Representation of Gaussian fields in series with independent coefficients

Author(s):
Gittelson, Claude Jeffrey

Publication Date:
2010

Permanent Link:
https://doi.org/10.3929/ethz-a-010403532

Rights / License:
In Copyright - Non-Commercial Use Permitted
Representation of Gaussian fields in series with independent coefficients

C.J. Gittelson

Research Report No. 2010-15
June 2010

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland
REPRESENTATION OF GAUSSIAN FIELDS IN SERIES WITH INDEPENDENT COEFFICIENTS

CLAUDE J. GITTELSON

Abstract. The numerical discretization of problems with stochastic data or stochastic parameters generally involves the introduction of coordinates that describe the stochastic behavior, such as coefficients in a series expansion or values at discrete points. The series expansion of a Gaussian field with respect to any orthonormal basis of its Cameron–Martin space has independent standard normal coefficients. A standard choice for numerical simulations is the Karhunen–Loève series, which is based on eigenfunctions of the covariance operator. We suggest an alternative, the hierarchic discrete spectral expansion, which can be constructed directly from the covariance kernel. The resulting basis functions are often well localized, and the convergence of the series expansion seems to be comparable to that of the Karhunen–Loève series. We provide explicit formulas for particular cases, and general numerical methods for computing exact representations of such bases. Finally, we relate our approach to numerical discretizations based on replacing a random field by its values on a finite set.

INTRODUCTION

The numerical discretization of problems with stochastic data or stochastic parameters requires that the random inputs are approximated by finite quantities. This is generally done in one of two ways. Either the random data is expanded in a series, which can be truncated for numerical computations, or it is replaced by a finite dimensional random variable, describing for example the value of a random field on a discrete set of points.

A standard approach, falling strictly into the first category, is to expand a random field into its Karhunen–Loève series. For Gaussian fields, the coefficients in this series are independent standard normal random variables.

The independence of these coefficients is crucial to many numerical methods. For example, in Monte Carlo simulation, coefficient sequences can be generated by independent draws of pseudorandom numbers. The construction of polynomial chaos bases as tensor products of orthonormal bases with respect to the distributions of the coefficients also requires that these are independent. Similarly, in collocation and quasi Monte Carlo methods, constructions of collocation points make use of the product structure of the joint distribution of the coefficients.

Nevertheless, the Karhunen–Loève series is often ill-suited for numerical computations, as it requires eigenfunctions of the covariance operator. These are usually not known exactly, and are expensive to approximate numerically. Furthermore, the eigenfunctions generally have global supports.

We suggest an alternative to the Karhunen–Loève series for general continuous Gaussian fields on bounded domains, which we call the hierarchic discrete spectral expansion. Assuming that the covariance kernel is given, the basis functions in our series expansion can be constructed exactly. As these form an orthonormal basis of
the Cameron–Martin space, independence of the coefficients in the series expansion is preserved.

Our method does not assume any particular structure of the covariance kernel or of the underlying domain. Alternative methods based on the fast Fourier transform exist for stationary Gaussian fields on discrete rectangular domains, [7, 10].

In Section 1, we provide a brief overview of the theory of Gaussian measures on Banach spaces, and lay the theoretical foundation for our basis in Proposition 1.9. We refer to [12, Chapter III] and [1, Chapters 2,3] for further details.

We consider the case of an exponential covariance in one dimension in Section 2. For this setting, we derive explicit formulas for a sequence of basis functions. In Section 2.3, we apply our approach to the Wiener measure, giving an alternative derivation of the well-known piecewise linear construction of Brownian motion.

The numerical construction of a hierarchic sequence of basis functions is discussed in Section 3. We consider in particular two special cases of our general algorithm. Gram–Schmidt orthogonalization in the Cameron–Martin space is closely related to the Cholesky decomposition of the covariance matrix. Hierarchical spectral decomposition of the covariance matrix can also be used to exactly construct an orthonormal basis of the Cameron–Martin space, providing an alternative interpretation to the naive approximation of the Karhunen–Loève series given by eigenvectors and eigenvalues of the covariance matrix.

Finally, in Section 4, we provide numerical examples of hierarchical discrete spectral bases computed by the two aforementioned special cases of our algorithm for several covariance kernels. We study in particular the decay of these basis functions.

1. The Cameron–Martin space of a Gaussian distribution

1.1. Gaussian measures on Banach spaces. Let $X$ be a real separable Banach space with Borel $\sigma$-algebra $\mathcal{B}(X)$, and let $\gamma$ be a Gaussian measure on $(X, \mathcal{B}(X))$, i.e. for all $\varphi \in X^*$, the image measure $\varphi(\gamma)$ on $\mathbb{R}$ is Gaussian. By [1, Thm. 3.2.3], there is a unique element $a_\gamma \in X$, called the mean of $\gamma$, such that

$$\varphi(a_\gamma) = \int_X \varphi(h) \gamma(dh) \quad \forall \varphi \in X^*. \quad (1)$$

The covariance operator $R_\gamma$ is given formally by

$$(R_\gamma \varphi, \psi) = \int_X \varphi(h - a_\gamma) \psi(h - a_\gamma) \gamma(dh) \quad \forall \varphi, \psi \in X^*. \quad (2)$$

Again by [1, Thm. 3.2.3], (2) defines a unique linear operator $R_\gamma : X^* \to X$. We define

$$\sigma(\varphi) := \sqrt{(R_\gamma \varphi, \varphi)}, \quad \varphi \in X^*. \quad (3)$$

Lemma 1.1. The assignment $\|h\| := \sigma(\varphi)$ if $h = R_\gamma \varphi$ for $\varphi \in X^*$ defines a norm on range($R_\gamma$) $\subset X$.

Proof. Let $\varphi, \psi \in X^*$ with $R_\gamma \varphi = R_\gamma \psi$. Using the symmetry of $(R_\gamma \cdot, \cdot)$,

$$\langle R_\gamma \varphi, \varphi \rangle = \langle R_\gamma \psi, \varphi \rangle = \langle R_\gamma \varphi, \psi \rangle = \langle R_\gamma \psi, \psi \rangle.$$

Therefore, $\|R_\gamma \varphi\|$ is well-defined on range($R_\gamma$). It is a norm since it coincides with $\|\varphi(\cdot - a_\gamma)\|_{L^2(\gamma)}$.

We define the Cameron–Martin space $H(\gamma)$ as the completion of range($R_\gamma$) with respect to the norm from Lemma 1.1. By [1, Thm. 3.2.7], $H(\gamma)$ is a separable Hilbert space, and due to [1, Prop. 2.4.6], $H(\gamma)$ embeds continuously into $X$. Furthermore, by [1, Lem. 3.2.2], $H(\gamma)$ is independent of the space $X$ on which $\gamma$ is considered.
The last integral is finite as a consequence of Fernique’s theorem [3], Thm. 2.8.5. Therefore, by dominated convergence,

\[ (f, R_\gamma \varphi)_{H(\gamma)} = \varphi(f). \tag{4} \]

**Proof.** Let \( f = R_\gamma \psi \) for \( \psi \in X^* \). Then using the parallelogram identity,

\[ (f, R_\gamma \varphi)_{H(\gamma)} = (R_\gamma \psi, R_\gamma \varphi)_{H(\gamma)} = (R_\gamma \psi, \varphi) = \varphi(f). \]

This extends to all \( f \in H(\gamma) \) by density. \( \square \)

**Lemma 1.3.** Let \( \varphi \in X^* \) and let \( (\varphi_n)_{n \in \mathbb{N}} \subset X^* \) be bounded with \( \varphi_n(h) \to \varphi(h) \) for all \( h \in X \). Then \( R_\gamma \varphi_n \to R_\gamma \varphi \) in \( H(\gamma) \).

**Proof.** We assume without loss of generality that \( \varphi = 0 \). By definition,

\[ \sigma(\varphi_n)^2 = (R_\gamma \varphi_n, \varphi_n) = \int_X \varphi_n(h - a_\gamma)^2 \gamma(\mathrm{d}h) \leq \|\varphi_n\|_X^2 \cdot \int_X \|h - a_\gamma\|^2 \gamma(\mathrm{d}h). \]

The last integral is finite as a consequence of Fernique’s theorem [3], cf. also [1, Thm. 2.8.5]. Therefore, by dominated convergence, \( \sigma(\varphi_n) \to \sigma(\varphi) = 0 \), and thus \( R_\gamma \varphi_n \to 0 \) in \( H(\gamma) \). \( \square \)

The conditions of Lemma 1.3 are satisfied e.g. if \( \varphi_n \to \varphi \) since weakly convergent sequences are bounded. Furthermore, the statement of Lemma 1.3 implies in particular that \( R_\gamma \varphi_n \to R_\gamma \varphi \) in \( X \).

1.2. **Orthonormal bases of the Cameron–Martin space.** The Gaussian measure \( \gamma \) on \( X \) is uniquely characterized by its mean \( a_\gamma \) and its covariance operator \( R_\gamma \). The covariance operator, in turn, is determined by the Cameron–Martin space \( H(\gamma) \). It can be expressed in terms of an orthonormal basis of \( H(\gamma) \).

**Proposition 1.4.** Let \( (e_m)_{m \in \Theta} \) be an orthonormal basis of \( H(\gamma) \). Then

\[ R_\gamma \varphi = \sum_{m \in \Theta} \varphi(e_m) e_m \quad \forall \varphi \in X^* \] \( \tag{5} \)

with unconditional convergence in \( H(\gamma) \) and in \( X \). Furthermore,

\[ \langle R_\gamma \varphi, \psi \rangle = \sum_{m \in \Theta} \varphi(e_m) \psi(e_m) \quad \forall \varphi, \psi \in X^* \] \( \tag{6} \)

with unconditional convergence in \( \mathbb{R} \).

