Size of Edge Separators in Geometric Inhomogeneous Random Graphs Depends on Underlying Geometry

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Publication Date: 2016

Permanent Link: https://doi.org/10.3929/ethz-a-010725415

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Size of Edge Separators in Geometric Inhomogeneous Random Graphs Depends on Underlying Geometry

Master Thesis
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Saturday 10th September, 2016

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Abstract

In Euclidean Geometric Inhomogeneous Random Graphs (GIRG), a model for real-networks recently presented by Bringmann, Keusch, and Lengler, each vertex is equipped with a weight and a position. The weights are drawn from a power-law distribution (with a fixed exponent $2 < \beta < 3$), while the positions are drawn uniformly from the $d$-dimensional torus $T^d$. Then, the edge probability of two vertices is roughly polynomial in the product of their weights, and the inverse of their Euclidean distance. The Euclidean GIRG model has a wide range of interesting properties, such as a small diameter, small average distance, small separators, and a large clustering coefficient.

In this thesis, we study a variant of the GIRG model that uses minimum component distance (MCD) instead of Euclidean distance, but is otherwise defined in the same way as its Euclidean counterpart. The main result of the thesis is that the MCD model (for $d > 1$) has no small (edge) separators, setting it apart from Euclidean GIRGs.
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Chapter 1

Introduction

In [2], Bringmann, Keusch and Lengler introduced a very general model for random graphs which are both scale-free (i.e. the degree distribution follows a power law) and have an underlying geometry. It is an abstraction of well-known models such as Chung-Lu random graphs [3] (section 5.3) and hyperbolic random graphs (HRGs) [7]. This abstraction ensures a certain degree of simplicity — especially compared to HRGs — by making heavy use of asymptotic notation. Even at full generality, important properties such as the existence of a giant component, polylogarithmic diameter of said giant component, as well as very small average distance can be shown. On the other hand, analyzing the clustering coefficient or the size of separators necessitates specializing the model. In [1], Bringmann, Keusch and Lengler define such a model, called Euclidean GIRG (geometric inhomogeneous random graphs), and prove that it has constant clustering coefficient and small separators. These properties are indeed not general; neither holds for the Chung-Lu model. In this thesis, we show that a significantly smaller change to the Euclidean GIRG model’s underlying geometry is sufficient to break the small-separator property (chapter 2). This change amounts to replacing the Euclidean norm used for GIRGs with minimum component distance (MCD) (section 1.1). The resulting model appears as Example 7.2 in [2]. In a loose sense, the model is similar to a union of one-dimensional Euclidean GIRG instances on the same vertex set, though these GIRG instances are not independent. We will further explore this intuition in Section 2.2. The MCD model generates graphs in which vertices are likely to be connected if they are close in at least one dimension, whereas in Euclidean GIRGs vertices have to be close along all dimensions. This might make it well-suited for modeling social networks in which participants are likely to connect if they are similar in at least one aspect (for example if they share the same hobby). However, as real-life networks commonly have small separators [4], the result that the MCD model does not indicates that it might be somewhat less
suitable for modeling such networks. On the other hand, the separator question remains open for models that interpolate between MCD graphs and Euclidean GIRGs. We briefly discuss the potential usefulness of such variations in the concluding section 2.5.

1.1 Model

We start by introducing the distance model from [2], Section 7; it is a natural generalization of GIRGs that also encompasses the modified model we study in this thesis. Each of the following paragraphs indicates which section of [2] it is based on.

(See section 2.1 of [2].) We consider the vertex set $V = \{1, \ldots, n\}$. Each vertex $v \in V$ has two values attached to it: a weight $w_v$ and a position $x_v$. The weight is a positive real number that we consider to be fixed (i.e. non-random) for a given $n$. However, it is possible to extend the model to randomized weights; see Lemma 4.6 in [2]. On the other hand, positions are drawn independently at random from a ground space for each vertex. For each pair $u \neq v \in V$, we denote the event that there is an edge between $u$ and $v$ as “$u \sim v$“. The edge between $u$ and $v$ is formed independently with a probability that depends on the positions of $u$ and $v$, and on $w := (w_1, \ldots, w_n)$, the tuple of all weights. Let us denote this probability by $p_{uv}$, i.e.

$$P[u \sim v \mid x_u, x_v, w, n] =: p_{uv}(x_u, x_v, w, n) =: p_{uv}(x_u, x_v)$$  \hspace{1cm} (1.1)

Before we can formulate the requirements we impose on $p_{uv}$, we need to elaborate on vertex positions and the ground space they are drawn from.

**Vertex positions** (See Section 7 of [2].) We define our ground space to be $T^d$, the $d$-dimensional torus, where $d \geq 1$ is a constant parameter of the model. The torus $T^d$ is the $d$-dimensional hypercube $[0, 1]^d$ with opposite boundaries identified, i.e. $T^d = \mathbb{R}^d / \mathbb{Z}^d$. Vertex positions $x_v$ are drawn independently and uniformly at random from $T^d$. The $i$-th component of $x_v$ is denoted by $x_{vi}$, i.e. $x_v = (x_{v1}, x_{v2}, \ldots, x_{vd})$.

