ell-1)}, \check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_2^{(\ell-1)}))$ by σ-subdividing $Q \in \widetilde{\mathcal{R}}_2^{(\ell-1)}$ towards the corner (assumed to coincide with the origin $(0,0,0)$)[4] for $\sigma \in (0,1)$: here, the set of open subdomains

$$\{(0,\sigma a),(\sigma a, a)\} \times \{\, (0,\sigma b),(\sigma b, b)\} \times \{\, (0,\sigma c),(\sigma c, c)\}$$

is called a σ-subdivision of the hexahedron $Q = (0,a) \times (0,b) \times (0,c),\ a,b,c > 0$, towards the corner $\boldsymbol{c} = (0,0,0)$. Thereby, the cube Q is split isotropically into 8 new cubes; cf. Figure 3.3 (left). The elemental co-polynomial degree vectors in each of the newly generated cubes abutting at $\boldsymbol{c}$ are set to be $(1,1,1)$. Furthermore, on the remaining cubes, the co-polynomial degrees are increased isotropically by $\boldsymbol{s}$ in each coordinate direction as described in (3.13).

(Ex3) $(\sigma, \boldsymbol{s})$*-Extension towards an edge* $\boldsymbol{e}$. Here, $(\widetilde{\mathcal{R}}_3^{(\ell)}, \check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_3^{(\ell)}))$ is obtained from $(\widetilde{\mathcal{R}}_3^{(\ell-1)}, \check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_3^{(\ell-1)}))$ by σ-subdividing the unique element $Q \in \widetilde{\mathcal{R}}_3^{(\ell-1)}$ abutting at the edge $\boldsymbol{e}$ anisotropically towards this edge $\boldsymbol{e}$ into 4 new hexahedral elements. Here, the set of open subdomains

$$\{(0,\sigma a),(\sigma a, a)\} \times \{(0,\sigma b),(\sigma b, b)\} \times \{\, (0,c)\}$$

is called a σ-subdivision of $Q = (0,a) \times (0,b) \times (0,c)$, $a,b,c > 0$ towards the edge $\boldsymbol{e} = \{\, (0,0,x_3) \,:\, 0 < x_3 < c\,\}$; three of these elements, denoted by Q_1', Q_2', Q_3', say, do not abut the edge $\boldsymbol{e}$, while the fourth one, denoted by Q'', does; cf. Figure 3.3 (center). We then define

$$\widetilde{\mathcal{R}}_3^{(\ell)} := (\widetilde{\mathcal{R}}_3^{(\ell-1)} \backslash \{Q\}) \cup \{Q_1', Q_2', Q_3', Q''\}, \qquad \ell \geq 1.$$

In the three elements Q_1', Q_2' and Q_3' not abutting edge $\boldsymbol{e}$, the co-polynomial degrees corresponding to the two directions *transversal to the edge* $\boldsymbol{e}$, denoted by $\check{\boldsymbol{p}}_{Q_i'}^{\perp}$, are set

[4]Throughout, Q denotes an open cuboid given in a local coordinate system:

$$Q = \{\, (x_1,x_2,x_2) \,:\, 0 < x_1 < a,\ 0 < x_2 < b,\ 0 < x_3 < c\,\}, \qquad a,b,c > 0.$$

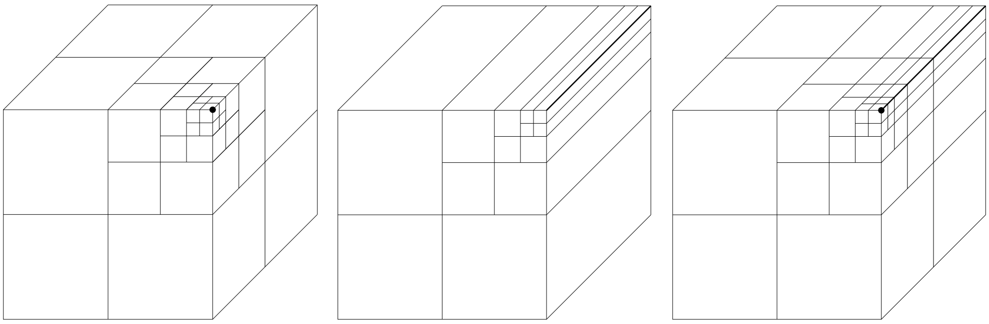

FIG. 3.3. *Examples of three basic geometric mesh subdivisions in reference patch $\widetilde{Q}$ with subdivision ratio $\sigma = \frac{1}{2}$: isotropic towards corner* $\mathbf{c}$ *(left), anisotropic towards edge* $\mathbf{e}$ *(center), and anisotropic towards one corner-edge pair* $\mathbf{ce}$ *(right). The sets* $\mathbf{c}, \mathbf{e}, \mathbf{ce}$ *are shown in boldface.*

to $(1,1)$, while the co-polynomial degree in the direction parallel to edge $\boldsymbol{e}$, denoted by $\check{p}^{\|}_{Q'_i}$, is given by the corresponding co-polynomial degree in the parent element increased by $\boldsymbol{s}$. That is, $\check{p}^{\|}_{Q'_i} := \check{p}^{\|}_{Q} + \boldsymbol{s}$. Similarly, for the element Q'', we set $\check{p}^{\|}_{Q''} := \check{p}^{\|}_{Q} + \boldsymbol{s}$ and $\check{\boldsymbol{p}}^{\perp}_{Q''} := (1,1)$. The remaining hexahedra $Q \in \widetilde{\mathcal{R}}_3^{(l-1)}$ which do not abut edge $\boldsymbol{e}$ are not subdivided, but the co-polynomial degree vectors $\check{\boldsymbol{p}}_Q$ in these elements are increased isotropically by $\boldsymbol{s}$ as described in (3.13).

(Ex4) $(\sigma, \boldsymbol{s})$-*Extension towards a corner-edge pair* $\boldsymbol{ce}$. Here, the singular corner $\boldsymbol{c}$ is contained in the closure of the singular edge $\boldsymbol{e}$, referred to as corner-edge pair $\boldsymbol{ce}$. Then, $\widetilde{\mathcal{R}}_4^{(\ell)}$ results from $\widetilde{\mathcal{R}}_4^{(l-1)}$ by σ-subdividing the mesh isotropically towards the corner as in (Ex2) and anisotropically towards the edge as in (Ex3); see Figure 3.3 (right). In addition, the co-polynomial degrees are chosen in correspondence with the previous rules for these refinements.

The mesh-degree combinations $(\widetilde{\mathcal{R}}_i^{(\ell)}, \boldsymbol{p}(\widetilde{\mathcal{R}}_i^{(\ell)}))$ which will be used for the construction of the *hp*-finite element spaces for the numerical approximation of (1.1)–(1.2) are now obtained from the sequences of mesh patches $\widetilde{\mathcal{R}}_i^{(\ell)}$ and corresponding (co-) polynomial degree vectors $\check{\boldsymbol{p}}(\widetilde{\mathcal{R}}_i^{(\ell)})$ resulting from the $(\sigma, \boldsymbol{s})$-extensions (Ex1)–(Ex2), where $\boldsymbol{p}(\widetilde{\mathcal{R}}_i^{(\ell)})$ is defined by (3.12).

REMARK 3.4. *Elements $Q \in \widetilde{\mathcal{R}}_3^{(\ell)}$ or $Q \in \widetilde{\mathcal{R}}_4^{(\ell)}$ are hexahedra with axiparallel faces whose aspect ratio increases exponentially as $\ell \to \infty$.*[5] *Some care is therefore required when speaking about the mesh width h_Q for elements $Q \in \widetilde{\mathcal{R}}_3^{(\ell)} \cup \widetilde{\mathcal{R}}_4^{(\ell)}$: we introduce for elements $Q \in \widetilde{\mathcal{R}}_3^{(\ell)} \cup \widetilde{\mathcal{R}}_4^{(\ell)}$ the* diameters $h^{\|}_Q$ and $h^{\perp}_Q$ of Q parallel respectively perpendicular to the nearest singular edge $\boldsymbol{e}$ on $\widetilde{K}$. *If $K = G_j(Q)$ under a patch map G_j as in* (3.6)*, we define the quantities $h^{\|}_K$ and $h^{\perp}_K$ as $h^{\|}_Q$ and $h^{\perp}_Q$, respectively. In addition, we denote by h_K the diameter of element K.*

3.3. σ-Geometric Mesh Families $\mathfrak{M}_\sigma$ in Ω. Using the basic $(\sigma, \boldsymbol{s})$-extensions in the reference patch and the four mesh-degree combinations $(\widetilde{\mathcal{R}}_i^{(\ell)}, \boldsymbol{p}(\widetilde{\mathcal{R}}_i^{(\ell)}))_{\ell \geq 0}$ obtained from them, corresponding mesh-degree combinations are obtained on the polyhedron Ω by proper combination of these four basic combinations in the (hexahedral) basic patches $Q_j \in \mathcal{M}^0$. More precisely, the resulting geometric mesh in Ω is obtained

[5]As our *hp*-convergence analysis in [30] will show, this loss of elemental shape-regularity is necessary to achieve exponential convergence.

by transporting the corresponding geometric subdivisions on $\widetilde{Q}$ into Ω by means of the patch mappings G_j. It will be referred to as (hexahedral) *σ-geometric mesh*.

Definition 3.5 (σ-Geometric mesh $\mathcal{M}_\sigma^{(\ell)}$). *Consider a refinement level $\ell \geq 0$. Then, a σ-geometric mesh $\mathcal{M}_\sigma^{(\ell)}$ with grading factor $\sigma \in (0,1)$ and with ℓ layers on Ω is defined by mapping finitely many copies of the reference σ-geometric meshes $\widetilde{\mathcal{R}}_i^{(\ell)}$, $i = 1,2,3,4$, on $\widetilde{Q}$ onto the basic hexahedral mesh $\mathcal{M}^0 = \{Q_j\}_{j=1}^J$ in Ω by applying the patch mappings G_j. More precisely, the reference mesh combinations are chosen as follows (cf. Lemma 3.2): If the intersection $\overline{Q}_j \cap \mathcal{S}$ is*

1. *empty, then the (single element) mesh $\widetilde{\mathcal{R}}_1^{(\ell)}$ is mapped;*
2. *exactly one corner $\boldsymbol{c}$ of Ω, then $\widetilde{\mathcal{R}}_2^{(\ell)}$ (with isotropic geometric refinement towards $\boldsymbol{c}$, cf. Figure 3.3 (left)) is mapped;*
3. *a closed subset of an edge $\mathbf{e} \in \mathcal{E}$ of Ω not containing any corner $\boldsymbol{c}$ of Ω, then $\widetilde{\mathcal{R}}_3^{(\ell)}$ is mapped (with anisotropic refinement towards the edge, cf. Figure 3.3 (middle));*
4. *a closed subset of an edge $\mathbf{e} \in \mathcal{E}$ of Ω containing exactly one corner $\boldsymbol{c}$ of Ω, then $\widetilde{\mathcal{R}}_4^{(\ell)}$ is mapped (with refinement towards both, corner $\boldsymbol{c}$ and edge $\mathbf{e}$; cf. Figure 3.3 (right));*
5. *exactly one point on an edge $\mathbf{e} \in \mathcal{E}$ of Ω (which is not a corner of Ω), then $\widetilde{\mathcal{R}}_1^{(\ell)}$ is mapped (with σ-geometric refinement towards this intersection point).*

Remark 3.6. *Note that the reference configurations on $\widetilde{Q}$ need to be suitably oriented before being mapped to $\mathcal{M}^0$.*

In the sequel, we shall be working with sequences of σ-geometrically refined meshes $\mathcal{M}_\sigma^{(0)}, \mathcal{M}_\sigma^{(1)}, \mathcal{M}_\sigma^{(2)}, \ldots,$ where $\mathcal{M}_\sigma^{(0)} := \mathcal{M}^0$, and, for $\ell \geq 1$, there holds: if $K \in \mathcal{M}_\sigma^{(\ell)}$, then there exists $K' \in \mathcal{M}_\sigma^{(\ell-1)}$ such that $K \subset K'$. As before, we shall refer to the index ℓ as *refinement level* and to the sequence $\{\mathcal{M}_\sigma^{(\ell)}\}_{\ell\geq 1}$ of σ-geometric meshes in Ω as *σ-geometric mesh family*. It will be denoted by $\mathfrak{M}_\sigma$.

