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On the Burst Erasure Correctability of Spatially Coupled LDPC Ensembles

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***Abstract*—Spatially-Coupled LDPC (SC-LDPC) ensembles achieve the capacity of binary memoryless channels, asymptotically, under belief-propagation decoding. In this work, we are interested in the finite-length performance of these ensembles on binary channels with memory. We study the average performance of random regular SC-LDPC ensembles on single-burst-erasure channels and provide tight bounds for the block erasure probability. Further, we show the effect of expurgation on the performance by analyzing the minimal stopping sets.**

I. Introduction

Low-density parity-check (LDPC) codes are widely used due to their outstanding performance under low-complexity belief propagation (BP) decoding. However, an error probability exceeding that of maximum-a-posteriori (MAP) decoding has to be tolerated with (sub-optimal) BP decoding. Recently, it has been empirically observed for spatially coupled LDPC (SC-LDPC) codes—first introduced by Jiminez Felström and Zigangirov as convolutional LDPC codes [1]—that the BP performance of these codes can improve dramatically towards the MAP performance of the underlying LDPC code under many different settings and conditions, e.g. [2]. This phenomenon, termed *threshold saturation*, has been proven rigorously in [3], [4]. In particular, the BP threshold of a coupled LDPC ensemble tends to its MAP threshold on any binary memoryless symmetric channel (BMS).

Besides their excellent performance on the BEC and AWGN channels, much less is known about the burst error correctability of SC-LDPC codes. In [5], the authors consider SC-LDPC ensembles over a block erasure channel (BLEC) where the channel erases complete spatial positions instead of individual bits. This block erasure model mimics block-fading channels frequently occurring in wireless communications. The authors give asymptotic lower and upper bounds for the bit and block erasure probabilities obtained from density evolution. In [6], the authors construct protograph-based codes that maximize the correctable burst lengths, while the authors in [7] apply interleaving (therein denoted band splitting) to a protograph-based SC-LDPC code to increase the correctable burst length.

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If windowed decoding is used, this approach results in an increased required window length and thus complexity. Recently, it has been shown that protograph-based LDPC codes can increase the diversity order of block fading channels and are thus good candidates for block erasure channels [8], [9]; however, they require large syndrome former memories if the burst length becomes large.

In this paper, we consider the (d_v, d_c, w, L, M) code ensemble introduced in [3] and derive tight lower bounds on the correctability of a long burst of erasures. First, we consider the case when a complete spatial position is erased and then generalize the expression to the case where the burst can occur at any position within a codeword. We show that estimating the capability of correcting long burst erasures reduces to the problem of finding small stopping sets in the code structure. Also, we demonstrate that if we properly expurgate the ensemble, then a random code from the ensemble has very good average burst erasure capabilities. We focus on the general (d_v, d_c, w, L, M) code ensemble as the common protograph-based approach contains unavoidable small stopping sets in each spatial position, which are not recoverable if erased [10].

II. Preliminaries

A. The Regular (d_v, d_c, w, L, M) SC-LDPC Ensemble

We now briefly review how to sample a code from a random regular (d_v, d_c, w, L, M) SC-LDPC ensemble [3]. We first lay out a set of positions indexed from $z = 1$ to L on a *spatial dimension*. At each spatial position (SP), z, there are M variable nodes (VNs) and $M\frac{d_v}{d_c}$ check nodes (CNs), where $M\frac{d_v}{d_c} \in \mathbb{N}$ and d_v and d_c denote the variable and check node degrees, respectively. Let $w > 1$ denote the smoothing (coupling) parameter. Then, we additionally consider $w-1$ sets of $M\frac{d_v}{d_c}$ CNs in SPs $L+1, \ldots, L+w-1$. Every CN is assigned with d_c "sockets" and made to impose an even parity constraint on its d_c neighboring VNs. Each VN in SP z is connected to d_v CNs in SPs $z, \ldots, z+w-1$ as follows: each of the d_v edges of this VN is allowed to randomly and uniformly connect to any of the wMd_v sockets arising from the CNs in SPs $z, \ldots, z+w-1$, such that parallel edges are avoided in the resultant bipartite graph. This graph represents the code so that we have $n = LM$ code bits, over L SPs.