**Proof.** Let \( \varphi \in X^* \). By Lemma 1.2,

\[ (R_\gamma \varphi, e_m)_{H(\gamma)} = \varphi(e_m) \quad \forall m \in \Theta. \]

Expanding \( R_\gamma \varphi \) in the orthonormal basis \( (e_m)_{m \in \Theta} \) of \( H(\gamma) \), we have

\[ R_\gamma \varphi = \sum_{m \in \Theta} (R_\gamma \varphi, e_m)_{H(\gamma)} e_m = \sum_{m \in \Theta} \varphi(e_m) e_m \]

with convergence in \( H(\gamma) \), and thus also in \( X \) since \( H(\gamma) \) embeds continuously into \( X \). The convergence is unconditional since all of the above is independent of any ordering of the basis \( (e_m)_{m \in \Theta} \). In particular, for any \( \psi \in X^* \),

\[ \langle R_\gamma \varphi, \psi \rangle = \psi(R_\gamma \varphi) = \psi \left( \sum_{m \in \Theta} \varphi(e_m) e_m \right) = \sum_{m \in \Theta} \varphi(e_m) \psi(e_m), \]

again with unconditional convergence. \( \square \)
An orthonormal basis \((e_m)_{m \in \Theta}\) of \(H(\gamma)\) is useful not only for computing the covariance \(R_\gamma\), but also for sampling the distribution \(\gamma\) on \(X\).

We define the product Gaussian measure \(\pi\) on \((\mathbb{R}^\Theta, \mathcal{B}(\mathbb{R}^\Theta))\),

\[
\pi := \bigotimes_{m \in \Theta} \pi_m, \tag{7}
\]

where each \(\pi_m\) is a standard Gaussian measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). In principle, \(\pi\) can be sampled numerically by independent standard Gaussian draws for each index \(m \in \Theta\). If, in some sense, these values are of decreasing importance, then the sequence can be truncated to finitely many random values. Using an orthonormal basis \((e_m)_{m \in \Theta}\) of \(H(\gamma)\), we parametrize \(\gamma\) by \(\pi\), allowing \(\gamma\) to be sampled by mapping a sample of \(\pi\) from \(\mathbb{R}^\Theta\) to \(X\). Also, series expansions of the form (8) are a prerequisite for many stochastic Galerkin and collocation methods.

**Theorem 1.5.** Let \((e_m)_{m \in \Theta}\) be an orthonormal basis of \(H(\gamma)\). Then the series in

\[
T : \mathbb{R}^\Theta \to X, \quad \xi = (\xi_m)_{m \in \Theta} \mapsto a_\gamma + \sum_{m \in \Theta} \xi_m e_m \tag{8}
\]

converges unconditionally in \(X\) for \(\pi\)-a.e. \(\xi = (\xi_m)_{m \in \Theta}\) in \(\mathbb{R}^\Theta\), and the distribution of \(T\) is \(T(\pi) = \gamma\).

For a proof of Theorem 1.5, we refer to [1, Thm. 3.5.1]; cf. also [5, 6, 8] and [12, Sec. V.5.5].

**Remark 1.6.** Theorem 1.5 applies directly to \(R_\gamma\)-Wiener processes. For any orthonormal basis \((e_m)_{m \in \Theta}\) of \(H(\gamma)\), let \((\beta^m_t)_{m \in \Theta}\) be independent scalar Wiener processes. Then a \(R_\gamma\)-Wiener processes \(W_t\) in \(X\) is given by

\[
W_t = \sum_{m \in \Theta} \beta^m_t e_m, \tag{9}
\]

with unconditional convergence in \(X\) a.s. for all \(t \geq 0\). Convergence of this series follows by applying Theorem 1.5 for every \(t > 0\). Equation (9) defines a \(H(\gamma)\)-Wiener processes since for all \(t > s\),

\[
W_t - W_s = \sum_{m \in \Theta} (\beta^m_t - \beta^m_s)e_m = \sqrt{t-s} \sum_{m \in \Theta} \xi_m e_m \tag{10}
\]

with independent standard normal \((\xi_m)_{m \in \Theta}\), and the last sum represents a \(X\)-valued centered Gaussian random variable with covariance \(R_\gamma\). In particular, if \(t_n \to t\), then (10) with \(s = t_n\) implies that \(W_t\) defined by (9) has continuous paths.

**Remark 1.7.** The series representation (8) also allows conditional simulation of the distribution \(\gamma\). For a finite set \(\Theta_n \subset \Theta\), the distribution of the series

\[
a_\gamma + \sum_{m \in \Theta_n} y_m e_m + \sum_{m \in \Theta \setminus \Theta_n} \xi_m e_m \tag{11}
\]

serves as a conditional probability of \(\gamma\), conditioned on \(\xi_m = y_m\) for \(m \in \Theta_n\), where \((\xi_m)_{m \in \Theta \setminus \Theta_n}\) are independent standard normal random variables, cf. [1, Corollary 3.5.2]. In particular, if \(\gamma\) describes a prior model for a random element of \(X\) and \(y_m\) are measurements of \(\xi_m\) for \(m \in \Theta_n\), then (11) is the resulting posterior model. This is the foundation for the interpolation techniques known in geostatistics as kriging, cf. e.g. [11, 2].
1.3. Continuous Gaussian fields. We consider the case \( X = C(D) \) for a compact set \( D \subset \mathbb{R}^d \). Then \( \gamma \) describes a Gaussian field on \( D \) that is almost surely continuous.

For all \( x \in D \), the Dirac functional \( \delta_x(f) := f(x) \) is in \( X^* \). We define
\[
k_x := R_\gamma \delta_x , \quad x \in D .
\]
Then the covariance kernel of \( \gamma \) is
\[
k(x, y) := k_x(y) = \langle R_\gamma \delta_x , \delta_y \rangle , \quad x, y \in D .
\]
The function \( k(\cdot, \cdot) \) is symmetric since
\[
k(x, y) = \langle R_\gamma \delta_x , \delta_y \rangle = \langle R_\gamma \delta_y , \delta_x \rangle = k(y, x) \quad \forall x, y \in D .
\]
By the Riesz representation theorem for \( X = C(D) \), \( X^* \) can be identified with the space of signed measures \( \mu \) on the Borel \( \sigma \)-algebra of \( D \) with the total variation norm. Accordingly, we will use the same symbol, e.g. \( \mu \), for the signed measure on \( (D, \mathcal{B}(D)) \) and for the element of \( X^* \) given by integration against this measure.

**Proposition 1.8.** Let \( \mu \) and \( \nu \) be signed measures on \( (D, \mathcal{B}(D)) \). Then
\[
\langle R_\gamma \mu, \nu \rangle = \int_D k(x, y) \mu(dx) \nu(dy) .
\]

**Proof.** For all \( x \in D \), by symmetry of \( \langle R_\gamma \cdot, \cdot \rangle \),
\[
\langle R_\gamma \mu, \delta_x \rangle = \langle R_\gamma \delta_x , \mu \rangle = \int_D k_x \mu(dx) .
\]
Consequently,
\[
\langle R_\gamma \mu, \nu \rangle = \int_D \langle R_\gamma \mu, \delta_x \rangle \nu(dx) = \int_D \int_D k(x, y) \mu(dx) \nu(dy) .
\]
In particular, if \( \mu(dx) = f(x) \, dx \) and \( \nu(dx) = g(x) \, dx \) for functions \( f \) and \( g \) in \( L^1(D) \), we recover the well-known formula describing the covariance operator \( R_\gamma \) as integration against \( k(\cdot, \cdot) \),
\[
\langle R_\gamma f, g \rangle = \int_D \int_D k(x, y) f(y) g(x) \, dy \, dx .
\]
Here, \( f \) and \( g \) are interpreted as the elements of \( X^* \), respectively.

**Proposition 1.9.** Let \( (p_i)_{i \in A} \) be a dense subset of \( D \). Then the linear span of \( (k_{p_i})_{i \in A} \) is dense in \( H(\gamma) \).

**Proof.** Let \( f \in H(\gamma) \) be orthogonal to \( k_{p_i} \) for all \( i \in A \). Lemma 1.2 implies
\[
f(p_i) = \delta_{p_i}(f) = \langle f, R_\gamma \delta_{p_i} \rangle_{H(\gamma)} = \langle f, k_{p_i} \rangle_{H(\gamma)} = 0 \quad \forall i \in A .
\]
Since \( (p_i)_{i \in A} \) is dense in \( D \) by assumption, and \( f \) is continuous due to \( H(\gamma) \subset X = C(D) \), it follows that \( f = 0 \).