Together with a distance function $\| \cdot \| : [-1, 1]^d \to \mathbb{R}_{\geq 0}$, the torus $T^d$ defines the underlying geometry. The function $\| \cdot \|$ does not need to be a norm or seminorm; however, it should be measurable and fulfill $\|x\| = \|-x\|$ and $\|0\| = 0$. For $r \geq 0$ and $x \in T^d$, let $B_r(x) := \{y \in T^d \mid \|x - y\| \leq r\}$ denote the $r$-ball around $x$ with respect to $\| \cdot \|$, and $V(r)$ the volume of $B_r(x)$. (Note that on $T^d$ this does not depend on $x$). The ball volume function $V : \mathbb{R}^d \to [0, 1]$ has to be surjective.

**Edge probabilities** (See Section 7 of [2].) We can now state the requirements imposed on the edge probabilities. Let $\alpha > 1$ be a parameter of the model
1.1. Model

and $W := \sum_{v \in V} w_v$ the total weight. The edge probability function $p_{uv}(x_u, x_v)$ must fulfill, for all vertex pairs $u \neq v \in V$

$$p_{uv}(x_u, x_v) = \Theta \left( \min \left\{ 1, V(\|x_u - x_v\|)^{-\alpha} \cdot \left( \frac{w_u w_v}{W} \right)^{\alpha} \right\} \right).$$

(EP)

(Note that compared to [2], we drop a max{.} from the exponent and demand $\alpha > 1$ instead of $\alpha \neq 1$. This is done for simplicity and does not significantly change any arguments.) The bounds represented by $\Theta$ are assumed to be global, i.e. there are constants $c_L, c_U > 0$ such that for sufficiently large $n$ the inequalities

$$p_{uv}(x_u, x_v) \geq c_L \min \left\{ 1, V(\|x_u - x_v\|)^{-\alpha} \cdot \left( \frac{w_u w_v}{W} \right)^{\alpha} \right\} \quad \text{and} \quad \text{(EPL)}$$

$$p_{uv}(x_u, x_v) \leq c_U \min \left\{ 1, V(\|x_u - x_v\|)^{-\alpha} \cdot \left( \frac{w_u w_v}{W} \right)^{\alpha} \right\} \quad \text{(EPU)}$$

hold for all $u \neq v \in V$ and all $x_u, x_v \in \mathbb{T}^d$. Note the only random variables $p_{uv}$ depends on are $x_u$ and $x_v$. In this thesis, we focus on two distance functions: The maximum norm and minimum component distance (MCD) (Examples 7.1 and 7.2 of [2]). While the former takes the maximum over the absolute differences in the coordinates, the latter takes the minimum. Since we are dealing with tori, we define the absolute difference for $a, b \in [0, 1)$ as

$$|a - b|_T := \min\{ |a - b|, 1 - |a - b| \}. \quad (1.2)$$

For $x, y \in \mathbb{T}^d$, maximum norm and MCD can then be expressed in term of $|.|_T$ as

$$\|x - y\|_{\max} := \max_{1 \leq i \leq d} |x_i - y_i|_T \quad \text{and}$$

$$\|x - y\|_{\min} := \min_{1 \leq i \leq d} |x_i - y_i|_T, \quad (1.3)$$

respectively. Note that the two distance functions coincide on $\mathbb{T}^1$ and that MCD is not a norm for $d > 1$. The ball volume function for the maximum norm is simply $V_{\max}(r) = (2r)^d$. (We limit $V_{\max}(r)$ to $r \in [0, 1/2]$ for simplicity, since that is the range of $\|\|_{\max}$.) For MCD, the volume function can be computed by using an inclusion-exclusion formula; it is given by $1 - (1 - 2r)^d$ (again, we restrict to $r \in [0, 1/2]$). As this is somewhat unwieldy, we observe that $2r \leq 1 - (1 - 2r)^d \leq 2dr$. Therefore, if we set $V_{\min}(r) := r$, we are off by at most a constant factor; this does not change the model at all, because such constant factors are accounted for by the $\Theta$ in EP.

With the maximum norm (or in fact, any norm — all norms on $\mathbb{T}^d$ are equivalent), we obtain the Euclidean GIRG model. Using the MCD, however, yields different properties; in particular, the resulting model has no small separators, as we will see in section 2.
While we now have everything we need to start sampling some graphs, we lack one crucial part which ensures most beneficial properties of the model: The power law the weights should follow.

Weights (See section 2.1 of [2].) Let $2 < \beta < 3$ be a constant parameter of the model. We demand that

$$w_{\min} := \min\{w_v \mid v \in V\} = \Omega(1),$$

(PL1)

and that there exists $\bar{w} = \bar{w}(n) \geq n^{\omega(1/\log \log n)}$ such that for all constants $\eta > 0$ there are $c_1, c_2 > 0$ with

$$c_1 \frac{n}{w^{\beta-1+\eta}} \leq \#\{v \in V \mid w_v \geq \bar{w}\} \leq c_2 \frac{n}{w^{\beta-1-\eta}}.$$  

(PL2)

The second inequality holds for all $w \geq w_{\min}$, while the first one only holds for all $\bar{w} \geq w \geq w_{\min}$. These conditions directly imply that the total weight $W := \sum_{v \in V} w_v$ is in $\Theta(n)$; this is proved in lemma 4.2 of [2]. Note that the only explicit use of (PL2) in this thesis is to prove variations of a much weaker statement: For each constant $c \geq w_{\min}$, there are constants $c'$ and $c''$ such that

$$c' n \leq \#\{v \in V \mid w_v \geq c\} \leq c'' n.$$  

(1.5)

However, behind the scenes, (PL2) is quite important. It is used not only for showing $W = \Theta(n)$, but also in the proof of Theorem 1.3. Both statements are crucial to the proof of the main theorem of this thesis.