Because of additional check nodes in SPs $z > L$, the code rate $r = 1 - \frac{d_v}{d_c} - \delta$, where $\delta = O(\frac{w}{L})$. Throughout this paper, we assume that $d_v \geq 3$ and $wM > 2(d_v + 1)d_c$.

B. Single-Burst-Erasure Channel Models

We introduce two channel models for computing the burst erasure recoverability. First, the *Single Position Burst Channel* (SPBC) erases all M VNs of exactly one SP in the transmitted codeword and leaves all other bits undisturbed.

The second model is the more general *Random Burst Channel* (RBC) whose burst pattern is denoted by RBC(ℓ,s,b) where $s \in \{1, \ldots, M\}$ is the starting bit index of the burst in SP $\ell \in \{1, \ldots, L\}$, indicating the offset from the first VN of the SP ℓ, and b is the length of the burst. Note that in general $0 < b \leq (L-\ell)M - s$. As for the SPBC, all VNs in the random burst are erased while all other VNs are received correctly. We sometimes omit the SP ℓ when referring to the RBC for the following reason: neglecting boundary effects in the limit of large enough L, all SPs are structured identically. With some abuse of terminology, we will use the same notation to refer to the channel itself, rather than the burst introduced by it.

While multiple models exist for a correlated erasure channel, like the Gilbert-Elliott model used in [6], we use this model because it is sufficient to describe the scenarios that we consider: for instance, the SPBC can be used to model a slotted-ALOHA multiple access scheme where each user transmits an SC-LDPC codeword over L time slots, but one SP might be erased in the case of a collision. Additionally, long burst erasures might occur in block fading scenarios, or in optical communications with, e.g., polarization dependent loss, where long burst erasures are common. Another scenario is optical storage, where long erasure bursts may occur as well.

III. Error Analysis on the SPBC

Let $P_{\mathrm{B}}^{\mathrm{SPBC}}(d_v, d_c, w, L, M)$ denote the average block erasure (decoding error) probability of the (d_v, d_c, w, L, M) ensemble on the SPBC under BP decoding, i.e., the probability that the iterative decoder fails to recover the codeword. For large enough M, size-2 stopping sets (each of which also form a codeword) are the dominant structures in the graph that cause the BP decoder to fail [10]. *Stopping sets* are subsets $\mathcal{A}$ of the VNs such that every neighbor of the VNs in $\mathcal{A}$ connects to $\mathcal{A}$ at least twice [11, Def. 3.138]. A *minimal stopping set* is one which does not contain a smaller size non-empty stopping set within itself. Hence, the number of size-2 stopping sets per SP, denoted $\mathbb{N}_2^{\mathrm{SP}}$, is a good starting point for analyzing the performance of the ensemble. We have

$$\begin{aligned} P_{\mathrm{B}}^{\mathrm{SPBC}} &= \text{Prob [At least one stopping set in a SP]} \\ &\geq \text{Prob } [\mathbb{N}_2^{\mathrm{SP}} \geq 1] \\ &\overset{(a)}{\geq} \frac{\mathbb{E}[\mathbb{N}_2^{\mathrm{SP}}]^2}{\mathbb{E}[{\mathbb{N}_2^{\mathrm{SP}}}^2]} \overset{(b)}{\geq} \mathbb{E}[\mathbb{N}_2^{\mathrm{SP}}] \left(1 - \frac{M^2}{(\frac{w}{d_c}M - 3)^{d_v}}\right) \\ &= \mathbb{E}[\mathbb{N}_2^{\mathrm{SP}}] \left(1 - O\left(\frac{1}{M^{d_v - 2}}\right)\right) \approx \mathbb{E}[\mathbb{N}_2^{\mathrm{SP}}] \doteq \lambda_{\mathrm{SP}}, \quad (1) \end{aligned}$$

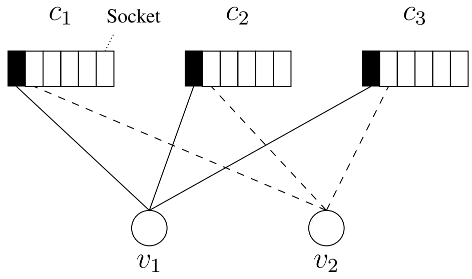

Fig. 1. A size-2 stopping set from a $(3, 6)$ random ensemble. CNs $\{c_1, c_2, c_3\}$ and VNs $\{v_1, v_2\}$ have been labeled for convenience. CNs have been expanded to show all their $d_c = 6$ sockets. The solid edges indicate definite connections and the dashed edges complete one configuration to form a stopping set. Parallel edges are not allowed in the ensemble.

