Diss. ETH No. 23120

Geometry and Structure of Metric Injective Hulls

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES OF ETH ZURICH

(Dr. sc. ETH Zurich)

presented by

Maël Pavón

MSc ETH Math., ETH Zurich

born September 23rd, 1989

citizen of Haut-Intyamon (FR) and Spain

accepted on the recommendation of

Prof. Dr. Urs Lang, Examiner

Prof. Dr. Victor Chepoi, Co-Examiner

2016

Abstract

In this thesis, we prove various results on metric injective hulls and injective metric spaces. In Chapter II, we show that the operator given by assigning to a metric space the corresponding metric space defined by Isbell's injective hull is 2-Lipschitz in the Gromov-Hausdorff sense when defined on the class of general metric spaces and 1-Lipschitz when restricted to the class of tree-like metric spaces. These estimates are optimal. In Chapter III, we characterize effectively the injective affine subspaces of the finite dimensional injective normed spaces in any dimension and go on characterizing effectively injective convex polyhedra, this characterization provides in particular a concrete verification algorithm. We then make use of this result to prove that the solution set of a system of linear inequalities with at most two variables per inequality is injective if it is non-empty. Turning to injective hulls with the structure of a polyhedral complex, we extend in Chapter IV the canonical decomposition theory of Bandelt and Dress to infinite metric spaces with integer-valued metric. We consider infinite totally split-decomposable metric spaces with integer-valued metric and satisfying a local rank condition. We then give a characterization for Isbell's injective hull of such metric spaces to be combinatorially equivalent to a cube complex satisfying the CAT(0) link condition. We apply this, among others, to injective hulls of cycle graphs. In Chapter V, we give an alternative characterization of finite combinatorial dimension for metric spaces. We consider the canonical decomposition of the collection of extremal functions that induce admissible graphs. We prove an optimal bound on the diameters of the elements of this canonical decomposition for discretely path-connected metric spaces. We conclude by proving for different classes of metric spaces including any proper metric space, that such a space is injective if and only if it is 4-hyperconvex and possesses a geodesic bicombing.

Zusammenfassung

In dieser Dissertation beweisen wir Resultate über metrische injektive Hüllen und injektive metrische Räume. In Kapitel II beweisen wir, dass die Isbellsche injektive Hülle ein 2-Lipschitz-Operator ist im Sinne von Gromov-Hausdorff, wenn man sie auf der Klasse aller metrischen Räume definiert. Die Isbellsche injektive Hülle wird zu einem 1-Lipschitz-Operator, wenn man sie auf die Klasse aller baumartigen metrischen Räume einschränkt. In Kapitel III charakterisieren wir effektiv die injektiven affinen Teilmengen der endlich dimensionalen normierten Räume beliebiger Dimension. Ausserdem beweisen wir noch eine effektive Charakterisierung aller injektiven konvexen Polyeder. Diese Charakterisierung liefert insbesondere einen konkreten Verifikationsalgorithmus. Wir verwenden dieses Resultat, um zu beweisen, dass ein System linearer Ungleichungen mit höchstens zwei Variablen pro Ungleichung injektiv ist, wenn es nicht leer ist. Desweiteren betrachten wir diejenigen injektiven Hüllen, die die Struktur eines polyedrischen Komplexes haben und in Kapitel IV verallgemeinern wir die kanonische Zerlegungstheorie von Bandelt und Dress, indem wir sie für unendliche metrische Räume mit ganzzahliger Metrik entwickeln. Wir betrachten dann diejenigen unendlichen, vollständig split-zerlegbaren metrischen Räumen mit ganzzahliger Metrik, die eine lokale Rangbedingung erfüllen. Für diese Klasse metrischer Räume charakterisieren wir diejenigen, für die ihre Isbellsche injektive Hülle kombinatorisch äquivalent zu einem Würfelkomplex ist, welcher die CAT(0)-Link-Bedingung erfüllt. Wir wenden dieses Kriterium insbesondere auf Zykelgraphen an. In Kapitel V geben wir eine alternative Charakterisierung der endlichen kombinatorischen Dimension. Wir betrachten die kanonische Zerlegung der Familie aller Extremalfunktionen, die zulässige Graphen induzieren. Wir beweisen eine optimale Schranke an die Durchmesser der Elemente dieser kanonischen Zerlegung für diskret wegzusammenhängende metrische Räume. Schliesslich beweisen wir für verschiedene Klassen metrischer Räume, insbesondere für jeden sogenannten proper metrischen Raum, dass dieser genau dann injektiv ist, wenn er 4-hyperkonvex ist und ein Bicombing besitzt.

Acknowledgements

I would like to thank my advisor Prof. Urs Lang for his guidance and for the time he dedicated to me all along my PhD studies. I would like to thank Prof. Victor Chepoi for being my Co-Referee. I gratefully acknowledge the support of the Swiss National Science Foundation. Finally, I would like to thank all the members of the research team of Prof. Urs Lang at ETH Zürich.

Contents

Chapter I

Introduction

The present thesis consists of four main components, corresponding to works [31, 37, 38] together with a collection of further results, each one of these components focusing on a precise topic of the theory of metric spaces. The common goal of these four works is to develop tools for the study of injective hulls and injective metric spaces in view of many different applications. For instance, in phylogenetic analysis where injective hulls are used to compare general metrics to tree-like ones, cf. [19, 20]. Furthermore, injective hulls appear to be relevant in theoretical computer science in particular for online algorithms in relation with the k-server problem, cf. [13, 14]. Additionally, injective hulls provide a source of new techniques and approaches for purely mathematical questions with geometric elements.

As an illustration, injectivity can be regarded as a general metric notion of global weak non-positive curvature since injective metric spaces share common features with CAT(0) spaces like, for instance, the existence of a geodesic bicombing. Developing tools for injective metric spaces and hulls can therefore lead for example to sharpen and understand better known results of CAT(0) geometry.

A metric space (X, d) is called *injective* if for any isometric embedding $i : A \to B$ of metric spaces and any 1-Lipschitz (equivalently distance-nonincreasing) map $f : A \to X$, there exists a 1-Lipschitz map $g \colon B \to X$, so that $g \circ i = f$. Examples of such spaces include the real line $\mathbb{R}$, $l_\infty(I)$ for any index set I, and all complete metric trees. As can be deduced from the definition, injective metric spaces are in particular non-empty, complete, geodesic and every triple of points has at least one median point. Moreover, injective metric spaces are abolute 1-Lipschitz retracts and reciprocally every absolute 1-Lipschitz retract is injective. In addition, all injective metric spaces are contractible. The terminology coincides with the definition of an injective object in category theory. Accordingly and unless otherwise stated, by injective we mean in the category of metric spaces and 1-Lipschitz maps.

As a matter of fact, proving injectivity is often achieved indirectly via an equivalent but more handy criterion. One such equivalent criterion, which essentially amounts to a point by point extension property, is hyperconvexity, which was introduced in [2]. A metric space (X, d) is called *hyperconvex* if any collection $\{(x_i, r_i)\}_{i\in I} \subset X \times [0, \infty)$ with the property that $d(x_i, x_j) \leq r_i + r_j$ for all pair of indices $i, j \in I$, satisfies $\bigcap_{i\in I} B(x_i, r_i) \neq \emptyset$ where $B(x, r) := \{y \in X : d(x, y) \leq r\}$. With the help of Zorn's lemma, it is not difficult to prove equivalence between injectivity and hyperconvexity. By the hyperconvexity criterion together with a theorem of Nachbin proved in [36], it follows that a normed space is linearly injective (i.e., injective in the category of normed spaces and linear 1-Lipschitz maps) if and only if it is injective in the metric category.

Ubiquity of injectivity in the theory of metric spaces is best appreciated when considering *Isbell's injective hull* construction. Indeed, Isbell showed in [29], that every metric space (X, d) possesses an *injective hull* $(\mathrm{e}, \mathrm{E}(X))$, by which is meant that two properties hold, namely that $\mathrm{E}(X)$ is an injective metric space (endowed with a canonical metric which is for conciseness absent from the notation) and $\mathrm{e}\colon X \to \mathrm{E}(X)$ is an isometric embedding such that every isometric embedding of X into another injective metric space factors through e.

Interestingly, Isbell's injective hull is a general object which can be used in different places as an alternative to *ad hoc* constructions, as an example, see [30] in relation with Gromov hyperbolic metric spaces and [6]. Therefore, Isbell's injective hull construction offers a general framework to prove many different results on metric spaces.

To be precise, Isbell's injective hull is defined as the set $\mathrm{E}(X)$ of so-called extremal functions which is given by

$$\mathrm{E}(X) := \left\{ f \in \mathbb{R}^X : f(x) = \sup\nolimits_{y\in X}(d(x, y) - f(y)) \text{ for all } x \in X \right\}.$$

It is not difficult to see that the difference between two elements of $\mathrm{E}(X)$ has finite supremum norm and $\mathrm{E}(X)$ is defined as being endowed with the metric

$$d_\infty(f, g) := \|f - g\|_\infty .$$

Furthermore, the canonical isometric embedding $\mathrm{e}\colon X \to \mathrm{E}(X)$ is given by the assignement $x \mapsto d_x$ where the map $d_x : y \mapsto d(x, y)$ is meant.

We now continue with a description of the main results contained in the four chapters following this introduction. Each chapter is written in a self-contained way and possesses its own introduction recalling all the facts and definitions needed for the development of the chapter. This allows reading a chapter independently of the others.

Metric Stability of Trees and Tight spans. We start by considering Isbell's injective hull as an operator $X \mapsto \mathrm{E}(X)$ on the family of all metric spaces. Our goal is to determine the Lipschitz constant of this operator for the relevant notions of distance. Besides, we wish to obtain an improved Lipschitz constant in case the operator is restricted to metric spaces with properties similar to those of metric trees. In this framework, dissimilarity between two metric spaces X and Y is modeled by the *Gromov-Hausdorff* comparison. The effective computation of the quantity $d_{\mathrm{GH}}(X, Y)$ consists in measuring the minimal distortion, namely minimizing the quantity

$$\frac{1}{2} \sup_{(x,y),(x',y') \in R} |d_X(x, x') - d_Y(y, y')|$$

over all relations R in the product set $X \times Y$. In this context, a *relation* is defined as a subset $R \subset X \times Y$ such that $\pi_X(R) = X$ and $\pi_Y(R) = Y$. Our procedure consists in proving extension results for relations between metric spaces. For the first extension result, we let X and Y be injective metric spaces and we consider subsets $R \subset X \times Y$ such that $\pi_X(R)$ is a spanning set of X. We show that there exists an extension $\bar{R}$ with same distortion such that $\pi_X(\bar{R})$ is now an α-net in X where α denotes the distortion of R. For the second extension result, we let X be a metric tree and Y be an injective metric space and we consider subsets $R \subset X \times Y$ such that $\pi_X(R)$ is this time a strictly spanning set of X. We show that in this case, there exists an extension $\bar{R}$ with same distortion such that $\pi_X(\bar{R}) = X$.

From the first extension result, we deduce that $X \mapsto \mathrm{E}(X)$ is 2-Lipschitz in the Gromov-Hausdorff sense on the class of general metric spaces and we show that this Lipschitz constant is optimal. Since injective hulls of Gromov δ-hyperbolic metric spaces are themselves δ-hyperbolic (cf. [30]), $X \mapsto \mathrm{E}(X)$ restricts to an operator on the class of tree-like metric spaces. From the second extension result, we deduce that Isbell's injective hull is 1-Lipschitz when restricted to the class of tree-like metric spaces, and the constant is clearly optimal. These results are in particular relevant to recovery issues as encountered for instance in phylogenetics.

This chapter corresponds to [31], which is joint work with Urs Lang and Roger Züst.

I. INTRODUCTION

Injective Convex Polyhedra. There are many reasons for the relevance of injective convex polyhedra in l^n_∞, which denotes the set of real n-tuples $x := (x_1, \dots, x_n)$ endowed with the norm $\|x\|_\infty = \max_{1\leq i\leq n} |x_n|$. It was shown by Nachbin in [36] that an n-dimensional normed space is injective if and only if it is linearly isometric to l^n_∞. Therefore, in order to study injectivity, it is natural to focus in the first place on non-empty subsets of the *model spaces* l^n_∞. Moreover, by Nachbin's characterization, if a subset of an n-dimensional normed space is injective and has non-empty interior, it follows by a rescaling argument that the ambient space has to be isometric to l^n_∞. With the goal of obtaining a simple elegant and concrete *combinatorial characterization*, we consequently consider convex polyhedra in l^n_∞. They indeed build the largest class for which our expectations have a chance to be fulfilled. Furthermore, as in CAT(0) geometry, convexity is important in our considerations and in addition, Gähler and Murphy showed in [22] that there is a unique geodesic bicombing σ on l^n_∞, namely the one given by the linear geodesics. By a (geodesic) *bicombing* on a geodesic metric space (X, d) is meant a map $\sigma\colon X \times X \times [0, 1] \to X$ such that for any $x, y \in X$, the induced mapping $\sigma_{xy} := \sigma(x, y, \cdot)\colon [0, 1] \to X$ satisfies that σ_{xy} is a geodesic from x to y, that is $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$ and $d(\sigma_{xy}(t), \sigma_{xy}(t')) = |t - t'|\, d(x, y)$, together with the properties $\sigma_{yx}(t) = \sigma_{xy}(1 - t)$ and $d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1 - t)\, d(x, x') + t\, d(y, y')$. Injective convex subsets of l^n_∞ are thus exactly the injective subsets of l^n_∞ whose unique (cf. [16]) convex geodesic bicombing coincides with the geodesic bicombing of l^n_∞. Finally, considering polyhedral sets is needed in order to be able to obtain combinatorial or computationally usable results. Since l^n_∞ is injective, for any subset S, there is a subset S' of l^n_∞ containing S, with the property that S' is isometric to $\mathrm{E}(S)$. Note that S is injective if and only if for any such S' one has $S = S'$. Hence injective subsets are exactly the ones stable under taking injective hulls.

We start by giving a proof of the already known characterization of injective affine subsets of l^n_∞ of any dimension. We go on proving a *local-to-global* injectivity criterion for convex polyhedra in l^n_∞. In order to do so, we consider the family of *tangent cones* to a given convex polyhedron. This family is a collection of convex polyhedral cones which encodes the complete local information of the associated convex polyhedron. Precisely, we prove that a polyhedron in l^n_∞ is injective if and only each of its tangent cones is injective.

We apply this local-to-global criterion to prove an effective characterization of injective convex polyhedra in l^n_∞. This characterization is effective in the sense that for any given polyhedron, the procedure to check whether it is injective or not involves computing finitely many times the intersection of finitely many affine subspaces of l_∞^n. This gives in particular an implementable algorithm to perform injectivity tests. The characterization involves looking at the intersection pattern of convex polyhedral cones centered at the origin with the facets of the unit cube $[-1, 1]^n$. From this characterization follows in particular the following known result: if Isbell's injective hull has a canonical polyhedral structure parametrized by the admissible graphs, then its cells are themselves injective.

As a concrete application of our characterization of injective convex polyhedra, we prove that the solution set to a system of linear inequalities with at most two variables per inequality is injective as soon as it is non-empty. In this context, we make use of tools from *linear programming* and theoretical computer science which were developed by Shostak in [39]. Note that the class of convex polyhedra given by such systems is stable under non-empty intersections, unlike general injective convex polyhedra as illustrated by an example given at the beginning of the chapter.

Injective Hulls of Infinite Totally Split-Decomposable Metric Spaces. We now look for a way to decompose Isbell's injective hull into simpler components. In order to do so, we extend a construction of Bandelt and Dress (cf. [4]) to infinite metric spaces. We start with a *split* (also called *cut*) $S := \{A, B\}$ of a set X which is a pair of non-empty subsets of X such that $A \cap B = \emptyset$ and $X = A \cup B$. For $x \in X$, we denote by $S(x)$ the element of S that contains x. The *split (pseudo-)metric* associated to S is then a pseudometric δ_S on X given by

$$\delta_S(x, y) \;=\; \begin{cases} 1 & \text{if } S(x) \neq S(y), \\ 0 & \text{if } S(x) = S(y). \end{cases}$$

We consider infinite metric spaces with integer-valued metric as in [30] and generalize the *canonical decomposition* of Bandelt and Dress which applies to finite metric spaces. In this context, we need to consider injective hulls of pseudometric spaces and we rather use the notation $\mathrm{E}(X, d)$ instead of $\mathrm{E}(X)$ for Isbell's injective hull, if (X, d) is a pseudometric space. Sometimes, when there is no ambiguity regarding the underlying space considered, we simply write $\mathrm{E}(d)$. Nevertheless, all three denote the same space.

The split-decomposition of a pseudometric d is divided into two components, the so-called *split-prime* component d_0 which is split-indecomposable and the *totally split-decomposable* component $d - d_0$ which can be decomposed as a weighted sum

$$\sum_{S \in \mathcal{S}} \alpha_S^d \delta_S$$

of simpler components, namely split pseudometrics δ_S, over the collection $\mathcal{S}$ of all so-called d-splits of (X, d). Each real nonnegative weight α_S^d is given by a particular formula depending on the d-split S. When the metric space (X, d) has integer-valued metric and satisfies the *local rank condition (LRC)*, the split-decomposition of d is characterized by the property that the inclusion

$$\mathrm{E}(X, d) \subset \mathbb{R}^X \cap \left(\mathrm{E}(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \mathrm{E}(X, \delta_S) \right).$$

holds regardless of the choices $\lambda_S \in [0, \alpha_S^d]$ and where $d_1 := d - \sum_{S \in \mathcal{S}} \lambda_S \delta_S$.

Extending the split decomposition theory to infinite metric spaces presents several important advantages. First, there is then the possibility to apply the new theory to the setting of injective hulls of finitely generated groups endowed with a word metric. The second advantage is that infinite metric spaces model homogeneous spaces in general. Furthermore, the class of infinite metric spaces is closed under taking direct limits.

We give a necessary and sufficient condition for Isbell's injective hull of any infinite totally split-decomposable metric space with integer-valued metric satisfying the (LRC) to be combinatorially equivalent to a *CAT(0) cube complex*. Cube complexes constitute a natural object of study in parallel to convex polyhedra especially in analogy with CAT(0) geometry. Note that the necessary and sufficient condition we prove enables us to characterize Isbell's injective hull of the totally split-decomposable part, if it satisfies the (LRC), of the split-decomposition of *any* integer-valued metric. This motivates, in further investigations, to look for other kinds of decompositions for the split-prime part d_0 which might involve classes of pseudometrics other than the split ones.

As an application, we give a complete combinatorial description of Isbell's injective hull of cycle graphs of any size by showing that Isbell's injective hulls of odd cycles are combinatorially equivalent to their associated Buneman complex and thus in particular to a cube complex satisfying the CAT(0) link condition. The *Buneman complex* is a well-known construction used in the analysis of data structures as can be done for instance in computational biology cf. [8, 19]. Moreover, as a further application, we give a closed formula for the number of maximal cells of this complex in the case of cycle graphs. This illustrates that the split decomposition theory is also effective for computations with concrete families of metric spaces.

Further Results on Metric Injectivity. Paralleling the case of convex polyhedra and of metric spaces with integer-valued metric satisfying the

(LRC), we then extend the classes of metric spaces under consideration by looking in the first case at metric spaces of finite combinatorial dimension and in the second case at integer-valued metric spaces in general, specializing when required, to discretely path-connected ones.

Regarding finite combinatorial dimension, it is well-known that Isbell's injective hull of a finite metric space is a finite dimensional polyhedral complex. The *combinatorial dimension* $\dim_{\mathrm{comb}}(X)$ of a metric space (X, d) is then the supremum over the dimensions of the polyhedral complexes $\mathrm{E}(Y)$ over all finite subsets $Y \subset X$. We give a new characterization of combinatorial dimension for metric spaces and as an application, an alternative proof of *Dress' theorem*. The goal is to provide a more natural proof using the same tools as we did for convex polyhedra, namely linear programming for system of inequalities with at most two variables per inequality. Moreover, this alternative viewpoint is expected to deepen the understanding of combinatorial dimension and to provide new examples of spaces as studied for instance in [16, 17]. As an illustrative feature, it is not difficult to see that l_1^n and l_∞^n have finite combinatorial dimension equal to 2^{n-1} and n respectively (cf. [30]), we can thus split the family of spaces l_p^n where $p \in [1, \infty]$ and $n \in \mathbb{N}$ into three categories according to their metric properties as follows

p	$\{2\}$	$(1,2) \cup (2,\infty)$	$\{1,\infty\}$
l_p^n	CAT(0)	Busemann	$\dim_{\mathrm{comb}}(l_p^n) < \infty$

Regarding integer-valued metric spaces, we prove new results on the structure of Isbell's injective hull. Precisely, we already know that there is a distinguished subset $\mathrm{E}'(X) \subset \mathrm{E}(X)$ which admits a decomposition $\{P(A)\}_{A \in \mathscr{A}(X)}$ parametrized by the collection $\mathscr{A}(X)$ of admissible edge sets where to each subset $P(A)$ corresponds a graph (X, A) possibly with a self-loop. An edge $\{x, y\} \in A$ corresponds to a pair $\{x, y\}$ satisfying $f(x) + f(y) = d(x, y)$ for any $f \in P(A)$, in particular $P(A) \subset P(A')$ if $A' \subset A$. We then prove a characterization due to Urs Lang, of those elements $P(A)$ of finite rank which are maximal in the sense that there is no $A \supsetneq A' \in \mathscr{A}(X)$. Later, we prove an optimal bound on the diameters of the elements of this decomposition, where optimality is proved to hold in every dimension.

Finally, we conclude with an outlook, intended to suggest further directions of investigation, which consists in proving new criteria for injectivity of certain classes of metric spaces. In particular, we prove that proper metric spaces are injective if and only if they are 4-*hyperconvex* and possess a geodesic bicombing.

Chapter II

Metric Stability of Trees and Tight Spans

II.1 Introduction

Our goal is to provide an optimal stability result, in terms of the Gromov–Hausdorff distance, for Isbell's [29] injective hull construction $X \mapsto \mathrm{E}(X)$ for metric spaces. Roughly speaking, $\mathrm{E}(X)$ is a smallest injective metric space containing an isometric copy of X (all relevant definitions will be reviewed later in this paper). Here, a metric space Y is called *injective* if for any isometric embedding $i\colon A \to B$ of metric spaces and any 1-Lipschitz (i.e., distance-nonincreasing) map $f\colon A \to Y$ there exists a 1-Lipschitz map $g\colon B \to Y$ of f, so that $g \circ i = f$ (see [1, Section 9] for the general categorical notion). Examples of injective metric spaces include the real line $\mathbb{R}$, $l_\infty(I)$ for any index set I, and all complete metric trees; however, by Isbell's result, this list is by far not exhaustive. Injective metric spaces are complete, geodesic, and contractible and share a number of remarkable properties. We refer to [30, Sections 2 and 3] for a recent survey of injective metric spaces and hulls.

An alternative, but equivalent, description of $\mathrm{E}(X)$ was given later by Dress [18], who called it the tight span of X. If X is compact, then so is $\mathrm{E}(X)$, and if X is finite, $\mathrm{E}(X)$ has the structure of a finite polyhedral complex of dimension at most $|X|/2$ with cells isometric to polytopes in some finite-dimensional l_∞ space. If every quadruple of points in X admits an isometric embedding into some metric tree, then so does X itself, and $\mathrm{E}(X)$ provides the minimal complete such tree. This last property makes the injective hull/tight span construction a useful tool in phylogenetic analysis. Based on genomic differences an evolutionary distance between similar species is defined, and the construction may then be applied to this finite metric space. Due to noise in the measurements or systematic errors, the process will rarely yield a tree, but (the 1-skeleton of) the resulting polyhedral complex may still give a good indication on the phylogenetic tree one tries to reconstruct (compare [19, 20] and the references there).

In view of these applications, and also from a purely geometric perspective, it is interesting to know how strongly the injective hull is affected by small changes of the underlying metric space. The dissimilarity of two metric spaces A, B is conveniently measured by their Gromov–Hausdorff distance $d_{\mathrm{GH}}(A, B)$. Moezzi [35, Theorem 1.55] observed that $d_{\mathrm{GH}}(\mathrm{E}(A), \mathrm{E}(B))$ is not larger than eight times $d_{\mathrm{GH}}(A, B)$. Here it is now shown that in fact

$$d_{\mathrm{GH}}(\mathrm{E}(A), \mathrm{E}(B)) \leq 2\, d_{\mathrm{GH}}(A, B),$$

and an example is constructed to demonstrate that the factor two is optimal (see Section 3). Furthermore, we prove that if both $\mathrm{E}(A)$ and $\mathrm{E}(B)$ are metric trees (in the most general sense of $\mathbb{R}$-trees), then

$$d_{\mathrm{GH}}(\mathrm{E}(A), \mathrm{E}(B)) \leq d_{\mathrm{GH}}(A, B),$$

without a factor two. In particular, this implies that if X, Y are two finite simplicial metric trees with sets of terminal vertices A, B, respectively, then $d_{\mathrm{GH}}(X, Y) \leq d_{\mathrm{GH}}(A, B)$. This result (which we have not been able to find in the literature) is not as obvious as it may appear at first glance. A complication arises from the fact that for the respective vertex sets V_X, V_Y, it is not true in general that $d_{\mathrm{GH}}(V_X, V_Y) \leq d_{\mathrm{GH}}(A, B)$, not even for combinatorially equivalent binary trees. For instance, consider the two trees X, Y depicted below, with the indicated edge lengths.

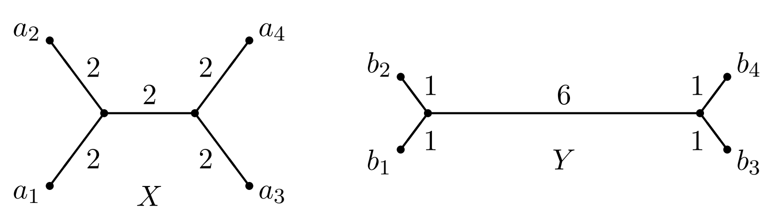

Figure II.1: Two metric trees X and Y with $d_{\mathrm{GH}}(X, Y) = 1$.

The correspondence between $A := \{a_1, \ldots, a_4\}$ and $B := \{b_1, \ldots, b_4\}$ that relates a_i to b_i distorts all distances by an additive error of two. Since the diameters of A and B also differ by two, no correspondence (i.e., left- and right-total relation) between A and B has (maximal) distortion less than two. The Gromov–Hausdorff distance equals one half this minimal number (see Section 3), so $d_{\mathrm{GH}}(A, B) = 1$. Similar considerations show that $d_{\mathrm{GH}}(V_X, V_Y) = 2$. Yet, $d_{\mathrm{GH}}(X, Y) = 1$. For the proof, points in X and Y need to be related in a non-canonical way.

II.2 Extension of Roughly Isometric Relations

As just indicated, the Gromov–Hausdorff distance may be characterized in terms of the additive distortion of relations between the two given metric spaces. Therefore, in this section, we begin by studying the possibility of extending relations without increasing the distortion.

Let X, Y be two metric spaces. We write $|xx'|$ for the distance of two points $x, x' \in X$ and, likewise, $|yy'|$ for the distance of $y, y' \in Y$. Given a relation R between X and Y, i.e., a subset of $X \times Y$, the *distortion* of R is defined as the (possibly infinite) number

$$\operatorname{dis}(R) := \sup\bigl\{\bigl||xx'| - |yy'|\bigr| : (x,y),(x',y') \in R\bigr\}.$$

In case R is given by a map $f\colon X \to Y$, we write $\operatorname{dis}(f)$ for $\operatorname{dis}(R)$. If $\operatorname{dis}(f) \le \varepsilon$ for some $\varepsilon \ge 0$, then f is called *ε-roughly isometric*. This means that

$$|xx'| - \varepsilon \le |f(x)f(x')| \le |xx'| + \varepsilon$$

for every pair of points $x, x' \in X$. See [9, Chapter 7] and [10, Chapter 7] for this terminology. We denote by $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ the canonical projections. For a set $A \subset X$, we say that A *spans* X if, for every pair $(x, x') \in X \times X$,

$$|xx'| = \sup_{a \in A}\bigl(|xa| - |x'a|\bigr);$$

equivalently, for all $\varepsilon > 0$ there is an $a_\varepsilon \in A$ such that $|xx'| + |x'a_\varepsilon| \le |xa_\varepsilon| + \varepsilon$. The definition is motivated by the fact that the injective hull of a metric space A may be characterized as an injective metric extension $X \supset A$ spanned by A, see Proposition 3.3 below. For a constant $\alpha \ge 0$, a set $S \subset X$ is called an *α-net* in X if for every $x \in X$ there exists a $z \in S$ such that $|xz| \le \alpha$.

2.1 Proposition. *Suppose that X, Y are two injective metric spaces. If $R \subset X \times Y$ is a set with $\alpha := \operatorname{dis}(R)/2 < \infty$ and the property that $\pi_X(R)$ spans X, then there exists an extension $R \subset \bar{R} \subset X \times Y$ such that $\pi_X(\bar{R})$ is an α-net in X and* $\operatorname{dis}(\bar{R}) = \operatorname{dis}(R)$.

In particular, every ε-roughly isometric map $f\colon A \to Y$ defined on a set $A \subset X$ that spans X admits an ε-roughly isometric extension $\bar{f}\colon S \to Y$ to some $\varepsilon/2$-net S in X and, hence, also a 2ε-roughly isometric extension $\hat{f}\colon X \to Y$. Below we shall use the simple fact that every injective metric space Y is *hyperconvex* [2] (the converse is true as well). This means that for every family $\{(y_i, r_i)\}_{i\in I}$ in $Y \times \mathbb{R}$ with the property that $r_i + r_j \geq |y_i y_j|$ for all pairs of indices $i, j \in I$, there is a point $y \in Y$ such that $|yy_i| \leq r_i$ for all $i \in I$.

Proof. It suffices to show that for every set $R \subset X\times Y$ with $\alpha := \mathrm{dis}(R)/2 < \infty$ and the property that $\pi_X(R)$ spans X and for every $\bar{x} \in X$ there exists a pair $(x_0, y_0) \in X \times Y$ such that $|\bar{x}x_0| \leq \alpha$ and

$$\mathrm{dis}\bigl(R \cup \{(x_0, y_0)\}\bigr) = \mathrm{dis}(R).$$

The general result then follows by an application of Zorn's lemma.

Let such R and $\bar{x}$ be given, and put $\alpha := \mathrm{dis}(R)/2$. For all $(x, y), (x', y') \in R$,

$$\bigl||xx'| - |yy'|\bigr| \leq 2\alpha$$

and $(|x\bar{x}|+\alpha)+(|x'\bar{x}|+\alpha) \geq |xx'|+2\alpha \geq |yy'|$. Hence, since Y is hyperconvex, there is a point $y_0 \in Y$ such that for all $(x, y) \in R$,

$$|yy_0| \leq |x\bar{x}| + \alpha.$$

Furthermore, since $\pi_X(R)$ spans X, for every $(x, y) \in R$ and $\varepsilon > 0$ there exists $(x_\varepsilon, y_\varepsilon) \in R$ such that $|x\bar{x}| + |\bar{x}x_\varepsilon| \leq |xx_\varepsilon| + \varepsilon$ and, hence,

$$|yy_0| \geq |yy_\varepsilon| - |y_0y_\varepsilon| \geq (|xx_\varepsilon| - 2\alpha) - (|\bar{x}x_\varepsilon| + \alpha) \geq |x\bar{x}| - 3\alpha - \varepsilon.$$

Since this holds for all $\varepsilon > 0$, it follows that $|yy_0| \geq |x\bar{x}| - 3\alpha$. For every $(x, y) \in R$, put $r(x, y) := |yy_0| + 2\alpha$, and set $r(\bar{x}) := \alpha$. We have $r(x, y) + r(\bar{x}) = |yy_0|+3\alpha \geq |x\bar{x}|$ and $r(x, y)+r(x', y') \geq |yy'|+4\alpha \geq |xx'|+2\alpha \geq |xx'|$, for all $(x, y), (x', y') \in R$. Thus, since X is hyperconvex, there exists a point $x_0 \in X$ such that

$$|xx_0| \leq r(x, y) = |yy_0| + 2\alpha$$

and $|\bar{x}x_0| \leq r(\bar{x}) = \alpha$ for all $(x, y) \in R$. Then also

$$|yy_0| \leq |x\bar{x}| + \alpha \leq |xx_0| + |\bar{x}x_0| + \alpha \leq |xx_0| + 2\alpha$$

and so $\bigl||xx_0| - |yy_0|\bigr| \leq 2\alpha = \mathrm{dis}(R)$ for all $(x, y) \in R$. □

Now we focus on trees. A metric space X is called *geodesic* if for every pair of points $x, x' \in X$ there is a geodesic segment $xx' \subset X$ connecting the two points, i.e., the image of an isometric embedding of the interval $[0, |xx'|]$ that sends 0 to x and $|xx'|$ to x'. By a *metric tree* X we mean a geodesic metric space with the property that for any triple (x, y, z) of points in X and any geodesic segments xy, xz, yz connecting them, $xy \subset xz \cup yz$. Thus, geodesic triangles in X are isometric to tripods, and geodesic segments are uniquely determined by their endpoints. For the next result we need to sharpen the above assumption that $\pi_X(R)$ spans X. We say that a subset A of a metric space X *strictly spans* X if for every pair $(x, x') \in X \times X$ there exists an $a \in A$ such that $|xx'| + |x'a| = |xa|$.

2.2 Proposition. *Suppose that X is a metric tree and Y is an injective metric space. If $R \subset X \times Y$ is a set with the property that $\pi_X(R)$ strictly spans X, there exists an extension $R \subset \bar{R} \subset X \times Y$ such that $\pi_X(\bar{R}) = X$ and* $\mathrm{dis}(\bar{R}) = \mathrm{dis}(R)$.

In particular, every ε-roughly isometric map $f \colon A \to Y$ defined on a set $A \subset X$ that strictly spans X admits an ε-roughly isometric extension $\bar{f} \colon X \to Y$.

Proof. It suffices to show that for every set $R \subset X \times Y$ with $\mathrm{dis}(R) < \infty$ and the property that $\pi_X(R)$ spans X and for every $\bar{x} \in X$ there exists a point $\bar{y} \in Y$ such that

$$\mathrm{dis}\big(R \cup \{(\bar{x}, \bar{y})\}\big) = \mathrm{dis}(R).$$

As above, the general result then follows by an application of Zorn's lemma.

Thus let such R and $\bar{x}$ be given. Put $\alpha := \mathrm{dis}(R)/2$. As in the proof of Proposition 2.1, there exists a point $y_0 \in Y$ with the property that

$$|yy_0| \le |x\bar{x}| + \alpha$$

for all $(x, y) \in R$. Let S be the set of all $(x, y) \in R$ with $|yy_0| < |x\bar{x}| - \alpha$. If $S = \emptyset$, then $\big||x\bar{x}| - |yy_0|\big| \le \alpha \le \mathrm{dis}(R)$ for all $(x, y) \in R$; in particular, $\bar{y} := y_0$ has the desired property. Suppose now that $S \neq \emptyset$, and fix an arbitrary $(x_1, y_1) \in S$. Since $\pi_X(R)$ strictly spans X, there exists a pair $(x_2, y_2) \in R$ such that $|x_1\bar{x}| + |\bar{x}x_2| = |x_1x_2|$. Now choose $\bar{y} \in Y$ so that $|\bar{y}y_0| \le \alpha$ and $|\bar{y}y_2| \le |y_0y_2| - \alpha$. Note that $|y_0y_2| \le |\bar{x}x_2| + \alpha$, so $|\bar{y}y_2| \le |\bar{x}x_2|$. For all $(x, y) \in R$,

$$|y\bar{y}| \le |yy_0| + |\bar{y}y_0| \le |yy_0| + \alpha \le |x\bar{x}| + 2\alpha.$$

To estimate $|\bar{y}y|$ from below, note first that if $(x, y) \in R \setminus S$, then

$$|y\bar{y}| \ge |yy_0| - |\bar{y}y_0| \ge |yy_0| - \alpha \ge |x\bar{x}| - 2\alpha.$$

Secondly, let $(x,y) \in S$. Consider the tripod $xx_1 \cup xx_2 \cup x_1x_2$, and note that $\bar{x} \in x_1x_2$. Since $(x,y),(x_1,y_1) \in S$, the strict inequality

$$|xx_1| \leq |yy_1| + 2\alpha \leq |yy_0| + |y_0y_1| + 2\alpha < |x\bar{x}| + |\bar{x}x_1|$$

holds, so $\bar{x} \notin xx_1$ and therefore $\bar{x} \in xx_2$. We conclude that

$$|y\bar{y}| \geq |yy_2| - |\bar{y}y_2| \geq (|xx_2| - 2\alpha) - |\bar{x}x_2| = |x\bar{x}| - 2\alpha.$$

This shows that $\big||x\bar{x}| - |y\bar{y}|\big| \leq 2\alpha = \mathrm{dis}(R)$ for all $(x,y) \in R$. □

The following example shows that Proposition 2.2 is no longer true in general if the word "strictly" is omitted.

2.3 Example. Let X be the interval $[0,2]$, and put $x_0 := 0$ and $x_n := 2 - 2^{-n}$ for all integers $n \geq 1$. The set $A := \{x_0, x_1, \dots\}$ spans X, but A does not strictly span X, because $2 \notin A$. Let Y be the simplicial metric tree with a single interior vertex y_1 and the countably many edges y_0y_1 and y_1y_n for $n = 2, 3, \dots$, where $|y_0y_1| = 2^{-1}$ and $|y_1y_n| = 2^{-1} - 2^{-n}$. Note that Y is complete, hence injective. The map $f\colon A \to Y$ defined by $f(x_n) := y_n$ for $n = 0, 1, 2, \dots$ is 1-roughly isometric, as is easily checked. Since there is no pair of points at distance one in Y, f does not admit a 1-roughly isometric extension $\bar{f}\colon X \to Y$.

However, the following holds.

2.4 Lemma. *Let X be a metric tree, and suppose that $A \subset X$ is a set that spans X. Then there exists a dense subtree $\Sigma \subset X$ such that $A \subset \Sigma$ and A strictly spans Σ.*

Proof. Let Σ be the union of all geodesic segments with both endpoints in A. Since X is a metric tree, it is easily seen that for every pair of points $x, x' \in \Sigma$ the geodesic segment xx' in X is part of a geodesic segment aa' with $a, a' \in A$. In particular, Σ is a geodesic subspace of X, hence a metric tree, and A strictly spans Σ. It remains to show that Σ is dense in X. Let $x \in X$. Fix an arbitrary $a \in A$. Since A spans X, for every $\varepsilon > 0$ there is an $a_\varepsilon \in A$ so that $|ax| + |xa_\varepsilon| \leq |aa_\varepsilon| + \varepsilon$. Consider the geodesic segment aa_ε. Let x_ε be the point on aa_ε nearest to x. Then

$$2|xx_\varepsilon| = |ax| + |xa_\varepsilon| - |aa_\varepsilon| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and $x_\varepsilon \in \Sigma$, x lies in the closure of Σ. □

II.3 Gromov–Hausdorff Distance Estimates

In this section we prove the results stated in the introduction. First we recall the definition of the Gromov–Hausdorff distance. Let (Z, d^Z) be a metric space. The usual *Hausdorff distance* $d^Z_{\mathrm{H}}(X, Y)$ of two subsets X, Y of Z is the infimum of all $\varrho > 0$ such that X is contained in the (open) ϱ-neighborhood of Y and, vice versa, Y lies in the ϱ-neighborhood of X. More generally, if X and Y are two metric spaces, their *Gromov–Hausdorff distance* $d_{\mathrm{GH}}(X, Y)$ is defined as the infimum of all $\varrho > 0$ for which there exist a metric space (Z, d^Z) and isometric copies $X', Y' \subset Z$ of X and Y, respectively, such that $d^Z_H(X', Y') < \varrho$. The distance is always finite if X and Y are bounded, and for general metric spaces X_1, X_2, X_3 the triangle inequality $d_{\mathrm{GH}}(X_1, X_2) + d_{\mathrm{GH}}(X_2, X_3) \geq d_{\mathrm{GH}}(X_1, X_3)$ holds. Furthermore, d_{GH} induces an honest metric on the set of isometry classes of compact metric spaces.

The Gromov–Hausdorff distance of two metric spaces X, Y may alternatively be characterized as follows. A *correspondence* R between X and Y is a subset of $X \times Y$ such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are surjective when restricted to R. Then

$$d_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_R \mathrm{dis}(R),$$

where the infimum is taken over all correspondences R between X and Y (see [9, Theorem 7.3.25]). In view of this characterization, the following two theorems are now easy consequences of the results in the previous section.

3.1 Theorem. *Suppose that X, Y are two injective metric spaces, $A \subset X$ is a set that spans X, and $B \subset Y$ is a set that spans Y. Then*

$$d_{\mathrm{GH}}(X, Y) \leq 2\, d_{\mathrm{GH}}(A, B).$$

Proof. Suppose that $R \subset A \times B$ is a correspondence between A and B with $\alpha := \mathrm{dis}(R)/2 < \infty$. By Proposition 2.1, there is an extension $R \subset R_1 \subset X \times Y$ such that $\pi_X(R_1)$ is an α-net in X and $\mathrm{dis}(R_1) = \mathrm{dis}(R)$, and there is a further extension $R_1 \subset R_2 \subset X \times Y$ such that $\pi_Y(R_2)$ is an α-net in Y and $\mathrm{dis}(R_2) = \mathrm{dis}(R_1)$. It is then easy to see how to extend R_2 to a correspondence $\bar{R}$ between X and Y so that $\mathrm{dis}(\bar{R}) \leq \mathrm{dis}(R_2) + 2\alpha = 2\,\mathrm{dis}(R)$. Hence,

$$d_{\mathrm{GH}}(X, Y) \leq \frac{1}{2}\,\mathrm{dis}(\bar{R}) \leq \mathrm{dis}(R),$$

and taking the infimum over all correspondences R between A and B with finite distortion we obtain the result. □

3.2 Theorem. *Suppose that X, Y are two metric trees, $A \subset X$ is a set that spans X, and $B \subset Y$ is a set that spans Y. Then*

$$d_{\mathrm{GH}}(X, Y) \le d_{\mathrm{GH}}(A, B).$$

Proof. Note that the completions $\bar{X}, \bar{Y}$ of X, Y satisfy $d_{\mathrm{GH}}(\bar{X}, \bar{Y}) = d_{\mathrm{GH}}(X, Y)$, and A, B span $\bar{X}, \bar{Y}$, respectively. We thus assume, without loss of generality, that the metric trees X, Y are complete, hence injective. Let $R \subset A \times B$ be a correspondence between A and B. By Lemma 2.4, A strictly spans a tree $X' \supset A$ that is dense in X. Hence, by Proposition 2.2, there is an extension $R \subset R_1 \subset X' \times Y$ such that $\pi_{X'}(R_1) = X'$ and $\mathrm{dis}(R_1) = \mathrm{dis}(R)$. We have $B \subset B' := \pi_Y(R_1)$, and so B' also spans Y. Again, B' strictly spans a tree $Y' \supset B'$ that is dense in Y, and there is an extension $R_1 \subset R_2 \subset X \times Y'$ such that $\pi_{Y'}(R_2) = Y'$ and $\mathrm{dis}(R_2) = \mathrm{dis}(R_1)$. Since $\pi_X(R_2) \supset X'$ is dense in X, and Y' is dense in Y, we obtain that

$$d_{\mathrm{GH}}(X, Y) = d_{\mathrm{GH}}(\pi_X(R_2), Y') \le \frac{1}{2}\,\mathrm{dis}(R_2) = \frac{1}{2}\,\mathrm{dis}(R).$$

As this holds for all correspondences R between A and B, this gives the result. □

Next, in order to relate these results to the discussion in the introduction, we recall Isbell's explicit construction of the injective hull $\mathrm{E}(X)$ of a metric space X. We denote by $\mathbb{R}^X$ the vector space of all real functions on X. As a set, $\mathrm{E}(X)$ is defined as

$$\mathrm{E}(X) := \bigl\{ f \in \mathbb{R}^X : f(x) = \sup\nolimits_{y \in X}(|xy| - f(y)) \text{ for all } x \in X \bigr\},$$

the set of the so-called *extremal functions* on X. For every $z \in X$, the distance function d_z, defined by $d_z(x) := |xz|$ for $x \in X$, belongs to $\mathrm{E}(X)$. In general, for every $f \in \mathrm{E}(X)$ and $z \in X$, the inequalities

$$d_z - f(z) \le f \le d_z + f(z)$$

hold, and it follows that $\|f - d_z\|_\infty := \sup |f - d_z| = f(z)$. In particular, $\|f - g\|_\infty$ is finite for every pair of functions $f, g \in \mathrm{E}(X)$, and this equips $\mathrm{E}(X)$ with a metric. The map $\mathrm{e}\colon X \to \mathrm{E}(X)$ that takes x to d_x is then a canonical isometric embedding of X into $\mathrm{E}(X)$, as $\|d_x - d_y\|_\infty = |xy|$ for all $x, y \in X$. Isbell proved that $(\mathrm{e}, \mathrm{E}(X))$ is indeed an *injective hull* of X, i.e., $\mathrm{E}(X)$ is an injective metric space, and $(\mathrm{e}, \mathrm{E}(X))$ is a minimal such extension of X in that no proper subspace of $\mathrm{E}(X)$ containing $\mathrm{e}(X)$ is injective. Furthermore, if (i, Y) is another injective hull of X, then there exists a unique isometry

$I\colon \mathrm{E}(X) \to Y$ with the property that $I \circ \mathrm{e} = i$. The following result explains how injective hulls are related to spanning subsets of (injective) metric spaces, in the sense of this paper.

3.3 Proposition. *(i) For every metric space A, the image $\mathrm{e}(A)$ of the canonical isometric embedding $\mathrm{e}\colon A \to \mathrm{E}(A)$ spans $\mathrm{E}(A)$.*

(ii) If X is an injective metric space and $A \subset X$ is a set that spans X, then X is isometric to $\mathrm{E}(A)$ via the map that sends $x \in X$ to the restricted distance function $d_x|_A$.

Proof. For (i), let a pair (f, g) of elements of $\mathrm{E}(A)$ be given, and let $\varepsilon > 0$. There exists either a point $b \in A$ such that $\|f - g\|_\infty \le f(b) - g(b) + \varepsilon/2$ or a point $a \in A$ such that $\|f - g\|_\infty \le g(a) - f(a) + \varepsilon/2$. Then, by the definition of $\mathrm{E}(A)$, we may choose $a \in A$ with $f(b) \le |ab| - f(a) + \varepsilon/2$ in the first case and $b \in A$ with $g(a) \le |ab| - g(b) + \varepsilon/2$ in the second. In either case, this gives

$$\|f - g\|_\infty \le |ab| - f(a) - g(b) + \varepsilon.$$

Since $|ab| - f(a) \le f(b) = \|f - d_b\|_\infty$ and $g(b) = \|g - d_b\|_\infty$, we obtain that $\|f - g\|_\infty \le \|f - d_b\|_\infty - \|g - d_b\|_\infty + \varepsilon$. As $d_b = \mathrm{e}(b) \in \mathrm{e}(A)$, this shows the claim.

For the proof of (ii), let $x, y \in X$. Since A spans X, we have first that for every $a \in A$, $d_x(a) = \sup_{b \in A}(|ab| - d_x(b))$, so $d_x|_A \in \mathrm{E}(A)$. Secondly, $|xy| = \sup_{a \in A}(|ax| - |ay|)$, which implies that the inequality

$$\big\|d_x|_A - d_y|_A\big\|_\infty = \sup_{a \in A}\big||ax| - |ay|\big| \le |xy|$$

is in fact an equality. Hence, the map that takes x to $d_x|_A$ is an isometric embedding of X into $\mathrm{E}(A)$. Since X is injective, so is the image of this map. Because no proper subspace of $\mathrm{E}(A)$ containing $\mathrm{e}(A)$ is injective, the image agrees with $\mathrm{E}(A)$. □

In view of Proposition 3.3, Theorem 3.1 is equivalent to saying that for any metric spaces A and B,

$$d_{\mathrm{GH}}(\mathrm{E}(A), \mathrm{E}(B)) \le 2\, d_{\mathrm{GH}}(A, B),$$

as stated in the introduction. We now show that the factor two is optimal.

3.4 Example. First we show that if $f\colon \mathbb{R}\times[0,4]\to\mathbb{R}$ is an ε-roughly isometric map, where $\mathbb{R}\times[0,4]\subset\mathbb{R}^2$ is endowed with the l_1 metric, then $\varepsilon\geq 4$. For any integer $n\geq 1$, consider the subset

$$Z_n := \bigl(\{0,8,\dots,8n\}\times\{0\}\bigr)\cup\bigl(\{4,12,\dots,8n-4\}\times\{4\}\bigr)$$

of $\mathbb{R}\times[0,4]$ of cardinality $2n+1$. Note that, with respect to the l_1 distance, distinct points in Z_n are at distance at least eight from each other, and the diameter of Z_n equals $8n$. Let $\{z_1,z_2,\dots,z_{2n+1}\}$ be an enumeration of Z_n so that $f(z_1)\leq f(z_2)\leq\dots\leq f(z_{2n+1})$. We have $f(z_{i+1})-f(z_i)\geq\|z_{i+1}-z_i\|_1-\varepsilon\geq 8-\varepsilon$, hence taking the sum from $i=1$ to $2n$ we obtain $f(z_{2n+1})-f(z_1)\geq 2n(8-\varepsilon)$. On the other hand, $f(z_{2n+1})-f(z_1)\leq\operatorname{diam}(Z_n)+\varepsilon=8n+\varepsilon$. It follows that $\varepsilon\geq 8n/(2n+1)$. This holds for any $n\geq 1$, thus $\varepsilon\geq 4$.

Now, for any $N>0$, consider the two metric spaces $A=\{a_1,\dots,a_4\}$ and $B=\{b_1,\dots,b_4\}$, where $|a_1a_2|=|a_3a_4|=4$, $|a_1a_3|=|a_2a_4|=N$, $|a_1a_4|=|a_2a_3|=N+4$, $|b_1b_2|=|b_3b_4|=2$ and $|b_ib_j|=N+2$ ($i\neq j$) otherwise. The correspondence $\{(a_1,b_1),\dots,(a_4,b_4)\}$ has distortion two, and since $\operatorname{diam}(A)=\operatorname{diam}(B)+2$ there is no correspondence between A and B with distortion less than two. So $d_{\mathrm{GH}}(A,B)=1$. The injective hull $\mathrm{E}(A)$ is isometric to $[0,N]\times[0,4]\subset\mathbb{R}^2$ with the l_1 distance, and $\mathrm{E}(B)$ is a metric tree with a central edge of length N and two edges of length one attached at each of its endpoints (like the tree Y depicted in the introduction). Let $\varepsilon_0<4$ be given. If N is chosen big enough, depending on ε_0, essentially the same argument as above shows that there is no ε-roughly isometric map $f\colon\mathrm{E}(A)\to\mathrm{E}(B)$ with $\varepsilon<\varepsilon_0$. In particular, every correspondence between $\mathrm{E}(A)$ and $\mathrm{E}(B)$ has distortion at least $\varepsilon_0/2$. In other words, for every $\delta_0<2$ we find a pair of four-point metric spaces A,B so that $\mathrm{E}(A)$ is two-dimensional, $\mathrm{E}(B)$ is a metric tree, $d_{\mathrm{GH}}(A,B)=1$, and $d_{\mathrm{GH}}(\mathrm{E}(A),\mathrm{E}(B))\geq\delta_0$.

Chapter III

Injective Convex Polyhedra

III.1 Introduction

We call a metric space X *injective* if for any metric spaces A, B such that there exists an isometric embedding $i\colon A \to B$ and for any 1-Lipschitz (i.e., distance nonincreasing) map $f\colon A \to X$, there is a 1-Lipschitz map $g\colon B \to X$ satisfying $g \circ i = f$ (cf. [1, Section 9] for the general categorical definition). In particular, it follows from a result of Nachbin that a real normed space X is injective in the the category of metric spaces if and only if X is injective in the category of normed spaces.

The purpose of the present chapter is to provide an effective characterization of injective convex polyhedra in l_∞^n by proving an easy combinatorial criterion. It is important to note that only the case of the l_∞-metric is relevant since if a convex polyhedron $P \subset \mathbb{R}^n$ with non-empty interior is injective for some norm $\|\cdot\|$ on $\mathbb{R}^n$, then considering an increasing sequence of rescalings of P whose union is equal to $\mathbb{R}^n$, it follows by Lemma 3.1 that the space $(\mathbb{R}^n, \|\cdot\|)$ is itself injective and by [36, Theorem 3], which states that an n-dimensional normed space X is injective if and only if X is linearly isometric to l_∞^n, it follows that $(\mathbb{R}^n, \|\cdot\|)$ is isometric to l_∞^n.

Note at this point that linear subspaces of injective normed spaces need not be injective. A straightforward example is the plane

$$V := \{x \in l_\infty^3 : x_1 + x_2 + x_3 = 0\} \tag{1.1}$$

which is not injective since it can be easily seen that the unit ball of V is an hexagon and thus V cannot be isometric to l_∞^2. Furthermore, Example 1.4 exhibits a non-injective convex polyhedron with injective supporting hyperplanes and Example 1.5 an injective convex polyhedron with a non-injective face.

III.1. INTRODUCTION

It was noted in [30] that a good characterization of injective polytopes is missing. The present chapter gives a solution to this problem. We start by giving in Section III.2 a characterization of injective affine subspaces of l_∞^n and as a consequence we obtain an easy injectivity criterion for hyperplanes, namely if $\nu \in \mathbb{R}^n \setminus \{0\}$, then the hyperplane

$$X := \{x \in \mathbb{R}^n : x \cdot \nu = 0\} \subset l_\infty^n \tag{1.2}$$

(where $x \cdot y$ denotes the standard scalar product on $\mathbb{R}^n$) is injective if and only if

$$\|\nu\|_1 \leq 2\,\|\nu\|_\infty\,. \tag{1.3}$$

For $\alpha \in \mathbb{R}$ and $\emptyset \neq A, B \subset \mathbb{R}^n$, we define $\alpha A, A+B, A-B \subset \mathbb{R}^n$ in the obvious way and we set $[a,b]A := \bigcup_{\alpha \in [a,b]} \alpha A$. For a convex polyhedron $\emptyset \neq P \subset \mathbb{R}^n$ and a point $p \in P$, the *tangent cone* $\mathrm{T}_p P$ is given by

$$\mathrm{T}_p P := \bigcup_{m \in \mathbb{N}} P_{p,m} \quad \text{where} \quad P_{p,m} := p + m(P - p).$$

The effective characterization we are aiming at is obtained in two steps. First, we prove that injectivity follows from a local injectivity property namely injectivity of tangent cones. It is no restriction to assume that the interior of P satisfies $\mathrm{int}(P) \neq \emptyset$ in the next theorem which is proved in Section III.3:

1.1 Theorem. *Let $P \subset l_\infty^n$ be a convex polyhedron such that* $\mathrm{int}(P) \neq \emptyset$*. Then, the following two conditions are equivalent:*

(*i*) *P is injective.*

(*ii*) *$\mathrm{T}_p P$ is injective for every $p \in \partial P$.*

By a *convex polyhedron* in $\mathbb{R}^n$ we mean a finite intersection of closed half-spaces. Closed half-spaces are just called half-spaces when no ambiguity arises. A *convex polytope* is then a compact convex polyhedron. A *cone* C is a subset of $\mathbb{R}^n$ such that $x \in C$ implies $\lambda x \in C$ for any $\lambda \geq 0$. Convex polyhedra which are additionally cones are called *convex polyhedral cones.* If C is a convex polyhedral cone and $x \in \mathbb{R}^n$, the *apex* $\mathrm{apex}(x + C)$ of a translate of C is defined as the affine space $x + V$ where V is the biggest linear subspace of $\mathbb{R}^n$ contained in C. It is easy to see that $\mathrm{T}_p P - p$ is a convex polyhedral cone. In the sequel, the relative interior of a subset S is denoted by $\mathrm{relint}(S)$. The *dimension* of a convex polyhedron $P \subset \mathbb{R}^n$ is the dimension of its affine hull. One has $\mathrm{int}(P) \neq \emptyset$ if and only if $\dim(P) = n$ and in this case, F is a *facet* of P if and only if F is a face of P and $\dim(F) = n - 1$. Let us denote by $\mathrm{Faces}(P)$ and $\mathrm{Facets}(P)$ the set of non-empty faces and the set of facets of P respectively, for any subset $S \subset \mathbb{R}^n$ let $\mathrm{Faces}(P, S) := \{F \in \mathrm{Faces}(P) : F \cap S \neq \emptyset\}$ and let $\mathrm{Faces}(P, S)^c$ be the complement of $\mathrm{Faces}(P, S)$ in $\mathrm{Faces}(P)$. Moreover, $\mathrm{Facets}^*(P, S) := \{F \in \mathrm{Facets}(P) : \mathrm{relint}(F) \cap S \neq \emptyset\}$. Note that the closed unit ball $B(0, 1) \subset l^n_\infty$ is nothing but the n-hypercube $[-1, 1]^n$ endowed with the l_∞-metric. The following theorem, which is proved in Section III.6, characterizes injective convex polyhedral cones:

1.2 Theorem. *A convex polyhedral cone $C \subsetneq l^n_\infty$ with $\mathrm{int}(C) \neq \emptyset$ is injective if and only if the following hold:*

(*i*) *$\mathrm{T}_p C$ is injective for every $p \in \partial C \setminus \mathrm{apex}(C)$.*

(*ii*) *There is $F \in \mathrm{Facets}^*([-1, 1]^n, C)$ such that $-F \notin \mathrm{Facets}^*([-1, 1]^n, C)$.*

It follows from Theorem 1.2 in the case where $\partial C \setminus \mathrm{apex}(C) = \emptyset$ or equivalently when C is a half-space, that (1.3) is an injectivity criterion for the half-spaces having the hyperplane X as in (1.2) as boundary. For $p \in \partial C \setminus \mathrm{apex}(C) \neq \emptyset$, the dimension of $\mathrm{apex}(\mathrm{T}_p C)$ is strictly bigger than that of $\mathrm{apex}(C)$ and making repeated use of Theorem 1.2 on tangent cones, one thus easily obtains:

1.3 Corollary. *A convex polyhedron $P \subsetneq l^n_\infty$ with $\mathrm{int}(P) \neq \emptyset$ is injective if and only if for every $p \in \partial P$, the convex polyhedral cone $K := \mathrm{T}_p P - p$ satisfies (ii) in Theorem 1.2, which means that there is a facet $F \in \mathrm{Facets}^*([-1, 1]^n, K)$ such that $-F \notin \mathrm{Facets}^*([-1, 1]^n, K)$.*

There are several equivalent characterizations of injective metric spaces and one of them is *hyperconvexity* (cf. [2]). We call a metric space X *hyperconvex* if for every family $\{(x_i, r_i)\}_{i \in I}$ in $X \times \mathbb{R}$ satisfying $r_i + r_j \geq d(x_i, x_j)$ for all $(i, j) \in I \times I$, one has $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$ (with the convention that the intersection equals X itself if $I = \emptyset$) where $B(x, r)$ denotes throughout the text, a closed ball in the contextually relevant metric (whereas open balls are denoted by $U(x, r)$). Furthermore, if $Y \subset Z$ with Z being injective and if there is a 1-Lipschitz retraction $r \colon Z \to Y$ (i.e., $r \in \mathrm{Lip}_1(Z, Y)$ and $r|_Y = \mathrm{id}_Y$), then Y is injective (this follows immediately from the definition of injectivity given above). The following two examples show that the characterization we are looking for requires more effort than one would think at first sight:

1.4 Example. Consider the half-spaces

$$H := \{x \in l^4_\infty : x_1 \geq 0\}$$

and

$$H' := \{x \in l_\infty^4 : x_1 \leq \frac{1}{3}(x_2 + x_3 + x_4)\}.$$

Note that it is easy to see that both H and H' are injective by considering in each case the 1-Lipschitz retraction given by mapping each point in the complement to the unique corresponding point on the boundary so that all coordinates but the first remain unchanged and then extending by the identity. Moreover, both ∂H and $\partial H'$ are injective by (1.3). However, it is easy to see that $P := H \cap H' \subset l_\infty^4$ is not injective by considering the three points

$$\{p, p', p''\} := \{(0,0,0,0), (0,0,-2,2), (0,-2,0,2)\} \subset \partial H \cap \partial H' \subset P,$$

note that

$$I := B(p,1) \cap B(p',1) \cap B(p'',1) = \{(t,-1,-1,1) : t \in [-1,1]\},$$

hence $I \cap P = \emptyset$. Thus P is not hyperconvex and therefore not injective.

