Derivation of a Linear, Robust $\mathcal{H}_2$ Controller for Systems with Parametric Uncertainty

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Abstract—In this short paper, we derive a linear, robust $\mathcal{H}_2$ controller for a linear system with parametric uncertainty as introduced in [1]. We prove that the linear, robust control law proposed in [1] minimizes the $\mathcal{H}_2$ norm of the linear, uncertain system. Thus we provide the proof for Theorem 1 in the mentioned paper.

I. PROBLEM DEFINITION

Let $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the system states and inputs of a linear, discrete-time system. In the general framework of robust control (see Fig. 1), the objective is to minimize an error signal, $z$, caused by a disturbance, $w \in \mathbb{R}^l$, for all possible uncertainties introduced via an uncertain signal, $p \in \mathbb{R}^l$, while maintaining stability (cf. [2]). This uncertain system can be represented by

$$\begin{align*}
x_{k+1} &= Ax_k + Bu_k + B_w w_k + B_p p_k \quad (1)
z_k &= C_z x_k + D_z u_k \quad (2)
q_k &= C_q x_k + D_q u_k \quad (3)
p_k &= \alpha \Delta q_k, \quad (4)
\end{align*}$$

where the subscript $k$ indicates the time step. The diagonal matrix $\Delta = \text{diag}(\delta_1, \ldots, \delta_f)$ with uncertain elements $|\delta_i| \leq 1$, $i = 1, \ldots, f$, together with $C_q$, $D_q$, and $B_p$ represent the uncertainty in the system. The matrices $A$ and $B$ are the nominal system matrices, and $\alpha$ is a scalar, which is nominally set to $\alpha = 1$, but can be used to reduce the uncertainty if no robust controller can be found for $\alpha = 1$. The matrices $B_w$, $C_z$ and $D_z$ define the control objective to be minimized by the robust controller, see [2].

A popular measure for the error signal, $z$, is the $\mathcal{H}_2$ system norm. The corresponding optimization problem is

$$\min_K \max_{\Delta} \|T_{zw}\|_2, \quad (5)$$

where the transfer function from $w_k$ to $z_k$, $T_{zw}$, depends on the controller $K$ and on the uncertainty matrix $\Delta$. This problem is a generalization of the well-known Linear Quadratic Regulator problem, which also minimizes the $\mathcal{H}_2$ system norm for systems without uncertainty.

Methods to solve such problems are given in [2]. The computations in this paper are based on [3], which derives a controller that guarantees stability and performance for all possible $\Delta$. While the method in [3] is potentially more conservative than others in [2], it obtains the discrete-time controller by solving a convex optimization problem in terms of linear matrix inequalities (LMIs). Efficient solvers for LMIs exist [4], making this technique applicable to online applications as considered in this paper. In fact, in [5] this technique was applied to find a model predictive controller, which required solving an LMI at every time step.

II. ROBUST CONTROLLER

In this section, we derive a robust controller of the form $u_k = K x_k$ that minimizes the $\mathcal{H}_2$ norm according to (5). We modify the uncertainty representation in [3] to fit our system definition in (1)-(4).

Optimization variables in the resulting optimization problem are colored in blue. The sizes of the identity matrix, $I$, and the matrix with all elements equal to zero, $0$, are omitted for legibility and can be extracted from the context. Symmetric matrix elements are denoted by $\bullet$ and $P > 0$ is a positive-definite matrix.

Theorem 1 (cf. [1]): System (1)-(4) is robustly stable under the state feedback controller $u_k = K x_k$ with $K = RQ^{-1}$ and $\|T_{zw}\|_2 < \gamma$ if the following optimization problem is feasible:

$$\min_{W = W^T, Q = Q^T, R, \Lambda = \text{diag}(\tau_1, \ldots, \tau_f), \gamma, \beta = 1} \gamma$$

subject to

$$\text{trace}(W) < \gamma, \begin{bmatrix} W & C_z Q + D_z R & 0 & 0 & 0 \end{bmatrix} > 0, \begin{bmatrix} Q & \bullet & \bullet & \bullet & \bullet \\ 0 & I & \bullet & \bullet & \bullet \\ 0 & 0 & \Lambda & \bullet & \bullet \\ \Lambda Q + B R & 0 & 0 & Q & \bullet \\ C_q Q + D_q R & 0 & 0 & 0 & \beta \Lambda \end{bmatrix} > 0.$$
Proof: We assume that the controller has already been implemented and has been absorbed into the new system matrices,
\[
A_c = A + BK, \quad C_c = C_x + D_xK \quad M_c = C_q + D_qK .
\]