**Remark 1.10.** Proposition 1.9 suggests a construction for an orthonormal basis of \( H(\gamma) \). Given a dense sequence \( (p_i)_{i \in \mathbb{N}} \) in \( D \), we can apply Gram–Schmidt orthogonalization to \( (k_{p_i})_{i \in \mathbb{N}} \). This is explored in Section 3.2. Note that by Lemma 1.2, \( f \in H(\gamma) \) is orthogonal to \( k_x \) for an \( x \in D \) if and only if \( f(x) = 0 \) since
\[
\langle f, k_x \rangle_{H(\gamma)} = \langle f, R_\gamma \delta_x \rangle_{H(\gamma)} = \delta_x(f) = f(x) .
\]
Therefore, constructing an orthonormal basis of \( H(\gamma) \) can reduce to finding functions in the span of \( (k_{p_i})_{i=1}^{n} \) with certain zeros.
For any sequence \( p = (p_i)_{i \in \Lambda} \) in \( D \), let \( K \) denote the covariance matrix of the functionals \( \langle \delta_{p_i}, \cdot \rangle \) \( i \in \Lambda \), i.e.
\[
K = \begin{bmatrix}
\langle R_i \delta_{p_i}, \delta_{p_j} \rangle \end{bmatrix}_{i,j \in \Lambda} = [k(p_i,p_j)]_{i,j \in \Lambda} .
\]
(18)

For a finitely supported vector \( a = (a_i)_{i \in \Lambda} \in \mathbb{R}^n \), we define the functional
\[
a \delta_p := \sum_{i \in \Lambda} a_i \delta_{p_i} \in C(D)^* .
\]
(19)

and the function
\[
a k_p := R_n a \delta_p = \sum_{i \in \Lambda} a_i k_{p_i} \in H(\gamma) \subset C(D) .
\]
(20)

Note that no convergence issues emerge since by assumption, \( a_i = 0 \) for all but finitely many \( i \in \Lambda \). Then, using Lemma 1.2, for all finitely supported \( a \) and \( b \) in \( \mathbb{R}^A \),
\[
(a k_p, b k_p)_H(\gamma) = \langle R_n a \delta_p, b \delta_p \rangle = a^\top K b .
\]
(21)

Therefore, orthogonality of the functions \( a k_p \) and \( b l_p \) in \( H(\gamma) \), which is equivalent to the functionals \( a \delta_p \) and \( b l_p \) being uncorrelated, is characterized by orthogonality of the vectors \( a \) and \( b \) with respect to \( K \).

**Corollary 1.11.** Let \( p = (p_i)_{i \in \Lambda} \) be a sequence in \( D \), and let \( (a^m)_{m \in \Theta} \subset \mathbb{R}^A \) such that \( a^m \) is finitely supported for all \( m \in \Theta \) and
\[
(a^m)^\top K a^{m'} = \delta_{mm'} \quad \forall m, m' \in \Theta .
\]
(22)

Then \( (a^m k_p)_{m \in \Theta} \) is an orthonormal system in \( H(\gamma) \). If \( p \) is dense in \( D \) and for all \( i \in \Lambda \), \( k_{p_i} \) is in the span of \( \{(a^m k_p)_{m \in \Theta} \in H(\gamma), \) then \( (a^m k_p)_{m \in \Theta} \) is an orthonormal basis of \( H(\gamma) \).

**Proof.** Orthonormality of \( (a^m k_p)_{m \in \Theta} \) follows from (22) due to (21). Density is a consequence of Proposition 1.9. \( \square \)

**Remark 1.2.** The assumption that coefficient vectors are finitely supported can be weakened to a more general summability condition. In the interest of a concise presentation, and since all numerically representable coefficient vectors are finitely supported, we consider only this setting.

2. **Examples**

2.1. **Exponential covariance in one dimension.** Let \( I \subset \mathbb{R} \) be a compact interval. We consider a Gaussian measure \( \gamma \) on \( C(I) \) with an exponential covariance kernel
\[
k(x,y) = \sigma^2 e^{-|x-y|/\lambda} , \quad x, y \in I ,
\]
(23)

for constants \( \sigma > 0 \) and \( \lambda > 0 \).

We define a hierarchic sequence of grids on \( I \). For all \( \ell \in \mathbb{N} \), let \( \Delta_\ell \subset I \) be a finite set with \( \Delta_\ell \cap \Delta_{\ell'} = \emptyset \) if \( \ell \neq \ell' \). We define the unions
\[
\Lambda_\ell := \bigcup_{j=1}^\ell \Delta_j \quad \text{and} \quad \Lambda := \bigcup_{\ell=1}^\infty \Delta_\ell = \bigcup_{\ell=1}^\infty \Lambda_\ell .
\]
(24)

Then \( n_\ell := \# \Delta_\ell < \infty \). For all \( \ell \in \mathbb{N} \), we enumerate the elements of \( \Lambda_\ell \) in increasing order,
\[
\Lambda_\ell = \{ p_{\ell,1}, \ldots, p_{\ell,n_\ell} \} , \quad p_{\ell,i} < p_{\ell,i+1} .
\]
(25)

We assume that new nodes on each level are never adjacent in the ordering (25).

**Assumption 2.A.** For all \( \ell \in \mathbb{N} \), if \( p_{\ell,i}, p_{\ell,j} \in \Delta_\ell \) and \( i \neq j \), then \( |i-j| \geq 2 \).
Assumption 2.A implies in particular
\[ n_1 \leq 1 \quad \text{and} \quad n_{\ell+1} \leq 2n_\ell + 1 \quad \forall \ell \in \mathbb{N}. \]  
(26)

It is always satisfied if \( \# \Delta_\ell = 1 \) for all \( \ell \in \mathbb{N} \).

To each \( p \in A \), we associate a function \( a^p_\ell \) in \( H(\gamma) \) with \( a^p \in \mathbb{R}^A \) finitely supported. Let \( \ell \in \mathbb{N} \) and \( p \in \Delta_\ell \) such that \( p = p^\ell \) in the ordering (25). We define the coefficient vector \( a^p \) by allowing \( a^p_q \) to be different from zero only if \( q \) is adjacent to \( p \) in \( \Delta_\ell \), and requiring that \( a^p k_p \) is zero at all such adjacent nodes \( q \). Finally, we normalize the coefficients such that \( a^p k_p \) has unit norm in \( H(\gamma) \).

If \( i \) is different from 1 and \( n_\ell \), i.e. if \( p \) is an interior point of \( A_\ell \), this results in a \( 3 \times 3 \) linear system with the following solution. For \( h_+ := (p^\ell_{i+1} - p^\ell_i)/\lambda \) and \( h_- := (p^\ell_i - p^\ell_{i-1})/\lambda \), let

\[
\tilde{a}_0 := \frac{e^{2h_- + 2h_+} - 1}{(e^{2h_-} - 1)(e^{2h_+} - 1)}, \quad \tilde{a}_{-1} := -\frac{e^{h_-}}{e^{2h_-} - 1}, \quad \tilde{a}_1 := -\frac{e^{h_+}}{e^{2h_+} - 1},
\]

and define \( a^p = (a^p_q)_{q \in A} \) by

\[
a^p_{\ell j} := \frac{\tilde{a}_j - i}{\sigma \sqrt{\tilde{a}_0}}, \quad j \in \{i-1, i, i+1\},
\]

and \( a^p_q := 0 \) for all other \( q \in A \).

If \( p \) is a boundary point of \( A_\ell \) and \( n_\ell \geq 2 \), the above conditions lead to a \( 2 \times 2 \) linear system. If \( i = 1 \), then \( h := (p^\ell_p - p^\ell_1)/\lambda \) and

\[
\tilde{a}_0 := 1, \quad \tilde{a}_1 := -e^{-h} \quad \text{and} \quad a^p_{\ell 1} := \frac{\tilde{a}_1}{\sigma \sqrt{1 - e^{-2h}}}. \quad (29)
\]

Similarly, if \( i = n_\ell \), we set \( h := (p^\ell_{n_\ell} - p^\ell_{n_\ell-1})/\lambda \) and

\[
\tilde{a}_0 := 1, \quad \tilde{a}_{-1} := -e^{-h} \quad \text{and} \quad a^p_{\ell -1} := \frac{\tilde{a}_{-1}}{\sigma \sqrt{1 - e^{-2h}}}. \quad (30)
\]

In both cases, \( a^p_q := 0 \) for all other \( q \in A \). Finally, if \( p \) is the only element of \( A_\ell \), then

\[
\tilde{a}_0 := 1 \quad \text{and} \quad a^p := \frac{1}{\sigma} ,
\]

and \( a^p_q := 0 \) for all \( q \neq p \). Some basis functions are plotted in Figure 2.1.

**Figure 2.1.** The first few functions \( a^p k_p \) on \( I = [0, 1] \), constructed on dyadic grids (circles)
Lemma 2.2. This implies \( x \). Proof. If \( c > 0 \),

\[
\begin{align*}
\ell(a^p k_p)(x) &= \tilde{a}_0 e^{-t}+\tilde{a}_1 e^{-(t-h_+)} \\
&= -\frac{e^{-t}}{e^{2h_-}-1} + \frac{e^{2h_-+2h_+} - 1}{(e^{2h_-}-1)(e^{2h_+}-1)} + e^{-t} e^{2h_+} \\
&= \frac{e^{-t} - e^{2h_-+1} + e^{2h_-+2h_+} - 1 - e^{2h_-+2h_+} + e^{2h_+}}{(e^{2h_-}-1)(e^{2h_+}-1)} = 0.
\end{align*}
\]

This implies \( a^p k_p(x) = 0 \) for all \( x \geq p_{\ell+1}^\ell \). By symmetry, also \( a^p k_p(x) = 0 \) for all \( x \leq p_{\ell-1}^\ell \). Similar, but shorter, computations lead to the same result for boundary points.

Proposition 2.1. Let \( \ell \in \mathbb{N} \) with \( n_\ell \geq 2 \). Then for all \( p = p^\ell_\ell \in A_\ell \), the support of \( a^p k_p \) in \( I \) is

\[
\text{supp } a^p k_p = [p^\ell_{\ell-1}, p^\ell_{\ell+1}],
\]

where we set \( p^\ell_{\ell} := \min(I) \) and \( p^\ell_{n_\ell+1} := \max(I) \).

