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A Beta-Beta Achievability Bound with Applications

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We consider an abstract channel that consists of an input set \(A\), an output set \(B\), and a random transformation \(P_{Y|X} : A \rightarrow B\). An \((M, \epsilon)\) code for the channel \((A, P_{Y|X}, B)\) comprises a message set \(M \triangleq \{1, \ldots, M\}\), an encoder \(f : M \rightarrow A\), and a decoder \(g : B \rightarrow M \cup \{\epsilon\}\), where \(\epsilon\) denotes an error event, that satisfies the average error probability constraint

\[
\frac{1}{M} \sum_{j=1}^{M} \left(1 - P_{Y|X}(g^{-1}(j) | f(j))\right) \leq \epsilon. \tag{1}
\]

Here, \(g^{-1}(j) \triangleq \{y \in Y : g(y) = j\}\). In Theorem 1 below, we provided a novel lower bound (i.e., achievability bound) on the largest number of codewords for which an \((M, \epsilon)\) code exists.

Theorem 1 (\(\beta\beta\) achievability bound): For every \(0 < \epsilon < 1\) and every input distribution \(P_X\), there exists an \((M, \epsilon)\) code for the channel \((A, P_{Y|X}, B)\) satisfying

\[
\frac{M}{2} \geq \sup_{0 < r < \epsilon} \frac{1}{Q_{Y}} \sum_{w \in \mathcal{W}} \beta_r(P_Y, Q_Y) \tag{2}
\]

Here, \(P_Y \triangleq P_{Y|X} \circ P_X\), and \(\beta_r(P, Q)\) is defined as

\[
\beta_r(P, Q) \triangleq \min \int P_{Z|W}(1 | w)Q(dw) \tag{3}
\]

where the minimum is over all conditional probability distributions \(P_{Z|W} : \mathcal{W} \rightarrow \{0, 1\}\) satisfying

\[
\int P_{Z|W}(1 | w)P(dw) \geq \alpha \tag{4}
\]

and \(\mathcal{W}\) denotes the support of \(P\) and \(Q\).

The proof of Theorem 1, which can be found in [1], relies on Shannon’s random coding technique and on a suboptimal decoder that is based on the Neyman-Pearson test [2] between \(P_{XY}\) and \(P_X Q_Y\). Hypothesis testing is used twice in the proof: to relate the decoding error probability to \(\beta_{1-\epsilon+r}(P_{XY}, P_X Q_Y)\), and to perform a change of measure from \(P_Y\) to \(Q_Y\).

The bound (2) is the dual of a converse bound recently established by Polyanskiy and Verdù [3, Th. 15]. Furthermore, both (2) and [3, Th. 15] can be viewed as a finite-blocklength analog of the following identity for mutual information (also known as the golden formula) [4, Eq. (8.7)], which is exceedingly useful for computing or bounding capacity [5]–[8]:

\[
I(X; Y) = D(P_X P_{Y|X} || P_X Q_Y) - D(P_Y || Q_Y). \tag{5}
\]

The connections between (2) and existing achievability bounds in the literature are discussed in [1].

The bound (2) is useful in situations where \(P_Y\) is not a product distribution (although the underlying channel law \(P_{Y|X}\) is stationary memoryless), for example due to cost constraints, or structural constraints on the channel input, such as orthogonality or constant composition. In such cases, traditional achievability bounds such as Feinstein’s bound [9] and the dependence testing bound [10, Th. 18], which are explicit in \(dP_{Y|X}/dP_Y\), become difficult to evaluate. In contrast, the \(\beta\beta\) bound (2) requires the evaluation of \(dP_{Y|X}/dP_Y\), which factorizes for product \(Q_Y\). This allows for an analytical computation of the bound (2). Furthermore, the term \(\beta_r(P_Y, Q_Y)\)—which captures the cost of the change of measure from \(P_Y\) to \(Q_Y\)—can be evaluated or bounded even when a closed-form expression for \(P_Y\) is not available. Applications of the bound (2), which illustrate these properties, are provided in [1].

REFERENCES