3.4. Mesh Layers. It will be convenient to partition the σ-geometric mesh family $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell\geq 1}$ defined above into certain subsets of elements with identical scaling properties in terms of their relative distance to the sets $\mathcal{C}$ and $\mathcal{E}$. To this end, we use the concept of *mesh layers* introduced in [8, Section 3]. The geometric mesh families $\mathfrak{M}_\sigma$ defined by the construction above admit such a decomposition into layers.

Proposition 3.7. *Any σ-geometric mesh family $\mathfrak{M}_\sigma$ obtained by iterating the basic extensions (Ex1)–(Ex4) above can be partitioned into a countable sequence of disjoint mesh layers $\mathfrak{L}_\sigma^\ell$, $\ell \geq 0$, and a corresponding nested sequence of terminal layers $\mathfrak{T}_\sigma^\ell$, $\ell = 1,2,3,\ldots$ such that each $\mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma$, $\ell \geq 1$ can be written as*

$$\mathcal{M}_\sigma^{(\ell)} = \mathfrak{L}_\sigma^0 \,\dot\cup\, \mathfrak{L}_\sigma^1 \,\dot\cup \ldots \dot\cup\, \mathfrak{L}_\sigma^{\ell-1} \,\dot\cup\, \mathfrak{T}_\sigma^\ell. \tag{3.14}$$

Proof. We proceed by induction with respect to the number $\ell \geq 1$ of layers. If $\ell = 1$, the assertion follows with the observation that $\mathcal{M}_\sigma^{(1)}$ is obtained by one σ-geometric refinement of the initial mesh $\mathcal{M}^0$ of hexahedra K. We partition $\mathcal{M}_\sigma^{(1)}$ into elements K for which $\overline{K} \cap (\mathcal{E} \cup \mathcal{C}) = \emptyset$; these constitute $\mathfrak{L}_\sigma^0$, and collect the remaining elements $K \in \mathcal{M}_\sigma^{(1)}$ in $\mathfrak{T}_\sigma^1$. Then $\mathcal{M}_\sigma^{(0)} = \mathfrak{L}_\sigma^0 \,\dot\cup\, \mathfrak{T}_\sigma^1$ satisfies (3.14) for $\ell = 1$.

Next, assume that the assertion has been established for some $\ell_0 \geq 1$. Then, for $\ell = \ell_0 + 1$, all elements K not belonging to $\mathfrak{T}_\sigma^{\ell_0}$ are left unaltered. All elements in the terminal layer $\mathfrak{T}_\sigma^{\ell_0}$ are σ-subdivided once towards the singular points in $\mathcal{E} \cup \mathcal{C}$. The elements K resulting from this σ-subdivision with $\overline{K} \cap (\mathcal{E} \cup \mathcal{C}) = \emptyset$ are collected in the new mesh layer $\mathfrak{L}_\sigma^{\ell_0}$, the remaining elements K resulting from the σ-subdivision of $Q \in \mathfrak{T}_\sigma^{\ell_0}$ constitute $\mathfrak{T}_\sigma^{\ell_0+1}$. Then, by construction, $\mathcal{M}_\sigma^{(\ell_0+1)} = \mathfrak{L}_\sigma^0 \,\dot\cup\, \mathfrak{L}_\sigma^1 \,\dot\cup \dots \dot\cup\, \mathfrak{L}_\sigma^{\ell_0} \,\dot\cup\, \mathfrak{T}_\sigma^{\ell_0+1}$ which proves (3.14) for all $\ell \geq 1$. $\square$

We shall require notation for those elements of the geometric mesh $\mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma$ which are not abutting at a corner or an edge. To this end, we define with the interior mesh layers $\mathfrak{L}_\sigma^l$ the submesh $\mathfrak{O}_\sigma^\ell \subset \mathcal{M}_\sigma^{(\ell)}$ of elements $K \in \mathcal{M}_\sigma^{(\ell)}$ not intersecting $\mathcal{C} \cup \mathcal{E}$:

$$\mathfrak{O}_\sigma^\ell := \mathfrak{L}_\sigma^0 \,\dot\cup\, \mathfrak{L}_\sigma^1 \,\dot\cup \dots \dot\cup\, \mathfrak{L}_\sigma^{\ell-1} \subset \mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma, \qquad \ell \geq 1. \tag{3.15}$$

Evidently, $\mathcal{M}_\sigma^{(\ell)} = \mathfrak{O}_\sigma^\ell \cup \mathfrak{T}_\sigma^\ell$ for $\ell \geq 1$.

We partition the terminal layer $\mathfrak{T}_\sigma^\ell$ according to the construction of the basic hexahedral mesh $\mathcal{M}^0$, cf. Lemma 3.2 and Definition 3.5:

$$\begin{aligned}
\mathcal{V}_{\mathcal{C}}^\ell &= \left\{ K \in \mathfrak{T}_\sigma^\ell : \overline{K} \cap \mathcal{S} = \boldsymbol{c} \text{ for some } \boldsymbol{c} \in \mathcal{C} \right\}, \\
\mathcal{V}_{\mathcal{E},1}^\ell &= \left\{ K \in \mathfrak{T}_\sigma^\ell : \overline{K} \cap \mathcal{C} = \emptyset \text{ and } \exists \boldsymbol{e} \in \mathcal{E} \text{ such that } \overline{K} \cap \mathcal{S} \subset \overline{\boldsymbol{e}} \text{ is a corner of } K \right\}, \\
\mathcal{V}_{\mathcal{E},2}^\ell &= \left\{ K \in \mathfrak{T}_\sigma^\ell : \overline{K} \cap \mathcal{C} = \emptyset \text{ and } \exists \boldsymbol{e} \in \mathcal{E} \text{ such that } \overline{K} \cap \mathcal{S} \subset \overline{\boldsymbol{e}} \text{ is an edge of } K \right\}, \\
\mathcal{V}_{\mathcal{C}\mathcal{E}}^\ell &= \left\{ K \in \mathfrak{T}_\sigma^\ell : \exists \boldsymbol{c} \in \mathcal{C} \text{ and } \boldsymbol{e} \in \mathcal{E}_{\boldsymbol{c}} \text{ such that } \boldsymbol{c} \in \overline{K} \cap \mathcal{S} \subset \overline{\boldsymbol{e}} \right\}.
\end{aligned}$$

REMARK 3.8. *We notice that elements in $\mathcal{V}_{\mathcal{C}}^\ell$, $\mathcal{V}_{\mathcal{E},1}^\ell$ and $\mathcal{V}_{\mathcal{C}\mathcal{E}}^\ell$ are isotropic with element diameter denoted by $h_K \simeq h_K^\perp \simeq h_K^\parallel$. Furthermore, elements in $\mathcal{V}_{\mathcal{E},2}^\ell$ might be anisotropic.*

3.5. Finite Element Spaces. We are now ready to introduce the *hp*-version discontinuous Galerkin finite element spaces. To this end, let $\mathcal{M}$ be a geometric mesh of a σ-geometric mesh family $\mathfrak{M}_\sigma$ in Ω. Let $\boldsymbol{\Phi}(\mathcal{M})$ and $\boldsymbol{p}(\mathcal{M})$ be the associated mapping and polynomial degree vectors, as introduced in (3.9) and (3.11), respectively. We then introduce the finite element space

$$V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p}) = \left\{ u \in L^2(\Omega) \,:\, u|_K \in \mathbb{Q}^{\boldsymbol{p}_K}(K),\ K \in \mathcal{M} \right\}. \tag{3.16}$$

Here, we define the local polynomial approximation space $\mathbb{Q}^{\boldsymbol{p}_K}(K)$ as follows: first, on the reference element $\widehat{Q}$ and for a polynomial degree vector $\boldsymbol{p} = (p_1, p_2, p_3) \in \mathbb{N}_0^3$, we introduce the following, anisotropic polynomial space:

$$\mathbb{Q}^{\boldsymbol{p}}(\widehat{Q}) = \mathbb{P}^{p_1}(\widehat{I}) \otimes \mathbb{P}^{p_2}(\widehat{I}) \otimes \mathbb{P}^{p_3}(\widehat{I}) = \operatorname{span}\left\{ \hat{\boldsymbol{x}}^{\boldsymbol{\alpha}} \,:\, \alpha_i \leq p_i,\ 1 \leq i \leq 3 \right\}. \tag{3.17}$$

Here, for $p \in \mathbb{N}_0$, we denote by $\widehat{\mathbb{P}}^p(\widehat{I})$ the space of all polynomials of degree at most p on the reference interval $\widehat{I} = (-1, 1)$. Then, if K is a hexahedral element of $\mathcal{M}$ with associated elemental mapping $\Phi_K : \widehat{Q} \to K$ and polynomial degree vector $\boldsymbol{p}_K = (p_{K,1}, p_{K,2}, p_{K,3})$, we define

$$\mathbb{Q}^{\boldsymbol{p}_K}(K) = \left\{ u \in L^2(K) : (u|_K \circ \Phi_K) \in \mathbb{Q}^{\boldsymbol{p}_K}(\widehat{Q}) \right\}. \tag{3.18}$$

In the case, where the polynomial degree vector $\boldsymbol{p}_K$ associated with K is isotropic, i.e., $p_{K,1} = p_{K,2} = p_{K,3} = p_K$, we simply write $\mathbb{Q}^{\boldsymbol{p}_K}(K) = \mathbb{Q}^{p_K}(K)$, i.e., we replace the vector $\boldsymbol{p}_K$ by the scalar p_K.

We now introduce two families of *hp*-finite element spaces for the discontinuous Galerkin methods; both yield exponentially convergent approximations and are based on the σ-geometric mesh families $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell\geq 1}$. The first family has uniform polynomial degree distributions, while the second (smaller) family will have linearly increasing polynomial degree vectors. The *first family of hp-dG subspaces* is defined by

$$V_\sigma^\ell := V(\mathcal{M}_\sigma^{(\ell)}, \mathbf{\Phi}(\mathcal{M}_\sigma^{(\ell)}), \boldsymbol{p}_1(\mathcal{M}_\sigma^{(\ell)})), \qquad \ell \geq 1, \tag{3.19}$$

where the elemental polynomial degree vectors p_K in $\boldsymbol{p}_1(\mathcal{M}_\sigma^{(\ell)})$ are isotropic and uniform, given on each element K as

$$p_K = \ell \qquad \forall\, K \in \mathcal{M}_\sigma^{(\ell)}. \tag{3.20}$$

The *second family of hp-dG subspaces* is chosen as

$$V_{\sigma,\boldsymbol{s}}^\ell := V(\mathcal{M}_\sigma^{(\ell)}, \mathbf{\Phi}(\mathcal{M}_\sigma^{(\ell)}), \boldsymbol{p}_2(\mathcal{M}_\sigma^{(\ell)})), \qquad \ell \geq 1, \tag{3.21}$$

for an increment parameter $\boldsymbol{s} > 0$. Here the polynomial degree vectors $\boldsymbol{p}_2(\mathcal{M}_\sigma^{(\ell)})$ are the anisotropic ones obtained from the $(\sigma, \boldsymbol{s})$-extensions described in Section 3.2. Note that both families of *hp*-spaces defined above are nested: for $0 < \sigma < 1$ and $\boldsymbol{s} = 1$, we have

$$V_{\sigma,1}^\ell \subset V_\sigma^\ell \qquad \text{and} \qquad V_{\sigma,1}^{\ell-1} \subset V_{\sigma,1}^\ell, \qquad \ell \geq 1. \tag{3.22}$$

Our *hp*-dG analysis will require certain uniformity conditions for the vectors $\mathbf{\Phi}(\mathcal{M})$ and $\boldsymbol{p}_2(\mathcal{M})$ of element mappings and element polynomial degrees, respectively. These will be specified next.