where (a) is the application of the second moment method and (b) can be shown as follows: Define $U_{ij} = 1$ if VNs i and j form a stopping set, otherwise $U_{ij} = 0$. Then $\mathbb{N}_2^{\mathrm{SP}} = \sum_{1 \leq i < j \leq M} U_{ij}$ where the summation is over all $\binom{M}{2}$ pairs of VNs from a SP. We can see that $\lambda_{\mathrm{SP}} = \mathbb{E}[\mathbb{N}_2^{\mathrm{SP}}] = \binom{M}{2}p$ where $p = \mathbb{E}[U_{ij}]$ is the probability of forming a size-2 stopping set. Furthermore,

$$\begin{aligned} \mathbb{E}\left[{\mathbb{N}_2^{\mathrm{SP}}}^2\right] &= \mathbb{E}\left[\left(\sum_{1 \leq i < j \leq M} U_{ij}\right)^2\right] \\ &= \sum_{1 \leq i < j \leq M} \mathbb{E}[U_{ij}^2] \quad + \sum_{\substack{(i,j) \neq (k,l) \\ i<j, k<l}} \mathbb{E}[U_{ij}U_{kl}], \end{aligned}$$

where in the last step, $\sum_{1 \leq i < j \leq M} \mathbb{E}[U_{ij}^2] = \binom{M}{2}p$ as $U_{ij} \in \{0, 1\}$ and the second term is over the remaining $\binom{M}{2}(\binom{M}{2} - 1)$ combinations. Using some combinatorial arguments, we can show that $\mathbb{E}[U_{ij}U_{kl}] = \mathbb{P}(U_{ij} = 1)\mathbb{P}(U_{kl} = 1 | U_{ij} = 1) \leq 2p / \binom{wM\frac{d_v}{d_c} - 2d_v}{d_v}$. As a result, we have

$$\begin{aligned} \mathbb{E}\left[{\mathbb{N}_2^{\mathrm{SP}}}^2\right] &< \mathbb{E}[\mathbb{N}_2^{\mathrm{SP}}] \left(1 + \frac{2\binom{M}{2}}{\binom{wM\frac{d_v}{d_c} - 2d_v}{d_v}}\right) \\ &< \mathbb{E}[\mathbb{N}_2^{\mathrm{SP}}] \left(1 + \frac{M^2}{(\frac{w}{d_c}M - 3)^{d_v}}\right), \end{aligned}$$

which eventually implies (1). Note that following standard arguments [10], [11, Appendix C], we can also approximate the bound on $P_{\mathrm{B}}^{\mathrm{SPBC}}$ by a Poisson distribution with mean λ_{SP}, for a large M, so that

$$P_{\mathrm{B}}^{\mathrm{SPBC}} \approx 1 - e^{-\lambda_{\mathrm{SP}}} \approx \lambda_{\mathrm{SP}}. \quad (2)$$

Both (1) and (2) are very tight when $w \geq d_v$ (which is a prerequisite for constructing capacity-achieving codes [3]) as otherwise, we have observed that the contribution of larger stopping sets becomes non-negligible.

A. Calculation of p

We now calculate the probability p of finding a size-2 stopping set within an SP of a code uniformly sampled from

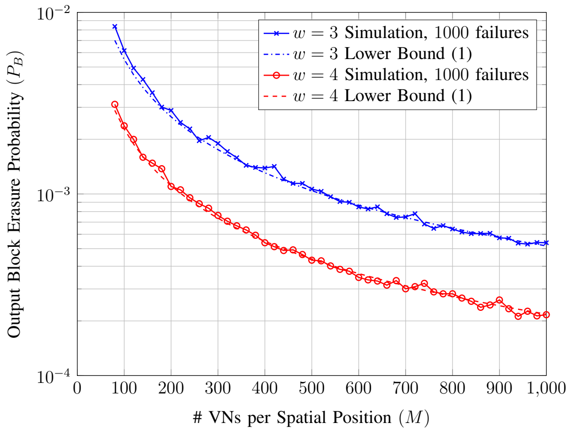

Fig. 2. Monte Carlo simulations on the SPBC with a $(3, 6)$ random ensemble for $w = 3$ and $w = 4$, along with their respective theoretical lower bound (1). The bound becomes tight very quickly with M.