1.2 Our Result

The main part of this thesis is the analysis of the stability of the MCD model under the removal of edges. In many real-world networks, it is sufficient to remove a small number of edges to split the network (more precisely, its giant component) into at least two large (i.e. linear-sized) disconnected parts [4]. Such networks are said to have small (edge) separators. As established in Theorem 2.5 of [1], Euclidean GIRGs have small separators. In contrast, we show that the opposite is true for the MCD model for $d > 1$, making MCD the first non-trivial distance function under which the distance model proposed in [2] has no small separators. In particular, this demonstrates that the existence of small separators depends crucially on the underlying geometry.

In order to be able to conduct a precise discussion, we need to introduce some terminology.

Definition 1.1 (from [9], page 21) For a graph $G = (V, E)$ and $\delta, \eta > 0$, a $(\delta, \eta)$-cut is a partition of $V$ into two sets of size at least $\delta |V|$ such that there are at most $\eta |V|$ cross-edges (i.e. edges that have one endpoint in each of the sets).
1.3. Proof Outline and Comparison with Erdős-Rényi

Usually, we will consider $(\delta, \eta)$-cuts of the largest component of $G$. Somewhat abusing notation, the two sets into which that component is partitioned still need to have size at least $\delta |V|$, and the cut must have at most $\eta |V|$ cross-edges (i.e. we do not use the size of the component instead of $|V|$).

Somewhat loosely speaking, we say that a graph family has small separators if for a graph with $n$ vertices it is possible to split its giant component into parts of linear size by removing only $o(n)$ edges. Note that we always refer to splitting the giant component and not the whole graph; this is due to the fact that a Euclidean GIRG (or MCD graph) of size $n$ has $\Omega(n)$ isolated vertices w.h.p., yielding a trivial cut between isolated and non-isolated vertices. As mentioned before, Theorem 2.5 in [1] established that Euclidean GIRGs have small separators; it states that there is a $c < 1$ such that removing $n^c = o(n)$ edges suffices to split the giant component of a Euclidean GIRG into linear-sized parts. We will show that the MCD model for $d > 1$ has no small separators by proving that for all $\delta > 0$ there exists a $\eta > 0$ such that an MCD graph has no $(\delta, \eta)$-cut w.h.p; this is the statement of the following main theorem.

**Theorem 1.2 (main theorem)** Let $G$ be a graph on $n$ vertices drawn according to the MCD model from section 1.1, i.e. with vertex weight distribution obeying (PL1) and (PL2), and with edge probabilities obeying (EP) with MCD as the distance function (in (EP), set $\|\cdot\| = \|\cdot\|_{\min}$ and $V(r) = r$). Then, for every $\delta > 0$ there is an $\eta > 0$ such that w.h.p, the largest component of $G$ has no $(\delta, \eta)$-cut.

Note that according to Theorem 1.3, w.h.p. the largest component is the only one with size $\Omega(n)$; this justifies using the term “giant” for the largest component.

1.3 Proof Outline and Comparison with Erdős-Rényi

The proof of the main theorem was inspired by the one given in [9]. There, Luczak and McDiarmid prove a statement (Lemma 2 in [9]) to Theorem 1.2 for Erdős-Rényi graphs with edge probability $p = c/n$, where $c > 1$ is a constant — for $c < 1$, the graph does not have a giant.

We will now present proof outlines for both the Erdős-Rényi (Lemma 2 in [9]) and MCD (Theorem 1.2) case, including a brief discussion why the (significantly simpler) approach for Erdős-Rényi cannot be applied to the MCD model directly.

1.3.1 Erdős-Rényi

(The argument in this section is adapted from [9].) Let $G = (V, E)$ be a random graph on $n$ vertices generated from the Erdős-Rényi model $G_{n, c/n}$ for a constant $c > 1$. Our goal is to show that for every $\delta > 0$, there is
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an $\eta > 0$ such that w.h.p. the giant of $G$ has no $(2\delta, \eta)$-cut (the factor 2 is arbitrary and chosen for reasons of convenience). Note that $c > 1$ guarantees that w.h.p. $G$ has a single component of linear size, namely the giant (Lemma 3 in [9]). Loosely speaking, we independently generate two random graphs $G' = (V, E')$ and $G'' = (V, E'')$ on the same vertex set $V$. We draw $G'$ from $G_{n, c_1/n}$ and $G''$ from $G_{n, c_2/n}$, where $c_1 > 1$ is a constant very close to (but still smaller than) $c$, and $c_2$ is a very small constant. Furthermore, we require

$$ c_2 = \frac{c - c_1}{1 - c_1/n} = (c - c_1) + O \left( \frac{1}{n} \right), \quad (1.6) $$

(Note that, as desired, $c_2$ ends up being very small when $c_1$ is close to $c$.) A quick calculation shows that with this choice of edge probabilities, the union of $G'$ and $G''$, denoted by $G' \cup G'' := (V, E' \cup E'')$, has the same distribution as $G$. This means that for every vertex pair $u \neq v \in V$ we have

$$ \mathbb{P} [u \sim v \in G] = \mathbb{P} [u \sim v \in G', \text{ or } u \sim v \in G'']. \quad (1.7) $$

Therefore, it is sufficient to show that for every $\delta > 0$, there is an $\eta$ such that $G' \cup G''$ has no $(2\delta, \eta)$-cut. The general idea of how to do this can be described as follows: We first draw $G'$. As we chose $c_1 > 1$, we know that $G'$ has a giant, i.e. a component of linear size. The giant might have some $(\delta, \eta)$-cuts, but all of these are destroyed by the additional edges from $G''$; these edges also enlarge the giant, possibly creating new “bad” cuts. However, we will argue that w.h.p. the resulting giant still has no $(2\delta, \eta)$-cuts as desired. That argument hinges on the fact $c_2$ can be made arbitrarily small, making $G''$ have very few edges. In the following, we will elaborate on this idea, explaining how to choose $c_1$ (and thus $c_2$), and $\eta$.