3.6. Degree Vectors of μ-Bounded Variation. From the analysis of *hp*-FEM in the one-dimensional setting in [11] it is known that exponential convergence for singular solutions can be achieved upon combination of σ-geometric mesh refinement with either *uniform polynomial degree increase* or with *$\boldsymbol{s}$-linear polynomial degree increase, i.e., the polynomial degrees within each element increase linearly with the number of mesh layers between that element and the component of the singular set $\mathcal{S}$ nearest to it*, with the factor of proportionality ("slope" in the terminology of [11]) being the parameter $\boldsymbol{s} > 0$.

In the three-dimensional situation under consideration here, an anisotropic version of this concept, namely *polynomial degree vectors $\boldsymbol{p}(\mathcal{M})$ of μ-bounded variation on meshes $\mathcal{M} \in \mathfrak{M}_\sigma$* will be used. To define it, for any $\mathcal{M} \in \mathfrak{M}_\sigma$, we denote the set of all interior faces in $\mathcal{M}$ by

$$\mathcal{F}_{\mathcal{I}}(\mathcal{M}) = \left\{ (\partial K_\sharp \cap \partial K_\flat)^\circ \,:\, K_\sharp, K_\flat \in \mathcal{M},\ \partial K_\sharp \cap \partial K_\flat \neq \emptyset \right\},$$

and, similarly, the set of all boundary faces by $\mathcal{F}_{\mathcal{B}}(\mathcal{M})$. In addition, let $\mathcal{F}(\mathcal{M}) = \mathcal{F}_{\mathcal{I}}(\mathcal{M}) \cup \mathcal{F}_{\mathcal{B}}(\mathcal{M})$ denote the set of all (smallest) faces of $\mathcal{M}$. If the mesh $\mathcal{M} \in \mathfrak{M}_\sigma$ is clear from the context, we shall omit the dependence of these sets on $\mathcal{M}$. Furthermore, for an element $K \in \mathcal{M}$, we denote the set of its faces by $\mathcal{F}_K = \{ f \in \mathcal{F} : f \subset \partial K \}$. In local coordinates on $K \in \mathcal{M}$, *hp*-dG solutions are tensor products of univariate polynomials of possibly anisotropic polynomial degrees which are collected in the vector $\boldsymbol{p}_K$, cf. (3.18).

The notion of μ-bounded degree vectors pertains to changes in polynomial degrees across faces $f \in \mathcal{F}(\mathcal{M})$. To precise this, we denote for any element $K \in \mathcal{M}$ and any $f \in \mathcal{F}_K$, by $p_{K,f}^{\parallel,(1)}$, $p_{K,f}^{\parallel,(2)}$ the two components of $\boldsymbol{p}_K$ parallel to f, and by $p_{K,f}^{\perp}$ the polynomial degree of $\boldsymbol{p}_K$ transversal to f (defined as the corresponding components on $\Phi_K^{-1}(K)$, cf. (3.18)). *A degree vector $\boldsymbol{p}(\mathcal{M})$ is said to be of μ-bounded variation if there is a constant $\mu \in (0,1)$ such that*

$$\mu \leq p^{\perp}_{K_\sharp,f} / p^{\perp}_{K_\flat,f} \leq \mu^{-1}, \tag{3.23}$$

uniformly for all interior faces $f = (\partial K_\sharp \cap \partial K_\flat)^\circ \in \mathcal{F}_{\mathcal{I}}(\mathcal{M})$. A family of degree vectors is of μ-bounded variation if each vector in the family is of μ-bounded variation (uniformly for the entire family).

REMARK 3.9. *For a mesh $\mathcal{M}$ in a σ-geometric family $\mathfrak{M}_\sigma$ obtained by a sequence of $(\sigma, \boldsymbol{s})$-extensions, the number of faces $f \in \mathcal{F}_K$ contained in the boundary of any element $K \in \mathcal{M}$ is uniformly bounded, i.e., there exists a constant $C(\mathfrak{M}_\sigma) < \infty$ such that $\sup_{\mathcal{M}\in\mathfrak{M}_\sigma} \sup_{K\in\mathcal{M}} |\mathcal{F}_K| \leq C$. Also, the degree vectors in the sequences of mesh-degree combinations produced by the $(\sigma, \boldsymbol{s})$-extensions (Ex1)–(Ex4) are of μ-bounded variation. Moreover, the elemental polynomial degrees in the corresponding degree vectors are $\boldsymbol{s}$-linearly increasing in the mesh layers between elements $K \in \mathcal{M}$, away from the singular support $\mathcal{S}$ of the solution u of (1.1)–(1.2).*

4. Discontinuous Galerkin Discretization.

We present the hp-dG discretizations of (1.1)–(1.2) for which we establish stability and error bounds on $\mathfrak{M}_\sigma$.

4.1. Face Operators.

In order to define a dG formulation on a given mesh $\mathcal{M}$ for the model problem (1.1)–(1.2), we shall first recall some element face operators. For this purpose, consider an interior face $f = (\partial K_\sharp \cap \partial K_\flat)^\circ \in \mathcal{F}_{\mathcal{I}}(\mathcal{M})$ shared by two elements $K_\sharp$ and $K_\flat$ in $\mathcal{M}$; cf. Section 3.6. Furthermore, let $v, \boldsymbol{w}$ be a scalar- respectively a vector-valued function that is sufficiently smooth inside the elements $K_\sharp, K_\flat$. Then we define the following jumps and averages of v and $\boldsymbol{w}$ along f:

$$[\![v]\!] = v|_{K_\sharp}\boldsymbol{n}_{K_\sharp} + v|_{K_\flat}\boldsymbol{n}_{K_\flat} \qquad \langle\!\langle v\rangle\!\rangle = \frac{1}{2}\left(v|_{K_\sharp} + v|_{K_\flat}\right)$$
$$[\![\boldsymbol{w}]\!] = \boldsymbol{w}|_{K_\sharp}\cdot\boldsymbol{n}_{K_\sharp} + \boldsymbol{w}|_{K_\flat}\cdot\boldsymbol{n}_{K_\flat} \qquad \langle\!\langle \boldsymbol{w}\rangle\!\rangle = \frac{1}{2}\left(\boldsymbol{w}|_{K_\sharp} + \boldsymbol{w}|_{K_\flat}\right).$$

Here, for an element $K \in \mathcal{M}$, we denote by $\boldsymbol{n}_K$ the outward unit normal vector on ∂K. For a boundary face $f = (\partial K \cap \partial\Omega)^\circ \in \mathcal{F}_{\mathcal{B}}(\mathcal{M})$ for $K \in \mathcal{M}$, and sufficiently smooth functions $v, \boldsymbol{w}$ on K, we let $[\![v]\!] = v|_K\boldsymbol{n}_\Omega$, $[\![\boldsymbol{w}]\!] = \boldsymbol{w}|_K\cdot\boldsymbol{n}_\Omega$, and $\langle\!\langle v\rangle\!\rangle = v|_K$, $\langle\!\langle \boldsymbol{w}\rangle\!\rangle = \boldsymbol{w}|_K$, where $\boldsymbol{n}_\Omega$ is the outward unit normal vector on $\partial\Omega$.

4.2. Interior Penalty Discretizations.

The problem (1.1)–(1.2) will be discretized using an interior penalty (IP) discontinuous Galerkin method. More precisely, we consider the subspaces $V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p}) = V^\ell_\sigma$ respectively $V^\ell_{\sigma,\boldsymbol{s}}$ defined in (3.19), (3.21) with a σ-geometric mesh $\mathcal{M} \in \mathfrak{M}_\sigma$ and an increment parameter $\boldsymbol{s} > 0$. For a fixed parameter $\theta \in [-1,1]$, we define the hp-discontinuous Galerkin solution u_{DG} by

$$u_{\mathrm{DG}} \in V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p}): \qquad a_{\mathrm{DG}}(u_{\mathrm{DG}}, v) = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} \qquad \forall\, v \in V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p}), \tag{4.1}$$

where the form $a_{\mathrm{DG}}(u,v)$ is given by

$$a_{\mathrm{DG}}(w,v) = A_{\mathrm{DG}}(w,v) - F_{\mathrm{DG}}(w,v) + \theta F_{\mathrm{DG}}(v,w) + \gamma J_{\mathrm{DG}}(v,w), \tag{4.2}$$

with

$$A_{\rm DG}(w,v)=\int_\Omega((\boldsymbol{A}\nabla_h w)\cdot\nabla_h v+cwv)\,{\rm d}\boldsymbol{x},\quad F_{\rm DG}(w,v)=\int_{\mathcal{F}(\mathcal{M})}\langle\!\langle\boldsymbol{A}\nabla_h w\rangle\!\rangle\cdot[\![v]\!]\,{\rm d}s,$$
$$J_{\rm DG}(w,v)=\int_{\mathcal{F}(\mathcal{M})}\alpha[\![w]\!]\cdot[\![v]\!]\,{\rm d}s.$$

Here, ∇_h is the elementwise gradient, and $\gamma>0$ is a stabilization parameter that will be specified later. Furthermore, $\alpha\in L^\infty(\mathcal{F})$ is a discontinuity stabilization function which is defined as follows:

$$\alpha(\boldsymbol{x})=\begin{cases}\dfrac{\max\left(p^\perp_{K_\sharp,f},p^\perp_{K_\flat,f}\right)^2}{\min\left(h^\perp_{K_\sharp,f},h^\perp_{K_\flat,f}\right)} & \text{if } \boldsymbol{x}\in f=(\partial K_\sharp\cap\partial K_\flat)^\circ\in\mathcal{F}_\mathcal{I} \\ & \text{for } K_\sharp,K_\flat\in\mathcal{M}, \\ \dfrac{(p^\perp_{K,f})^2}{h^\perp_{K,f}} & \text{if } \boldsymbol{x}\in f=(\partial K\cap\partial\Omega)^\circ\in\mathcal{F}_\mathcal{B} \text{ for } K\in\mathcal{M}.\end{cases}\tag{4.3}$$

We recall from Section 3.6 that, for $K\in\mathcal{M}$ and $f\in\mathcal{F}_K$, we denote by $p^\perp_{K,f}$ the component of $\boldsymbol{p}_K$ in the direction transversal to f (defined as the corresponding component on $\Phi_K^{-1}(K)$; cf. (3.18)). Analogously, $h^\perp_{K,f}$ is the height of K over the face f, i.e., the diameter of element K in the direction transversal to f, defined as the corresponding quantity on the axiparallel cuboid $G_j^{-1}(K)$; see (3.6). The parameter θ allows us to describe a whole range of interior penalty methods: for $\theta=-1$ we obtain the standard symmetric interior penalty (SIP) method while for $\theta=1$ the non-symmetric (NIP) version is obtained; cf. [2] and the references therein.