an ensemble. As example, we randomly choose two VNs v_1 and v_2 from an SP of the $(d_v = 3, d_v = 6, w, L, M)$ ensemble. First, we connect the $d_v = 3$ edges of v_1 to randomly chosen empty sockets of d_v distinct CNs as described in Section II-A. Let c_1, c_2, c_3 denote the CNs adjacent to v_1. A stopping set (and in this case, also a low-weight codeword) is formed if and only if the edges of v_2 are connected to the same CNs, i.e., c_1, c_2, c_3. This situation is shown in Fig. 1: once we have assigned d_v CNs to v_1, we have $d_c - 1 = 5$ free distinct sockets each for CNs c_1, c_2, c_3. Thus, the first edge of v_2 has $d_v(d_c - 1) = 15$ ways to attach to these sockets, the second edge has $(d_v - 1)(d_c - 1) = 10$ ways and the last edge has $(d_v - 2)(d_c - 1) = 5$ ways. In general, the edges of v_2 can be connected to any of the $wMd_v - d_v$ possible sockets.

By a counting argument, we can compute $p = \frac{T_{ss}}{T}$ where T_{ss} is the total number of combinations by which the edges of v_2 can form a stopping set with v_1 and T is the total number of combinations by which the edges of v_2 can be fit to the possible CN sockets without forming parallel edges.

Hence, for a general (d_v, d_c, w, M) ensemble we can calculate $p = \frac{T_{ss}}{T}$ with

$$T_{ss} = \prod_{i=0}^{d_v-1} (d_v - i)(d_c - 1) = d_v!(d_c - 1)^{d_v},$$

$$T = \sum_{i=0}^{d_v} \frac{(d_c - 1)^i d_v!}{(d_v - i)!} \binom{d_v}{i} \left[\prod_{k=0}^{d_v-1-i} (wMd_v - (d_v + k)d_c) \right].$$

For large M, T can be well approximated by the dominating summand $(i = 0)$ leading to

$$p \approx \prod_{i=0}^{d_v-1} \frac{(d_v - i)(d_c - 1)}{(wMd_v - (d_v + i)d_c)} \approx \frac{d_v!(d_c - 1)^{d_v}}{((wM - d_c)d_v)^{d_v}}. \quad (3)$$

We observe that $\lambda_{SP} = \binom{M}{2} p \sim O(M^{2-d_v})$.

B. Simulations

We performed Monte-Carlo simulations where we randomly selected a spatial position from the middle of the graph (to avoid boundary effects) to be erased, for each transmitted codeword. At the receiver we performed BP decoding and averaged over the ensemble. We counted 1000 decoding failures for each M to assess the average block erasure probability $P_{\mathrm{B}}^{\mathrm{SPBC}}$. The simulation results for a $(3, 6)$ random ensemble with $w = 3$ and $w = 4$ are shown in Fig. 2 along with their respective lower bounds calculated using (1) and (3). We observe that the bound indeed becomes a good approximation for large M, since large-size stopping sets (larger than 2) vanish. The simulation curve is slightly unstable because counting 1000 failures is not enough to keep the sample variance small as $P_{\mathrm{B}}^{\mathrm{SPBC}}$ decreases by $O(M^{2-d_v})$.

IV. Error Analysis on the RBC

We now generalize our results to the RBC, where a burst can span multiple spatial positions and can be of arbitrary length. Besides the stopping sets within a single spatial position, we first have to derive an expression for stopping sets that span multiple SPs.

A. Size-2 Stopping Sets across Coupled SPs

The results from Sec. III can be extended when the channel is a RBC, i.e., the burst occurs at arbitrary location and is of arbitrary length. This means that size-2 stopping sets formed across coupled SPs will also contribute to decoding failures. Hence, we will now calculate the probability that two VNs chosen each from two coupled spatial positions form a stopping set.