Next, we have:

1.5 Example. Consider the injective half-space H' defined above, let further $H'' := \{x \in l_\infty^4 : x_1 \leq 0\}$ and

$$P' := H' \cap H'' \subset l_\infty^4.$$

Note that the face

$$F := \partial H' \cap \partial H'' \subset l_\infty^4$$

of P' is not injective since

$$F = \{x \in l_\infty^4 : x_1 = 0, x_2 + x_3 + x_4 = 0\}$$

is isometric to (1.1) which is not injective as we already noted. Let us now however show that P' is injective by defining an explicit 1-Lipschitz retraction r of l_∞^4 onto P'. Let $\varrho \in \mathrm{Lip}_1(l_\infty^4, \mathbb{R})$ be the map

$$(x_1, \ldots, x_4) \mapsto \frac{1}{3}(x_2 + x_3 + x_4).$$

Now, let $r \colon l_\infty^4 \to P'$ be given by

$$(x_1, \ldots, x_4) \mapsto (\min\{x_1, 0, \varrho(x)\}, x_2, x_3, x_4)$$

and note that r is the desired 1-Lipschitz retraction.

Finally, in Sections III.7 and III.8, we introduce and use a theorem of Shostak (cf. [39]), in order to prove:

1.6 Corollary. *Consider* $f, g\colon \{1,\dots,m\} \to \{1,\dots,n\}$ *and for* $i \in \{1,\dots,m\}$, $a_i, b_i, c_i \in \mathbb{R}$ *so that*

$$P := \bigcap_{i\in\{1,\dots,m\}} \left\{x \in \mathbb{R}^n : a_i x_{f(i)} + b_i x_{g(i)} \geq c_i\right\}$$

verifies $\operatorname{int}(P) \neq \emptyset$ *and* $P \neq \mathbb{R}^n$. *Then,* $P \subset l_\infty^n$ *satisfies the injectivity criterion stated in Corollary 1.3 and is therefore injective.*

III.2 Injective Linear Subspaces in l_∞^n

We start this section with a characterization of injective linear subspaces in l_∞^n. For each $i \in I_n := \{1,\dots,n\}$, we have the linear isometry

$$\mu_i\colon l_\infty^n \to l_\infty^n,\ (x_1,\dots,x_n) \mapsto (x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_n)$$

and the 1-Lipschitz linear map

$$\pi_i\colon l_\infty^n \to \mathbb{R},\ (x_1,\dots,x_n) \mapsto x_i.$$

Moreover, let us denote by $\{e_1,\dots,e_n\}$ the standard basis of $\mathbb{R}^n$. Injective convex polyhedra were also studied in [35]. Note that Theorem 2.1 and 2.2 as well as Lemma 3.1 in Section III.3 already appear in [35]. Our proof of Theorem 2.1 is however more elementary.

2.1 Theorem. *Let* $\emptyset \neq X \subset l_\infty^n$ *be a linear subspace and let* $k := \dim(X)$. *Then, the following are equivalent:*

(*i*) *X is injective.*

(*ii*) *There is a subset* $J \subset I_n$ *with* $|J| = k$ *such that for any* $i \in I_n \setminus J$ *there exist real numbers* $\{c(i,j)\}_{j\in J}$ *such that* $\sum_{j\in J}|c(i,j)| \leq 1$ *and such that*

$$X = \left\{x \in l_\infty^n : \forall i \in I_n \setminus J\ ,\ x_i = \sum_{j\in J} c(i,j)x_j\right\}.$$

Proof. Assume first that (*ii*) holds. Assume for simplicity that $J = \{1,\dots,k\}$. Let us define the map $L\colon l_\infty^k \to l_\infty^n$ such that for any $(y_1,\dots,y_k) \in l_\infty^k$,

$$L(y) := \left(y_1,\dots,y_k,\sum_{j=1}^k c(k+1,j)y_j,\dots,\sum_{j=1}^k c(n,j)y_j\right).$$

It is then easy to see that L is an isometric embedding with $L(l_\infty^k) = X$. It follows that X and l_∞^k are isometric and thus X is injective.

Assume now that (i) holds, there consequently exists a linear isometric embedding $L\colon l_\infty^k \to X \subset l_\infty^n$ (see the Introduction). In particular,

$$\|L(e_j)\|_\infty = 1 \tag{2.1}$$

and

$$\|L(\sigma e_j + \tau e_l)\|_\infty = 1 \tag{2.2}$$

for $(j,l) \in I_k \times I_k$ with $j \neq l$ (where $I_k := \{1,\dots,k\}$) and $\sigma, \tau \in \{\pm 1\}$. Now, (2.1) implies for $j \in I_k$ the existence of some $f(j) \in I_n$ such that $|(\pi_{f(j)} \circ L)(e_j)| = 1$; replacing L by $L \circ \mu_j$ if necessary, we can assume without loss of generality that

$$(\pi_{f(j)} \circ L)(e_j) = 1 \tag{2.3}$$

for any $j \in I_k$. Therefore, (2.3) together with (2.2) imply that $(\pi_{f(j)} \circ L)(e_l) = 0$ for $(j,l) \in I_k \times I_k$ with $j \neq l$ and thus f is injective. We summarize by writing $(\pi_{f(j)} \circ L)(e_l) = \delta_{jl}$. Now, we can assume for simplicity that $f(j) = j$ for any $j \in I_k$ hence in particular $J := f(I_k) = \{1,\dots,k\}$ and

$$(\pi_j \circ L)(e_l) = \delta_{jl}. \tag{2.4}$$

It follows that there are $c(k+1,j),\dots,c(n,j) \in \mathbb{R}$ such that

$$L(e_j) = \big(0,\dots,0,1,0,\dots,0,c(k+1,j),\dots,c(n,j)\big),$$

where the first k entries of $L(e_j)$ are zero except the j-th one. For any $(\sigma_1,\dots,\sigma_k) \in \{\pm 1\}^k$, one has by linearity

$$\Big\|\sum_{j=1}^k \sigma_j L(e_j)\Big\|_\infty = \Big\|\sum_{j=1}^k \sigma_j e_j\Big\|_\infty = 1.$$

Inserting successively appropriate values for $(\sigma_1,\dots,\sigma_k)$ in the above equality, one obtains for any $i \in I_n \setminus J = \{k+1,\dots,n\}$,

$$\sum_{j=1}^k |c(i,j)| \leq 1.$$

Since $X = L(l_\infty^k)$, there are for any $x \in X$ real numbers $c_1,\dots,c_k \in \mathbb{R}$ such that $x = \sum_{l=1}^k c_l L(e_l)$. For any $j \in I_k$, it follows from (2.4) that

$$x_j = \pi_j(x) = \sum_{l=1}^k c_l (\pi_j \circ L)(e_l) = \sum_{l=1}^k c_l \delta_{jl} = c_j.$$

Hence finally

$$x = \sum_{j=1}^{k} x_j L(e_j) = \Big(x_1, \dots, x_k, \sum_{j=1}^{k} c(k+1, j)x_j, \dots, \sum_{j=1}^{k} c(n, j)x_j\Big).$$

This proves that (ii) holds and concludes the proof. □

The next theorem is an immediate consequence of Theorem 2.1:

2.2 Theorem. *Let $\nu \in \mathbb{R}^n \setminus \{0\}$. The hyperplane $X = \{x \in \mathbb{R}^n : x \cdot \nu = 0\} \subset l_\infty^n$ is injective if and only if $\|\nu\|_1 \leq 2\|\nu\|_\infty$.*

Proof. Assume first that X is injective. By Theorem 2.1, there is some $i \in I$ such that

$$X = \Big\{x \in l_\infty^n : -x_i + \sum_{j \in I\setminus\{i\}} c(i, j)x_j = 0\Big\}.$$

with $\sum_{j\in I\setminus\{i\}} |c(i, j)| \leq 1$. Define now ν so that $\nu_j := c(i, j)$ if $j \neq i$ and $\nu_i := -1$. Note that ν is a normal vector of X and satisfies $\|\nu\|_1 \leq 2\|\nu\|_\infty$.

For the other implication, let ν a normal vector of X satisfying $\|\nu\|_1 \leq 2\|\nu\|_\infty$ and assume without loss of generality that $\|\nu\|_\infty = 1$; hence, $\|\nu\|_1 \leq 2$. There is $i \in I$ such that $|\nu_i| = 1$ and assume additionally without loss of generality that $\nu_i = -1$. Thus $\sum_{j\in I\setminus\{i\}} |\nu_j| \leq 1$ and $x \cdot \nu = -x_i + \sum_{j\in I\setminus\{i\}} \nu_j x_j$, hence we can apply Theorem 2.1 to

$$X = \Big\{x \in l_\infty^n : -x_i + \sum_{j \in I\setminus\{i\}} \nu_j x_j = 0\Big\},$$

to obtain that X is injective. This concludes the proof of the theorem. □

III.3 Tangent cones of Injective Convex Polyhedra in l_∞^n

We start this section with a lemma and then go on to prove Theorem 1.1. Throughout the text, we call a sequence of sets $(X_m)_{m\in\mathbb{N}}$ increasing if and only if $X_m \subset X_{m+1}$ for $m \in \mathbb{N}$ whereas it is called decreasing if the reverse inclusions hold.

3.1 Lemma. *Let $\emptyset \neq S \subset l_\infty^n$ be a closed subset. Then, the following are equivalent:*

(i) *S is injective.*

(ii) *There is $x \in S$ such that $S \cap B(x,r)$ is injective for any $r \in (0,\infty)$.*

(iii) *There is an increasing sequence $(X_m)_{m\in\mathbb{N}}$ of injective subsets of S such that $S = \bigcup_m X_m$.*

Proof. We only prove that (iii) implies (i) since the other implications follow immediately from the definitions. In order to do so, we prove that (iii) implies that S is hyperconvex. Consider a family $\{(x_\alpha, r_\alpha)\}_{\alpha\in A}$ in $S \times \mathbb{R}$ such that $r_\alpha + r_\beta \geq \|x_\alpha - x_\beta\|_\infty$ for any $(\alpha,\beta) \subset A \times A$. Pick $\gamma \in A$ arbitrarily and let $m_0 \in \mathbb{N}$ be such that $x_\gamma \in X_{m_0}$. Consider a sequence $(A_m, y_m)_{m\in\mathbb{N}}$ such that

$$A_m := \left\{\alpha \in A : x_\alpha \in X_{m+m_0}\right\}$$

and

$$y_m \in S \cap \bigcap_{\alpha\in A_m} B(x_\alpha, r_\alpha),$$

noting that $X_{m+m_0} \cap \bigcap_{\alpha\in A_m} B(x_\alpha, r_\alpha) \neq \emptyset$ hence $S \cap \bigcap_{\alpha\in A_m} B(x_\alpha, r_\alpha) \neq \emptyset$. Since S is closed and $(y_m) \subset S \cap B(x_\gamma, r_\gamma)$, it follows that there is a convergent subsequence (y_{m_l}) such that $y_{m_l} \to y \in S \cap B(x_\gamma, r_\gamma)$. Thus, $y \in S \cap \bigcap_{\alpha\in A} B(x_\alpha, r_\alpha)$. This proves that S is hyperconvex and finishes the proof of the lemma. □

We also make use in the proof of Theorem 1.1 of the following (cf. [42]):

3.2 Theorem. *$S \subset \mathbb{R}^n$ is a convex polyhedron if and only if there is a convex polytope Q and a convex polyhedral cone C such that*

$$S = Q + C.$$

For an n-dimensional polyhedron P and for $k \in \{0, 1, \ldots, n-1\}$, let $\mathrm{Faces}_k(P)$ denote the set of k-dimensional faces of P and let $\partial^k P$ be the union of all elements of $\mathrm{Faces}_k(P)$. We use the notation $d(A,B) := \inf_{(a,b)\in A\times B} \|a-b\|_\infty$ for two subsets $\emptyset \neq A, B \subset l_\infty^n$. The open δ-neighborhood $\bigcup_{a\in A} U(a,\delta)$ of A is denoted by $N(A,\delta)$.

Proof of Theorem 1.1. By Lemma 3.1 and by definition of $\mathrm{T}_p P$ it immediately follows that (i) implies (ii). Assume now that (ii) holds. Let us consider an enumeration $\{F_j\}_{j\in\{1,\ldots,N\}}$ of $\mathrm{Faces}(P) \setminus \{P\}$. For $j \in \{1, \ldots, N\}$ we consider an arbitrary point $p^j \in \mathrm{relint}(F_j)$ and a corresponding 1-Lipschitz retraction $\varrho_j \colon l_\infty^n \to \mathrm{T}_{p^j} P$. For $p \in \partial P$, let

$$\varepsilon_p := \sup\{\varepsilon \in (0,\infty] : U(p,\varepsilon) \cap \mathrm{T}_p P = U(p,\varepsilon) \cap P\}.$$

Note that if $\varepsilon_p = \infty$ for some p, then $P = \mathrm{T}_pP$ and thus P is injective. Otherwise, we proceed inductively on the dimension of faces of P to show that there is a $\delta > 0$ such that

$$P \cup N(\partial P, \delta) \subset P \cup \bigcup_{p \in \partial P} U(p, \varepsilon_p).$$

Suppose $F \in \mathrm{Faces}_k(P) \setminus \{P\}$, for $k = 0$ set $c(F) := F$ and for $k \geq 1$, set

$$c(F) := F \setminus N(\partial^0 P \cup \partial^1 P \cup \cdots \cup \partial^{k-1} P, \tfrac{\delta^{(k)}}{2}),$$
$$\varepsilon^{(k+1)} := \min_{F \in \mathrm{Faces}_k(P)} \left[\frac{1}{2} \min_{\mathrm{Faces}(P) \ni F' \not\supseteq F} d(c(F), F') \right].$$

By the Separation Theorem for Polyhedra, cf. [41, Theorem 10.4], one has $d(P', P'') > 0$ for any two disjoint convex polyhedra $\emptyset \neq P', P'' \subset l_\infty^n$ and thus $\varepsilon^{(k+1)} > 0$. Furthermore, let $\delta^{(0)} := \varepsilon^{(1)}$ and

$$\delta^{(k+1)} := \min \left\{ \varepsilon^{(k+1)}, \tfrac{\delta^{(k)}}{2} \right\}.$$

Moreover, we set $A^0 := \partial^0 P$ and for $k \geq 1$:

$$A^k := \bigcup_{F \in \mathrm{Faces}_k(P)} c(F) = \partial^k P \setminus N(\partial^0 P \cup \partial^1 P \cup \cdots \cup \partial^{k-1} P, \tfrac{\delta^{(k)}}{2}).$$

It follows by construction that for any $p \in A^k$ and any $F \in \mathrm{Faces}(P)$, one has $U(p, \delta^{(k+1)}) \cap F \neq \emptyset$ if and only if $p \in F$. Hence $U(p, \delta^{(k+1)}) \cap \mathrm{T}_pP = U(p, \delta^{(k+1)}) \cap P$ for any $p \in A^k$. It follows by induction that

$$\bigcup_{p \in \partial P} U(p, \delta^{(n)}) = N(\partial^{n-1} P, \delta^{(n)}) \subset \bigcup_{k=0}^{n-1} N(A^k, \delta^{(k+1)}).$$

This shows that $\delta := \delta^{(n)} > 0$ satisfies $P \cup N(\partial P, \delta) \subset P \cup \bigcup_{p \in \partial P} U(p, \varepsilon_p)$. It is now easy to see that we obtain a 1-Lipschitz retraction $\varrho \colon N(P, \delta) \to P$ by setting $\varrho := \widetilde{\varrho}|_{N(P,\delta)}$ where $\widetilde{\varrho} := \varrho_1 \circ \cdots \circ \varrho_N$. By Theorem 3.2, there is a convex polytope Q and a polyhedral cone C such that $P = Q + C$. We can assume without loss of generality that $0 \in \mathrm{int}(Q)$. We can set $\kappa := 1 + \frac{\delta}{2\,\mathrm{diam}(Q)}$ and since $\kappa P = \kappa Q + C$ it follows that $\kappa P \subset N(P, \delta)$. By iteration, we obtain a sequence $\{(\varrho^m, P^m)\}_{m \in \mathbb{N}}$ of rescalings $P^m := \kappa^m P$ of P and corresponding 1-Lipschitz retractions $\varrho^m \colon P^m \to P^{m-1}$ by setting $\varrho^m(\kappa x) := \kappa \varrho^{m-1}(x)$ for $m \geq 2$ and $\varrho^1 := \varrho|_{\kappa P}$. Finally, we can define the 1-Lipschitz retraction $r \colon l_\infty^n \to P$ as an inverse limit map for the system $\{(\varrho^m, P^m)\}_{m \in \mathbb{N}}$, that is $r(x) := (\varrho^1 \circ \cdots \circ \varrho^m)(x)$ where m is the smallest natural such that $x \in P^m$. It follows that P is injective. $\square$

Let us consider a simple example to show that it is necessary in the above proof to argue locally before extending to increasing rescalings.

3.3 Example. Consider $Q := [-2,0] \times [-2,0] = B((-1,-1),1) \subset l_\infty^2$. We enumerate the tangent cones of Q as follows; for $k \in \{1,2,3,4\}$:

$$C_{2k-1} := \mathrm{T}_{p_k} Q \text{ where } (p_1, \ldots, p_4) := ((-2,-2),(-2,0),(0,0),(0,-2)),$$

$$C_{2k} := \mathrm{T}_{q_k} Q \text{ where } (q_1, \ldots, q_4) = ((-2,-1),(-1,0),(0,-1),(-1,-2)).$$

Consider corresponding 1-Lipschitz retractions such that

$$\varrho_2(x_1,x_2) := (-x_1-4,x_2) \text{ if } x_1 < -2 \text{ and } \varrho_6(x_1,x_2) := (-x_1,x_2) \text{ if } x_1 > 0,$$
$$\varrho_8(x_1,x_2) := (x_1,-x_2-4) \text{ if } x_2 < -2 \text{ and } \varrho_4(x_1,x_2) := (x_1,-x_2) \text{ if } x_2 > 0$$

and extend ϱ_2, ϱ_4, ϱ_6 and ϱ_8 by the identity. Finally, we set for odd indices: $\varrho_1 := \varrho_2 \circ \varrho_8$ and $\varrho_{2k-1} := \varrho_{2k} \circ \varrho_{2k-2}$ for $k \neq 1$. It is then easy to see that $(\varrho_8 \circ \cdots \circ \varrho_1)((-10,-10)) = (-6,2) \notin Q$.

3.4 Remark. Note that it is enough to assume that the minimal (for the inclusion) tangent cones of $P \subset l_\infty^n$ are injective. Hence, letting P be a convex polyhedron with non-empty interior, the following are equivalent:

(*i*) P is injective.

(*ii*) All minimal tangent cones of P are injective.

III.4 Systems of Inequalities

The proposition that we prove in this section is used in the proof of Lemma 5.1 in Section III.5. For each $i \in I_n := \{1, \ldots, n\}$, let $\widehat{\pi}_i \in \mathrm{Lip}_1(l_\infty^n, l_\infty^{n-1})$ denote the map

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, \widehat{x}_i, \ldots, x_n)$$

and recall that $\pi_i \in \mathrm{Lip}_1(l_\infty^n, \mathbb{R})$ denotes the map $(x_1, \ldots, x_n) \mapsto x_i$. The following fact was already observed by D. Descombes (personal communication):

4.1 Proposition. *Let* $I \subset I_n$, $\mathfrak{R} := \{\underline{r}_i : i \in I\} \cup \{\overline{r}_i : i \in I\} \subset \mathrm{Lip}_1(l_\infty^{n-1}, \mathbb{R})$ *and*

$$Q := \Big\{ x \in l_\infty^n : \text{ for all } i \in I \text{ one has } (\underline{r}_i \circ \widehat{\pi}_i)(x) \leq x_i \leq (\overline{r}_i \circ \widehat{\pi}_i)(x) \Big\}.$$

Assume that:

(i) $Q \neq \emptyset$*;*

(ii) for any $i \in I$*,* $\underline{r}_i \leq \overline{r}_i$*.*

It follows that Q *is injective.*

Proof. We first show the statement in the case $\mathfrak{R} \subset \mathrm{Lip}_\lambda(l_\infty^{n-1}, \mathbb{R})$ for some $\lambda \in [0, 1)$. For $i \in I$, let us define $\varrho_i \in \mathrm{Lip}_1(l_\infty^n, l_\infty^n)$ by setting

$$\varrho_i(x) := \Big(x_1, \ldots, x_{i-1}, \min\big\{(\overline{r}_i \circ \widehat{\pi}_i)(x), \max\{x_i, (\underline{r}_i \circ \widehat{\pi}_i)(x)\}\big\}, x_{i+1}, \ldots, x_n\Big)$$

for any $x \in l_\infty^n$. Consider an enumeration $I = \{i_1, \ldots, i_N\}$. Moreover, set $G_0 := \mathrm{id}_{l_\infty^n}$ as well as $G_j := \varrho_{i_j} \circ \cdots \circ \varrho_{i_1}$ and

$$T := G_N = \varrho_{i_N} \circ \cdots \circ \varrho_{i_1}.$$

Fix now $x \in l_\infty^n$. We show that $(T^m(x))_{m \in \mathbb{N}}$ converges to a fixed point of T. Let us define the maps $\{f_{i_j}\}_{i_j \in I} \subset \mathrm{Lip}_\lambda(l_\infty^n, \mathbb{R})$ by

$$f_{i_j} \ : \ y \mapsto \min\Big\{(\overline{r}_{i_j} \circ \widehat{\pi}_{i_j})(y), \max\big\{\alpha_{i_j}, (\underline{r}_{i_j} \circ \widehat{\pi}_{i_j})(y)\big\}\Big\},$$

where $\alpha_{i_j} := (\pi_{i_j} \circ G_{j-1} \circ T^m)(x) = (\pi_{i_j} \circ G_j \circ T^{m-1})(x)$. We further set

$$\beta_{i_j} := \Big|\pi_{i_j}\Big((G_j \circ T^m)(x) - T^m(x)\Big)\Big|$$

for any $i_j \in I$ and observe that

$$\begin{aligned}
\beta_{i_j} &= \Big|\pi_{i_j}\Big((G_j \circ T^m)(x) - (G_j \circ T^{m-1})(x)\Big)\Big| \\
&= \Big|\pi_{i_j}\Big((G_j \circ T^m)(x) - (\varrho_{i_j} \circ G_j \circ T^{m-1})(x)\Big)\Big| \\
&= \big|(f_{i_j} \circ G_{j-1} \circ T^m)(x) - (f_{i_j} \circ G_j \circ T^{m-1})(x)\big| \\
&\leq \lambda \left\|(G_{j-1} \circ T^m)(x) - (G_j \circ T^{m-1})(x)\right\|_\infty \\
&\leq \lambda \left\|(G_{j-1} \circ T^m)(x) - (G_{j-1} \circ T^{m-1})(x)\right\|_\infty \\
&\leq \lambda \left\|T^m(x) - T^{m-1}(x)\right\|_\infty .
\end{aligned}$$

Thus

$$\left\|T^{m+1}(x) - T^m(x)\right\|_\infty \leq \max_{i_j \in I} \beta_{i_j} \leq \lambda \left\|T^m(x) - T^{m-1}(x)\right\|_\infty .$$

It easily follows that $(T^m(x))_{m \in \mathbb{N}}$ is a Cauchy sequence and thus converging to a fixed point x^* of T. This implies in particular that $x^* \in Q$.

We now prove the statement in case only $\mathfrak{R} \subset \mathrm{Lip}_1(l_\infty^{n-1}, \mathbb{R})$ is assumed. Moreover, assume without loss of generality that $0 \in Q$. By Lemma 3.1, it is enough to show that for any $R > 0$, the set $Q \cap B(0,R) \subset l_\infty^n$ is injective. For each $i \in I$, we set

$$(\underline{s}_i \circ \widehat{\pi}_i)(x) := \min\{\max\{(\underline{r}_i \circ \widehat{\pi}_i)(x), -R\}, R\},$$
$$(\overline{s}_i \circ \widehat{\pi}_i)(x) := \min\{\max\{(\overline{r}_i \circ \widehat{\pi}_i)(x), -R\}, R\}.$$

It is easy to see that $\underline{r}_i \leq \overline{r}_i$ implies $-R \leq \underline{s}_i \leq \overline{s}_i \leq R$. Hence, if one sets

$$P := \Big\{ x \in B(0,R) : \text{ for all } i \in I, \text{ one has } (\underline{s}_i \circ \widehat{\pi}_i)(x) \leq x_i \leq (\overline{s}_i \circ \widehat{\pi}_i)(x) \Big\},$$

it is easy to see that since the functions $\underline{s}_i$ and $\overline{s}_i$ are 1-Lipschitz and since we assumed that $0 \in Q$, one has $P = Q \cap B(0,R)$. We can thus for $k \in \mathbb{N}$ and $i \in I$, set $\lambda_k := 1 - \frac{1}{k}$, as well as

$$(\underline{s}_i^k \circ \widehat{\pi}_i)(x) := \lambda_k \big[(\underline{s}_i \circ \widehat{\pi}_i)(x) + R \big] - R,$$
$$(\overline{s}_i^k \circ \widehat{\pi}_i)(x) := \lambda_k \big[(\overline{s}_i \circ \widehat{\pi}_i)(x) - R \big] + R.$$

Note that $-R \leq \underline{s}_i^k \leq \underline{s}_i \leq \overline{s}_i \leq \overline{s}_i^k \leq R$. For any $k \in \mathbb{N}$, we now set

$$Q_k := \Big\{ x \in B(0,R) : \text{ for all } i \in I, \text{ one has } (\underline{s}_i^k \circ \widehat{\pi}_i)(x) \leq x_i \leq (\overline{s}_i^k \circ \widehat{\pi}_i)(x) \Big\}.$$

The functions in $\mathfrak{R}^k := \{\underline{s}_i^k : i \in I\} \cup \{\overline{s}_i^k : i \in I\}$ are all λ_k-Lipschitz. Hence, we can apply the above argument and define the 1-Lipschitz retraction $r^k \colon B(0,R) \to Q_k$ to be the pointwise limit of the sequence $(T^{m,k})_{m \in \mathbb{N}}$. It follows that Q_k is injective. Finally, since the sequence $(Q_k)_{k \in \mathbb{N}}$ is decreasing for the inclusion and $Q \cap B(0,R) = \bigcap_{k \in \mathbb{N}} Q_k$, it follows that $Q \cap B(0,R)$ is injective (cf. for instance [21, Theorem 5.1]). □

We later need a statement which is slightly more general than Proposition 4.1 and whose proof is a direct analogue of the above proof. Let $I^1, I^2, I^3 \subset I_n$ with $I^i \cap I^j = \emptyset$ if $i \neq j$ and

$$\mathfrak{R}^1 := \{\underline{r}_i : i \in I^1\} \cup \{\overline{r}_i : i \in I^1\}, \;\; \mathfrak{R}^2 := \{\underline{r}_i : i \in I^2\}, \;\; \mathfrak{R}^3 := \{\overline{r}_i : i \in I^3\}$$

such that $\mathfrak{R}^1, \mathfrak{R}^2, \mathfrak{R}^3 \subset \mathrm{Lip}_1(l_\infty^{n-1}, \mathbb{R})$. Set moreover

$$Q^1 := \Big\{ x \in \mathbb{R}^n : \text{ for all } i \in I^1 \text{ one has } (\underline{r}_i \circ \widehat{\pi}_i)(x) \leq x_i \leq (\overline{r}_i \circ \widehat{\pi}_i)(x) \Big\},$$
$$Q^2 := \Big\{ x \in \mathbb{R}^n : \text{ for all } i \in I^2 \text{ one has } (\underline{r}_i \circ \widehat{\pi}_i)(x) \leq x_i \Big\},$$
$$Q^3 := \Big\{ x \in \mathbb{R}^n : \text{ for all } i \in I^3 \text{ one has } x_i \leq (\overline{r}_i \circ \widehat{\pi}_i)(x) \Big\},$$

so that $Q^1, Q^2, Q^3 \subset l_\infty^n$. Assume finally that $Q := Q^1 \cap Q^2 \cap Q^3 \neq \emptyset$ and that for any $i \in I^1$, $\underline{r}_i \leq \overline{r}_i$. It follows that Q is injective.

III.5 The Cone K_C

In this section, we prove Lemmas 5.1 and 5.3. Both are used in the proof of Theorem 1.2. For $j \in I_n = \{1, \dots, n\}$, let us define the cone

$$C_j := \{x \in \mathbb{R}^n : x_j = \|x\|_\infty\},$$

note that

$$\mathrm{int}(C_j) = \{x \in \mathbb{R}^n : x_j > \max_{i \in I_n \setminus \{j\}} |x_i|\}$$

and set

$$\mathcal{C} := \{-C_j : j \in I_n\} \cup \{C_j : j \in I_n\}.$$

Let $\emptyset \neq C \subset \mathbb{R}^n$ be a convex polyhedral cone; in particular, $0 \in \mathrm{apex}(C)$ and $C = C + C = \lambda C$ for $\lambda > 0$. Define

$$\mathcal{S}_C := \{C' \in \mathcal{C} : \mathrm{int}(C') \cap C = \emptyset\}.$$

Finally, set

$$\begin{aligned} \bar{\mathrm{K}}_C &:= \{p \in \mathbb{R}^n : \exists\, a \in \mathrm{apex}(C) \text{ such that } \{C' \in \mathcal{C} : p \in a + C'\} \subset \mathcal{S}_C\} \\ &= \mathrm{apex}(C) + \left(\mathbb{R}^n \setminus \bigcup_{C' \in \mathcal{C} \setminus \mathcal{S}_C} C' \right) \end{aligned}$$

and

$$\mathrm{K}_C := \mathbb{R}^n \setminus \bar{\mathrm{K}}_C, \tag{5.1}$$

noting in particular that K_C is a cone, $C \subset \mathrm{K}_C$ and $\mathrm{apex}(C) + \mathrm{K}_C = \mathrm{K}_C$. Although we use the above expression in the proof of Lemma 5.1, note that K_C also admits the expression

$$\mathrm{K}_C := \bigcap_{a \in \mathrm{apex}(C)} \bigcup_{C' \in \mathcal{C} \setminus \mathcal{S}_C} (a + C'). \tag{5.2}$$

For a $\nu \in \mathbb{R}^n \setminus \{0\}$, let us denote by

$$H_\nu := \{x \in \mathbb{R}^n : x \cdot \nu \geq 0\}$$

the corresponding inner half-space at the origin with normal vector ν. Moreover, we again denote the standard basis of $\mathbb{R}^n$ by $\{e_1, \dots, e_n\}$. Note that in this notation and for any $j \in I_n$,

$$C_j = \bigcap_{(i,\sigma) \in (I_n \setminus \{j\}) \times \{\pm 1\}} H_{e_j + \sigma e_i} \quad \text{and} \quad -C_j = \bigcap_{(i,\sigma) \in (I_n \setminus \{j\}) \times \{\pm 1\}} H_{-e_j + \sigma e_i}.$$

We now prove that the cone $K_C \subset l_\infty^n$ is injective. The purpose of introducing K_C is that we are able to construct in the proof of Theorem 1.2 a 1-Lipschitz retraction of K_C onto C. As a remark, it follows from Lemma 5.3 that K_C consists of the union of C and points $p \in l_\infty^n$ that are contained in a finite intersection $\bigcap_i B(x_i, r_i)$ of balls centered at points $x_i \in C$ such that

$$\mathrm{apex}(C) \cap \bigcap_i B(x_i, r_i) = \emptyset.$$

5.1 Lemma. *Let $C \subset \mathbb{R}^n$ be a convex polyhedral cone such that* $\mathrm{int}(C) \neq \emptyset$*, then $K_C \subset l_\infty^n$ is injective.*

Proof. We want to use Proposition 4.1. We set

$$I^1 := \{j \in I_n : \exists \sigma \in \{\pm 1\} \text{ such that } \sigma C_j \in \mathcal{S}_C \text{ and } -\sigma C_j \notin \mathcal{S}_C\},$$
$$I^2 := \{j \in I_n : \{C_j, -C_j\} \subset \mathcal{S}_C\}.$$

Whenever $j \in I^1$ and $\tau C_j \in \mathcal{S}_C$, set

$$I^1_{(j,\tau)} := \{(i,\sigma) \in (I_n \setminus \{j\}) \times \{\pm 1\} : \sigma C_i \in \mathcal{S}_C\}$$

and whenever $j \in I^2$, let

$$I^2_{(j,\tau)} := I^2_j := \{(i,\sigma) \in (I_n \setminus \{j\}) \times \{\pm 1\} : \{C_i, -C_i\} \subset \mathcal{S}_C\}.$$

For $\alpha \in \{1,2\}$ and $j \in I^\alpha$, we define the cones

$$\widetilde{C}_j^\tau := \bigcap_{(i,\sigma) \in [(I_n \setminus \{j\}) \times \{\pm 1\}] \setminus I^\alpha_{(j,\tau)}} H_{\tau e_j - \sigma e_i}$$

with $\widetilde{C}_j^\tau := H_{\tau e_j}$ if $[(I_n \setminus \{j\}) \times \{\pm 1\}] \setminus I^\alpha_{(j,\tau)} = \emptyset$ and define for $a \in \mathrm{apex}(C)$ and $x \in l_\infty^n$ corresponding 1-Lipschitz functions by

$$r_a^{j,\tau}(x) := a_j + \tau \max_{(i,\sigma) \in [(I_n \setminus \{j\}) \times \{\pm 1\}] \setminus I^\alpha_{(j,\tau)}} \sigma(x_i - a_i).$$

If $\tau = 1$, then $y \in a + \widetilde{C}_j^1$ if and only if $y_j \geq r_a^{j,1}(y)$. We set $(\overline{r}_j \circ \widehat{\pi}_j)(x) := \inf_{a \in \mathrm{apex}(C)} r_a^{j,1}(x)$ and

$$N_{(j,1)} := \bigcap_{a \in \mathrm{apex}(C)} \left[\mathbb{R}^n \setminus \mathrm{int}(a + \widetilde{C}_j^1)\right] = \Big\{x \in \mathbb{R}^n : x_j \leq (\overline{r}_j \circ \widehat{\pi}_j)(x)\Big\}.$$

If $\tau = -1$, then $y \in a + \widetilde{C}_j^{-1}$ if and only if $y_j \leq r_a^{j,-1}(y)$. We set $(\underline{r}_j \circ \widehat{\pi}_j)(x) := \sup_{a \in \mathrm{apex}(C)} r_a^{j,-1}(x)$ and

$$N_{(j,-1)} := \bigcap_{a \in \mathrm{apex}(C)} \big[\mathbb{R}^n \setminus \mathrm{int}(a + \widetilde{C}_j^{-1})\big] = \Big\{x \in \mathbb{R}^n : x_j \geq (\underline{r}_j \circ \widehat{\pi}_j)(x)\Big\}.$$

If $j \in I^1$ and $\tau C_j \in \mathcal{S}_C$, we set $N_{(j,-\tau)} := \mathbb{R}^n$. Now, if $j \in I^2$ we need to show that

$$\underline{r}_j \circ \widehat{\pi}_j \leq \overline{r}_j \circ \widehat{\pi}_j, \tag{5.3}$$

before we can apply the statement after Proposition 4.1. Let us set

$$A_j := C_j \cup \bigcup_{(l,\eta) \in I^2_{(j,1)}} \eta C_l.$$

It is easy to see that $\mathrm{apex}(C) \cap \mathrm{int}(A_j) = \emptyset$ since $\mathrm{int}(C) \neq 0$. Furthermore, $\widetilde{C}_j^1 \subset A_j$ since for $x \in \widetilde{C}_j^1$, if $(i,\sigma) \in [(I_n \setminus \{j\}) \times \{\pm 1\}] \setminus I^2_{(j,1)}$, then $x_j \geq \sigma x_i$. Hence, either $x_j = \|x\|_\infty$ or there is $(l,\eta) \in I^2_{(j,1)}$ such that $\eta x_l = \|x\|_\infty$. It follows in particular that $\mathrm{apex}(C) \cap \mathrm{int}(\widetilde{C}_j^1) = \emptyset$. One then easily deduces (noting that $\widetilde{C}_j^{-1} = -\widetilde{C}_j^1$) that

$$\big[\mathrm{apex}(C) + \mathrm{int}(\widetilde{C}_j^1)\big] \cap \big[\mathrm{apex}(C) + \widetilde{C}_j^{-1}\big] = \emptyset$$

and this implies that $\underline{r}_j \circ \widehat{\pi}_j \leq \overline{r}_j \circ \widehat{\pi}_j$. Indeed, if $r_a^{j,1}(y) < r_{a'}^{j,-1}(y)$ for some $y \in \mathbb{R}^n$, it follows that $[a + \mathrm{int}(\widetilde{C}_j^1)] \cap [a' + \mathrm{int}(\widetilde{C}_j^{-1})] \neq \emptyset$. Now, on the one hand, it is easy to see that setting

$$N_C := \bigcap_{(i,\sigma) \in (I^1 \cup I^2) \times \{\pm 1\}} N_{(i,\sigma)} \tag{5.4}$$

it follows that $\mathrm{K}_C = N_C$. Indeed, note that $\mathbb{R}^n \setminus \mathrm{K}_C \subset \mathbb{R}^n \setminus N_C$ since if a face F of $[-1,1]^n$ which satisfies

$$\begin{aligned} F \in \mathcal{F} := \big\{F' \in \mathrm{Faces}([-1,1]^n) \setminus \{[-1,1]^n\} :\\ \forall (i,\sigma) \in I_n \times \{\pm 1\},\ \text{if } F' \subset \sigma C_i \text{ then } \sigma C_i \in \mathcal{S}_C\big\}, \end{aligned}$$

then $\mathrm{relint}(F) \cap N_C = \emptyset$. Indeed, in the asymmetric case where F is such that $-F \notin \mathcal{F}$, there is then $j \in I^1$ such that $F \subset \sigma C_j \subset \mathcal{S}_C$ for some $\sigma \in \{\pm 1\}$ and thus $\mathrm{relint}(F)$ is in the complement of $N_{(j,\sigma)}$. In the symmetric case where both F and $-F$ are in $\mathcal{F}$, there is then $j \in I^2$ such that $F \subset \sigma C_j$, $-F \subset -\sigma C_j$

and $\{C_j, -C_j\} \subset \mathcal{S}_C$ for some $\sigma \in \{\pm 1\}$, thus $\mathrm{relint}(F)$ is in the complement of $N_{(j,\sigma)}$. Hence in both cases and for any $\lambda > 0$, one has:

$$\mathrm{relint}\Big(\lambda F + \mathrm{apex}(C)\Big) \cap N_C = \emptyset$$

and thus $\mathbb{R}^n \setminus \mathrm{K}_C \subset \mathbb{R}^n \setminus N_C$. Now, note that if $x \in \mathbb{R}^n \setminus N_C$, then $x \in a + \mathrm{int}(\widetilde{C}_j^\tau)$ for some $j \in I^\alpha$ verifying $\tau C_j \in \mathcal{S}_C$ for some $\tau \in \{\pm 1\}$. Hence

$$x \in a + \mathrm{int}\left(\tau C_j \cup \bigcup_{(l,\eta)\in I^\alpha_{(j,\tau)}} \eta C_l\right)$$

and thus $x \notin \mathrm{K}_C$ by (5.1). Finally, by (5.3) we can, using (5.4), apply the statement following the proof of Proposition 4.1 to $N_C \subset l_\infty^n$ in order to obtain that N_C is injective and thus so is $\mathrm{K}_C \subset l_\infty^n$, which finishes the proof. □

To illustrate Lemma 5.1, consider the case where $C = H_{e_n}$. It follows that $\mathrm{apex}(C) = \partial H_{e_n}$ and $\mathcal{S}_C = \{-C_n\}$. Thus $\mathrm{K}_C = C = H_{e_n}$ is injective by Lemma 5.1, which we already know from the statement following the proof of Proposition 4.1. In the case where $C = C_n$, one has $\mathrm{apex}(C) = \{0\}$ and $\mathcal{S}_C = \mathcal{C} \setminus \{C_n\}$. Hence $\mathrm{K}_C = C = C_n$ is injective as we already know. Finally, if

$$C = C_n^\varepsilon := \left\{x \in \mathbb{R}^n : x_n \geq \max_{i \in I_n \setminus \{n\}} (1+\varepsilon)|x_i|\right\} \subset \mathrm{int}(C_n) \cup \{0\}$$

for some $\varepsilon > 0$, then we again have $\mathrm{apex}(C) = \{0\}$, $\mathcal{S}_C = \mathcal{C} \setminus \{C_n\}$ and $\mathrm{K}_C = C_n$. We moreover denote by

$$\mathrm{aff}(X) := \left\{\sum_{i=1}^{l} \alpha^i x^i : \{\alpha^1, \dots, \alpha^l\} \subset \mathbb{R},\ \{x^1, \dots, x^l\} \subset X, \sum_{i=1}^{l} \alpha^i = 1\right\} \subset \mathbb{R}^n$$

the *affine hull* of a subset $\emptyset \neq X \subset \mathbb{R}^n$. We now define a class of polytopes that can be obtained as a finite intersection of balls in l_∞^n.

5.2 Definition. Set $\mathcal{I}_0 := [-1,1]^n$ and for $1 \leq k \leq n$, let

$$\mathcal{I}_k := \left\{[-1,1]^n \cap \bigcap_{j=1}^{k} \big(T_{p^j} F_j - p^j\big) \subset \mathbb{R}^n : \text{for all } j \in \{1, \dots, k\},\ F_j \in \mathrm{Facets}([-1,1]^n) \text{ and } p^j \in F_j\right\}.$$

The next lemma enables us to find for any $p \in \mathrm{K}_C \setminus C$ a face $F \in \mathrm{Faces}(P, \mathrm{apex}(C))^c$ of a polytope $P \in \mathcal{I}_k$ such that for some $\bar{\gamma} \in (0, \infty)$ and $\bar{a} \in \mathrm{apex}(C)$, $F_p := \bar{\gamma}F + \bar{a}$ contains p. The interesting feature of F_p is that it is stable under any 1-Lipschitz retraction of l_∞^n onto a set containing C. It is key that the set $\mathcal{I}_k$ is finite for every k and that $\mathrm{Faces}(P)$ is finite for any $P \in \mathcal{I}_k$.

5.3 Lemma. *Let $C \subset l_\infty^n$ be a convex polyhedral cone such that $\mathrm{int}(C) \neq \emptyset$ and $0 \leq k := \dim(\mathrm{apex}(C)) < n$. Define $\Delta\colon \mathrm{apex}(C) \times (0, \infty) \to \mathbb{R}$ by*

$$\Delta(a, \gamma) := \min_{P \in \mathcal{I}_k} \min_{F \in \mathrm{Faces}(P, \mathrm{apex}(C))^c} d(\gamma F + a, \mathrm{apex}(C))$$

with $\min_{F \in \mathrm{Faces}(P, \mathrm{apex}(C))^c} d(\gamma F + a, \mathrm{apex}(C)) := \infty$ if $\mathrm{Faces}(P, \mathrm{apex}(C))^c = \emptyset$. Then, for each $p \in \mathrm{K}_C$ so that $d(p, \mathrm{apex}(C)) = \eta > 0$, there are $(\bar{a}, \bar{\gamma}) \in \mathrm{apex}(C) \times [\eta, \infty)$, $P \in \mathcal{I}_k$ and $F \in \mathrm{Faces}(P, \mathrm{apex}(C))^c$ such that for $F_p := \bar{\gamma}F + a$, one has:

- *(i)* *$p \in F_p$ as well as*
- *(ii)* *$d(F_p, \mathrm{apex}(C))$ is positive, $\Delta(0, 1) \neq \infty$ and $\Delta(0, 1)$ is positive as well. In addition:*

$$d(F_p, \mathrm{apex}(C)) \geq \Delta(a, \bar{\gamma}) = \bar{\gamma}\Delta(0, 1) \geq \eta\Delta(0, 1).$$

- *(iii)* *Moreover, for any set $C \subset X \subset l_\infty^n$ and any retraction $r \in \mathrm{Lip}_1(l_\infty^n, X)$ onto X, one has $r(F_p) \subset F_p$.*

In the proof of Lemma 5.3, we consider for given points p and q in l_∞^n, the set

$$\bigcup_{m \in \mathbb{N} \cap [n_0, \infty)} B(mq, \|mq - p\|_\infty) \subset l_\infty^n.$$

It is not difficult to see that there is a threshold $n_0 \in \mathbb{N}$ as well as $\bar{q} \in \mathbb{R}q$ such that $\|mq - p\|_\infty = \|mq - \bar{q}\|_\infty$ for any $m \geq n_0$ and such that the sequence of balls $(B(mq, \|mq - p\|_\infty))_{m \in \mathbb{N} \cap [n_0, \infty)}$ is increasing. Altogether, this implies that the above union can be written as the tangent cone $\mathrm{T}_{\bar{q}} B(n_0 q, \|n_0 q - p\|_\infty)$. Before proceeding to the formal proof of Lemma 5.3, we give a brief outline of the main ideas. For a fixed point $p \in \mathrm{K}_C \setminus C$, our strategy to prove Lemma 5.3 is to iterate the above observation as many times as the dimension k of $\mathrm{apex}(C)$. Going from step j to step $j + 1$, we consider a particular increasing sequence of balls with centers on a line in $\mathrm{apex}(C)$ and whose union is the tangent cone $\mathrm{T}_{\bar{z}} B(z, R_1)$ as described above.

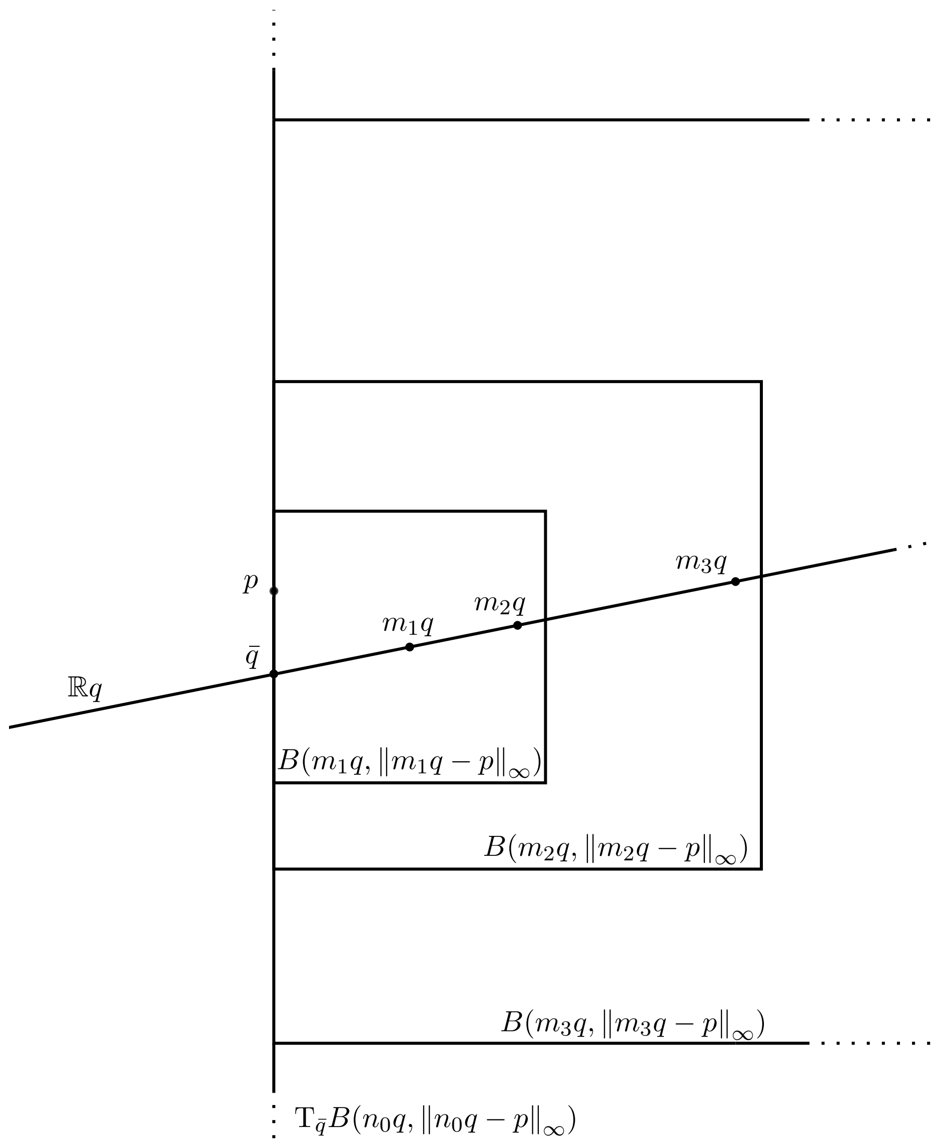

Figure III.1: Illustration of the definition of $T_{\bar{q}}B(n_0q, \|n_0q - p\|_\infty)$ in the case of l^2_∞. Each of m_1, m_2 and m_3 is bigger or equal than the threshold n_0 and hence, each of the three corresponding l_∞-balls have tangent cone at $\bar{q}$ equal to $T_{\bar{q}}B(n_0q, \|n_0q - p\|_\infty)$, which contains p in its boundary.

Following an easy criterion described in the proof, we consider a corresponding sequence of balls centered on a ray in C in the interior of a cone $\sigma_i C_{j_i} \in \mathcal{C}$ and once more, it follows as above, that their union can be written as a tangent cone to a ball, namely in this case $p + H_{\sigma_i e_{j_i}}$. These two tangent cones are defined in such a way that their intersection G_{j+1} (which is then by definition an increasing union of intersection of balls centered in C) is $(n-1)$-dimensional. Hence, $\bigcap_{l=0}^k G_l$ is of a similar form and we show that

$$\mathrm{apex}(C) \cap \bigcap_{l=0}^{k} G_l = \{\bar{a}\}.$$

Finally, we consider the polytope $P := B(\bar{a}, \bar{\gamma}) \cap \bigcap_{l=0}^k G_l$ which is a translated rescaling of a polytope in $\mathcal{I}_k$ (cf. Definition 5.2) and we show that P has a face $p \in F_p$ which is disjoint from $\mathrm{apex}(C)$ and which can be written as a finite intersection of balls centered in C. In particular, F_p is stable under any 1-Lipschitz retraction of l_∞^n onto a subset containing C. Finally, note that if C is injective (hence hyperconvex), then in particular $F_p \cap C \neq \emptyset$.

Proof of Lemma 5.3. Fix $p \in \mathrm{K}_C \subset l_\infty^n$ such that $\eta := d(p, \mathrm{apex}(C)) > 0$. We set $A_0 := \mathrm{apex}(C)$, $G_0 := l_\infty^n$, $D_0 := l_\infty^n$. We continue inductively and define for $1 \leq j+1 \leq k$ the following

$$A_{j+1} := \mathrm{apex}(C) \cap \bigcap_{l=0}^{j+1} \mathrm{apex}(G_l) \quad \text{and} \quad D_{j+1} := \bigcap_{l=0}^{j+1} \mathrm{aff}(G_l)$$

as well as the sets $G_1, \dots, G_k$ along the following procedure: for each $0 \leq j \leq k-1$, choose arbitrarily $a \in A_j$ and set $Y_j := B(a, 1) \cap D_j$. Next, pick $q \in A_j$ such that the following hold:

1) If there is a facet F of Y_j such that $A_j \cap \mathrm{relint}(F) \neq \emptyset$, then $q \in A_j \cap \mathrm{relint}(F)$.

2) If for any facet F' of Y_j, one has $A_j \cap \mathrm{relint}(F') = \emptyset$, then there is a face F'' of Y_j with $\dim(F'') \leq \dim(Y_j) - 2$ such that $A_j \subset \mathrm{aff}(F'' \cup \{a\})$ and then $q \in A_j \cap \mathrm{relint}(F'')$.

It is not difficult to see that exactly one of these two cases occur. Let us now set $q^m := a + m(q-a)$ for $m \in \mathbb{N}$. There exists $m_1 > 0$ such that one can find $\mathfrak{I} := \{(j_1, \sigma_1), \dots, (j_N, \sigma_N)\} \subset I_n \times \{\pm 1\}$ so that $\|p - q^m\|_\infty = \sigma_i(p_{j_i} - q^m_{j_i})$ for $m \geq m_1$ if and only if $(j_i, \sigma_i) \in \mathfrak{I}$. Hence $p \in q^m + [\bigcap_{(j_i,\sigma_i)\in\mathfrak{I}} \sigma_i C_{j_i} \setminus \bigcup_{(l,\tau)\notin\mathfrak{I}} \tau C_l]$. Since $p \in \mathrm{K}_C$ and $q^m \in \mathrm{apex}(C)$, it follows that there is some $(j_i, \sigma_i) \in \mathfrak{I}$ such that $w \in C \cap \mathrm{int}(\sigma_i C_{j_i}) \neq \emptyset$. We then set $w^m := a + mw \in C \cap [a + \mathrm{int}(\sigma_i C_{j_i})]$. As we noted before the proof, one can find $z, \bar{z} \in a + \mathbb{R}(q - a)$, as well as $R_1 > 0$ and $m_2 \in \mathbb{N} \cap [m_1, \infty)$ such that:

$$\mathrm{T}_{\bar{z}} B(z, R_1) = \bigcup_{m \geq m_2} B(q^m, \|q^m - p\|_\infty)$$

and $v, \bar{v} \in a + \mathbb{R}w$ as well as $R_2 > 0$ such that

$$p + H_{\sigma_i e_{j_i}} = \mathrm{T}_{\bar{v}} B(v, R_2) = \bigcup_{m \geq m_2} B(w^m, \|w^m - p\|_\infty).$$

We then set

$$G_{j+1} := \mathrm{T}_{\bar{z}} B(z, R_1) \cap (p + H_{\sigma_i e_{j_i}})$$

which is a face of $\mathrm{T}_{\bar{z}} B(z, R_1)$ and thus in particular a cone with

$$\mathrm{apex}(G_{j+1}) = \mathrm{apex}(\mathrm{T}_{\bar{z}} B(z, R_1)).$$

By construction, we can define the re-indexing $1 \leq f(j+1) := j_i \leq n$ such that

$$\mathrm{aff}(G_{j+1}) = p + \partial H_{e_{f(j+1)}}.$$

There is $I(j) := \{f(1), \dots, f(j)\} \subset I_n$ such that for any $x, y \in D_j$ and for any $f(l) \in I(j)$, $x_{f(l)} = y_{f(l)}$. Therefore, since for $m \geq m_2$ both p and q^m are in D_j and $p \in q^m + \sigma_i C_{j_i}$ it follows in particular that $j_i \notin I(j)$. Hence $q^m \notin \mathrm{aff}(G_{j+1}) = p + \partial H_{e_{j_i}}$ and therefore $\emptyset \neq \mathrm{aff}(G_{j+1}) \cap A_j \neq A_j$. Now, it is easy to see that for $1 \leq j+1 \leq k$, one has:

$$G_{j+1} \cap A_j = \mathrm{apex}(G_{j+1}) \cap A_j = \mathrm{aff}(G_{j+1}) \cap A_j,$$

$\dim(A_{j+1}) = \dim(A_j) - 1$ and $A_k = \{\bar{a}\} \subset \mathrm{apex}(C)$. We finally set $\bar{\gamma} := \|\bar{a} - p\|_\infty \geq \eta$ and

$$P := B(\bar{a}, \bar{\gamma}) \cap \bigcap_{l=0}^{k} G_l.$$

Similarly to what we have argued before, since $p \in \mathrm{K}_C$ there is $b \in C \cap \mathrm{int}(\bar{a} + \tau C_{n_0})$ where $n_0 \notin I(k)$ such that setting $\beta := \|b - p\|_\infty$ and

$$Q := B(\bar{a}, \bar{\gamma}) \cap D_k,$$

one has that $\bar{F} := B(b, \beta) \cap Q$ is a facet of Q in D_k. Setting finally $F_p := \bar{F} \cap P = B(b, \beta) \cap P$, it follows that F_p has the desired properties, in particular it is a face of P (Remark that $F_p = \bar{F} \cap P = (\mathrm{aff}(\bar{F}) \cap Q) \cap P = \mathrm{aff}(\bar{F}) \cap P$

and there is a half-space H of D_k such that $\mathrm{rel}\partial H = \mathrm{aff}(\bar{F})$ and $P \subset Q \subset H$. Hence F_p is a face of P cf. [42, Chapter 2]) and note that P is a translated rescaling (with parameters $\bar{a}$ and $\bar{\gamma}$) of a polytope in $\mathcal{I}_k$. This proves (i).