REMARK 4.1. *The stabilization term $\alpha(\boldsymbol{x})$ is often chosen in dependence on $\boldsymbol{A}$. For the sake of simplicity, we will not consider this choice. As a result, the stabilization parameter γ will have to be selected as a function of (the upper and lower bounds of) $\boldsymbol{A}$; see also the coercivity result in Theorem 4.4 below.*

4.3. Trace and Inverse Inequalities. In order to analyze the numerical fluxes in the dG formulation, we shall require some inequalities on the faces of elements.

We begin by proving an anisotropic trace inequality. Recall from (3.6) that an element $K\in\mathfrak{M}_\sigma$ belonging to patch Q_j is the image of an axiparallel cuboid $\widetilde{K}$ via the patch map $G_j:\widetilde{Q}\to Q_j$, as in (3.6). Similarly, we denote the preimage of a face $f\in\mathcal{F}_K$ by $\widetilde{f}\in\mathcal{F}_{\widetilde{K}}$. Functions and gradients are then transformed from K to $\widetilde{K}$ via the patch maps according to

$$\widetilde{v}(\widetilde{\boldsymbol{x}})=v(\boldsymbol{x}),\qquad\widetilde{\nabla}\widetilde{v}(\widetilde{\boldsymbol{x}})=\mathsf{D}G_j^\top(\boldsymbol{x})\nabla v(\boldsymbol{x}),\tag{4.4}$$

where $\boldsymbol{x}=G_j(\widetilde{\boldsymbol{x}})$. Hence, from (3.4) and (3.5), we have

$$\|v\|_{L^t(f)}\simeq\|\widetilde{v}\|_{L^t(\widetilde{f})},\qquad\|\nabla v\|_{L^t(f)}\simeq\left\|\widetilde{\nabla}\widetilde{v}\right\|_{L^t(\widetilde{f})},\tag{4.5}$$

and

$$|\widetilde{v}|_{W^{1,t}(\widetilde{K})}\simeq|v|_{W^{1,t}(\widetilde{K})},\qquad|\widetilde{v}|_{W^{2,t}(\widetilde{K})}\simeq|v|_{W^{2,t}(K)},\tag{4.6}$$

for any $t\geq1$.

Lemma 4.2. *Let $\mathfrak{M}_\sigma$ be a geometric mesh family with $0 < \sigma < 1$, $\mathcal{M} \in \mathfrak{M}_\sigma$, $K \in \mathcal{M}$, $f \in \mathcal{F}_K$, and $t \geq 1$. Then for any $v \in W^{1,t}(K)$, we have the trace inequality*

$$\|v\|_{L^t(f)}^t \leq C_t \left(h_{K,f}^\perp\right)^{-1} \left(\|v\|_{L^t(K)}^t + \left(h_{K,f}^\perp\right)^t \left\|\widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp}(\widetilde{v})\right\|_{L^t(\widetilde{K})}^t\right).$$

The constant $C_t > 0$ depends on σ and the constant $C_{\mathcal{M}^0}$ in (3.4)–(3.5), *but is independent of the element size and element aspect ratio. Here, $\widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp}(\widetilde{v})$ signifies the partial derivative of $\widetilde{v}$ in direction transversal to $\widetilde{f} \in \mathcal{F}_{\widetilde{K}}$, expressed in the coordinates on the reference patch $\widetilde{Q}$.*

Proof. We first assume that K is an axiparallel cuboid. Due to density, we may further assume that $v \in C^\infty(\overline{K})$. Then, we introduce a local coordinate system in K such that all points in f satisfy $x_3 = 0$ and the x_3-axis is orthogonal to f. We have $h_3 = h_{K,f}^\perp$. Then, for $(x_1, x_2, 0) \in f$,

$$\begin{aligned}|v(x_1,x_2,0)| &= \left|\int_0^{h_3} \partial_{x_3}\left(\frac{x_3 - h_3}{h_3} v\right) \mathrm{d}x_3\right| = \left|\int_0^{h_3} \left(h_3^{-1} v + \frac{x_3 - h_3}{h_3}\partial_{x_3} v\right) \mathrm{d}x_3\right| \\ &\leq h_3^{-1}\int_0^{h_3} |v| \,\mathrm{d}x_3 + \int_0^{h_3} \left|\frac{x_3 - h_3}{h_3}\partial_{x_3} v\right| \mathrm{d}x_3.\end{aligned}$$

Then, using that $(|a| + |b|)^t \leq 2^{t-1}(|a|^t + |b|^t)$ for any $a, b \in \mathbb{R}$, it follows that

$$|v(x_1,x_2,0)|^t \leq 2^{t-1}\left(h_3^{-t}\left(\int_0^{h_3} |v| \,\mathrm{d}x_3\right)^t + \left(\int_0^{h_3} \left|\frac{x_3 - h_3}{h_3}\partial_{x_3} v\right| \mathrm{d}x_3\right)^t\right).$$

Applying Hölder's inequality, we have that

$$\int_0^{h_3} |v| \,\mathrm{d}x_3 \leq h_3^{\frac{t-1}{t}} \left(\int_0^{h_3} |v|^t \,\mathrm{d}x_3\right)^{\frac{1}{t}},$$

and

$$\begin{aligned}\int_0^{h_3} \left|\frac{x_3 - h_3}{h_3}\partial_{x_3} v\right| \mathrm{d}x_3 &\leq \left(\int_0^{h_3} \left|\frac{x_3 - h_3}{h_3}\right|^{\frac{t}{t-1}} \mathrm{d}x_3\right)^{\frac{t-1}{t}} \left(\int_0^{h_3} |\partial_{x_3} v|^t \,\mathrm{d}x_3\right)^{\frac{1}{t}} \\ &\leq C_t' h_3^{\frac{t-1}{t}} \left(\int_0^{h_3} |\partial_{x_3} v|^t \,\mathrm{d}x_3\right)^{\frac{1}{s}},\end{aligned}$$

with a constant $C_t' > 0$ only depending on t. Therefore,

$$|v(x_1,x_2,0)|^t \leq C_t h_3^{-1}\left(\int_0^{h_3} |v|^t \,\mathrm{d}x_3 + h_3^t \int_0^{h_3} |\partial_{x_3} v|^t \,\mathrm{d}x_3\right),$$

and thus,

$$\|v\|_{L^t(f)}^t = \int_f |v(x_1,x_2,0)|^t \,\mathrm{d}x_1\,\mathrm{d}x_2 \leq C_t h_3^{-1} \int_K \left(|v|^t + h_3^t |\partial_{x_3} v|^t\right) \mathrm{d}\boldsymbol{x}.$$

This shows the assertion for an axiparallel cuboid.

If now $K = G_j(\widetilde{K})$ for a cuboid $\widetilde{K}$, $f \in \mathcal{F}_K$ and $\widetilde{f} \in \mathcal{F}_{\widetilde{K}}$ the corresponding face on $\widetilde{K}$, then the equivalence properties in (4.5) and the result on $\widetilde{K}$ give

$$\|v\|^t_{L^t(f)} \simeq \|\widetilde{v}\|^t_{L^t(\widetilde{f})} \lesssim (h^{\perp}_{\widetilde{K},\widetilde{f}})^{-1} \left(\|\widetilde{v}\|^t_{L^t(\widetilde{K})} + (h^{\perp}_{\widetilde{K},\widetilde{f}})^t \left\| \widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp}(\widetilde{v}) \right\|^t_{L^t(\widetilde{K})} \right).$$

Noting that $h^{\perp}_{\widetilde{K},\widetilde{f}} = h^{\perp}_{K,f}$ by Remark 3.4, and scaling back the L^t-norm from $\widetilde{K}$ to K by using (4.6) yield the desired inequality on K. To bound gradients on faces, we proceed similarly. Scaling using (4.5) and applying Lemma 4.2 on the axiparallel element $\widetilde{K}$ result in

$$\|\nabla v\|_{L^t(f)} \simeq \left\| \widetilde{\nabla}\widetilde{v} \right\|_{L^t(\widetilde{f})} \lesssim (h^{\perp}_{K,f})^{-1} \left(\left\| \widetilde{\nabla}\widetilde{v} \right\|^t_{L^t(\widetilde{K})} + (h^{\perp}_{K,f})^t \left\| \widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp}(\widetilde{\nabla}\widetilde{v}) \right\|^t_{L^t(\widetilde{K})} \right).$$

Scaling back the L^t-norm of the gradient with (4.6) implies the following bound:

$$\|\nabla v\|^t_{L^t(f)} \leq C(h^{\perp}_{K,f})^{-1} \left(\|\nabla v\|^t_{L^t(K)} + (h^{\perp}_{K,f})^t \left\| \widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp}(\widetilde{\nabla}\widetilde{v}) \right\|^t_{L^t(\widetilde{K})} \right). \tag{4.7}$$

Moreover, if we scale back from $\widetilde{K}$ to K the second-order derivative on the right-hand side of (4.7), we obtain

$$\|\nabla v\|^t_{L^t(f)} \leq C(h^{\perp}_{K,f})^{-1} \left(\|\nabla v\|^t_{L^t(K)} + (h^{\perp}_{K,f})^t |v|^t_{W^{2,t}(K)} \right). \tag{4.8}$$

This completes the proof. ∎

Next, we establish various inequalities for discrete functions on faces. To that end, we denote by $|f|$ the surface measure of a face f.

Lemma 4.3. *Let $\mathcal{M} \in \mathfrak{M}_\sigma$ for $0 < \sigma < 1$, $\phi \in V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p})$, $K \in \mathcal{M}$, and $f \in \mathcal{F}_K$. Then, there exist constants $C_i > 0$, $i = 1, 2, 3$, (depending only on σ and on the constant $C_{\mathcal{M}^0}$ in (3.4)–(3.5)) such that we have*

a) the polynomial trace inequality:

$$\|\phi\|^2_{L^2(f)} \leq C_1 \left(p^{\perp}_{K,f}\right)^2 (h^{\perp}_{K,f})^{-1} \|\phi\|^2_{L^2(K)};$$

b) the inverse inequality:

$$\|\phi\|_{L^\infty(f)} \leq C_2 |f|^{-\frac{1}{2}} p^{\|,(1)}_{K,f} p^{\|,(2)}_{K,f} \|\phi\|_{L^2(f)};$$

c) and the bound:

$$\sum_{K\in\mathcal{M}} \sum_{f\in\mathcal{F}_K} \int_f \alpha^{-1} |\nabla\phi|^2 \, \mathrm{d}s \leq C_3 \|\nabla_h \phi\|^2_{L^2(\Omega)}.$$

Here, $C_1, C_2, C_3 > 0$ are constants independent of ϕ, $p^{\perp}_{K,f}$, and of $h^{\perp}_{K,f}$.

We note that related results have been proved earlier, for instance, in [6, 10].