Let us first consider two VNs chosen from two adjacent SPs: w.l.o.g, call them v_1 and v_2 chosen from SPs 1 and 2, respectively. We immediately notice that the check positions adjacent to v_1 are $1, 2, \ldots, w$ and to v_2 are $2, 3, \ldots, w + 1$. Hence, to form a stopping set, v_1 should not have any edge connected to check position 1. This restricts the number of favorable constellations [3] for v_1 to be $(w - 1)^{d_v}$. Using the same ideas as in Section III-A and restricting the constellations for v_1, we have

$$p_{(1,2)} = \frac{(w - 1)^{d_v}}{w^{d_v}} p,$$

where p can be approximated by (3). This idea can now be extended to VNs chosen from positions $(1, 3), (1, 4), \ldots, (1, w)$ by restricting the number of favorable constellations for v_1. Hereafter, we will refer to these as size-2 $(1, i)$-stopping sets. Hence, a (d_v, d_c, w, L, M) ensemble can be completely characterized, for large enough M, by the vector

$$\underline{p}(d_v, d_c, w, L, M) = (p_{(1,1)}, p_{(1,2)}, \ldots, p_{(1,w)}) \quad (4)$$
$$\text{with } p_{(1,i)} = \left(\frac{w - (i - 1)}{w} \right)^{d_v} p.$$

The average number of size-2 stopping sets of each type, $\lambda_{(1,i)}$, can be calculated as

$$\lambda_{(1,1)} = \binom{M}{2} p_{(1,1)} = \lambda_{SP} \;\; ; \;\; \lambda_{(1,i)} = M^2 p_{(1,i)}, \quad (5)$$

where $i = 2, 3, \ldots, w$. Again, we see that $\lambda_{(1,i)} \sim O(M^{2-d_v})$.

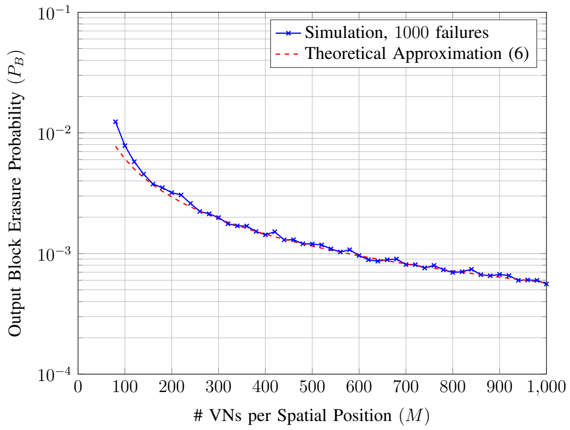

Fig. 3. Monte Carlo simulations for a $(3, 6, 3, 20, M)$ random ensemble on the RBC with burst length $b = 1.25M$, along with the theoretical approximation (6).

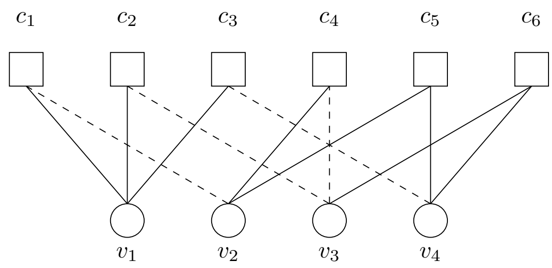

Fig. 4. A size-4 stopping set from an expurgated $(3, 6, w, L, M)$ random ensemble. CNs $\{c_1, c_2, c_3, c_4, c_5, c_6\}$ and VNs $\{v_1, v_2, v_3, v_4\}$ have been labeled for convenience. The solid edges indicate definite connections and the dashed edges complete one configuration to form a stopping set. Parallel edges are not allowed in the ensemble.