In this paragraph, we define $\eta$ and explain why the additional edges from $G''$ destroy all $(\delta, \eta)$-cuts in the giant of $G'$ with high probability. Here, Lemma 2.4 (originally Lemma 7 in [9]) plays a crucial role. We use it to obtain that for $\varepsilon := e^\delta c_2/16 - 1$, there are constants $\eta'$ and $n_0$ such that for all $n \geq n_0$, every connected graph on $n$ vertices has at most $(1 + \varepsilon)^n$ cuts with at most $n' n$ cross-edges. We will combine Lemma 2.4 with the observation that a partition of $V$ into two sets of size at least $\delta n$ has $\Omega(n)$ cross-edges in $G''$ in expectation and with high probability. More precisely, consider a partition of $V$ into $C_1$ and $C_2$, both of size at least $\delta n$. The expected number of cross-edges is at least $\delta n \cdot \delta n \cdot c_2/n = \delta^2 c_2 n$; a Chernoff bound (Theorem 2.1ii ) yields that with probability $1 - e^{-\delta^2 c_2 n/8}$, there are at least $\eta n' := \eta n'$ cross-edges. Finally, we set $\eta := \min(\eta', \eta'')$. Since by definition $\eta \leq \eta'$, the giant of $G'$ has at most $(1 + \varepsilon)^n$ distinct $(\delta, \eta)$-cuts. Each such cut has less than $\eta n$ cross-edges in $G''$ with probability at most $e^{-\delta^2 c_2 n/8} = (1 + \varepsilon)^{-2n}$. A union bound over all $(\delta, \eta)$-cuts in the giant of $G'$ shows that with probability $1 - (1 + \varepsilon)^{n - 2n} = 1 - 16n$, every such cut has at least $\eta n$ edges in $G''$. Thus,
after adding the edges of $G''$, there are no $(\delta, \eta)$-cuts left in the giant of $G'$ with high probability.

It remains to show that adding the edges of $G''$ does not create “bad” cuts in the giant. For this, it is sufficient to show that these edges increase the size of the giant by at most $\delta n$ vertices (as we will see shortly). Intuitively, choosing $c_2 \ll \delta$ should ensure that the size of the giant does not increase by more than $\delta n$, as the expected number of edges in $G''$ is $c_2 n$, and all other connected components are much smaller than the giant. More formally, this follows from Lemma 3 in [9] (originally from [6]). This lemma states that there is a continuous function $\gamma(t)$ such that the size of the giant of graphs from $G_{n,t/n}$ is tightly concentrated around $\gamma(t)n$. In particular, we can choose $c_1$ close enough to $c$ such that

$$\gamma(c) - \gamma(c_1) \leq \delta/2. \quad (1.8)$$

Using concentration bounds provided in Lemma 3 in [9], one can then show that indeed the giant of $G' \cup G''$ has at most $\delta n$ vertices more than the giant of $G'$. Finally, assume that there is a $(2\delta, \eta)$-cut in the giant of $G' \cup G''$. Since $G''$ added at most $\delta n$ vertices to the giant, such a cut induces a $(\delta, \eta)$-cut in the giant of $G'$. However, we have shown that such a cut does not exist w.h.p., leading to a contradiction. Therefore, the giant of $G' \cup G''$ has no $(2\delta, \eta)$-cut with high probability.

### 1.3.2 MCD Model

While the proof of Theorem 1.2 was inspired by the approach for Erdős-Rényi graphs, various complications had to be addressed, resulting in large differences. In this section, let $G = (V, E)$ denote a graph on $n$ vertices drawn from an MCD model. Unlike in the Erdős-Rényi case, there is no natural way to have independent random graphs $G'$ and $G''$ such that their union $G' \cup G''$ has the same distribution as $G$. However, we can still obtain a weaker form of (1.7) which is just a lower bound instead of an equality:

$$\mathbb{P}[u \sim v \text{ in } G] \geq \mathbb{P}[u \sim v \text{ in } G' \text{, or } u \sim v \text{ in } G''] \quad (1.9)$$

This lower bound is established by inequalities (LB1) and (LB2) in Section 2.2. It relies on the fact that for each vertex $u \in V$, its $d$-th coordinate $x_{ud}$ is independent of its other coordinates (i.e. $x_{ui}$ for $1 \leq i < d$). Correspondingly, Inequality (LB1) is likely to be satisfied when two vertices are close along one of the first $d - 1$ coordinates, whereas (LB2) is likely to be satisfied when two vertices are close along the $d$-th coordinate. It is possible use this lower bound to show that the edges of $G''$ (i.e. the graph induced by (LB2)) destroy all “bad” cuts in the giant of $G'$ (induced by (LB1)). However, unlike in the Erdős-Rényi case, there is unfortunately no way to make $G''$ have arbitrarily few edges. Furthermore, there is no statement similar to Lemma 3 in [9]
that would help us determine the exact size of the giant in the MCD model. Finally, we need to compensate for the fact that (1.9) underestimates edge probabilities by adding additional edges. Therefore, the edges added by $G''$, or for compensating (1.9), might well increase the size of the giant by a significant amount.