Proof. We first consider an interior point \( p = p^\ell_\ell \) of \( A_\ell \). Any \( x \geq p_{\ell+1} \) is of the form \( x = p^\ell_\ell + t \lambda \) for a \( t \geq h_+ = (p^\ell_{\ell+1} - p^\ell_\ell)/\lambda \). Then using the notation from (27), for a \( c > 0 \),

\[
\begin{align*}
\ell(a^p k_p)(x) &= \tilde{a}_0 e^{-t}+\tilde{a}_1 e^{-(t-h_+)} \\
&= -\frac{e^{-t}}{e^{2h_-}-1} + \frac{e^{2h_-+2h_+} - 1}{(e^{2h_-}-1)(e^{2h_+}-1)} + e^{-t} e^{2h_+} \\
&= \frac{e^{-t} - e^{2h_-+1} + e^{2h_-+2h_+} - 1 - e^{2h_-+2h_+} + e^{2h_+}}{(e^{2h_-}-1)(e^{2h_+}-1)} = 0.
\end{align*}
\]

This implies \( a^p k_p(x) = 0 \) for all \( x \geq p^\ell_{\ell+1} \). By symmetry, also \( a^p k_p(x) = 0 \) for all \( x \leq p^\ell_{\ell-1} \).

Lemma 2.2. For all \( p \in A \) and all \( x \in I \),

\[
0 \leq (a^p k_p)(x) \leq (a^p k_p)(p) = \frac{1}{\tilde{a}_p}. \tag{33}
\]

In particular, the maximal value of \( a^p k_p \) has the following form. If \( p \in A_\ell \) is an interior node of \( A_\ell \), then

\[
(a^p k_p)(p) = \sigma \sqrt{\frac{(e^{2h_-}-1)(e^{2h_+}-1)}{e^{2h_-+2h_+}-1}} \tag{34}
\]

with \( h_- \) and \( h_+ \) as above. If \( p \) is a boundary node of \( A_\ell \) with \( n_\ell \geq 2 \), then

\[
(a^p k_p)(p) = \sigma \sqrt{1 - e^{-2h}} \tag{35}
\]

with \( h \) as above, and if \( n_\ell = 1 \), then \( (a^p k_p)(p) = \sigma \).

Proof. We first compute \( (a^p k_p)(p) \). Let \( p \in A_\ell \) be an interior node of \( A_\ell \). Then

\[
(a^p k_p)(p) = \frac{1}{\sigma \sqrt{\tilde{a}_p}} (\tilde{a}_- \sigma^2 e^{-h_-} + \tilde{a}_0 \sigma + \tilde{a}_1 \sigma^2 e^{-h_+})
\]

\[
= \sigma \sqrt{\frac{(e^{2h_-}-1)(e^{2h_+}-1)}{e^{2h_-+2h_+}-1}}.
\]

If \( p \in A_\ell \) is a boundary node of \( A_\ell \) with \( n_\ell \geq 2 \),

\[
(a^p k_p)(p) = \frac{1}{\sigma \sqrt{1 - e^{-2h}}} \sigma^2 - \frac{e^{-h}}{\sigma \sqrt{1 - e^{-2h}}} \sigma^2 e^{-h} = \sigma \sqrt{1 - e^{-2h}}.
\]

Finally, if \( n_\ell = 1 \), then \( a^p k_p = \sigma^{-1} k_p \), and therefore \( (a^p k_p)(p) = \sigma \).
To prove (33), we show that the derivative of $a^pk_p$ is monotonic between elements of $\Lambda$. It suffices to show this on the interval $(p_i, p_{i+1})$ if $p = p_i$. For interior nodes, this is a consequence of

$$\frac{d}{dt} \left( \tilde{a}_1 e^{-h_1^2 h_1^+} + \tilde{a}_0 e^{-t} + \tilde{a}_1 e^{-(h_1^+ - t)} \right) = -\tilde{a}_1 e^{-t} - \tilde{a}_0 e^{-t} + \tilde{a}_1 e^{-h_1^+}$$

$$= -e^{-t} \left( e^{2h_1^+} + e^{2t} \right) \leq 0$$

for all $0 < t < h_+$. If $i = 1$, we consider

$$\frac{d}{dt} \left( \tilde{a}_0 e^{-t} + \tilde{a}_1 e^{-(h_1 - t)} \right) = -e^{-t} + e^{-2h_1^+} \leq 0,$$

for all $0 < t < h$, and if $i = n$, we have

$$\frac{d}{dt} \left( \tilde{a}_1 e^{-t} + \tilde{a}_0 e^{-t} \right) = -e^{-t} (1 - e^{-2h_1}) \leq 0,$$

for all $t > 0$. \hfill \Box

**Theorem 2.3.** The functions $(a^pk_p)_{p \in \Lambda}$ form an orthonormal system in $H(\gamma)$. If $\Lambda$ is dense in $I$, then $(a^pk_p)_{p \in \Lambda}$ is an orthonormal basis of $H(\gamma)$.

**Proof.** Let $p = p_i \in \Delta^c$ and $q \in \Delta \setminus \{p\}$. By Lemma 1.2 and Proposition 2.1,

$$(a^pk_p, a^pk_q)_{H(\gamma)} = 0$$

since $q$ is not in the interior of the support of $a^pk_p$.

Let $p_i \in \Delta \setminus \{p\}$. If $|i - j| \geq 2$, then $a^pk_p$ is a linear combination of $k_q$ for $q \in \Delta \setminus \{p\}$, and therefore $(a^pk_p, a^pk_p)_{H(\gamma)} = 0$. By Assumption 2.A, if $|i - j| \geq 2$, then $p_i \in \Delta \setminus \Delta_i$, i.e. $p_i \in \Delta^c$ for some $\ell \leq t - 1$. Consequently, $a^pk_p$ is a linear combination of $k_q$ for $q \in \Delta^c$, and thus again $(a^pk_p, a^pk_p)_{H(\gamma)} = 0$. Therefore, the functions $(a^pk_p)_{p \in \Lambda}$ are mutually orthogonal in $H(\gamma)$.

Using Lemma 2.2 and the orthogonality of $a^pk_p$ to $k_q$ for all $q \in \Delta \setminus \{p\}$,

$$\|a^pk_p\|_{H(\gamma)}^2 = (a^pk_p, a^pk_p)_{H(\gamma)} = a_p^p \delta_p(a^pk_p) = 1.$$

This shows that the functions $(a^pk_p)_{p \in \Lambda}$ are normalized in $H(\gamma)$.

Proposition 1.9 implies that if $\Lambda$ is dense in $I$, then $(k_p)_{p \in \Lambda}$ is dense in $H(\gamma)$. For every $\ell$, $(a^pk_p)_{p \in \Lambda}$ are $n_\ell$ linearly independent elements of the span of $(k_q)_{q \in \Lambda_i}$. Consequently, for all $q \in \Delta_i$, $k_q$ is a linear combination of $(a^pk_p)_{p \in \Lambda_i}$. \hfill \Box

### 2.2. Decay of basis elements.

**Proposition 2.4.** For all $p \in \Lambda$, if $p \in \Delta^c$ is an interior node of $\Lambda_i$, then

$$\|a^pk_p\|_{C(I)} \leq \sigma \min \left( e^{h_+} \sqrt{\frac{2h_- h_+}{h_- + h_+}}, 1 \right)$$

with $h_-$ and $h_+$ as above. If $p$ is a boundary node and $n_{\ell} \geq 2$, then

$$\|a^pk_p\|_{C(I)} \leq \sigma \sqrt{\min(2h, 1)}$$

with $h$ as above. If $n_{\ell} = 1$, then $\|a^pk_p\|_{C(I)} = \sigma$.

**Proof.** We use the elementary estimates

$$1 - e^{-t} \leq t \leq e^t - 1 \leq t e^t \quad \forall t \geq 0.$$

By Lemma 2.2, the maximum of $a^pk_p$ is attained at $p$ and is equal to $1/a_p^p$. The first part of (36) follows from (34) and

$$\frac{(e^{2h_+} - 1) (e^{2h_+} - 1)}{e^{2h_+} + 2h_+} \leq \frac{(2h_- e^{2h_-} - 2h_+ e^{2h_+})}{2h_- + 2h_+} = e^{2h_- + 2h_+} \frac{2h_- h_+}{h_- + h_+}.$$
For boundary nodes, (37) is a consequence of (35) and
\[ 1 - e^{-2h} \leq \sqrt{\min(2h, 1)} . \]

For the case \( n_\ell = 1 \), the claim is shown in Lemma 2.2.