Proof. We first prove the result of a) on the unit cube $\widehat{Q} = (-1, 1)^3$ in two steps: let $\hat{I} = (-1, 1)$ denote the unit interval and let $L_i(\hat{x})$ denote the i-th Legendre polynomial in $\hat{I}$, normalized such that $L_i(1) = 1$. Then, any $\phi_p \in \mathbb{P}_p(\hat{I})$ can be written as $\phi_p(\hat{x}) = \sum_{i=0}^p a_i L_i(\hat{x})$, and for $\hat{x} \in \hat{I}$, we have

$$|\phi_p(\hat{x})| \leq \|\phi_p\|_{L^\infty(\hat{I})} \leq 4p \|\phi_p\|_{L^2(\hat{I})}; \tag{4.9}$$

cf. [25]. Next, let H be a separable Hilbert space with norm $\| \circ \|_H$, inner product $(\cdot,\cdot)_H$ and countable orthonormal basis $\{\psi_\nu\}_{\nu\geq 1}$. Denote by $\mathbb{P}^p(\hat{I};H)$ the polynomial functions of degree p on $\hat{I}$ with coefficients in H. Then, we can write, for $\phi_p \in \mathbb{P}^p(\hat{I};H)$,

$$\phi_p = \sum_{\nu=1}^{\infty} \phi_{p\nu}\psi_\nu, \qquad \phi_{p\nu} = (\phi_p,\psi_\nu)_H \in \mathbb{P}^p(\hat{I}), \qquad \nu \geq 1.$$

Applying Parseval's equality in H, we get

$$\|\phi_p(\hat{x})\|_H^2 = \sum_{\nu=1}^{\infty} |\phi_{p\nu}(\hat{x})|^2 \leq 16p^2 \sum_{\nu=1}^{\infty} \|\phi_{p\nu}\|^2_{L^2(\hat{I})} = 16p^2 \|\phi_p\|^2_{L^2(\hat{I};H)}, \qquad \hat{x} \in \hat{I}.$$

We obtain the asserted result for the unit cube by choosing $\hat{x} = -1$ and H to be the space of polynomials $H = \mathbb{Q}^{\boldsymbol{p}}(f)$ with $\boldsymbol{p} = \boldsymbol{p}^{\|}_{K,f}$, equipped with the $L^2(f)$ inner product, and $p = p^{\perp}_{K,f}$. Axiparallel scaling readily shows the result on an axiparallel cuboid, and the case of a general element follows from scaling using (4.5)–(4.6).

Proof of b): Consider first $f = (-1,1)^2$, and $\phi \in \mathbb{P}^{p_1}(\hat{I}) \otimes \mathbb{P}^{p_2}(\hat{I})$. Furthermore, let $(\hat{x},\hat{y}) \in [-1,1]^2$ such that $|\phi(\hat{x},\hat{y})| = \|\phi\|_{L^\infty(f)}$. Then, using (4.9), we obtain

$$\|\phi\|^2_{L^\infty(f)} \leq Cp_1^2 \int_{-1}^{1} \phi(\tau,\hat{y})^2 \,\mathrm{d}\tau \leq Cp_1^2 \int_{-1}^{1} \sup_{\hat{y}\in\hat{I}} \phi(\tau,\hat{y})^2 \,\mathrm{d}\tau \leq Cp_1^2 p_2^2 \iint_{-1}^{1} \phi(\tau,\theta)^2 \,\mathrm{d}\theta \,\mathrm{d}\tau.$$

Then, applying scaling as in a) and taking the square root, results in the desired bound.

Proof of c): Let f be a face of $\mathcal{F}_K$. With scaling to the patch, the definition of α in (4.3), and estimate a) on $\widetilde{K}$, we obtain

$$(\alpha|_f)^{-1} \int_f |\nabla\phi|^2 \,\mathrm{d}s \leq C(\alpha|_f)^{-1} \int_{\widetilde{f}} |\widetilde{\nabla}\widetilde{\phi}|^2 \,\mathrm{d}\widetilde{s} \leq C \left\|\widetilde{\nabla}\widetilde{\phi}\right\|^2_{L^2(\widetilde{K})} \leq C \|\nabla\phi\|^2_{L^2(K)}.$$

Hence, summing over all elements $K \in \mathcal{M}$ and faces $f \in \mathcal{F}_K$, and taking into account Remark 3.9 complete the proof. □

4.4. Coercivity and Continuity. We shall now study the well-posedness of the *hp*-dGFEM. To this end, we shall use the standard dG norm given by

$$\|v\|^2_{\mathrm{DG}} = \int_\Omega \left(|\nabla_h v|^2 + cv^2\right) \mathrm{d}\boldsymbol{x} + \gamma \int_{\mathcal{F}} \alpha \,|[\![v]\!]|^2 \,\mathrm{d}s \tag{4.10}$$

for any $V(\mathcal{M},\boldsymbol{\Phi},\boldsymbol{p}) + H^1(\Omega)$.

Theorem 4.4. *For any σ-geometric mesh family $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell\geq 1}$ and family of degree vectors $\{\boldsymbol{p}(\mathcal{M}_\sigma^{(\ell)})\}_{\ell\geq 1}$ of μ-bounded variation, the dG bilinear form $a_{\mathrm{DG}}(\cdot,\cdot)$ is continuous and coercive uniformly in ℓ. More precisely, there exist constants $0 < C_1(\sigma,\mu,C_{\mathcal{M}^0},\theta,\gamma,\alpha_0) \leq C_2(\sigma,\mu,C_{\mathcal{M}^0},\theta,\gamma,\alpha_0) < \infty$ such that*

$$|a_{\mathrm{DG}}(v,w)| \leq C_1 \|v\|_{\mathrm{DG}} \|w\|_{\mathrm{DG}} \qquad \forall\, v,w \in V(\mathcal{M}_\sigma^{(\ell)},\boldsymbol{\Phi}(\mathcal{M}_\sigma^{(\ell)}),\boldsymbol{p}(\mathcal{M}_\sigma^{(\ell)})).$$

Moreover, for $\gamma > 0$ sufficiently large (except for the NIP method, $\theta = 1$ in (4.2), where any $\gamma > 0$ can be selected), independently of the refinement level ℓ, the element aspect ratios, the local mesh sizes, and the local polynomial degree vectors,

$$a_{\mathrm{DG}}(v,v) \geq C_2 \|v\|^2_{\mathrm{DG}} \qquad \forall\, v \in V(\mathcal{M}_\sigma^{(\ell)},\boldsymbol{\Phi}(\mathcal{M}_\sigma^{(\ell)}),\boldsymbol{p}(\mathcal{M}_\sigma^{(\ell)})).$$

In particular, there exists a unique solution u_{DG} of (4.1).

Proof. Due to the uniform ellipticity assumption (1.3) we may suppose, without loss of generality, that $\boldsymbol{A} = \mathbf{id}$. Then, recalling the splitting (4.2), employing the Cauchy-Schwarz inequality and using (1.3), there holds

$$|A_{\mathrm{DG}}(w,v)| + |J_{\mathrm{DG}}(w,v)| \leq \|w\|_{\mathrm{DG}} \|v\|_{\mathrm{DG}}.$$

To prove continuity of $F_{\mathrm{DG}}(\cdot,\cdot)$, we use Lemma 4.3 c), to obtain

$$\begin{aligned} |F_{\mathrm{DG}}(v,w)| &\leq \left\| \alpha^{-\frac{1}{2}} \langle\!\langle \nabla_h v \rangle\!\rangle \right\|_{L^2(\mathcal{F})} \left\| \alpha^{\frac{1}{2}} [\![w]\!] \right\|_{L^2(\mathcal{F})} \\ &\leq \left(\sum_{K \in \mathcal{M}} \sum_{f \in \mathcal{F}_K} \int_f \alpha^{-1} |\nabla v|^2 \,\mathrm{d}s \right)^{\frac{1}{2}} \left\| \alpha^{\frac{1}{2}} [\![w]\!] \right\|_{L^2(\mathcal{F})} \\ &\leq C_3 \gamma^{-\frac{1}{2}} \|v\|_{\mathrm{DG}} \|w\|_{\mathrm{DG}}. \end{aligned} \tag{4.11}$$

The same bound holds for $|F_{\mathrm{DG}}(w,v)|$ and hence, applying the triangle inequality to (4.2) and inserting the above bounds, yields the continuity of a_{DG}.

To show coercivity we use (1.3) and (4.11) (with $w = v$) to obtain

$$\begin{aligned} a_{\mathrm{DG}}(v,v) &\geq \|\nabla_h v\|^2_{L^2(\Omega)} + \left\| c^{\frac{1}{2}} v \right\|^2_{L^2(\Omega)} - (1+|\theta|) F_{\mathrm{DG}}(v,v) + \gamma \left\| \alpha^{\frac{1}{2}} [\![v]\!] \right\|^2_{L^2(\mathcal{F})} \\ &\geq \|v\|^2_{\mathrm{DG}} - C_3 (1+|\theta|) \gamma^{-\frac{1}{2}} \|v\|^2_{\mathrm{DG}}. \end{aligned}$$

Choosing $\gamma > 0$ sufficiently large (and independent of $v \in V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p})$) shows the coercivity of a_{DG}. ∎

4.5. Galerkin Orthogonality. The aim of this section is to prove that the dG formulation (4.1) satisfies the property of Galerkin orthogonality. We first establish the following auxiliary result:

LEMMA 4.5. *Let $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ be a multi-index, and $K \in \mathcal{M}_\sigma^{(\ell)}$. Then for $|\boldsymbol{\alpha}| \leq 2$ and $v \in M^{|\boldsymbol{\alpha}|}_{-1-\boldsymbol{\beta}}(K)$, with a weight vector $\boldsymbol{\beta}$ fulfilling* (2.8) *and* (2.11), *we have*

$$\|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^1(K)} \lesssim \begin{cases} |K|^{\frac{1}{2}} \|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^2(K)} & \text{if } K \in \mathfrak{O}^\ell_\sigma, \\ h_K^{\frac{5}{2}+\beta_{\boldsymbol{c}}-|\boldsymbol{\alpha}|} |v|_{M^{|\boldsymbol{\alpha}|}_{-1-\boldsymbol{\beta}}(K)} & \text{if } K \in \mathcal{V}^\ell_{\mathcal{C}} \cup \mathcal{V}^\ell_{\mathcal{C}\mathcal{E}}, \\ h_K^{\frac{5}{2}+\beta_{\boldsymbol{e}}-|\boldsymbol{\alpha}^\perp|} |v|_{M^{|\boldsymbol{\alpha}|}_{-1-\boldsymbol{\beta}}(K)} & \text{if } K \in \mathcal{V}^\ell_{\mathcal{E},1}, \\ \left(h_K^{\parallel}\right)^{\frac{1}{2}} \left(h_K^{\perp}\right)^{2+\beta_{\boldsymbol{e}}-|\boldsymbol{\alpha}^\perp|} |v|_{M^{|\boldsymbol{\alpha}|}_{-1-\boldsymbol{\beta}}(K)} & \text{if } K \in \mathcal{V}^\ell_{\mathcal{E},2}. \end{cases}$$

Proof. Let us first consider the case where $K \cap \mathcal{S} = \emptyset$, i.e. $K \in \mathfrak{O}^\ell_\sigma$. Then, the distance functions $r_{\boldsymbol{c}}, r_{\boldsymbol{e}}$ and $\rho_{\boldsymbol{ce}}$ from (2.2) occurring in (2.9) are strictly positive, and it follows that $\mathsf{D}^{\boldsymbol{\alpha}} v \in L^2(K)$. Hence, by Hölder's inequality, we have

$$\|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^1(K)} \leq \|1\|_{L^2(K)} \|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^2(K)} \lesssim |K|^{\frac{1}{2}} \|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^2(K)}.$$