The solution to this problem is based on randomly selecting a special subset of vertices from among the giant of $G'$, and considering its incident edges separately from the others. We call this set $F$, and generate it by including vertices of the giant of $G'$ (with weight bounded by a constant) independently with a constant probability $p_F$. This probability can be made arbitrarily small, which allows us to control the size of $F$ and upper-bound the number of incident edges (Lemma 2.12) — a crucial part of the argument that these edges do not grow the giant by too much (Lemma 2.8). We will generate $G$ using Algorithm 1 (see Section 2.3), which consists of several “phases”. It starts by first drawing the first $d-1$ coordinates for each vertex, which determines $G'$ (a.k.a. $G_1$ in the proof). Then, it selects $F$ from its giant. It continues by drawing the $d$-th coordinate for all vertices not in $F$. This allows it to determine all edges not incident to $F$, i.e. in this intermediate graph, all edges not incident to $F$ are present if and only if they are present in $G$. In the proof, this intermediate graph is denoted by $G_3$ (as $G_2$ is used for an earlier phase which we skip over in this outline). Finally, we draw the $d$-th coordinate for vertices in $F$, which allows us to add all edges incident to $F$, completing the graph. We show that these additional edges incident to $F$, which result from vertices being close along the $d$-th coordinate, destroy all $(\delta, \eta)$-cuts in the giant of $G_3$ (Lemma 2.5). We use (LB2) (and thus $G''$, which is induced by it) as a lower bound for edge probabilities in this step. Additionally, $\eta$ is chosen in a way similar to the Erdős-Rényi case. As the edge probabilities depend on the vertex positions, some of them are correlated; however, Lemma 2.6 shows that the vertices are spread uniformly enough over the $d$-th dimension to alleviate this issue. In fact, using its corollary (Corollary 2.7), the proof of Lemma 2.5 is quite similar to the analogous argument given above for the Erdős-Rényi case (i.e. that the edges of $G''$ destroy all $(\delta, \eta)$-cuts).

The remainder of the proof consists of showing that the edges incident to $F$ added in the previous step do not increase the size of the giant by too much (Lemma 2.8). This part is significantly more involved than in the Erdős-Rényi case, as we basically need to make up for the lack of an analogue to Lemma 3 in [9]. This means that we explicitly need to show that $F$ has few incident edges (Lemma 2.12), which for the MCD model requires a concentration bound more sophisticated than Chernoff — namely the Azuma-like bound given in Theorem 2.3 (originally from [2]). Lemma 2.9 helps completing the argument by stating that there are only few vertices in $G_3$ that are in connected components larger than some constant, but not in the giant.
1.4. Known Properties of GIRGs

This implies that on average, each edge incident to \( F \) adds only a constant number of vertices to the giant.

Finally, lemmas 2.8 and 2.5 enable the proof of Theorem 2.13 stating that the giant of \( G \) has no \((4\delta, \eta)\)-cut with high probability. In turn, this almost immediately implies the main theorem.

1.4 Known Properties of GIRGs

Various interesting properties of GIRGs have been proved; we reproduce some of these (taken from [2]) in this section. One of them, namely Theorem 1.3i) (existence of the giant component), is crucial to the proof of the main theorem (Theorem 1.2). Theorem 1.6 is of special interest as it states that Euclidean GIRGs have small separators, unlike the MCD model. Note that apart from Theorem 1.6, all these properties are proved in [2] for a more general model. However, Theorem 7.3 of that paper confirms that for any distance function, the distance model presented in the previous section is an instance of the general model. In particular, the following results hold for both the MCD model and Euclidean GIRGs.

**Giant component** A large portion of this paper is dedicated to studying the separator properties of the giant component under the MCD model. The term “giant” is justified by the following theorem; it states that there is a single large component with \( \Omega(n) \) vertices, while every other component has size \( o(n) \), thus being completely dwarfed by giant.

**Theorem 1.3 (Component sizes, Theorem 5.9 in [2])**

(i) There is a constant \( s_{\text{max}} \) such that with high probability, there is a connected component containing at least \( s_{\text{max}}n = \Omega(n) \) vertices.

(ii) With high probability, all other components have at most \( \log O(1) n \) vertices (polylogarithmic size).

Note that \( s_{\text{max}} \) depends on the parameters of the model, including the factors hidden by \( \Theta \) in the edge probability equation (EP).

**Scale-free** By definition, we require the vertex weights to follow a power law. The same holds for vertex degree; the model is said to be scale-free.

**Theorem 1.4 (Theorem 6.3 in [2])** For all \( \eta > 0 \) there exist constants \( c_3, c_4 > 0 \) such that w.h.p.

\[
c_3 \frac{n}{c^{\beta-1+\eta}} \leq \# \{ v \in V | \deg(v) \geq c \} \leq c_4 \frac{n}{c^{\beta-1-\eta}}.
\] (1.10)

where the second inequality holds for all \( 1 \leq c \) and the first one hold for all \( 1 \leq c \leq \bar{w} \), where \( \bar{w} \) is the same as in (PL2).
Note that the exponents are the same as for the weights (see (PL2)).