Moreover, $d(F_p, \mathrm{apex}(C))$ is positive since

$$F_p \cap \mathrm{apex}(C) = F_p \cap D_k \cap \mathrm{apex}(C) = F_p \cap \{\bar{a}\} = \emptyset.$$

The rest of (ii) is easily seen to hold. Indeed, $\Delta(0,1)$ is positive since $\mathcal{I}_k$ is a finite set and thus up to rescaling and translation along points of $\mathrm{apex}(C)$, there are only finitely many different intersections of a hyperplane of the form $p + H_{\sigma_i e_{j_i}}$ with a tangent cone to a ball like $\mathrm{T}_{\bar{z}} B(z, R_1)$ and thus there are only finitely many different outcomes for the sets $G_1, \ldots, G_k$ depending only on the dimension of l_∞^n and independently of the particular C.

Since P is bounded and looking at the definition of the sets $G_1, \ldots, G_k$; it is clear that the set P can be expressed as an intersection of closed balls centered in C that are pairwise intersecting and note that such balls are stable under r as given in (iii). This finally concludes the proof of the Lemma. $\square$

To illustrate Lemma 5.3, consider again the case where

$$C = C_n^\varepsilon := \{x \in \mathbb{R}^n : x_n \geq \max_{i \in I_n \setminus \{n\}} (1+\varepsilon)|x_i|\} \subset \mathrm{int}(C_n) \cup \{0\}$$

for some $\varepsilon > 0$ and consequently $\mathrm{apex}(C) = \{0\}$, $\mathcal{S}_C = \mathcal{C} \setminus \{C_n\}$ and $\mathrm{K}_C = C_n$. For any $p = (p_1, \ldots, p_n) \in C_n \setminus C_n^\varepsilon$, one has

$$F_p = B(0, \|p\|_\infty) \cap (p + H_{e_n}) = \{x \in l_\infty^n : \|x\|_\infty = \|p\|_\infty \text{ and } x_n = p_n\}.$$

Now, in the case $C = C_n^\varepsilon + \mathbb{R}e_{n-1}$, one has $\mathrm{apex}(C) = \mathbb{R}e_{n-1}$, $\mathrm{K}_C = C_n + \mathbb{R}e_{n-1}$. For any $p = (p_1, \ldots, p_n) \in \mathrm{K}_C \setminus C$, one has with $\bar{p} := (0, \ldots, 0, p_{n-1}, 0)$:

$$\begin{aligned} F_p &= \partial(p + H_{e_{n-1}}) \cap B(\bar{p}, p_n) \cap (p + H_{e_n}) \\ &= \{x \in l_\infty^n : \|x - \bar{p}\|_\infty = p_n,\ x_{n-1} = p_{n-1} \text{ and } x_n = p_n\}. \end{aligned}$$

III.6 Injective Convex Polyhedral Cones

In this section, we start by proving that (ii) in Theorem 1.2 is necessary for injectivity. We then go on with the proof of Theorem 1.2 with the help of Lemmas 5.1 and 5.3 from Section III.5. In the next lemma, we also make use of the observation before the proof of Lemma 5.3 (see also Fig. III.1).

6.1 Lemma. *Let $C \subset l_\infty^n$ be an injective convex polyhedral cone with non-empty interior such that for any $F \in \mathrm{Facets}^*([-1,1]^n, C)$, $-F \in \mathrm{Facets}^*([-1,1]^n, C)$ as well. Then $C = \mathbb{R}^n$.*

Proof. By assumption there is a subset $I \subset I_n = \{1, \dots, n\}$ such that

$$C \cap \operatorname{int}(\sigma C_j) \neq \emptyset \text{ if and only if } (j, \sigma) \in I \times \{\pm 1\}. \tag{6.1}$$

Let us assume for simplicity that $I = \{1, \dots, k\}$ with $I := \emptyset$ if $k = 0$. Note that by (6.1), for any $i \in I$ and any $x \in \mathbb{R}^n$ there is $(u^i, v^i) \in [\operatorname{int}(C) \cap \operatorname{int}(C_i)] \times [\operatorname{int}(C) \cap \operatorname{int}(-C_i)]$ such that $mu^i + x \in \operatorname{int}(C) \cap \operatorname{int}(C_i)$ and $mv^i + x \in \operatorname{int}(C) \cap \operatorname{int}(-C_i)$ for any $m \in \mathbb{N}$ as well as:

$$x + \partial H_{e_i} = \bigcup_{m \in \mathbb{N}} B(mu^i + x, \|mu^i\|_\infty) \cap \bigcup_{m \in \mathbb{N}} B(mv^i + x, \|mv^i\|_\infty)$$

(where H_ν is defined in Section III.5). Setting $U^i_m + x := B(mu^i + x, \|mu^i\|_\infty)$ and $V^i_m + x := B(mv^i + x, \|mv^i\|_\infty)$, we obtain

$$\bigcap_{i \in I}(x + \partial H_{e_i}) = \bigcap_{i \in I}\left(\bigcup_{m \in \mathbb{N}}[U^i_m + x] \cap \bigcup_{m \in \mathbb{N}}[V^i_m + x]\right).$$

It follows that there are $m_1, \dots, m_k, n_1, \dots, n_k \in \mathbb{N}$ such that

$$x \in \bigcap_{i \in I}\big([U^i_{m_i} + x] \cap [V^i_{n_i} + x]\big) =: S \subset \bigcap_{i \in I}(x + \partial H_{e_i}).$$

Note that S is an intersection of closed balls with centers in C and pairwise intersecting in l^n_∞ (since they all contain x), hence $S \cap C \neq \emptyset$ by hyperconvexity of C. We then deduce

$$\Big(\{x_1\} \times \cdots \times \{x_k\} \times \mathbb{R}^{n-k}\Big) \cap C \neq \emptyset \tag{6.2}$$

for any $\{x_i\}_{i \in I} \subset \mathbb{R}$. Set

$$\pi := \widehat{\pi}_{k+1} \circ \cdots \circ \widehat{\pi}_n$$

with $\pi \equiv 0$ if $k = 0$ and $\pi := \mathrm{id}_{\mathbb{R}^n}$ if $k = n$. From (6.2), it follows $\pi(C) = \mathbb{R}^k$. Assume now by contradiction that $\pi(\operatorname{apex}(C)) \neq \mathbb{R}^k$. Pick $p \in C$ such that $\pi(p) \notin \pi(\operatorname{apex}(C))$ and pick $q \in C \cap \pi^{-1}(\{-\pi(p)\})$. Remark that setting $z := q + p \in C \setminus \operatorname{apex}(C)$ one has $z \neq 0$ and $\pi(z) = 0$. Hence $\max_{1 \leq j \leq k} |z_j| = 0 < \max_{k+1 \leq j \leq n} |z_j|$ thus $\max_{1 \leq j \leq k} |z_j| < \|z\|_\infty$ and therefore $z \notin \cup_{1 \leq j \leq k}[C_j \cup (-C_j)]$. Since $\operatorname{int}(C) \neq \emptyset$ it follows that $C \cap \operatorname{int}(\sigma C_l) \neq \emptyset$ for some $(l, \sigma) \notin I \times \{\pm 1\}$ which contradicts (6.1). Thus $\pi(\operatorname{apex}(C)) = \mathbb{R}^k$. Hence, for any $y \in \mathbb{R}^k$, there is $w \in \operatorname{apex}(C)$ such that $\pi(w) = y$. Assume now by contradiction that there is $w' \in C$ such that

$\pi(w') = y$ and $w' \neq w$. Then $z := w' - w \in C \setminus \{0\}$ satisfies $\pi(z) = 0$ thus $\max_{1 \leq j \leq k} |z_j| = 0 < \max_{k+1 \leq j \leq n} |z_j|$ and this as before contradicts (6.1). It follows that $\pi\colon C \to \mathbb{R}^k$ is injective. By definition of π and since $\mathrm{int}(C) \neq \emptyset$, we deduce that $k = n$ thus $C = \mathbb{R}^n$. This proves the Lemma. □

Let $d_H(A, B)$ denote the Hausdorff distance of two subsets $\emptyset \neq A, B \subset l_\infty^n$, in other words

$$d_H(A, B) := \inf\{r \in (0, \infty) : A \subset N(B, r) \text{ and } B \subset N(A, r)\} \in [0, \infty]$$

with $\inf \emptyset := \infty$.

Before proceeding to the formal proof of Theorem 1.2, we give an outline of the main ideas in a few paragraphs. The strategy to show in the proof of Theorem 1.2 that (i) and (ii) imply the injectivity of C is to construct a 1-Lipschitz retraction r of K_C onto C. In order to do so, we consider an increasing sequence $(l\alpha q + C)_{l \in \mathbb{N}}$ of translates of C along $\mathbb{R}q$ with $\alpha > 0$. The direction q is chosen such that $-q \in \mathrm{int}(C)$, in order that $C \subset l\alpha q + C$ and $\cup_{l \in \mathbb{N}} (l\alpha q + C) = \mathbb{R}^n$. Moreover, q is chosen so that for a facet F of $[-1, 1]^n = B(0, 1) \subset l_\infty^n$ such that $F \notin \mathrm{Facets}^*([-1, 1]^n, C)$ and $-F \in \mathrm{Facets}^*([-1, 1]^n, C)$, one has $q \in \mathrm{relint}(F)$ which implies $d(q + \mathrm{apex}(C), \mathrm{K}_C) > 0$. We define r as the composition $r_2 \circ r_1$ of two 1-Lipschitz retractions. The points of $\mathrm{K}_C \setminus C$ that have distance to $\mathrm{apex}(C)$ greater than a fixed constant are mapped by r_1 to C. The purpose of r_2 is then to map the points situated in a neighborhood of $\mathrm{apex}(C)$ but which are outside $\mathrm{apex}(C)$, onto C.

Starting with the definition of r_1, we let r^l be the composition of retractions onto the tangent cones of $l\alpha q + C$ that are different from $l\alpha q + C$ itself and we let r_1 be the inverse limit of the system $(r^l)_{l \in \mathbb{N}}$, similarly to the proof of Theorem 1.1. After that, we define r_2 as the pointwise limit of the composition of a system of 1-Lipschitz retractions $(\varrho^k)_{k \in \mathbb{N}}$. The map ϱ^k is the composition of a fixed number of 1-Lipschitz retractions $\varrho^{k,l}$ defined (similarly as r^l above) as the composition of retractions onto the tangent cones of $l\frac{\alpha q}{2^k} + C$ (different from $l\frac{\alpha q}{2^k} + C$ itself).

To prove that $r := r_2 \circ r_1$ is the desired map, we note that the 1-Lipschitz retractions used to define r are all 1-Lipschitz retractions of l_∞^n onto a set containing C. Lemma 5.3 provides for any $p \in \mathrm{K}_C \setminus C$ a polytope F_p containing p, stable under r and such that $F_p \cap \mathrm{apex}(C) = \emptyset$.

In particular, r induces a 1-Lipschitz retraction of F_p onto $F_p \cap C$. To show that the image of r is exactly C, we consider in a particular neighborhood of $\mathrm{apex}(C)$, an arbitrary point $p \in C_{k,l_0+1} \cap (\mathrm{K}_C \setminus C_{k,l_0})$ where $C_{k,l_0} = l_0\frac{\alpha q}{2^k} + C$ and consider the map ϱ^{k,l_0} which consists of the composition of every 1-Lipschitz retraction onto the tangent cones of C_{k,l_0} (different from C_{k,l_0} itself).

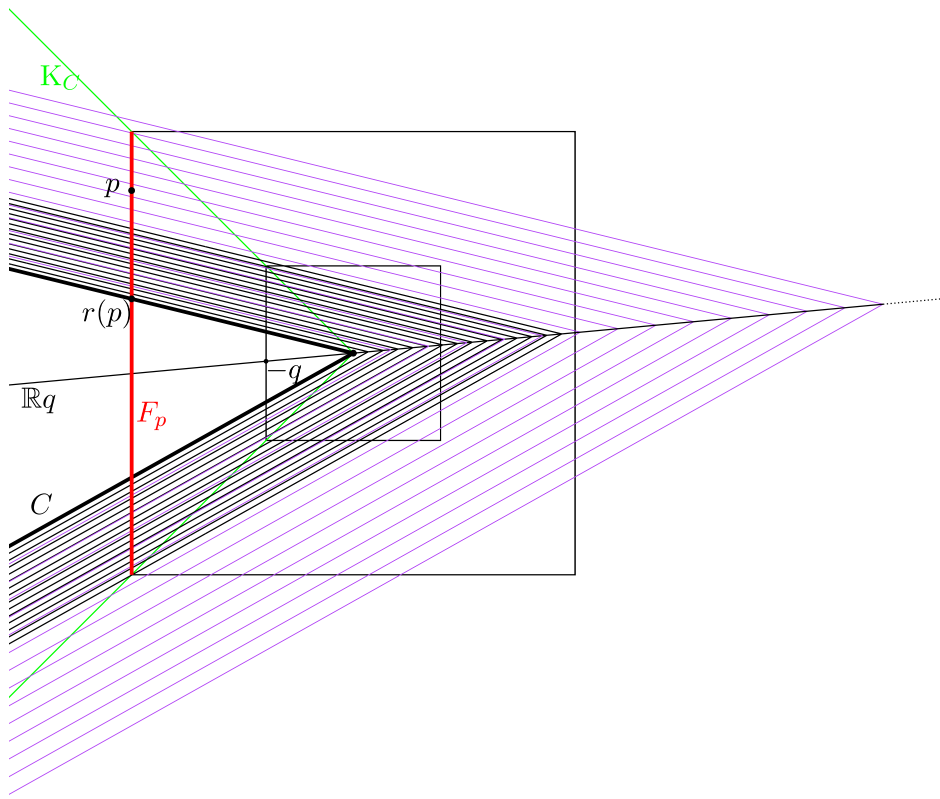

Figure III.2: Sketch of the construction of r in the proof of Theorem 1.2 for the especial case $n = 2$ and $\operatorname{apex}(C) = \{0\}$. The two squares are balls in l^2_∞ centered at the origin. For each point $p \in \mathrm{K}_C$, we consider a family $\{C_{k,l}\}_{l\in\{0,\dots,m\}}$ of translates of C (purple cones) such that $F_p \cap \mathrm{K}_C \subset \bigcup_{l\in\{0,\dots,m\}} C_{k,l}$ and which induce the image point $r(p) \in F_p \cap C$. The black cones represent the family of smaller scale translates $\{C_{k+1,l}\}_{l\in\{0,\dots,m\}}$, they induce the image under r of points in K_C that are closer to $\operatorname{apex}(C)$.

We show that there is a ball $U(p_0, \delta_{p_0})$ containing p and centered in C_{k,l_0} such that

$$U(p_0, \delta_{p_0}) \cap C_{k,l_0} = U(p_0, \delta_{p_0}) \cap \mathrm{T}_{p_0} C_{k,l_0}.$$

This step is similar to an argument in the proof of Theorem 1.1 with the key difference that it is here important that $p_0 \notin \operatorname{apex}(C_{k,l_0})$, in order that $C_{k,l_0} \subsetneq \mathrm{T}_{p_0} C_{k,l_0}$ and by definition of ϱ^{k,l_0} that consequently $\varrho^{k,l_0}(p) \subset C_{k,l_0}$. We can repeat this procedure until $l_0 = 0$ to obtain $\varrho^k(p) \in C$.

We use indifferently the notation $[-r, r]^n$ and $B(0, r)$ in the proof since both denote the same subset of l_∞^n.

Proof of Theorem 1.2. If C is injective, we know by Theorem 1.1 that its tangent cones are all injective. Furthermore, (ii) follows from Lemma 6.1.

Assume now that (i) and (ii) hold. Pick a facet F of $[-1,1]^n$ such that $F \notin \mathrm{Facets}^*([-1,1]^n, C)$ as well as $-F \in \mathrm{Facets}^*([-1,1]^n, C)$ and pick $q \in \operatorname{relint}(F)$ such that $-q \in \operatorname{relint}(-F) \cap \operatorname{int}(C)$. Remark that

$$\operatorname{int}([0,\infty)F) + \operatorname{apex}(C) \subset \mathbb{R}^n \setminus \mathrm{K}_C.$$

For $R > 0$, set

$$\Sigma_R := \mathrm{K}_C \cap \Big(B(0,R) + \operatorname{apex}(C) + [0,\infty)q \Big)^c.$$

Let us define the map $\bar{\Delta}\colon \operatorname{apex}(C) \times (0,\infty) \to \mathbb{R}$ by

$$\bar{\Delta}(a,\gamma) := \min_{P \in \mathcal{I}_k} \min_{F' \in \mathrm{Faces}(P, \operatorname{apex}(C))^c} d((\gamma F' + a) \cap \mathrm{K}_C, [0,\infty)q + \operatorname{apex}(C)). \quad (6.3)$$

where $k := \dim(\operatorname{apex}(C))$. It is easy to see with the help of Lemma 5.3 that $\varepsilon := \bar{\Delta}(0,1) > 0$ and thus by rescaling

$$\bar{\Delta}(a,\kappa) = \bar{\Delta}(0,\kappa) = \kappa \bar{\Delta}(0,1) = \kappa\varepsilon. \quad (6.4)$$

Furthermore, there is $\bar{\varepsilon} \in (0,\varepsilon)$ such that $C \cup \bigcup_{p \in \partial C} U(p, \bar{\varepsilon}) \subset C \cup \bigcup_{p \in \partial C} U(p, \varepsilon_p)$ where for any $p \in \partial C$, we set

$$\varepsilon_p := \sup\{\delta \in (0,\varepsilon) : U(p,\delta) \cap \mathrm{T}_p C = U(p,\delta) \cap C\}, \quad (6.5)$$

cf. proof of Theorem 1.1. Let us then choose $\alpha \in [0,\infty)$ such that

$$d_H(C, \alpha q + C) < \tfrac{\bar{\varepsilon}}{2}. \quad (6.6)$$

Since by definition, one has $[0,\infty)q + C = l_\infty^n$, there is $m \in \mathbb{N}$ so that

$$B(0,1) + \operatorname{apex}(C) \subset m\alpha q + C$$

which after rescaling becomes

$$B(0, \tfrac{1}{2^k}) + \mathrm{apex}(C) \subset \frac{m\alpha q}{2^k} + C. \tag{6.7}$$

Let $\{T_j\}_{j\in\{1,\dots,N\}}$ be an enumeration of the set:

$$\big\{\mathrm{T}_p(C) : \text{there is } F \in \mathrm{Faces}(C) \setminus \{\mathrm{apex}(C)\} \text{ such that } p \in \mathrm{relint}(F)\big\}.$$

If for each $j \in \{1, \dots, N\}$, we pick a 1-Lipschitz retraction $\varrho_j \colon l_\infty^n \to T_j$, then $\varrho := \varrho_N \circ \dots \circ \varrho_1$ defines a 1-Lipschitz retraction of $\alpha q + C$ onto C, cf. proof of Theorem 1.1. Let us now for $y \in X$ denote by τ_y the translation map $x \mapsto x + y$. For $l \in \mathbb{N}$, the map

$$r^l := \tau_{l\alpha q} \circ \varrho \circ \tau_{-l\alpha q}$$

is a 1-Lipschitz retraction of $(l+1)\alpha q + C$ onto $l\alpha q + C$. We then define

$$r_1(x) := (r^0 \circ r^1 \circ \dots \circ r^M)(x)$$

where M is the smallest natural such that $x \in M\alpha q + C$. Similarly, for any $j \in \{1, \dots, N\}$, $k \in \mathbb{N} \cup \{0\}$ and $l \in \{0, \dots, m\}$, we set

$$\varrho^{k,l} := \tau_{\frac{l\alpha q}{2^k}} \circ \varrho \circ \tau_{-\frac{l\alpha q}{2^k}}$$

as well as

$$\varrho^k := \varrho^{k,0} \circ \dots \circ \varrho^{k,m}.$$

We then define

$$r := r_2 \circ r_1$$

by setting for any $y \in r_1(\mathrm{K}_C)$:

$$r_2(y) := \lim_{k\to\infty} (\varrho^k \circ \varrho^{k-1} \dots \circ \varrho^1)(y).$$

We now show that r is well-defined, $r|_C = \mathrm{id}_C$ and $r \in \mathrm{Lip}_1(\mathrm{K}_C, C)$. This implies that C is injective by Lemma 5.1. Consider first $R \in (\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ with $k \in \mathbb{N} \cup \{0\}$ and let $p \in \mathrm{K}_C$ be a point at distance R from apex(C). Borrowing its notation, we can by Lemma 5.3 find a corresponding F_p containing p such that by (6.3) and (6.4), one has

$$F_p \cap \mathrm{K}_C \subset \Sigma_{\frac{\varepsilon}{2^{k+1}}}. \tag{6.8}$$

Assume that $p \notin C$. Note that by (*iii*) in Lemma 5.3 and since it is easy to see that $r(\mathrm{K}_C) \subset \mathrm{K}_C$, one has

$$r(F_p \cap \mathrm{K}_C) \subset F_p \cap \mathrm{K}_C.$$

Let us set $C_{k,l} := l\frac{\alpha q}{2^k} + C$ for any $l \in \{0, \dots, m\}$. By (6.7), there is then $l_0 \in \{0, \dots, m-1\}$ such that $p \in C_{k,l_0+1} \setminus C_{k,l_0}$ since p was chosen so that

$$p \in \partial\Big(B(0,R) + \mathrm{apex}(C)\Big) \cap \mathrm{K}_C \subset B(0, \tfrac{1}{2^k}) + \mathrm{apex}(C).$$

It follows by (6.6) that

$$d_H(C_{k,l_0}, C_{k,l_0+1}) = d_H(C, C_{k,1}) = \frac{1}{2^k} d_H(C, \alpha q + C) < \frac{\bar{\varepsilon}}{2^{k+1}}.$$

Therefore, noting that if $z \in C_{k,l_0}$ then $\sigma(z) := 2^k \left(z - l_0\frac{\alpha q}{2^k}\right) \in C$, one sees (cf. (6.5) for the definition of $\varepsilon_{\sigma(z)}$) that

$$p \in \bigcup_{z \in \partial C_{k,l_0}} U(z, \tfrac{\bar{\varepsilon}}{2^{k+1}}) \subset \bigcup_{z \in \partial C_{k,l_0}} U(z, \tfrac{\varepsilon_{\sigma(z)}}{2^k}).$$

Hence, there is $p_0 \in \partial C_{k,l_0}$ such that $p \in U(p_0, \delta_{p_0})$, $\delta_{p_0} < \frac{\bar{\varepsilon}}{2^{k+1}}$ and

$$U(p_0, \delta_{p_0}) \cap C_{k,l_0} = U(p_0, \delta_{p_0}) \cap \mathrm{T}_{p_0} C_{k,l_0}.$$

From $\delta_{p_0} < \frac{\bar{\varepsilon}}{2^{k+1}}$ and $\bar{\varepsilon} < \varepsilon$, it follows that $p_0 \notin \mathrm{apex}(C_{k,l_0})$ because by (6.8):

$$d(F_p \cap \mathrm{K}_C, \mathrm{apex}(C_{k,l_0})) \geq d(F_p \cap \mathrm{K}_C, [0,\infty)q + \mathrm{apex}(C)) \geq \frac{\varepsilon}{2^{k+1}} > \delta_{p_0}.$$

There is then $j \in \{1, \dots, N\}$ such that $\mathrm{T}_{p_0} C_{k,l_0} = l_0\frac{\alpha q}{2^k} + T_j$. Hence $\varrho^{k,l_0}(p) \in C_{k,l_0}$ and thus $\varrho^k(p) \in C$.

The case where $p \in \mathrm{K}_C$ is a point at distance $R \geq 1$ from $\mathrm{apex}(C)$ is similar. It follows that r is well-defined and it is then obviously a 1-Lipschitz retraction onto C. This finally concludes the proof. $\square$

III.7 Graph Representation of Linear Systems of Inequalities with at most Two Variables per Inequality

In this section, we introduce concepts that we later use in Section III.8 to prove Corollary 1.6. Let $\emptyset \neq Q \subset \mathbb{R}^n$ be an intersection of general half-spaces, that is half-spaces that are either closed or open. To a general half-space H containing Q, we assign its inner normal vector $\nu \in \mathbb{R}^n \setminus \{0\}$ in order that there is $p \in \mathbb{R}^n$ such that $H = p + H_\nu$ if H is closed and $H = p + \mathrm{int}(H_\nu)$ if H is open (recalling that $H_\nu := \{x \in \mathbb{R}^n : x \cdot \nu \geq 0\}$). For $n \in \mathbb{N}$, let us denote by $\mathcal{Z}_n$ the family of all Q for which there is a set $\emptyset \neq \mathcal{N}(Q) \subset \mathbb{R}^n \setminus \{0\}$ such that the following hold:

(a) $\mathcal{N}(Q)$ is finite and Q can be written as the intersection over all $\nu \in \mathcal{N}(Q)$ of a general half-space with inner normal vector ν.

(b) For every $\nu \in \mathcal{N}(Q)$, there exist $f_\nu, g_\nu \in \{0\} \cup \{e_1, \ldots, e_n\}$ and $a_\nu, b_\nu \in \mathbb{R}$ so that $f_\nu \neq g_\nu$ as well as

$$\nu = a_\nu f_\nu + b_\nu g_\nu.$$

We now describe a construction that was introduced in [39]. Every $Q \in \mathcal{Z}_n$ is the solution set of a linear system of inequalities of the form

$$\Sigma := \{a_\nu y_\nu + b_\nu z_\nu \succeq c_\nu\}_{\nu \in \mathcal{N}(Q)}$$

where $\succeq$ stands for $\geq$ in some inequalities and possibly for $>$ in some others and $y_\nu, z_\nu \in \{x_0, x_1, \ldots, x_n\}$ denote variables so that $y_\nu = x_i$ if $f_\nu = e_i$ as well as $z_\nu = x_j$ if $g_\nu = e_j$ and $y_\nu = x_0$ if $f_\nu = 0$. Conversely, to any system of linear inequalities as above, we can associate an element of $\mathcal{Z}_n$. Now, we can require all variables appearing in Σ to have nonzero coefficients except the *zero variable* x_0 which we additionally require to appear only with coefficient zero. We can associate to Σ an undirected labeled multigraph without self-loops $\Gamma_\Sigma := (\mathrm{V}_\Sigma, \mathrm{E}_\Sigma)$ where the vertex set V_Σ is given by $\{x_0, x_1, \ldots, x_n\}$ and the set $\mathrm{E}_\Sigma := \{E_\nu\}_{\nu \in \mathcal{N}(Q)}$ consists of all the labeled edges $E_\nu = \big(\{y_\nu, z_\nu\}, \Sigma_\nu\big)$ where Σ_ν denotes the inequality $a_\nu y_\nu + b_\nu z_\nu \succeq c_\nu$. Note that Γ_Σ does not contain any self-loop since we require $y_\nu \neq z_\nu$, that is all equations in Σ contain two different variables. Equations that contain only one variable different from x_0 are given by edges connecting to x_0 and remark that Σ does neither contain any trivial inequalities like for example $1 \geq 0$ nor trivial inequalities of the other type like for instance $-\frac{1}{3} > 0$. A *path* P in Γ_Σ is then given by

$$\big((v_1, \ldots, v_{m+1}), E_1, \ldots, E_m\big) \tag{7.1}$$

where $(v_1, \ldots, v_{m+1})$ is a sequence of vertices in V_Σ and $(E_1, \ldots, E_m)$ a sequence of labeled edges in E_Σ such that for each $l \in \{1, \ldots, m\}$, one has:

$$E_l = (\{v_l, v_{l+1}\}, a_l v_l + b_l v_{l+1} \succeq c_l)\,.$$

We call P *admissible* if for each $l \in \{1, \ldots, m-1\}$, the coefficients b_l and a_{l+1} have opposite signs (i.e., one is strictly positive and the other one is strictly negative). Note that if P is admissible, one has $v_l \neq x_0$ for each $l \in \{2, \ldots, m-1\}$ because we have required that x_0 appears only with zero coefficient. Admissible paths correspond to sequences of inequalities that form transitivity chains, the three inequalities $2x_1 - 3x_2 > -4$, $2x_2 + x_3 \geq 4$ and $-x_3 - x_1 \geq 0$ give e.g. rise to an admissible path. However, the three inequalities $x_1 - x_2 \geq 0$, $x_2 - x_3 \geq 0$ and $-x_3 - x_4 \geq 0$ cannot label an admissible path since the coefficients of x_3 have the wrong relative signs. A path is called a *loop* if its first and last vertices are identical and a loop is said to be *simple* as soon as its intermediate vertices are distinct. The reverse of an admissible loop is admissible and cyclic permutations of a loop P given by (7.1) are admissible if and only if a_1 and b_m have opposite signs, in which case P is called *permutable*. Note also that since x_0 appears in Σ only with zero coefficient, no admissible loop with initial vertex x_0 is permutable.

For an admissible path P given again by (7.1), let us define the *residue inequality* of P to be the inequality obtained by applying transitivity to the inequalities labeling the edges of P. The residue inequality of P is thus of the form $av_1 + bv_{m+1} \succeq c$, where $\succeq$ denotes a strict inequality if and only if at least one of the inequalities labeling the edges of P is strict. Consider for example a path P given by

$$\begin{aligned}\Big((x_1, x_2, x_3, x_4), (\{x_1, x_2\}, x_1 - 2x_2 \geq -1)\,,\\ (\{x_2, x_3\}, x_2 + 3x_3 > -2)\,,\\ (\{x_3, x_4\}, -x_3 - x_4 \geq 0)\Big),\end{aligned}$$

we have $x_1 > -1 + 2(-2 - 3x_3) = -5 - 6x_3 \geq -5 + 6x_4$ and thus the residue inequality of P is $x_1 - 6x_4 > -5$. In the case where P is a loop with initial vertex v, its residue inequality is of the form $(a+b)v \succeq c$. If it happens that $(a+b)v > c$, $a + b = 0$ and $c \geq 0$ or $(a+b)v \geq c$, $a+b = 0$ and $c > 0$, the residue inequality of P is false and we say that P is an *infeasible* loop. Note in particular that infeasibility implies admissibility. We define a *closure* $\overline{\Gamma}_\Sigma := (V_\Sigma, \overline{\mathrm{E}}_\Sigma)$ of Γ_Σ to be a graph $\overline{\Gamma}_\Sigma$ containing Γ_Σ and having the same vertex set, such that $\overline{\mathrm{E}}_\Sigma$ is obtained from E_Σ by adding for each simple admissible loop P (modulo permutations and reversals) of Γ_Σ, a *residue edge* which is a new edge labeled with the residue inequality of P. Let moreover $\mathrm{Nontrivial}(\overline{\mathrm{E}}_\Sigma)$ denote all the elements of $\overline{\mathrm{E}}_\Sigma$ that are no self-loop at x_0. Note that a closure is not necessarily unique since the initial vertex of each permutable loop can be chosen arbitrarily. We can now state the main theorem of [39]:

7.1 Theorem. *Σ is unsatisfiable if and only if $\overline{\Gamma}_\Sigma$ has an infeasible simple loop.*

As an example, consider the system

$$\Sigma = \{\Sigma_i\}_{i\in\{1,\dots,6\}} = \Big\{x_1 - x_2 \geq 0,\ 2x_1 + x_2 \geq -1,\ x_3 - x_1 \geq 0,\\ x_4 - x_3 \geq 0,\ x_3 - x_4 \geq -1,\ -x_3 \geq \tfrac{1}{2}\Big\}.$$

It is easy to see that the only loop of Γ_Σ contributing an edge to $\overline{\Gamma}_\Sigma$ is the loop

$$\Big((x_1, x_2, x_1), (\{x_1, x_2\}, \Sigma_1), (\{x_2, x_1\}, \Sigma_2)\Big)$$

having residue inequality $x_1 \geq -\frac{1}{3}$. Now note that the loop

$$\Big((x_0, x_1, x_3, x_0), (\{x_0, x_1\}, x_1 \geq -\tfrac{1}{3}), (\{x_1, x_3\}, \Sigma_3), (\{x_3, x_0\}, \Sigma_6)\Big) \subset \overline{\Gamma}_\Sigma$$

is infeasible and hence Σ must be unsatisfiable according to the theorem.

III.8 Injectivity of Linear Systems of Inequalities with at most Two Variables per Inequality

In this section, we prove Proposition 8.1 from which Corollary 1.6 follows immediately. For $j \in I_n = \{1, \dots, n\}$, let $F_j := [-1,1]^{j-1} \times \{1\} \times [-1,1]^{n-j}$ which is a facet of the unit cube $[-1,1]^n$. Note that $\mathrm{relint}(F_j) = (-1,1)^{j-1} \times \{1\} \times (-1,1)^{n-j}$ and this is used in the proof below.

8.1 Proposition. *Let $C \subset \mathcal{Z}_n$ be a convex polyhedral cone with* $\mathrm{int}(C) \neq \emptyset$ *satisfying*

$$C = \bigcap_{\nu\in\mathcal{N}(C)} \{x \in \mathbb{R}^n : x \cdot (a_\nu f_\nu + b_\nu g_\nu) \geq 0\}.$$

There is then $(j, \tau) \in I_n \times \{\pm 1\}$ such that

$$C \cap \mathrm{relint}(\tau F_j) \neq \emptyset = C \cap \mathrm{relint}(-\tau F_j).$$

Proof. We proceed by induction on n. It is easy to see that the result holds for $n = 1$ and $n = 2$. We assume that the result holds for $\{1, \dots, n-1\}$ and show that it consequently holds for n. Since $\mathrm{int}(C) \neq \emptyset$, there is $(s, \sigma) \in I_n \times \{\pm 1\}$

such that $C \cap \operatorname{relint}(\sigma F_s) \neq \emptyset$. If $C \cap \operatorname{relint}(-\sigma F_s) = \emptyset$, we are done. Hence, assume that

$$C \cap \operatorname{relint}(-F_s) \neq \emptyset \neq C \cap \operatorname{relint}(F_s) \tag{8.1}$$

which recalling the notation $\partial H_{e_s} = \{x \in \mathbb{R}^n : x_s = 0\}$ implies

$$\operatorname{relint}(C \cap \partial H_{e_s}) \neq \emptyset. \tag{8.2}$$

The map $\widehat{\pi}_s$ given by $(x_1, x_2, \ldots, x_s, \ldots, x_n) \mapsto (x_1, x_2, \ldots, \widehat{x}_s, \ldots, x_n)$ is, when restricted to ∂H_{e_s}, an isometry with the property that $C^0 := \widehat{\pi}_s(C \cap \partial H_{e_s}) \in \mathcal{Z}_{n-1}$. To see that the latter holds, assume without loss of generality that $f_\nu \neq e_s$ for every $\nu \in \mathcal{N}(C)$. We can write $\mathcal{N}(C) = \mathcal{N}(C)^{\not s} \sqcup \mathcal{N}(C)^s$ where $\mathcal{N}(C)^{\not s}$ is the set of all ν such that $f_\nu \neq e_s \neq g_\nu$ and $\mathcal{N}(C)^s$ the set of those such that $f_\nu \neq e_s = g_\nu$. We then write $C^{\not s} := \cap_{\nu \in \mathcal{N}(C)^{\not s}} H_\nu$ and $C^s := \cap_{\nu \in \mathcal{N}(C)^s} H_\nu$ which implies $C = C^s \cap C^{\not s}$. It is easy to see that

$$C \cap \partial H_{e_s} = C^{\not s} \cap \partial H_{e_s} \cap \bigcap_{\nu \in \mathcal{N}(C)^s} H_{a_\nu f_\nu}.$$

Applying $\widehat{\pi}_s$ on both sides, we get:

$$\begin{aligned} C^0 &= \widehat{\pi}_s(C^{\not s} \cap \partial H_{e_s}) \cap \widehat{\pi}_s \left(\partial H_{e_s} \cap \bigcap_{\nu \in \mathcal{N}(C)^s} H_{a_\nu f_\nu} \right) \\ &= \bigcap_{\nu \in \mathcal{N}(C)^{\not s}} H_{\widehat{\pi}_s(\nu)} \cap \bigcap_{\nu \in \mathcal{N}(C)^s} H_{\widehat{\pi}_s(a_\nu f_\nu)} \in \mathcal{Z}_{n-1}. \end{aligned}$$

It follows by the induction hypothesis that there is $(t, \tau) \in (I_n \setminus \{s\}) \times \{\pm 1\}$ such that

$$C^0 \cap \widehat{\pi}_s(\operatorname{relint}(\tau F_t) \cap \partial H_{e_s}) \neq \emptyset = C^0 \cap \widehat{\pi}_s(\operatorname{relint}(-\tau F_t) \cap \partial H_{e_s}). \tag{8.3}$$

Note moreover that $C^0 \cap \widehat{\pi}_s(\operatorname{relint}(\tau F_t) \cap \partial H_{e_s}) \neq \emptyset$ implies $C \cap \operatorname{relint}(\tau F_t) \neq \emptyset$. Furthermore, if $C^{\not s} \cap \operatorname{relint}(-\tau F_t) \cap \partial H_{e_s} = \emptyset$, then $C \cap \operatorname{relint}(-\tau F_t) = \emptyset$ and thus we are done. We thus assume that

$$C^{\not s} \cap \operatorname{relint}(-\tau F_t) \cap \partial H_{e_s} \neq \emptyset. \tag{8.4}$$

We now show that one can find $a, b \in \mathbb{R}$ with $b \neq 0$ such that $C \subset H_{a e_s + b e_t}$. We can assume without loss of generality that in addition to $f_\nu \neq e_s$, one has $f_\nu \neq e_t$ for any $\nu \in \mathcal{N}(C)$ since otherwise we can find the desired normal vector $ae_s + be_t$. Let $\mathcal{N}(C)^{t\!\!/}$ be the set of all ν such that $f_\nu \neq e_t \neq g_\nu$ and $C^{t\!\!/} := \cap_{\nu\in\mathcal{N}(C)^{t\!\!/}} H_\nu$. Let

$$W_t := (-1,1)^n \cup \mathrm{relint}(F_t) \cup \mathrm{relint}(-F_t).$$

Note that $W_t \cap C \in \mathcal{Z}_n$ and $\widehat{\pi}_t \circ \widehat{\pi}_s(C \cap \partial H_{e_s} \cap \mathrm{relint}(-\tau F_t)) \in \mathcal{Z}_{n-2}$. Let Σ and Σ^0 denote the respective associated systems induced by the supporting half-spaces. Remark that Σ^0 is obtained by plugging $x_s = 0$ and $x_t = -\tau$ in every inequality of Σ and deleting those loops corresponding to those inequalities associated to W_t that are made trivial. Note that Σ^0 is unsatisfiable by (8.3) and thus by Theorem 7.1, there is an infeasible (hence by definition admissible) simple loop L in every closure $\overline{\Gamma}_{\Sigma^0}$ of the graph Γ_{Σ^0} associated to the system Σ^0.

Let now $\Gamma_{\Sigma^s} := (\mathrm{V}_\Sigma \setminus \{x_s, x_t\}, \mathrm{E}_{\Sigma^s})$ where E_{Σ^s} consists of all labeled edges $E \in \mathrm{E}_{\Sigma^0}$ so that there is $(\{y_\mu, x_s\}, a_\mu y_\mu + b_\mu x_s \succeq c_\mu) \in \mathrm{E}_\Sigma$ such that $E = (\{y_\mu, x_0\}, a_\mu y_\mu \succeq c_\mu)$ (possibly with $y_\mu = x_0$). Analogously, $\Gamma_{\Sigma^t} := (\mathrm{V}_\Sigma \setminus \{x_s, x_t\}, \mathrm{E}_{\Sigma^t})$ where E_{Σ^t} consists of all labeled edges $E \in \mathrm{E}_{\Sigma^0}$ so that there is $(\{y_\mu, x_t\}, a_\mu y_\mu + b_\mu x_t \succeq c_\mu) \in \mathrm{E}_\Sigma$ such that $E = (\{y_\mu, x_0\}, a_\mu y_\mu \succeq c_\mu + \tau b_\mu)$ (possibly with $y_\mu = x_0$). Now, it is easy to see that for $u \in \{s,t\}$, one can choose closures satisfying:

$$\mathrm{Nontrivial}\Big(\overline{\mathrm{E}}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^u}\Big) \subset \mathrm{Nontrivial}\Big(\overline{\mathrm{E}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^u}}\Big). \tag{8.5}$$

Indeed, note that since $\mathrm{E}_{\Sigma^u} \subset \mathrm{E}_{\Sigma^0}$, it follows that

$$\overline{\mathrm{E}}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^u} = \big(\overline{\mathrm{E}}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^0}\big) \cup \big(\mathrm{E}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^u}\big). \tag{8.6}$$

By definition, $\mathrm{E}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^u} \subset \overline{\mathrm{E}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^u}}$. Now, consider an admissible loop $L_0 \subset \Gamma_{\Sigma^0}$. If L_0 contains an edge of Γ_{Σ^u}, then by admissibility (in particular x_0 does not arise as intermediate vertex), L_0 is a loop starting at x_0 and thus L_0 does not induce any nontrivial residue edge. Hence, if $L_0 \subset \Gamma_{\Sigma^0}$ induces one nontrivial residue edge in $\overline{\mathrm{E}}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^0}$, then in particular, the residue equation of L_0 does not contain x_0 alone and thus L_0 does not contain any edge of Γ_{Σ^u}. This thus implies that $\mathrm{Nontrivial}\Big(\overline{\mathrm{E}}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^0}\Big) \subset \mathrm{Nontrivial}\Big(\overline{\mathrm{E}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^u}}\Big)$. It finally follows by (8.6) that (8.5) holds.

Now, if $L \subset \overline{\Gamma}_{\Sigma^0}$ is nontrivial and does not contain any edge of Γ_{Σ^s}, we obtain in view of (8.5):

$$\begin{aligned} L &\subset \Big(\mathrm{V}_\Sigma \setminus \{x_s, x_t\}, \mathrm{Nontrivial}\Big(\overline{\mathrm{E}}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^s}\Big)\Big) \\ &\subset \Big(\mathrm{V}_\Sigma \setminus \{x_s, x_t\}, \mathrm{Nontrivial}\Big(\overline{\mathrm{E}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^s}}\Big)\Big) =: \Gamma. \end{aligned}$$

But Γ has $Q = \widehat{\pi}_t \circ \widehat{\pi}_s \left(C^{\not s} \cap \operatorname{relint}(-\tau F_t) \cap \partial H_{e_s}\right)$ as associated solution set. Thus Γ contains an infeasible simple loop and therefore its associated system is unsatisfiable by Theorem 7.1. Hence, $Q = \emptyset$ and thus $C^{\not s} \cap \operatorname{relint}(-\tau F_t) \cap \partial H_{e_s} = \emptyset$, which contradicts (8.4). It follows that L has to contain an edge of Γ_{Σ^s}.

Similarly, if L is nontrivial and does not contain any edge of Γ_{Σ^t}, we obtain in view of (8.5):

$$\begin{aligned} L &\subset \Big(\mathrm{V}_\Sigma \setminus \{x_s, x_t\}, \operatorname{Nontrivial}\Big(\overline{\mathrm{E}}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^t}\Big)\Big) \\ &\subset \Big(\mathrm{V}_\Sigma \setminus \{x_s, x_t\}, \operatorname{Nontrivial}\Big(\overline{\mathrm{E}_{\Sigma^0} \setminus \mathrm{E}_{\Sigma^t}}\Big)\Big) =: \Gamma. \end{aligned}$$

But Γ has $Q = \widehat{\pi}_t \circ \widehat{\pi}_s \left(C^{\not t} \cap W_t \cap \partial H_{e_s}\right)$ as associated solution set. Thus Γ contains an infeasible simple loop and therefore its associated system is unsatisfiable by Theorem 7.1. Hence, $Q = \emptyset$ and thus $C^{\not t} \cap W_t \cap \partial H_{e_s} = \emptyset$, which contradicts (8.2) as one can easily see by noting that $C^{\not t} \cap \partial H_{e_s}$ is a cone. It follows that L has to contain an edge of Γ_{Σ^t}.

Finally, note that a self-loop in $\overline{\Gamma}_{\Sigma^0}$ at x_0 cannot arise as intermediate segment on an admissible path and no self-loop at $x_i \neq x_0$ can be induced by a loop containing an edge in $\mathrm{E}_{\Sigma^s} \cup \mathrm{E}_{\Sigma^t}$. Hence the only remaining case is when L is a self-loop at x_0. But then, since Γ_{Σ^0} is defined so as not to contain any infeasible self-loop at x_0, it follows that L must be induced by a simple nontrivial admissible loop L_0 in Γ_{Σ^0} and as above, one has that L_0 needs to be containing an edge of E_{Σ^s} as well as an edge of E_{Σ^t}.

Thus, up to replacing the loop $L \subset \overline{\Gamma}_{\Sigma^0}$ by L_0 if necessary, we can assume that L contains an edge in E_{Σ^s} as well as an edge in E_{Σ^t}. It follows that L has starting or ending edge, let us say without loss of generality starting edge $E_r = (\{x_0, x_r\}, b_r x_r \succeq c_r) \in \mathrm{E}_{\Sigma^s}$ and accordingly final edge $E_u = (\{x_0, x_u\}, b_u x_u \succeq c_u) \in \mathrm{E}_{\Sigma^t}$ for some $x_r, x_u \in \mathrm{V} \setminus \{x_0, x_s, x_t\}$. For any edge E of L different from E_u and E_r, E does not contain x_0 as endpoint by admissibility of L, hence

$$E \in \mathrm{E}_{\Sigma^0} \setminus (\mathrm{E}_{\Sigma^s} \cup \mathrm{E}_{\Sigma^t})$$

which means that E has a corresponding edge $E^\Sigma \in \mathrm{E}_\Sigma$ that is labeled by the same equation as E and that has thus the same endpoints (these are thus different from x_s and x_t). Moreover, by definition of E_{Σ^u}, we have edges $E_r^\Sigma, E_u^\Sigma \in \mathrm{E}_\Sigma$ corresponding to E_r and E_u which satisfy $E_r^\Sigma = (\{x_r, x_s\}, a_r x_r + b_r x_s \succeq 0)$ and $E_u^\Sigma = (\{x_u, x_t\}, a_u x_u + b_u x_t \succeq 0)$. We thus obtain an admissible (simple) path $P \subset \Gamma_\Sigma$ from x_s to x_t. The residue inequality of P is then of the form $a x_s + b x_t \succeq 0$ and thus $C \subset H_\nu$ for $\nu := a e_s + b e_t$. By (8.1), it follows that

$|a| < |b|$ and thus by an easy argument $C \cap \text{relint}(-\text{sign}(b)F_t) = \emptyset$. Since we assumed that $C \cap \text{relint}(\tau F_t) \neq \emptyset$, it follows that $\text{sign}(b) = \tau$ and thus

$$C \cap \text{relint}(\tau F_t) \neq \emptyset = C \cap \text{relint}(-\tau F_t).$$

This proves the induction step and finishes the proof. □

Chapter IV

Injective Hulls of Infinite Totally Split-Decomposable Metric Spaces

IV.1 Introduction

In 1992, Bandelt and Dress (cf. [4]) introduced a decomposition theory for finite metric spaces which is canonical, namely it is the only one which is, in a sense, compatible with Isbell's injective hull.

Our first goal is to extend the canonical decomposition theory to the class of (possibly) infinite metric spaces with integer-valued totally split-decomposable metric and possessing an injective hull which has the structure of a polyhedral complex. For this class, we then provide necessary and sufficient conditions for the injective hull to be combinatorially equivalent to a CAT(0) cube complex.

The basic definitions of the canonical decomposition theory of Bandelt and Dress do not need to be modified to suit our more general situation. A *split* (also called *cut*) $S = \{A, B\}$ of a set X is a pair of non-empty subsets of X such that $A \cap B = \emptyset$ and $X = A \cup B$, or in other words $X = A \sqcup B$. For $x \in X$, we denote by $S(x)$ the element of S that contains x. The *split (pseudo-)metric* associated to S is then a pseudometric δ_S on X such that

$$\delta_S(x, y) := \begin{cases} 1 & \text{if } S(x) \neq S(y), \\ 0 & \text{if } S(x) = S(y). \end{cases}$$

For a pseudometric d on X, we call $S = \{A, B\}$ a *d-split* (of X) if the *isolation* *index* given by (1.3) satisfies

$$\alpha_S^d > 0.$$

The pseudometric d is said to be *totally split-decomposable* if $d = \sum_{S \in \mathcal{S}} \alpha_S^d \delta_S$ where $\mathcal{S}$ is the set of all d-splits.

A split subsystem $\bar{\mathcal{S}} \subset \mathcal{S}$ is called *octahedral* if and only if there is a partition of X into a disjoint union of six non-empty sets $X = Y_1^1 \sqcup Y_1^{-1} \sqcup Y_2^1 \sqcup Y_2^{-1} \sqcup Y_3^1 \sqcup Y_3^{-1}$ such that $\bar{\mathcal{S}}$ consists of the four splits

$$\begin{aligned} S_1 &:= \{Y_1^1 \sqcup Y_2^1 \sqcup Y_3^1, Y_1^{-1} \sqcup Y_2^{-1} \sqcup Y_3^{-1}\}, \\ S_2 &:= \{Y_1^1 \sqcup Y_2^1 \sqcup Y_3^{-1}, Y_1^{-1} \sqcup Y_2^{-1} \sqcup Y_3^1\}, \\ S_3 &:= \{Y_1^1 \sqcup Y_2^{-1} \sqcup Y_3^1, Y_1^{-1} \sqcup Y_2^1 \sqcup Y_3^{-1}\}, \\ S_4 &:= \{Y_1^1 \sqcup Y_2^{-1} \sqcup Y_3^{-1}, Y_1^{-1} \sqcup Y_2^1 \sqcup Y_3^1\}. \end{aligned}$$

$\mathcal{S}$ is called *octahedral-free* if it does not contain any octahedral split subsystem. Two splits $S := \{A, B\}$ and $S' := \{A', B'\}$ are said to be *compatible* if $A' \subset A$ (and thus $B \subset B'$) or alternatively $A \subset A'$ (and thus $B' \subset B$).

For general facts regarding injective hulls, we shall refer to [30]. Injective hulls can be characterized in several different ways. In the sequel, *the* injective hull refers to Isbell's injective hull construction $(X, d) \mapsto \mathrm{E}(X, d)$. Recall at this point that the injective hull $\mathrm{E}(X, d)$ of a pseudometric space (X, d) is given by

$$\mathrm{E}(X, d) = \left\{ f \in \mathbb{R}^X : f(x) = \sup_{y \in X} (d(x, y) - f(y)) \text{ for all } x \in X \right\}. \quad (1.1)$$

The difference between two elements of $\mathrm{E}(X, d)$ has finite $\|\cdot\|_\infty$-norm and $\mathrm{E}(X, d)$ is endowed with the metric

$$d_\infty(f, g) := \|f - g\|_\infty .$$

It is easy to see that for $f \in \mathrm{E}(X, d)$, if $d(x, x') = 0$ then $f(x) = f(x')$. Hence, if (X, d) is a pseudometric space and (Y, d') is the associated metric space obtained by collapsing every maximal set of diameter zero to a single point, then $\mathrm{E}(X, d)$ and $\mathrm{E}(Y, d')$ are isometric. Accordingly, the statements involving the injective hull will be stated for metric spaces instead of pseudometric spaces. As it is shown in [30, Theorem 4.5] and as we shall recall later in this introduction, as soon as (X, d) is a metric space with integer-valued metric verifying *the local rank condition (LRC)*, which is discussed below, there is a canonical locally finite dimensional polyhedral structure on $\mathrm{E}(X, d)$.

In the case where d is totally split-decomposable, our goal is to provide necessary and sufficient conditions ensuring that $\mathrm{E}(X, d)$ is combinatorially equivalent to a CAT(0) cube complex. Accordingly, we have:

1.1 Theorem. *Let (X, d) be a metric space with integer-valued totally split-decomposable metric satisfying the local rank condition. Let $\mathcal{S}$ be the set of all d-splits. Then, the following are equivalent:*

(i) $\mathcal{S}$ does not contain any octahedral split subsystem $\bar{\mathcal{S}}$ satisfying that for every $S = \{A, B\} \in \mathcal{S} \setminus \bar{\mathcal{S}}$, there is $S' := \{A', B'\} \in \bar{\mathcal{S}}$ such that S and S' are compatible.

(ii) Each cell of $\mathrm{E}(X, d)$ is a parallelotope.

If (i) or (ii) holds, there is a CAT(0) cube complex $\mathrm{K}(X, d)$ and a canonical bijective cell complex isomorphism $\sigma\colon \mathrm{E}(X, d) \to \mathrm{K}(X, d)$ mapping cells affinely to cells.

By a *parallelotope* we mean a Minkowski sum of a finite collection of linearly independent closed segments (see for instance [25]). When condition (i) in Theorem 1.1 holds, we say that the family of all d-splits of X has *no compatibly octahedral decomposition*. If the diameters of the cells of $\mathrm{E}(X, d)$ are uniformly bounded, σ in Theorem 1.1 can be chosen to be bi-Lipschitz.

For a metric space (X, d), let $I(x, y) := \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$. (X, d) is called *discretely geodesic* if the metric is integer-valued and for every pair of points $x, y \in X$ there exists an isometric embedding $\gamma\colon \{0, 1, \ldots, d(x, y)\} \to X$ such that $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. Moreover, we say that a discretely geodesic metric space X has *β-stable intervals*, for some constant $\beta \geq 0$, if for every triple of points $x, y, y' \in X$ such that $d(y, y') = 1$ we have

$$d_H(I(x, y), I(x, y')) \leq \beta$$

where d_H denotes the Hausdorff distance in X.

The injective hull has among other features, applications to geometric group theory. Let Γ be a finitely generated group, G a finite generating set, and let Γ be equipped with the word metric d_G with respect to the alphabet $G \cup G^{-1}$. It is shown in the proof of [30, Theorem 1.1] that if (Γ, d_G) has β-stable intervals, then (Γ, d_G) satisfies the (LRC) as well as $\mathrm{E}(\Gamma, d_G)$ is proper and has the structure of a polyhedral complex. The isometric action of Γ on (Γ, d_G) given by $(x, y) \mapsto L_x(y) := xy$ induces consequently a proper action by cell isometries of Γ on $\mathrm{E}(\Gamma, d_G)$ given by

$$(x, f) \mapsto \bar{L}_x(f) = f \circ L_x^{-1}.$$

Moreover, if (Γ, d_G) is δ-hyperbolic (in particular it has β-stable intervals), then $\mathrm{E}(\Gamma, d_G)$ has only finitely many isometry types of cells and the action is cocompact. As an immediate consequence of these observations, we thus obtain:

1.2 Theorem. *Let Γ be a finitely generated group and (Γ, d_G) the associated metric space with respect to the alphabet $G \cup G^{-1}$. Assume that d_G is totally split-decomposable and (i) in Theorem 1.1 holds. Then, the following hold:*

(i) if (Γ, d_G) has β-stable intervals, there is a proper action of Γ on $\mathrm{K}(\Gamma, d_G)$ given by

$$(x, y) \mapsto (\sigma \circ \bar{L}_x \circ \sigma^{-1})(y).$$

(ii) If (Γ, d_G) is δ-hyperbolic, the action of Γ on $\mathrm{K}(\Gamma, d_G)$ is proper as well as cocompact.

We give an outline of the structure of Isbell's injective hull and describe when it corresponds to that of a polyhedral complex, following [30]. Given a pseudometric space (X, d), let us consider the vector space $\mathbb{R}^X$ of real-valued functions on X and

$$\Delta(X, d) := \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X\}.$$

We call $f \in \Delta(X, d)$ *extremal* if there is no $g \leq f$ in $\Delta(X, d)$ distinct from f. The set $\mathrm{E}(X, d)$ of extremal functions is equivalently given by (1.1). To be able to describe the structure of $\mathrm{E}(X, d)$ further, one can assign to every $f \in \mathrm{E}(X, d)$ the undirected graph with vertex set X and edge set

$$A(f) := \big\{\{x, y\} : x, y \in X \text{ and } f(x) + f(y) = d(x, y)\big\}, \tag{1.2}$$

allowing self-loops $\{x, x\}$ which correspond to zeros of f. Furthermore, we let

$$\mathrm{E}'(X, d) := \big\{f \in \Delta(X, d) : \bigcup A(f) = X\big\}.$$

Note that if $f \in \mathrm{E}'(X, d)$, the graph $(X, A(f))$ has no isolated vertices (although it may be disconnected). A set A of unordered pairs of (possibly equal) points in X is called *admissible* if there exists an $f \in \mathrm{E}'(X, d)$ with $A(f) = A$, and we denote by $\mathscr{A}(X)$ the collection of admissible sets.

To every $A \in \mathscr{A}(X)$, we associate the affine subspace $H(A)$ of $\mathbb{R}^X$ given by

$$\begin{aligned} H(A) :&= \{g \in \mathbb{R}^X : A \subset A(g)\} \\ &= \{g \in \mathbb{R}^X : g(x) + g(y) = d(x, y) \text{ for all } x, y \in A\}. \end{aligned}$$

We define the *rank* of A to be the dimension of $H(A)$,

$$\operatorname{rank}(A) := \dim(H(A)) \in \mathbb{N} \cup \{0, \infty\}.$$

We can compute rank(A) as follows: if f, g are two elements of $H(A)$ and $\{x, y\} \in A$, one has $f(x) + f(y) = d(x, y) = g(x) + g(y)$, hence $f(y) - g(y) = -(f(x) - g(x))$, which means that the difference $f - g$ has alternating sign along all edge paths in the graph (X, A). Therefore, there is either none or exactly one degree of freedom for the values of $f \in H(A)$ on every connected component of (X, A), depending upon whether or not the component contains an odd cycle. We call such components (viewed as subsets of X) *odd* or *even* A*-components*, respectively.

If (X, d) is a finite metric space, $\mathrm{E}(X, d)$ is a finite polyhedral complex. If (X, d) is infinite, we say that (X, d) satisfies the *local rank condition (LRC)* if and only if for every $f \in \mathrm{E}(X, d)$, there exist $\varepsilon, N > 0$ such that for all $g \in \mathrm{E}'(X, d)$ with $\|f - g\|_\infty < \epsilon$, one has rank$(A(g)) \leq N$. Recall (cf. [30, Theorem 4.4]) that if (X, d) is a metric space with integer-valued metric and satisfying the (LRC), then $\mathrm{E}(X, d) = \mathrm{E}'(X, d)$. In this case, let

$$P(A) := \mathrm{E}'(X, d) \cap H(A) = \mathrm{E}(X, d) \cap H(A) = \Delta(X, d) \cap H(A).$$

The family $\{P(A)\}_{A \in \mathscr{A}(X)}$ then defines a polyhedral structure on $\mathrm{E}(X, d)$ and in particular $P(A')$ is a face of $P(A)$ if and only if $A \subset A'$.