Furthermore, elements belonging to $\mathcal{V}^\ell_{\mathcal{C}}$, $\mathcal{V}^\ell_{\mathcal{C}\mathcal{E}}$ and $\mathcal{V}^\ell_{\mathcal{E},1}$ are isotropic; cf. Remark 3.8. Hence, for $K \in \mathcal{V}^\ell_{\mathcal{C}}$ and $|\boldsymbol{\alpha}| < \frac{5}{2} + \beta_{\boldsymbol{c}}$, there holds

$$\begin{aligned} \|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^1(K)} &\leq \left\| r_{\boldsymbol{c}}^{1+\beta_{\boldsymbol{c}}-|\boldsymbol{\alpha}|} \right\|_{L^2(K)} \left\| r_{\boldsymbol{c}}^{-1-\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)} \\ &\lesssim h_K^{\frac{5}{2}+\beta_{\boldsymbol{c}}-|\boldsymbol{\alpha}|} \left\| r_{\boldsymbol{c}}^{-1-\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)}. \end{aligned}$$

Similarly, for $K \in \mathcal{V}^{\ell}_{\mathcal{C}\mathcal{E}}$, and $|\boldsymbol{\alpha}| < \frac{5}{2} + \beta_{\boldsymbol{c}}$ and $|\boldsymbol{\alpha}^{\perp}| < 2 + \beta_{\boldsymbol{e}}$,

$$\begin{aligned}\|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^1(K)} &\leq \left\| r_{\boldsymbol{c}}^{1+\beta_{\boldsymbol{c}}-|\boldsymbol{\alpha}|} \rho_{\boldsymbol{ce}}^{1+\beta_{\boldsymbol{e}}-|\boldsymbol{\alpha}^{\perp}|} \right\|_{L^2(K)} \left\| r_{\boldsymbol{c}}^{-1-\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \rho_{\boldsymbol{ce}}^{-1-\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^{\perp}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)} \\ &\lesssim h_K^{\frac{5}{2}+\beta_{\boldsymbol{c}}-|\boldsymbol{\alpha}|} \left\| r_{\boldsymbol{c}}^{-1-\beta_{\boldsymbol{c}}+|\boldsymbol{\alpha}|} \rho_{\boldsymbol{ce}}^{-1-\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^{\perp}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)}.\end{aligned}$$

Additionally, for $K \in \mathcal{V}^{\ell}_{\mathcal{E},1}$, we obtain

$$\begin{aligned}\|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^1(K)} &\leq \left\| r_{\boldsymbol{e}}^{1+\beta_{\boldsymbol{e}}-|\boldsymbol{\alpha}^{\perp}|} \right\|_{L^2(K)} \left\| r_{\boldsymbol{e}}^{-1-\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^{\perp}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)} \\ &\lesssim h_K^{\frac{5}{2}+\beta_{\boldsymbol{e}}-|\boldsymbol{\alpha}^{\perp}|} \left\| r_{\boldsymbol{e}}^{-1-\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^{\perp}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)}.\end{aligned}$$

Finally, for any (possibly anisotropic) $K \in \mathcal{V}^{\ell}_{\mathcal{E},2}$, we have

$$\begin{aligned}\|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{L^1(K)} &\leq \left\| r_{\boldsymbol{e}}^{1+\beta_{\boldsymbol{e}}-|\boldsymbol{\alpha}^{\perp}|} \right\|_{L^2(K)} \left\| r_{\boldsymbol{e}}^{-1-\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^{\perp}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)} \\ &\lesssim \left(h_K^{\parallel} \right)^{\frac{1}{2}} \left(h_K^{\perp} \right)^{2+\beta_{\boldsymbol{e}}-|\boldsymbol{\alpha}^{\perp}|} \left\| r_{\boldsymbol{e}}^{-1-\beta_{\boldsymbol{e}}+|\boldsymbol{\alpha}^{\perp}|} \mathsf{D}^{\boldsymbol{\alpha}} v \right\|_{L^2(K)}.\end{aligned}$$

This shows the desired bounds. ◻

Remark 4.6. *We point out that all the norms on the right-hand sides of the estimates in Lemma 4.5 are part of the full norm* $\|\circ\|_{M^{|\boldsymbol{\alpha}|}_{-1-\boldsymbol{\beta}}(\Omega)}$. *Particularly, all of these expressions remain bounded under the regularity assumptions of Proposition 2.2.*

Lemma 4.7. *Let* $u \in M^2_{-1-\boldsymbol{\beta}}(\Omega)$ *and assume that* $f \in \mathcal{F}_{\mathcal{I}}(\mathcal{M})$ *for* $\mathcal{M} \in \mathfrak{M}_{\sigma}$, $0 < \sigma < 1$. *Then, there holds* $[\![\boldsymbol{A}\nabla u]\!]|_f = 0$.

Proof. Using Lemma 4.5, the trace inequality (4.8) with $t = 1$, and the boundedness of $\boldsymbol{A}$, it follows that $\|\boldsymbol{A}\nabla u\|_{L^1(f)}$ is bounded, albeit with mesh dependent constants. Furthermore, noting that $\boldsymbol{A}\nabla u$ belongs to H^1 away from the singular points $\mathcal{S}$, implies that $[\![\boldsymbol{A}\nabla u]\!]|_{f'} = 0$ for all subsets $f' \subset f$ with $\overline{f'} \subset f$. The result follows using the dominated convergence theorem as in [31, Lemma 3]. ◻

Lemma 4.8. *Let* $u \in M^2_{-1-\boldsymbol{\beta}}(\Omega)$ *and* $v \in C^1(\mathcal{M}^{(\ell)}_{\sigma}, \Omega)$, *where*

$$C^1(\mathcal{M}^{(\ell)}_{\sigma}, \Omega) = \left\{ v \in L^2(\Omega) \,:\, v|_K \in C^1(\overline{K}) \ \forall K \in \mathcal{M}^{(\ell)}_{\sigma} \right\}.$$

Then, there holds the Green's formula

$$\int_K v \,\mathsf{L} u \,\mathrm{d}\boldsymbol{x} = \int_K (\boldsymbol{A}\nabla u) \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_K cuv \,\mathrm{d}\boldsymbol{x} - \int_{\partial K} ((\boldsymbol{A}\nabla u) \cdot \boldsymbol{n}_K)\, v \,\mathrm{d}s \tag{4.12}$$

for all $K \in \mathcal{M}^{(\ell)}_{\sigma}$, *where* L *is the operator from* (1.1).

We remark that in light of Lemma 4.5 and (4.8) with $t = 1$, the boundary integral on the right-hand side of (4.12) is well-defined as a continuous bilinear form on $L^1(\partial K) \times L^{\infty}(\partial K)$.

Proof. Let $\{\phi_n\}_{n \geq 0} \subset C_0^{\infty}(\overline{\Omega})$ with $\lim_{n\to\infty} \|u - \phi_n\|_{M^2_{-1-\boldsymbol{\beta}}(\Omega)} = 0$. Then, recalling Lemma 4.5 and the trace inequality (4.8) with $t = 1$, there holds

$$\begin{aligned}&\|\mathsf{L}(u - \phi_n)\|_{L^1(K)} + \|\boldsymbol{A}\nabla(u - \phi_n)\|_{L^1(K)} + \|c(u - \phi_n)\|_{L^1(K)} \\ &\qquad\qquad \leq C \sum_{|\boldsymbol{\alpha}| \leq 2} \|\mathsf{D}^{\boldsymbol{\alpha}}(u - \phi_n)\|_{L^1(K)} \leq C \,\|u - \phi_n\|_{M^2_{-1-\boldsymbol{\beta}}(\Omega)} \overset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

and

$$\|\boldsymbol{A}\nabla(u-\phi_n)\|_{L^1(\partial K)} \leq C \sum_{f\in\mathcal{F}_K} \|\nabla(u-\phi_n)\|_{L^1(f)} \leq C\,\|u-\phi_n\|_{M^2_{-1-\boldsymbol{\beta}}(\Omega)} \stackrel{n\to\infty}{\longrightarrow} 0.$$

As before, the (generic) constant $C > 0$ may depend on the mesh. Thence, using the fact that the Green formula (4.12) holds for ϕ_n for all $n \geq 0$, and applying the above estimates, results in

$$\begin{aligned}
&\left| \int_K v\,\mathsf{L}u\,\mathrm{d}\boldsymbol{x} - \int_K (\boldsymbol{A}\nabla u)\cdot\nabla v\,\mathrm{d}\boldsymbol{x} - \int_K cuv\,\mathrm{d}\boldsymbol{x} + \int_{\partial K} (\boldsymbol{A}\nabla u)\cdot\boldsymbol{n}_K)v\,\mathrm{d}s \right| \\
&\qquad = \left| \int_K v\,\mathsf{L}(u-\phi_n)\,\mathrm{d}\boldsymbol{x} - \int_K (\boldsymbol{A}\nabla(u-\phi_n))\cdot\nabla v\,\mathrm{d}\boldsymbol{x} - \int_K c(u-\phi_n)v\,\mathrm{d}\boldsymbol{x} \right. \\
&\qquad\qquad \left. + \int_{\partial K} (\boldsymbol{A}\nabla(u-\phi_n))\cdot\boldsymbol{n}_K)v\,\mathrm{d}s \right| \\
&\qquad \leq \|v\|_{L^\infty(K)}\,\|\mathsf{L}(u-\phi_n)\|_{L^1(K)} + \|\boldsymbol{A}\nabla(u-\phi_n)\|_{L^1(K)}\,\|\nabla v\|_{L^\infty(K)} \\
&\qquad\quad + \|c(u-\phi_n)\|_{L^1(K)}\,\|v\|_{L^\infty(K)} + \|\boldsymbol{A}\nabla(u-\phi_n)\|_{L^1(\partial K)}\,\|v\|_{L^\infty(\partial K)} \stackrel{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This implies (4.12) for $u \in M^2_{-1-\boldsymbol{\beta}}(\Omega)$. ☐

We can now prove the Galerkin orthogonality of the *hp*-dGFEM on rather general families of *hp*-dG spaces which include, in particular, the dG spaces V^ℓ_σ and $V^\ell_{\sigma,\boldsymbol{s}}$ in (3.19) and (3.21), respectively.