B. Performance on the RBC

Now let us see the effect of RBC(s, b) on the ensemble in terms of the average block erasure probability, $P_{\mathrm{B}}^{\mathrm{RBC}}$. For keeping the expressions simple, let us assume in the example that $w = 3$ and $0 < b \leq 2M$. This means that the burst can span a maximum of 3 SPs. Applying the same argument as in Section III and assuming all values for s are equally likely,

$$P_{\mathrm{B}}^{\mathrm{RBC}} \approx \sum_{s=1}^{M} \frac{1 - P_{(1,1)}P_{(2,2)}P_{(3,3)}P_{(1,2)}P_{(2,3)}P_{(1,3)}}{M}; \quad (6)$$

$$P_{(k,k)} = 1 - \binom{m_k}{2} p_{(1,1)} \quad \text{for } k = 1, 2, 3,$$

$$P_{(k,k+1)} = 1 - m_k m_{k+1} p_{(1,2)} \quad \text{for } k = 1, 2,$$

$$P_{(k,k+2)} = 1 - m_k m_{k+2} p_{(1,3)} \quad \text{for } k = 1,$$

where $m_1 = (M - s), m_2 = \min(b - m_1, M), m_3 = (b - m_1 - m_2)$ are the lengths of the burst in each SP that it affects, progressing from left to right. If any of these lengths is zero, all probabilities involving that length are 1, i.e., the probability of forming no size-2 stopping sets involving the SP corresponding to this (zero) length is 1. For general w and longer bursts, this strategy can be extended for finding a very good approximation for the average block erasure probability for the ensemble.

To verify the tightness of (6), we again performed Monte-Carlo simulations and counted 1000 decoding failures for each M to assess the average block erasure probability $P_{\mathrm{B}}^{\mathrm{RBC}}$. For the sake of example, we fixed the burst length to be $b = 1.25M$. We selected a value for s, uniformly from $\{1, \ldots, M\}$, for each codeword. The simulation results for the $(3, 6, 3, 20, M)$ ensemble are shown in Fig. 3 along with (6). We see that (6) is indeed a tight approximation.

V. Effects of Expurgation

A. Minimal Stopping Set Size

As the performance is mainly dominated by size-2 stopping sets, we can improve the burst erasure correction capability by expurgating the ensemble and thereby removing all small stopping sets. Observing that a size-2 stopping set, as shown in Fig. 1, is built around 4-cycles, we can reduce the size of the minimal stopping sets by removing small cycles from the graph. For example, increasing the girth of the graph to 6 leads to minimal stopping sets (i.e., of smallest size) of size $d_v + 1$ [12].

B. Performance on the SPBC

We can use the same approach as in Section III-A to calculate the probability of occurrence of the stopping set shown in Fig. 4 within a spatial position of a code sampled uniformly from the ensemble. Once again we have $p = \frac{T_{ss}}{T}$, where T_{ss} is the total number of combinations of the edges of v_1, v_2, v_3, v_4 that form a stopping set and T is the total number of combinations by which these edges can fit to the available CN sockets. Since T is the total number of combinations in which the edges of $(d_v + 1)$ VNs can be assigned to sockets ensuring no 4-cycles, we can again approximate it by its dominant term as

$$T \approx \prod_{j=0}^{d_v(d_v+1)-1} (wMd_v - jd_c).$$

For a general (d_v, d_c, w, M) random ensemble, the expression for T_{ss} can be calculated as

$$T_{ss} = \prod_{i=0}^{d_v} \left[\prod_{j=1}^{i} j(d_c - 1)(d_v - i + 1) \right] \times \left[\prod_{k=\sum_{m=0}^{i-1}(d_v-m)}^{\sum_{m=0}^{i-1}(d_v-m)+(d_v-i-1)} (wMd_v - kd_c) \right] \binom{d_v}{i}.$$

It can be verified that the last value for k in the above expression is $k = \frac{d_v(d_v+1)}{2} - 1$. Then, we can simplify and rearrange the expression as

$$T_{ss} = T_{1/2} \prod_{i=1}^{d_v} [(d_c - 1)(d_v - i + 1)]^i \frac{d_v!}{(d_v - i)!},$$

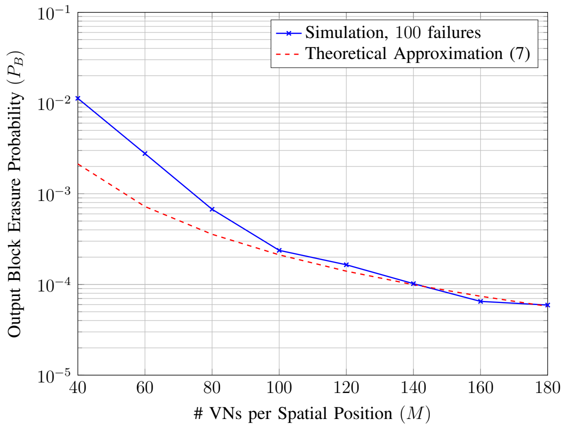

Fig. 5. Monte Carlo simulations on the SPBC with an expurgated $(3,6)$ random ensemble for $w=3$ along with the theoretical approximation. The approximation becomes tight very quickly with M.

