Two Applications of the Gaussian Poincaré Inequality in the Shannon Theory

Silas L. Fong
Department of ECE,
National University of Singapore,
Email: silas_fong@nus.edu.sg

Vincent Y. F. Tan
Department of ECE/ Mathematics,
National University of Singapore
Email: vtan@nus.edu.sg

Abstract—We employ the Gaussian Poincaré inequality for two tasks in the Shannon theory. First, we show that the Gaussian broadcast channel admits a strong converse. Second, we demonstrate that the empirical output distribution of a delay-limited code for the AWGN channel with quasi-static fading and with non-vanishing probability of error converges to the maximum mutual information output distribution (in the normalized relative entropy sense).

I. INTRODUCTION

The Poincaré inequality for Gaussian measures [1] is one of the most prominent results in the theory of concentration of measure. Roughly speaking, it states that if \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth function and \( \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \) is the standard Gaussian density, then the variance of \( f \) can be bounded in terms of the expectation of the squared derivative of \( f \), i.e.,
\[
\text{Var}_{\phi}[f] \leq \mathbb{E}_{\phi}[\|\nabla f\|^2]. \tag{1}
\]

In the present work, we employ a modification of the Gaussian Poincaré inequality for two tasks in Shannon theory. These are described briefly in the following sections.

II. GAUSSIAN BROADCAST CHANNELS

The Gaussian broadcast channel [2, Ch. 5] is a basic model for the downlink of a communication system. Two messages \( W_1 \in [2^nR_1] \) and \( W_2 \in [2^nR_2] \) are to be encoded into a codeword \( X^n = f^{(n)}(W_1, W_2) \). This codeword is power constrained, i.e., \( \|X^n\|_2^2 \leq nP \). It is transmitted through two AWGN channels with variances \( \sigma_1^2 \) and \( \sigma_2^2 \) respectively, i.e.,
\[
Y_1^n = X^n + Z_1^n, \quad \text{and} \quad Y_2^n = X^n + Z_2^n. \tag{2}
\]

Decoder \( j \), which observes \( Y_j^n \), is required to estimate message \( W_j \) where \( j = 1, 2 \). The average probability of error is defined to be \( \text{Pr}(W_1, W_2) \neq (W_1, W_2) \) where \( W_j \) is decoder \( j \)'s estimate of \( W_j \). The capacity region \( C_{BC} \) is well known and is given by
\[
C_{BC} = \bigcup_{\alpha \in [0,1]} \left\{ (R_1, R_2) \in \mathbb{R}_+^2 \left| \begin{array}{l}
R_1 \leq C \left( \frac{\sigma_1^2}{\alpha P} \right)
R_2 \leq C \left( \frac{(1-\alpha)P}{\sigma_2^2 + \frac{\sigma_1^2}{\alpha P}} \right)
\end{array} \right\}, \tag{3}
\]
where \( C(x) := \frac{1}{2} \log(1 + x) \). This region is achieved using superposition coding [3]. Recall that the capacity region is the set of all rate pairs for which the error probability vanishes. The central question of our investigation in [4] is whether the region in (3) is enlarged if we relax the condition that the error probability vanishes. We allow the error probability to be upper bounded by a non-vanishing constant \( \varepsilon \in (0, 1) \). We show that the \( \varepsilon \)-capacity region is precisely the region in (3).

The main technicality in the proof involves bounding a certain variance of the log-likelihood of the messages using (1).

III. GOOD DELAY-LIMITED CODES

In [5], we used (1) to investigate quasi-static fading channels [6, Sec. 5.4.1] where the fading coefficient \( H \) is random but remains constant during the course of transmission. We are interested in the so-called delay-limited capacity [7], which is the maximum achievable rate under the assumption that the maximal error probability over all non-zero fading coefficients vanishes as the blocklength grows.

We adopt a long-term power constraint [8] and the max-over-messages error criterion for delay-limited decoding. It is known (e.g., [7, Sec. III-B]) that the delay-limited capacity is \( C(P_{DL}) \) where \( P_{DL} := \frac{P}{n^\alpha} \). We show in [5] that for any sequence of codes that is capacity-achieving and whose error probability is upper bounded by some \( \varepsilon \in (0, 1) \), such that sequence of induced output distributions \( \{p_{Y^n_j}(\cdot)\}_{n=1}^\infty \) satisfies
\[
\lim_{n \to \infty} \frac{1}{nD(p_{Y^n_j}, p_{Y^n})} = 0 \tag{4}
\]
where \( p_{Y^n_j}(y) = \mathcal{N}(y; 0, 1 + P_{DL}) \).

REFERENCES