THEOREM 4.9. *Suppose that the solution u of* (1.1)–(1.2) *belongs to* $M^2_{-1-\boldsymbol{\beta}}(\Omega)$, *where $\boldsymbol{\beta}$ is the weight vector from* (2.8) *and* (2.11). *Then, every dG approximation* $u_{\mathrm{DG}} \in V(\mathcal{M},\boldsymbol{\Phi},\boldsymbol{p})$ *satisfies the Galerkin orthogonality property*

$$a_{\mathrm{DG}}(u-u_{\mathrm{DG}},v) = 0 \qquad \forall\, v \in V(\mathcal{M},\boldsymbol{\Phi},\boldsymbol{p}),$$

where u_{DG} is the dG solution from (4.1) *and where $V(\mathcal{M},\boldsymbol{\Phi},\boldsymbol{p})$ is a hp-dG space with a σ-geometric mesh of hexahedra and a μ-bounded polynomial degree vector.*

Proof. Due to the fact that $u \in H^1_0(\Omega)$ we have that $[\![u]\!] = 0$ on $\mathcal{F}$. Hence, for $v \in V(\mathcal{M},\boldsymbol{\Phi},\boldsymbol{p})$, there holds

$$a_{\mathrm{DG}}(u,v) = \sum_{K\in\mathcal{M}} \int_K (\nabla v\cdot\boldsymbol{A}\nabla u + cuv)\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}} \langle\!\langle \boldsymbol{A}\nabla u\rangle\!\rangle\cdot[\![v]\!]\,\mathrm{d}s.$$

We remark here that the last integral over the faces in $\mathcal{F}$ is well-defined due to the smoothness of $\boldsymbol{A}$ (cf. Section 2), Lemma 4.5, (4.8) with $t = 1$, and the fact that $v \in V(\mathcal{M},\boldsymbol{\Phi},\boldsymbol{p})$. Upon integrating by parts, Lemma 4.8, we arrive at

$$a_{\mathrm{DG}}(u,v) = \int_\Omega v\,\mathsf{L}u\,\mathrm{d}\boldsymbol{x} + \sum_{K\in\mathcal{M}} \int_{\partial K} (\boldsymbol{A}\nabla u\cdot\boldsymbol{n}_K)v\,\mathrm{d}s - \int_{\mathcal{F}} \langle\!\langle \boldsymbol{A}\nabla u\rangle\!\rangle\cdot[\![v]\!]\,\mathrm{d}s.$$

Furthermore, noticing that

$$\sum_{K\in\mathcal{M}} \int_{\partial K} (\boldsymbol{A}\nabla u\cdot\boldsymbol{n}_K)v\,\mathrm{d}s = \int_{\mathcal{F}} \langle\!\langle \boldsymbol{A}\nabla u\rangle\!\rangle\cdot[\![v]\!]\,\mathrm{d}s + \int_{\mathcal{F}_\mathcal{I}} [\![\boldsymbol{A}\nabla u]\!]\langle\!\langle v\rangle\!\rangle\,\mathrm{d}s$$

results in

$$a_{\mathrm{DG}}(u,v) = \int_\Omega v\,\mathsf{L}u\,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{F}_\mathcal{I}} [\![\boldsymbol{A}\nabla u]\!]\langle\!\langle v\rangle\!\rangle\,\mathrm{d}s.$$

Then, using Lemma 4.7, we obtain

$$a_{\mathrm{DG}}(u,v) = \int_\Omega v\, \mathsf{L}u\, \mathrm{d}\boldsymbol{x} = \int_\Omega f v\, \mathrm{d}\boldsymbol{x} = a_{\mathrm{DG}}(u_{\mathrm{DG}}, v).$$

This completes the proof. □

4.6. Error Estimates. We will now analyze the error of the dG method (4.1) and show that it can be bounded by a certain interpolation error of the exact solution in the dG subspace. We proceed in a standard way and split the error $e_{\mathrm{DG}} = u - u_{\mathrm{DG}}$ of the dG method, where u is the solution of (1.1)–(1.2) and $u_{\mathrm{DG}} \in V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p})$ is the dG solution from (4.1), into two parts η and ξ: $e_{\mathrm{DG}} = \eta + \xi$, where

$$\eta = u - \Pi u \in H_0^1(\Omega) + V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p}), \qquad \xi = \Pi u - u_{\mathrm{DG}} \in V(\mathcal{M}, \boldsymbol{\Phi}, \boldsymbol{p}). \tag{4.13}$$

Here, $\Pi : M^2_{-1-\boldsymbol{\beta}}(\Omega) \to V(\mathcal{M}, \boldsymbol{\Phi}(\mathcal{M}), \boldsymbol{p})$ is a suitable hp-(quasi)interpolant.

To state our abstract error estimates of the hp-dGFEM, we introduce the functionals

$$\begin{aligned}\Upsilon_{\mathfrak{O}^\ell_\sigma}[\zeta] = &\sum_{K\in\mathfrak{O}^\ell_\sigma} \left(\max_{f\in\mathcal{F}_K} \left(h^\perp_{K,f}\right)^{-2} \|\zeta\|^2_{L^2(K)} + \|\nabla\zeta\|^2_{L^2(K)} \right) \\ &+ \sum_{K\in\mathfrak{O}^\ell_\sigma} \sum_{f\in\mathcal{F}_K} \left(h^\perp_{K,f}\right)^2 \left\| \widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp} \widetilde{\nabla}\widetilde{\zeta} \right\|^2_{L^2(\widetilde{K})},\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\Upsilon_{\mathfrak{T}^\ell_\sigma}[\zeta] = &\sum_{K\in\mathfrak{T}^\ell_\sigma} \left(\max_{f\in\mathcal{F}_K} \left(h^\perp_{K,f}\right)^{-2} \|\zeta\|^2_{L^2(K)} + \|\nabla\zeta\|^2_{L^2(K)} \right) \\ &+ \sum_{K\in\mathfrak{T}^\ell_\sigma} \sum_{f\in\mathcal{F}_K} |f|^{-1} h^\perp_{K,f} \|\nabla\zeta\|^2_{L^1(f)}.\end{aligned} \tag{4.15}$$

Here, recall that $|f|$ is the surface measure of a face f. Furthermore, $\widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp}\widetilde{\nabla}\widetilde{\zeta}$ in (4.14) is the patch derivative of the gradient of ζ, pulled back to the axiparallel element $\widetilde{K} = G_j^{-1}(K)$ as in (4.7) and (4.4). In light of the results of the previous subsection, the above functionals are well-defined provided that $\zeta \in M^2_{-1-\boldsymbol{\beta}}(K)$ for all $K \in \mathcal{M}^{(\ell)}_\sigma$; in particular, $\widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp}\widetilde{\nabla}\widetilde{\zeta} \in L^2(\widetilde{K})$ for any $K = G_j(\widetilde{K}) \in \mathfrak{O}^\ell_\sigma$ since functions in $M^2_{-1-\boldsymbol{\beta}}(K)$ (together with their pull-backs to the reference patch) belong to H^2 away from the singular set $\mathcal{S}$.

We can now state our error estimate.

THEOREM 4.10. *Let the solution of* (1.1)–(1.2) *satisfy* $u \in M^2_{-1-\boldsymbol{\beta}}(\Omega)$, *with a weight vector fulfilling* (2.8) *and* (2.11). *Assume that on any* σ*-geometric mesh family* $\mathfrak{M}_\sigma = \{\mathcal{M}^{(\ell)}_\sigma\}_{\ell\geq 1}$ *and for any family of polynomial degree vectors* $\{\boldsymbol{p}(\mathcal{M}^{(\ell)}_\sigma)\}_{\ell\geq 1}$ *of* μ*-bounded variation, the hp-interpolant is stable: there exists a finite constant* B *(depending only on* σ, μ *and on the patch maps* $\mathfrak{G}$*) such that*

$$\forall v \in M^2_{-1-\boldsymbol{\beta}}(\Omega): \quad \Upsilon_{\mathfrak{O}^\ell_\sigma}[\Pi v] + \Upsilon_{\mathfrak{T}^\ell_\sigma}[\Pi v] \leq B < \infty. \tag{4.16}$$

Then, there holds the quasioptimality bound

$$|\!|\!| u - u_{\mathrm{DG}} |\!|\!|^2_{\mathrm{DG}} \leq C \boldsymbol{p}^4_{\max} \left(\Upsilon_{\mathfrak{O}^\ell_\sigma}[\eta] + \Upsilon_{\mathfrak{T}^\ell_\sigma}[\eta] \right), \tag{4.17}$$

where η is the interpolation term from (4.13)*, and* $u_{\mathrm{DG}} \in V(\mathcal{M}_\sigma^{(\ell)}, \Phi(\mathcal{M}_\sigma^{(\ell)}), \boldsymbol{p}(\mathcal{M}_\sigma^{(\ell)}))$ *is the numerical solution from* (4.1)*. Here,* $C = C(\sigma, \mu, C_{\mathcal{M}^0}, \theta, \gamma, \alpha_0) > 0$ *is a constant independent of the refinement level ℓ, the aspect ratios, the local mesh sizes, and the local polynomial degree vectors. Furthermore,* $\boldsymbol{p}_{\max} = \max_{K\in\mathcal{M}_\sigma^{(\ell)}} \max \boldsymbol{p}_K$.

Proof. From the triangle inequality we have that

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \le \|\eta\|_{\mathrm{DG}} + \|\xi\|_{\mathrm{DG}}. \tag{4.18}$$

We bound the two norms on the right-hand side of this inequality separately.

Bounding $\|\xi\|_{\mathrm{DG}}$. Galerkin orthogonality (Theorem 4.9) and coercivity (Theorem 4.4) imply the existence of a constant $C > 0$ such that

$$C\|\xi\|_{\mathrm{DG}}^2 \le a_{\mathrm{DG}}(\xi,\xi) = -a_{\mathrm{DG}}(\eta,\xi). \tag{4.19}$$

Recalling (4.2), we can write

$$|a_{\mathrm{DG}}(\eta,\xi)| \le |A_{\mathrm{DG}}(\eta,\xi)| + |F_{\mathrm{DG}}(\eta,\xi)| + |\theta||F_{\mathrm{DG}}(\xi,\eta)| + \gamma|J_{\mathrm{DG}}(\eta,\xi)| \tag{4.20}$$

Using the Cauchy-Schwarz inequality, we obtain

$$|A_{\mathrm{DG}}(\eta,\xi)| \le \|\boldsymbol{A}\nabla_h\eta\|_{L^2(\Omega)} \|\nabla_h\xi\|_{L^2(\Omega)} + \left\|\sqrt{c}\eta\right\|_{L^2(\Omega)} \left\|\sqrt{c}\xi\right\|_{L^2(\Omega)} \lesssim \|\eta\|_{H^1(\Omega)} \|\xi\|_{\mathrm{DG}}.$$

Furthermore, special care has to be taken in dealing with $\nabla\eta$ on faces close to $\mathcal{S}$. We first notice that

$$|F_{\mathrm{DG}}(\eta,\xi)| = \left|\int_{\mathcal{F}} \langle\!\langle \boldsymbol{A}\nabla_h\eta\rangle\!\rangle \cdot [\![\xi]\!] \,\mathrm{d}s\right| \lesssim \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \sum_{f\in\mathcal{F}_K} \int_f |\nabla\eta|\, |[\![\xi]\!]| \,\mathrm{d}s.$$

Then, on $\mathfrak{D}_\sigma^\ell$, there holds:

$$\begin{aligned}
&\sum_{K\in\mathfrak{D}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \int_f |\nabla\eta|\, |[\![\xi]\!]| \,\mathrm{d}s \\
&\quad\lesssim \left(\sum_{K\in\mathfrak{D}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \int_f \alpha^{-1} |\nabla\eta|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\sum_{K\in\mathfrak{D}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \gamma \int_f \alpha\, |[\![\xi]\!]|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad\lesssim \left(\sum_{K\in\mathfrak{D}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \int_f \alpha^{-1} |\nabla\eta|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \|\xi\|_{\mathrm{DG}}.
\end{aligned}$$

Moreover, on $\mathfrak{T}_\sigma^\ell$, we apply Lemma 4.3 b)

$$\begin{aligned}
&\sum_{K\in\mathfrak{T}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \int_f |\nabla\eta|\, |[\![\xi]\!]| \,\mathrm{d}s \\
&\qquad\le \sum_{K\in\mathfrak{T}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \|\nabla\eta\|_{L^1(f)} \|[\![\xi]\!]\|_{L^\infty(f)} \\
&\qquad\lesssim \boldsymbol{p}_{\max}^2 \sum_{K\in\mathfrak{T}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} |f|^{-\frac{1}{2}} \|\nabla\eta\|_{L^1(f)} \|[\![\xi]\!]\|_{L^2(f)}.
\end{aligned}$$