In order to prove Theorem 1.1, we need to be able to decompose any pseudometric d on a set X in a way that is coherent with the structure of $\mathrm{E}(X, d)$. The *isolation index* of a pair $S := \{A, B\}$ of non-empty subsets with respect to a pseudometric d on X is the non-negative number α_S^d (equivalently $\alpha_{\{A,B\}}^d$ or simply α_S) given by

$$\alpha_S^d := \frac{1}{2} \inf_{\substack{a,a' \in A \\ b,b' \in B}} \Big[\max\big\{ d(a,b) + d(a',b'), d(a',b) + d(a,b'), d(a,a') + d(b,b') \big\} - d(a,a') - d(b,b') \Big]. \tag{1.3}$$

Moreover, we call a pseudometric d_0 on X *split-prime* if $\alpha_S^{d_0} = 0$ for any split S of X. Note that by Lemma 2.4, there are for any integer-valued pseudometric only finitely many d-splits separating any pair of points.

1.3 Theorem. *Let (X, d) be a pseudometric space with integer-valued pseudometric, let $\mathcal{S}_X$ be the set of all splits of X and let $\mathcal{S}$ be the set all d-splits.*

Let

$$\begin{cases} \lambda_S \in [0, \alpha_S^d] & \textit{if } S \in \mathcal{S}, \\ \lambda_S = 0 & \textit{if } S \in \mathcal{S}_X \setminus \mathcal{S}. \end{cases}$$

Then,

$$\widetilde{d} := d - \sum_{S \in \mathcal{S}} \lambda_S \delta_S$$

is a pseudometric such that for every split $S \in \mathcal{S}_X$, one has

$$\alpha_S^{\widetilde{d}} = \alpha_S^d - \lambda_S.$$

In particular, there is a split-prime pseudometric d_0 such that

$$d = d_0 + \sum_{S \in \mathcal{S}} \alpha_S^d \delta_S.$$

The decomposition given by Theorem 1.3 can be characterized uniquely in a corollary to Theorem 2.10:

1.4 Corollary. *Let (X, d) be a metric space with integer-valued metric satisfying the (LRC). Let $\mathcal{S}$ be the family of all d-splits of X so that $d = d_0 + \sum_{S \in \mathcal{S}} \alpha_S \delta_S$ and let $\lambda_S \in [0, \alpha_S]$ for every $S \in \mathcal{S}$. Then, setting $d_1 := d - \sum_{S \in \mathcal{S}} \lambda_S \delta_S$, we have*

$$\mathrm{E}(X, d) \subset \mathbb{R}^X \cap \left(\mathrm{E}(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \mathrm{E}(X, \delta_S) \right). \tag{1.4}$$

Moreover, for any split $S = \{A, B\}$ of X and any $\lambda_S > 0$ such that $d = d_1 + \lambda_S \delta_S$, if $\mathrm{E}(X, d) \subset \mathrm{E}(X, d_1) + \lambda_S \mathrm{E}(X, \delta_S)$, then the following hold:

(*i*) *S is a d-split of X and*

(*ii*) *$\lambda_S \leq \alpha_S$.*

Note that the above relates in particular to the question raised in [3, Section 8.3] and regarding the structure of general cellular graphs.

The present chapter is divided into three main parts: Section 2 deals with the generalization of the decomposition theory of Bandelt and Dress (cf. [4]) to (possibly) infinite metric spaces with integer-valued metric. Section 3 deals with the proof of Theorem 1.1, we adapt and generalize the arguments of [28] to infinite metric spaces and infinite split systems. Section 4 starts with the observation that the Buneman complex $B(\mathcal{S}, \alpha)$, which is a well-known object in discrete mathematics (cf. [19] and the references there) satisfies the CAT(0) link condition and continues with the proof that $\mathrm{E}(X, d)$ satisfies this same condition. Finally, Section 5 deals with several examples.

IV.2 Decomposition Theory

It is easy to see that $\mathrm{E}(X,d)$ as defined at the beginning of the Introduction is a subset of

$$\Delta_1(X,d) := \{f \in \Delta(X,d) : f \text{ is 1-Lipschitz}\}.$$

Note that a function $f \in \mathbb{R}^X$ belongs to $\Delta_1(X,d)$ if and only if

$$\|f - d_x\|_\infty = f(x) \quad \text{for all } x \in X. \tag{2.1}$$

The metric $d_\infty(f,g) := \|f - g\|_\infty$ on $\Delta_1(X,d)$ is thus well-defined since for any $x \in X$, one has

$$\|f - g\|_\infty \leq \|f - d_x\|_\infty + \|g - d_x\|_\infty = f(x) + g(x) < \infty.$$

The set $\mathrm{E}(X,d) \subset \Delta_1(X,d)$ is equipped with the induced metric and one has the canonical isometric map

$$\mathrm{e}\colon (X,d) \to \mathrm{E}(X,d), \quad \mathrm{e}(x) = d_x.$$

In case (X,d) is a metric space, e is an isometric embedding and Isbell showed that $(\mathrm{e}, \mathrm{E}(X,d))$ is indeed an injective hull of X. That is, $\mathrm{E}(X,d)$ is an injective metric space, and every isometric embedding of X into another injective metric space factors through e. A metric space (X,d) is called *injective* if for any isometric embedding $i\colon Y \to Z$ of metric spaces and any 1-Lipschitz (i.e., distance nonincreasing) map $f\colon Y \to X$ there exists a 1-Lipschitz map $g\colon Z \to X$, so that $g \circ i = f$ (cf. [1, Section 9] for the general categorical definition). For a recent survey of injective metric spaces, we refer to [30, Section 2].

Let (X,d) be any pseudometric space. A *partial split* $S = \{A,B\}$ of X is a pair of non-empty subsets of X such that $A \cap B = \emptyset$. If in addition $X = A \cup B$ holds, then $S = \{A,B\}$ is a split of X. A *partial d-split* is a partial split $S = \{A,B\}$ for which $\alpha_S^d > 0$. For any $\{a,a'\} \subset A$ and $\{b,b'\} \subset B$ (where if $x = y$, $\{x,y\}$ denotes $\{x\}$), let

$$\beta^d_{\{\{a,a'\},\{b,b'\}\}} := \frac{1}{2}\big[\max\{d(a,b) + d(a',b'), d(a',b) + d(a,b'), d(a,a') + d(b,b')\} \\ - d(a,a') - d(b,b')\big]$$

and when the reference to the pseudometric is unnecessary, we shall omit it and write simply $\alpha_{\{A,B\}}$ as well as $\beta_{\{\{a,a'\},\{b,b'\}\}}$. Note that for any pseudometric d on X, one has

$$\alpha^d_{\{\{r,s\},\{t,u\}\}} = \beta^d_{\{\{r,s\},\{t,u\}\}}. \tag{2.2}$$

Indeed,

$$d(r,t)+d(s,u)-d(r,s)-d(t,u) \le d(r,t)+d(u,r)-d(t,u) = 2\beta^d_{\{\{r\},\{t,u\}\}}$$

and

$$\begin{aligned} d(r,t)+d(s,u)-d(r,s)-d(t,u) \le d(r,t)+d(u,r)-d(t,u) &\le 2d(r,t) \\ &= 2\beta^d_{\{\{r\},\{t\}\}}. \end{aligned}$$

The proof of the next lemma follows the one of [4, Theorem 1].

2.1 Lemma. *Let (X,d) be a pseudometric space and $\{A_0,B_0\}$ a partial d-split of X. For any $\{a,a'\}\subset A_0$, $\{b,b'\}\subset B_0$ and $x\in X\setminus(A_0\cup B_0)$, one has*

$$\beta_{\{\{a,a'\},\{b,b'\}\}} \ge \alpha_{\{\{a,a',x\},\{b,b'\}\}} + \alpha_{\{\{a,a'\},\{b,b',x\}\}}. \tag{2.3}$$

Proof. Assume that (2.3) fails for some a_1,a_2,b_1,b_2,x, then all three quantities must be positive. For simplicity, we write xy for $d(x,y)$. Let $\{i,j\}=\{1,2\}$ be so that

$$\beta_{\{\{a_1,x\},\{b_1,b_2\}\}} = \tfrac{1}{2}\Big(a_1b_j + xb_i - a_1x - b_1b_2\Big).$$

It follows that

$$\begin{aligned} &\tfrac{1}{2}\Big(a_1b_j + xb_i - a_1x - b_1b_2 + \max\{a_1x+a_2b_i, a_1b_i+a_2x\} - a_1a_2 - xb_i\Big) = \\ &\beta_{\{\{a_1,x\},\{b_1,b_2\}\}} + \beta_{\{\{a_1,a_2\},\{x,b_i\}\}} \ge \\ &\alpha_{\{\{a_1,a_2,x\},\{b_1,b_2\}\}} + \alpha_{\{\{a_1,a_2\},\{b_1,b_2,x\}\}} > \\ &\beta_{\{\{a_1,a_2\},\{b_1,b_2\}\}} = \\ &\tfrac{1}{2}\Big(\max\{a_1b_1+a_2b_2, a_1b_2+a_2b_1\} - a_1a_2 - b_1b_2\Big). \end{aligned}$$

Hence

$$\delta := a_1b_j - a_1x + \max\{a_1x+a_2b_i, a_1b_i+a_2x\} > \max\{a_1b_1+a_2b_2, a_1b_2+a_2b_1\}.$$

The above strict inequality can only hold if $a_1b_i + a_2x > a_1x + a_2b_i$ which implies that $\delta = a_1b_j - a_1x + a_1b_i + a_2x$. Therefore, one has

$$a_1b_1 + a_1b_2 - a_1x + a_2x = \delta > \max\{a_1b_1+a_2b_2, a_1b_2+a_2b_1\}.$$

Hence for each $k\in\{1,2\}$, one has

$$a_1b_k + a_2x > a_1x + a_2b_k.$$

By interchanging the role of a_1 and a_2, we also obtain the reverse strict inequality and this is a contradiction. This proves (2.3). $\square$

The next theorem corresponds to [4, Theorem 1].

2.2 Theorem. *Let (X, d) be a finite pseudometric space and let $\{A_0, B_0\}$ be a partial d-split. Then*

$$\sum_{\{A,B\}\in\mathcal{E}_0} \alpha_{\{A,B\}} \le \alpha_{\{A_0,B_0\}}$$

where $\mathcal{E}_0$ denotes the set of all d-splits $\{A, B\}$ extending $\{A_0, B_0\}$, namely d-splits satisfying $A_0 \subset A$ and $B_0 \subset B$.

Proof. Note that since X is finite, there are $\{a, a'\} \subset A_0$ and $\{b, b'\} \subset B_0$ such that

$$\alpha_{\{A_0,B_0\}} = \beta_{\{\{a,a'\},\{b,b'\}\}}$$

and thus we obtain

$$\alpha_{\{A_0,B_0\}} \ge \alpha_{\{A_0\cup\{x\},B_0\}} + \alpha_{\{A_0,B_0\cup\{x\}\}}. \tag{2.4}$$

By induction, the desired result follows. □

We go on with the following more general situation.

2.3 Theorem. *Let (X, d) be a pseudometric space such that for every $x, y \in X$, there are only finitely many d-splits S satisfying $S(x) \neq S(y)$. Let moreover $\{A_0, B_0\}$ be a partial d-split. Then,*

$$\sum_{\{A,B\}\in\mathcal{E}_0} \alpha_{\{A,B\}} \le \alpha_{\{A_0,B_0\}}$$

where $\mathcal{E}_0$ denotes the set of all d-splits $\{A, B\}$ extending $\{A_0, B_0\}$.

Proof. Note that for any $\varepsilon > 0$, there are $\{a, a'\} \subset A_0$ and $\{b, b'\} \subset B_0$ such that

$$\alpha_{\{A_0,B_0\}} + \varepsilon \ge \beta_{\{\{a,a'\},\{b,b'\}\}}$$

and thus using (2.3), we obtain

$$\alpha_{\{A_0,B_0\}} + \varepsilon \ge \alpha_{\{A_0\cup\{x\},B_0\}} + \alpha_{\{A_0,B_0\cup\{x\}\}}. \tag{2.5}$$

Since there are only finitely many d-splits separating any pair of points in X, there are only finitely many d-splits $\{S_i\}_{i\in\{1,\dots,m\}}$ extending $\{A_0, B_0\}$, i.e. such that one has $S_i := \{A_i, B_i\}$ where $A_0 \subset A_i$ and $B_0 \subset B_i$. For every $1 \le i < j \le m$, choose $x_{ij} \in X \setminus (A_0 \cup B_0)$ such that either $x_{ij} \in A_i \cap B_j$ or $x_{ij} \in B_i \cap A_j$ (in particular neither $x_{ij} \in A_i \cap A_j$ nor $x_{ij} \in B_i \cap B_j$). Writing

$Z := (x_{ij})_{1\leq i<j\leq m}$ and applying (2.5) recursively for each element of Z, we obtain for some big enough constant $C > 0$:

$$\alpha_{\{A_0,B_0\}} + C\varepsilon \geq \sum_{Z_A\subset Z} \alpha_{\{A_0\cup Z_A, B_0\cup(Z\setminus Z_A)\}}.$$

Note that by choice of Z, there is for any $\{A_i, B_i\}$ a unique $Z_A \subset Z$ (namely $Z_A := Z \cap A_i$) such that $\{A_i, B_i\}$ extends $\{A_0 \cup Z_A, B_0 \cup (Z \setminus Z_A)\}$ which implies

$$\alpha_{\{A_0\cup Z_A, B_0\cup(Z\setminus Z_A)\}} \geq \alpha_{\{A_i,B_i\}}.$$

It follows that

$$\alpha_{\{A_0,B_0\}} + C\varepsilon \geq \sum_{i\in\{1,\ldots,m\}} \alpha_{\{A_i,B_i\}}.$$

and letting ε tend to zero, this concludes the proof. □

Note now that if (X, d) is a pseudometric space with integer-valued pseudometric, then for any split S, one has

$$\alpha_S \in [0,\infty) \cap \tfrac{1}{2}\mathbb{Z}. \tag{2.6}$$

2.4 Lemma. *Let (X, d) be a pseudometric space with integer-valued pseudometric. For every $x, y \in X$, there are at most $2d(x, y)$ distinct d-splits S satisfying $S(x) \neq S(y)$.*

Proof. Assume by contradiction that one can find $m := 2d(x, y) + 1$ distinct d-splits S_i such that $S_i(x) \neq S_i(y)$. For every $1 \leq i < j \leq m$, choose $z_{ij} \in X$ such that either $z_{ij} \in S_i(x)\cap S_j(y)$ or $z_{ij} \in S_i(y)\cap S_j(x)$. Furthermore, setting $Z := \{x, y\} \cup \{z_{ij} : 1 \leq i < j \leq m\}$, one has $S_i(x) \cap Z \neq S_j(x) \cap Z$ if $i \neq j$, hence

$$d(x,y) + \tfrac{1}{2} \leq \sum_{i\in\{1,\ldots,m\}} \alpha^d_{S_i} \leq \sum_{i\in\{1,\ldots,m\}} \alpha^{d|_{Z\times Z}}_{\{S_i(x)\cap Z, S_i(y)\cap Z\}}$$

and by Theorem 2.2, applied to the partial split $A_0 := \{x\}$ and $B_0 := \{y\}$ of the finite set Z, the right-hand side is less than or equal to $\alpha^{d|_{Z\times Z}}_{\{\{x\},\{y\}\}} = d(x, y)$, which is a contradiction. □

Furthermore, it is easy to see that if $S = \{A, B\}$ is a d-split of X such that $S(x) \neq S(y)$ for some pair of points x, y such that $d(x, y) = 1$, then one has $C(x, y) := \{z \in X : d(x, y) + d(y, z) = d(x, z)\} \subset S(y)$ and $C(y, x) \subset S(x)$. Note however that one might have $X = C(x, y) \cup C(y, x)$ without $\{C(x, y), C(y, x)\}$ being a d-split, this is the case for instance if X is the set of all vertices of the plane tessellation by hexagons, endowed with the shortest-path metric. Indeed, in this case, X is bipartite and thus $X = C(x,y) \cup C(y,x)$. However, d is in this case a split-prime metric. The next theorem corresponds to [4, Theorem 2] and the same proof works.

2.5 Lemma. *Let (X,d) be a pseudometric space with integer-valued pseudometric. Let $\mathcal{S}$ be the set of all d-splits and let $\widetilde{\mathcal{S}} \subset \mathcal{S}$ be any finite subset. If $\lambda_S \in (0, \alpha^d_S]$ for every $S \in \widetilde{\mathcal{S}}$ and $\lambda_S := 0$ for every other split, then $\widetilde{d} := d - \sum_{S \in \widetilde{\mathcal{S}}} \lambda_S \delta_S$ is a pseudometric such that for every split S of X, one has*

$$\alpha^{\widetilde{d}}_S = \alpha^d_S - \lambda_S.$$

Proof. We first prove the assertion for

$$\widetilde{d} := d - \lambda \delta_{\{A_0,B_0\}}$$

where $\{A_0, B_0\} \in \mathcal{S}$ and $\lambda \leq \alpha^d_{\{A_0,B_0\}}$. The result then follows by induction. We verify that $\widetilde{d}$ is indeed a pseudometric by showing the triangle inequality. Let $x, y, z \in X$ and assume without loss of generality that $x, y \in A_0$. For simplicity, we shall denote $d(x,y)$ simply by xy. If $x, y, z \in A_0$, then d and $\widetilde{d}$ agree on $\{x,y,z\}$ and we are done. Otherwise, $z \in B_0$, in which case we get

$$\widetilde{d}(x,z) = xz - \lambda \leq xy + yz - \lambda = \widetilde{d}(x,y) + \widetilde{d}(y,z).$$

On the other side, since $\lambda \leq \alpha^d_{\{A_0,B_0\}} \leq \beta^d_{\{\{x,y\},\{z\}\}} = \frac{1}{2}(xz + yz - xy)$, we obtain by rearranging

$$xy \leq xz - \lambda + yz - \lambda = \widetilde{d}(x,z) + \widetilde{d}(y,z).$$

Thus, $\widetilde{d}$ is a pseudometric.

Let $\{x,y\}$ and $\{z,w\}$ be two disjoint subsets of X. If $\{\{x,y\},\{z,w\}\}$ extends to $\{A_0, B_0\}$, then clearly

$$\beta^{\widetilde{d}}_{\{\{x,y\},\{z,w\}\}} = \beta^d_{\{\{x,y\},\{z,w\}\}} - \lambda. \tag{2.7}$$

Now, we prove that if $\{\{x,y\},\{z,w\}\}$ does not extend to $\{A_0, B_0\}$, then

$$\beta^{\widetilde{d}}_{\{\{x,y\},\{z,w\}\}} = \beta^d_{\{\{x,y\},\{z,w\}\}}. \tag{2.8}$$

First, if either A_0 or B_0 contains at least three of x, y, z, w, then (2.8) clearly holds. We may thus assume without loss of generality that $x, z \in A_0$ and $y, w \in B_0$. Since

$$\beta^d_{\{\{x,z\},\{y,w\}\}} = \tfrac{1}{2}\Big(\max\{xy + zw, xw + yz\} - xz - yw\Big) \geq \alpha^d_{\{A_0,B_0\}} \geq \lambda,$$

we obtain that $\max\{xy+zw-2\lambda, xw+yz-2\lambda\} \geq xz+yw$. Hence

$$\begin{aligned}
&\beta^{\widetilde{d}}_{\{\{x,y\},\{z,w\}\}}\\
&= \tfrac{1}{2}\Big(\max\{xz+yw, xw+yz-2\lambda, xy+zw-2\lambda\} - xy - zw + 2\lambda\Big)\\
&= \tfrac{1}{2}\Big(\max\{xw+yz-2\lambda, xy+zw-2\lambda\} - xy - zw + 2\lambda\Big)\\
&= \tfrac{1}{2}\Big(\max\{xz+yw, xw+yz, xy+zw\} - xy - zw\Big)\\
&= \beta^{d}_{\{\{x,y\},\{z,w\}\}},
\end{aligned}$$

as required. Finally, it remains to prove that for every split $\{A,B\}$ of X, one has

$$\alpha^{\widetilde{d}}_{\{A,B\}} = \begin{cases} \alpha^{d}_{\{A,B\}} - \lambda & \text{if } \{A,B\} = \{A_0,B_0\},\\ \alpha^{d}_{\{A,B\}} & \text{otherwise.}\end{cases}$$

By (2.7), one has $\alpha^{\widetilde{d}}_{\{A_0,B_0\}} = \alpha^{d}_{\{A_0,B_0\}} - \lambda$. Let now $\{A,B\}$ be a split of X different from $\{A_0,B_0\}$. Since the metric d takes only a discrete set of values, by (2.2) there are $a,a' \in A$ and $b,b' \in B$ such that

$$\alpha^{d}_{\{A,B\}} = \alpha^{d}_{\{\{a,a'\},\{b,b'\}\}} = \beta^{d}_{\{\{a,a'\},\{b,b'\}\}}.$$

Since $\{A_0,B_0\}$ is a d-split, if $\alpha^{d}_{\{A,B\}} = 0$, then $\{A_0,B_0\}$ cannot extend $\{\{a,a'\},\{b,b'\}\}$, and if $\alpha^{d}_{\{A,B\}} > 0$, then $\{A_0,B_0\}$ cannot extend $\{\{a,a'\},\{b,b'\}\}$ either by Theorem 2.3. Hence by (2.8), one has

$$\alpha^{d}_{\{A,B\}} = \beta^{d}_{\{\{a,a'\},\{b,b'\}\}} = \beta^{\widetilde{d}}_{\{\{a,a'\},\{b,b'\}\}} \geq \alpha^{\widetilde{d}}_{\{A,B\}}.$$

To prove the reverse inequality, assume that $a,a' \in A$ and $b,b' \in B$ are such that

$$\alpha^{\widetilde{d}}_{\{A,B\}} = \alpha^{\widetilde{d}}_{\{\{a,a'\},\{b,b'\}\}} = \beta^{\widetilde{d}}_{\{\{a,a'\},\{b,b'\}\}}.$$

If $\{A_0,B_0\}$ does not extend $\{\{a,a'\},\{b,b'\}\}$, one has by (2.8) that

$$\alpha^{d}_{\{A,B\}} \leq \beta^{d}_{\{\{a,a'\},\{b,b'\}\}} = \beta^{\widetilde{d}}_{\{\{a,a'\},\{b,b'\}\}}.$$

Now, if $\{A_0,B_0\}$ extends $\{\{a,a'\},\{b,b'\}\}$, it follows from Theorem 2.3 and (2.7) that

$$\begin{aligned}
\alpha^{d}_{\{A,B\}} \leq \alpha^{d}_{\{A,B\}} + \alpha^{d}_{\{A_0,B_0\}} - \lambda \leq \alpha^{d}_{\{\{a,a'\},\{b,b'\}\}} - \lambda \leq \beta^{d}_{\{\{a,a'\},\{b,b'\}\}} - \lambda&\\
= \beta^{\widetilde{d}}_{\{\{a,a'\},\{b,b'\}\}}&\\
= \alpha^{\widetilde{d}}_{\{A,B\}}.&
\end{aligned}$$

Using induction, this concludes the case $|\widetilde{S}| < \infty$. $\square$

We can now proceed to the more general case.

Proof of Theorem 1.3. Let $\widetilde{\mathcal{S}} \subset \mathcal{S}$ be the set of all d-splits S such that $\lambda_S > 0$. For any $Y := \{x, y, z\} \subset X$, we can consider the set $\widetilde{\mathcal{S}}_Y \subset \widetilde{\mathcal{S}}$ of all splits in $\widetilde{\mathcal{S}}$ that restrict to a split of Y. Since $\widetilde{\mathcal{S}} \subset \mathcal{S}$, $\widetilde{\mathcal{S}}_Y$ is a finite set by Lemma 2.4. We set

$$\widetilde{d}_Y := d - \sum_{S \in \widetilde{\mathcal{S}}_Y} \lambda_S \delta_S. \tag{2.9}$$

Now, $\widetilde{d}|_{Y \times Y}$ coincides with $\widetilde{d}_Y|_{Y \times Y}$ and the latter satisfies the triangle inequality as we proved in the case $|\widetilde{S}| < \infty$, hence so does the former. It follows that $\widetilde{d}$ is a pseudometric. Let now $S_0 := \{A_0, B_0\}$ be any split of X, we consider three cases:

(a) Assume that $S_0 \in \widetilde{\mathcal{S}}$. Note that $\widetilde{d}$ is in general not integer-valued. For any $\varepsilon > 0$, we can choose $\widetilde{a}, \widetilde{a}' \in A_0$ and $\widetilde{b}, \widetilde{b}' \in B_0$ such that

$$\alpha^{\widetilde{d}}_{\{A_0, B_0\}} \geq \alpha^{\widetilde{d}}_{\{\{\widetilde{a}, \widetilde{a}'\}, \{\widetilde{b}, \widetilde{b}'\}\}} - \varepsilon.$$

Moreover, we can choose $a, a' \in A_0$ and $b, b' \in B_0$ such that

$$\alpha^{d}_{\{A_0, B_0\}} = \alpha^{d}_{\{\{a, a'\}, \{b, b'\}\}}.$$

By Theorem 2.3, $\{A_0, B_0\}$ is the unique d-split extending $\{\{a, a'\}, \{b, b'\}\}$. Furthermore, set

$$Y := \{\widetilde{a}, \widetilde{a}', \widetilde{b}, \widetilde{b}', a, a', b, b'\}.$$

We can assume without loss of generality that $S_0 \subset \widetilde{\mathcal{S}}_Y$. We have

$$\begin{aligned} \alpha^{\widetilde{d}}_{\{A_0, B_0\}} \geq \alpha^{\widetilde{d}}_{\{\{\widetilde{a}, \widetilde{a}'\}, \{\widetilde{b}, \widetilde{b}'\}\}} - \varepsilon &\geq \alpha^{\widetilde{d}}_{\{\{\widetilde{a}, \widetilde{a}', a, a'\}, \{\widetilde{b}, \widetilde{b}', b, b'\}\}} - \varepsilon \\ &= \alpha^{\widetilde{d}_Y}_{\{\{\widetilde{a}, \widetilde{a}', a, a'\}, \{\widetilde{b}, \widetilde{b}', b, b'\}\}} - \varepsilon \end{aligned}$$

Note that d-splits of X that restrict to splits of Y are $d|_{Y \times Y}$-splits. Since $S_0 \subset \widetilde{\mathcal{S}}_Y$, S_0 is a d-split and it is the unique d-split of X that restricts to $\{\{\widetilde{a}, \widetilde{a}', a, a'\}, \{\widetilde{b}, \widetilde{b}', b, b'\}\}$ on Y, we can apply the Lemma 2.5 with $\widetilde{d}_Y$ given by (2.9) to deduce that

$$\alpha^{\widetilde{d}_Y|_{Y \times Y}}_{\{\{\widetilde{a}, \widetilde{a}', a, a'\}, \{\widetilde{b}, \widetilde{b}', b, b'\}\}} = \alpha^{d|_{Y \times Y}}_{\{\{\widetilde{a}, \widetilde{a}', a, a'\}, \{\widetilde{b}, \widetilde{b}', b, b'\}\}} - \lambda_{\{A_0, B_0\}}.$$

Finally, the right-hand side is equal to

$$\alpha^{d}_{\{\{a, a'\}, \{b, b'\}\}} - \lambda_{\{A_0, B_0\}} = \alpha^{d}_{\{A_0, B_0\}} - \lambda_{\{A_0, B_0\}}.$$

Hence, $\alpha^{\widetilde{d}}_{\{A_0,B_0\}} \geq \alpha^{d}_{\{A_0,B_0\}} - \lambda_{\{A_0,B_0\}} - \varepsilon$. Since this holds for any $\varepsilon > 0$, we get $\alpha^{\widetilde{d}}_{\{A_0,B_0\}} \geq \alpha^{d}_{\{A_0,B_0\}} - \lambda_{\{A_0,B_0\}}$. The other inequality is obtained similarly, noting that

$$\begin{aligned}\alpha^{\widetilde{d}}_{\{A_0,B_0\}} \leq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}'\},\{\widetilde{b},\widetilde{b}'\}\}} &\leq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} + \varepsilon \\ &= \alpha^{\widetilde{d}_Y}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} + \varepsilon.\end{aligned}$$

(b) Assume that $S_0 \in \mathcal{S} \setminus \widetilde{\mathcal{S}}$. Let Y and $\mathcal{S}_Y$ be defined as in the former case. Similarly to the former case, we have

$$\begin{aligned}\alpha^{\widetilde{d}}_{\{A_0,B_0\}} \geq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}'\},\{\widetilde{b},\widetilde{b}'\}\}} - \varepsilon &\geq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} - \varepsilon \\ &= \alpha^{\widetilde{d}_Y}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} - \varepsilon.\end{aligned}$$

as well as

$$\begin{aligned}\alpha^{\widetilde{d}}_{\{A_0,B_0\}} \leq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}'\},\{\widetilde{b},\widetilde{b}'\}\}} &\leq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} + \varepsilon \\ &= \alpha^{\widetilde{d}_Y}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} + \varepsilon.\end{aligned}$$

Since $\widetilde{\mathcal{S}}_Y \subset \widetilde{\mathcal{S}}$, we have $S_0 \notin \widetilde{\mathcal{S}}_Y$ by assumption. Since S_0 is the unique d-split of X that restricts to $\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}$ on Y, it follows that no element of $\widetilde{\mathcal{S}}_Y$ restricts to $\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}$ on Y. We can apply Lemma 2.5 with $\widetilde{d}_Y$ given by (2.9) to obtain

$$\alpha^{\widetilde{d}_Y|_{Y\times Y}}_{\{\{a,a',\bar{a},\bar{a}'\},\{b,b',\bar{b},\bar{b}'\}\}} = \alpha^{d|_{Y\times Y}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}}.$$

The right-hand side is now equal to

$$\alpha^{d}_{\{\{a,a'\},\{\bar{b},\bar{b}'\}\}} = \alpha^{d}_{\{A_0,B_0\}}.$$

Since $\varepsilon > 0$ can be taken arbitrarily small, we thus get $\alpha^{\widetilde{d}}_{\{A_0,B_0\}} = \alpha^{d}_{\{A_0,B_0\}}$.

(c) Assume that $S_0 \notin \mathcal{S}$, let Y and $\mathcal{S}_Y$ be defined as in the former case, we get

$$\begin{aligned}\alpha^{\widetilde{d}}_{\{A_0,B_0\}} \geq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}'\},\{\widetilde{b},\widetilde{b}'\}\}} - \varepsilon &\geq \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} - \varepsilon \\ &= \alpha^{\widetilde{d}_Y}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} - \varepsilon.\end{aligned}$$

as well as

$$\alpha^{\widetilde{d}}_{\{A_0,B_0\}} \le \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}'\},\{\widetilde{b},\widetilde{b}'\}\}} \le \alpha^{\widetilde{d}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} + \varepsilon$$
$$= \alpha^{\widetilde{d}_Y}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} + \varepsilon.$$

Note that S_0 restricts to $\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}$ on Y and since

$$\alpha^{d}_{\{A_0,B_0\}} = \alpha^{d}_{\{\{a,a'\},\{b,b'\}\}} = 0,$$

it follows that $\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}$ is not a $d|_{Y\times Y}$-split of Y. Hence by the finite case with $\widetilde{d}_Y$ given by (2.9), it follows that

$$\alpha^{\widetilde{d}_Y|_{Y\times Y}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} = \alpha^{d|_{Y\times Y}}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}} = \alpha^{d}_{\{\{\widetilde{a},\widetilde{a}',a,a'\},\{\widetilde{b},\widetilde{b}',b,b'\}\}}$$
$$= \alpha^{d}_{\{A_0,B_0\}}$$
$$= 0.$$

Since $\varepsilon > 0$ can be taken arbitrarily small, we thus get $\alpha^{\widetilde{d}}_{\{A_0,B_0\}} = 0 = \alpha^{d}_{\{A_0,B_0\}}$.

This concludes the proof. □

The next definition is the same as in the finite case.

2.6 Definition. We say that a collection $\mathcal{S}$ of splits of X is *weakly compatible* if there are no four points $\{x_0,x_1,x_2,x_3\} \subset X$ and three splits $\{S_1,S_2,S_3\} \subset \mathcal{S}$ such that for any $i,j \in \{1,2,3\}$, one has

$$S_i(x_0) = S_i(x_j) \Longleftrightarrow i = j.$$

It is clear from (2.2) that for a pseudometric space (X,d) and every set of four different points $\{x_0,x_1,x_2,x_3\} \subset X$ at least one of

$$\alpha_{\{\{x_0,x_1\},\{x_2,x_3\}\}},\ \alpha_{\{\{x_0,x_2\},\{x_1,x_3\}\}} \text{ and } \alpha_{\{\{x_0,x_3\},\{x_1,x_2\}\}}$$

is equal to zero. From Theorem 2.3, it thus follows that the d-splits with respect to any integer-valued pseudometric d on X are weakly compatible. Now, for pseudometric spaces with integer-valued pseudometric, the next theorem is proved as [4, Theorem 3].

2.7 Theorem. *The d-splits with respect to any integer-valued pseudometric d on a set X are weakly compatible. Conversely, let $\mathcal{S}_0$ be any collection of weakly compatible splits of X. For each $S \in \mathcal{S}_0$, choose some $\lambda_S \in (0, \infty)$ such that*

$$d := \sum_{S \in \mathcal{S}_0} \lambda_S \delta_S \colon X \times X \to \mathbb{Z} \cap [0, \infty).$$

Then, $\mathcal{S}_0$ is the set of all d-splits and for each $S \in \mathcal{S}_0$, the isolation index $\alpha_S = \alpha_S^d$ equals λ_S.

Proof. Let $S := \{A, B\} \in \mathcal{S}_0$. Pick $x, y \in A$ and $z, w \in B$ such that

$$\alpha^d_{\{\{x,y\},\{z,w\}\}} = \alpha^d_{\{A,B\}}.$$

By weak compatibility of $\mathcal{S}_0$, we can assume that there is no split in $\mathcal{S}_0$ extending (without loss of generality) $\{\{x, w\}, \{y, z\}\}$. Let

$$\begin{aligned} \mathcal{S}_1 &:= \{S \in \mathcal{S}_0 : S \text{ extends } \{\{x, y\}, \{z, w\}\}\}, \\ \mathcal{S}_2 &:= \{S \in \mathcal{S}_0 : S \text{ extends } \{\{x, z\}, \{y, w\}\}\}, \end{aligned}$$

noting that $S \subset \mathcal{S}_1$. All splits in $\mathcal{S}_0 \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$ equally contribute to each of the three distance sums involving x, y, z, w in $\beta^d_{\{\{x,y\},\{z,w\}\}}$, so that by (2.2), we get

$$\begin{aligned} \alpha^d_{\{\{x,y\},\{z,w\}\}} = \beta^d_{\{\{x,y\},\{z,w\}\}} &= \frac{1}{2}\left(\max\{xz + yw, xw + yz\} - xy - zw\right) \\ &= \max\left\{\sum_{S \in \mathcal{S}_1} \lambda_S, \sum_{S \in \mathcal{S}_1 \cup \mathcal{S}_2} \lambda_S\right\} - \sum_{S \in \mathcal{S}_1} \lambda_S \\ &= \sum_{S \in \mathcal{S}_1} \lambda_S \\ &\geq \lambda_{\{A,B\}} \\ &> 0. \end{aligned}$$

Therefore, $\{A, B\}$ is a d-split. Let us denote by $\mathcal{S}$ the set of all d-splits of X, we have just proved that $\mathcal{S}_0 \subset \mathcal{S}$. We can decompose d according to Theorem 1.3 to obtain

$$d = d_0 + \sum_{S \in \mathcal{S}} \alpha_S^d \delta_S \geq \sum_{S \in \mathcal{S}_0} \alpha_S^d \delta_S \geq \sum_{S \in \mathcal{S}_0} \lambda_S \delta_S = d,$$

which implies that equality holds throughout. This finally yields $d_0 \equiv 0$ as well as $\mathcal{S}_0 = \mathcal{S}$. Furthermore, for each $S \in \mathcal{S}$, one has $\alpha_S^d = \lambda_S$ and for each $S \notin \mathcal{S}$, one has $\alpha_S^d = 0$. This concludes the result. □

In the next lemma, assumptions ensure that $\mathrm{E}(X,d)$ is a polyhedral complex. We denote by $\Sigma^0(\mathrm{E}(X,d))$ the set of vertices of the polyhedral complex $\mathrm{E}(X,d)$. Equivalently, it is the set of all functions $f \in \mathrm{E}'(X,d)$ such that $\mathrm{rank}(A(f)) = 0$.

2.8 Lemma. *Let (X,d) be a metric space with integer-valued metric satisfying the (LRC) and let $f \in \Sigma^0(\mathrm{E}(X,d))$. Let $\mathcal{S}_{<\infty}$ be any finite subset of the set $\mathcal{S}$ of all d-splits of X. If for every $S \in \mathcal{S}_{<\infty}$, one picks $\lambda_S \in [0, \alpha_S^d]$, then there are functions $f_S \in \mathrm{E}(X, \delta_S)$ such that*

$$f - \sum_{S \in \mathcal{S}_{<\infty}} \lambda_S f_S \in \Delta\left(X, d - \sum_{S \in \mathcal{S}_{<\infty}} \lambda_S \delta_S\right).$$

Proof. Let $S := \{A, B\} \in \mathcal{S}_{<\infty}$ where $\mathcal{S}_{<\infty}$ is any finite subset $\mathcal{S}$. Since $\mathrm{rank}(A(f)) = 0$ and thus $A(f)$ is in particular not bipartite, there are $a, a' \in X$ such that $\{a, a'\} \in A(f)$ and either $a, a' \in A$ or $a, a' \in B$. Assume without loss of generality that $a, a' \in A$. Note that if there are $b, b' \in B$ such that one has $\{b, b'\} \in A(f)$, then

$$\begin{aligned} \max\{d(a,b) + d(a',b'), d(a,b') + d(a',b)\} &\leq f(a) + f(a') + f(b) + f(b') \\ &= d(a,a') + d(b,b'), \end{aligned}$$

and thus $\alpha_S^d = 0$, which contradicts our assumption. Hence for any $b, b' \in B$, one has $\{b, b'\} \notin A(f)$. We set

$$f_S(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

We show that $f^{(1)} := f - \lambda_S f_S$ and $d^{(1)} := d - \lambda_S f_S$ satisfy $f^{(1)} \in \Delta\left(X, d^{(1)}\right)$. We denote distances $d(x,y)$ simply by xy and we distinguish three cases:

(a) if $x, y \in A$, then

$$f^{(1)}(x) + f^{(1)}(y) = f(x) + f(y) \geq xy = d^{(1)}(x,y),$$

(b) if $x \in A$ and $y \in B$, then

$$f^{(1)}(x) + f^{(1)}(y) = f(x) + f(y) - \lambda_S \geq xy - \lambda_S = d^{(1)}(x,y),$$

(*c*) if $x, y \in B$, there are $\{a, a'\} \subset A$ such that

$$f(a) + f(a') - aa' = 0$$

hence,

$$\begin{aligned} f^{(1)}(x) + f^{(1)}(y) &= f(x) + f(y) - 2\lambda_S \\ &\geq [f(x) + f(y) - 2\lambda_S] + \left[f(a) + f(a') - aa'\right] \\ &\geq \max\left\{xa + ya', xa' + ya\right\} - aa' - 2\lambda_S \\ &\geq xy. \end{aligned}$$

Hence $f^{(1)} \in \Delta\left(X, d^{(1)}\right)$ as desired. Now, note that if $\{x, y\} \in A^d(f)$, then either $x, y \in A$ or $x \in A$ and $y \in B$. In both cases one sees that equalities hold in (*a*) and (*b*) above and thus that $\{x, y\} \in A^{d^{(1)}}(f^{(1)})$ (where $A^{d^{(1)}}(f^{(1)})$ refers to those $\{x, y\}$ such that $f^{(1)}(x) + f^{(1)}(y) = d^{(1)}(x, y)$), thus

$$A^d(f) \subset A^{d^{(1)}}(f^{(1)}), \tag{2.10}$$

hence in particular $f^{(1)} \in \Sigma^0(\mathrm{E}(X, d^{(1)}))$. For $S' := \{A', B'\} \in \mathcal{S} \setminus \{S\}$, we can proceed as above finding $c, c' \in A'$ such that $\{c, c'\} \in A^{d^{(1)}}(f^{(1)})$. By Theorem 1.3, it follows that $\alpha_{S'}^{d^{(1)}} = \alpha_{S'}^{d} > 0$ which implies that for $e, e' \in B'$, one has $\{e, e'\} \notin A^{d^{(1)}}(f^{(1)})$. We set

$$f_{S'}^{(1)}(x) = \begin{cases} 0 & \text{if } x \in A', \\ 1 & \text{if } x \in B' \end{cases}$$

and as before, we obtain $f^{(1)} - \lambda_{S'} f_{S'}^{(1)} \in \Delta\left(X, d^{(1)} - \lambda_{S'}\delta_{S'}\right)$. By (2.10), it follows that $f_{S'} = f_{S'}^{(1)}$. Hence, we obtain

$$f - \lambda_S f_S - \lambda_{S'} f_{S'} = f^{(1)} - \lambda_{S'} f_{S'}^{(1)} \in \Delta\left(X, d^{(1)} - \lambda_{S'}\delta_{S'}\right) = \Delta\left(X, d^{(2)}\right)$$

where $d^{(2)} = d - \lambda_S\delta_S - \lambda_{S'}\delta_{S'}$. By induction, we now get the desired result. □

The next lemma is proved as the corresponding assertion in [4, Theorem 7].

2.9 Lemma. *Let (X, d) be a metric space with integer-valued metric satisfying the (LRC). For any split $S = \{A, B\}$ of X and any $\lambda_S > 0$ such that $d = d_1 + \lambda_S\delta_S$, if $\Delta(X, d) = \Delta(X, d_1) + \lambda_S\Delta(X, \delta_S)$, then the following hold:*

(*i*) *S is a d-split of X and*

(ii) $\lambda_S \leq \alpha_S$.

Proof. Assume that there is a split $S = \{A, B\}$ of X and $\lambda_S > 0$ such that $d = d_1 + \lambda_S \delta_S$. If $\Delta(X, d) = \Delta(X, d_1) + \lambda_S \Delta(X, \delta_S)$, we show that for any $A_0 := \{a, a'\} \subset A$ and $B_0 := \{b, b'\} \subset B$ and writing simply xy instead of $d(x, y)$, one has

$$\frac{1}{2}\Big(\max\{ab + a'b', a'b + ab'\} - aa' - bb'\Big) \geq \lambda_S.$$

Note that for any $Y \subset X$ and any split S of X, one has

$$\Delta(d|_{Y\times Y}) = \Delta(d_1|_{Y\times Y}) + \lambda_S \Delta(\delta_S|_{Y\times Y})$$

and thus in particular for $Y := A_0 \cup B_0$ and the split $S := \{A_0, B_0\}$ of Y. Define now the map $f\colon Y \to \mathbb{R}$ as follows

$$\begin{aligned} f(a) &:= \tfrac{1}{2}(aa' + ab - a'b), \\ f(a') &:= \tfrac{1}{2}(aa' + a'b - ab), \\ f(b) &:= \tfrac{1}{2}(ab + a'b - aa'), \\ f(b') &:= \max\{ab' - f(a), a'b' - f(a'), bb' - f(b)\}. \end{aligned}$$

It is then easy to see that

$$\begin{aligned} f(a) + f(a') &= aa', \\ f(a) + f(b) &= ab, \\ f(a') + f(b) &= a'b. \end{aligned}$$

Furthermore, $f(a)$, $f(a')$ and $f(b)$ are clearly non-negative. From

$$ab' - f(a) + a'b' - f(a') = ab' + a'b' - aa' \geq 0$$

we deduce that $f(b') \geq 0$. Therefore, $f \in \mathrm{E}(Y, d) \subset \Delta(Y, d)$. By assumption, there exist $f_1 \in \Delta(Y, d_1)$ and $f_S \in \Delta(Y, \delta_S)$ such that

$$f = f_1 + \lambda_S f_S.$$

Since $\delta_S(a, a') = 0$, we have

$$d_1(a, a') = aa' = f(a) + f(a') \geq f_1(a) + f_1(a')$$

hence $f_1(a) + f_1(a') = aa'$ and thus $f_S(a) = f_S(a') = 0$ which implies

$$f_S(b) \geq 1 \quad \text{and} \quad f_S(b') \geq 1.$$

Moreover, we have

$$\begin{aligned} f_1(a) + f_1(b) + \lambda_S f_S(b) = f(a) + f(b) = ab = d_1(a, b) + \lambda_S \\ \leq f_1(a) + f_1(b) + \lambda_S. \end{aligned}$$

Therefore

$$f_S(b) = 1.$$

Similarly,

$$\begin{aligned} f_1(a') + f_1(b) + \lambda_S f_S(b) = f(a') + f(b) = a'b = d_1(a', b) + \lambda_S \\ \leq f_1(a') + f_1(b) + \lambda_S, \end{aligned}$$

and thus

$$f(a') + f(b) = f_1(a') + f_1(b) + \lambda_S = a'b.$$

Observe that since $\lambda_S f_S(b) = \lambda_S > 0$, one has

$$f(b) + f(b') > f_1(b) + f_1(b') \geq d_1(b, b') = bb'.$$

Since we may interchange the role of a and a' in the following, we can assume that $f(a) + f(b') = ab'$ since $f \in \mathrm{E}(Y, d)$. Hence

$$\begin{aligned} f_1(a) + f_1(b') + \lambda_S f_S(b) = f(a) + f(b') = ab' = d_1(a, b') + \lambda_S \\ \leq f_1(a) + f_1(b') + \lambda_S. \end{aligned}$$

and therefore

$$f_S(b') = 1.$$

Since

$$f_1(b) + f_1(b') \geq d_1(b, b') = bb',$$

we finally obtain

$$\begin{aligned} &\tfrac{1}{2}\Big(\max\{ab + a'b', a'b + ab'\} - aa' - bb'\Big) \\ &\geq \tfrac{1}{2}\Big(a'b + ab' - aa' - bb'\Big) \\ &= \tfrac{1}{2}\Big(f(a') + f(b) + f(a) + f(b') - f(a) - f(a') - bb'\Big) \\ &= \tfrac{1}{2}\Big(f_1(a') + f_1(b) + \lambda + f_1(a) + f_1(b') + \lambda - f_1(a) - f_1(a') - bb'\Big) \\ &= \tfrac{1}{2}\Big(f_1(b) + f_1(b') - bb'\Big) + \lambda_S \\ &\geq \lambda_S, \end{aligned}$$

i.e. $\alpha_S^d \geq \lambda_S$ and this is the desired result. □

Note that
$$\Delta(X, d_0) + \sum_{S \in \mathcal{S}} \alpha_S \Delta(X, \delta_S) \subset [0, \infty]^X,$$
and since $\mathcal{S}$ is possibly infinite, we cannot replace $[0,\infty]^X$ by $\mathbb{R}^X$ in general.

2.10 Theorem. *Let (X, d) be a metric space with integer-valued metric satisfying the (LRC). Let $\mathcal{S}$ be the family of all d-splits of X so that $d = d_0 + \sum_{S \in \mathcal{S}} \alpha_S \delta_S$, let $\lambda_S \in [0, \alpha_S]$ for every $S \in \mathcal{S}$ and set $d_1 := d - \sum_{S \in \mathcal{S}} \lambda_S \delta_S$. Then*
$$\Delta(X, d) = \mathbb{R}^X \cap \left(\Delta(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \Delta(X, \delta_S) \right).$$

Proof. Let $\mathcal{S}$ be the set of all d-splits and let $S \in \mathcal{S}$. As in the proof of Lemma 2.8, for $f \in \Sigma^0(\mathrm{E}(X,d))$, since $\mathrm{rank}(A(f)) = 0$ and thus $A(f)$ is in particular not bipartite, there are $a, a' \in X$ such that $\{a, a'\} \in A(f)$ and without loss of generality that $a, a' \in A$. Then, for any $b, b' \in B$, one has $\{b, b'\} \notin A(f)$. We set
$$f_S(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$
We first show that for every $x \in X$, one has $\sum_{S \in \mathcal{S}} \lambda_S f_S(x) \neq \infty$. Note that for every $x \in X$, there exists $x' \in X$ such that $\{x, x'\} \in A^d(f)$. Furthermore, we have
$$d(x, x') = d_0(x, x') + \sum_{\substack{S \in \mathcal{S} \\ S(x) \neq S(x')}} \alpha_S^d \delta_S(x, x').$$
Since $\alpha_S^d \in [0, \infty) \cap \frac{1}{2}\mathbb{Z}$, we deduce that the set
$$\mathcal{S}_{xx'} := \{S \in \mathcal{S} : S(x) \neq S(x')\}$$
is finite. Moreover, for every $S := \{A, B\} \in \mathcal{S} \setminus \mathcal{S}_{xx'}$, since $\{x, x'\} \in A^d(f)$, one has $x, x' \in A$ and $f_S(x) = 0 = f_S(x')$ by definition. It follows that $\sum_{S \in \mathcal{S}} \lambda_S f_S \in \mathbb{R}^X$ as well as
$$f_1 := f - \sum_{S \in \mathcal{S}} \lambda_S f_S \in \mathbb{R}^X.$$
For $d_1 := d - \sum_{S \in \mathcal{S}} \lambda_S \delta_S$, it now remains to show that $f_1 \in \Delta(X, d_1)$. For every $x, y \in X$, there are $x', y' \in X$ such that $\{x, x'\}, \{y, y'\} \in A^d(f)$. Since for every $S \in \mathcal{S}_{xy}$, one has $f_S(x) + f_S(y) = 1$, it follows that $\mathcal{S}_{xy} \subset \mathcal{S}_{xx'} \cup \mathcal{S}_{yy'} =:$

$\mathcal{S}_{<\infty}$ where $|\mathcal{S}_{<\infty}| < \infty$. By Lemma 2.8 and setting $d_1^{<\infty} := d - \sum_{S \in \mathcal{S}_{<\infty}} \lambda_S \delta_S$, it follows that

$$f_1^{<\infty} := f - \sum_{S \in \mathcal{S}_{<\infty}} \lambda_S f_S \in \Delta\left(X, d_1^{<\infty}\right)$$

and thus

$$f_1(x) + f_1(y) = f_1^{<\infty}(x) + f_1^{<\infty}(y) \geq d_1(x, y).$$

This shows that

$$\Sigma^0(\mathrm{E}(X,d)) \subset \left(\Delta(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \Delta(X, \delta_S)\right).$$

Now, using that (X, d) satisfies the (LRC), we can take for each cell of $\mathrm{E}(X, d)$, all finite convex combinations of its vertices and by convexity of $\Delta(X, d_1)$ and $\Delta(X, \delta_S)$ for every $S \in \mathcal{S}$, we deduce that

$$\mathrm{E}(X,d) \subset \left(\Delta(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \Delta(X, \delta_S)\right).$$

Adding finally $[0, \infty)^X$ on both sides and intersecting with $\mathbb{R}^X$, we get

$$\Delta(X,d) \subset \mathbb{R}^X \cap \left(\Delta(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \Delta(X, \delta_S)\right).$$

Since the other inclusion is easy to see, we obtain the desired result. □

Let $f \in \mathrm{E}(X, d) \subset \Delta(X, d)$. From

$$\Delta(X,d) = \mathbb{R}^X \cap \left(\Delta(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \Delta(X, \delta_S)\right), \tag{2.11}$$

we have a decomposition $f = f_1 + \sum_{S \in \mathcal{S}} \lambda_S f_S$. Note that if there are for $S \in \mathcal{S}$, functions $f_S \geq g_S \in \Delta(X, \delta_S)$ and $f_1 \geq g_1 \in \Delta(X, d_1)$ where not all inequalities are equalities, then $g := g_1 + \sum_{S \in \mathcal{S}} \lambda_S g_S \in \Delta(X, d)$ by (2.11). Since $g \leq f \in \mathrm{E}(X, d)$, this contradicts the minimality of f. We must therefore have that

$$f \in \mathbb{R}^X \cap \left(\mathrm{E}(X, d_1) + \sum_{S \in \mathcal{S}} \lambda_S \mathrm{E}(X, \delta_S)\right).$$

We have thus shown that (2.11) implies

$$\mathrm{E}(X,d) \subset \mathbb{R}^X \cap \left(\mathrm{E}(X,d_1) + \sum_{S\in\mathcal{S}} \lambda_S \mathrm{E}(X,\delta_S) \right).$$

Hence, we obtain:

2.11 Corollary. *Let (X,d) be a metric space with integer-valued metric satisfying the (LRC). Let $\mathcal{S}$ be the family of all d-splits of X such that $d = d_0 + \sum_{S\in\mathcal{S}} \alpha_S\delta_S$ and let $\lambda_S \in [0,\alpha_S]$ for every $S \in \mathcal{S}$ so that $d_1 := d - \sum_{S\in\mathcal{S}} \lambda_S\delta_S$. Then*

$$\mathrm{E}(X,d) \subset \mathbb{R}^X \cap \left(\mathrm{E}(X,d_1) + \sum_{S\in\mathcal{S}} \lambda_S \mathrm{E}(X,\delta_S) \right). \tag{2.12}$$

It is also easy to see that (2.12) implies (2.11). Remember that from [30], there is a 1-Lipschitz map $p\colon \Delta(X,d) \to \mathrm{E}(X,d)$ such that for every $f \in \Delta(X,d)$, one has $p(f) \le f$. Then $p(f) =: g \in \mathrm{E}(X,d)$. From (2.12), we obtain a decomposition of g as $g := g_1 + \sum_{S\in\mathcal{S}} \lambda_S g_S$. Moreover, $f - g \in [0,\infty)^X$, hence $g_1 + (f-g) \in \Delta(X,d_1)$ and thus

$$f = g_1 + (f-g) + \sum_{S\in\mathcal{S}} \lambda_S g_S \in \mathbb{R}^X \cap \left(\Delta(X,d_1) + \sum_{S\in\mathcal{S}} \lambda_S \Delta(X,\delta_S) \right)$$

which is the desired result.

We continue with a remark relating to the case of finite metric spaces (with a nonnegative real-valued metric).

2.12 Remark. Consider for any finite metric space (X,d) the sequence of metric spaces $((X,d^n))_{n\in\mathbb{N}}$ where $d^n\colon X\times X \to \frac{1}{n}\mathbb{Z}$ is given by

$$d^n(x,x') := \min\{t \in \tfrac{1}{n}\mathbb{Z} : t \ge d(x,x')\}.$$

It is easy to see that one has $d^n \to d$ pointwise and d^n defines a metric for any $n \in \mathbb{N}$. For any $f \in \mathrm{E}(X,d)$, let $f^n := f + \frac{1}{2n}$. One has

$$f^n \in \Delta(X,d^n) = \Delta(X,d_0^n) + \sum_{S\in\mathcal{S}} \alpha_S^{d^n} \Delta(X,\delta_S)$$

and thus we can write $f^n = f_0^n + \sum_{S\in\mathcal{S}} \alpha_S^{d^n} f_S^n$ where each function appearing on the right-hand side is bounded by $\mathrm{diam}(X)$. From the formula for the isolation indices, we see that for $S \in \mathcal{S}$, one has $\alpha_S^{d^n} \to \alpha_S^d$ and by properness of $\Delta(X, \delta_S)$ endowed with the supremum norm, considering subsequences if necessary, $f_S^n \to f_S \in \Delta(X, \delta_S)$, similarly $f_0^n \to f_0 \in \Delta(X, d_0)$. Since $f^n \to f$ in the supremum norm, and by uniqueness of the limit, we obtain

$$f = f_0 + \sum_{S \in \mathcal{S}} \alpha_S^d f_S \in \Delta(X, d_0) + \sum_{S \in \mathcal{S}} \alpha_S^d \Delta(X, \delta_S).$$

But since $f \in \mathrm{E}(X, d)$, it follows by minimality that

$$f_0 + \sum_{S \in \mathcal{S}} \alpha_S^d f_S \in \mathrm{E}(X, d_0) + \sum_{S \in \mathcal{S}} \alpha_S^d \mathrm{E}(X, \delta_S)$$

and this closes the remark.

We conclude this section with a final remark.

2.13 Remark. Let (X, d) be a metric space with integer-valued metric. By Theorem 1.3, one can consider the associated split decomposition

$$d = d_0 + \sum_{S \in \mathcal{S}} \alpha_S^d \delta_S.$$

Note that the pseudometric $\sum_{S \in \mathcal{S}} \alpha_S^d \delta_S$ is now only known to be $\frac{1}{2}\mathbb{Z}$-valued by (2.6). Nevertheless, it is easy to see that $\mathrm{E}(X, \gamma d) = \gamma \mathrm{E}(X, d)$ in $\mathbb{R}^X$ for any $\gamma > 0$. Hence

$$\begin{aligned} \mathrm{E}(X, \sum_{S \in \mathcal{S}} \alpha_S^d \delta_S) = \tfrac{1}{2}\mathrm{E}(X, \sum_{S \in \mathcal{S}} 2\alpha_S^d \delta_S) &\subset \tfrac{1}{2}\left(\mathbb{R}^X \cap \sum_{S \in \mathcal{S}} 2\alpha_S^d \mathrm{E}(X, \delta_S)\right) \\ &= \mathbb{R}^X \cap \sum_{S \in \mathcal{S}} \alpha_S^d \mathrm{E}(X, \delta_S). \end{aligned}$$

Thus, by rescaling the situation is similar to the integer-valued case.

IV.3 The Buneman Complex and Related Topics

If $\mathcal{S}$ is a split system (on a set X) and $\alpha\colon \mathcal{S} \to (0, \infty)$ is any map $S \mapsto \alpha_S$, the weighted split system given by the pair $(\mathcal{S}, \alpha)$ is called a *split system pair (of X)*. If $\mathcal{S}$ is weakly compatible as in Definition 2.6, then $(\mathcal{S}, \alpha)$ is called a weakly compatible split system pair. Let now $\mathcal{S}$ be a weakly compatible split system on a pseudometric space with integer-valued pseudometric (X, d) and assume that

$$d = \sum_{S \in \mathcal{S}} \alpha_S \delta_S.$$

By Theorem 2.7, $\mathcal{S}$ is the set of all d-splits of X and d is thus totally split-decomposable. The weakly compatible split system pair $(\mathcal{S}, \alpha)$ is called *the split system pair associated to* (X, d). Unless otherwise stated, this is the split system pair that we refer to in the sequel, when considering a totally split-decomposable pseudometric space (X, d). We give some statements that hold for general cell complexes which we define as follows.