It remains to be shown that (34) is bounded by \( \sigma \). To this end, we compute the derivative
\[ \frac{d}{dh_-} \frac{(e^{2h_-} - 1)(e^{2h_-} - 1)}{e^{2h_-} + 2h_+ - 1} = \frac{2e^{2h_-} - (e^{2h_-} - 1)^2}{(e^{2h_-} + 2h_+ - 1)^2} \geq 0 , \]
and similarly with \( h_+ \) and \( h_- \) switched. Let \( h := h_- + h_+ \) and \( \vartheta := h_-/h \). Then by monotonicity, for any fixed \( \vartheta \in (0, 1) \),
\[ \sup_{h \to \infty} \frac{(e^{2h} - 1)(e^{2(1-\vartheta)h} - 1)}{e^{2h} - 1} = \lim_{h \to \infty} \frac{(e^{2h} - 1)(e^{2(1-\vartheta)h} - 1)}{e^{2h} - 1} = \lim_{h \to \infty} 2\vartheta(1 - e^{-2(1-\vartheta)h}) + 2(1 - \vartheta)(1 - e^{-2h}) = 1 , \]
which concludes the proof. \( \square \)

**Proposition 2.5.** For all \( p \in A \), if \( p \in \Delta_\ell \) is an interior node of \( \Lambda_\ell \), then
\[ \|a^{p_{k_\ell}}\|_{L^2(I)} \leq \sigma \min \left(e^{h_- + h_+}, \sqrt{2h_- + h_+}, \sqrt{h_- + h_+} \right) \]  
with \( h_- \) and \( h_+ \) as above. If \( p \) is a boundary node and \( n_\ell \geq 2 \), then
\[ \|a^{p_{k_\ell}}\|_{L^2(I)} \leq \sigma \min(2h, 1)(h + h_0) , \]
with \( h \) as above and \( h_0 := (p - \min(I))/\lambda \) if \( p = p_{k_\ell} \) and \( h_0 := (\max(I) - p)/\lambda \) if \( p = p_{n_\ell} \). If \( n_\ell = 1 \), then
\[ \|a^{p_{k_\ell}}\|_{L^2(I)} \leq \sigma \sqrt{|I|} . \]

**Proof.** The claim follows from Proposition 2.1 and Proposition 2.4 using the estimate
\[ \|a^{p_{k_\ell}}\|_{L^2(I)} \leq \sqrt{\sup |a^{p_{k_\ell}}|} \|a^{p_{k_\ell}}\|_{C(I)} \]  
For small correlation lengths \( \lambda \), i.e. large values of \( h, h_- \), \( h_+ \) and \( h_0 \), the estimate in Proposition 2.5 is quite crude, but sufficient for our purposes.

**Example 2.6.** Let \( I := [0, 1] \), \( A_0 := \{1/2\} \) and \( \Delta_\ell := \{2^{-\ell} : i = 0, 1, \ldots, 2^\ell\} \) for all \( \ell \geq 1 \), as in Figure 2.1. Then for all \( \ell \in \mathbb{N} \) and all \( p \in \Delta_\ell \), \( h = h_- = h_+ = 2^{-\ell} \), and \( h_0 = 0 \). In particular, Propositions 2.4 and 2.5 imply
\[ \|a^{p_{k_\ell}}\|_{C(I)} \leq C_\sigma \frac{1}{\sqrt{n_\ell}} \text{ and } \|a^{p_{k_\ell}}\|_{L^2(I)} \leq C_\sigma \frac{1}{n_\ell} \quad \forall p \in \Delta_\ell . \]
Thus the convergence rate in \( L^2(I) \) of (8) and (9) coincides with that of the Karhunen–Loève series. At any given point \( x \in [0, 1] \), since only at most two basis functions per level are nonzero, the convergence of the series is exponential if all other basis functions are disregarded.

**2.3. The Wiener measure.** The same approach as in Section 2.1 can be used to construct an orthonormal basis of the Cameron–Martin space of the Wiener measure. Let \( I = [0, 1] \) and
\[ b(x, y) := \min(x, y) , \quad x, y \in I . \]
Furthermore, let \( A_\ell = \{p_1^\ell, \ldots, p_{n_\ell}^\ell\} \), \( A_\ell \uparrow A =: p \), be a hierarchic sequence of grids satisfying Assumption 2.2A, and not containing \( 0 \).

Proceeding as in the case of an exponential covariance kernel, we construct for each \( p \in A \) a function \( a^{p_{k_\ell}} \) that is a linear combination of \( k_q \) for at most three
q ∈ Λ. Since k_q is piecewise linear for all q ∈ Λ, the orthogonality conditions on a^p k_p imply that a^p k_p is a multiple of a hat function.

For each p ∈ Λ, we define a hat function ζ_p as follows. For p ∈ ∆_ℓ, ζ_p is the piecewise linear function on I subordinate to the nodes A_ℓ ∪ {0, 1} such that ζ_p(q) = δpq for all q ∈ A_ℓ, ζ_p(0) = 0 and ζ_p(1) = ζ_p(p_{ℓ-1}).

**Proposition 2.7.** For all p ∈ Λ,

\[ a^p k_p = \frac{1}{a_p} \zeta_p. \tag{43} \]

**Proof.** We consider the case p = p_i ∈ ∆_ℓ with 2 ≤ i ≤ n_ℓ - 1. The case of boundary points is similar. Then, since a^p k_p is piecewise linear, the condition

\[ 0 = (a^p k_p, k_q)_{H(\gamma)} = (a^p k_p)(q) \]

for q = p_i±1 implies that a^p k_p is a multiple of ζ_p. The third defining condition,

\[ 1 = \|a^p k_p\|_{H(\gamma)}^2 = (a^p k_p, a^p k_p)_{H(\gamma)} = a_p^2 (a^p k_p)(p) \]

leads to the claim. □

The values of a_p can be determined by explicitly solving the linear system defining a^p. Let p = p_i ∈ ∆_ℓ. If i = n_ℓ = 1, then

\[ a_p = \frac{1}{\sqrt{p}}. \tag{44} \]

If n_ℓ ≥ 2 and i = 1, for h := p_2 - p_1,

\[ a_p = \sqrt{\frac{p + h}{ph}}, \tag{45} \]

whereas if i = n_ℓ, for h := p_{n_ℓ} - p_{n_ℓ-1},

\[ a_p = \frac{1}{\sqrt{h}}. \tag{46} \]

For interior points, i.e. if 2 ≤ i ≤ n_ℓ - 1,

\[ a_p = \sqrt{\frac{h_- + h_+}{h_- h_+}}, \tag{47} \]

for h_- := p_i - p_{i-1} and h_+ := p_{i+1} - p_i.

This method provides an alternative derivation of the well-known piecewise linear basis for Brownian motion due to Lévy and Ciesielski, which is more commonly derived by integrating L^2([0, 1])-orthonormal Haar wavelets.

The statements of Proposition 2.1 and Lemma 2.2 for the Wiener measure are a direct consequence of Proposition 2.7. In particular, Theorem 2.3 applies, as the proof requires only these two statements and is not specific to exponential covariances. Bounds similar to those given in Section 2.2 also hold in the case of the Wiener measure, and follow from Proposition 2.7 and the above values of a_p.

### 3. Numerical construction of hierarchic bases

#### 3.1. A general algorithm

Let (e_m)_{m∈Θ} be an orthonormal system in H(γ) for a finite, possible empty, set Θ. We wish to determine a finite sequence (e_m)_{m∈Θ}, in H(γ) such that (e_m)_{m∈Θ} is an orthonormal system in H(γ).

Let p := (p_i)_{i∈A} be a sequence in D. We assume that, for a finite set A_0 ⊂ A, e_m is in the span of (k_{p_i})_{i∈A_0} for all m ∈ Θ, and allow e_m to be in the span of
\((k_p)_i \in A_0 \cup A_1\) for a second finite set \(A_1 \subset A\). The coefficients of \((e_m)_{m \in \Theta_0}\) with respect to \((k_p)_i \in A_0\) can be stored as the columns of a matrix,

\[
A_0 := [a^m]_{m \in \Theta_0}, \quad e_m = \sum_{i \in A_0} a_i^m k_p, \quad m \in \Theta_0.
\]  

(48)

We also consider the covariance matrix of \((e_m)_{m \in \Theta_0}\) and \((k_p)_i \in A_0\),

\[
F_0 := [(k_p_i, e_m)_{H(\gamma)}]_{m \in \Theta_0, i \in A_0} = [e_m(p_i)]_{m \in \Theta_0, i \in A_0},
\]

where \(m\) indexes the rows and \(i\) the columns of \(F_0\). Although this matrix is not required in order to augment the orthonormal system \((e_m)_{m \in \Theta_0}\), it is computed as a byproduct of our algorithm.

\[
\text{AUGMENTBasis}[k, p, A_0, A_1, A_0, F_0] \mapsto [A, F]
\]

\[
K_0 \leftarrow [k(p_i, p_j)]_{i \in A_0, j \in A_1}, \quad K_1 \leftarrow [k(p_i, p_j)]_{i, j \in A_1},
\]

\[
C \leftarrow A_1 K_0, \quad [V, D] \leftarrow \text{eig}(K_1 - C^\top C)
\]

\[
B \leftarrow V D^{-1/2}, \quad A \leftarrow \begin{pmatrix} A_0 & -A_0CB \\ 0 & B \end{pmatrix}, \quad F \leftarrow \begin{pmatrix} F_0 & C \\ 0 & D^{1/2}V^\top \end{pmatrix}
\]

In \text{AUGMENTBasis}, \text{eig}, with \(\epsilon \geq 0\) is a function that computes some nonzero eigenpairs of a matrix. We assume that the largest eigenvalue not computed by \text{eig} has magnitude less than or equal to \(\epsilon\).

Remark 3.1. If \(A_0 \cap A_1 \neq \emptyset\), then the matrices \(A\) and \(F\) constructed by the algorithm \text{AUGMENTBasis} do not have the block structure indicated in the formulation of the algorithm. Rather, the rows of \(A\) corresponding to the same index \(i \in A_0 \cap A_1\) are assumed to be summed. In \(F\), the new value of a column corresponding to \(i \in A_0 \cap A_1\) overwrites the old value.