**Average distance and diameter** Despite the fact that the giant encompasses a linear number of vertices, its diameter is much smaller. This approximates the well-known “small world”-property popularized by the phrase “six degrees of separation” [10].

**Theorem 1.5 (Average distance and diameter, Theorem 5.9 in [2])** With high probability

(i) the diameter of the giant, i.e. the largest distance between any pair of vertices, is bounded by \( \log^{O(1)} n \).

(ii) the average distance in the giant, i.e. the expected distance between two vertices chosen uniformly and independently at random, is \( (2 \pm o(1)) \frac{\log \log n}{\log(\beta - 2)} \) almost surely and in expectation.

The question whether the diameter bound can be improved to \( \log n \) from \( \log^{O(1)} n \) remains open [8].

**Small separators in the Euclidean GIRG model** As mentioned before, the Euclidean GIRGs have small separators, i.e. it suffices to remove \( o(n) \) edges to split the giant into two parts of linear size.

**Theorem 1.6 (Theorem 2.5 in [1])** For any \( 0 < \varepsilon \leq \min\{\alpha - 1, \beta - 2, 1/d\} \), it almost surely suffices to delete \( n^{1-\varepsilon+o(1)} \) edges of a Euclidean GIRG to split its giant into two parts of linear size each.

This can be achieved by splitting \( \mathbb{T}^d \) into two halves with two hyperplanes perpendicular to an axis, and then deleting all edges crossing these hyperplanes; the proof of the theorem shows that there are only \( n^{1-\min\{\alpha - 1, \beta - 2, 1/d\} + o(1)} \) such edges. Note that due to symmetry, this will roughly split the giant in half.

We will gradually proceed to proving that the converse is true for the MCD model. First, we introduce a specific view on the sampling process which will prove quite useful later on.
Chapter 2

Proof of non-separability

In this chapter, we will develop the proof of the main theorem, Theorem 1.2, stating that the MCD model has no small separators. It relies on two crucial lemmas, namely 2.5 and 2.8. We need to introduce various tools: Auxiliary results from other papers (Section 2.1), specific lower bounds for edge probabilities (Section 2.2), and a suitable random graph generation procedure (Section 2.3).

2.1 Tools

Before we can delve into the specifics of the proof, we need to introduce some more general auxiliary results. Concentration bounds are the bread and butter of the proof, especially the following Chernoff bounds:

**Lemma 2.1 ("Weak" Chernoff bounds (e.g. Theorem 1.1 in [5]))** Let \( X := \sum_{i=1}^{n} X_i \) where for all \( 1 \leq i \leq n \), the random variables \( X_i \) are independently distributed in \([0, 1] \). Then

(i) \( \Pr[X > (1 + \varepsilon) E[X]] \leq \exp \left(-\frac{\varepsilon^2}{2} E[X]\right) \), for all \( 0 < \varepsilon < 1 \),

(ii) \( \Pr[X < (1 - \varepsilon) E[X]] \leq \exp \left(-\frac{\varepsilon^2}{2} E[X]\right) \), for all \( 0 < \varepsilon < 1 \), and

(iii) \( \Pr[X > t] \leq 2^{-t} \) for all \( t > 2E[X] \).

In particular, this holds when the \( X_i \) are i.i.d. indicator random variables.

In one case, these simple bounds will not do; instead we will have to make use of a more general form:

**Lemma 2.2 ("Strong" Chernoff bound (e.g. Theorem 2.15 in [3]))** Let \( X := \sum_{i=1}^{n} X_i \) where for all \( 1 \leq i \leq n \), the random variables \( X_i \) are independently dis-
2. Proof of non-separability

tributed in \{0, 1\} (i.e. they are indicator variables). Then, for any \(\varepsilon > 0\) we have

\[
P[X \geq (1 + \varepsilon)\mathbb{E}[X]] \leq \left(\frac{e^\varepsilon}{(1 + \varepsilon)^{1 + \varepsilon}}\right)^{\mathbb{E}[X]} \leq \left(\frac{e}{1 + \varepsilon}\right)^{(1 + \varepsilon)\mathbb{E}[X]}.
\]  

(2.1)

The second inequality is a trivial simplification that will come in handy when we apply this lemma.

Finally, we will need a more sophisticated concentration bound to handle random functions that cannot easily be represented as a sum of independent random variables.

**Theorem 2.3 (Theorem 3.3 in [2])**  Let \(Z_1, \ldots, Z_m\) be independent random variables over \(\Omega_1, \ldots, \Omega_m\). Let \(Z = (Z_1, \ldots, Z_m), \Omega = \prod_{k=1}^m \Omega_k\) and let \(g : \Omega \to \mathbb{R}\) be measurable with \(0 \leq g(\omega) \leq M\) for all \(\omega \in \Omega\). Let \(B \subseteq \Omega\) (the subset of “bad” events) such that for some \(c > 0\) and for all \(\omega, \omega' \in B\) that differ in at most two components we have

\[
|g(\omega) - g(\omega')| \leq c.
\]  

(2.2)

Then for all \(t > 0\)

\[
P[|g(Z) - \mathbb{E}[g(Z)]| \geq t] \leq 2e^{-\frac{t^2}{32mc^2}} + (2\frac{mM}{c} + 1)\mathbb{P}[B].
\]  

(2.3)

Apart from the concentration bounds, we need the following graph-theoretic lemma to bound the number of small cuts in connected graphs. Its main use stems from the fact that while the bound it provides is still exponential, the base of the exponentiation can be chosen almost freely.