3.1 Definition. We call K a (real) cell complex if K is a subset of a real vector space endowed with a family of convex subsets $\{C_i\}_{i \in I}$ of K such that the collection $\Delta_{\mathrm{K}} := \{C_i\}_{i \in I}$ verifies for any $C_i, C_j \in \Delta_{\mathrm{K}}$ that $C_i \cap C_j \in \Delta_{\mathrm{K}} \cup \{\emptyset\}$ and $\bigcup_i C_i = \mathrm{K}$. The sets C_i are called the *cells* of K and the dimension of C_i is the dimension of its affine hull, which might be infinite.

Let now $(\mathcal{S}, \alpha)$ be any split system pair on a set X, let

$$U(\mathcal{S}) := \{A \subset X : \text{there is } S \in \mathcal{S} \text{ such that } A \in S\}$$

and for $\mu \colon U(\mathcal{S}) \to [0, \infty)$, let

$$\mathrm{supp}(\mu) := \{A \in U(\mathcal{S}) : \mu(A) > 0\}.$$

For $A \subset X$, we usually denote its complement $X \setminus A$ by $\bar{A}$. If $A \in U(\mathcal{S})$, we denote the split $\{A, \bar{A}\} \in \mathcal{S}$ by S_A. Moreover, we define a (possibly) infinite dimensional hypercube

$$H(\mathcal{S}, \alpha) := \{\mu \colon U(\mathcal{S}) \to [0, \infty) : \text{for all } A \in U(\mathcal{S}) \\ \text{one has } \mu(A) + \mu(\bar{A}) = \tfrac{\alpha_{S_A}}{2}\}.$$

Note that $H(\mathcal{S}, \alpha)$ has a natural cell complex structure, cells are sets of the form

$$[\mu] := \{\mu' \in H(\mathcal{S}, \alpha) : \mathrm{supp}(\mu') \subset \mathrm{supp}(\mu)\}$$

where $\mu \in H(\mathcal{S}, \alpha)$. The cells of $H(\mathcal{S}, \alpha)$ are (possibly) infinite dimensional hypercubes. The *Buneman complex* is the subcomplex of $H(\mathcal{S}, \alpha)$ given by

$$B(\mathcal{S}, \alpha) := \\ \{\mu \in H(\mathcal{S}, \alpha) : \text{if } A, B \in \mathrm{supp}(\mu) \text{ and } A \cup B = X, \text{ then } A \cap B = \emptyset\}.$$

Furthermore, let

$$\bar{T}(\mathcal{S}, \alpha) := \\ \left\{ \mu \in H(\mathcal{S}, \alpha) : \text{if } \{A_i\}_{i \in I} \subset \mathrm{supp}(\mu) \text{ and } \bigcup_{i \in I} A_i = X, \text{ then } \bigcap_{i \in I} A_i = \emptyset \right\}.$$

It is easy to see that $\bar{T}(\mathcal{S},\alpha)$ is a subcomplex of $B(\mathcal{S},\alpha)$, as in the finite case [19, Section 4]. Indeed, $\bar{T}(\mathcal{S},\alpha) \subset B(\mathcal{S},\alpha)$ and $\bar{T}(\mathcal{S},\alpha)$ is a subcomplex of $H(\mathcal{S},\alpha)$ since if $\mu \in \bar{T}(\mathcal{S},\alpha)$, then for any $\mu' \in H(\mathcal{S},\alpha)$ such that $\mathrm{supp}(\mu') \subset \mathrm{supp}(\mu)$, one obviously has $\mu' \in \bar{T}(\mathcal{S},\alpha)$. We can thus denote by $[\psi]$ the smallest cell of $H(\mathcal{S},\alpha)$ containing $\psi \in H(\mathcal{S},\alpha)$ and thus if $\psi \in B(\mathcal{S},\alpha)$, then $[\psi] \subset B(\mathcal{S},\alpha)$ and similarly if $\psi \in \bar{T}(\mathcal{S},\alpha)$, then $[\psi] \subset \bar{T}(\mathcal{S},\alpha)$.

Now, for $x \in X$ consider the map $\phi_x \colon U(\mathcal{S}) \to [0,\infty)$ which is defined as

$$\phi_x(A) := \begin{cases} \frac{\alpha_{S_A}}{2} & \text{if } x \notin A, \\ 0 & \text{if } x \in A. \end{cases}$$

Moreover, let $d_1 \colon \mathbb{R}^{U(\mathcal{S})} \times \mathbb{R}^{U(\mathcal{S})} \to [0,\infty]^X$ be given by

$$(\mu,\psi) \mapsto \sum_{A \in U(\mathcal{S})} |\mu(A) - \psi(A)|.$$

The map $\kappa \colon \mathbb{R}^{U(\mathcal{S})} \to [0,\infty]^X$ given by $\mu \mapsto \kappa(\mu)$ where for $x \in X$, one has

$$\kappa(\mu)(x) = d_1(\mu,\phi_x).$$

The next lemma summarizes several useful properties which follow from the above definitions.

3.2 Lemma. *Let $(\mathcal{S},\alpha)$ be a split system pair on a set X and assume $d := \sum_{S \in \mathcal{S}} \alpha_S \delta_S$ defines a pseudometric on X. It follows easily that the following hold:*

(i) For every $x, y \in X$, one has

$$d_1(\phi_x,\phi_y) = d(x,y).$$

(ii) For every $x \in X$, one has

$$\phi_x \in \bar{T}(\mathcal{S},\alpha) \subset B(\mathcal{S},\alpha).$$

(iii) For every $\mu, \phi \in \mathbb{R}^{U(\mathcal{S})}$, one has

$$\sup_{x \in X} |\kappa(\mu)(x) - \kappa(\phi)(x)| \leq d_1(\mu,\phi)$$

where each side might be infinite.

For $x, y \in X$ and $S = \{A, \bar{A}\} \in \mathcal{S}$, let $S(x) := A$ if $x \in A$ and $S(x) := \bar{A}$ if $x \in \bar{A}$. Define $\mathcal{S}_{xy} := \{S \in \mathcal{S} : S(x) \neq S(y)\}$. Under the assumptions of Lemma 3.2, for $\psi\colon \mathcal{S} \to \mathbb{R}$ one has

$$\begin{aligned}
\kappa(\psi)(x) + \kappa(\psi)(y) = &\sum_{S \in \mathcal{S}_{xy}} \Big[\big|\psi(S(x)) - \phi_x(S(x))\big| + \big|\psi(S(x)) - \phi_y(S(x))\big| \\
&\qquad + \big|\psi(\overline{S(x)}) - \phi_x(\overline{S(x)})\big| + \big|\psi(\overline{S(x)}) - \phi_y(\overline{S(x)})\big| \Big] \\
&+ \sum_{S \notin \mathcal{S}_{xy}} \Big[\big|\psi(S(x)) - \phi_x(S(x))\big| + \big|\psi(S(x)) - \phi_y(S(x))\big| \\
&\qquad + \big|\psi(\overline{S(x)}) - \phi_x(\overline{S(x)})\big| \\
&\qquad + \big|\psi(\overline{S(x)}) - \phi_y(\overline{S(x)})\big| \Big] \\
\geq &\sum_{S \in \mathcal{S}_{xy}} \alpha_S \delta_S(x, y) \\
&+ \sum_{S \notin \mathcal{S}_{xy}} \Big[2\big|\psi(S(x))\big| + 2\big|\psi(\overline{S(x)}) - \tfrac{\alpha_S}{2}\big| \Big],
\end{aligned}$$

Hence, it follows that

$$\frac{1}{2}[\kappa(\psi)(x) + \kappa(\psi)(y) - d(x, y)] \geq \sum_{\substack{S \in \mathcal{S} \\ S(x) = S(y)}} \Big[\big|\psi(S(x))\big| + \big|\psi(\overline{S(x)}) - \tfrac{\alpha_S}{2}\big| \Big], \quad (3.1)$$

and equality holds if $\psi \in H(\mathcal{S}, \alpha)$.

For simplicity, we denote the injective hull $\mathrm{E}(X, d)$ by $\mathrm{E}(d)$ when the underlying space X is clear (unlike in [30] where it is denoted by $\mathrm{E}(X)$) and $\mathrm{E}'(X, d)$ by $\mathrm{E}'(d)$. Analogously, $\Delta(X, d)$ is denoted by $\Delta(d)$ (it corresponds to the space $\Delta(X)$ in [30]).

3.3 Lemma. *Let $(\mathcal{S}, \alpha)$ be a split system pair on a set X and assume that $d := \sum_{S \in \mathcal{S}} \alpha_S \delta_S$ defines a pseudometric on X. For every $\mu \in H(\mathcal{S}, \alpha)$, the following are equivalent:*

(i) $\kappa(\mu) \in \mathrm{E}'(d)$.

(ii) $\mu \in \bar{T}(\mathcal{S}, \alpha)$.

Proof. Consider $\mu \in H(\mathcal{S}, \alpha)$. Let us first show that *(ii)* implies *(i)*. Let $\mu \in H(\mathcal{S}, \alpha)$ and assume that $\kappa(\mu) \notin \mathrm{E}'(d)$. For $x, y \in X$ we have $\kappa(\mu)(x) +$

$\kappa(\mu)(y) = d_1(\mu, \phi_x) + d_1(\mu, \phi_y) \geq d_1(\phi_x, \phi_y) = d(x, y)$, in particular $\kappa(\mu) \in [0, \infty]^X$. By our contradiction assumption, there is $x \in X$ such that for every $y \in X$, one has $\kappa(\mu)(x) + \kappa(\mu)(y) > d(x, y)$ where the left-hand side is possibly infinite. Since $\mu \in H(\mathcal{S}, \alpha)$, equality holds in (3.1), hence there is $S_y \in \mathcal{S}$ such that $S_y(x) = S_y(y)$ and $\mu(S_y(x)) > 0$. Therefore $X = \bigcup_{y \in X} S_y(x)$ where $\{S_y\}_{y \in X} \subset \operatorname{supp}(\mu)$. Moreover, $x \in \bigcap_{y \in X} S_y(x) \neq \emptyset$. It follows that $\mu \notin \bar{T}(\mathcal{S}, \alpha)$. This shows that (ii) implies (i).

To show the other implication, assume that $\kappa(\mu) \in \mathrm{E}'(d)$. For every $x \in X$, there is $w \in X$ such that $\kappa(\mu)(x) + \kappa(\mu)(w) = d(x, w)$. By (3.1), for any $S \in \mathcal{S}$ one has

$$\text{if } S(x) = S(w), \text{ then } \mu(S(x)) = 0 = \mu(S(w)). \tag{3.2}$$

Note that for any $A_i \in \operatorname{supp}(\mu)$, there exists by definition $S_i \in \mathcal{S}$ as well as $x_i \in X$ such that $A_i = S_i(x_i)$. Now, if $X = \bigcup_{i \in I} A_i = \bigcup_{i \in I} S_i(x_i)$ and if by contradiction we assume that $\bigcap_{i \in I} S_i(x_i) \neq \emptyset$, we can pick an arbitrary $z \in \bigcap_{i \in I} S_i(x_i)$. Then, for every $i \in I$ one has $S_i(x_i) = S_i(z)$ and thus $X = \bigcup_{i \in I} S_i(z)$. It is now easy to see that the existence of z contradicts (3.2). Indeed, for any $y \in X$, there is $S_j \in \{S_i\}_{i \in I} \subset \mathcal{S}$ such that $y \in S_j(z)$ and hence

$$S_j(y) = S_j(z). \tag{3.3}$$

However,

$$S_j(z) = S_j(x_j) = A_j \in \operatorname{supp}(\mu) \tag{3.4}$$

and y can be chosen so that $\kappa(\mu)(z) + \kappa(\mu)(y) = d(z, y)$. Thus by (3.2) and (3.3), one has $\mu(S_j(y)) = 0 = \mu(S_j(z))$ which is a contradiction to (3.4). This finishes the proof. □

3.4 Lemma. *Let (X, d) be a totally split-decomposable metric space with integer-valued metric satisfying the (LRC). Then, the map $\bar{\kappa} := \kappa|_{\bar{T}(\mathcal{S},\alpha)} \colon \bar{T}(\mathcal{S}, \alpha) \to \mathrm{E}(d)$ is surjective.*

Proof. Let $f \in \mathrm{E}(d)$. By Corollary 2.11, we have $\mathrm{E}(d) \subset \sum_{S \in \mathcal{S}} \alpha_S \mathrm{E}(\delta_S)$. Thus, we have a decomposition

$$f = \sum_{S \in \mathcal{S}} \alpha_S f_S$$

where $f_S \in \mathrm{E}(\delta_S)$ implies that if $S = \{A, \bar{A}\}$, then for any $x \in A$ and $y \in \bar{A}$ one has that f_S is constantly equal to $f_S(x)$ on A, constantly equal to $f_S(y)$ on $\bar{A}$ and $f_S(x), f_S(y) \geq 0$ as well as $f_S(x) + f_S(y) = 1$. Define the map

$\psi_f\colon U(\mathcal{S}) \to [0,\infty)$ by setting for every $S = \{A, \bar{A}\}$ as well as for arbitrarily chosen $x \in A$ and $y \in \bar{A}$:

$$\psi_f(A) := \frac{\alpha_S}{2} f_S(x) \quad \text{and} \quad \psi_f(\bar{A}) := \frac{\alpha_S}{2} f_S(y).$$

It is clear that ψ_f is well-defined, i.e., the above definition does not depend on the particular choice of x and y. Furthermore, it is easy to see that

(*a*) $\psi_f \in H(\mathcal{S}, \alpha)$ and

(*b*) $\kappa(\psi_f) = f \in \mathrm{E}(d) = \mathrm{E}'(d)$.

It then follows from Lemma 3.3 that $\psi_f \in \bar{T}(\mathcal{S}, \alpha)$ and this finishes the proof. □

For a map $\phi\colon \mathcal{S} \to [0,\infty)$, let

$$\mathcal{S}(\phi) := \{S \in \mathcal{S} : S \subset \mathrm{supp}(\phi)\}.$$

Let us define for a cell $[\phi]$ of $\bar{T}(\mathcal{S}, \alpha)$ and $x \in X$, the map $\gamma^x_{[\phi]}\colon U(\mathcal{S}) \to [0,\infty)$ given by

$$\gamma^x_{[\phi]}(A) = \begin{cases} \phi_x(A) & \text{if } A \in U(\mathcal{S}(\phi)), \\ \phi(A) & \text{if } A \in U(\mathcal{S} \setminus \mathcal{S}(\phi)). \end{cases}$$

Note that one has $\psi \in [\phi]$ if and only if $\mathrm{supp}(\psi) \subset \mathrm{supp}(\phi)$ and it follows that if $A \in U(\mathcal{S} \setminus \mathcal{S}(\phi))$, then

$$\psi(A) = \phi(A) = \gamma^x_{[\phi]}(A).$$

Hence $\gamma^x_{[\phi]} \in [\phi]$ since $\mathrm{supp}(\gamma^x_{[\phi]}) \subset \mathrm{supp}(\phi)$.

3.5 Definition. Let $i\colon (X, d_X) \to (Y, d_Y)$ be an isometric map of pseudometric spaces. We say that $Z \subset Y$ is X-*gated* (for i and with respect to d_Y) if and only if for every $x \in X$, there is $y_x \in Z$ such that for every $z \in Z$, one has

$$d_Y(i(x), z) = d_Y(i(x), y_x) + d_Y(y_x, z).$$

The proof of the next lemma follows the proof of [28, Lemma 3.1].

3.6 Lemma. *Let $(\mathcal{S}, \alpha)$ be a split system pair on a set X and assume that $d := \sum_{S \in \mathcal{S}} \alpha_S \delta_S$ defines a pseudometric on X. Then, every cell $[\phi]$ of $\bar{T}(\mathcal{S}, \alpha)$ is X-gated with respect to the restriction of d_1 to $\bar{T}(\mathcal{S}, \alpha)$.*

Proof. We already noted that $\phi_x \in \bar{T}(\mathcal{S}, \alpha)$ for every $x \in X$ and that the map $x \mapsto \phi_x$ is an isometric embedding of X into $\bar{T}(\mathcal{S}, \alpha)$. By Lemma 3.3, if $\psi \in \bar{T}(\mathcal{S}, \alpha)$, then for any $x \in X$, there is $y \in X$ such that $\kappa(\psi)(x)+\kappa(\psi)(y) = d(x, y)$ and thus $d_1(\phi_x, \psi) = \kappa(\psi)(x) < \infty$. In particular, the restriction of d_1 to $\bar{T}(\mathcal{S}, \alpha)$ is a metric. Now, for $x \in X$ and $\psi \in [\phi]$, one has:

$$\begin{aligned} d_1(\phi_x, \psi) &= \sum_{A \in U(\mathcal{S})} |\phi_x(A) - \psi(A)| \\ &= \sum_{A \in U(\mathcal{S} \setminus \mathcal{S}(\phi))} |\phi_x(A) - \psi(A)| + \sum_{A \in U(\mathcal{S}(\phi))} |\phi_x(A) - \psi(A)| \\ &= \sum_{A \in U(\mathcal{S} \setminus \mathcal{S}(\phi))} |\phi_x(A) - \gamma^x_{[\phi]}(A)| + \sum_{A \in U(\mathcal{S}(\phi))} |\gamma^x_{[\phi]}(A) - \psi(A)| \\ &= d_1(\phi_x, \gamma^x_{[\phi]}) + d_1(\gamma^x_{[\phi]}, \psi), \end{aligned}$$

which shows that $\gamma^x_{[\phi]}$ is a gate for ϕ_x in $[\phi]$ with respect to the metric d_1. □

3.7 Lemma. *Let $(\mathcal{S}, \alpha)$ be a split system pair on a set X and assume that $d := \sum_{S \in \mathcal{S}} \alpha_S \delta_S$ defines a pseudometric on X. Then, for every $\phi \in \bar{T}(\mathcal{S}, \alpha)$, the split system $\mathcal{S}(\phi) \subset \mathcal{S}$ is antipodal, which means that for any $x \in X$, there is $y \in X$ such that*

$$\text{for every } S \in \mathcal{S}(\phi), \text{ one has } S(x) \neq S(y). \tag{3.5}$$

For $x, y \in X$, if $d(x, y) = \kappa(\phi)(x) + \kappa(\phi)(y)$, then x and y satisfy (3.5).

Proof. By Lemma 3.3, $\kappa(\phi) \in \mathrm{E}'(d)$. Thus for any $x \in X$, there is $y \in X$ such that

$$d(x, y) = \kappa(\phi)(x) + \kappa(\phi)(y)$$

which can be rewritten as

$$\sum_{S \in \mathcal{S}} \alpha_S \delta_S(x, y) = d_1(\phi, \phi_x) + d_1(\phi, \phi_y). \tag{3.6}$$

It is easy to see that for every $S \in \mathcal{S}$, one has

$$\alpha_S \delta_S(x, y) = \sum_{A \in S} |\phi_x(A) - \phi_y(A)| \leq \sum_{A \in S} \big[|\phi_x(A) - \phi(A)| + |\phi(A) - \phi_y(A)|\big]$$

which together with (3.6) imply

$$\alpha_S \delta_S(x, y) = \sum_{A \in S} \big[|\phi_x(A) - \phi(A)| + |\phi(A) - \phi_y(A)|\big]. \tag{3.7}$$

Assume now that there is $S \in \mathcal{S}(\phi)$ such that $S(x) = S(y)$, then by (3.7), one has

$$0 = \alpha_S \delta_S(x, y) = \sum_{A \in S} \big[|\phi_x(A) - \phi(A)| + |\phi(A) - \phi_y(A)| \big] = 4\phi(S(x)),$$

which implies $S(x) \notin \operatorname{supp}(\phi)$ and thus $S \notin \mathcal{S}(\phi)$, which is a contradiction. This finishes the proof. □

By Lemma 3.6 every cell $[\phi]$ of $\bar{T}(\mathcal{S}, \alpha)$ is X-gated. Let $(\Gamma([\phi]), d_1)$ denote the set of all X-gates of $[\phi]$ endowed with the restriction of d_1. A pseudometric space (X, d) is called *antipodal* if there exists an involution $\sigma \colon X \to X$ such that for every $x, y \in X$, one has

$$d(x, \sigma(x)) = d(x, y) + d(y, \sigma(x)).$$

The proof of the next lemma follows the proof of [28, Lemma 4.2].

3.8 Lemma. *Let $(\mathcal{S}, \alpha)$ be a split system pair on a set X and assume that $d := \sum_{S \in \mathcal{S}} \alpha_S \delta_S$ defines a pseudometric on X. Then, for every cell $[\phi]$ of $\bar{T}(\mathcal{S}, \alpha)$, the metric space $(\Gamma([\phi]), d_1)$ is antipodal.*

Proof. Let $x \in X$. Lemma 3.7 implies that there is $y \in X$ such that for every $S \in \mathcal{S}(\phi)$, one has $S(x) \neq S(y)$. Now, define the map $\sigma \colon \Gamma([\phi]) \to \Gamma([\phi])$ by picking for each $\gamma^u_{[\phi]} \in \Gamma([\phi])$ an arbitrary element

$$v \in \bigcap_{S \in \mathcal{S}(\phi)} \overline{S(u)}$$

and letting

$$\gamma^u_{[\phi]} \mapsto \gamma^v_{[\phi]}.$$

First note that if $\gamma^u_{[\phi]} = \gamma^{u'}_{[\phi]}$, one then has for each $A \in U(\mathcal{S}(\phi))$:

$$\phi_u(A) = \gamma^u_{[\phi]}(A) = \gamma^{u'}_{[\phi]}(A) = \phi_{u'}(A)$$

that is

$$\bigcap_{S \in \mathcal{S}(\phi)} \overline{S(u)} = \bigcap_{S \in \mathcal{S}(\phi)} \overline{S(u')}.$$

Note moreover that σ is well-defined since if $v \neq v'$ are such that $S(u) \neq S(v)$ and $S(u) \neq S(v')$ for all $S \in \mathcal{S}(\phi)$, then $S(v) = S(v')$ for all $S \in \mathcal{S}(\phi)$ and thus $\gamma^{v}_{[\phi]} = \gamma^{v'}_{[\phi]}$ by definition of $\gamma^{v}_{[\phi]}$ and $\gamma^{v'}_{[\phi]}$. Hence σ is a well-defined map. It is easy to see that σ is also an involution since

$$v \in \bigcap_{S \in \mathcal{S}(\phi)} \overline{S(u)}$$

implies that for every $S \in \mathcal{S}(\phi)$, one has $S(v) = \overline{S(u)}$ that is $\overline{S(v)} = S(u)$ and thus

$$u \in \bigcap_{S \in \mathcal{S}(\phi)} \overline{S(v)}.$$

Furthermore, it is easy to deduce from $S(x) \neq S(y)$ for all $S \in \mathcal{S}(\phi)$ that for $z \in X$ and $A \in U(\mathcal{S}(\phi))$, one has

$$|\phi_x(A) - \phi_y(A)| = |\phi_x(A) - \phi_z(A)| + |\phi_z(A) - \phi_y(A)|. \tag{3.8}$$

Indeed, if $A = S(x)$ then both sides are equal to $\frac{\alpha_S}{2}$ and the same happens if $A = \overline{S(x)}$. Finally, since $\gamma^{x}_{[\phi]}, \gamma^{y}_{[\phi]}, \gamma^{z}_{[\phi]} \in [\phi]$, it follows that for every $A \in U(\mathcal{S} \setminus \mathcal{S}(\phi))$, one has:

$$\gamma^{x}_{[\phi]}(A) = \gamma^{z}_{[\phi]}(A) = \gamma^{y}_{[\phi]}(A)$$

which together with (3.8) imply

$$d_1(\gamma^{x}_{[\phi]}, \gamma^{y}_{[\phi]}) = d_1(\gamma^{x}_{[\phi]}, \gamma^{z}_{[\phi]}) + d_1(\gamma^{z}_{[\phi]}, \gamma^{y}_{[\phi]}).$$

Since x and z were chosen arbitrarily in X, it follows that $(\Gamma([\phi]), d_1)$ is antipodal and this concludes the proof. □

For $\psi \in \bar{T}(\mathcal{S}, \alpha)$, we know from Lemma 3.3, that $\kappa(\psi) \in \mathrm{E}'(d)$. Recalling that $H(A) := \{g \in \mathbb{R}^X : A \subset A(g)\}$ from the introduction and setting $A := A(\kappa(\psi)))$, let us then denote by $[\kappa(\psi)]$ the set $H(A) \cap \mathrm{E}'(d)$. If (X, d) has integer-valued metric and satisfies the (LRC), $\mathrm{E}(d)$ is a cell complex where all cells are of this form.

3.9 Lemma. *Let $(\mathcal{S}, \alpha)$ be a split system pair on a set X and assume that $d := \sum_{S \in \mathcal{S}} \alpha_S \delta_S$ defines a pseudometric on X. Then, for every cell $[\psi]$ of $\bar{T}(\mathcal{S}, \alpha)$, one has*

$$\kappa([\psi]) \subset [\kappa(\psi)].$$

Proof. For each $x \in X$, there is $y \in X$ such that $\kappa(\psi)(x) + \kappa(\psi)(y) = d(x, y)$. It follows from (3.7) that

$$\sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi)} \alpha_S \delta_S(x, y) = \sum_{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi))} \big[|\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)| \big]$$

and from the definitions of $\gamma^x_{[\psi]}, \gamma^y_{[\psi]}$ it follows that

$$d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) = \sum_{S \in \mathcal{S}(\psi)} \alpha_S \delta_S(x, y).$$

We thus obtain using again the definitions of $\gamma^x_{[\psi]}, \gamma^y_{[\psi]}$ for the last equality below:

$$\begin{aligned} d_1(\phi_x, \phi_y) &= d(x, y) \\ &= \sum_{S \in \mathcal{S}} \alpha_S \delta_S(x, y) \\ &= \sum_{S \in \mathcal{S}(\psi)} \alpha_S \delta_S(x, y) + \sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi)} \alpha_S \delta_S(x, y) \\ &= d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) + \sum_{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi))} \big[|\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)| \big] \\ &= d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) + d_1(\phi_x, \gamma^x_{[\psi]}) + d_1(\phi_y, \gamma^y_{[\psi]}). \end{aligned} \tag{3.9}$$

Besides, using Lemma 3.7 we obtain for every $\mu \in [\psi]$:

$$\begin{aligned} d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) &= \sum_{S \in \mathcal{S}(\psi)} \alpha_S \delta_S(x, y) \\ &= \sum_{S \in \mathcal{S}(\psi)} \alpha_S \\ &= \sum_{S \in \mathcal{S}(\psi)} \Big[(\mu(S(x)) - 0) + (\tfrac{\alpha_S}{2} - \mu(S(x))) \\ &\qquad + (\tfrac{\alpha_S}{2} - \mu(\overline{S(x)})) + (\mu(\overline{S(x)}) - 0) \Big] \\ &= \sum_{S \in U(\mathcal{S}(\psi))} \big[|\phi_x(A) - \mu(A)| + |\phi_y(A) - \mu(A)| \big] \\ &= d_1(\gamma^x_{[\psi]}, \mu) + d_1(\mu, \gamma^y_{[\psi]}) \end{aligned} \tag{3.10}$$

where the last equality follows from the fact that since $\mu, \gamma^x_{[\psi]}, \gamma^y_{[\psi]} \in [\psi]$ one has for every $A \in U(\mathcal{S} \setminus \mathcal{S}(\psi))$ that $\psi(A) = \mu(A)$ as well as $\gamma^x_{[\psi]}(A) = \psi(A) = \gamma^y_{[\psi]}(A)$. We thus deduce from (3.9) and (3.10) that

$$d_1(\phi_x, \phi_y) = d_1(\phi_x, \gamma^x_{[\psi]}) + d_1(\gamma^x_{[\psi]}, \mu) + d_1(\mu, \gamma^y_{[\psi]}) + d_1(\gamma^y_{[\psi]}, \phi_y).$$

It thus follows that

$$d(x, y) = d_1(\phi_x, \phi_y) = d_1(\phi_x, \mu) + d_1(\mu, \phi_y) = \kappa(\mu)(x) + \kappa(\mu)(y).$$

Since $\{x, y\}$ was an arbitrary edge of $A(\kappa(\psi))$, we obtain that $A(\kappa(\psi)) \subset A(\kappa(\mu))$ and this is equivalent to $\kappa(\mu) \in [\kappa(\psi)]$. This finishes the proof. $\square$

For $|X| < \infty$, it is easy to see using Theorem 2.10 that $\mathrm{E}'(d) = \mathrm{E}(d)$ is a subcomplex of the zonotope $\sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)$. Indeed, for every cell $[f]$ of $\mathrm{E}(d)$, we set $H(A(f)) := \bigcap_{\{x,y\}\in A(f)} \partial H_{\{x,y\}}$ where $\partial H_{\{x,y\}} := \{g \in \mathbb{R}^X : g(x) + g(y) = d(x, y)\}$ and we have cf. [30]:

$$[f] = \mathrm{E}(d) \cap H(A(f)) = \Delta(d) \cap H(A(f)).$$

It follows that every cell of $\mathrm{E}(d)$ is a face of the convex polyhedron $\Delta(d) = \sum_{S\in\mathcal{S}} \alpha_S \Delta(\delta_S)$. Thus, we can write $[f] := \partial H \cap \Delta(d)$ for every cell $[f]$ of $\mathrm{E}(d)$, where $H \subset \mathbb{R}^X$ is a half-space containing $\Delta(d)$. It easily follows that $[f] = \partial H \cap \sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)$ and thus $[f]$ is a face of $\sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)$. In case $|X| = \infty$, one can make the following observations:

3.10 Remark. Assume (X, d) has integer-valued metric, is totally split-decomposable, and satisfies the (LRC). In the proof Lemma 3.15 below, we only need that every cell $[f]$ of $\mathrm{E}(d)$ can be written as

$$[f] = \sum_{S\in\mathcal{S}_{[f]}} \alpha_S \mathrm{E}(\delta_S) + \sum_{S\in\mathcal{S}\setminus\mathcal{S}_{[f]}} \alpha_S p_S \tag{3.11}$$

where for each $S \in \mathcal{S} \setminus \mathcal{S}_{[f]}$, one has $p_S \in \{0, 1\}^X$. To see that this holds, note first that as above in the finite case, one has:

$$[f] = \mathrm{E}(d) \cap H(A(f)) = \Big(\sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)\Big) \cap H(A(f)).$$

Set

$$Z_{\{x,y\}} := \Big(\sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)\Big) \cap \partial H_{\{x,y\}}$$

where as above

$$H_{\{x,y\}} = \Big\{g \in \mathbb{R}^X : g(x) + g(y) \geq d(x, y) = \sum_{\substack{S\in\mathcal{S}\\ S(x)\neq S(y)}} \alpha_S\Big\}.$$

It is then easy to see that by definition of the sets $\mathrm{E}(\delta_S)$ and since one has a decomposition $f = \sum_{S\in\mathcal{S}} \alpha_S f_S$ with $f_S \in \mathrm{E}(\delta_S)$ for every $S \in \mathcal{S}$, it follows that:

$$Z_{\{x,y\}} = \sum_{\substack{S\in\mathcal{S}\\ S(x)\neq S(y)}} \alpha_S \mathrm{E}(\delta_S) + \sum_{\substack{S\in\mathcal{S}\\ S(x)=S(y)}} \alpha_S p_S$$

where $p_S|_{S(x)} \equiv 0$ and $p_S|_{\overline{S(x)}} \equiv 1$ and additionally, for every $\sum_{S\in\mathcal{S}} \alpha_S g_S \in Z_{\{x,y\}}$, it follows that $g_S = p_S$ for every $S \in \mathcal{S}$ satisfying $S(x) = S(y)$. In other words, one has the stronger property:

$$Z_{\{x,y\}} = \Big(\sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)\Big) \setminus \Big\{ \sum_{S\in\mathcal{S}} \alpha_S g_S \in \sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S) : \\ \exists S \in \mathcal{S} \text{ such that } S(x) = S(y) \text{ and } g_S \neq p_S \Big\}.$$

Taking repeatedly intersections of $Z_{\{x,y\}}$ with sets of the form $\partial H_{\{x',y'\}}$ where $\{x',y'\} \in A(f)$, we obtain after finitely many steps, the set $\mathcal{S}_{xy} := \{S \in \mathcal{S} : S(x) \neq S(y)\}$ being finite, and setting $\mathcal{S}' := \{S \in \mathcal{S} :$ for all $\{x,y\} \in A(f)$ one has $S(x) \neq S(y)\}$ as well as $\mathcal{S}'' := \{S \in \mathcal{S} :$ there is $\{x,y\} \in A(f)$ so that $S(x) = S(y)\}$

$$[f] = \Big(\sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)\Big) \cap \bigcap_{\{x,y\}\in A(f)} \partial H_{\{x,y\}} = \sum_{S\in\mathcal{S}'} \alpha_S \mathrm{E}(\delta_S) + \sum_{S\in\mathcal{S}''} \alpha_S p_S \quad (3.12)$$

where $p_S|_{S(x)} \equiv 0$ and $p_S|_{\overline{S(x)}} \equiv 1$ for $\{x,y\} \in A(f)$ satisfying $S(x) = S(y)$. As before, the following stronger property holds:

$$[f] = \Big(\sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S)\Big) \setminus \bigcup_{\{x,y\}\in A(f)} \Big\{ \sum_{S\in\mathcal{S}} \alpha_S g_S \in \sum_{S\in\mathcal{S}} \alpha_S \mathrm{E}(\delta_S) : \\ \exists S \in \mathcal{S} \text{ so that } S(x) = S(y) \text{ and } g_S \neq p_S \Big\}. \quad (3.13)$$

We go on with a more concrete description of the representation of the cells of $\mathrm{E}(X,d)$ in the case where each of them is a combinatorial hypercube.

3.11 Remark. It is not difficult to see if (X,d) is as in Remark 3.10 and if every cell $[f]$ of $\mathrm{E}(X,d)$ is a combinatorial hypercube, then the representation (3.11) verifies

$$k := \dim([f]) = |\mathcal{S}_{[f]}|. \quad (3.14)$$

Indeed, for every 1-cell $[g]$ of $\mathrm{E}(X,d)$, one can represent $[g]$ as in (3.11), namely:

$$[g] = \sum_{S\in\mathcal{S}_{[g]}} \alpha_S \mathrm{E}(\delta_S) + \sum_{S\in\mathcal{S}\setminus\mathcal{S}_{[g]}} \alpha_S p_S^g.$$

Now, note that the affine hull $\mathrm{aff}([g])$ is a 1-dimensional affine subspace of $\mathbb{R}^X$ which contains for every $S \in \mathcal{S}_{[g]}$, a translate of $\mathrm{aff}(\mathrm{E}(\delta_S))$. Hence, if $|\mathcal{S}_{[g]}| \geq 2$, then for $S, S' \in \mathcal{S}_{[g]}$ with $S' \neq S''$, $\mathrm{aff}(\mathrm{E}(\delta_{S'}))$ and $\mathrm{aff}(\mathrm{E}(\delta_{S''}))$ have colinear directional vectors. This is however impossible since for any $S \in \mathcal{S}$, the directional vector of $\mathrm{E}(\delta_S)$ with $S = \{A, B\}$ is a scalar multiple of the function $h \in \mathbb{R}^X$ that satisfies $h|_A \equiv 1$ and $h|_B \equiv -1$.

It is easy to see by induction that if Z is a sum of finitely many Minkowski segments which is combinatorially equivalent to an n-hypercube, then for any vertex z of Z, it follows that Z is the Minkowski sum of translates of all its edges incident to z. Indeed, assume z is any vertex of Z. By the combinatorial n-hypercube equivalence of Z, z is incident to exactly n different edges. All facets of Z incident to z are again zonotopes and are combinatorially equivalent to $(n-1)$-hypercubes. There are n such facets and each of them is by induction the sum of $n-1$ edges among those n edges incident to z. Since Z is a zonotope, it is centrally symmetric. Thus, the symmetric image of each facet of Z incident to z is again a facet of Z that can be written as a sum of edges. Hence, there are $2n$ facets of Z that can be written as a Minkowski sum of edges of Z. Since Z must have exactly $2n$ facets, the result follows.

Now, since every cell $[f]$ is a combinatorial hypercube as well as a zonotope, it is thus equal to the Minkowski sum of its edges which are in turn 1-cells of $\mathrm{E}(X, d)$. It follows that for $i \in \{1, \ldots, k\}$, we can pick 1-cells $[f_i]$ of $\mathrm{E}(X, d)$, so that they all intersect in the vertex f_0 of $[f]$, and we can write:

$$[f] = f_0 + \sum_{i \in \{1,\ldots,k\}} ([f_i] - f_0)$$

Using the representation (3.11) for $[f]$ and for each $[f_i]$, we can thus write:

$$[f] = f_0 + \sum_{i \in \{1,\ldots,k\}} \left(\alpha_{S_i} \mathrm{E}(\delta_{S_i}) + \sum_{S \in \mathcal{S} \setminus \{S_i\}} \alpha_S p_S - f_0 \right).$$

Since $f_0 = \sum_{S \in \mathcal{S}} \alpha_S p_S = \sum_{S \in \mathcal{S} \setminus \{S_1, \ldots, S_k\}} \alpha_S p_S + \sum_{i \in \{1,\ldots,k\}} \alpha_{S_i} p_{S_i}$, it follows:

$$\begin{aligned} [f] &= f_0 + \sum_{i \in \{1,\ldots,k\}} \left(\alpha_{S_i} \mathrm{E}(\delta_{S_i}) - \alpha_{S_i} p_{S_i} \right) \\ &= \sum_{S \in \mathcal{S} \setminus \{S_1,\ldots,S_k\}} \alpha_S p_S + \sum_{i \in \{1,\ldots,k\}} \alpha_{S_i} \mathrm{E}(\delta_{S_i}), \end{aligned}$$

which implies that $\mathcal{S}_{[f]} = \{S_i\}_{i \in \{1,\ldots,k\}}$ since it is already clear that $\{S_i\}_{i \in \{1,\ldots,k\}} \subset \mathcal{S}_{[f]}$ and thus they must be equal by the above (i.e., in $\mathrm{aff}(P + P')$, if $P + P' = P$, then $P' = \{0\}$).

3.12 Remark. If (X,d) is again as in Remark 3.10 and if every cell $[f]$ of $\mathrm{E}(X,d)$ is a combinatorial hypercube, it is easy to see that if $f=\sum_{S\in\mathcal{S}}\alpha_S f_S$ as usual with $\alpha_S>0$ and $f_S\in\mathrm{E}(\delta_S)$ and setting

$$\mathcal{S}_f := \{S\in\mathcal{S} : f_S(X)\subset(0,1)\},$$

then

$$[f]=\sum_{S\in\mathcal{S}_f}\alpha_S\mathrm{E}(\delta_S)+\sum_{S\in\mathcal{S}\setminus\mathcal{S}_f}\alpha_S p_S,$$

i.e. $\mathcal{S}_{[f]}=\mathcal{S}_f$. Let $[f]$ be given by its representation as in (3.11). Note first that for any $S\in\mathcal{S}_f$ and for any pair of points $x,y\in X$ such that $f(x)+f(y)=d(x,y)$, one has equivalently

$$\sum_{S\in\mathcal{S}}\alpha_S f_S(x)+\sum_{S\in\mathcal{S}}\alpha_S f_S(y)=\sum_{S\in\mathcal{S}}\alpha_S\delta_S(x,y),$$

hence one necessarily has $S(x)\neq S(y)$ and thus by (3.12), it follows that $\mathcal{S}_f\subset\mathcal{S}_{[f]}$. Now, for the other inclusion, assume that $\mathcal{S}_f\subsetneq\mathcal{S}_{[f]}$. Since $[f]$ is a hypercube, it follows that

$$\sum_{S\in\mathcal{S}_f}\alpha_S\mathrm{E}(\delta_S)+\sum_{S\in\mathcal{S}\setminus\mathcal{S}_f}\alpha_S p_S$$

is a strict subcell of $[f]$ containing f and this contradicts the definition of $[f]$.

3.13 Remark. For (X,d) as in Remark 3.10, let us define the map

$$\lambda\colon \mathrm{E}(X,d)\to\bar{T}(\mathcal{S},\alpha)$$

by the assignement $f\mapsto\psi_f$ where ψ_f (as defined in the proof of Lemma 3.4) is depending on a choice of a representation $\sum_{S\in\mathcal{S}}\alpha_S f_S$ for f, and this choice is not unique in general. Furthermore, note that one always has $\kappa\circ\lambda=\mathrm{id}_{\mathrm{E}(X,d)}$. It follows that κ is surjective. In general however, $\lambda\circ\kappa\neq\mathrm{id}_{\bar{T}(\mathcal{S},\alpha)}$.

We go on with a more concrete description of the maps κ and λ:

3.14 Remark. Again, if (X,d) is as in Remark 3.10 and if every cell $[f]$ of $\mathrm{E}(X,d)$ is a combinatorial hypercube, note that $\kappa\colon\bar{T}(\mathcal{S},\alpha)\to\mathrm{E}(X,d)$ is given by

$$\psi\mapsto\left(\kappa(\psi):x\mapsto\sum_{S\in\mathcal{S}}2\psi(S(x))=\sum_{S\in\mathcal{S}}\alpha_S\frac{\psi(S(x))}{\alpha_S/2}=:\sum_{S\in\mathcal{S}}\alpha_S f_S(x)\right)\quad(3.15)$$

and $\lambda\colon \mathrm{E}(X,d) \to \bar{T}(\mathcal{S},\alpha)$ is given by $f \mapsto \psi_f$ where recall that ψ_f is given for any $A \in U(\mathcal{S})$ and for an arbitrarily chosen $x \in A$ by

$$\psi_f : A \mapsto \frac{\alpha_{S_A}}{2} f_{S_A}(x).$$

By Remark 3.12, it follows that one has:

$$[\kappa(\psi)] = \sum_{S\in\mathcal{S}_{\kappa(\psi)}} \alpha_S \mathrm{E}(\delta_S) + \sum_{S\in\mathcal{S}\setminus\mathcal{S}_{\kappa(\psi)}} \alpha_S p_S^{\kappa(\psi)}$$

and thus:

(*a*) $\lambda([\kappa(\psi)]) = [\psi]$,

(*b*) moreover,
$$[\kappa(\psi)] = (\kappa\circ\lambda)([\kappa(\psi)]) = \kappa([\psi])$$
where the first equality was already noted in Remark 3.13 and the second equality follows from (*a*).

(*c*) To see that κ is injective, assume $g := \kappa(\psi) = \kappa(\psi') =: g'$. Then, in particular, one has $[g] = [g']$. Thus, by (3.13) one has $\mathcal{S}_g = \mathcal{S}_{g'}$ as well as $p_S^g = p_S^{g'}$ for each $S \in \mathcal{S} \setminus \mathcal{S}_g$. It follows that $\sum_{S\in\mathcal{S}_g} \alpha_S g_S = \sum_{S\in\mathcal{S}_{g'}} \alpha_S g'_S$. Since $\sum_{S\in\mathcal{S}_g} \alpha_S \mathrm{E}(\delta_S)$ is a finite dimensional zonotope combinatorially equivalent to a hypercube, it follows that $g_S = g'_S$ for every $S \in \mathcal{S}_g$ and thus by (3.15), it follows $\psi = \psi'$. Hence together with Remark 3.13, it follows that κ is bijective with inverse λ.

(*d*) In addition:
$$\lambda([f]) = \lambda([(\kappa\circ\lambda)(f)]) = (\lambda\circ\kappa)[\lambda(f)] = [\lambda(f)]$$
where the first equality was already noted in Remark 3.13, the second equality follows from (*b*) and the last equality follows from (*c*).

(*e*) By Remarks 3.11 and 3.12, we have that $\dim([f]) = k$ implies $\dim([\lambda(f)]) = k$.

(*f*) Finally, if $\dim([\psi]) = k$, then $\dim([\kappa(\psi)]) = k$ since if we assume that $\dim([\kappa(\psi)]) > k$, it then follows as in (*e*) from Remarks 3.11 and 3.12 that $\dim(\lambda([\kappa(\psi)])) > k$ and by (*a*) it follows that $\dim([\psi]) > k$ which is a contradiction.

Hence, the map
$$\kappa\colon \bar{T}(\mathcal{S},\alpha) \to \mathrm{E}(X,d)$$
defines a bijection as well as an isomorphism of cell complexes.

In view of Remark 3.10, we have:

3.15 Lemma. *Let (X,d) be a totally split-decomposable metric space with integer-valued metric which satisfies the (LRC). For every cell $[f]$ of $\mathrm{E}(d)$, there is $\bar{f} \in [f]$ such that $[\bar{f}] = [f]$ and so that for $\psi_{\bar{f}} \in \bar{T}(S,\alpha)$ as defined in the proof of Lemma 3.4, one has*
$$[\kappa(\psi_{\bar{f}})] \subset \kappa([\psi_{\bar{f}}]).$$

Proof. By the above remark, there is $\mathcal{S}_{[f]} \subset \mathcal{S}$ such that
$$[f] = \sum_{S \in \mathcal{S}_{[f]}} \alpha_S \mathrm{E}(\delta_S) + \sum_{S \in \mathcal{S} \setminus \mathcal{S}_{[f]}} \alpha_S p_S$$
where for each $S \in \mathcal{S} \setminus \mathcal{S}_{[f]}$, one has $p_S \in \{0,1\}^X$. We can thus write
$$f = \sum_{S \in \mathcal{S}_{[f]}} \alpha_S f_S + \sum_{S \in \mathcal{S} \setminus \mathcal{S}_{[f]}} \alpha_S p_S$$
where for every $S \in \mathcal{S}$, one has $f_S \in \mathrm{E}(\delta_S)$. Let us moreover define
$$\mathcal{S}_f := \{S \in \mathcal{S} : f_S(X) \subset (0,1)\}.$$
It is clear that $\mathcal{S}_f \subset \mathcal{S}_{[f]}$. Moreover, $\mathcal{S}(\psi_f) = \mathcal{S}_f$ for ψ_f defined as in the proof of Lemma 3.4. Let us now define $\bar{f} \in [f]$ as
$$\bar{f} := \sum_{S \in \mathcal{S}_{[f]}} \alpha_S \bar{f}_S + \sum_{S \in \mathcal{S} \setminus \mathcal{S}_{[f]}} \alpha_S p_S$$
such that for any $S \in \mathcal{S}_{[f]}$, $\bar{f}_S$ is constantly equal to $\frac{1}{2}$ on X. It is then clear that $\mathcal{S}_{\bar{f}} = \mathcal{S}_{[f]}$. Since $\mathcal{S}(\psi_{\bar{f}}) = \mathcal{S}_{\bar{f}} = \mathcal{S}_{[f]} \supset \mathcal{S}_f = \mathcal{S}(\psi_f)$ and because for every $A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_{\bar{f}})) = U(\mathcal{S} \setminus \mathcal{S}_{[f]})$, one has
$$\psi_{\bar{f}}(A) = \psi_f(A),$$
it follows that $\mathrm{supp}(\psi_f) \subset \mathrm{supp}(\psi_{\bar{f}})$ and thus $\psi_f \in [\psi_{\bar{f}}]$. Now for any $g \in [f]$, one similarly has
$$\mathcal{S}(\psi_g) = \mathcal{S}_g \subset \mathcal{S}_{[f]} = \mathcal{S}(\psi_{\bar{f}})$$

as well as for every $A \in U(\mathcal{S} \setminus \mathcal{S}_{[f]}) = U(\mathcal{S} \setminus \mathcal{S}(\psi_{\bar{f}}))$

$$\psi_g(A) = \psi_{\bar{f}}(A).$$

Thus, $\mathrm{supp}(\psi_g) \subset \mathrm{supp}(\psi_{\bar{f}})$ and thus $\psi_g \in [\psi_{\bar{f}}]$, that is $g = \kappa(\psi_g) \in \kappa([\psi_{\bar{f}}])$. It follows that

$$[\kappa(\psi_{\bar{f}})] = [\bar{f}] = [f] \subset \kappa([\psi_{\bar{f}}]),$$

which finishes the proof. □

Consider the isometric embedding $\mathrm{e}\colon X \to \mathrm{E}(d)$ given by

$$x \mapsto (d_x \colon y \mapsto d(x,y))$$

where $\mathrm{E}(d)$ is endowed with the metric $d_\infty(f,g) := \|f - g\|_\infty$. Assume that (X,d) satisfies the assumptions of Lemma 3.15. We say that $\mathrm{E}(d)$ is *cell-decomposable* if every cell C of $\mathrm{E}(d)$ is X-gated (cf. Definition 3.5). Now, we have the following:

3.16 Lemma. *Let (X,d) be a totally split-decomposable metric space with integer-valued metric satisfying the (LRC). Then, $\mathrm{E}(d)$ is cell-decomposable.*

Proof. Let C be a cell of $\mathrm{E}(d) = \mathrm{E}'(d)$. By Lemmas 3.9 and 3.15, there is $\bar{f} \in C$ such that $[\bar{f}] = C$ and such that $\psi_{\bar{f}} \in \bar{T}(S,\alpha)$ (as defined in the proof of Lemma 3.4) satisfies

$$C = [\bar{f}] = [\kappa(\psi_{\bar{f}})] = \kappa([\psi_{\bar{f}}]).$$

as well as for every $g \in [\bar{f}]$:

$$\mathrm{supp}(\psi_g) \subset \mathrm{supp}(\psi_{\bar{f}}). \tag{3.16}$$

Let $x \in X$ be chosen arbitrarily. We want to show that

$$\kappa(\gamma^x_{[\psi_{\bar{f}}]}) \text{ is a gate for } d_x \text{ in } [\bar{f}].$$

For an arbitrarily chosen $f \in [\bar{f}]$, let us set $\psi := \psi_f$ which by (3.16) satisfies $\psi_f \in [\psi_{\bar{f}}]$. Now, by Lemma 3.3, there must exist $y \in X$ such that

$$\kappa(\psi_{\bar{f}})(x) + \kappa(\psi_{\bar{f}})(y) = \bar{f}(x) + \bar{f}(y) = d(x,y).$$

Since $\kappa(\psi_f) = f \in [\bar{f}]$, one has $A(\bar{f}) \subset A(f)$, cf. (1.2) and thus $\kappa(\psi)(x) + \kappa(\psi)(y) = d(x, y)$, hence

$$\begin{aligned} d_1(\phi_x, \phi_y) &= d(x, y) \\ &= \kappa(\psi)(x) + \kappa(\psi)(y) \\ &= d_1(\phi_x, \psi) + d_1(\psi, \phi_y) \\ &= d_1(\phi_x, \gamma^x_{[\psi_{\bar{f}}]}) + d_1(\gamma^x_{[\psi_{\bar{f}}]}, \psi) + d_1(\psi, \gamma^y_{[\psi_{\bar{f}}]}) + d_1(\gamma^y_{[\psi_{\bar{f}}]}, \phi_y) \end{aligned} \tag{3.17}$$

and since κ is 1-Lipschitz, it follows that

$$\begin{aligned} \|\kappa(\phi_x) - \kappa(\phi_y)\|_\infty \leq\ & \left\|\kappa(\phi_x) - \kappa(\gamma^x_{[\psi_{\bar{f}}]})\right\|_\infty + \left\|\kappa(\gamma^x_{[\psi_{\bar{f}}]}) - \kappa(\psi)\right\|_\infty \\ & + \left\|\kappa(\psi) - \kappa(\gamma^y_{[\psi_{\bar{f}}]})\right\|_\infty + \left\|\kappa(\gamma^y_{[\psi_{\bar{f}}]}) - \kappa(\phi_y)\right\|_\infty \\ \leq\ & d_1(\phi_x, \gamma^x_{[\psi_{\bar{f}}]}) + d_1(\gamma^x_{[\psi_{\bar{f}}]}, \psi) \\ & + d_1(\psi, \gamma^y_{[\psi_{\bar{f}}]}) + d_1(\gamma^y_{[\psi_{\bar{f}}]}, \phi_y) \\ =\ & d(x, y). \end{aligned} \tag{3.18}$$

It is easy to see that $\kappa(\phi_x) = d_x$ and $\kappa(\phi_y) = d_y$ as well as $\|d_x - d_y\|_\infty = d(x, y)$ which implies that both inequalities above are actual equalities. Since $\kappa(\psi_f) = f$, we thus obtain

$$\begin{aligned} \|d_x - d_y\|_\infty =\ & \left\|d_x - \kappa(\gamma^x_{[\psi_{\bar{f}}]})\right\|_\infty + \left\|\kappa(\gamma^x_{[\psi_{\bar{f}}]}) - f\right\|_\infty \\ & + \left\|f - \kappa(\gamma^y_{[\psi_{\bar{f}}]})\right\|_\infty + \left\|\kappa(\gamma^y_{[\psi_{\bar{f}}]}) - d_y\right\|_\infty . \end{aligned}$$

In particular

$$\left\|d_x - \kappa(\gamma^x_{[\psi_{\bar{f}}]})\right\|_\infty + \left\|\kappa(\gamma^x_{[\psi_{\bar{f}}]}) - f\right\|_\infty = \|d_x - f\|_\infty$$

and this proves that $\kappa(\gamma^x_{[\psi_{\bar{f}}]})$ is a gate for d_x in $[\bar{f}]$. This is the desired result. □

It is easy to see that [27, Theorem 1.1] generalizes to the case where $|X| = \infty$ as long as $\mathrm{E}'(X, d) = \mathrm{E}(X, d)$. To be self-contained, we give a proof of the theorem.

3.17 Theorem. *Let (X, d) be a metric space with integer-valued metric satisfying the (LRC). If $f \in \mathrm{E}(X, d)$ is such that $[f]$ is X-gated, then the following hold:*

(i) $(G([f]), d_\infty)$ *is a finite antipodal metric space.*

(ii) *The map* $\Phi\colon ([f], d_\infty) \to \mathrm{E}(G([f]), d_\infty)$ *given by*

$$g \mapsto \left(\gamma^x_{[f]} \mapsto g(x) - \gamma^x_{[f]}(x)\right) \tag{3.19}$$

is a bijective isometry as well as an isomorphism of polytopes.

Proof. If $\operatorname{rank}(A(f)) = 0$, the result clearly holds, hence let $f \in \mathrm{E}(d)$ be such that $\operatorname{rank}(A(f)) \geq 1$.

We first show that $(G([f]), d_\infty)$ is an antipodal metric space. For each $x \in X$, consider

$$\varrho(x) := \{y \in X : \{x, y\} \in A(f)\}.$$

We define the map $\sigma\colon G([f]) \to G([f])$ by sending every gate $\gamma^x_{[f]}$ to the gate $\gamma^y_{[f]}$ where $y \in \varrho(x)$ is chosen arbitrarily. To see that σ is well-defined, note that for every $g \in [f]$ and if $\{x, y\} \in A(g)$, one has

$$\begin{aligned} d_\infty(d_x, d_y) &= d(x, y) \\ &= g(x) + g(y) \\ &= d_\infty(d_x, g) + d_\infty(d_y, g) \\ &= d_\infty(d_x, \gamma^x_{[f]}) + d_\infty(\gamma^x_{[f]}, g) + d_\infty(g, \gamma^y_{[f]}) + d_\infty(\gamma^y_{[f]}, d_y) \end{aligned} \tag{3.20}$$

which rearranging and using the triangle inequality gives that for any $y \in \varrho(x)$ and any $g \in [f]$, one has

$$d_\infty(\gamma^x_{[f]}, \gamma^y_{[f]}) = d_\infty(\gamma^x_{[f]}, g) + d_\infty(g, \gamma^y_{[f]}). \tag{3.21}$$

It follows in particular from (3.21) that $(X, A(f))$ has no odd $A(f)$-component. Now, let $x' \in X$ be such that $\gamma^{x'}_{[f]} = \gamma^x_{[f]}$. For any $y' \in \varrho(x')$, one can use (3.21) to obtain

$$\begin{aligned} d_\infty(\gamma^{x'}_{[f]}, \gamma^{y'}_{[f]}) = d_\infty(\gamma^x_{[f]}, \gamma^{y'}_{[f]}) &= d_\infty(\gamma^x_{[f]}, \gamma^y_{[f]}) - d_\infty(\gamma^y_{[f]}, \gamma^{y'}_{[f]}) \\ &= d_\infty(\gamma^{x'}_{[f]}, \gamma^y_{[f]}) - d_\infty(\gamma^y_{[f]}, \gamma^{y'}_{[f]}) \\ &= d_\infty(\gamma^{x'}_{[f]}, \gamma^{y'}_{[f]}) - 2d_\infty(\gamma^y_{[f]}, \gamma^{y'}_{[f]}) \end{aligned}$$

which implies that $\gamma^{y'}_{[f]} = \gamma^y_{[f]}$, and this is the desired result. This proves that σ is well-defined. It is now clear that σ is an involution which turns $(G([f]), d_\infty)$ into an antipodal metric space. It is finite since $(X, A(f))$ has only finitely many even $A(f)$-components.