Proposition 3.2. Let \(A_0\) and \(F_0\) be as in (48) and (49) for an orthonormal system \((e_m)_{m \in \Theta_0}\) in \(H(\gamma)\). Let the set \(\Theta\) index the columns of \(A\) generated by \text{AUGMENTBasis}, and \(\Theta_1 := \Theta \setminus \Theta_0\). For all \(m \in \Theta\), let

\[
e_m := \sum_{i \in A_0 \cup A_1} a_i^m k_p, \quad (50)
\]

where \(a^m := (a_i^m)\) is the \(m\)-th column of \(A\). Then the sequence \((e_m)_{m \in \Theta}\) is an orthonormal system in \(H(\gamma)\). For all \(m \in \Theta_0\), (50) coincides with (48). Furthermore, for all \(m \in \Theta\), the \(m\)-th row of \(F\) is the vector \((e_m(p_i))_{i \in A_0 \cup A_1}\). In particular, if \(m \in \Theta_1\) and \(i \in A_0 \setminus A_1\), then \(e_m(p_i) = 0\).

Proof. We can assume without loss of generality that \(A_0 \cap A_1 = \emptyset\) since \(p_i = p_j\) is possible for \(i \neq j\). Identifying such indices leads to the compression described in Remark 3.1.

By definition of \(A\) and (48), the columns of

\[
E := \begin{pmatrix} I & -CB \\ 0 & B \end{pmatrix}
\]
represent \((e_m)_{m \in \Theta}\) with respect to \((e_m)_{m \in \Theta_0} \cup \{(p_i)_{i \in A_1}\). Furthermore, the covariance matrix of \((e_m)_{m \in \Theta_0} \cup \{(p_i)_{i \in A_1}\} is

\[
L := \begin{pmatrix} I & A_0^\top K_0 \\ A_0 \otimes K_1 \end{pmatrix} = \begin{pmatrix} I & C \\ C^\top & K_1 \end{pmatrix}.
\]

Thus the covariance matrix of \((e_m)_{m \in \Theta}\) is

\[
E^\top LE = \begin{pmatrix} I & -B^\top C \cdot \gamma \\ -B^\top C \cdot \gamma & 0 \end{pmatrix} \begin{pmatrix} I & C \\ C^\top & K_1 \end{pmatrix} \begin{pmatrix} I & -B \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B^\top(K_1 - C^\top C)B \end{pmatrix}.
\]

By definition of \(B\),

\[
B^\top(K_1 - C^\top C)B = I.
\]

Therefore, \((e_m)_{m \in \Theta}\) is an orthonormal system in \(H(\gamma)\), cf. Corollary 1.11.

Finally, we have

\[
FA = \begin{pmatrix} F_0 A_0 & -F_0 A_0 CB + CB \\ 0 & D_{1/2} V^\top B \end{pmatrix} = I,
\]

since \(F_0 A_0 = I\) by orthonormality of \((e_m)_{m \in \Theta_0}\) in \(H(\gamma)\). Therefore, \(FA\) is the matrix representation of

\[
h = \sum_{m \in \Theta} (h, e_m)_{H(\gamma)} e_m \quad \forall h \in \text{span}(e_m)_{m \in \Theta}
\]

in the basis \((k_{p_i})_{i \in A_0 \cup A_1}\). This implies that the \(m\)-th row of \(F\) is the vector \(((k_{p_i}, e_m)_{H(\gamma)})_{i \in A_0 \cup A_1}\), and the claim follows using Lemma 1.2.

\[\square\]

Remark 3.3. Let \((p_i)_{i \in A}\) be a finite sequence in \(D\) such that \((k_{p_i})_{i \in A}\) are linearly independent, and let

\[
K := [(R, \delta p_i, \delta p_j)]_{i,j \in A} = [k(p_i, p_j)]_{i,j \in A} = [(k_{p_i}, k_{p_j})_{H(\gamma)}]_{i,j \in A}
\]

be the covariance matrix of \((\delta p_i)_{i \in A}\). Then \(K\) is symmetric positive definite, and thus allows a factorization of the form

\[
K = F^\top F
\]

with a regular matrix \(F\). Defining \(A := F^{-1}\), we have

\[
A^\top KA = A^\top F^\top FA = I,
\]

and by Theorem 1.11, the columns of \(A\) are the coefficients with respect to \((k_{p_i})_{i \in A}\) of an orthonormal system in \(H(\gamma)\). Since \(A\) is square, this orthonormal system is an orthonormal basis of the linear span of \((k_{p_i})_{i \in A}\) in \(H(\gamma)\). Generalizing to rank deficient \(K\), we assume \(F\) to have full row rank, and define \(A\) as the right inverse of \(F\). The routine \textsc{AugmentBasis} with \(\epsilon = 0\) hierarchically constructs such \(F\) and \(A\). With positive \(\epsilon\), it adds a compression of the covariance matrix, and (52) only holds approximately.

We consider the following hierarchically constructed \((e_m)_{m \in \Theta} \subset H(\gamma)\). Let \(p = (p_i)_{i \in A}\) be a sequence in \(D\), and let \((e_m)_{m \in \Theta_0}\) be an arbitrary finite orthonormal system in \(H(\gamma)\) in the linear hull of \((k_{p_i})_{i \in A_0}\) for a finite set \(A_0 \subset A\). For all \(\ell \in \mathbb{N}\), let \(A_\ell \subset A\) be a finite set, and let \((e_m)_{m \in \Theta_\ell}\) be the sequence in \(H(\gamma)\) constructed by applying \textsc{AugmentBasis} to \((e_m)_{m \in \Theta_\ell}, \epsilon \leq \ell - 1\) as in Proposition 3.2 with \(\epsilon = \epsilon_\ell\). We define \(\Theta := \bigcup_{\ell \in \mathbb{N}_0} \Theta_\ell\).
Corollary 3.6.

Corollary 3.7.

Proof.

Therefore, using Lemma 3.4 and the definition of Proposition 3.5.

Proof.

For all \( \ell \in \mathbb{N}_0 \), let \( P_\ell \) be the orthonormal projection in \( H(\gamma) \) onto the span of \( e_m \) for \( m \in \Theta_{\ell'} \) with \( \ell' \leq \ell \). It has the form

\[
P_\ell h = \sum_{\ell' = 0}^{\ell} \sum_{m \in \Theta_{\ell'}} \langle h, e_m \rangle_H(\gamma) e_m, \quad h \in H(\gamma) .
\]  

(53)

For all \( \ell \in \mathbb{N}_0 \), let

\[
K_\ell := K|_{\Lambda_1 \times \Lambda_1} = [(k_{p_i}, k_{p_j})_{H(\gamma)}]_{i,j \in A_\ell}
\]  

(54)

be the submatrix of \( K \) from (51) on the index set \( A_\ell \). Using \( P_\ell \), we define the approximation

\[
\tilde{K}_\ell := [(P_\ell k_{p_i}, P_\ell k_{p_j})_{H(\gamma)}]_{i,j \in A_\ell}
\]  

(55)

so \( \tilde{K}_\ell \) is computable directly from \( e_m \) for \( m \in \Theta_{\ell'}, \ell' \leq \ell \).

Lemma 3.4. Using the notation from \textsc{AugmentBasis},

\[
\tilde{K}_1 = (F^T F)|_{\Lambda_1 \times \Lambda_1} = C^T C + V D V^T .
\]  

(57)

Proof. By Proposition 3.2 and (53),

\[
F^T F = \sum_{\ell \in \Theta_0 \cup \Theta_1} (k_{p_i}, e_m)_{H(\gamma)} (k_{p_j}, e_m)_{H(\gamma)}
\]  

\[= [(P_\ell k_{p_i}, P_\ell k_{p_j})_{H(\gamma)}]_{i,j} ,
\]

where \( i, j \in A_0 \cup A_1 \). We compute

\[
F^T F = \begin{pmatrix} F_0^T & 0 \\ C^T V D^{1/2} \\ 0 & D^{1/2} V \end{pmatrix} \begin{pmatrix} F_0 \\ C \\ 0 \end{pmatrix} = \begin{pmatrix} F_0^T F_0 & F_0^T C \\ C^T F_0 & C^T C + V D V^T \end{pmatrix} .
\]

Therefore, for all \( i, j \in A_1 \), \([\tilde{K}_1]_{ij} = [F^T F]_{ij} = [C^T C + V D V^T]_{ij} .
\]

\( \square \)

Proposition 3.5. For all \( \ell \in \mathbb{N} \) and all \( b, c \in \mathbb{R}^{\Lambda_1} \subset \mathbb{R}^d \),

\[
\left| \langle R, b \delta_p, c \delta_p \rangle - b^T \tilde{K}_1 c \right| \leq \epsilon_\ell \|b\|_{\ell^2} \|c\|_{\ell^2} .
\]  

(58)

Proof. We assume without loss of generality that \( \ell = 1 \). By (20),

\[
\langle R, b \delta_p, c \delta_p \rangle = b^T K_1 c .
\]

Therefore, using Lemma 3.4 and the definition of \( \epsilon_\ell \),

\[
\left| \langle R, b \delta_p, c \delta_p \rangle - b^T \tilde{K}_1 c \right| = |b^T (K_1 - C^T C - V D V^T) c| \leq \epsilon_\ell \|b\|_{\ell^2} \|c\|_{\ell^2} .
\]

\( \square \)

Corollary 3.6. For all \( \ell \in \mathbb{N} \) and all \( i, j \in A_\ell \),

\[
|k(p_i, p_j) - (P_\ell k_{p_i}, P_\ell k_{p_j})_{H(\gamma)}| \leq \epsilon_\ell .
\]  

(59)

Proof. The claim follows from Proposition 3.5 and (13).