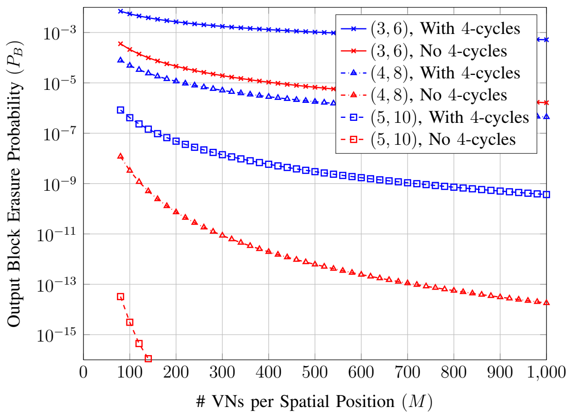

Fig. 6. The theoretical approximation (1) and (7) on P_B^{SPBC} for various ensembles in both the unexpurgated and expurgated scenarios.

where, $T_{1/2} = \prod_{k=0}^{\frac{d_v(d_v+1)}{2}-1}(wMd_v - kd_c)$ is the first half of the products in T which can be canceled while calculating p, so that

$$\frac{T}{T_{1/2}} \approx \prod_{j=\frac{d_v(d_v+1)}{2}}^{d_v(d_v+1)-1}(wMd_v - jd_c).$$

Hence, for a general (d_v, d_c, w, L, M) ensemble, the probability of forming such a minimal stopping set of size (d_v+1) can be shown to be

$$p = \frac{T_{ss}}{T} \approx \frac{\prod_{i=1}^{d_v}\left[(d_c-1)(d_v-i+1)\right]^i \frac{d_v!}{(d_v-i)!}}{\prod_{j=\frac{d_v(d_v+1)}{2}}^{d_v(d_v+1)-1}(wMd_v - jd_c)} \qquad (7)$$

which means the expected number of such stopping sets within a SP of the code is $\lambda_{SP} = \binom{M}{d_v+1}p$. Using similar arguments as in Section III, we have $P_{\text{B,exp}}^{\text{SPBC}} \approx \lambda_{SP}$.

C. Comparison of Ensembles

We now compare the average performance of different SC-LDPC ensembles on the SPBC. We fix the asymptotic design code rate as $r = \frac{1}{2}$, the smoothing parameter as $w = d_v$ and plot the (tight) approximations on P_B^{SPBC} of three ensembles, namely $(3,6)$, $(4,8)$ and $(5,10)$, for both the unexpurgated and the expurgated cases in Fig. 6.

For the unexpurgated case, the average block erasure probability varies as $P_{\text{B}}^{\text{SPBC}} \sim O(M^{2-d_v})$. When the ensemble is expurgated, the improvement is by an order of $\frac{d_v+1}{2}$ in M and we have $P_{\text{B,exp}}^{\text{SPBC}} \sim O(M^{(d_v+1)(2-d_v)/2})$. Therefore, for a fixed (asymptotic design) rate of $\frac{1}{2}$, a unit increase in d_v improves the performance by a factor of about M^{-d_v}.

VI. Conclusion

We have analyzed random regular SC-LDPC ensembles on the burst erasure channel and provided insights into improving the block erasure probability by increasing VN degree and expurgating the code. We have shown, through these results, that the vector in (4) completely characterizes the average ensemble performance on the erasure channel.

Future work will focus on, among others, finding good approximations for the expurgated ensembles in the case of the RBC model and to extend the considerations to the case where we have independent random erasures besides the burst erasures. We note that the vector in (4) will play a significant role in the analysis of random erasures too.

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