We now show that Φ defines a bijective isometry. Note that Φ can also be expressed as

$$\Phi(g) : \gamma^x_{[f]} \mapsto d_\infty(g, d_x) - d_\infty(\gamma^x_{[f]}, d_x) \tag{3.22}$$

and also as

$$\Phi(g) = d^g_\infty : \gamma^x_{[f]} \mapsto d_\infty(g, \gamma^x_{[f]}) \tag{3.23}$$

It is easy to see by (3.23) that Φ is well-defined, i.e. it does not depend on the choice of x or x' as long as $\gamma^x_{[f]} = \gamma^{x'}_{[f]}$. Moreover, for every $g \in [f]$, one clearly has $\Phi(g) \in \Delta(G([f]), d_\infty)$ by (3.23). As a consequence of (3.21) and (3.23), for every $x \in X$ one has

$$\Phi(g)(\gamma^x_{[f]}) + \Phi(g)(\sigma(\gamma^x_{[f]})) = d(\gamma^x_{[f]}, \sigma(\gamma^x_{[f]})),$$

which shows that $\Phi(g) \in \mathrm{E}(G([f]), d_\infty)$. To see that Φ is surjective, let us define for any $h \in \mathrm{E}(G([f]), d_\infty)$, the associated function

$$g' : x \mapsto h(\gamma^x_{[f]}) + \gamma^x_{[f]}(x).$$

We clearly have $\Phi(g') = h$ and thus we only need to show that $g' \in [f]$. We have

$$\begin{aligned} g'(x) + g'(y) &= h(\gamma^x_{[f]}) + \gamma^x_{[f]}(x) + h(\gamma^y_{[f]}) + \gamma^y_{[f]}(y) \\ &\geq d_\infty(\gamma^x_{[f]}, \gamma^y_{[f]}) + d_\infty(\gamma^x_{[f]}, d_x) + d_\infty(\gamma^y_{[f]}, d_y) \\ &\geq d_\infty(d_x, d_y) \\ &= d(x, y), \end{aligned} \tag{3.24}$$

hence in particular $g' \in \Delta(d)$. If $\{x, y\} \in A(f)$, one has by (3.21)

$$\begin{aligned} &d_\infty(\gamma^x_{[f]}, \gamma^y_{[f]}) + d_\infty(\gamma^x_{[f]}, d_x) + d_\infty(\gamma^y_{[f]}, d_y) \\ &= d_\infty(\gamma^x_{[f]}, f) + d_\infty(f, \gamma^y_{[f]}) + d_\infty(\gamma^x_{[f]}, d_x) + d_\infty(\gamma^y_{[f]}, d_y) \\ &= d_\infty(f, d_x) + d_\infty(f, d_y) \\ &= f(x) + f(y) \\ &= d(x, y), \end{aligned}$$

and since $h \in \mathrm{E}(G([f]), d_\infty)$, one has

$$h(\gamma^x_{[f]}) + h(\sigma(\gamma^x_{[f]})) = d_\infty(\gamma^x_{[f]}, \sigma(\gamma^x_{[f]}))$$

(recall that for any extremal function f, if $\{x, y\} \in A(f)$ and $xy + yz = xz$, then $\{x, z\} \in A(f)$). Hence if $\{x, y\} \in A(f)$, one can replace all inequalities in

(3.24) by equalities. This shows that $g' \in [f]$ and thus Φ is surjective. Now, it is easy to see that Φ preserves distances since for every $g, h \in [f]$, one has by (3.22) that

$$g(x) - h(x) = g(x) - \gamma^x_{[f]}(x) - \Big(h(x) - \gamma^x_{[f]}(x)\Big) = \Phi(g)(\gamma^x_{[f]}) - \Phi(h)(\gamma^x_{[f]})$$

and thus Φ is a bijective isometry. Note now that by [30] there is an affine isometry $\alpha\colon ([f], d_\infty) \to l^n_\infty$ and since $\mathrm{E}(G([f]), d_\infty)$ consists of a unique maximal cell, another affine isometry $\beta\colon \mathrm{E}(G([f]), d_\infty) \to l^n_\infty$. It follows that the map

$$\beta \circ \Phi \circ \alpha^{-1}\colon \alpha([f]) \to \beta(\mathrm{E}(G([f]), d_\infty))$$

is a bijective isometry between convex subsets (with non-empty interior) of finite dimensional normed spaces. It follows by an extension of Mazur-Ulam Theorem (cf. [33]) that $\beta \circ \Phi \circ \alpha^{-1}$ is the restriction of an affine bijective isometry. It follows that Φ has the same property and is thus in particular a polytope isomorphism. □

Let (X, d) be a totally split-decomposable metric space with integer-valued metric satisfying the (LRC). By Lemma 3.16, we know that every cell $[f] \subset \mathrm{E}'(d) = \mathrm{E}(d)$ is X-gated. Hence if $(G([f]), d_\infty)$ denotes the set of all X-gates of $[f]$, and if d_∞ denotes the metric $d_\infty(f, g) = \|f - g\|_\infty$ (we adopt the same notation for restrictions of d_∞), we obtain by Theorem 3.17 that the following hold:

1) $(G([f]), d_\infty)$ is an antipodal metric space.

2) $[f]$ and $\mathrm{E}(G([f]), d_\infty)$ are combinatorially equivalent polytopes.

If we assume that $\dim([f]) = n$, then since $G([f]) \subset [f]$, it follows (cf. [30, Proposition 3.5 and Theorem 4.3 (1)]) that $\mathrm{E}(G([f]), d_\infty)$ isometrically embeds into $([f], d_\infty)$ through

$$\mathrm{E}(G([f]), d_\infty) \hookrightarrow \mathrm{E}([f]) \cong [f].$$

Thus in particular, one has

$$\dim(\mathrm{E}(G([f]), d_\infty)) \leq n \tag{3.25}$$

Finally, this implies by [27, Theorem 1.2] that

$$|G([f])| \leq 2n.$$

Indeed, if $|G([f])| \geq 2(n+1)$ (possibly $|G([f])| = \infty$), then since $(G([f]), d_\infty)$ is antipodal, we can select $(n+1)$ pairs of antipodal points in $G([f])$ to obtain an antipodal metric space (A, d_∞) with $A \subset G([f])$ and $|A| = 2(n+1)$. It follows by [30, Proposition 3.5] that $\mathrm{E}(A, d_\infty)$ is isometrically embedded in $\mathrm{E}(G([f]), d_\infty)$ and thus again by [27, Theorem 1.2], one has

$$n + 1 \leq \dim(\mathrm{E}(A, d_\infty)) \leq \dim(\mathrm{E}(G([f]), d_\infty)$$

which contradicts (3.25). Hence, with (2) above, this proves that

$$|G([f])| \leq 2\dim([f]).$$

From Lemma 3.18, the following follows.

3.18 Lemma. *Let $\kappa\colon (A, d) \to (A', d')$ be a map of metric spaces such that the following hold:*

(i) κ is 1-Lipschitz,

(ii) κ is surjective,

(iii) (A, d) is an antipodal metric space and

(iv) for any $x \in A$, there is $y \in A$ antipodal to x such that

$$d(x, y) = d'(\kappa(x), \kappa(y)).$$

Then, it follows that κ is an isometry.

Proof. Let $x, z \in A$ be chosen arbitrarily. By *(iv)*, there is y antipodal to x such that $d(x, y) = d'(\kappa(x), \kappa(y))$. Hence, one has:

$$\begin{aligned} d(x, z) + d(z, y) &= d(x, y) \\ &= d'(\kappa(x), \kappa(y)) \\ &\leq d'(\kappa(x), \kappa(z)) + d'(\kappa(z), \kappa(y)) \\ &\leq d(x, z) + d(z, y) \\ &= d(x, y). \end{aligned}$$

It follows that the above inequalities are actual equalities and using again that κ is 1-Lipschitz, it follows that

$$d'(\kappa(x), \kappa(z)) = d(x, z).$$

Since x and z were chosen arbitrarily, this proves the result. □

3.19 Lemma. *Let (X,d) be a totally split-decomposable metric space with integer-valued metric satisfying the (LRC). Let $[f]$ be any positive dimensional cell of $\mathrm{E}(d)$ and let $\psi_{\bar{f}}$ as well as $\bar{f}$ be defined as in the proofs of Lemmas 3.4 and 3.15. Then, the map*

$$\bar{\kappa} := \kappa|_{\Gamma([\psi_{\bar{f}}])} \colon (\Gamma([\psi_{\bar{f}}]), d_1) \to (G([f]), d_\infty)$$

is an isometry.

Proof. We already know that $\bar{\kappa}$ is 1-Lipschitz and it is surjective by the proof of Lemma 3.16. Now, for any $x \in X$, there is $y \in X$ such that

$$\bar{f}(x) + \bar{f}(y) = d(x,y).$$

By the proofs of Lemmas 3.7 and 3.8, it follows (by definition $\kappa(\psi_{\bar{f}}) = \bar{f}$) that $\gamma^x_{[\psi_{\bar{f}}]}$ and $\gamma^y_{[\psi_{\bar{f}}]}$ are antipodal in $(\Gamma([\psi_{\bar{f}}]), d_1)$. Furthermore, by (3.17) and (3.18), one has

$$d_1(\gamma^x_{[\psi_{\bar{f}}]}, \gamma^y_{[\psi_{\bar{f}}]}) = \left\| \kappa(\gamma^x_{[\psi_{\bar{f}}]}) - \kappa(\gamma^y_{[\psi_{\bar{f}}]}) \right\|_\infty .$$

We can thus apply Lemma 3.18 to deduce that $\bar{\kappa}$ is an isometry. □

Under the assumptions of Lemma 3.19. For $x, y \in X$ arbitrarily chosen, it follows from the definitions of $\gamma^x_{[\psi_{\bar{f}}]}$ and $\gamma^y_{[\psi_{\bar{f}}]}$, that one has

$$d_1(\gamma^x_{[\psi_{\bar{f}}]}, \gamma^y_{[\psi_{\bar{f}}]}) = \sum_{S \in \mathcal{S}(\psi_{\bar{f}})} \alpha_S \delta_S(x,y) \tag{3.26}$$

where $\mathcal{S}(\psi_{\bar{f}})$ is weakly compatible. It follows by Theorem 2.7 that

$$(\Gamma([\psi_{\bar{f}}]), d_1) \text{ is a totally split-decomposable metric space.}$$

Moreover, for any metric space (X,d), the *underlying graph* $\mathrm{UG}(X,d)$ of (X,d) is the graph (X,E) where $\{x,y\} \in E$ if and only if $d(x,z) + d(z,y) > d(x,y)$ for any $z \in X \setminus \{x,y\}$. Furthermore, let C_6 denote the 6-cycle metric graph and let $K_{3\times 2}$ denote the complete graph on six vertices with 3 disjoint edges taken away (i.e., the 1-skeleton of the octahedron).

3.20 Remark. Note that if $\mathcal{S}$ is an antipodal split system on (X,d), then for any $(A_i)_{i\in I}$, if $\bigcup_{i\in I} A_i = X$, it follows that $\bigcap_{i\in I} A_i = \emptyset$. Indeed, if $x \in \bigcap_{i\in I} A_i$, there is a subsystem of pairwise different splits $\{S_i\}_{i\in I} \subset \mathcal{S}$ such that $A_i = S_i(x)$. Now, there is $y \in X$ such that $y \in \bigcap_{S\in\mathcal{S}} \overline{S(x)} \subset \bigcap_{i\in I} \overline{S_i(x)} = \bigcap_{i\in I} A_i^c = (\bigcup_{i\in I} A_i)^c$, which implies that $\bigcup_{i\in I} A_i \neq X$. The octahedral split system is an example of antipodal split system.

We conclude this section with a proof.

Proof of Theorem 1.1. The second part of Theorem 1.1, namely the existence of $\mathrm{K}(X,d)$ and σ follows immediately from Theorem 4.4 (which is proved in the next section). Indeed, Theorem 4.4 implies that if we remetrize $\mathrm{E}(X,d)$ by identifying each cell (which is a parallelotope by the first part of Theorem 1.1) with a corresponding unit hypercube (of same dimension) endowed with the euclidean metric, and considering the induced length metric, we obtain a complex $\mathrm{K}(X,d)$ which satisfies the CAT(0) link condition. Since (X,d) satisfies the (LRC), it follows that $\mathrm{K}(X,d)$ is complete and locally CAT(0) (analogue to I.7.13 Theorem and II.5.2 Theorem in [7]). By the (LRC), it also follows that $\mathrm{K}(X,d)$ is locally bi-Lipschitz equivalent to $\mathrm{E}(X,d)$, the topology induced by the length metric on $\mathrm{K}(X,d)$ is therefore the same as the topology on $\mathrm{E}(X,d)$ and thus $\mathrm{K}(X,d)$ is contractible as well. By Cartan-Hadamard Theorem, it follows that $\mathrm{K}(X,d)$ is globally CAT(0).

We now prove the first part of Theorem 1.1. As an introductory remark, note that by Lemma 3.8, Lemma 3.19 and (3.26), it follows that $(G([f]),d_\infty)$ is an antipodal totally split-decomposable metric space with $2\dim([f])$ elements. By Theorem 3.17, $([f],d_\infty)$ is combinatorially equivalent to $\mathrm{E}(G([f]),d_\infty)$, which is by [27, Theorem 1.2] an n-dimensional combinatorial hypercube if $|G([f])| = 2n \geq 8$. Moreover, if $|G([f])| \leq 4$, then $\mathrm{E}(G([f]),d_\infty)$ is clearly a combinatorial hypercube as well. Now, assume that $|G([f])| = 6$. Since $(G([f]),d_\infty)$ is antipodal, it follows by [26, Corollary 3.3] that $\mathrm{UG}(G([f]),d_\infty)$ is either $K_{3\times 2}$ or C_6. If $\mathrm{UG}(G([f]),d_\infty) = C_6$, then by [27, Theorem 1.2 (a)], $\mathrm{E}(G([f]),d_\infty)$ is a 3-dimensional combinatorial hypercube.

Assume now that $\mathrm{E}(G([f]),d_\infty)$ is a combinatorial rhombic dodecahedron, i.e. (ii) in Theorem 1.1 does not hold, then $\mathrm{UG}(G([f]),d_\infty) = K_{3\times 2}$ and it follows by the proof of [26, Theorem 5.1, Case 2] that

$$d_\infty(\kappa(\gamma^x_{[\psi_{\bar f}]}),\kappa(\gamma^y_{[\psi_{\bar f}]})) = \sum_{S\in\{S_1,S_2,S_3,S_4\}} \beta_S\delta_S(x,y)$$

where $\{S_1,S_2,S_3,S_4\}$ is weakly compatible and the coefficients β_S are all positive. Moreover, by (3.26) and Lemma 3.19, we have

$$d_\infty(\kappa(\gamma^x_{[\psi_{\bar f}]}),\kappa(\gamma^y_{[\psi_{\bar f}]})) = \sum_{S\in\mathcal{S}(\psi_{\bar f})} \alpha_S\delta_S(x,y)$$

where $\mathcal{S}(\psi_{\bar f})$ is weakly compatible and consists of d-splits of X. Note that the metric d_∞ on $G([f])$ induces a pseudometric $\bar d$ on X by setting

$$\bar d(x,y) := d_\infty(\kappa(\gamma^x_{[\psi_{\bar f}]}),\kappa(\gamma^y_{[\psi_{\bar f}]})).$$

It follows by Theorem 2.7 and approximation by rescalings of integer-valued pseudometrics that the split systems $\mathcal{S}(\psi_{\bar{f}})$ and $\{S_1, S_2, S_3, S_4\}$ each consist of all the $\bar{d}$-splits of X, which implies that $\mathcal{S}(\psi_{\bar{f}}) = \{S_1, S_2, S_3, S_4\}$. Therefore, the split system $\bar{\mathcal{S}} := \{S_1, S_2, S_3, S_4\}$ consists of d-splits of X and thus it is an octahedral split subsystem of $\mathcal{S}$. We can write the splits in $\bar{S}$ as

$$\begin{aligned}
S_1 &:= \{Y_1^1 \sqcup Y_2^1 \sqcup Y_3^1, Y_1^{-1} \sqcup Y_2^{-1} \sqcup Y_3^{-1}\},\\
S_2 &:= \{Y_1^1 \sqcup Y_2^1 \sqcup Y_3^{-1}, Y_1^{-1} \sqcup Y_2^{-1} \sqcup Y_3^1\},\\
S_3 &:= \{Y_1^1 \sqcup Y_2^{-1} \sqcup Y_3^1, Y_1^{-1} \sqcup Y_2^1 \sqcup Y_3^{-1}\},\\
S_4 &:= \{Y_1^1 \sqcup Y_2^{-1} \sqcup Y_3^{-1}, Y_1^{-1} \sqcup Y_2^1 \sqcup Y_3^1\},
\end{aligned} \tag{3.27}$$

where $X = Y_1^1 \sqcup Y_1^{-1} \sqcup Y_2^1 \sqcup Y_2^{-1} \sqcup Y_3^1 \sqcup Y_3^{-1}$ and all sets being non-empty. From our introductory remark and since we have assumed that $[f]$ is a combinatorial rhombic dodecahedron, then $[f]$ must be a maximal cell since as we have already seen, in dimensions strictly higher than three, a cell must be a hypercube and the same holds for all of its faces. Since our assumptions imply $\mathrm{E}(d) = \mathrm{E}'(d)$ and $[f]$ is a three dimensional maximal cell, it follows from Theorem V.1.2 that the graph $(X, A(f))$ consists of three complete bipartite connected components that are given by their respective partitions, namely $Y_1^1 \sqcup Y_1^{-1}$, $Y_2^1 \sqcup Y_2^{-1}$ and $Y_3^1 \sqcup Y_3^{-1}$. For each $S := \{A, B\} \in \mathcal{S} \setminus \bar{\mathcal{S}}$, there is $\{x, y\} \in A(f)$ such that $S(x) = S(y)$ by bipartiteness, let us say $\{x, y\} \subset A$. It follows from (3.1) that $\psi := \psi_{\bar{f}} \in \bar{T}(\mathcal{S}, \alpha)$ (where $\psi_{\bar{f}}$ is as defined in the proof of Lemma 3.4, in particular $[\bar{f}] = [f]$ where $\kappa(\psi) = \bar{f}$) satisfies then $\psi(A) = 0$. Hence, $\psi(B) = \frac{\alpha_S}{2}$ and thus for every further $\{x', y'\} \in A(f)$, one has $\{x', y'\} \not\subset B$. This implies by bipartite completeness of $Y_i^1 \sqcup Y_i^{-1}$ that there are $\sigma, \tau, \theta \in \{\pm 1\}$ such that $Y_1^\sigma \cup Y_2^\tau \cup Y_3^\theta \subsetneq A$ which is equivalent to $\{A, B\}$ and $\{A', B'\} = \{Y_1^\sigma \cup Y_2^\tau \cup Y_3^\theta, Y_1^{-\sigma} \cup Y_2^{-\tau} \cup Y_3^{-\theta}\} \in \bar{\mathcal{S}}$ being compatible (i.e., $A' \subset A$). It follows that (i) in Theorem 1.1 does not hold.

Conversely, assume that (i) does not hold and thus there exists such a split subsystem $\bar{\mathcal{S}}$ with the properties stated in (i). Define $\psi \in H(\mathcal{S}, \alpha)$ so that $\mathcal{S}(\psi) = \bar{\mathcal{S}}$ and $\bar{\mathcal{S}}$ consists of four splits as given in (3.27) and is a converse to (i). We can choose ψ so that for any $S := \{A, B\} \in \mathcal{S} \setminus \bar{\mathcal{S}}$, one additionally has $\psi(A) = 0 \quad \text{for} \quad Y_1^\sigma \cup Y_2^\tau \cup Y_3^\theta \subsetneq A$ and accordingly $\psi(B) = \frac{\alpha_S}{2}$. One has $\psi \in \bar{T}(\mathcal{S}, \alpha)$ since for any $(C_i)_{i \in I} \subset \mathrm{supp}(\psi)$, we can consider for each $i \in I$, a corresponding $S_i' = \{A_i', B_i'\} \in \bar{\mathcal{S}}$ such that $C_i \subset B_i' =: D_i$. It follows that if $\cup_{i \in I} C_i = X$, then $\cup_{i \in I} D_i = X$ and thus by Remark 3.20, it follows that $\cap_{i \in I} D_i = \emptyset$ which implies $\cap_{i \in I} C_i = \emptyset$. It is then easy to see that $[\kappa(\psi)]$ is a combinatorial rhombic dodecahedron since for any $(x, y) \in Y_i^1 \times Y_i^{-1}$, one has that $S(x) = S(y)$ implies that $S \in \mathcal{S} \setminus \mathcal{S}(\psi)$ and by definition of ψ we have $\psi(S(x)) = 0 = \psi(S(y))$ but since $\psi \in H(\mathcal{S}, \alpha)$, we have equality in (3.1), which implies that $\{x, y\} \in A(\kappa(\psi))$. This means that $(X, A(\kappa(\psi)))$ consists of the three complete bipartite connected components $X = \cup_{i\in\{1,2,3\}}(Y_i^1 \sqcup Y_i^{-1})$ which implies that $\dim([\kappa(\psi)]) = 3$. Setting $f := \kappa(\psi) \in \mathrm{E}(d)$, it is easy to see that we have the decomposition $f := \sum_{S\in\mathcal{S}} \alpha_S f_S$ so that for $S := \{A, B\} \in \mathcal{S}$, one has

$$f_S(z) = \begin{cases} \frac{\psi(A)}{\alpha_S/2} & \text{if} \quad z \in A, \\ \\ \frac{\psi(B)}{\alpha_S/2} & \text{if} \quad z \in B, \end{cases} \tag{3.28}$$

and $\psi_f = \psi$ holds (ψ_f is defined in the proof of Lemma 3.4). By (3.12) and (3.13), we have (in the notation of the proof of Lemma 3.15) that $\mathcal{S}_f \subset \mathcal{S}_{[f]}$ and $|\mathcal{S}_{[f]}| = 4$. But (3.28) shows that $|\mathcal{S}_f| = 4$ since $|\mathcal{S}(\psi)| = 4$. It follows that $\mathcal{S}_f = \mathcal{S}_{[f]}$ and thus we can set $\bar{f} := f$. We then have $\psi_{\bar{f}} = \psi_f = \psi$ and with Lemma 3.19 we obtain that $[f]$ is a combinatorial rhombic dodecahedron and thus (ii) in Theorem 1.1 does not hold either. This finishes the proof. □

IV.4 The CAT(0) Link Condition for the Buneman Complex and the Cubical Injective Hull

We start by considering $B(\mathcal{S}, \alpha)$ which displays some similarities with the CAT(0) cube complex that is constructed in [11] and denoted by X. The next definition is a combinatorial characterization of the local CAT(0) condition for cube complexes, cf. [12].

4.1 Definition. A cell complex K as in Definition 3.1, whose finite dimensional cells are combinatorial hypercubes, is said to satisfy the CAT(0) *link condition* if for every set of seven cells $C, C_1^1, C_1^2, C_1^3, C_2^1, C_2^2, C_2^3$ of K, such that the following hold:

(A) $C = \bigcap_{i\in\{1,2,3\}} C_2^i$,

(B) $C_1^j = \bigcap_{i\in\{1,2,3\}\setminus\{j\}} C_2^i$,

(C) $\dim(C) = k \geq 0$ and for each $i \in \{1, 2, 3\}$, one has $\dim(C_1^i) = k + 1$ as well as $\dim(C_2^i) = k + 2$,

there exists a cell $\bar{C}$ of K such that $\dim(\bar{C}) = k + 3$ and $\bigcup_{i\in\{1,2,3\}} C_2^i \subset \bar{C}$.

We proceed to the next observation which is a direct consequence of the definitions.

4.2 Lemma. *Let $(\mathcal{S}, \alpha)$ be a split system pair on a set X. Then, the Buneman complex $B(\mathcal{S}, \alpha)$ satisfies the CAT(0) link condition.*

Proof. Let

$$[\mu], [\mu_1^1], [\mu_1^2], [\mu_1^3], [\mu_2^1], [\mu_2^2], [\mu_2^3] \subset B(\mathcal{S}, \alpha)$$

be cells of $B(\mathcal{S}, \alpha)$ such that the following hold:

(i) $\text{supp}(\mu) = \bigcap_{i \in \{1,2,3\}} \text{supp}(\mu_2^i)$,

(ii) $\text{supp}(\mu_1^j) = \bigcap_{i \in \{1,2,3\} \setminus \{j\}} \text{supp}(\mu_2^i)$,

(iii) $\dim([\mu]) = |\mathcal{S}(\mu)| = k \geq 0$ and for each $j \in \{1, 2, 3\}$, one has

$$\dim([\mu_1^j]) = |\mathcal{S}(\mu_1^j)| = k + 1,$$

as well as

$$\dim([\mu_2^j]) = |\mathcal{S}(\mu_2^j)| = k + 2.$$

This implies that there are splits $\{S_1, S_2, S_3\} \subset \mathcal{S} \setminus \mathcal{S}(\mu)$ such that

(i) for $j \in \{1, 2, 3\}$, one has $\text{supp}(\mu_1^j) = \text{supp}(\mu) \cup S_j$ and

(ii) for $i \in \{1, 2, 3\}$, $\text{supp}(\mu_2^i) = \text{supp}(\mu) \cup \bigcup_{j \in \{1,2,3\} \setminus \{i\}} S_j$.

But now, pick $\psi \in H(\mathcal{S}, \alpha)$ such that

$$\text{supp}(\psi) = \text{supp}(\mu) \cup \bigcup_{j \in \{1,2,3\}} S_j = \bigcup_{j \in \{1,2,3\}} \text{supp}(\mu_2^j).$$

It is then very easy to check that $\psi \in B(\mathcal{S}, \alpha)$ and thus $[\psi]$ is a cell of $B(\mathcal{S}, \alpha)$. Moreover, one has by definition:

(i) $\bigcup_{i \in \{1,2,3\}} [\mu_2^i] \subset [\psi]$ and

(ii) $\dim([\psi]) = |\mathcal{S}(\psi)| = k + 3$.

This finishes the proof of the CAT(0) link condition for $B(\mathcal{S}, \alpha)$. □

Recall that a split system $\mathcal{S}$ is called antipodal if for every $x \in X$, there is $y \in X$ such that for every $S \in \mathcal{S}$, one has

$$S(x) \neq S(y).$$

As a preliminary to the proof of Theorem 4.4, we have the following:

4.3 Lemma. *Let $\mathcal{S}$ be a split system on a set X. Then,*

(i) Assume that $\mathcal{S}$ is a weakly compatible split system and assume that for all $i \in \{1, 2, 3\}$, the split system $\mathcal{S}(\mu_2^i) = \mathcal{S}(\mu) \cup [\{S_1, S_2, S_3\} \setminus \{S_i\}]$ is antipodal. Then, $\mathcal{S}(\psi) := \mathcal{S}(\mu) \cup \{S_1, S_2, S_3\}$ is antipodal as well.

(ii) Let (X, d) be a totally split-decomposable metric space (hence in particular, $\mathcal{S}$ is weakly compatible). Let $\{f_2^i\}_{i\in\{1,2,3\}} \subset \mathrm{E}'(d)$ be such that for the split systems given in (i), one has: $\mathcal{S}(\mu_2^i) = \mathcal{S}(\psi_{f_2^i})$. Then, for every $x \in X$, one can find $y \in X$ so that the following hold:

(a) For some $i \in \{1, 2, 3\}$, one has $\{x, y\} \in A(f_2^i)$.

(b) For every $S \in \mathcal{S}(\psi)$, one has $S(x) \neq S(y)$.

Proof. Let $x \in X$ be arbitrarily chosen. Since for every $i \in \{1, 2, 3\}$, $\mathcal{S}(\mu_2^i)$ is antipodal, there is $y_2^i \in X$ such that for every $S \in \mathcal{S}(\mu_2^i)$, one has

$$S(x) \neq S(y_2^i).$$

Note now that if $S_i(x) \neq S_i(y_2^i)$, then for every $S \in \mathcal{S}(\psi)$, one has

$$S(x) \neq S(y_2^i).$$

Moreover, if $y_2^i = y_2^j$ with $i \neq j$, then

$$S_i(x) \neq S_i(y_2^j) = S_i(y_2^i)$$

and hence as above it follows that for every $S \in \mathcal{S}(\psi)$, one has

$$S(x) \neq S(y_2^i).$$

If we now assume that there are pairwise different points $\{x, y_2^1, y_2^2, y_2^3\}$ such that

$$S_i(x) = S_i(y_2^j) \Longleftrightarrow i = j,$$

then it follows that the points $\{x, y_2^1, y_2^2, y_2^3\}$ and the splits $\{S_1, S_2, S_3\}$ contradict the weak compatibility of $\mathcal{S}$. This proves the first assertion.

The second assertion follows from the fact that by the last part of the statement of Lemma 3.7, if for each $i \in \{1, 2, 3\}$, we pick $y_2^i \in X$ such that $\{x, y_2^i\} \in A(f_2^i)$, then $S(x) \neq S(y_2^i)$ for every $S \in \mathcal{S}(\mu_2^i)$, and thus by the above proof, we deduce that for some $i \in \{1, 2, 3\}$, one has $S(x) \neq S(y_2^i)$ for every $S \in \mathcal{S}(\psi)$ which implies that if we set $y := y_2^i$, the second assertion follows. □

We now have:

4.4 Theorem. *Let (X, d) be a metric space with integer-valued totally split-decomposable metric satisfying the (LRC) and such that each cell of* $\mathrm{E}(X, d)$ *is a combinatorial hypercube. Then,* $\mathrm{E}(X, d)$ *satisfies the CAT(0) link condition.*

Proof. Assume that there are cells

$$[f], [f_1^1], [f_1^2], [f_1^3], [f_2^1], [f_2^2], [f_2^3] \subset \mathrm{E}'(d)$$

such that

(*i*) $[f] = \bigcap_{i \in \{1,2,3\}} \operatorname{supp}[f_2^i]$,

(*ii*) $[f_1^j] = \bigcap_{i \in \{1,2,3\} \setminus \{j\}} \operatorname{supp}[f_2^i]$,

(*iii*) $\dim([f]) = k \geq 0$ and for each $i \in \{1, 2, 3\}$, one has $\dim([f_1^i]) = k + 1$ as well as $\dim([f_2^i]) = k + 2$.

Since all cells of $\mathrm{E}'(d)$ are hypercubes, it is easy to see (using Remark 3.11 to see that $\dim([\psi_f]) = |\mathcal{S}(\psi_f)| = k$ as well as for the other similar equalities in (*iii*)) that one has

(*i*) $[\psi_f] = \bigcap_{i \in \{1,2,3\}} \operatorname{supp}[\psi_{f_2^i}]$,

(*ii*) $[\psi_{f_1^j}] = \bigcap_{i \in \{1,2,3\} \setminus \{j\}} \operatorname{supp}[\psi_{f_2^i}]$,

(*iii*) $\dim([\psi_f]) = |\mathcal{S}(\psi_f)| = k$ and for each $j \in \{1, 2, 3\}$, one has $\dim([\psi_{f_1^j}]) = |\mathcal{S}(\psi_{f_1^j})| = k + 1$ as well as $\dim([\psi_{f_2^i}]) = |\mathcal{S}(\psi_{f_2^i})| = k + 2$.

By Lemma 4.2, there is $\psi \in B(\mathcal{S}, \alpha)$ such that

(*a*) $\operatorname{supp}(\psi) = \operatorname{supp}(\psi_f) \cup \bigcup_{i \in \{1,2,3\}} S_i$ and

(*b*) $\mathcal{S}(\psi) = \mathcal{S}(\psi_f) \cup \{S_1, S_2, S_3\}$.

Let $x \in X$ be chosen arbitrarily, by Lemma 4.3 there is $y \in X$ such that for every $S \in \mathcal{S}(\psi)$, one has $S(x) \neq S(y)$ and without loss of generality $\{x, y\} \in A(f_2^1)$. For the sake of simplicity, we set $g := f_2^1$. It follows from (3.9) that

$$d_1(\phi_x, \phi_y) = d_1(\phi_x, \gamma^x_{[\psi_g]}) + d_1(\gamma^x_{[\psi_g]}, \gamma^y_{[\psi_g]}) + d_1(\gamma^y_{[\psi_g]}, \phi_y)$$

which is easily seen to imply

$$\sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi_g)} \alpha_S \delta_S(x,y) = d_1(\phi_x, \gamma^x_{[\psi_g]}) + d_1(\phi_y, \gamma^y_{[\psi_g]}). \tag{4.1}$$

On the other hand, starting from the definition of ϕ_x and $\gamma^x_{[\psi_g]}$, we have

$$d_1(\phi_x, \gamma^x_{[\psi_g]}) = \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x \in A}} |0 - \psi_g(A)| + \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x \notin A}} \left| \frac{\alpha_{S_A}}{2} - \psi_g(A) \right|$$

and analogously

$$d_1(\phi_y, \gamma^y_{[\psi_g]}) = \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ y \in A}} |0 - \psi_g(A)| + \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ y \notin A}} \left| \frac{\alpha_{S_A}}{2} - \psi_g(A) \right|.$$

Hence, using the fact that $g \in \mathrm{E}'(d)$ and thus $\psi_g \in H(\mathcal{S}, \alpha)$, cf. proof of Lemma 3.4, we obtain a second expression for the right-hand side of (4.1), namely

$$\begin{aligned} d_1(\phi_x, \gamma^x_{[\psi_g]}) + d_1(\phi_y, \gamma^y_{[\psi_g]}) = & \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x,y \in A}} 2\psi_g(A) \\ & + \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x \in A, y \notin A}} \frac{\alpha_{S_A}}{2} \\ & + \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x \notin A, y \in A}} \frac{\alpha_{S_A}}{2} \\ & + \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x,y \notin A}} 2 \left| \frac{\alpha_{S_A}}{2} - \psi_g(A) \right|. \end{aligned} \tag{4.2}$$

Since the sum of the second and third term of the right-hand side of (4.2) amounts to $\sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi_g)} \alpha_S \delta_S(x,y)$, comparing (4.1) and (4.2) we obtain

$$\sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x,y \in A}} 2\psi_g(A) + \sum_{\substack{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g)) \\ x,y \notin A}} 2 \left| \frac{\alpha_{S_A}}{2} - \psi_g(A) \right| = 0$$

hence for every $A \in U(\mathcal{S} \setminus \mathcal{S}(\psi_g))$ such that $x, y \in A$, one has $\psi_g(A) = 0$ or in other words, for any $S \in \mathcal{S} \setminus \mathcal{S}(\psi_g)$ such that $S(x) = S(y)$, one has $\psi_g(S(x)) =$

$0 = \psi_g(S(y))$. Now, note that by definition of ψ, one has $\mathrm{supp}(\psi_g) \subset \mathrm{supp}(\psi)$ as well as $\mathcal{S} \setminus \mathcal{S}(\psi) \subset \mathcal{S} \setminus \mathcal{S}(\psi_g)$ and thus for every $S \in \mathcal{S} \setminus \mathcal{S}(\psi)$ such that $S(x) = S(y)$, one has $\psi(S(x)) = 0$. It is easy to see that this implies that for every $S \in \mathcal{S} \setminus \mathcal{S}(\psi)$, since $\psi \in H(\mathcal{S}, \alpha)$, one has

$$\alpha_S \delta_S(x, y) = 2\big[\psi(S(x)) + |\phi_y(S(x)) - \psi(S(x))|\big]. \tag{4.3}$$

Moreover, one easily obtains

$$\begin{aligned}
&\sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi)} 2\big[\psi(S(x)) + |\phi_y(S(x)) - \psi(S(x))|\big] \\
&= \sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi)} \big[|\phi_x(S(x)) - \psi(S(x))| + |\phi_y(S(x)) - \psi(S(x))| \\
&\qquad + |\phi_x(\overline{S(x)}) - \psi(\overline{S(x)})| + |\phi_y(\overline{S(x)}) - \psi(\overline{S(x)})|\big] \\
&= \sum_{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi))} \big[|\phi_x(A) - \psi(A)| + |\phi_y(A) - \psi(A)|\big].
\end{aligned} \tag{4.4}$$

Furthermore, y was chosen so that for every $S \in \mathcal{S}(\psi)$, one has $S(x) \neq S(y)$. Thus for every $\bar{\psi} \in [\psi]$, one obtains:

$$\begin{aligned}
&\alpha_S \delta_S(x, y) \\
&= \alpha_S \\
&= [0 + \bar{\psi}(S(x))] + [\tfrac{\alpha_S}{2} - \bar{\psi}(S(x))] + [\tfrac{\alpha_S}{2} - \bar{\psi}(\overline{S(x)})] + [\bar{\psi}(\overline{S(x)}) - 0] \\
&= \sum_{A \in S} \big[|\phi_x(A) - \bar{\psi}(A)| + |\phi_y(A) - \bar{\psi}(A)|\big]
\end{aligned} \tag{4.5}$$

and thus since $d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) = \sum_{S \in \mathcal{S}(\psi)} \alpha_S \delta_S(x, y)$ together with (4.5) and since for every $A \in U(\mathcal{S} \setminus \mathcal{S}(\psi))$, one has $\bar{\psi}(A) = \psi(A)$, it follows that

$$\begin{aligned}
d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) &= \sum_{A \in U(\mathcal{S}(\psi))} \big[|\phi_x(A) - \bar{\psi}(A)| + |\phi_y(A) - \bar{\psi}(A)|\big] \\
&= d_1(\gamma^x_{[\psi]}, \bar{\psi}) + d_1(\bar{\psi}, \gamma^y_{[\psi]}).
\end{aligned} \tag{4.6}$$

Hence

$$\begin{aligned} d_1(\phi_x, \phi_y) &= \sum_{S \in \mathcal{S}} \alpha_S \delta_S(x,y) \\ &= \sum_{S \in \mathcal{S}(\psi)} \alpha_S \delta_S(x,y) + \sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi)} \alpha_S \delta_S(x,y) \\ &= d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) + \sum_{S \in \mathcal{S} \setminus \mathcal{S}(\psi)} 2\big[\psi(S(x)) + |\phi_y(S(x)) - \psi(S(x))|\big] \\ &= d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) + \sum_{A \in U(\mathcal{S} \setminus \mathcal{S}(\psi))} \big[|\phi_x(A) - \psi(A)| + |\phi_y(A) - \psi(A)|\big] \\ &= d_1(\phi_x, \gamma^x_{[\psi]}) + d_1(\gamma^x_{[\psi]}, \gamma^y_{[\psi]}) + d_1(\gamma^y_{[\psi]}, \phi_y). \end{aligned} \tag{4.7}$$

where the third equality follows from (4.3), the fourth one from (4.4) and the last one by our definitions. Hence, inserting (4.6) into (4.7), one has:

$$d_1(\phi_x, \phi_y) = d_1(\phi_x, \gamma^x_{[\psi]}) + d_1(\gamma^x_{[\psi]}, \psi) + d_1(\psi, \gamma^y_{[\psi]}) + d_1(\gamma^y_{[\psi]}, \phi_y). \tag{4.8}$$

It follows from $d(x,y) = d_1(\phi_x, \phi_y)$ and (4.8) that

$$d(x,y) = d_1(\phi_x, \psi) + d_1(\psi, \phi_y) = \kappa(\psi)(x) + \kappa(\psi)(y). \tag{4.9}$$

Since for any $x \in X$, there is such an $y \in X$, it follows that $\kappa(\psi) \in \mathrm{E}'(d)$. Moreover, by definition of ψ, one has

$$\bigcup_{i \in \{1,2,3\}} \kappa([\psi_{f_2^i}]) = \kappa\left(\bigcup_{i \in \{1,2,3\}} [\psi_{f_2^i}]\right) \subset \kappa([\psi]) \subset [\kappa(\psi)]$$

where the last inclusion follows from Lemma 3.9. Now, since $[\kappa(\psi)]$ is a hypercube, this proves that $\mathrm{E}(d) = \mathrm{E}'(d)$ satisfies the CAT(0) link condition. □

Two splits $S := \{A, B\}$ and $S' = \{A', B'\}$ of X are called *incompatible* if

$$A \cap A', A \cap B', B \cap A', B \cap B' \neq \emptyset.$$

A split system $\mathcal{S}$ is called *incompatible* if any pair of splits in $\mathcal{S}$ is incompatible.

4.5 Remark. For particular split system pairs $(\mathcal{S}, \alpha)$ and particular sets X, the Buneman complex $B(\mathcal{S}, \alpha)$ displays some similarities with the CAT(0) cube complex that is constructed in [11]. There, a split system pair is obtained by considering a wall space $\mathcal{W}$ (corresponding to $\mathcal{S}$) on a set Y (corresponding to X) and taking the function α chosen to be constantly equal to one on $\mathcal{S}$.

For a Coxeter group, its Cayley graph is endowed with the standard word metric and there is a canonical decomposition (in general not weakly compatible, e.g. the tessellation of the plane by hexagons) of this metric given by the splits of the form $S := \{C(x,y), C(y,x)\}$ where $C(x,y) := \{z \in X : d(x,y) + d(y,z) = d(x,z)\}$ and where $\{x,y\}$ is an edge in the Cayley graph, together with α constantly equal to one.

The set K^0 (denoted by X^0 in [11]) is defined to be consisting of all the *admissible sections*, i.e. the maps $\sigma\colon \mathcal{S} \to U(\mathcal{S})$ such that for any $S \neq S'$: $\sigma(S) \cap \sigma(S') \neq \emptyset$. Next, K^1 is the graph with vertex set K^0, where two vertices σ and σ' are connected by an edge if and only if there is a unique $S \in \mathcal{S}$ such that $\sigma(S) \neq \sigma'(S)$. For an arbitrarily fixed point $p \in X$, one then lets Γ_p be the path-connected component of σ_p in K^1 where for any S, one lets $\sigma_p(S) := S(p)$.

Let us define $B^p(\mathcal{S},\alpha) := \{\psi \in B(\mathcal{S},\alpha) : d_1(\psi,\phi_p) < \infty\}$, let $\Sigma^0(\Gamma_p)$ be the 0-skeleton of Γ_p, and let $\Sigma^0(B^p(\mathcal{S},\alpha))$ and $\Sigma^1(B^p(\mathcal{S},\alpha))$ be the 0- and 1-skeleton of $B^p(\mathcal{S},\alpha)$. We define

$$M\colon (\Sigma^0(\Gamma_p), d_1) \to (\Sigma^0(B^p(\mathcal{S},\alpha)), d_1)$$

by sending every admissible section $\sigma\colon \mathcal{S} \to U(\mathcal{S})$ to a function $M(\sigma)\colon U(\mathcal{S}) \to \mathbb{R}$ defined by assigning (recall that $S_A := \{A, A^c\}$):

$$A \mapsto \begin{cases} \frac{\alpha_{S_A}}{2} & \text{if } A = \sigma(S)^c, \\ 0 & \text{if } A = \sigma(S). \end{cases}$$

Now assume that $\{\sigma,\tau\}$ is an edge of Γ_p which means that there is a unique $S' := \{A', B'\} \in \mathcal{S}$ such that $\sigma(S') \neq \tau(S')$. This is equivalent to the fact that $M(\sigma)(A') = M(\tau)(B')$, $M(\sigma)(B') = M(\tau)(A')$ and for any $A \in U(\mathcal{S} \setminus \{S'\})$, one has $M(\sigma)(A) = M(\tau)(A)$. It is easy to see, that this is in turn equivalent to the fact that there is a function $\psi \in \Sigma^1(B^p(\mathcal{S},\alpha))$ such that $\dim([\psi]) = 1$, $S' \subset \mathrm{supp}(\psi)$ and so that for every $A \in U(\mathcal{S} \setminus \{S'\})$, one has $M(\sigma)(A) = \psi(A) = M(\tau)(A)$. Therefore, M extends bijectively to an isometric isomorphism of cell complexes

$$M_1\colon (\Gamma_p, d_1) \to (\Sigma^1(B^p(\mathcal{S},\alpha)), d_1).$$

Let us denote an edge e_j of Γ_p by its corresponding labeling split S_j (the unique one on which the endpoints of e_j differ). Now, *k-corners* $(\sigma, \{e_1,\dots,e_k\})$ (in the terminology of [11]) are simply pairs of the form $(\sigma, \{S_1,\dots,S_k\})$ where

$\sigma \in \Gamma_p$ and where the split system $\{S_1, \dots, S_k\}$ is incompatible. The complex K is then obtained by gluing a k-cube to every k-corner (one shows that the existence of a k-corner implies the existence of the 1-skeleton of a k-hypercube contained in Γ_p and containing this k-corner as a vertex). It is now easy to see that there is an isomorphism of cell complexes

$$i \colon \mathrm{K} \to B^p(\mathcal{S}, \alpha)$$

which extends M_1. To any k-dimensional cube C giving rise to a cell of K corresponds by construction a k-corner $(\sigma, \{S_1, \dots, S_k\})$. It is then easy to see that defining $\psi \in H(\mathcal{S}, \alpha)$ so that $\psi(A) := \sigma(A)$ for $A \in U(\mathcal{S} \setminus \{S_1, \dots, S_k\})$ and $\psi(A) := \frac{\alpha_{S_A}}{4}$ otherwise, we obtain that $\psi \in B^p(\mathcal{S}, \alpha)$ and $[\psi]$ is a k-dimensional cell of $B^p(\mathcal{S}, \alpha)$, hence a k-dimensional combinatorial hypercube. We can thus extend M_1 and map bijectively C to $[\psi]$ with i. Conversely, let $\sigma \in \Sigma^0(B^p(\mathcal{S}, \alpha))$ be a vertex of a k-dimensional cell $[\psi]$ of $B^p(\mathcal{S}, \alpha)$. The pair $(\sigma, \mathcal{S}(\psi))$ has to be a k-corner in Γ_p by incompatiblity of $\mathcal{S}(\psi)$ and thus the inverse image of $[\psi]$ under i is the k-dimensional cube in K glued to the k-corner $(\sigma, \mathcal{S}(\psi))$.

IV.5 Examples

5.1 Example. Let (X, d) be an infinite connected graph endowed with the shortest-path metric. It is easy to see that if X is bipartite, then the system $\mathcal{S}$ of all d-splits of (X, d) is octahedral-free. Indeed, assume on the contrary that $X = Y_1^1 \sqcup Y_1^{-1} \sqcup Y_2^1 \sqcup Y_2^{-1} \sqcup Y_3^1 \sqcup Y_3^{-1}$ is a partition of X into six non-empty subsets such that

$$\begin{aligned} S_1 &:= \{Y_1^1 \sqcup Y_2^1 \sqcup Y_3^1, Y_1^{-1} \sqcup Y_2^{-1} \sqcup Y_3^{-1}\}, \\ S_2 &:= \{Y_1^1 \sqcup Y_2^1 \sqcup Y_3^{-1}, Y_1^{-1} \sqcup Y_2^{-1} \sqcup Y_3^1\}, \\ S_3 &:= \{Y_1^1 \sqcup Y_2^{-1} \sqcup Y_3^1, Y_1^{-1} \sqcup Y_2^1 \sqcup Y_3^{-1}\}, \\ S_4 &:= \{Y_1^1 \sqcup Y_2^{-1} \sqcup Y_3^{-1}, Y_1^{-1} \sqcup Y_2^1 \sqcup Y_3^1\}, \end{aligned} \tag{5.1}$$

and $\{S_1, \dots, S_4\} \subset \mathcal{S}$. Since X is a connected graph, there must be an edge $\{x, y\}$ between two non-antipodal sets in the partition, for instance an edge joining $x \in Y_1^1$ to $y \in Y_1^{-1}$. Now, we see that both $S_3(x) \neq S_3(y)$ and $S_4(x) \neq S_4(y)$. This is a contradiction to the fact that since X is bipartite, there is for any edge $\{x, y\}$ in X, at most one d-split S separating x and y (i.e. such that $S(x) \neq S(y)$) namely the split given by $\{C(x, y), C(y, x)\}$ where $C(x, y) := \{z \in X : d(x, y) + d(y, z) = d(x, z)\}$.

5.2 Example. For $n \in \mathbb{N}$, let

$$C_{2n+1} := \big(\{x_1, \dots, x_{2n+1}\}, \{\{x_i, x_{i+1}\}\}_{i \in \{1,\dots,2n+1\}}\big) \quad \text{where} \quad x_{2n+2} := x_1.$$

The graph C_{2n+1} is the odd cycle with $2n+1$ vertices and we endow it with the shortest path metric d. We use the fact that the metric d is totally split-decomposable to give an explicit description of the injective hull $\mathrm{E}(C_{2n+1}, d)$.

One can easily verify that $d = \frac{1}{2}\sum_{S \in \mathcal{S}} \delta_S$ where $\mathcal{S}$ is the set of all d-splits of X, hence $\alpha \colon \mathcal{S} \to (0, \infty)$ can be chosen to be constantly equal to $\frac{1}{2}$. One has $\mathcal{S} = \{S_1, \dots, S_{2n+1}\}$ where for $i \in \{1, \dots, 2n+1\}$, with indices taken modulo $2n+1$, S_i is given by where

$$S_i = \{A_i, B_i\} = \{\{x_{i+1}, \dots, x_{i+n}\}, \{x_{i+n+1}, \dots, x_i\}\}.$$

It is now not difficult to prove that the assumptions of Theorem 1.1 are fulfilled for $(X, d) = (C_{2n+1}, d)$. Since we are in the case of a finite metric space, the (LRC) is trivially satisfied. Moreover,

$$d = \frac{1}{2}\sum_{S \in \mathcal{S}} \delta_S,$$

where $\mathcal{S}$ is the family of all d-splits of X. Finally, it is not difficult to see that $\mathcal{S}$ is octahedral-free and thus that (i) in Theorem 1.1 holds as well. Indeed, if a subsystem $\bar{\mathcal{S}} := \{\bar{S}_1, \bar{S}_2, \bar{S}_3, \bar{S}_4\} \subset \mathcal{S}$ is octahedral, that is, it is induced by a partition into six non-empty subsets

$$C_{2n+1} = Y_1^1 \sqcup Y_1^{-1} \sqcup Y_2^1 \sqcup Y_2^{-1} \sqcup Y_3^1 \sqcup Y_3^{-1}$$

as in (5.1), then for any $x \in Y_i^\sigma$, for any $y \in Y_i^{-\sigma}$ and for every $S \in \bar{\mathcal{S}}$, one has $S(x) \neq S(y)$ (we say that S *cuts* the edge $\{x, y\}$). In order for such a pair $\{x, y\}$ to exist for every $x \in C_{2n+1}$, it follows that if $S_j \in \bar{\mathcal{S}}$ with $1 \leq j \leq n+1$ (the case $n+2 \leq j \leq 2n+1$ is similar), then $S_{j+n}, S_{j+n+1} \notin \bar{\mathcal{S}}$. Indeed, there are exactly two splits that cut one of the edges $\{\{x_j, x_{j+1}\}, \{x_{j+n}, x_{j+n+1}\}\}$ which are cut by S_j, namely S_{j+n} which cuts $\{x_{j+n}, x_{j+n+1}\}$ and S_{j+n+1} which cuts $\{x_j, x_{j+1}\}$. Since we are considering C_{2n+1}, it follows that $\bar{\mathcal{S}}$ must induce a partition into eight non-empty subsets

$$C_{2n+1} = Z_1^1 \sqcup Z_1^{-1} \sqcup Z_2^1 \sqcup Z_2^{-1} \sqcup Z_3^1 \sqcup Z_3^{-1} \sqcup Z_4^1 \sqcup Z_4^{-1}$$

such that $S(x) \neq S(y)$ for every $S \in \bar{\mathcal{S}}$ if and only if $x \in Z_i^\sigma$ and $y \in Z_i^{-\sigma}$. It follows that $\bar{\mathcal{S}}$ is not octahedral and thus $\mathcal{S}$ must be octahedral-free, which shows in particular that (i) in Theorem 1.1 holds. We deduce that for every

$n \in \mathbb{N} \cup \{0\}$, $\mathrm{E}(C_{2n+1}, d)$ is a finite cube complex satisfying the CAT(0) link-condition (and simply connected since it is an injective hull).

We can furthermore describe explicitely the dimension and gluing pattern of the maximal cells of $\mathrm{E}(C_{2n+1}, d)$ by studying the different split susbsystems of $\mathcal{S}$. Note first that by finiteness, we have $B(\mathcal{S}, \alpha) = \bar{T}(\mathcal{S}, \alpha)$ (see [19]) and by Remark 3.14 the map

$$\kappa\colon \bar{T}(\mathcal{S}, \alpha) \to \mathrm{E}(C_{2n+1}, d)$$

is in particular an isomorphism of cell complexes. The family of maximal cells of $\mathrm{E}(C_{2n+1}, d)$ is thus in bijection with the family of maximal cells of $B(\mathcal{S}, \alpha)$ which in turn bijectively corresponds with the family $\mathfrak{M}$ of maximal incompatible split subsystems $\mathcal{M} \subset \mathcal{S}$.

Observe that for any such $\mathcal{M} \in \mathfrak{M}$, we can consider a corresponding element $\psi \in \bar{T}(\mathcal{S}, \alpha)$ such that $\mathcal{S}(\psi) = \mathcal{M}$. We have

$$\kappa(\psi)(x) + \kappa(\psi)(y) = d(x, y)$$

if and only if $S(x) \neq S(y)$ for every $S \in \mathcal{M}$. Since $\mathcal{M}$ is a maximal incompatible split subsystem of $\mathcal{S}$, it follows that for any $S = \{A, B\} \in \mathcal{S} \setminus \mathcal{M}$ where $|A| = k$ and $|B| = k+1$, one has $\psi(A) = \frac{\alpha_S}{2} = \frac{1}{4}$ and $\psi(B) = 0$.

For the gluing pattern, we have for any two maximal cells $[\psi], [\mu] \subset \bar{T}(\mathcal{S}, \alpha)$, $[\psi] \cap [\mu] \neq 0$ if and only if $\psi(A) = \mu(A)$ for every $A \in U(\mathcal{S} \setminus (\mathcal{S}(\psi) \cup \mathcal{S}(\mu)))$ and in this case, $[\psi] \cap [\mu]$ is the set of all functions $\phi \in B(\mathcal{S}, \alpha)$ such that $\phi(A) = \psi(A)$ for $A \in U(\mathcal{S}(\psi) \setminus \mathcal{S}(\mu)))$, $\phi(A) = \mu(A)$ for $A \in U(\mathcal{S}(\mu) \setminus \mathcal{S}(\psi)))$ and $\mathcal{S}(\phi) = \mathcal{S}(\psi) \cap \mathcal{S}(\mu)$.

Note that for any $\bar{S}_j \in \bar{\mathcal{S}}$, the only two splits that are not incompatible with $\bar{S}_j$ are the only two splits that cut an edge already cut by $\bar{S}_j$.

To compute $|\mathfrak{M}|$, it is easier to describe the split system $\mathcal{S}$ in a different way, by assigning for $i \in \{1, \dots, n+1\}$, to every edge $\{x_i, x_{i+1}\}$ of C_{2n+1}, the pair of splits $S_i^1 = \{A_i^1, B_i^1\}$ and $S_i^{-1} = \{A_i^{-1}, B_i^{-1}\}$ which cut the edge $\{x_i, x_{i+1}\}$ (i.e. $S_i^1(x_i) \neq S_i^1(x_{i+1})$ and $S_i^{-1}(x_i) \neq S_i^{-1}(x_{i+1})$) and which are determined by the requirements $x_i \in A_i^1$ and $|A_i^1| = n+1$ as well as $x_i \in A_i^{-1}$ and $|A_i^{-1}| = n$. Moreover, $S_1^1 = S_{n+1}^{-1}$.

We divide the family $\mathcal{M}$ of maximal incompatible split subsystems of $\mathcal{S}$ into three subfamilies

$$\mathfrak{M} = \mathfrak{M}^a \cup \mathfrak{M}^b \cup \mathfrak{M}^c$$

so that for

$$\mathcal{M} := \{S_{j_1}^{\sigma_1}, \dots, S_{j_k}^{\sigma_k}\} \in \mathfrak{M}$$

with $1 \leq j_1 < j_2 < \dots < j_{k-1} < j_k \leq 2n+1$, one has the following three cases:

(*a*) $\mathcal{M} \in \mathfrak{M}^a$ if and only if $S_{j_1}^{\sigma_1} = S_1^1$. In this case, we have

$$|\mathfrak{M}^a| = \Sigma^a_{(n-1,n-1)} + \Sigma^a_{(n-1,n)}$$

where $\Sigma_{(n-1,n-1)} := \Sigma^a_{(n-1,n-1)}$ stands for the case where $S_{j_k}^{\sigma_k} = S_n^1$ and $\Sigma_{(n-1,n)} := \Sigma^a_{(n-1,n)}$ for the case where $S_{j_k}^{\sigma_k} = S_n^{-1}$. The notation

$$\Sigma^a_{(n-1,n-1)}$$

refers to the fact that

$$|B_{j_1}^{\sigma_1} \cap A_{j_k}^{\sigma_k}| = n-1$$

and

$$|A_{j_1}^{\sigma_1} \cap B_{j_k}^{\sigma_k}| = n-1$$

(i.e., starting with x_2 and going counterclockwise we count $n-1$ points until we hit the line representing $S_{j_k}^{\sigma_k}$, similarly when starting with x_{n+2} we count $n-1$ points as well until we hit the line representing $S_{j_k}^{\sigma_k}$.)

(*b*) $\mathcal{M} \in \mathfrak{M}^b$ if and only if $S_{j_1}^{\sigma_1} = S_1^{-1}$. In this case, we have

$$|\mathfrak{M}^b| = \Sigma^b_{(n-1,n-1)} + \Sigma^b_{(n,n-1)}$$

where by symmetry

$$\Sigma^b_{(n-1,n-1)} = \Sigma_{(n-1,n-1)}$$

stands for the case where $S_{j_k}^{\sigma_k} = S_n^{-1}$ and

$$\Sigma^b_{(n,n-1)} = \Sigma_{(n-1,n)}$$

for the case where $S_{j_k}^{\sigma_k} = S_{n+1}^1$.

(*c*) $\mathcal{M} \in \mathfrak{M}^c$ if and only if $S_{j_1}^{\sigma_1} = S_2^1$. In this case, we have

$$|\mathfrak{M}^c| = \Sigma^c_{(n-1,n-1)}$$

where by symmetry

$$\Sigma^c_{(n-1,n-1)} = \Sigma_{(n-1,n-1)}$$

stands for the case where $S_{j_k}^{\sigma_k} = S_{n+1}^1$.

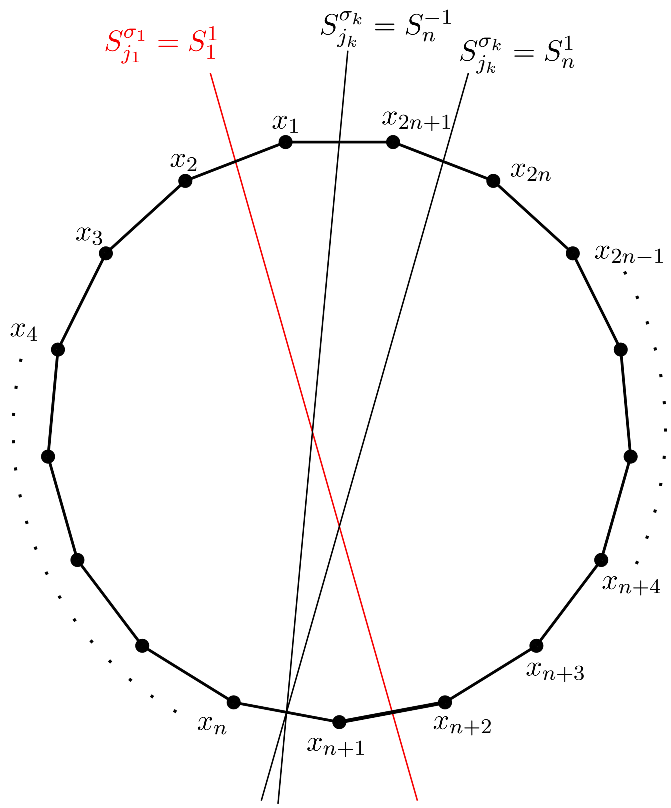

Figure IV.1: $\Sigma^a_{(n-1,n-1)}$ corresponds to the number of maximal incompatible split systems $\{S^{\sigma_1}_{j_1}, \dots, S^{\sigma_k}_{j_k}\}$ with $S^{\sigma_1}_{j_1} = S^1_1$ and $S^{\sigma_k}_{j_k} = S^1_n$. $\Sigma^a_{(n-1,n)}$ is the number of those with $S^{\sigma_1}_{j_1} = S^1_1$ and $S^{\sigma_k}_{j_k} = S^{-1}_n$.