\( \square \)

Corollary 3.7. For all \( \ell \in \mathbb{N} \) and all \( i, j \in A_\ell \),

\[
|k(p_i, p_j)|_{H(\gamma)} \leq \sqrt{\epsilon_\ell}.
\]  

(60)

Proof. Using orthogonality of the projection \( P_\ell \) and Corollary 3.6 with \( j = i \),

\[
|k(p_i, P_\ell k_{p_i})|^2_{H(\gamma)} = |k(p_i)|^2_{H(\gamma)} - |P_\ell k_{p_i}|^2_{H(\gamma)} \leq \epsilon_\ell .
\]

\( \square \)
Theorem 3.8. If \( p = (p_i)_{i \in A} \) is dense in \( D \), \( \bigcup_{\ell \in \mathbb{N}_0} A_\ell = A \) and \( \epsilon_\ell \to 0 \) as \( \ell \to \infty \), then \( (e_m)_{m \in \Theta} \) is an orthonormal basis of \( H(\gamma) \).

Proof. Orthonormality of \( (e_m)_{m \in \Theta} \) follows from Proposition 3.2. To prove density, it suffices by Proposition 1.9 to show that \( k_x \) is in the span of \( (e_m)_{m \in \Theta} \) for all \( x \in D \).

Let \( x \in D \). Then there is a sequence \( (i_n)_{n \in \mathbb{N}} \) in \( A \) such that \( x_n := p_{i_n} \to x \) and \( i_n \neq i_{n'} \) if \( n \neq n' \). Since \( \|\delta_{x_n}\|_{C(D)} = 1 \) for all \( n \), Lemma 1.3 implies \( k_{x_n} \to k_x \) in \( H(\gamma) \). For all \( n \in \mathbb{N} \), let \( \ell_n \in \mathbb{N}_0 \) with \( i_n \in A_{\ell_n} \). Then \( \ell_n \to \infty \) since \( A_\ell \) is finite for all \( \ell \in \mathbb{N}_0 \). Due to Corollary 3.7, \( P_{i_n} k_{x_n} \to k_x \) by the assumption \( \epsilon_\ell \to 0 \). \( \square \)

3.2. Gram–Schmidt orthogonalization in the Cameron–Martin space. We consider AUGMENTBASIS with \( \epsilon = 0 \) for \( A_1 = \{j\} \), where \( j \) is assumed to be an element of \( A \setminus A_0 \). In this case, AUGMENTBASIS reduces to Gram–Schmidt orthogonalization in the Cameron–Martin space, and can be found in [9, Chapter 9]. An explicit formulation is given in GRAM SchmidtStep.

GramSchmidtStep\([k, p, A_0, j, A_0, F_0] \to [A, F]\]

\[ k_0 \leftarrow [k(p_i, p_j)]_{i \in A_0} \]
\[ c \leftarrow A_0 k_0 ; \]
\[ d \leftarrow k(p_j, p_j) - c^\top c ; \]
\[ \text{if } d = 0 \text{ then} \]
\[ A \leftarrow \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \]
\[ F \leftarrow \begin{pmatrix} F_0 \\ c \end{pmatrix} \]
\[ \text{else} \]
\[ A \leftarrow \begin{pmatrix} A_0 & -s^{-1} A_0 c \\ 0 & s^{-1} \end{pmatrix} \]
\[ F \leftarrow \begin{pmatrix} F_0 \\ c \end{pmatrix} \]

Remark 3.9. Let \( (p_i)_{i=1}^n \subset D \) such that \( (k_{p_i})_{i=1}^n \) are linearly independent, and let \( (e_i)_{i=1}^n \) be constructed by recursive application of GRAM SchmidtStep. Furthermore, let \( K \) be the covariance matrix of \( (\delta_{p_i})_{i=1}^n \), and let \( F \) be the last output of GRAM SchmidtStep. By Proposition 3.2,

\[ F^\top F = \left[ \sum_{m=1}^n (k_{p_m}, e_m)_{H(\gamma)} (k_{p_j}, e_m)_{H(\gamma)} \right]_{i,j=1}^n = \left[ (k_{p_i}, k_{p_j})_{H(\gamma)} \right]_{i,j=1}^n = K. \]

Furthermore, it follows by induction that \( F \) is an upper triangular matrix with positive diagonal entries. Therefore, \( F \) is the right Cholesky factor of \( K \).

Remark 3.10. The basis functions \( (e_m)_{m \in \mathbb{N}} \) can be characterized independently of each other. For a fixed sequence \( (p_i)_{i \in \mathbb{N}} \in D \), \( e_m \) is in the span of \( (k_{p_i})_{i=1}^m \) and orthogonal to \( k_{p_i} \) for all \( i \leq m - 1 \). This defines \( e_m \) uniquely up to a scalar factor, which is determined by normalization in \( H(\gamma) \).

3.3. Hierarchic spectral decomposition of the covariance matrix. As a second particular case of AUGMENTBASIS, we assume that the sets \( A_\ell \) are nested,

\[ A_0 \subset A_1 \subset \cdots \subset A_{\ell} \subset A_{\ell+1} \subset \cdots \subset A, \]
and, in view of Theorem 3.8,
\[ A = \bigcup_{\ell=0}^{\infty} A_\ell, \]  
\[ i.e., A_\ell \uparrow A. \] For example, \( (p_i)_{i \in A_\ell} \) could be the nodes of nested grids of \( D \). In this setting, Proposition 3.5 applies to all nodes \( p_i \) visited up to level \( \ell \) for any \( \ell \in \mathbb{N} \).

The routine \texttt{AugmentBasis} is customized to (61) in \texttt{AugmentSpectralBasis}. In particular, \texttt{AugmentSpectralBasis} takes into account Remark 3.1.

\texttt{AugmentSpectralBasis}[k, p, A_0, A_1, A_0] \rightarrow [A, F]

\begin{align*}
K &\leftarrow [k(p_i, p_j)]_{i,j \in A_1} \\
K_0 &\leftarrow K\big|_{A_0 \times A_1} \\
C &\leftarrow A_1^T K_0 \\
[V, D] &\leftarrow \text{eig}(K - C^T C) \\
B &\leftarrow \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} \leftarrow V D^{-1/2} ; \quad // \ B_0 = B\big|_{\Lambda_0 \times A_1} \text{ and } B_1 = B\big|_{(A_1 \setminus A_0) \times A_1} \\
A &\leftarrow \begin{pmatrix} A_0 & B_0 - A_0 C B \\ 0 & B_1 \end{pmatrix} \\
F &\leftarrow \begin{pmatrix} C \\ D^{1/2} V^T \end{pmatrix}
\end{align*}

Remark 3.11. A common but seemingly naive approximation to the Karhunen–Loève series consists of computing some or all eigenpairs of the covariance matrix \( K \), e.g. on the vertices of a finite element mesh, in place of more precise representations of eigenfunctions of the covariance operator. Let \( p = (p_i)_{i \in A} \) be a finite sequence in \( D \). Then this procedure is similar to \texttt{AugmentSpectralBasis} with \( A_0 = \emptyset \) and \( A_1 = A \). However, we provide a different interpretation. Instead of being approximations to the eigenfunctions of the covariance operator, the eigenvectors of the covariance matrix define an orthonormal system in \( H(\gamma) \) in their own right,
\[ e_m := \lambda_m^{-1/2} \sum_{i \in A} v_i^m k_{p_i}, \]  
\[ (63) \]
where \( v^m = (v_i^m)_{i \in A} \) is a normalized eigenvector of \( K \) with eigenvalue \( \lambda_m \). Thus the functions \( e_m \) are defined on all of \( D \), not just at the discrete points \( (p_i)_{i \in A} \). This decouples their construction from any other discretization of \( D \), such as a finite element mesh.

4. Numerical examples

4.1. Exponential kernels. We consider covariance kernels of the form
\[ k(x, y) := e^{-|x-y|^\alpha/\lambda^\alpha}, \quad x, y \in [0, 1]^d, \]  
\[ (64) \]
with parameters \( \alpha \in [1, 2] \) and \( \lambda > 0 \), on the \( d \)-dimensional unit hypercube \( D := [0, 1]^d \). We assume throughout that \( \lambda = 1/4 \).

Figures 4.1 and 4.2 show a few basis functions generated by the Gram–Schmidt method from Section 3.2 and the hierarchic spectral method from Section 3.3 in one dimension, with \( \alpha = 1 \) and \( \alpha = 1.5 \), respectively. The sets \( (p_i)_{i \in A_\ell} \) are hierarchic dyadic grids,
\[ (p_i)_{i \in A_\ell} = \{ i2^{-\ell}; \ i = 0, 1, \ldots, 2^\ell \}. \]  
\[ (65) \]
The new points on each level are marked in the plots by circles. The Gram–Schmidt method adds the points from left to right. The functions generated by Gram–Schmidt for \( \alpha = 1 \), shown in Figure 4.1a, coincide with those derived in
Section 2.1. In particular, by Proposition 2.1, they have compact support. This is not the case for the functions generated by the hierarchic spectral method, cf. Figure 4.1b. For $\alpha = 1.5$, though the basis functions generated by both methods have global support, those generated by Gram–Schmidt appear to be more localized in Figure 4.2.

![Figure 4.1. The first few functions for $\alpha = 1$](image1)

![Figure 4.2. The first few functions for $\alpha = 1.5$](image2)

In higher dimensions, the dyadic grids are given by

$$\{(p_i)_{i \in \Lambda_L} = \{i2^{-\ell} \; i \in \{0, 1, \ldots, 2^\ell\}^d\},$$

and the Gram–Schmidt method adds points in lexicographic order.