**Lemma 2.4 (Cut bound (Lemma 7 in [9]))**  For any \(\varepsilon > 0\) there exist \(\eta_0(\varepsilon) > 0\) and \(n_0\) such that the following holds. For all \(n \geq n_0\), and for all connected graphs \(G\) with \(n\) vertices, there are at most \((1 + \varepsilon)^n\) bipartitions of \(G\) with at most \(\eta_0n\) cross-edges.

### 2.2 Lower bounds

The definitions laid out in the previous chapter give rise to a simple sampling procedure: For a given \(n \in \mathbb{N}\) and weight sequence \(w = (w_1, w_2, \ldots, w_n)\), sample the position \(x_v \in \mathbb{T}_d\) uniformly and independently at random for each vertex \(v \in V\). Given the positions and weights, the edge probabilities \(p_{uv}\) are completely defined; therefore, for every vertex pair \(u \neq v \in V\), insert the edge between \(u\) and \(v\) independently with probability \(p_{uv}\). One way to implement this is by drawing a random variable \(Y_{uv}\) independently and uniformly from \([0, 1]\), and inserting the edge if \(Y_{uv} < p_{uv}(x_u, x_v)\). This point of view will help us derive lower bounds crucial to the proof that there are no small separators.
2.2. Lower bounds

Inequality (EPL) is a lower bound for $p_{uv}$; this means that

$$Y_{uv} < c_L \min \left\{ 1, V(|x_u - x_v|)^{-a} \cdot \left( \frac{w_u w_v}{W} \right)^a \right\}$$

implies $Y_{uv} < p_{uv}$. Therefore, it is a sufficient condition for edge insertion, i.e. the edge between $u$ and $v$ is inserted with probability one if it is fulfilled.

We are particularly interested in applying this to the MCD model, where we have defined $V_{\text{min}}(r) = r$. For the MCD model, the sufficient condition is thus given as

$$Y_{uv} < c_L \min \left\{ 1, \|x_u - x_v\|_{\text{min}}^{-a} \cdot \left( \frac{w_u w_v}{W} \right)^a \right\}.$$  

Note that if $\|x_u - x_v\|_{\text{min}} \leq \frac{w_u w_v}{W}$, the $\min \{ \ldots \}$ term becomes 1. In that case, having $Y_{uv} < c_L$ guarantees that the edge between $u$ and $v$ is inserted, i.e.

$$\mathbb{P} \left[ u \sim v \mid \|x_u - x_v\|_{\text{min}} \leq \frac{w_u w_v}{W} \right] \geq \min\{c_L, 1\}.$$  

This statement (with a slight modification) will be quite handy for proving a key lemma, namely Lemma 2.6.

It is important to note that given $Y_{uv}$, $x_u$ and $x_v$, the existence of the edge between $u$ and $v$ is completely deterministic; one simply checks whether $Y_{uv} < p_{uv}(x_u, x_v)$ holds. Since the three random variables are independent, it is not necessary to stick to the already mentioned “standard” procedure of first sampling the positions and then the $Y_{uv}$. For example, one might also draw all $Y_{uv}$ first and then proceed to draw the positions. This view can be fruitful when dealing with the MCD model. Consider the lower bound (2.5).

Conditioned on $Y_{uv}$, there are two possibilities: If $Y_{uv} \geq c_L$, the inequality will not be satisfied; the distance does not matter. Otherwise, the condition is met if the following inequality is true:

$$\min_{1 \leq i \leq d} \|x_{ui} - x_{vi}\|_T = \|x_u - x_v\|_{\text{min}} < c_L^{1/a} \cdot \frac{1}{Y_{uv}^{1/a}} \cdot \left( \frac{w_u w_v}{W} \right).$$

By the definition of MCD, it is satisfied if and only if for at least one coordinate $1 \leq i \leq d$, the absolute difference $|x_{ui} - x_{vi}|_T$ is smaller than the right
2. Proof of non-separability

hand side. The coordinate values are i.i.d. from $[0, 1]$; conditioned on $Y_{uv}$ we can consider drawing the positions as $d$ independent “attempts” to fulfil (2.8). Under that view, a $d$-dimensional MCD graph is lower-bounded by a union of $d$ one-dimensional (Euclidean GIRG) instances which all use the same $Y_{uv}$ and are thus not independent.

A lower bound over completely independent instances would be a very useful tool, since it would allow us to use powerful concentration bounds (mainly Chernoff). This can be achieved by “splitting” each $Y_{uv}$ into two independent random variables in a particular way. Let $Y_{uv}^1$ and $Y_{uv}^2$ be i.i.d. from $[0, 1]$, with the following CDF for $c \in [0, 1]$:

\[
P \left[ Y_{uv}^1 < c \right] = P \left[ Y_{uv}^2 < c \right] = 1 - \sqrt{1 - c}. \tag{2.9}
\]

A quick calculation shows that the following two statements hold for each $c \in [0, 1]$:

\[
c / 2 \leq P \left[ Y_{uv}^1 < c \right] = P \left[ Y_{uv}^2 < c \right] \leq c \tag{2.10}
\]

\[
P \left[ \min \{Y_{uv}^1, Y_{uv}^2\} < c \right] = P \left[ Y_{uv} < c \right] = c. \tag{2.11}
\]