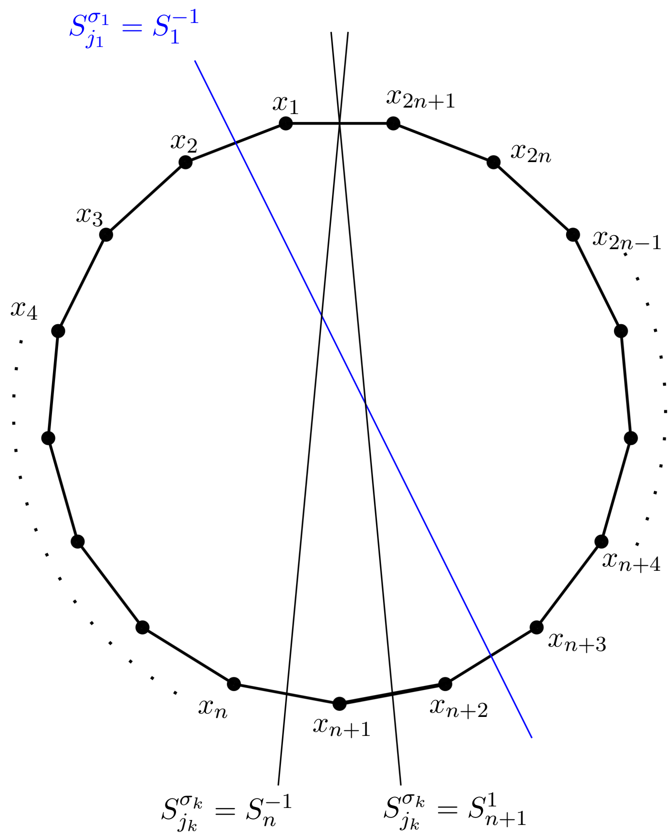

Figure IV.2: $\Sigma^b_{(n-1,n-1)}$ corresponds to the number of maximal incompatible split systems $\{S^{\sigma_1}_{j_1}, \ldots, S^{\sigma_k}_{j_k}\}$ with $S^{\sigma_1}_{j_1} = S^{-1}_1$ and $S^{\sigma_k}_{j_k} = S^{-1}_n$. $\Sigma^b_{(n,n-1)}$ is the number of those with $S^{\sigma_1}_{j_1} = S^{-1}_1$ and $S^{\sigma_k}_{j_k} = S^{1}_{n+1}$.

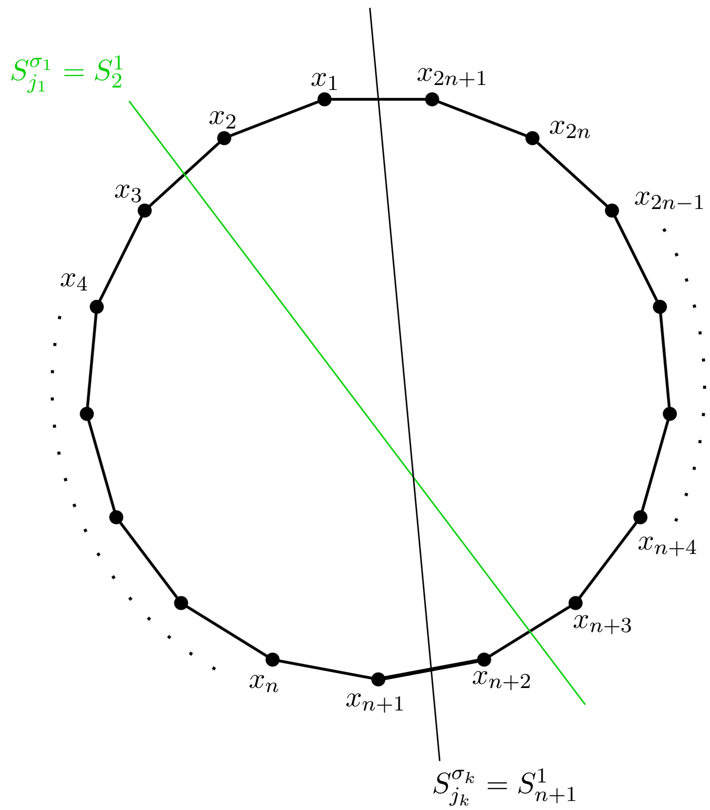

Figure IV.3: $\Sigma^c_{(n-1,n-1)}$ corresponds to the number of maximal incompatible split systems $\{S^{\sigma_1}_{j_1}, \ldots, S^{\sigma_k}_{j_k}\}$ with $S^{\sigma_1}_{j_1} = S^1_2$ and $S^{\sigma_k}_{j_k} = S^1_{n+1}$.

Summing up, we obtain the formula

$$|\mathfrak{M}| = \Theta_n = 3\Sigma_{(n-1,n-1)} + 2\Sigma_{(n-1,n)}$$

and it is easy to see that we furthermore have the following recurrence relations: $\Sigma_{(n-1,n-1)} = \Sigma_{(n-2,n-2)} + \Sigma_{(n-3,n-2)}$ and $\Sigma_{(n-1,n)} = \Sigma_{(n-2,n-1)} + \Sigma_{(n-2,n-2)}$ and the initial conditions $\Sigma_{(0,0)} := 1$, $\Sigma_{(0,1)} := 0$ and $\Sigma_{(1,1)} := 1$. If one considers the three roots $\{\sigma_1, \sigma_2, \sigma_3\}$ of the equation $x^3 - x - 1 = 0$ given by

$$\sigma_1 = \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}} + \frac{\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{\frac{1}{3}}}{3^{\frac{2}{3}}},$$

$$\sigma_2 = -\frac{1}{6}\left(1+i\sqrt{3}\right)\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}} - \frac{\left(1-i\sqrt{3}\right)\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{\frac{1}{3}}}{2\cdot 3^{\frac{2}{3}}},$$

$$\sigma_3 = -\frac{1}{6}\left(1-i\sqrt{3}\right)\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}} - \frac{\left(1+i\sqrt{3}\right)\left(\frac{1}{2}\left(9+\sqrt{69}\right)\right)^{\frac{1}{3}}}{2\cdot 3^{\frac{2}{3}}}$$

and the three roots $\{\tau_1, \tau_2, \tau_3\}$ of the equation $x^3 - 2x^2 + x - 1 = 0$ given by

$$\tau_1 = \frac{1}{3}\left(2 + \left(\frac{25}{2} - \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}} + \left(\frac{1}{2}\left(25+3\sqrt{69}\right)\right)^{\frac{1}{3}}\right),$$

$$\tau_2 = \frac{2}{3} - \frac{1}{6}\left(1+i\sqrt{3}\right)\left(\frac{25}{2} - \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}} - \frac{1}{6}\left(1-i\sqrt{3}\right)\left(\frac{25}{2} + \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}},$$

$$\tau_3 = \frac{2}{3} - \frac{1}{6}\left(1-i\sqrt{3}\right)\left(\frac{25}{2} - \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}} - \frac{1}{6}\left(1+i\sqrt{3}\right)\left(\frac{25}{2} + \frac{3\sqrt{69}}{2}\right)^{\frac{1}{3}},$$

one can verify that

$$\Theta_n = \sigma_1\tau_1^n + \sigma_3\tau_2^n + \sigma_2\tau_3^n.$$

Note that Θ_0 is one. The first values of Θ_n for $n \geq 1$ are listed in the following table:

n	1	2	3	4	5	6	7	8	9	10
C_{2n+1}	C_3	C_5	C_7	C_9	C_{11}	C_{13}	C_{15}	C_{17}	C_{19}	C_{21}
Θ_n	3	5	7	12	22	39	68	119	209	367

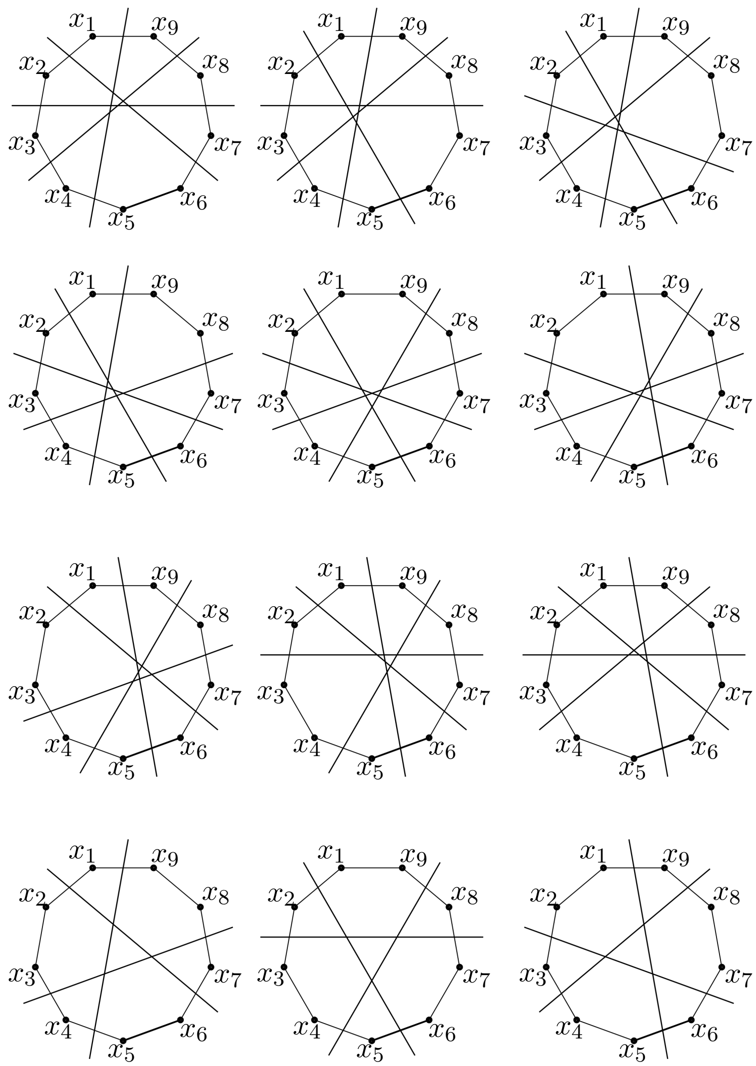

Figure IV.4: List of all 12 maximal incompatible split subsystems for $C_{2n+1} = C_9$. The first three lines correspond to the nine 4-dimensional maximal cells of $\mathrm{E}(C_{2n+1})$ and the last line corresponds to the three 3-dimensional maximal cells.

The numbers Θ_n can also be proved to correspond to the coefficient of z^n in the power series expansion in a neighborhood of the origin of the analytic function

$$z \mapsto \frac{3z - z^2}{1 - 2z + z^2 - z^3}.$$

Finally, it is not difficult to compute the number of k-dimensional cells of $\mathrm{E}(C_{2n+1})$ as a function of k and n as it is done in [40]. Indeed, note that for a 0-cell $\psi \in \bar{T}(\mathcal{S}, \alpha)$, it is easy to see that ψ is in the maximal cell $[\mu]$ if and only if for any split $S = \{A, B\} \in \mathcal{S}$ where $|A| = n + 1$ and $A \in \mathrm{supp}(\psi)$, one has $S \in \mathcal{S}(\mu)$. More generally, for every $k \in \{0, \dots, n\}$, there is a bijection between the k-dimensional cells of $\mathrm{E}(C_{2n+1})$ and the set of pairs $(\bar{\mathcal{S}}, \bar{\mathcal{S}}')$ where $\bar{\mathcal{S}}$ is a split subsystem of $\mathcal{S}$ with k elements and $\bar{\mathcal{S}} \subset \bar{\mathcal{S}}'$. The correspondence between $[\psi]$ and $(\bar{\mathcal{S}}, \bar{\mathcal{S}}')$ is given by picking for every $S = \{A, B\} \in \mathcal{S}$ where $|A| = n + 1$, the function ψ in such a way that $\mathcal{S}(\psi) := \bar{\mathcal{S}}$, $\psi(B) = 0$ if $S \in \bar{\mathcal{S}}' \setminus \bar{\mathcal{S}}$ and $\psi(A) = 0$ if $S \in \mathcal{S} \setminus \bar{\mathcal{S}}'$.

5.3 Example. Let (X, d) be an infinite connected bipartite $(4, 4)$-graph as defined in [3] endowed with the shortest-path metric. Let $S = \{A, B\}$ be an alternating split on X (cf. [3]). Assume by contradiction that S has isolation index $\alpha_S^d \in \{0, \frac{1}{2}\}$. We show that $\alpha_S = 1$. Since all isolation indices of splits on X are in $\frac{1}{2}\mathbb{Z}$, it follows that one can find four points $r, s, u, v \in X$ such that $r, s \in A$, $u, v \in B$ and $\alpha^d_{\{\{r,s\},\{u,v\}\}} = \alpha^d_S$. We can now consider a finite subgraph Y of X such that

(a) Y is a bipartite $(4, 4)$-graph and

(b) $I(r, s) \cup I(r, u) \cup I(r, v) \cup I(s, u) \cup I(s, v) \cup I(u, v) \subset Y$.

In general, the restriction $d|_{Y \times Y}$ does not coincide with the shortest path metric d_Y on Y. However, by (b), it follows that for any $a, b \in \{r, s, u, v\}$, one has $d_Y(a, b) = d|_{Y \times Y}(a, b)$. Note that $S_Y := \{A \cap Y, B \cap Y\}$ is an alternating split of Y (restrictions of alternating splits to $(4, 4)$-subgraphs are easily seen to be alternating again) and

$$\alpha^{d_Y}_{S_Y} \leq \alpha^{d_Y}_{\{\{r,s\},\{u,v\}\}} = \alpha^{d|_{Y \times Y}}_{\{\{r,s\},\{u,v\}\}} = \alpha^d_S \leq \tfrac{1}{2}.$$

However, by (a), (Y, d_Y) is a finite bipartite $(4, 4)$-graph and thus by [3, Proposition 8.8], it follows that $\alpha^{d_Y}_{S_Y} = 1$ which contradicts the above. If now $\mathcal{S}_A$ denotes the system of all alternating splits of (X, d), note that for any $x, y \in X$ such that $d(x, y) = 1$, there is a unique $S \in \mathcal{S}_A$ such that $S(x) \neq S(y)$. Note that $d_0 := d - \sum_{S \in \mathcal{S}_A} \delta_S$ is a pseudometric by Theorem 1.3. Hence for any $a, b \in X$, consider a path $a = x_0,\ x_1,\ \ldots,\ x_{m-1},\ x_m = b$ in (X, d), we have

$$d_0(a,b) \leq \sum_{i=0}^{m-1} d_0(x_i, x_{i+1}) = 0.$$

It follows that d_0 is identically zero and thus d is totally split-decomposable. Moreover, (X, d) satisfies the (LRC). Indeed, the isometric cycles in (X, d) are gated (cf. [3, Theorem 8.7]), it follows that (X, d) has 1-stable intervals and thus by the proof of [30, Theorem 1.1], we obtain the desired result. Examples of such infinite bipartite $(4,4)$-graphs are given for $m \geq 4$, for σ any element of the symmetric group $\mathfrak{S}_m$ and for $\{r_{\sigma(i)\sigma(i+1)}\}_{i \in \{1,\ldots,m\}} \subset \mathbb{N} \cap [2, \infty) \cup \{\infty\}$ by the Cayley graph of Coxeter groups of the form

$$C = \langle s_1, \ldots, s_m |\, (s_{\sigma(1)} s_{\sigma(2)})^{r_{\sigma(1)\sigma(2)}} = 1, \ldots, (s_{\sigma(m-1)} s_{\sigma(m)})^{r_{\sigma(m-1)\sigma(m)}} = 1, \\ (s_{\sigma(m)} s_{\sigma(1)})^{r_{\sigma(m)\sigma(1)}} = 1 \,\rangle \ .$$

The restriction on the number of relations ensures that the Cayley graph is planar, $m \geq 4$ ensures that the degree is at least four and the condition $r_{\sigma(i)\sigma(i+1)} \geq 2$ for every i ensures that each face contains at least four vertices.

Chapter V

Further Results on Metric Injectivity

V.1 Introduction

The goal of this chapter is to develop structural tools for Isbell's injective hull of infinite metric spaces with integer valued-metric. In case X is separable, $\mathrm{E}(X)$ isometrically embeds into $l_\infty(\mathbb{N})$. With this in mind, we obtain criteria for the injectivity of subsets of $l_\infty(\mathbb{N})$.

Recall that a metric space (X, d) is called *injective* if for any isometric embedding $i\colon A \to B$ of metric spaces and any 1-Lipschitz (equivalently distance-nonincreasing) map $f\colon A \to Y$, there exists a 1-Lipschitz extension $g\colon B \to X$ of f, so that $g \circ i = f$. Examples of injective metric spaces include the real line $\mathbb{R}$, $l_\infty(I)$ for any index set I, and all complete metric trees. Isbell showed that every metric space X possesses an *injective hull* $(\mathrm{e}, \mathrm{E}(X))$ which means that two properties hold: $\mathrm{E}(X)$ is an injective metric space and $\mathrm{e}\colon X \to \mathrm{E}(X)$ is an isometric embedding such that every isometric embedding of X into some injective metric space factors through e. Following [30], we start with an outline of the construction of Isbell's injective hull. Later, we use injective and hyperconvex indifferently, they are equivalent properties.

Given a metric space (X, d), let us consider the vector space $\mathbb{R}^X$ of real-valued functions on X and

$$\Delta(X) := \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X\}.$$

We call $f \in \Delta(X)$ *extremal* if there is no $g \leq f$ in $\Delta(X)$ distinct from f. The set $\mathrm{E}(X)$ of extremal functions is equivalently given by

$$\mathrm{E}(X) = \left\{f \in \mathbb{R}^X : f(x) = \sup\nolimits_{y \in X}(d(x, y) - f(y)) \text{ for all } x \in X\right\}. \quad (1.1)$$

The difference between two elements of $\mathrm{E}(X)$ has finite supremum norm and $\mathrm{E}(X)$ is endowed with the metric

$$d_\infty(f,g) := \|f-g\|_\infty .$$

To be able to describe the structure of $\mathrm{E}(X)$ in further details, one can assign to every $f \in \mathbb{R}^X$ the undirected graph with vertex set X and edge set

$$A(f) := \big\{\{x,y\} : x,y \in X \text{ and } f(x)+f(y) = d(x,y)\big\}, \tag{1.2}$$

allowing self-loops $\{x,x\}$ which correspond to zeros of f. Furthermore, one lets

$$\mathrm{E}'(X) := \big\{f \in \Delta(X) : \bigcup A(f) = X\big\}.$$

Note that whenever $f \in \mathrm{E}'(X)$, the graph $(X, A(f))$ has no isolated vertices (although it may be disconnected). A set A of unordered pairs of (possibly equal) points in X is called *admissible* if there exists an $f \in \mathrm{E}'(X)$ with $A(f) = A$, and we denote by $\mathscr{A}(X)$ the collection of admissible sets. To every $A \in \mathscr{A}(X)$, we associate the affine subspace $H(A)$ of $\mathbb{R}^X$ given by

$$\begin{aligned} H(A) &:= \{g \in \mathbb{R}^X : A \subset A(g)\} \\ &= \{g \in \mathbb{R}^X : g(x)+g(y) = d(x,y) \text{ for all } \{x,y\} \in A\}. \end{aligned}$$

We define the *rank* of A to be the dimension of $H(A)$, namely

$$\operatorname{rank}(A) := \dim(H(A)) \in \mathbb{N} \cup \{0,\infty\}.$$

Furthermore, let

$$P(A) := \mathrm{E}'(X) \cap H(A) = \{g \in \mathrm{E}'(X) : A \subset A(g)\}$$

and note that

$$P(A) = \mathrm{E}(X) \cap H(A) = \Delta(X) \cap H(A).$$

If (X,d) is a finite metric space, $\mathrm{E}(X)$ is a finite polyhedral complex and its dimension is the maximum of the dimensions of its cells. The *combinatorial dimension* $\dim_{\mathrm{comb}}(X)$ of a metric space (X,d) is the supremum of the dimensions of the polyhedral complexes $\mathrm{E}(Y)$ over all finite subsets $Y \subset X$. For any set S such that $|S| \leq 2$, let S_2 denote the collection of all subsets of S of cardinality exactly two. For Z a set of even cardinality, every involution $i\colon Z \to Z$ selects a subset $Z_i := \{\{z, i(z)\} : z \in Z\}$ of Z_2 of $|Z|/2$ disjoint pairs. Using the main theorem of [39], we obtain the following criterion in the spirit of Dress' Theorem, cf. [16, Theorem 4.1].

1.1 Theorem. *Let (X, d) be a metric space and let $n \in \mathbb{N}$. The inequality $\dim_{\text{comb}}(X) \leq n$ holds if and only if for every set $Z \subset X$ satisfying $|Z| = 2(n+1)$ and for every fixed-point free involution $i \colon Z \to Z$, there exists a weight function $w \colon Z_2 \to \frac{1}{2}\mathbb{Z} \cap [-2, 2]$ such that the following hold:*

(1) $w \not\equiv 0$, $w \leq 0$ on $Z_i := \{\{z, i(z)\} : z \in Z\}$ and $w \geq 0$ on $Z_2 \setminus Z_i$,

(2) for every $z \in Z$, one has $\sum_{z' \in Z \setminus \{z\}} w(\{z, z'\}) = 0$ and

(3) $S(w) := \sum_{\{z,z'\} \in Z_2} w(\{z, z'\}) d(z, z') \geq 0$.

Now, an *A-path* in X of length $l \in \mathbb{N} \cup \{0\}$ is an $(l+1)$-tuple $(v_0, \ldots, v_l) \in X^{l+1}$ with $\{v_{i-1}, v_i\} \in A$ for $i \in \{1, \ldots, l\}$. An *A-cycle* is an A-path $(v_0, \ldots, v_l)$ with $v_l = v_0$. Note that (x, x) is an A-cycle of length 1 if $\{x, x\} \in A$. We define the *A-component* $[x]^A$ of a point $x \in X$ to be the set

$$[x]^A := \{y \in X : \text{there exists an } A\text{-path from } x \text{ to } y\}.$$

Later, if no ambiguity arises, we write simply $[x]$ instead of $[x]^A$. Now, if g and h are two elements in $H(A)$ and $\{v, v'\} \in A$, we have $g(v) + g(v') = d(v, v') = h(v) + h(v')$ hence $g(v') - h(v') = -(g(v) - h(v))$, it thus follows that

$$g(y) - h(y) = (-1)^l (g(x) - h(x)) \tag{1.3}$$

whenever there is an A-path of length l from x to y. As a consequence, if there exists an A-cycle of odd length in $[x]^A$, for all $g, h \in H(A)$, one then has

$$g|_{[x]^A} = h|_{[x]^A}. \tag{1.4}$$

We call $[x]^A$ an *odd A-component* in this case. In the other case, i.e. if $[x]^A$ contains no A-cycle of odd length, $[x]^A$ is called an *even A-component.* We have the following important property holding for integer-valued metric spaces, which is due to Urs Lang and which we prove in Section V.3.

1.2 Theorem. *Let (X, d) be a metric space with integer-valued metric and let $A \in \mathscr{A}(X)$ be such that $1 \leq k := \operatorname{rank}(A) < \infty$. Then, $P(A)$ is maximal in the sense that there is no $A \supsetneq A' \in \mathscr{A}(X)$ if and only if all connected components of (X, A) are complete bipartite, in particular, (X, A) has no odd A-component.*

Back to a general metric space (X, d), we note that the set $\{g|_{[x]^A} : g \in H(A)\}$ is a one-dimensional real parameter family. In fact, every even A-component admits a unique partition

$$[x]^A = [x]^A_1 \cup [x]^A_{-1} \tag{1.5}$$

such that $x \in [x]_1^A$ and every edge $\{v, v'\} \in A$ with $\{v, v'\} \subset [x]^A$ connects $[x]^A$ and $[x]_{-1}^A$, in other words, the subgraph of (X, A) induced by $[x]^A$ is bipartite. If one chooses reference points $x_1, \dots, x_n \in X$ such that $[x_1]^A, \dots, [x_n]^A$ are precisely the n even A-components of X, there is then a unique induced canonical partition associated to A, namely

$$X = X_0^A \cup \bigcup_{i=1}^{n} \left([x_i]_1^A \cup [x_i]_{-1}^A\right) \tag{1.6}$$

where X_0^A (which we will simply denote by X_0^A if no ambiguity arises) is the union of all non-bipartite components of (X, A). Later, for each $(i, \sigma) \in \{1, \dots, n\} \times \{\pm 1\}$ we denote by $\langle x_i^\sigma \rangle^A$ a choice of a reference point in $[x_i]_\sigma^A$ in which case we can rewrite (1.6) as

$$X = X_0^A \cup \bigcup_{i=1}^{n} \left(\langle x_i^\sigma \rangle^A \cup \langle x_i^{-\sigma} \rangle^A\right). \tag{1.7}$$

By (1.3), $g(y) - h(y) = \sigma(g(x) - h(x))$ whenever $g, h \in H(A)$, $\sigma \in \{1, -1\}$, and $y \in [x]_\sigma^A$. It is now clear that $\operatorname{rank}(A)$ is exactly the number of even A-components of X. If $\operatorname{rank}(A) = 0$, $H(A)$ consists of a single function. This occurs in particular if $A = A(d_x)$ for some $x \in X$ in which case $\{x, y\} \in A$ for every $y \in X$, so X is A-connected, and (x, x) is an A-cycle of length 1. Now, a metric space (X, d) is called *discretely path-connected* if it has integer-valued metric and if for every pair of points $x, y \in X$, there exists a *discrete path* $\gamma\colon \{0, 1, \dots, l\} \to X$ from x to y namely γ satisfies $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(k-1), \gamma(k)) = 1$ for every $k \in \{1, \dots, l\}$. If γ can in addition be chosen to be an isometric embedding $\gamma\colon \{0, 1, \dots, d(x, y)\} \to X$, then (X, d) is called *discretely geodesic*. In the theorem below, let

$$\operatorname{diam}(P(A)) := \sup\{\|f - g\|_\infty : f, g \in P(A)\}$$

which is finite. In this case, one has

1.3 Theorem. *Let (X, d) be a discretely path-connected metric space, $A \in \mathscr{A}(X)$ and let $X = X_0 \cup \bigcup_{j=1}^k [x_j]$ be the associated decomposition with reference points $\{x_1, \dots, x_k\}$ as defined in* (1.6)*. Then, the following hold:*

(i) If $X_0 = \emptyset$, then $\operatorname{diam}(P(A)) \le 2\operatorname{dim}(P(A)) - 1$.

(ii) If $X_0 \neq \emptyset$, then $\operatorname{diam}(P(A)) \le 2\operatorname{dim}(P(A))$.

We also exhibit metric spaces showing that the above bounds are optimal. If X has integer-valued metric, and if $y \in X_0 := X \setminus \bigcup_{k=1}^{n} [x_k]$, then there is an A-path from y to itself of odd length, so $f(y) \in \mathbb{Z} - f(y)$ and thus $f(y) \in \frac{1}{2}\mathbb{Z}$. Hence, if $\operatorname{rank}(f) = 0$, then $X = X_0$ and therefore

$$f(X) \subset \frac{1}{2}\mathbb{Z}. \tag{1.8}$$

For (X, d) a general metric space again, suppose now that $A \in \mathscr{A}(X)$, and $1 \leq n := \operatorname{rank}(A) < \infty$. Then, the difference of any two elements of $H(A)$ is uniformly bounded on X, so the supremum norm gives a metric on $H(A)$, and there exists an affine isometry from $H(A)$ onto l_∞^n. In particular $H(A)$ is injective. Indeed, let $I \colon H(A) \to l_\infty^n$ be the affine map defined by

$$I(g) := (g(x_1), \ldots, g(x_n)). \tag{1.9}$$

It follows from (1.3) that $\|g - h\|_\infty = \max_{1 \leq k \leq n} |g(x_k) - h(x_k)| = \|I(g) - I(h)\|_\infty$ for all $g, h \in H(A)$.

Finally, we switch to metric spaces with bicombings and subsets of $l_\infty(\mathbb{N})$.

1.4 Definition. A metric space (X, d) *with a bicombing* is a geodesic metric space with a map

$$\sigma \colon X \times X \times [0, 1] \to X$$

such that for any $x, y \in X$, the map $\sigma_{xy} := \sigma(x, y, \cdot) \colon [0, 1] \to X$ satisfies the following three properties:

(i) σ_{xy} is a geodesic from x to y. This namely means that $\sigma_{xy}(0) = x$, $\sigma_{xy}(1) = y$, and $d(\sigma_{xy}(t), \sigma_{xy}(t')) = |t - t'| \, d(x, y)$ for $t, t' \in [0, 1]$,

(ii) $\sigma_{yx}(t) = \sigma_{xy}(1 - t)$ for $t \in [0, 1]$,

(iii) $d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1 - t) \, d(x, x') + t \, d(y, y')$ for $t \in [0, 1]$.

A map σ with those properties is called a *bicombing.*

Note that separable metric spaces embed isometrically via the Kuratowski embedding into $l_\infty(\mathbb{N})$. Now, recall from [30] that every injective metric space admits a bicombing and remember that hyperconvexity and injectivity are equivalent. Recall that the dual $l_1(\mathbb{N})^*$ and $l_\infty(\mathbb{N})$ are isomorphic as normed spaces. Note that a weak* closed subset of $l_\infty(\mathbb{N})$ is norm closed and thus complete for the standard metric of $l_\infty(\mathbb{N})$. From Section V.5, we obtain

1.5 Theorem. *Let (X, d) be a proper metric space or a weak* closed subset of $l_\infty(\mathbb{N})$ endowed with standard metric of $l_\infty(\mathbb{N})$. Then, the following are equivalent:*

(i) X is a hyperconvex metric space,

(ii) X is a 4-hyperconvex metric space with a bicombing.

In the above, a metric space X is *proper* if closed bounded subsets are compact. Finally, we note that the constant four in the above statement cannot be decreased.

V.2 Dress' Theorem

In order for this chapter to be self-contained, we describe the same construction as in Section III.7 and which was introduced in [39]. Let Σ be a linear system of inequalities of the form

$$\Sigma := \{a_i y_i + b_i z_i \succeq c_i\}_{i \in I}$$

where the generic notation $\succeq$ is used to regroup both $\geq$ and $>$ under a common symbol. For $i \in \{1, \ldots, n\}$, each variable y_i and z_i is drawn from a finite set $\{w_0, w_1, \ldots, w_n\}$. As a matter of convention, we can require for every $i \in \{1, \ldots, n\}$ that $a_i \neq 0$ unless $y_i = w_0$, in which case $a_i := 0$ where w_0 is the *zero variable*, similarly $b_i \neq 0$ if $z_i \neq w_0$ and $b_i := 0$ otherwise. We can associate to Σ an undirected labeled multigraph without self-loops $\Gamma_\Sigma := (\mathrm{V}_\Sigma, \mathrm{E}_\Sigma)$ where the vertex set V_Σ is given by $\{w_0, w_1, \ldots, w_n\}$ and the set $\mathrm{E}_\Sigma := \{E_i\}_{i \in I}$ consists of all the labeled edges $E_i = \big(\{y_i, z_i\}, \Sigma_i\big)$ where Σ_i denotes the inequality $a_i y_i + b_i z_i \succeq c_i$. Note that Γ_Σ does not contain any self-loop since we require $y_i \neq z_i$, that is all equations in Σ contain two different variables. Equations that contain only one variable different from w_0 are given by edges connecting to w_0. Note that by definition, Σ does neither contain any trivial inequalities like for example $1 \geq 0$ nor trivial inequalities of the other type like for instance $-\frac{1}{3} > 0$. A *path* P in Γ_Σ is then given by

$$\big((v_1, \ldots, v_{m+1}), E_1, \ldots, E_m\big) \tag{2.1}$$

where $(v_1, \ldots, v_{m+1})$ is a sequence of vertices in V_Σ and $(E_1, \ldots, E_m)$ a sequence of labeled edges in E_Σ such that for each $l \in \{1, \ldots, m\}$, one has

$$E_l = (\{v_l, v_{l+1}\}, a_l v_l + b_l v_{l+1} \succeq c_l)\,.$$

We call P *admissible* if for each $l \in \{1, \ldots, m-1\}$, the coefficients b_l and a_{l+1} have opposite signs (i.e., one is strictly positive and the other one is strictly negative). Note that if P is admissible, one has $v_l \neq w_0$ for each $l \in \{2, \ldots, m-1\}$ since w_0 only appears with zero coefficient. Admissible paths correspond to sequences of inequalities that form transitivity chains. A path is called a *loop* if its first and last vertices are identical and a loop is said to be *simple* as soon as its intermediate vertices are distinct. The reverse of an admissible loop is admissible and cyclic permutations of a loop P given by (2.1) are admissible if and only if a_1 and b_m have opposite signs, in which case P is called *permutable*. Note also that since w_0 only appears with zero coefficient, no admissible loop with initial vertex w_0 is permutable.

For an admissible path P given again by (2.1), let us define the *residue inequality* of P to be the inequality obtained by applying transitivity to the inequalities labeling the edges of P. The residue inequality of P is thus of the form $av_1 + bv_{m+1} \succeq c$, where $\succeq$ denotes a strict inequality if and only if at least one of the inequalities labeling the edges of P is strict.

In the case where P is a loop with initial vertex v, its residue inequality is of the form $(a+b)v \succeq c$. If it happens that $(a+b)v > c$, $a+b = 0$ and $c \geq 0$ or $(a+b)v \geq c$, $a+b = 0$ and $c > 0$, the residue inequality of P is false and we say that P is an *infeasible* loop. By definition, infeasibility implies admissibility. We define a *closure* $\overline{\Gamma}_\Sigma := (V_\Sigma, \overline{\mathrm{E}}_\Sigma)$ of Γ_Σ to be a graph $\overline{\Gamma}_\Sigma$ containing Γ_Σ and having the same vertex set, such that $\overline{\mathrm{E}}_\Sigma$ is obtained from E_Σ by adding for each simple admissible loop P (modulo permutations and reversals) of Γ_Σ, a *residue edge*, that is a new edge labeled with the residue inequality of P. Let moreover $\mathrm{Nontrivial}(\overline{\mathrm{E}}_\Sigma)$ denote all the elements of $\overline{\mathrm{E}}_\Sigma$ that are no self-loop at w_0. Note that a closure is not necessarily unique since the initial vertex of each permutable loop can be chosen arbitrarily. We can now state the main theorem of [39]:

2.1 Theorem. *Σ is unsatisfiable if and only if $\overline{\Gamma}_\Sigma$ has an infeasible simple loop.*

In the proof of Theorem 1.1, we will use the correspondence between admissible loops and weight functions satisfying properties like (1)-(3) in Theorem 1.1. Later, we often use the shorthand notations w_{xy} and d_{xy} to denote $w(\{x,y\})$ and $d(x,y)$ respectively.

Proof of Theorem 1.1. Let us first assume that such a weight function w satisfying the assumptions of the theorem exists. Let $Z_i := \{\{z, i(z)\} : z \in Z\}$. Let $E \subset Z_2$ and let $\bar{w}$ be any weight function on Z_2, we define the map

$\sigma_E^{\bar{w}} \colon Z \to \mathbb{R}$ by the assignment

$$z \mapsto \sum_{\{z,z'\}\in E} \bar{w}_{zz'}.$$

By (2) in Theorem 1.1, one has $\sigma_{Z_2}^{w}(z) = 0$. Hence in particular for every $z \in Z$, since i is an involution, one has

$$\sigma^w_{Z_2\setminus Z_i}(z) = \sigma^{-w}_{Z_i}(z) = -w_{zi(z)} = \sigma^{-w}_{Z_i}(i(z)) = \sigma^w_{Z_2\setminus Z_i}(i(z)) \tag{2.2}$$

Assume now that $f \in \Delta(Z)$ and $Z_i \subset A(f)$, one has $f(z) + f(z') \geq d_{zz'}$ for any $\{z, z'\} \in Z_2$ and

$$f(z) + f(i(z)) = d_{zi(z)} \text{ for any } \{z, i(z)\} \in Z_i. \tag{2.3}$$

Now,

$$\sum_{z\in Z} \sigma^w_{Z_2\setminus Z_i}(z) f(z) = \sum_{\{z,z'\}\in Z_2\setminus Z_i} w_{zz'}[f(z)+f(z')] \geq \sum_{\{z,z'\}\in Z_2\setminus Z_i} w_{zz'} d_{zz'} \tag{2.4}$$

We can thus apply (3) in Theorem 1.1 to obtain that the last quantity is bounded below by $\sum_{\{z,z'\}\in Z_i} -w_{zz'}d_{zz'}$ which is again the same as $\sum_{z\in Z} \sigma^w_{Z_2\setminus Z_i}(z) f(z)$ by (2.2) and by (2.3). It follows that the inequality in (2.4) is an actual equality, which implies

$$f(z') + f(z'') = d_{z'z''} \text{ for every } \{z', z''\} \in Z_2 \setminus Z_i \text{ such that } w_{z'z''} \neq 0.$$

Now, since $w \not\equiv 0$, there is $\{\bar{z}, i(\bar{z})\} \in Z_i$ such that $w_{\bar{z}i(\bar{z})} < 0$ and there is therefore $\{z', z''\} \in Z_2 \setminus Z_i$ such that $w_{z'z''} > 0$. Hence, if $f \in \Delta(Z)$ and $Z_i \subset A(f)$ then $Z_i \subsetneq A(f)$ and thus $\mathrm{rank}(f) \leq n$. It follows that $\dim_{\mathrm{comb}}(X) \leq n$.

Conversely, assume that $\dim_{\mathrm{comb}}(X) \leq n$ and let $Z \subset X$ with $|Z| = 2(n+1)$. Let moreover $i \colon Z \to Z$ be a fixed-point free involution. Let Σ be the linear system of inequalities associated to $\Delta(Z)$ and to the edge set Z_i, in other words $f \in \mathbb{R}^X$ is a solution to Σ if and only if $f \in \Delta(X)$ and $A(f) = Z_i$. We now want to apply Theorem 2.1 and proceed as follows.

The system Σ induces an associated self-loop free labeled multigraph $\Gamma_\Sigma := (\{f(z_0), f(z_1), \ldots, f(z_{2n+2})\}, \mathrm{E}_\Sigma)$ as in Table V.1 where the zero variable is denoted by $f(z_0)$. By assumption, the system Σ is unsatisfiable and thus the arbitrarily chosen closure $\bar{\Gamma}_\Sigma$ contains an infeasible simple loop L by Theorem 2.1. It is easy to see that by admissibility, L cannot contain any of the labeled edges in (b) which are of the form $(\{f(z_i), f(z_j)\}, f(z_i) + f(z_j) \geq d_{ij})$.

Σ	E_Σ
(a) $f(z_i)+f(z_j)>d_{ij}$	$(\{f(z_i),f(z_j)\},f(z_i)+f(z_j)>d_{ij})$
(b) $f(z_i)+f(z_j)=d_{ij}$	$(\{f(z_i),f(z_j)\},-f(z_i)-f(z_j)\geq -d_{ij})$ and $(\{f(z_i),f(z_j)\},f(z_i)+f(z_j)\geq d_{ij})$
(c) $f(z_i)>d_{ii}=0$	$(\{f(z_0),f(z_i)\},f(z_i)>0)$

Table V.1: Description of assignments of labeled edges in Γ_Σ to inequalities Σ.

Furthermore, we can assume without loss of generality that L is not a self-loop at $f(z_0)$. We introduce now

$$\mathrm{E}_{Z_i} := \Big\{(\{f(z),f(i(z)\},-f(z)-f(i(z))\geq -d_{ij})\Big\}_{z\in Z}$$

and

$$\mathrm{E}_{Z_2\setminus Z_i} := \Big\{(\{f(z_i),f(z_j\},f(z_i)+f(z_j)>d_{ij})\Big\}_{\{z_i,z_j\}\in Z_2\setminus Z_i}.$$

We distinguish two cases:

Case 1: *L does not contain the variable $f(z_0)$.*

Then by admissibility, there is a permutation θ such that L is a simple loop given by $\Big((f(z_{\theta(1)}),\ldots,f(z_{\theta(2k)})),E_1,\ldots,E_{2k}\Big)$ and we can assume without loss of generality that

$$E_1=(\{f(z_{\theta(1)}),f(z_{\theta(2)})\},-f(z_{\theta(1)})-f(z_{\theta(2)})\geq -d_{\theta(1)\theta(2)})\in \mathrm{E}_{Z_i},$$

and

$$E_{2k}=\{f(z_{\theta(2k)}),f(z_{\theta(1)})\},f(z_{\theta(2k)})+f(z_{\theta(1)})>d_{\theta(2k)\theta(1)})\in \mathrm{E}_{Z_2\setminus Z_i},$$

therefore in particular $i(z_{\theta(2k-1)})=z_{\theta(2k)}$, that is $E_{2k-1}\in \mathrm{E}_{Z_i}$. Hence, the residue inequality of L is of the form

$$0>\sum_{j=1}^{2k}(-1)^j d_{\theta(j)\theta(j+1)}.$$

It follows by infeasibility of L that $\sum_{j=1}^{2k}(-1)^j d_{\theta(j)\theta(j+1)}\geq 0$. Setting now

$$w_{z,z'} := \begin{cases} -1 & \text{if } \{z,z'\}\in\{\{z_{\theta(2l-1)}z_{\theta(2l)}\}\}_{l\in\{1,\ldots,k\}},\\ 1 & \text{if } \{z,z'\}\in\{\{z_{\theta(2l)}z_{\theta(2l+1)}\}\}_{l\in\{1,\ldots,k\}},\\ 0 & \text{otherwise.}\end{cases}$$

we obtain the desired weight function w.

Case 2: *L contains the variable $f(z_0)$.*

Under this assumption, L is a simple loop given by

$$\Big((f(z_0), f(z_{\theta(1)}), \dots, f(z_{\theta(2k)})), E_1, \dots, E_{2k+1}\Big).$$

We can assume without loss of generality that the starting and ending edges of L have labels induced by simple admissible loops

$$L^\sigma := \Big((f(z_{\sigma(1)}), \dots, f(z_{\sigma(2k_1-1)})), E_1^\sigma, \dots, E_{2k_1}^\sigma\Big)$$

and

$$L^\tau := \Big((f(z_{\tau(1)}), \dots, f(z_{\tau(2k_2-1)})), E_1^\tau, \dots, E_{2k_2}^\tau\Big)$$

instead of having starting and ending edges of the form

$$(\{f(z_0), f(z_{\theta(1)})\}, f(z_{\theta(1)}) > 0) \text{ and } (\{f(z_0), f(z_{\theta(2k)})\}, f(z_{\theta(2k)}) > 0).$$

Indeed, since if for instance $L' := ((f(z_1), f(z_2), f(z_3)), E_1, E_2, E_3)$ is an admissible loop with $E_2 \in \mathrm{E}_{Z_i}$, then by the triangle inequality, we can replace the edge $(\{f(z_0), f(z_1)\}, f(z_1) > 0)$ by substituting the label $f(z_1) > 0$ by the residue equation $2f(z_1) > d_{12} - d_{23} + d_{13}$ of L'. By the triangle inequality, L' remains infeasible after this replacement. Note that since Z_i is a set of disjoint edges and since the variable $f(z_{\sigma(1)})$ must appear with the same sign in both E_1^σ and $E_{2k_1}^\sigma$ and $f(z_{\tau(1)})$ with the same sign in both E_1^τ and $E_{2k_2}^\tau$, one must have $E_1^\sigma, E_{2k_1}^\sigma, E_1^\tau, E_{2k_2}^\tau \in \mathrm{E}_{Z_2 \setminus Z_i}$. Hence, the residue inequality of L is without loss of generality of the form

$$\frac{1}{2}\sum_{j=1}^{2k_1-1}(-1)^{j+1} d_{\sigma(j)\sigma(j+1)} + \sum_{j=1}^{2k-1}(-1)^j d_{\theta(j)\theta(j+1)} + \frac{1}{2}\sum_{j=1}^{2k_2-1}(-1)^{j+1} d_{\tau(j)\tau(j+1)} \geq 0 \tag{2.5}$$

with

(A) $\sigma(2k_1) = \sigma(1) = \theta(1)$ and $\tau(2k_2) = \tau(1) = \theta(2k)$ as well as

(B) $i(z_{\theta(2l-1)}) = z_{\theta(2l)}$, $i(z_{\sigma(2l)}) = z_{\sigma(2l+1)}$ and $i(z_{\tau(2l)}) = z_{\tau(2l+1)}$.

If $w_{z_i z_j}$ is defined as the coefficient of d_{ij} in (2.5), then (1) in the statement of the Theorem holds by (b) in Table V.1. Furthermore, we have

$$\sum_{z \in Z \setminus \{z_{\theta(1)}\}} w_{z_{\theta(1)} z} = w_{z_{\sigma(1)} z_{\sigma(2)}} + w_{z_{\sigma(2k_1-1)} z_{\sigma(2k_1)}} + w_{z_{\theta(1)} z_{\theta(2)}} = \tfrac{1}{2} + \tfrac{1}{2} - 1 = 0$$

and similarly

$$\sum_{z\in Z\setminus\{z_{\theta(2k)}\}} w_{z_{\theta(2k)}z} = w_{z_{\theta(2k-1)}z_{\theta(2k)}} + w_{z_{\tau(1)}z_{\tau(2)}} + w_{z_{\tau(2k_2-1)}z_{\tau(2k_2)}} = 0.$$

Moreover, for $z' \in Z \setminus \{z_{\theta(1)}, z_{\theta(2k)}\}$, it is clear that our definition implies $\sum_{z\in Z\setminus\{z'\}} w_{z'z} = 0$, thus (2) in the statement of the Theorem holds. By definition of w, (2.5) is equivalent to $S(w) \geq 0$ and thus (3) holds as well. This concludes the proof. □

Finally, we have the following variant of Dress' Theorem. The equivalence between the first two properties is [16, Theorem 4.1]. Our proof uses Theorem 1.1.

2.2 Theorem. *Let (X, d) be a metric space and let $n \geq 1$ be an integer. The following are equivalent:*

- *(i)* $\dim_{\text{comb}}(X) \leq n$.
- *(ii) For every set $Z \subset X$ satisfying $|Z| = 2(n+1)$ and for every fixed-point free involution $i\colon Z \to Z$, there exists a fixed-point free bijection $j\colon Z \to Z$ different from i such that*

$$\sum_{z\in Z} d(z, i(z)) \leq \sum_{z\in Z} d(z, j(z)). \tag{2.6}$$

- *(iii) For $Z \subset X$ with $|Z| = 2(n+1)$ and for every fixed-point free involution $i\colon Z \to Z$, there exists a weight function $w\colon Z_2 \to \{-2, -1, 0, 1, 2\}$ such that properties (1) − (3) in Theorem 1.1 hold.*

Proof. Recall that $Z_i := \{\{z, i(z)\} : z \in Z\}$. We start by showing the equivalence between *(ii)* and *(iii)*. Assume first that *(ii)* holds. Define $w_{zz'}$ to be the coefficient of $d_{zz'}$ in the expression

$$\sum_{z\in Z} d_{zj(z)} - \sum_{z\in Z} d_{zi(z)}.$$

It is easy to see that w then satisfies *(iii)*. Conversely, assume that *(iii)* holds. We define the undirected edge sets E_i and E_j by giving weight -1 to each edge of E_i and weight 1 to each edge of E_j and choosing multiplicity in order for the sum to match the weights given by w following Table V.2. Since w

satisfies (2) in Theorem 1.1, it follows that the graph (Z, E_j) has degree zero or two at each vertex since

$$\sum_{z' \in Z\setminus\{z\}:\{z,z'\}\in Z_2\setminus Z_i} w_{zz'} = -\sum_{z' \in Z\setminus\{z\}:\{z,z'\}\in Z_i} w_{zz'} = -w_{zi(z)} \leq 2$$

and thus we can decompose it into disjoint cycles by Veblen's theorem for multigraphs, cf. [23]. Finally, on each of these cycles, we can pick an arbitrary orientation which induces then a bijection $j\colon Z \to Z$. It is then easy to see that j satisfies the requirement of (ii) and this concludes the proof.

$w_{zz'}$	-2	-1	$w_{zz'} = 0$ and $z' = i(z)$	$w_{zz'} = 0$ and $z' \neq i(z)$	1	2
E_i	2	2	2	0	0	0
E_j	0	1	2	0	1	2

Table V.2: Multiplicity choices according to weights.

The fact that (ii) implies (i) is the easy implication in the statement [16, Theorem 4.1]. To be self-contained, we repeat the proof. Let $Y \subset X$ be a finite set. If $|Y| \leq 2n+1$, then $\dim(\mathrm{E}(Y)) \leq n$. Suppose now that $|Y| \geq 2n+2$ as well as $f \in \mathrm{E}(Y)$, and let $Z \subset Y$ be a set with $|Z| = 2n+2$ so that there is a fixed point free involution $i\colon Z \to Z$ such that for every $z \in Z$, one has $\{z, i(z)\} \in A(f)$. By assumption, there exists a fixed point free bijection $j\colon Z \to Z$ such that $j \neq i$ and

$$\sum_{z\in Z} d(z, i(z)) \leq \sum_{z\in Z} d(z, j(z)).$$

Since $f(z) + f(i(z)) = d(z, i(z))$ and $d(z, j(z)) \leq f(z) + f(j(z))$, this gives

$$\sum_{z\in Z} f(z) + f(i(z)) \leq \sum_{z\in Z} f(z) + f(j(z)).$$

Since both i and j are bijections, these two sums agree so that each of the inequalities $d(z, j(z)) \leq f(z) + f(j(z))$ must in fact be an equality. Therefore $\{z, j(z)\} \in A(f)$. There is at least one $z \in Z$ such that $j(z) \neq i(z)$, so the graph with vertex set Z and edge set $\{\{z, i(z)\}\}_{z\in Z} \cup \{\{z, j(z)\}\}_{z\in Z}$ has at most n connected components. As this holds for every Z and i as above, we conclude that the graph $(Y, A(f))$ has no more than n components. Since $f \in \mathrm{E}(Y)$ was chosen arbitrarily, this shows that the dimension of $\mathrm{E}(Y)$ is less than or equal to n.

We now show that (i) implies (ii). With the help of Theorem 1.1, we obtain a shorter and more natural argument than the one given in the proof of [16, Theorem 4.1]. Thus, assume that (i) holds. Let us consider the set $W(0)$ of all $w\colon Z_2 \to \mathbb{R}$ such that $w \leq 0$ on Z_i, $w \geq 0$ on $Z_2 \setminus Z_i$, and

$$\sum_{z' \in Z\setminus\{z\}} w_{zz'} = 0$$

for all $z \in Z$. By Theorem 1.1, it follows that we can pick $w \in W(0) \setminus \{0\}$ such that $S(w) := \sum_{\{z,z'\}\in Z_2} w_{zz'} d(z,z') \geq 0$ and so that no $w' \in W(0) \setminus \{0\}$ with $S(w') \geq 0$ has strictly smaller support (edges where the weight function is nonzero). It follows that

$$\{v \in W(0) : \operatorname{spt}(v) \subset \operatorname{spt}(w)\} \subset \{v \in W(0) : S(v) \geq 0\}. \tag{2.7}$$

To see that (2.7) holds, let $\lambda > 0$ be the maximal number with the property that $|\lambda v| \leq |w|$. Then, $v' := w - \lambda v$ belongs to $W(0)$ and $\operatorname{spt}(v')$ is a strict subset of $\operatorname{spt}(w)$. If either $v' \neq 0$, then $S(v') < 0$ and $\lambda S(v) = S(w) - S(v') > 0$ and otherwise $v' = 0$ which implies $S(v') = 0$ and thus $\lambda S(v) = S(w) - S(v') = S(w) \geq 0$. This proves (2.7).

Now we can proceed as just as in [16]. Since for fixed z, the sum of the weights $w_{zz'}$ is zero, we claim that there exist $m \in \mathbb{N}$ and pairwise distinct points $\{z_0, z_1, \ldots, z_m\}$ such that for each $k \in \{0, \ldots, m\}$, one has

$$w_{z_k i(z_k)} < 0 < w_{i(z_k) z_{k+1}}$$

where $z_{m+1} := z_0$. We do not exclude that for $l > k$ one might for instance have $z_l = i(z_k)$. To see that such points $\{z_0, z_1, \ldots, z_m\}$ exist, one can proceed inductively starting with $z'_0 \in Z$ such that $w_{z'_0 i(z'_0)} < 0$. Therefore, there is $z'_1 \in Z$ such that $w_{i(z'_0) z'_1} > 0$. Since $|Z| < \infty$, we can continue this process until we eventually reach $z'_l \in Z$ with $w_{z'_l i(z'_l)} < 0$ and so that the only $z \in Z$ such that $w_{i(z'_l) z} > 0$ satisfies $z = z'_k$ for some $0 \leq k \leq l-1$. In this case, the result follows by setting $z_0 := z'_k$, $z_1 := z'_{k+1}$, …, $z_m := z'_l$ and this proves the claim.

Finally, the function

$$v := \sum_{k=0}^{m} \Big(-\delta_{\{z_k, i(z_k)\}} + \delta_{\{i(z_k), z_{k+1}\}} \Big)$$

defined on Z_2 belongs to $W(0) \setminus \{0\}$ and $\operatorname{spt}(v) \subset \operatorname{spt}(w)$. Hence $S(v) \geq 0$ by (2.7). This means that

$$\sum_{k=0}^{m} d(z_k, i(z_k)) \leq \sum_{k=0}^{m} d(i(z_k), z_{k+1})$$

and (ii) easily follows, extending j by i if necessary. □

V.3 Barriers, Graphs of Maximal Elements and Applications

Let (X, d) be a metric space. For $A \in \mathscr{A}(X)$, let $P(A)$ be called a *barrier* if $\mathrm{E}(X) \setminus P(A)$ is disconnected. In the proposition below, we use the notation $I(f, f') := \{g \in \mathrm{E}(X) : d_\infty(f, g) + d_\infty(g, f') = d_\infty(f, f')\}$.

3.1 Proposition. *Let (X, d) be a metric space with integer-valued metric. Let $A \in \mathscr{A}(X)$ be such that $0 \leq \mathrm{rank}(A) < \infty$ and let $X' := X \setminus \mathrm{e}^{-1}(P(A))$. Then, $P(A)$ is a barrier if and only if the following hold:*

(i) There are $\emptyset \neq B, C \subset X'$ with $B \cap C = \emptyset$ and $X' = B \cup C$ so that for every $b \in B$ and for every $c \in C$, there is $h \in P(A)$ such that $\{b, c\} \in A(h)$ or equivalently

$$I(d_b, d_c) \cap P(A) \neq \emptyset.$$

(ii) For every $g \in \mathrm{E}(X, d) \setminus P(A)$ and for some $Y \in \{B, C\}$, one has for every $y \in Y$ that

$$I(g, d_y) \cap P(A) \neq \emptyset.$$

Proof. It is not difficult to see that if $P(A)$ is a barrier, then (i) and (ii) follow. Assume now that (i) and (ii) hold. Since $P(A)$ is injective by [30, Theorem 4.3], there is a 1-Lipschitz retraction $\varrho\colon \mathrm{E}(X, d) \to P(A)$. For any $z \in \mathrm{e}^{-1}(P(A))$, we have

$$d_\infty(\varrho(g), d_z) = d_\infty(\varrho(g), \varrho(d_z)) \leq d_\infty(g, d_z).$$

Now, by (ii), there is for any $y \in Y$, an element $h_y \in I(g, d_y) \cap P(A)$. Therefore

$$\begin{aligned} \varrho(g)(y) = d_\infty(\varrho(g), d_y) &\leq d_\infty(\varrho(g), h_y) + d_\infty(h_y, d_y) \\ &= d_\infty(\varrho(g), \varrho(h_y)) + d_\infty(h_y, d_y) \\ &\leq d_\infty(g, h_y) + d_\infty(h_y, d_y) \\ &= d_\infty(g, d_y) \\ &= g(y). \end{aligned}$$

For each $Y \in \{B, C\}$, we set $\mathrm{K}(Y) := F_Y^{-1}((-\infty, 0])$ where $F_Y\colon \mathrm{E}(X, d) \to \mathbb{R}$ is the Lipschitz function defined by

$$g \mapsto \sup_{z \in X \setminus Y} (\varrho(g)(z) - g(z)).$$

The set $\mathrm{K}(Y)$ is closed by continuity of F_Y. Furthermore, we have $d_b \in \mathrm{K}(B)$ and $d_c \in \mathrm{K}(C)$ and thus $\mathrm{K}(B), \mathrm{K}(C) \neq \emptyset$. In addition,

$$\mathrm{E}(X,d) \setminus P(A) = (\mathrm{E}(X,d) \setminus \mathrm{K}(B)) \sqcup (\mathrm{E}(X,d) \setminus \mathrm{K}(C))$$

Moreover, $\mathrm{K}(B) \cap \mathrm{K}(C) \subset P(A)$ since $F_B(g), F_C(g) \leq 0$ implies $\varrho(g) \leq g$ on all of X and thus $\varrho(g) = g$ by minimality of g, which implies that $g \in P(A)$. It follows that $\mathrm{E}(X,d) \setminus \mathrm{K}(B)$ and $\mathrm{E}(X,d) \setminus \mathrm{K}(C)$ induce a disjoint open non-empty partition of $\mathrm{E}(X,d) \setminus P(A)$ which is thus disconnected. □

Now we proceed to

Proof of Theorem 1.2. We know by [30, Theorem 4.3], that $P(A)$ is a k-dimensional polytope. Consider the family $\{f_v\}_{v \in V} \subset P(A)$ of all vertices of $P(A)$, i.e. for every $v \in V$, one has $\mathrm{rank}(A(f_v)) = 0$. Since $P(A)$ is a finite dimensional polytope, one has $|V| < \infty$. Furthermore, by (1.8), we already know that $f_v(X) \subset \frac{1}{2}\mathbb{Z}$ for every $v \in V$. It follows that the convex combination

$$f := \frac{1}{|V|} \sum_{v \in V} f_v \tag{3.1}$$

satisfies $f \in \mathrm{E}'(X)$ and $f(X) \subset \frac{1}{2|V|}\mathbb{Z}$ as well as $A(f) = A$. Hence in particular

$$\delta := \frac{1}{3} \min_{\{y,y'\} \notin A} |f(y) + f(y') - d(y,y')| > 0. \tag{3.2}$$

Assume that $A \in \mathscr{A}(X)$ is such that $1 \leq k = \mathrm{rank}(A) < \infty$ and that $P(A)$ is maximal. Assume by contradiction that there is a bipartite component $[w] = [w]_1 \cup [w]_{-1}$ of (X, A) (with $w \in [w]_1$) which is not complete bipartite, hence there is $(x, \bar{x}) \in [w]_1 \times [w]_{-1}$ such that $\{x, \bar{x}\} \notin A$. Let us define

$$Y := \{y \in [w]_{-1} : \{x, y\} \in A\}$$

as well as

$$Z := \{z \in [w]_1 : \text{for all } \bar{z} \in [w]_{-1} \setminus Y,\ \{z, \bar{z}\} \notin A\}.$$

Hence $[w] = Y \sqcup ([w]_{-1} \setminus Y) \sqcup Z \sqcup ([w]_1 \setminus Z)$ noting that each of these four sets is non-empty, $x \in Z$ and $\bar{x} \in [w]_{-1} \setminus Y$. For $f \in \mathrm{E}'(X)$ and δ as above, we can now set

$$g(y) := \begin{cases} f(y) - \delta & \text{if } \ y \in Z, \\ f(y) + \delta & \text{if } \ y \in Y, \\ f(y) & \text{otherwise.} \end{cases}$$

It is easy to see that $g \in \mathrm{E}'(X)$, since

(a) if $y \in Z$, then $\{y, \bar{y}\} \in A(g)$ for some $\bar{y} \in Y$,

(b) if $y \in Y$, then $\{x, y\} \in A(g)$ and we already noted that $x \in Z$,

(c) if $y \in [w]_1 \setminus Z$, then $\{y, \bar{y}\} \in A(g)$ for some $\bar{y} \in [w]_{-1} \setminus Y$ and finally

(d) if $y \in [w]_{-1} \setminus Y$, then $\{y, \bar{y}\} \in A(g)$ for some $\bar{y} \in [w]_1 \setminus Z$ by definition of Z.