Using (7)-(9) with (1)-(4) yields a new system with simpler dynamics,
\[
\begin{align*}
x_{k+1} &= A_c x_k + B_w w_k + B_p p_k , \quad (10) \\
z_k &= C_c x_k , \quad (11) \\
p_k &= \alpha M_c x_k . \quad (12)
\end{align*}
\]

From here onwards the time indices \( k \) are dropped in order to increase legibility. The uncertain signal \( p \) in (10)-(12) can be rewritten in elementwise form to obtain
\[
\begin{align*}
p(i) &\leq \alpha \delta_1(M_c) x(i) , \quad (13) \\
\iff p^T(i) p(i) &\leq \alpha^2 x^T(M_c)^T(M_c)(i,i) x , \quad (14) \\
\iff \begin{bmatrix} x & w & p(i) \end{bmatrix}^T\begin{bmatrix} -\alpha^2(M_c)^T(M_c)(i,i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ w \\ p(i) \end{bmatrix} &\leq 0,
\end{align*}
\]
where \( p(i) \) is the \( i \)-th element of the vector \( p \) and \( M_c(i,i) \) refers to the \( i \)-th row of the matrix \( M \).

The three LMI constraints in (6) encode the \( H_2 \) performance criterion. The first two are standard results for linear systems without uncertainty, cf. [6]. They correspond to the calculation of the \( H_2 \) norm of system (10)-(12) without uncertainty; that is, \( \Delta = 0 \). The \( H_2 \) norm for this system is given by \( \text{trace}(C_c Q C_c^T \Gamma) \), where \( Q \) is the controllability Gramian, which can be calculated using dissipativity theory. The controllability Gramian corresponds to the ellipsoid \( x^T Q^{-1} x \), which is reachable with unit energy [4]. Thus, the third constraint can be obtained from
\[
V(x_{k+1}) - V(x_k) - w^T w_k < 0, \quad (16)
\]
where \( V(x) = x^T P x \), and \( P = Q^{-1} \) is a positive-definite, symmetric matrix. Plugging \( V(x) \) and the system dynamics (10)-(12) into (16) yields
\[
\begin{align*}
\begin{bmatrix} x^T & w^T \end{bmatrix} &\begin{bmatrix} A_c^T P A_c - P & A_c^T P B_w \\ B_c^T P A_c & B_c^T P B_w - I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0.
\end{align*}
\]

Using the S-Procedure (cf. [4]), the uncertainty definition (15) and the stability/performance condition (17) can be combined to a single equivalent matrix inequality: the inequality (17) is true for all uncertainties that satisfy (15) if and only if
\[
\begin{align*}
A_c^T P A_c - P + \alpha^2 M_c^T M_c &\quad A_c^T P B_w \quad A_c^T P B_p \\ B_c^T P A_c &\quad B_c^T P B_w - I \quad B_c^T P B_p \\
\begin{bmatrix} A_c^T P A_c & A_c^T P B_w & A_c^T P B_p \\ B_c^T P A_c & B_c^T P B_w & B_c^T P B_p \end{bmatrix} &< 0, \quad (18)
\end{align*}
\]
where \( \Gamma = \text{diag}(\tau_i, \ldots, \tau_f) \), and \( \tau_i > 0 \) for all \( i = 1, \ldots, f \).

Using Schur complements (cf. [4]), the previous matrix inequality (18) is equal to
\[
\begin{bmatrix} P & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix} > 0. \quad (19)
\]
This matrix inequality is nonlinear in \( P \) and \( \Gamma \). It can be linearized by multiplying (19) by \( \text{diag}(Q, I, \Gamma^{-1}, I, I) \) from the right and by the transpose of the same matrix from the left. Here \( Q = P^{-1} \) is the controllability Gramian, \( \Lambda = \Gamma^{-1} \) and \( \alpha^{-2} = \beta \). The resulting matrix inequality is
\[
\begin{align*}
\begin{bmatrix} Q & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \end{bmatrix} &< 0, \quad (20)
\end{align*}
\]
which is linear in all unknown variables for a given \( \beta \). Using (7)-(9) in (20) yields
\[
\begin{align*}
\begin{bmatrix} Q & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \end{bmatrix} &< 0, \quad (21)
\end{align*}
\]
The matrix inequality (21) is nonlinear, because both \( Q \) and \( K \) are optimization variables. An invertible, nonlinear change of variables can be used in order to regain linearity. By defining \( R = K Q \), the problem remains linear and the control matrix \( K \) can be extracted via the inverse transformation \( K = R Q^{-1} \). The resulting LMI is given by
\[
\begin{align*}
\begin{bmatrix} Q & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast \end{bmatrix} &< 0, \quad (22)
\end{align*}
\]
which is the same as in (6).