We note in passing that for $f \in \mathcal{F}_K$, the jump $[\![\xi]\!]|_f$ might couple over two mesh patches. In this case, the compatibility condition (3.3) ensures that $[\![\xi]\!]|_f$ is a mapped polynomial with respect to element K. Applying the Cauchy-Schwarz inequality then yields

$$\begin{aligned}
&\sum_{K\in\mathfrak{T}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}\int_f |\nabla\eta|\,|[\![\xi]\!]|\,\mathrm{d}s \\
&\lesssim \boldsymbol{p}_{\max}^2\left(\sum_{K\in\mathfrak{T}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}|f|^{-1}\left\|\alpha^{-\frac{1}{2}}\nabla\eta\right\|_{L^1(f)}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathfrak{T}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}\gamma\left\|\alpha^{\frac{1}{2}}[\![\xi]\!]\right\|_{L^2(f)}^2\mathrm{d}s\right)^{\frac{1}{2}} \\
&\lesssim \boldsymbol{p}_{\max}^2\left(\sum_{K\in\mathfrak{T}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}|f|^{-1}\,h_{K,f}^{\perp}\,\|\nabla\eta\|_{L^1(f)}^2\right)^{\frac{1}{2}}|\!|\!|\xi|\!|\!|_{\mathrm{DG}}.
\end{aligned}$$

Adding the above estimates for $\mathfrak{O}_\sigma^\ell$ and $\mathfrak{T}_\sigma^\ell$, we arrive at

$$\begin{aligned}
&|F_{\mathrm{DG}}(\eta,\xi)| \\
&\lesssim \boldsymbol{p}_{\max}^2\left(\sum_{K\in\mathfrak{O}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}\left\|\alpha^{-\frac{1}{2}}\nabla\eta\right\|_{L^2(f)}^2+\sum_{K\in\mathfrak{T}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}|f|^{-1}\,h_{K,f}^{\perp}\,\|\nabla\eta\|_{L^1(f)}^2\right)^{\frac{1}{2}}|\!|\!|\xi|\!|\!|_{\mathrm{DG}}.
\end{aligned}$$

In addition, we have

$$|F_{\mathrm{DG}}(\xi,\eta)| \lesssim \left(\sum_{K\in\mathcal{M}_\sigma^{(\ell)}}\sum_{f\in\mathcal{F}_K}\int_f \alpha^{-1}|\nabla\xi|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_{\mathcal{F}}\alpha\,|[\![\eta]\!]|^2\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

Thus, employing Lemma 4.3 c) leads to

$$|F_{\mathrm{DG}}(\xi,\eta)| \lesssim \|\nabla_h\xi\|_{L^2(\Omega)}\left(\int_{\mathcal{F}}\alpha\,|[\![\eta]\!]|^2\,\mathrm{d}s\right)^{\frac{1}{2}} \lesssim |\!|\!|\xi|\!|\!|_{\mathrm{DG}}\left(\sum_{K\in\mathcal{M}_\sigma^{(\ell)}}\sum_{f\in\mathcal{F}_K}\left\|\alpha^{\frac{1}{2}}\eta\right\|_{L^2(f)}^2\right)^{\frac{1}{2}}.$$

Furthermore, applying the Cauchy-Schwarz inequality, we conclude

$$\begin{aligned}
|J_{\mathrm{DG}}(\eta,\xi)| &\le \left(\int_{\mathcal{F}}\alpha\,|[\![\eta]\!]|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_{\mathcal{F}}\alpha\,|[\![\xi]\!]|^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{K\in\mathcal{M}_\sigma^{(\ell)}}\sum_{f\in\mathcal{F}_K}\left\|\alpha^{\frac{1}{2}}\eta\right\|_{L^2(f)}^2\right)^{\frac{1}{2}}|\!|\!|\xi|\!|\!|_{\mathrm{DG}}.
\end{aligned}$$

Combining (4.19)–(4.20) with the above estimates gives

$$\begin{aligned}
|\!|\!|\xi|\!|\!|_{\mathrm{DG}}^2 \lesssim \|\eta\|_{H^1(\Omega)}^2 &+ \sum_{K\in\mathcal{M}_\sigma^{(\ell)}}\sum_{f\in\mathcal{F}_K}\left\|\alpha^{\frac{1}{2}}\eta\right\|_{L^2(f)}^2 \\
&+\boldsymbol{p}_{\max}^4\sum_{K\in\mathfrak{O}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}\left\|\alpha^{-\frac{1}{2}}\nabla\eta\right\|_{L^2(f)}^2 \\
&+\boldsymbol{p}_{\max}^4\sum_{K\in\mathfrak{T}_\sigma^\ell}\sum_{f\in\mathcal{F}_K}|f|^{-1}\,h_{K,f}^{\perp}\,\|\nabla\eta\|_{L^1(f)}^2.
\end{aligned}$$

Bounding $\|\eta\|_{\mathrm{DG}}$. There holds:

$$\begin{aligned}\|\eta\|_{\mathrm{DG}}^2 &\lesssim \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \|\eta\|_{H^1(K)}^2 + \gamma \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \sum_{f\in\mathcal{F}_K} \left\| \alpha^{\frac12} [\![\eta]\!] \right\|_{L^2(f)}^2 \\ &\lesssim \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \|\eta\|_{H^1(K)}^2 + \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \sum_{f\in\mathcal{F}_K} \left\| \alpha^{\frac12} \eta \right\|_{L^2(f)}^2 .\end{aligned}$$

Bounding $\|e_{\mathrm{DG}}\|_{\mathrm{DG}}$. Referring to (4.18) and utilizing the estimates for $\|\xi\|_{\mathrm{DG}}$ and $\|\eta\|_{\mathrm{DG}}$, we obtain

$$\begin{aligned}&\|u-u_{\mathrm{DG}}\|_{\mathrm{DG}}^2 \\ &\lesssim \|\eta\|_{H^1(\Omega)}^2 + \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \sum_{f\in\mathcal{F}_K} \left\| \alpha^{\frac12} \eta \right\|_{L^2(f)}^2 \\ &\quad + \boldsymbol{p}_{\max}^4 \sum_{K\in\mathfrak{D}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \left\| \alpha^{-\frac12} \nabla\eta \right\|_{L^2(f)}^2 + \boldsymbol{p}_{\max}^4 \sum_{K\in\mathfrak{T}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} |f|^{-1} h_{K,f}^{\perp} \|\nabla\eta\|_{L^1(f)}^2 .\end{aligned}$$

Applying the trace inequality from Lemma 4.2 (with $s=2$) implies that

$$\begin{aligned}&\sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \sum_{f\in\mathcal{F}_K} \left\| \alpha^{\frac12} \eta \right\|_{L^2(f)}^2 \\ &\qquad\lesssim \boldsymbol{p}_{\max}^2 \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \sum_{f\in\mathcal{F}_K} h_{K,f}^{-1} \left(\left(h_{K,f}^{\perp}\right)^{-1} \|\eta\|_{L^2(K)}^2 + h_{K,f}^{\perp} \left\| \partial_{\widetilde{K},\widetilde{f},\perp} \widetilde{\eta} \right\|_{L^2(\widetilde{K})}^2 \right) \\ &\qquad\lesssim \boldsymbol{p}_{\max}^2 \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \sum_{f\in\mathcal{F}_K} \left(\left(h_{K,f}^{\perp}\right)^{-2} \|\eta\|_{L^2(K)}^2 + \left\| \widetilde{\nabla} \widetilde{\eta} \right\|_{L^2(\widetilde{K})}^2 \right) \\ &\qquad\lesssim \boldsymbol{p}_{\max}^2 \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \left(\max_{f\in\mathcal{F}_K} \left(h_{K,f}^{\perp}\right)^{-2} \|\eta\|_{L^2(K)}^2 + \|\nabla\eta\|_{L^2(K)}^2 \right),\end{aligned}$$

where in the last step, we have employed scaling from $\widetilde{K}$ to K with (4.6).

Similarly, the trace estimate (4.7) with $t=2$ yields

$$\sum_{K\in\mathfrak{D}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \left\| \alpha^{-\frac12} \nabla\eta \right\|_{L^2(f)}^2 \lesssim \sum_{K\in\mathfrak{D}_\sigma^\ell} \sum_{f\in\mathcal{F}_K} \left(\|\nabla\eta\|_{L^2(K)}^2 + \left(h_{K,f}^{\perp}\right)^2 \left\| \widetilde{\partial}_{\widetilde{K},\widetilde{f},\perp} \widetilde{\nabla} \widetilde{\eta} \right\|_{L^2(\widetilde{K})}^2 \right).$$

Finally, noticing that

$$\|\eta\|_{H^1(\Omega)}^2 \lesssim \sum_{K\in\mathcal{M}_\sigma^{(\ell)}} \left(\max_{f\in\mathcal{F}_K} \left(h_{K,f}^{\perp}\right)^{-2} \|\eta\|_{L^2(K)}^2 + \|\nabla\eta\|_{L^2(K)}^2 \right)$$

completes the proof. ▯

5. Concluding Remarks. We have introduced a class of *hp*-version discontinuous Galerkin discretizations of second-order linear elliptic problems in polyhedra in $\mathbb{R}^3$. We have considered families $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell\geq 1}$ of irregular geometric meshes of mapped hexahedral elements with subdivision factor $\sigma \in (0,1)$ (not necessarily equal to 1/2). Since (interior penalty) dG discretizations do not require conformity of the meshes, anisotropic geometric mesh refinements towards edges and vertices of the domain are possible with only hexahedral elements in fairly general polyhedra. We have presented an algorithm to generate, for given $\sigma \in (0,1)$ and any Lipschitz polyhedron Ω with a finite number of plane faces, families $\mathfrak{M}_\sigma$ of *σ-geometric meshes*, i.e., there is a constant $\sigma \in (0,1)$ such that any two elements $K, K' \in \mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma$ abutting at a common face $f = (\overline{K} \cap \overline{K}')^\circ$ have diameters perpendicular to f which are bounded from above and below by an absolute multiple of $\min\{\sigma/(1-\sigma), (1-\sigma)/\sigma\}$ respectively of $\max\{\sigma/(1-\sigma), (1-\sigma)/\sigma\}$.

In local coordinates on elements $K \in \mathcal{M}_\sigma^{(\ell)}$, the approximate solutions belong to a tensor product polynomial space $\mathbb{Q}^{\boldsymbol{p}_K}(K)$ of elemental polynomial degrees $\boldsymbol{p}_K$. In particular, we have admitted nonuniform, possibly anisotropic and $\boldsymbol{s}$-linear elemental polynomial degree distributions of *μ-bounded variation*. The hexahedral elements $K \in \mathcal{M}_\sigma^{(\ell)}$ are mapped images of the unit cube in $\mathbb{R}^3$, i.e., each element $K \in \mathcal{M}_\sigma^{(\ell)}$ is, upon some anisotropic dilation-translation, the image of the unit cube under analytic element mappings with Jacobians which are uniformly bounded from below and above over the whole geometric mesh family $\mathfrak{M}_\sigma$.

We have proved that the *hp*-dG finite element approximation is *well-defined and stable* on these meshes, independent of the level ℓ of refinement. Although here we have considered only the scalar model problem (1.1)–(1.2), we mention that analogous *hp*-dGFEM could be readily defined for second-order elliptic systems (see [23, 22] for the required regularity). In our subsequent work [30], we shall show that, on the σ-geometric meshes and $\boldsymbol{s}$-linearly increasing polynomial degrees constructed in this article, the error bounds in Theorem 4.10 yield *exponential convergence rates* for *hp*-dG FEM in polyhedra if the data in (1.1)–(1.2) is piecewise analytic.

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