The decay of the $C(D)$ and $L^2(D)$ norms of the basis functions generated by Gram–Schmidt and the hierarchic spectral method for $\alpha = 1$ are plotted in Figures 4.3, 4.4 and 4.5, for one, two and three dimensions, respectively. Figure 4.6 shows the same for $\alpha = 1.5$ and $d = 1$. The decay is compared to that of the spectral basis computed directly on the finest level. For the hierarchic spectral method, tolerances $\epsilon_L = 0$ and $\epsilon_L = 5 \cdot 2^{-\alpha \ell}$ are used. In the latter case, the hierarchic spectral method generates fewer basis functions than the other methods.

In each case, the rate of decay in the $L^2(D)$-norm is the same for all basis functions considered. It is approximately

$$\|e_m\|_{L^2(D)} \sim m^{-\alpha}.$$
A staircase effect is apparent, in particular for the Gram–Schmidt method. This is due to the uneven spacing of points between levels of the hierarchic dyadic grids. Within each level, the relative position of a node to all previous nodes is very similar for all nodes. Between levels, the distances scale by a factor of two. In two and three dimensions, the functions constructed by the hierarchic spectral method display jumps in the opposite direction between levels. At these points, the covariance matrix is refined, and the subsequent basis functions correspond to eigenvectors with large eigenvalues of the difference between the coarse and fine covariance matrices.

The rate of decay in $C(D)$ coincides with that in $L^2(D)$ for the hierarchic spectral method. However, the decay of the functions generated by the Gram–Schmidt method is slower in $C(D)$ than in $L^2(D)$. For $\alpha = 1$ and $d = 1$, this is shown in Example 2.6.

The slower decay in $C(D)$ seems to be the cost for the better localization in space of the basis functions generated by the Gram–Schmidt method. For $\alpha = 1$ and $d = 1$, these functions have compact support, and the size of the support decreases at the rate $m^{-1}$. In other cases, though their supports are not compact, the functions generated by Gram–Schmidt are still almost local. Figure 4.7 illustrates the decay of basis functions generated by the Gram–Schmidt method for $\alpha = 1.5$ and $d = 1$. 

![Graph](image-url)
The Gaussian kernel on $D = [0, 1]^d$ with correlation length $\lambda > 0$ is

$$k(x, y) := e^{-|x-y|^2/\lambda^2}, \quad x, y \in D.$$  

The main difference between the Gaussian kernel (69) and the exponential kernels (64) with $1 \leq \alpha < 2$ is that the eigenvalues of the covariance operator associated to the Gaussian kernel decay exponentially, opposed to algebraic decay for exponential kernels. Again, we set $\lambda$ to $1/4$ in all computations. Figure 4.8 shows the first few basis functions generated by the Gram–Schmidt method and the hierarchic spectral method in one dimension.

The decay of the basis functions generated by both of the above methods for Gaussian kernels is plotted in Figures 4.9 and 4.10 in one and two dimensions, respectively. These are compared to the spectral basis computed directly on the
Figure 4.7. localization of functions generated by Gram–Schmidt for $\alpha = 1.5, d = 1$

Figure 4.8. the first few functions for a Gaussian kernel

Figure 4.9. decay of the basis functions for $d = 1$

finest level. In all cases, the decay is exponential, with approximately the same rate. However, the Gram–Schmidt method suffers from an instability, limiting its convergence. It can be stabilized by replacing the Cholesky decomposition by a
suitable generalization of a pivoted Cholesky decomposition, as was studied in [4] for finite sets of points.

The hierarchic spectral method is shown with tolerances $\epsilon_\ell = 0$ and $\epsilon_\ell = 2^{-2\ell}$. In the latter case, only very few basis functions $e_m$ are constructed, and their norms are very close to those of the basis constructed with no truncation. Figure 4.11 shows the number of new basis functions constructed on each level, i.e. after each refinement of the covariance matrix. The Gram–Schmidt method and the hierarchic spectral method with $\epsilon_\ell = 0$ construct one basis function for each point at which the covariance kernel is evaluated. Accordingly, the number of new basis functions per level increases exponentially. Conversely, the hierarchic spectral method with positive tolerances $\epsilon_\ell = 2^{-2\ell}$ constructs far fewer basis functions. The number of new basis functions constructed on each level seems to be bounded independently of the level.

4.3. **Spherical covariance kernels.** The spherical covariance kernel is given by

$$k(x, y) := \sigma^2 \rho_3 \left( \frac{|x - y|}{\lambda} \right)$$

in three dimensions or less, with positive parameters $\sigma$ and $\lambda$, where

$$\rho_3(z) := 1 - \frac{3}{2} z^2 + \frac{1}{2} z^3, \quad z \in [0, 1],$$
and $\rho_3(z) := 0$ if $z \geq 1$ is, up to a factor, the volume of intersection of two spheres with diameter 1 and midpoints separated by $z$. Similar constructions exist for balls of dimension different from three.

We consider the covariance kernel (70) on $[0,1]^d$, $d \in \{1,2,3\}$, with $\sigma = \lambda = 1$. We use the hierarchic dyadic grids $(p_i)_{i \in \Lambda}$ defined in Section 4.1. The first few basis functions generated by the Gram–Schmidt and hierarchic spectral methods are shown in Figure 4.12.

**Figure 4.12.** the first few functions for a spherical covariance

![Figure 4.12](image)

**Figure 4.13.** decay of the basis functions for a spherical covariance and $d = 1$

Figures 4.13, 4.14 and 4.15 show the decay of the basis functions generated by Gram–Schmidt and the hierarchic spectral method in one, two and three dimensions, respectively. The behavior is very similar to that of the exponential covariance kernel with $\alpha = 1$, discussed in Section 4.1.

**Conclusion**

Orthonormal bases of the Cameron–Martin space of a Gaussian measure on $C(D)$ can be constructed explicitly, without resorting to eigenfunctions of the covariance operator. Their construction uses only the covariance kernel, which is readily available in many stochastic models. No assumptions are made on the structure of the kernel.
The covariance kernel is evaluated on an unstructured discrete set of points. Generally, one basis function can be computed for every evaluation point. The general algorithm for constructing such bases can be formulated on the level of numerical linear algebra involving the covariance matrix, and, as such, is amenable to implementation. In exact arithmetic, it constructs exact representations of the basis functions.

The bases constructed in this manner are hierarchic. Elements computed on an initial coarse grid of evaluation points are left unchanged when additional basis functions are computed on the same grid or on a refined grid. This flexible construction of the basis elements may lend itself to adaptive or iterative methods that require representations of a Gaussian field with various degrees of accuracy. The representation can be refined locally by selectively adding evaluation points of the covariance kernel. Also, the initial basis functions represent the most dominant components of the random field, and constructing these on a coarse grid of evaluation points leads to simple representations of these functions in terms of the covariance kernel at only a few points.

Numerical experiments indicate that the decay of the basis functions in $L^2(D)$ is comparable to that of the square root of the eigenvalues of the covariance operator, which is known to be optimal. This is confirmed by explicit computations in the
case of an exponential covariance kernel in one dimension. The hierarchic spectral method for constructing basis functions is particularly close to optimal in this respect, and stable in the case of ill-conditioned covariance matrices, e.g. resulting from Gaussian covariance kernels.

Bases constructed by Gram–Schmidt orthogonalization in the Cameron–Martin space are often spatially localized. In some settings, e.g. an exponential covariance kernel in one dimension or the Wiener measure, the basis functions have compact supports with diameter tending to zero. Furthermore, the basis functions can be characterized independently of each other as functions in certain finite dimensional spaces with given zeros. Thus, if the covariance kernel is modified, the basis functions can be updated independently of each other.

References


Seminar für Angewandte Mathematik, Eidgenössische Technische Hochschule Zürich, Rämistrasse 101, CH-8092 Zurich, Switzerland
E-mail address: claude.gittelson@sam.math.ethz.ch
<table>
<thead>
<tr>
<th>No.</th>
<th>Authors/Title</th>
</tr>
</thead>
</table>
| 10-15| *C.J. Gittelson*  
Representation of Gaussian fields in series with independent coefficients |
| 10-14| *R. Hiptmair, J. Li and J. Zou*  
Convergence analysis of Finite Element Methods for $H(\text{div};\Omega)$-elliptic interface problems |
| 10-13| *M.H. Gutknecht and J.-P.M. Zemke*  
Eigenvalue computations based on IDR |
| 10-12| *H. Brandsmeier, K. Schmidt and Ch. Schwab*  
A multiscale $hp$-FEM for 2D photonic crystal band |
| 10-11| *V.H. Hoang and C. Schwab*  
Sparse tensor Galerkin discretizations for parametric and random parabolic PDEs. I: Analytic regularity and gpc-approximation |
| 10-10| *V. Gradinaru, G.A. Hagedorn, A. Joye*  
Exponentially accurate semiclassical tunneling wave functions in one dimension |
| 10-09| *B. Pentenrieder and C. Schwab*  
$hp$-FEM for second moments of elliptic PDEs with stochastic data. Part 2: Exponential convergence |
| 10-08| *B. Pentenrieder and C. Schwab*  
$hp$-FEM for second moments of elliptic PDEs with stochastic data. Part 1: Analytic regularity |
| 10-07| *C. Jerez-Hanckes and J.-C. Nédélec*  
Asymptotics for Helmholtz and Maxwell solutions in 3-D open waveguides |
| 10-06| *C. Schwab and O. Reichmann*  
Numerical analysis of additive, Lévy and Feller processes with applications to option pricing |
| 10-05| *C. Schwab and R. Stevenson*  
Fast evaluation of nonlinear functionals of tensor product wavelet expansions |
| 10-04| *B.N. Khoromskij and C. Schwab*  
Tensor-structured Galerkin approximation of parametric and stochastic elliptic PDEs |