It follows that $A(g) \subsetneq A(f) = A$ and this contradicts the maximality of $P(A)$.

Now, assume again that $A \in \mathscr{A}(X)$ is such that $1 \leq k = \operatorname{rank}(A) < \infty$ and that $P(A)$ is maximal. Suppose by contradiction that there is at least one odd A-component, in other words, X_0 as in (1.6) satisfies $X_0 \neq \emptyset$. Pick $x \in X_0$ such that $A \neq A(d_x)$. Let X_0^x be the connected component of X_0 containing x. Define

$$\mathcal{S} := \{S \subset X_0' : x \in S \text{ and for every } a, b \in S \text{ one has } \{a, b\} \notin A\}.$$

The set $\mathcal{S}$ is non-empty since it contains $\{x\}$ and it is endowed with a partial ordering given by the inclusion. Moreover, if $\{T_i\}_{i \in I}$ is a totally ordered subset of $\mathcal{S}$, then it is easy to see that $T := \cup_{i \in I} T_i$ satisfies $x \in T$ and if $a, b \in T$ then $\{a, b\} \notin A$, therefore $T \in \mathcal{S}$. Finally, one easily sees that T is an upper bound for $\{T_i\}_{i \in I}$. It follows that we can apply Zorn's lemma to deduce that $\mathcal{S}$ contains a maximal element M. By maximality of M, it is easy to see that for every $y \in X_0^x \setminus M$, there is $z \in M$ such that $\{y, z\} \in A$. Altogether, M satisfies

1) If $a, b \in M$ then $\{a, b\} \notin A$.

2) For every $y \in X_0^x \setminus M$, there is $z \in M$ such that $\{y, z\} \in A$.

Hence we can let $f \in \mathrm{E}'(X)$ be defined by (3.1) so that $A(f) = A$ and let $\delta > 0$ be given by (3.2), we can set

$$g(y) := \begin{cases} f(y) - \delta & \text{if } \ y \in M, \\ f(y) + \delta & \text{if } \ y \in X_0^x \setminus M, \\ f(y) & \text{if } \ y \in X \setminus X_0^x. \end{cases}$$

It is easy to see that $g \in \mathrm{E}'(X)$ and $A(g) \subsetneq A(f) = A$. Hence, this contradicts the maximality of $P(A)$. So we must have $X_0 = \emptyset$.

Conversely, assume now that all connected components of (X, A) are complete bipartite. This implies that for every connected component $[w]$ of X, for every pairs $(x,\bar{x}),(y,\bar{y}) \in [w]_1 \times [w]_{-1}$ and for $f \in \mathrm{E}'(X)$ such that $A(f) = A$, one has

$$d(x,\bar{x}) + d(y,\bar{y}) = f(x) + f(\bar{x}) + f(y) + f(\bar{y}) = d(x,\bar{y}) + d(\bar{x},y). \tag{3.3}$$

Let now $g \in \mathrm{E}'(X)$ be such that $A(g) \subset A$. Pick arbitrarily $(u,\bar{u}) \in [w]_1 \times [w]_{-1}$. Either $\{u,\bar{u}\} \in A(g)$ or there is $(v,\bar{v}) \in ([w]_1 \setminus \{u\}) \times ([w]_{-1} \setminus \{\bar{u}\})$ such that $\{u,\bar{v}\},\{\bar{u},v\} \in A(g) \subset A$ which implies together with (3.3) that

$$g(u) + g(v) + g(\bar{u}) + g(\bar{v}) = d(u,\bar{v}) + d(\bar{u},v) = d(u,\bar{u}) + d(v,\bar{v})$$

and thus $\{u,\bar{u}\},\{v,\bar{v}\} \in A(g)$ since $d(u,\bar{u}) \le g(u) + g(\bar{u})$ and $d(v,\bar{v}) \le g(v) + g(\bar{v})$. Consequently, we have $A(g) = A$. It follows that $P(A)$ is maximal and this concludes the proof. □

Let (X,d) be a metric space with integer-valued metric as well as $A \in \mathscr{A}(X)$ and assume that its associated decomposition (1.7) with reference points $\langle x_i^\sigma \rangle_{(i,\sigma)\in\{1,\ldots,n\}\times\{\pm 1\}}$ satisfies $X_0 = \emptyset$. Define

$$Y := \{x_1^1,\ldots,x_n^1,x_1^{-1},\ldots,x_n^{-1}\} \subset X \tag{3.4}$$

and let Y_2 denote the family of subsets of cardinality two of Y. Note that A induces a perfect matching in Y given by

$$A_Y := \big\{\{x_1^1,x_1^{-1}\},\ldots,\{x_n^1,x_n^{-1}\}\big\}.$$

Let us set

$$Y_2' := \big\{\{x_i^\sigma,x_j^\tau\} : (i,\sigma),(j,\tau) \in \{1,\ldots,n\} \times \{\pm 1\} \text{ and } i \neq j\big\}$$

Furthermore, for all $(i,\sigma),(j,\tau) \in \{1,\ldots,n\} \times \{\pm 1\}$ so that $\{x_i^\sigma,x_j^\tau\} \in Y_2'$, let

$$c(x_i^{-\sigma},x_j^{-\tau}) := \min_{(x,y)\in\langle x_i^\sigma\rangle\times\langle x_j^\tau\rangle} \Big[d(x,x_i^{-\sigma}) + d(y,x_j^{-\tau}) - d(x,y)\Big]. \tag{3.5}$$

Similarly, for all $(i,\sigma) \in \{1,\ldots,n\} \times \{\pm 1\}$, let further

$$c(x_i^{-\sigma}) := \frac{1}{2} \min_{(x,y)\in\langle x_i^\sigma\rangle\times\langle x_i^\sigma\rangle} \Big[d(x,x_i^{-\sigma}) + d(y,x_i^{-\sigma}) - d(x,y)\Big]. \tag{3.6}$$

3.2 Proposition. *Let (X,d) be a metric space with integer-valued metric. Let $A \in \mathscr{A}(X)$ be such that $1 \le n := \operatorname{rank}(A) < \infty$. Then, there is no $A \supsetneq A' \in \mathscr{A}(X)$ if and only if the decomposition of X given by (1.7) satisfies $X_0 = \emptyset$ and the following hold:*

(*i*) *For all* $(i,\sigma) \in I_n \times \{\pm 1\}$, $x \in \langle x_i^\sigma \rangle$ *and* $y \in \langle x_i^{-\sigma} \rangle$, *one has*

$$d(x,y) + d(x_i^{-\sigma}, x_i^\sigma) = d(x, x_i^{-\sigma}) + d(y, x_i^\sigma). \tag{3.7}$$

(*ii*) *There is a solution* $g \in \mathbb{R}^Y$ *to the system of linear inequalities with at most two variables per inequality given by*

$$g(x_i^1) + g(x_i^{-1}) = d(x_i^1, x_i^{-1}) \quad \text{for } i \in \{1,\dots,n\}, \tag{3.8}$$

$$g(x_i^\sigma) + g(x_j^\tau) < c(x_i^\sigma, x_j^\tau) \quad \text{for } \{x_i^\sigma, x_j^\tau\} \in Y_2', \tag{3.9}$$

$$g(x_i^\sigma) < c(x_i^\sigma) \quad \text{for } (i,\sigma) \in \{1,\dots,n\} \times \{\pm 1\}. \tag{3.10}$$

Proof. First, remember that by Theorem 1.2, if $A \in \mathscr{A}(X)$, $P(A)$ is maximal if and only if all connected components of (X, A) are complete bipartite, in particular $X_0 = \emptyset$. As argued at the beginning of the proof of Theorem 1.2, there is by definition a function $f \in \mathrm{E}'(X)$ such that $A(f) = A$ and such that $f(X) \subset \frac{1}{2N}\mathbb{Z}$ where N denotes the number of vertices of $P(A)$. Considering then the restriction $A_Y := A_Y(f_Y)$ and $g := f_Y$, it is easy to see that (*i*) and (*ii*) are fulfilled. Conversely, assume that (*i*) and (*ii*) hold. It follows in particular that $g \in P(A_Y)$. Now, for every $(i,\sigma) \in I_n \times \{\pm 1\}$ and every $x \in \langle x_i^\sigma \rangle$, let $\bar{g} \in \mathbb{R}^X$ be the function given by

$$\bar{g}(x) := d(x, x_i^{-\sigma}) - g(x_i^{-\sigma}).$$

By (3.7) and (3.8), one has for all $(x,y) \in \langle x_i^\sigma \rangle \times \langle x_i^{-\sigma} \rangle$, that

$$\bar{g}(x) + \bar{g}(y) = d(x, x_i^{-\sigma}) - g(x_i^{-\sigma}) + d(y, x_i^\sigma) - g(x_i^\sigma) = d(x,y).$$

For all $(i,\sigma), (j,\tau) \in I_n \times \{\pm 1\}$ satisfying either $i \neq j$ or satisfying $i = j$ and $\sigma = \tau$, one has for all $(x,y) \in \langle x_i^\sigma \rangle \times \langle x_j^\tau \rangle$ by (*b*):

$$\begin{aligned}
\bar{g}(x) + \bar{g}(y) &= d(x, x_i^{-\sigma}) - g(x_i^{-\sigma}) + d(y, x_j^{-\tau}) - g(x_j^{-\tau}) \\
&> d(x, x_i^{-\sigma}) + d(y, x_j^{-\tau}) \\
&\quad - \min_{(z,w) \in \langle x_i^\sigma \rangle \times \langle x_j^\tau \rangle} \Big[d(z, x_i^{-\sigma}) + d(w, x_j^{-\tau}) - d(z,w) \Big] \\
&\geq d(x,y).
\end{aligned}$$

Furthermore, for every $(i,\sigma) \in I_n \times \{\pm 1\}$ and $x \in \langle x_i^\sigma \rangle$, one has by (*c*):

$$\begin{aligned}
2\bar{g}(x) &= 2(d(x, x_i^{-\sigma}) - g(x_i^{-\sigma})) \\
&> 2d(x, x_i^{-\sigma}) - \min_{(z,w) \in \langle x_i^\sigma \rangle \times \langle x_i^\sigma \rangle} \Big[d(z, x_i^{-\sigma}) + d(w, x_i^{-\sigma}) - d(z,w) \Big] \geq 0.
\end{aligned}$$

Hence, equations (3.8), (3.9) and (3.10) ensure that $\bar{g} \in \mathrm{E}'(X)$ as well as $A(\bar{g}) = A$. This finishes the proof. □

The next example shows in particular that the system of linear inequalities given in Proposition 3.2 is minimal in dimension two (it is trivially minimal in dimension one). Indeed, we exhibit a maximal cell $P(A)$ for which $I(P(A))$ is the solution set of a system where each inequality corresponds to a supporting half-space of $I(P(A))$ containing a facet in its boundary.

3.3 Example. Consider the finite metric space $X := \{x_i\}_{1\leq i\leq 4} \cup \{x'_i\}_{1\leq i\leq 4}$ given by $d(x_i, x'_i) := 2$ as well as

$$d(x_i, x_{i+1}) = d(x_i, x'_{i+1}) = d(x'_i, x_{i+1}) = d(x'_i, x'_{i+1}) := 3$$

and $d(x_i, x_{i+2}) := 6$. The function f constantly equal to three on all of X generates a cell $P(A(f))$ of dimension two which is isometric to an octagon in l^2_∞ with four sides of length two and parallel to the coordinate axes and four diagonal sides of length one. The complex $\mathrm{E}(X)$ consists of a central cell $P(A(f))$ and each side of length two is parametrized by a pair $\{x_i, x'_i\}$. Along each such side, two triangles isometric to the convex hull $\mathrm{conv}((0,0),(1,1),(2,0)) \subset l^2_\infty$ are glued along their longer side. The first is $P(A(f_i))$, containing d_{x_i} and generated by the function

$$f_i(y) := \begin{cases} 0.5 & \text{if } \ y = x_i, \\ 1.5 & \text{if } \ y = x'_i, \\ 5.5 & \text{if } \ y \in \{x_{i+2}, x'_{i+2}\}, \\ 3 & \text{otherwise.} \end{cases}$$

The second triangle being $P(A(f'_i))$, containing $d_{x'_i}$ and generated by the function f'_i defined by interchanging the roles of x_j and x'_j for every j in the definition of f_i. Define further

$$g_i(y) := \begin{cases} 1 & \text{if } y \in \{x_i, x'_i\}, \\ 5 & \text{if } y \in \{x_{i+2}, x'_{i+2}\}, \\ 3 & \text{otherwise.} \end{cases}$$

and

$$h_{ii+1}(y) := \begin{cases} 1.5 & \text{if } \ y \in \{x_i, x'_i, x_{i+1}, x'_{i+1}\}, \\ 4.5 & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathcal{F} := \mathcal{G} \cup \mathcal{H} := \{g_1, g_2, g_3, g_4\} \cup \{h_{12}, h_{23}, h_{34}, h_{41}\} \subset P(A(f))$. For every $g \in \mathcal{F}$, one has $\mathrm{rank}(A(g)) = 1$ and for every further $h \in \mathcal{F}$ such that $g \neq h$, one has $A(g) \neq A(h)$. It follows that $P(A(f))$ has eight different one-dimensional faces. Furthermore, if $g \in \mathcal{G}$, then $\mathrm{diam}(P(A(g))) = 2$ and if $h \in \mathcal{H}$, then $\mathrm{diam}(P(A(h))) = 1$.

For any metric space (X, d), since $f \in \mathbb{R}^X$ is extremal if and only if for all $x \in X$, one has

$$f(x) = \sup_{y \in X}(d(x,y) - f(y)), \tag{3.11}$$

it follows that each d_z is extremal. Applying (3.11) twice one obtains

$$f(x) \leq \sup_{y \in X}(d(x,x') + d(x',y) - f(y)) = d(x,x') + f(x')$$

for all $x, x' \in X$, so every $f \in \mathrm{E}(X)$ is 1-Lipschitz and thus

$$\mathrm{E}(X) \subset \Delta(X) \cap \mathrm{Lip}_1(X, \mathbb{R}). \tag{3.12}$$

Moreover, for any $f \in \mathrm{E}(X)$, one has

$$\|f - d_x\|_\infty = f(x) \tag{3.13}$$

since a function $f \in \mathbb{R}^X$ belongs to $\Delta(X) \cap \mathrm{Lip}_1(X, \mathbb{R})$ if and only if for all $x, y \in X$, one has $|f(x) - d(x,y)| \leq f(y)$. Moreover, note that if $f \in \mathrm{E}(X)$ satisfies $f(x) = 0$, then $f = d_x$ as can easily be seen from the inequalities

$$d(x,y) - f(x) \leq f(y) \leq d(x,y) + f(x)$$

which hold for any $y \in X$. We can now conclude this section by two remarks regarding the zero rank elements of $\mathrm{E}(X)$.

3.4 Remark. Let (X, d) be any integer-valued metric space. Let $A \in \mathscr{A}(X)$ be such that $\mathrm{rank}(A) = k$. Moreover, let $f \in \mathrm{E}'(X)$ be such that $A = A(f)$ and finally assume that $d_x \in P(A)$. Consider arbitrary reference points $x_1, \ldots, x_k \in X$ for the decomposition (1.6) associated to A. Furthermore, let $I_f \colon H(A) \to l_\infty^k$ denote the affine isometry defined by $g \mapsto \big(g(x_1) - f(x_1), \ldots, g(x_k) - f(x_k)\big)$. Let $g \in P(A)$, if $x \in X_0$, then by (1.4), one has $d_x|_{X_0} = g|_{X_0}$, we thus obtain $g(x) = d_x(x) = 0$ and hence $g = d_x$ by the above observation. Hence, assume now that $g \neq d_x$ and thus $x \notin X_0$, there is $j \in \{1, \ldots, k\}$ such that $x \in [x_j]_\sigma$. Further, pick $\bar{x} \in [x_j]_{-\sigma}$ so that $\{x, \bar{x}\} \in A(f) \subset A(g)$. One has

$$\|g - d_x\|_\infty = g(x) = d(x, \bar{x}) - g(\bar{x}) = d_x(\bar{x}) - g(\bar{x}). \tag{3.14}$$

By (1.3), one obtains

$$d_x(\bar{x}) - g(\bar{x}) = -(d_x(x_j) - g(x_j)) = -\pi_j(d_x(\bar{x}) - g(\bar{x})). \tag{3.15}$$

where π_j is the j-th coordinate map $\pi_j\colon l_\infty^k \to \mathbb{R}$ given by $y \mapsto y_j$. Let us define the cone $C_j := \{x \in l_\infty^k : x_j = \|x\|_\infty\}$. We obtain from (3.14) and (3.15) that

$$I_f(P(A)) \subset \{I_f(d_x)\} - C_j$$

where the right-hand side is a Minkowski sum of subsets of l_∞^k.

3.5 Remark. Note that in general, $(\Sigma^0 \circ \mathrm{E}')(X) \neq \mathrm{e}(X)$ where $(\Sigma^0 \circ \mathrm{E}')(X)$ denotes the 0-skeleton of $\mathrm{E}'(X)$, namely the set of function $f \in \mathrm{E}'(X)$ such that $\operatorname{rank}(A(f)) = 0$. The preceding remark only applies to vertices in $\mathrm{e}(X)$. It is a natural question to ask under which conditions on X, one has $(\Sigma^0 \circ \mathrm{E}')(X) = \mathrm{e}(X)$. The following example shows that even if X is a median metric space, $(\Sigma^0 \circ \mathrm{E}')(X) \neq \mathrm{e}(X)$ in general. First, looking at modular spaces is justified by noticing that if there is a triple of different points $x, y, z \in X$ such that $I(x,y) \cap I(x,z) \cap I(y,z) = \emptyset$, then $\mathrm{e}(X) \neq (\Sigma^0 \circ \mathrm{E}')(X) = \emptyset$.

The function $f\colon \{x,y,z\} \to \mathbb{R}$ defined by $f(x) := (y|z)_x$, $f(y) := (x|z)_y$ and $f(z) := (x|y)_z$ can be extended to an element of $g \in \mathrm{E}(X)$ by [30, Proposition 3.5]. For $g \in \mathrm{E}'(X)$, it follows that $\{x,y\}, \{x,z\}, \{y,z\} \in A(g)$ and thus all points in $P(A(g))$ are median points in $\mathrm{E}'(X)$ for x, y, z. Hence, any vertex of $P(A(g))$ is in $(\Sigma^0 \circ \mathrm{E}')(X) \setminus \mathrm{e}(X)$. The contrary is however not true, as one can see by letting X be the vertex set of the 3-cube $[0,1]^3 \subset l_1^3$ with the induced metric. Indeed, it is easy to see that the function

$$f(x) := \begin{cases} 1 & \text{if } x_1 + x_2 + x_3 \equiv 0 (\text{mod } 2), \\ 2 & \text{otherwise} \end{cases}$$

verifies $f \in (\Sigma^0 \circ \mathrm{E}')(X) \setminus \mathrm{e}(X)$.

V.4 The Diameter of Cells of $\mathrm{E}(X)$

We begin by summarizing a couple of properties which are proved in [30, Section 4].

4.1 Lemma. *Let $A \in \mathscr{A}(X)$ and let $X = X_0 \cup \bigcup_{j=1}^k [x_j]$ be the associated decomposition with reference points $\{x_1, \ldots, x_k\}$ as defined* (1.6)*. Now, for any pair $(g,h) \in P(A) \times P(A)$, set $F_{gh} := g - h$. Then, the following hold*

(i) F_{gh} is constant on each $[x_j]_\sigma$.

(ii) For any pair $(x,y) \in [x_j]_\sigma \times [x_j]_{-\sigma}$, one has

$$F_{gh}(x) = -F_{gh}(y).$$

(iii) *For any* $x \in X_0$, *one has* $F_{gh}(x) = 0$.

(iv) $F_{gh} \in \mathrm{Lip}_2(X, \mathbb{R})$.

Proof. Recall from (1.3) that if there is an A-path $(y_1, \ldots, y_N) \in X^N$, i.e. $\{y_i, y_{i+1}\} \in A$ for each $i \in \{1, \ldots, N-1\}$ and $(y_1, y_N) = (x, y)$, then it follows that $g(y_1) - h(y_1) = (-1)^{N-1}(g(y_N) - h(y_N))$, that is

$$F_{gh}(x) = (-1)^{N-1} F_{gh}(y) \tag{4.1}$$

and note that $N-1$ denotes the number of A-edges of $(y_1, \ldots, y_N)$. Hence if $x, y \in [x_j]_\sigma$, since $[x_j]$ is a bipartite component, $N-1$ can be chosen to be even and thus $F_{gh}(x) = F_{gh}(y)$ which implies (i).

If $(x, y) \in [x_j]_\sigma \times [x_j]_{-\sigma}$, one can choose $N-1$ to be odd and thus $F_{gh}(x) = -F_{gh}(y)$, which shows (ii).

If $x \in X_0$, then we may set $x = y$ in (4.1) and $N-1$ can be chosen to be odd, so (iii) follows easily.

To see that (iv) holds, note that $g, h \in \mathrm{E}(X) \subset \mathrm{Lip}_1(X, \mathbb{R})$ by (3.12) and thus (iv) follows. □

Recall that in the notation of (1.6), for any $z \in X \setminus X_0$, there is $j \in \{1, \ldots, k\}$ and $\sigma \in \{\pm 1\}$ such that $z \in [x_j]_\sigma$. We then set $[z]_1 := [x_j]_\sigma$ and $[z]_{-1} := [x_j]_{-\sigma}$.

Proof of Theorem 1.3. We first prove (i). We set $k := \dim(P(A))$ and assume that $X_0 = \emptyset$. Pick any $x \in X$ and assume without loss of generality that $x \in [x_l]_\tau$, pick now $y \in [x_l]_{-\tau}$. By discrete path-connectedness of X, there is a path $(y_1, \ldots, y_N) \in X^N$ (i.e. $d(y_i, y_{i+1}) = 1$ for $i \in \{1, \ldots, N-1\}$) such that $(y_1, y_N) = (x, y)$. Now, for any pair $(g, h) \in P(A) \times P(A)$, set $F_{gh} := g - h$.

From (i) and (ii) in Lemma 4.1, it is easy to see that there are $\{z_1, \ldots, z_M\} \subset \{y_1, \ldots, y_N\}$ with $(z_1, z_M) = (x, y) = (y_1, y_N)$ having the following two properties:

(1) For any $i, j \in \{1, \ldots, M\}$, if $i \neq j$, then $F_{gh}(z_i) \neq F_{gh}(z_j)$ or equivalently

$$[z_i]_1 = [x_j]_\sigma \neq [x_m]_\eta = [z_j]_1.$$

(2) For any $j \in \{1, \ldots, M-1\}$ there is $(w_j, w_{j+1}) \in [z_j]_1 \times [z_{j+1}]_1$ such that $d(w_j, w_{j+1}) = 1$ or equivalently

$$d([z_j]_1, [z_{j+1}]_1) = 1$$

where $d(A, B) := \inf_{a \in A, b \in B} d(a, b)$.

Note that $M \leq 2k$ by (1) above. Besides, F_{gh} is a 2-Lipschitz function (iv) in Lemma 4.1. It follows from (2) above that

$$|F_{gh}(z_j) - F_{gh}(z_{j+1})| = |F_{gh}(w_j) - F_{gh}(w_{j+1})| \leq 2.$$

Hence, one has:

$$\begin{aligned} 2|F_{gh}(x)| = |F_{gh}(x) - F_{gh}(y)| &= \left| \sum_{j=1}^{M-1} \Big[F_{gh}(z_j) - F_{gh}(z_{j+1}) \Big] \right| \\ &\leq \sum_{j=1}^{M-1} 2 \\ &\leq 2(2k-1). \end{aligned}$$

It follows that $\|F_{gh}\|_\infty \leq 2k-1$ since the same procedure applies for every $x \in X$. Since g and h were chosen arbitrarily in $P(A)$, we obtain that

$$\mathrm{diam}(P(A)) \leq 2\mathrm{dim}(P(A)) - 1.$$

Now, to prove (2) assume $X_0 \neq \emptyset$. From (iv) in Lemma 4.1, we know that F_{gh} is constantly equal to zero on X_0. Proceeding in a way similar to the above, we obtain $\|F_{gh}\|_\infty \leq 2k$. It follows that

$$\mathrm{diam}(P(A)) \leq 2\mathrm{dim}(P(A)),$$

as required. □

We now present the two examples showing that the above bounds are optimal. Both examples are not only discretely path-connected, but even discretely geodesic. This means that the above bounds remain optimal when restricting to the class of discretely geodesic metric spaces.

4.2 Example. For $n \in \mathbb{N}$, let (X, d) be given by the graph (X, E) endowed with the shortest-path metric where the vertex set $|X| = 6n - 4$ consists of the union $X := V_1 \cup V_2 \cup V_3$ with

$$V_i := \{v, v_1^i, v_2^i, \ldots, v_{2n-2}^i, \bar{v}\}$$

and the edge set E is given by $E := E_1 \cup E_2 \cup E_3$ where

$$E_i := \{v, v_1^i\} \cup \{\bar{v}, v_{2n-2}^i\} \cup \{\{v_j^i, v_{j+1}^i\} : 1 \leq j \leq 2n-3\}.$$

Let now $f\colon X \to \mathbb{R}$ stand for the constant function equal to $\frac{2n-1}{2}$. It is easy to see that $f \in \mathrm{E}(X)$ and that

$$f = \frac{1}{2}(d_v + d_{\bar{v}}).$$

Note now that $\dim(P(A(f))) = n$. Indeed, the set of all n even $A(f)$-components of the graph $(X, A(f))$ consists of the component

$$[v] = [v]_1 \cup [v]_{-1} = \{v\} \cup \{\bar{v}\}$$

together with the $n-1$ components given by

$$[v_j^1] = [v_j^1]_1 \cup [v_j^1]_{-1} = \{v_j^1, v_j^2, v_j^3\} \cup \{v_{2n-1-j}^1, v_{2n-1-j}^2, v_{2n-1-j}^3\}$$

for $j \in \{1, \dots, n-1\}$. On the other side, remark that $d_v, d_{\bar{v}} \in P(A(f))$ and $\|d_v - d_{\bar{v}}\|_\infty = 2n-1$, thus $\operatorname{diam}(P(A(f))) \geq 2n-1$. Therefore

$$\operatorname{diam}(P(A(f))) = 2n - 1$$

since $\operatorname{diam}(P(A(f))) \leq \operatorname{diam}(\mathrm{E}(X)) = \operatorname{diam}(X) = 2n-1$. This proves that (i) in Theorem 1.3 is optimal.

4.3 Example. For $n \in \mathbb{N}$, let (X, d) be given by the graph (X, E) endowed with the shortest-path metric where the vertex set $|X| = 6n - 1$ consists of the union $X := V_1 \cup V_2 \cup V_3$ with

$$V_i := \{v, v_1^i, v_2^i, \dots, v_{2n-1}^i, \bar{v}\},$$

noting that unlike Example 4.2, the sets V_i have odd cardinality and the edge set E is given by $E := E_1 \cup E_2 \cup E_3$ where

$$E_i := \{v, v_1^i\} \cup \{\bar{v}, v_{2n-1}^i\} \cup \{\{v_j^i, v_{j+1}^i\} : 1 \leq j \leq 2n-2\}.$$

Let now $f\colon X \to \mathbb{R}$ stand for the function constantly equal to n. It is easy to see that $f \in \mathrm{E}(X)$ and that

$$f = \frac{1}{2}(d_v + d_{\bar{v}}).$$

Note now that $\dim(P(A(f))) = n$. Indeed, the set of all n even $A(f)$-components of the graph $(X, A(f))$ consists of the component

$$[v] = [v]_1 \cup [v]_{-1} = \{v\} \cup \{\bar{v}\}$$

together with the $n-1$ components given by

$$[v_j^1] = [v_j^1]_1 \cup [v_j^1]_{-1} = \{v_j^1, v_j^2, v_j^3\} \cup \{v_{2n-j}^1, v_{2n-j}^2, v_{2n-j}^3\}$$

for $j \in \{1, \ldots, n-1\}$. Unlike Example 4.2, there is in addition an odd $A(f)$-component given by the three vertices $\{v_n^1, v_n^2, v_n^3\}$. Finally, remark that similarly to the above, one has $d_v, d_{\bar{v}} \in P(A(f))$ and $\|d_v - d_{\bar{v}}\|_\infty = 2n$ hence $\mathrm{diam}(P(A(f))) = 2n$. This proves that (ii) in Theorem 1.3 is optimal.

Finally, we exhibit an example of a discretely geodesic metric space showing that there is no counterpart to Theorem 1.3 namely, there is no function $f\colon \mathbb{R} \to \mathbb{R}$ such that for any cell $P(A)$ of $\mathrm{E}(X)$, one has

$$\dim(P(A)) \leq f(\mathrm{diam}(P(A))).$$

4.4 Example. For $n \in \mathbb{N}$, let (X, d) be given by the graph (X, E) endowed with the shortest-path metric, where the vertex set $|X| = 2n+2$ is given by the union

$$X := \{v\} \cup \{\bar{v}\} \cup W \cup \bar{W}$$

with $W := \{w_1, \ldots, w_n\}$ and $\bar{W} := \{\bar{w}_1, \ldots, \bar{w}_n\}$. The edge set E is then given by

$$E := \{v, \bar{v}\} \cup E_1 \cup E_2 \cup E_3$$

where $E_1 := \{\{w_i, v\} : 1 \leq i \leq n\}$, $E_2 := \{\{\bar{w}_i, \bar{v}\} : 1 \leq i \leq n\}$ and

$$E_3 := \cup_{1 \leq i \leq n} \{\{w_i, \bar{w}_j\} : 1 \leq j \leq n \text{ and } j \neq i\}.$$

It is easy to see that (X, E) is a bipartite graph of diameter equal to three since for instance $d(w_1, \bar{w}_1) = 3$. Let us now define $f\colon X \to \mathbb{R}$ by setting

$$f(v) := \tfrac{1}{2} - \tfrac{1}{3^n} \ \text{ and } \ f(\bar{v}) := \tfrac{1}{2} + \tfrac{1}{3}$$

as well as setting for every $i \in \{1, \ldots, n\}$,

$$f(w_i) := \tfrac{3}{2} - \tfrac{1}{3^i} \ \text{ and } \ f(\bar{w}_i) := \tfrac{3}{2} + \tfrac{1}{3^i}.$$

It is then easy to see that $f \in \mathrm{E}(X)$. Moreover, $\dim(P(A(f))) = n$ since the graph $(X, A(f))$ consists of the two even $A(f)$-components:

$$[w_1] = [w_1]_1 \cup [w_1]_{-1} = \{w_1\} \cup \{\bar{w}_1, \bar{v}\},$$

$$[w_n] = [w_n]_1 \cup [w_n]_{-1} = \{w_n, v\} \cup \{\bar{w}_n\},$$

as well as the $n-2$ even $A(f)$-components:

$$[w_i] = [w_i]_1 \cup [w_i]_{-1} = \{w_i\} \cup \{\bar{w}_i\} \text{ for } i \in \{2, \ldots, n-1\}.$$

We continue with a further example. Note that by Theorem 1.2, the graph associated to a maximal cell is a disjoint union of complete bipartite graphs, hence in particular $X_0 = \emptyset$. Therefore, the bound in (*ii*) of Theorem 1.3 is never attained in maximal cells. In the next example, it is shown that the bound in (*i*) of Theorem 1.3 is in maximal cells in general not attained either.

4.5 Example. Let (X, d) be a finite discretely path-connected metric space and let $A \in \mathscr{A}(X)$ be such that $P(A)$ is a maximal cell of $\mathrm{E}(X)$ and $\dim(P(A)) = 2$. Let

$$X = \bigcup_{j=1}^{2} ([x_j]_1 \cup [x_j]_{-1})$$

be the associated decomposition Assume now by contradiction that there are $g, h \in P(A)$ such that $\|g - h\|_\infty = 3$, i.e. by (*i*) Theorem 1.3 one has $\mathrm{diam}(P(A)) = 3$. By (*ii*) in Lemma 4.1 one has for $F_{gh} := g - h$ and $(z, w) \in [x_i]_\sigma \times [x_i]_{-\sigma}$ that $F_{gh}(z) = -F_{gh}(w)$. We can assume without loss of generality that F_{gh} is constantly equal to 3 on $[x_2]_1$ and thus it is constantly equal to -3 on $[x_2]_{-1}$. Moreover, since F_{gh} is 2-Lipschitz by (*iv*) in Lemma 4.1, it follows that F_{gh} is constantly equal to 1 on $[x_1]_1$ and constantly equal to -1 on $[x_1]_{-1}$. Furthermore, since (X, d) is discretely path connected and again because F_{gh} is 2-Lipschitz, there are $y \in [x_1]_1$ and $y' \in [x_1]_{-1}$ such that $d(y, y') = 1$. Since A has complete bipartite components, one has $\{y, y'\} \in A$. Since both g and h are 1-Lipschitz functions, it follows from

$$2 = |F_{gh}(y) - F_{gh}(y')| \leq |g(y) - g(y')| + |h(y) - h(y')| \leq 2$$

that each above inequality is an equality, hence

$$g(y') = g(y) - 1 = g(y) - d(y, y') = -g(y')$$

which implies $g(y') = 0$ and thus $g = d_{y'}$. Analogously, $h(y') = h(y) + 1 = h(y) + d(y, y') = 2h(y) + h(y')$ implies that $h(y) = 0$ and thus $h = d_y$. Hence, $\|g - h\|_\infty = \left\|d_{y'} - d_y\right\|_\infty = d(y, y') = 1$ and this is a contradiction to our assumption. It follows that if $P(A)$ is a maximal cell of $\mathrm{E}(X)$ and $\dim(P(A)) = 2$, then $\mathrm{diam}(P(A)) < 3$.

Example 4.5 shows together with 4.2 that the isometry classes realized as cells of Isbell's injective hull is different from the isometry classes realized as maximal cells. In particular, Example 4.5 shows for instance that the regular octagon with unit edge-length in l_∞^2 which has a diameter equal to three, cannot arise as a maximal cell in the injective hull of any finite discretely geodesic metric space. Similarly, the cells we construct in Example 4.2 do not arise as maximal cells in dimension two. It is already known [26] that if (X,d) is a finite metric space, the polyhedral structure of $\mathrm{E}(X)$ consists of a single maximal cell if and only if (X,d) is an antipodal metric space. It follows that the regular octagon with unit edge-length in l^2_∞ cannot be realized as Isbell's injective hull of a finite discretely geodesic metric space.

V.5 Bicombings

The following theorem is stated in [17]. Below, $\mathfrak{S}_m$ denotes the symmetric group of order m. Recall that we define bicombings in Definition 1.4.

5.1 Theorem. *Let (X,d) be a complete metric space with a bicombing σ. Then, for $m \in \mathbb{N}$, there exists a map $\mathrm{bar}_m \colon X^m \to X$ such that the following hold:*

(i) $\mathrm{bar}_m(x_1,\dots,x_m)$ *lies in the closed σ-convex hull of $\{x_1,\dots,x_m\}$;*

(ii) $d(\mathrm{bar}_m(x_1,\dots,x_m), \mathrm{bar}_n(y_1,\dots,y_m)) \leq \min_{\pi\in\mathfrak{S}_m} \frac{1}{m}\sum_{i=1}^m d(x_i, y_{\pi(i)})$*;*

(iii) $\varphi(\mathrm{bar}_m(x_1,\dots,x_m)) = \mathrm{bar}_m(\varphi(x_1),\dots,\varphi(x_m))$ *whenever φ is an isometry of X and σ is such that for any $(x,y) \in X \times X$ one has $\varphi \circ \sigma_{xy} = \sigma_{\varphi(x)\varphi(y)}$, i.e. σ is φ-equivariant.*

We then call bar_m *a barycenter map.*

The construction satisfies $\mathrm{bar}_1(x) := x$ and $\mathrm{bar}_2(x,y) := \sigma_{xy}(\frac{1}{2}) = \sigma_{yx}(\frac{1}{2})$ as well as for $m \geq 3$,

$$\mathrm{bar}_m(x_1,\dots,x_m) = \mathrm{bar}_m(\mathrm{bar}_{m-1}(\hat{x}^1),\dots,\mathrm{bar}_{m-1}(\hat{x}^m)),$$

where $\hat{x}^i := (x_1,\dots,x_{i-1},x_{i+1},\dots,x_m)$.

5.2 Definition. A metric space (X,d) is said to be *n-hyperconvex* if for any collection $\{(x_i,r_i)\}_{i\in\{1,\dots,n\}} \subset X \times [0,\infty)$ satisfying $r_i + r_j \geq d(x_i,x_j)$ for all $i,j \in \{1,\dots,n\}$, one has

$$\bigcap_{i\in\{1,\dots,n\}} B(x_i,r_i) \neq \emptyset.$$

where $B(x,r) := \{y \in X : d(x,y) \leq r\}$. If the same holds when replacing $\{1,\dots,n\}$ by any index set I, then (X,d) is said to be *hyperconvex*. (X,d) is said to be n-hyperconvex *up to α* respectively hyperconvex *up to α* for some $\alpha \in [0,\infty)$ if the above properties hold for balls with radii bounded from above by α.

The following proposition is a direct adaptation of [5, Lemma 2.13], see also [32]:

5.3 Proposition. *Let (X,d) be a complete 4-hyperconvex metric space with a bicombing σ. Then (X,d) is n-hyperconvex for any $n \in \mathbb{N}$.*

Proof. We prove by induction for $n \geq 4$ that if collections of n mutually intersecting balls have a common point, then the same is true for collections of $n+1$ balls. Let $\{B_i\}_{i\in\{1,\dots,n+1\}}$ be a collection of $n+1$ mutually intersecting closed balls. Fix any point $x_0 \in X$ and let

$$\Delta_0 := \max_{1\leq i\leq n+1} d(x_0, B_i).$$

We show that there is a point $x_1 \in X$ such that

$$\Delta_1 := \max_{1\leq i\leq n+1} d(x_1, B_i) \leq \tfrac{4}{5}\Delta_0$$

and such that

$$d(x_0, x_1) \leq \Delta_0.$$

Iterating this procedure we obtain a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that

$$\Delta_k := \max_{1\leq i\leq n+1} d(x_k, B_i) \leq \left(\tfrac{4}{5}\right)^k \Delta_0$$

and such that

$$d(x_k, x_{k+1}) \leq \left(\tfrac{4}{5}\right)^k \Delta_0.$$

It follows that $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence and its limit must be contained in $\bigcap_{i\in\{1,\dots,n+1\}} B_i$ since $\Delta_k \to 0$.

Let $N := \binom{n+1}{n-1}$ and let $\mathcal{A} := \{A_1, \dots, A_N\}$ be an enumeration of the collection of subsets $A \subset \{1, \dots, n+1\}$ of cardinality $n-1$. By the induction hypothesis, the collection of n balls consisting of $B(x_0, \Delta_0)$ and $\{B_i\}_{i\in A}$ has a common point y_A. By Theorem 5.1, there is a barycenter map $\mathrm{bar}_N \colon X^N \to X$. We set

$$x_1 := \mathrm{bar}_N(y_{A_1}, \dots, y_{A_N}).$$

Since $B(x_0, \Delta_0)$ is σ-convex by *(iii)* in Definition 1.4 and since $\{y_A\}_{A\in\mathcal{A}} \subset B(x_0, \Delta_0)$, we have $x_1 \in B(x_0, \Delta_0)$ by *(i)* in Theorem 5.1 and hence $d(x_0, x_1) \leq \Delta_0$. Let $i \in \{1, \dots, n+1\}$ be chosen arbitrarily, in order to estimate $d(x_1, B_i)$, note first that

$$d(y_A, B_i) \leq d(y_A, x_0) + d(x_0, B_i) \leq 2\Delta_0.$$

For any y_A, choose $z_A \in B_i$ such that $d(y_A, z_A) \leq d(y_A, B_i) + \varepsilon$. If $y_A \in B_i$, set $z_A := y_A$. Moreover, define

$$z := \mathrm{bar}_N(z_{A_1}, \dots, z_{A_N}).$$

By (i) in Theorem 5.1 applied to $\mathrm{bar}_N \colon X^N \to X$, we have

$$d(x_1, B_i) \leq d(x_1, z) \leq \frac{1}{N} \min_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^{N} d(y_{A_i}, z_{A_{\sigma(i)}}) \leq \frac{1}{N} \sum_{i=1}^{N} d(y_{A_i}, z_{A_i})$$
$$\leq \varepsilon + \frac{1}{N} \sum_{A \in \mathcal{A}, i \notin A} d(y_A, B_i).$$

Since there are exactly $N' := \binom{n}{n-2}$ sets $A \in \mathcal{A}$ such that $i \in A$, we obtain

$$\varepsilon + \frac{1}{N} \sum_{A \in \mathcal{A}, i \notin A} d(y_A, B_i) \leq \varepsilon + \frac{1}{N} \left(N - N'\right) 2\Delta_0 = \varepsilon + \frac{4}{n+1} \Delta_0 \leq \varepsilon + \frac{4}{5} \Delta_0.$$

Since this holds for any $\varepsilon > 0$ and $i \in \{1, \dots, n+1\}$ was chosen arbitrarily, we obtain $\max_{1 \leq i \leq n+1} d(x_1, B_i) \leq \frac{4}{5} \Delta_0$. This concludes the proof. □

It is noted in [5] that 4-hyperconvex in Proposition 5.3 is optimal. Indeed, note that it is not difficult to see that if (X, d) is a proper metric space, then (X, d) is hyperconvex if and only if it is hyperconvex for every $n \in \mathbb{N}$. Consider now l_1^n with $n \geq 1$, it is not difficult to see that l_1^n is a proper 3-hyperconvex metric space. However, l_1^n is not hyperconvex for $n \geq 3$. Now,

5.4 Lemma. *Let (X, d) be a metric space. Assume there is $\alpha > 0$ such that for any $x \in X$, the closed ball $B(x, \alpha)$ is hyperconvex. Then, X is hyperconvex up to $\frac{\alpha}{2}$.*

Proof. Let $\{(x_i, r_i)\}_{i \in I} \subset X \times [0, \frac{\alpha}{2}]$ be such that $d(x_i, x_j) \leq r_i + r_j$, hence in particular $d(x_i, x_j) \leq \alpha$. Fixing an $i \in I$, it follows that for any $j \in I$, one has $x_j \in B(x_i, \alpha)$. By hyperconvexity of $B(x_i, \alpha)$, it thus follows that $\bigcap_{i \in I} B(x_i, \frac{\alpha}{2}) \neq \emptyset$ and thus the result follows. □

Conversely, it is easy to see that if X is hyperconvex up to α, then for any $x \in X$ the closed ball $B(x, \frac{\alpha}{2})$ is hyperconvex. An analogue of the next lemma appears in [34].

5.5 Lemma. *Let (X, d) be a metric space with a bicombing σ and hyperconvex up to $\frac{\alpha}{2} > 0$. Then, (X, d) is hyperconvex up to α.*

Proof. Let $\{(x_i, r_i)\}_{i\in I} \subset X \times [0,\alpha]$ be such that $d(x_i, x_j) \leq r_i + r_j$. For any $i \in I$, note that by (iii) in Definition 1.4, one has $d(\sigma_{x_i x_j}(\frac{1}{2}), \sigma_{x_i x_k}(\frac{1}{2})) \leq \frac{d(x_j, x_k)}{2}$ and thus since $\{\frac{r_i}{2}\}_{i\in I} \subset [0, \frac{\alpha}{2}]$, we can pick

$$x_i' \in \bigcap_{j\in I} B(\sigma_{x_i x_j}(\tfrac{1}{2}), \tfrac{r_j}{2})$$

noticing that in particular

$$x_i' \in B(\sigma_{x_i x_i}(\tfrac{1}{2}), \tfrac{r_i}{2}) = B(x_i, \tfrac{r_i}{2}). \tag{5.1}$$

Now, we have by (ii) in Definition 1.4 that $\sigma_{x_i x_j}(\frac{1}{2}) = \sigma_{x_j x_i}(\frac{1}{2})$ and thus together with the triangle inequality, one has

$$d(x_i', x_j') \leq d(x_i', \sigma_{x_i x_j}(\tfrac{1}{2})) + d(\sigma_{x_i x_j}(\tfrac{1}{2}), x_j') \leq \frac{r_i}{2} + \frac{r_j}{2}.$$

Therefore, we can pick

$$x \in \bigcap_{i\in I} B(x_i', \tfrac{r_i}{2})$$

and together with (5.1), it follows that for any $i \in I$, one has $d(x, x_i) \leq r_i$. Consequently

$$x \in \bigcap_{i\in I} B(x_i, r_i)$$

and this concludes the proof. □

From Lemma 5.5 we immediately obtain:

5.6 Corollary. *Let (X, d) be a metric space with bicombing and hyperconvex up to $\alpha > 0$. Then, (X, d) is n-hyperconvex for any $n \in \mathbb{N}$.*

Recall that $l_1(\mathbb{N})^*$ and $l_\infty(\mathbb{N})$ are isomorphic in the category of normed spaces and thus so are $l_1(\mathbb{N})^{**}$ and $l_\infty(\mathbb{N})^*$. There is a canonical linear embedding $\Lambda' \colon l_1(\mathbb{N}) \to l_1(\mathbb{N})^{**}$ given by the evaluation map $f \mapsto (\varphi \mapsto \varphi(f))$ and let

$$\Lambda \colon l_1(\mathbb{N}) \to l_\infty(\mathbb{N})^* \tag{5.2}$$

be the associated linear embedding.

5.7 Lemma. *Let A be a weak* closed subset of $l_\infty(\mathbb{N})$. If A is n-hyperconvex for any $n \in \mathbb{N}$, then A is hyperconvex.*

Proof. Consider now $\{(x_i, r_i)\}_{i\in I} \subset A \times [0,\infty)$ such that $\|x_i - x_j\|_\infty \leq r_i + r_j$. Fix $i \in I$. By Banach-Alaoglu, $B(x_i, r_i)$ is weak* compact and since A is weak* closed, it follows that $A \cap B(x_i, r_i)$ is weak* compact. Now, for any finite index subset J of I and for any $j \in J$, the set $A \cap B(x_i, r_i) \cap B(x_j, r_j)$ is weak* closed in S. Furthermore,

$$A \cap B(x_i, r_i) \cap \bigcap_{j\in J} B(x_j, r_j) \neq \emptyset.$$

By the closed set criterion for compactness, we finally obtain that

$$A \cap \bigcap_{i\in I} B(x_i, r_i) \neq \emptyset.$$

This concludes the proof. □

From the above observation and an analogue one in the proper case as well as with Proposition 5.3, one easily obtains Theorem 1.5. Now, note that if $\{(f_\gamma, C_\gamma)\}_{\gamma\in\Gamma} \subset l_1(\mathbb{N}) \times \mathbb{R}$, then the convex set

$$P := \bigcap_{\gamma\in\Gamma} \Lambda(f_\gamma)^{-1}\big([C_\gamma, \infty)\big) \subset l_\infty(\mathbb{N}) \tag{5.3}$$

is weak* closed in $l_\infty(\mathbb{N})$.

5.8 Lemma. *Let P be given by* (5.3). *If P is hyperconvex, then for any $p \in P$, the tangent cone*

$$\mathrm{T}_p P := \bigcup_{n\in\mathbb{N}} \big(p + n(P - p)\big)$$

is hyperconvex.

Proof. Since

$$\mathrm{T}_p P = \bigcap_{\gamma\in\Gamma(p)} \Lambda(f_\gamma)^{-1}\big([C_\gamma, \infty)\big)$$

where $\Gamma(p) := \{\gamma \in \Gamma : \Lambda(f_\gamma)(p) = C_\gamma\}$, it suffices to show by Lemma 5.7 that $\mathrm{T}_p P$ is n-hyperconvex for any $n \in \mathbb{N}$. Let $\{(x_i, r_i)\}_{i\in I} \subset \mathrm{T}_p P \times [0,\infty)$ be such that $|I| < \infty$ and $\|x_i - x_j\|_\infty \leq r_i + r_j$. There is then $n \in \mathbb{N}$ such that $\{x_i\}_{i\in I} \subset p + n(P - p)$. By hyperconvexity of the latter, it follows that

$$\big(p + n(P - p)\big) \cap \bigcap_{i\in I} B(x_i, r_i) \neq \emptyset$$

hence a fortiori $\mathrm{T}_p P \cap \bigcap_{i\in I} B(x_i, r_i) \neq \emptyset$ and the result follows. □

Since every convex subset of $l_\infty(\mathbb{N})$ is endowed with a bicombing given by the linear geodesics, we deduce the following consequence:

5.9 Theorem. *Let P be given by* (5.3) *assuming additionally that $|\Gamma| < \infty$. Then, P is hyperconvex if and only if for every $p \in P$, the tangent cone $\mathrm{T}_p P$ is hyperconvex.*

Proof. Necessity follows from Lemma 5.8. Now, assume that for every $p \in P$, the tangent cone $\mathrm{T}_p P$ is hyperconvex. Similarly to the procedure involved in finding δ in the proof of Theorem III.1.1, one can find $\epsilon > 0$ such that for any $p \in P$, there is $q \in P$ such that

$$B(p,\varepsilon) \cap P = B(p,\varepsilon) \cap \mathrm{T}_q P.$$

By hyperconvexity of $\mathrm{T}_q P$, it follows that $B(p,\varepsilon) \cap \mathrm{T}_q P$ is hyperconvex as well, and thus so is $B(p,\varepsilon) \cap P$. It follows from Lemma 5.4, Lemma 5.5 and Corollary 5.6 that every finite collection of mutually intersecting closed balls in P has a common point in P. It finally follows from Lemma 5.7 that P is hyperconvex. This concludes the proof. □

From Lemma 5.7 applied to each tangent cone and using that in this particular case, the usual barycenter of a finite point set from $l_\infty(\mathbb{N})$ can be used in the proof of Proposition 5.3, we obtain

5.10 Corollary. *Let $\{(f_\gamma, C_\gamma)\}_{\gamma \in \{1,\dots,m\}} \subset l_1(\mathbb{N}) \times \mathbb{R}$ and consider the convex subset*

$$P := \bigcap_{\gamma \in \{1,\dots,m\}} \Lambda(f_\gamma)^{-1}\big([C_\gamma, \infty)\big) \subset l_\infty(\mathbb{N}).$$

Then, P is hyperconvex if and only if for every $p \in P$, the tangent cone $\mathrm{T}_p P$ is 4-hyperconvex.

Bibliography

[1] J. Adamek, H. Herrlich, G. E. Strecker, Abstract and concrete categories: the joy of cats, Reprint of the 1990 original [Wiley, New York], Repr. Theory Appl. Categ. No. 17 (2006), 1–507.

[2] N. Aronszajn, P. Panitchpakdi, Extension of uniformly continuous tranformations and hyperconvex metric spaces, Pacific J. Math. 6 (1956), 405–439.

[3] H.-J. Bandelt, V. Chepoi, Metric graph theory and geometry: a survey, Contemp. Math. 453 (2008), 49–86.

[4] H.-J. Bandelt, A. W. M. Dress, A Canonical Decomposition Theory for Metrics on a Finite Set, Adv. Math., 92 (1992), 47–105.

[5] Y. Benyamini, J. Lindenstrauss, Geometric nonlinear functional analysis, Vol. 1. American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000.

[6] M. Bonk, O. Schramm, Embeddings of Gromov hyperbolic spaces, Geom. Funct. Anal. 10 (2000), 266–306.

[7] M. R. Bridson, A. Haefliger, Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.

[8] P. Buneman, The recovery of trees from measures of dissimilarity, Mathematics in the Archaeological and Historical Sciences (F. Hodson et al., Eds.), 387–395, Edinburgh Univ. Press, Edinburgh, 1971.

[9] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001.

[10] S. Buyalo, V. Schroeder, Elements of asymptotic geometry, EMS Monographs in Mathematics, 2007.

[11] I. Chatterji, G. Niblo, From wall space to CAT(0) cube complexes, International Journal of Algebra and Computation, Vol. 15, Nos. 5 and 6 (2005), 875–885.

[12] V. Chepoi, Graphs of some CAT(0) complexes, Adv. in Appl. Math. 24 (2000), no. 2, 125–179.

[13] M. Chrobak, L. L. Larmore, A new approach to the server problem. SIAM J. Discrete Math. 4 (1991), no. 3, 323–328.

[14] M. Chrobak, L. L. Larmore, Generosity helps or an 11-competitive algorithm for three servers, J. Algorithms 16 (1994), 234–263.

[15] F. H. Clarke, Generalized gradients and applications, Trans. Amer. Math. Soc. 205 (1975), 247–262.

[16] D. Descombes, U. Lang, Convex geodesic bicombings and hyperbolicity, Geom. Dedicata 177 (2015), 367–384.

[17] D. Descombes, U. Lang, Flats in spaces with convex geodesic bicombings, arXiv:1508.02745.

[18] A. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, Adv. in Math. 53 (1984), 321–402.

[19] A. Dress, K. T. Huber, V. Moulton, An explicit computation of the injective hull of certain finite metric spaces in terms of their associated Buneman complex, Adv. in Math. 168 (2002), 1–28.

[20] A. Dress, V. Moulton, W. Terhalle, T-theory: an overview, Europ. J. Combinatorics 17 (1996), 161–175.

[21] R. Espínola, M. A. Khamsi, Introduction to Hyperconvex Spaces, Handbook of Metric Fixed Point Theory, (W. A. Kirk, B. Sims, Eds.), Kluwer Academic Publishers, Dordrecht, 2001.

[22] S. Gähler, G. Murphy, A metric characterization of normed linear spaces. Math. Nachr. 102 (1981), 297–309.

[23] R. Gould, Graph theory, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1988.

[24] R. L. Graham, D. E. Knuth, O. Patashnik, Concrete Mathematics. A foundation for computer science, Second edition. Addison-Wesley Publishing Company, Reading, MA, 1994.

[25] B. Grünbaum, Convex polytopes, Second edition. Graduate Texts in Mathematics, 221. Springer-Verlag, New York, 2003.

[26] K. T. Huber, J. Koolen, V. Moulton, The tight-span of an antipodal metric space: part I-combinatorial properties, Discrete Mathematics 303 (2005), 65–79.

[27] K. T. Huber, J. Koolen, V. Moulton, The tight-span of an antipodal metric space: part II-geometrical properties, Discrete and Computational Geometry 31 (4) (2004), 567–586.

[28] K. T. Huber, J. Koolen, V. Moulton, On the structure of the tight-span of a totally split-decomposable metric, European Journal of Combinatorics 27 (2006), 461–479.

[29] J. R. Isbell, Six theorems about injective metric spaces, Comment. Math. Helv. 39 (1964), 65–76.

[30] U. Lang, Injective hulls of certain discrete metric spaces and groups, J. Topol. Anal. 5 (2013), 297–331.

[31] U. Lang, M. Pavón and R. Züst, Metric stability of trees and tight spans, Arch. Math. (Basel) 101 (2013), 91–100.

[32] J. Lindenstrauss, Extension of compact operators, Mem. Amer. Math. Soc. No. 48, Providence, RI, 1964.

[33] P. Mankiewicz, On Extension of Isometries in Normed Linear Spaces, Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys. 20 (1972), 367–371.

[34] B. Miesch, The Cartan-Hadamard Theorem for Metric Spaces with Local Bicombings, arXiv:1509.07001.

[35] A. Moezzi, The Injective Hull of Hyperbolic Groups, Dissertation ETH Zürich, No. 18860, 2010.

[36] L. Nachbin, A theorem of the Hahn–Banach type for linear transformations, Trans. Amer. Math. Soc. 68 (1950), 28–46.

[37] M. Pavón, Injective Convex Polyhedra, arXiv:1410.7306.

[38] M. Pavón, Injective Hulls of Infinite Totally Split-Decomposable Metric Spaces, arXiv:1505.07807.

[39] R. Shostak, Deciding Linear Inequalities by Computing Loop Residues, J. Assoc. Comput. Mach. 28 (1981), no. 4, 769–779.

[40] R. Suter, Injective hulls of odd cycles, arXiv:1302.1329.

[41] R. J. Vanderbei, Linear programming. Foundations and extensions, Fourth edition, International Series in Operations Research & Management Science, 196. Springer, New York, 2014.

[42] G. M. Ziegler Lectures on Polytopes, Springer-Verlag, New York, 1995.