Because of the latter, instead of checking $Y_{uv} < p_{uv}(x_u, x_v)$ for including edges, we can equivalently use

\[
\min \{Y_{uv}^1, Y_{uv}^2\} < p_{uv}(x_u, x_v) \tag{EIC}
\]

as Edge Insertion Criterion. The lower bound (2.5) becomes

\[
\min \{Y_{uv}^1, Y_{uv}^2\} < p_{uv}^L(\|x_u - x_v\|_{\min}). \tag{2.12}
\]

Since we have

\[
\|x_u - x_v\|_{\min} = \min \{ \min_{1 \leq i < d} \{|x_{ui} - x_{vi}|\}, |x_{u\min} - x_{v\min}| \}, \tag{2.13}
\]

Inequality (2.12) is satisfied if

\[
Y_{uv}^1 < p_{uv}^L(\min_{1 \leq i < d} \{|x_{ui} - x_{vi}|\}) \quad \text{or} \quad (LB1)
\]

\[
Y_{uv}^2 < p_{uv}^L(|x_{ud} - x_{vd}|). \quad (LB2)
\]

Note that (LB1) and (LB2) are independent of each other; furthermore, each of the two implies (2.12) and thus EIC, giving a lower bound. Using (LB1) as the edge insertion criterion satisfies (EP) on $\mathbb{T}^{d-1}$; this enables us to apply the theorems from the previous section on the resulting graphs. For fixed $x_u$ and $x_v$, the probability of satisfying $Y_{uv}^1 < p_{uv}^L$ is bounded by

\[
p_{uv}^L / 2 \leq P \left[ Y_{uv}^1 < p_{uv}^L \right] \leq p_{uv}^L \tag{2.14}
\]
according to (2.10). Thus, given that $p_{Luv}^L$ satisfies (EP), so does (LB1).

Note that for $d = 1$, Inequality (LB1) is not defined. This is what sets the case $d = 1$ (which does have small separators) apart from $d > 1$ (which does not). It does not seem to be possible to lower-bound a one-dimensional MCD graph with two completely independent random graphs that fulfill (EP), which is crucial for the proof.

### 2.3 Generation Procedure

Let us describe the procedure we use for generating MCD model graphs. It is designed to facilitate the proof of the main theorem. A condensed version is included as Algorithm 1.

**Phase 1** We start by drawing $Y_{uv}^1$ and $x_{ui}$ for all $u, v \in V$ and all $1 \leq i < d$, i.e., leaving out all $Y_{uv}^2$ and $x_{ud}$. This is already sufficient to determine the graph induced by (LB1), which we call $G_1$ (note that it was called $G'$ in Section 1.3). By Theorem 1.3, there is a constant $s_{\text{max}}$ such that w.h.p. the largest component (the giant) of $G_1$ has at least $s_{\text{max}} n$ vertices; we will always assume that this holds. We denote the giant of $G_1$ by $K_{\text{max}}^1$.

**Phase 2** According to (PL2), there is a constant $B'$ such that w.h.p. at least $s_{\text{max}} n / 2$ vertices of $K_{\text{max}}^1$ have weight less than $B'$. This can be shown using the power-law requirements: In (PL2), set $\eta = 1$ and $B' > (2c_2 / s_{\text{max}})^{1/(\beta-2)}$. Then, at most $s_{\text{max}} n / 2$ vertices have weight at least $B'$. Even if all of them are in $K_{\text{max}}^1$, since we assume $|K_{\text{max}}^1| \geq s_{\text{max}} n$, at least $s_{\text{max}} n / 2$ vertices of $K_{\text{max}}^1$ must have weight smaller than $B'$. (As mentioned in the previous phase, that assumption holds with high probability.) We now draw a set of vertices $F'$ by including each vertex (not just those in the giant) with weight less than $B'$ independently with probability $4 f / s_{\text{max}}$, where $0 < f < \min \{\delta/12, s_{\text{max}}/12\}$ is a constant.

**Phase 3** We set $F := F' \cap K_{\text{max}}^1$. The expected size of $F$ is at least $4f / s_{\text{max}} \cdot s_{\text{max}} n / 2 = 2fn$, and at most $4f / s_{\text{max}} \cdot s_{\text{max}} n = 4fn$. Since vertices of $K_{\text{max}}^1$ with weight less than $B'$ are included independently in $F'$ (and thus $F$), we can use the Chernoff bounds provided by Theorem 2.1 (with $\epsilon = 1/2$) to show that w.h.p. we have $6fn \geq |F| \geq fn$; in our further considerations, we will assume this to be true. Note that the order of phases 1 and 2 is interchangeable, as they are independent (the choice of $B'$ depends only on $s_{\text{max}}$ and not on the outcome of $G_1$).

The final three phases consist of drawing $x_{ud}$ and $Y_{uv}^2$ for all vertices in a particular order that depends on the outcome of phases 1–3. We enumerate the vertices from 1 to $n$; the $k$-th vertex is denoted by $u_k$. The enumeration
has to be such that the vertices of $V \setminus K^1_{\max}$ come first, then the vertices of $K^1_{\max} \setminus F$, and finally the vertices of $F$. More precisely, we have $u_i \in V \setminus K^1_{\max}$ for all $1 \leq i \leq |V \setminus K^1_{\max}|$; for all $|V \setminus K^1_{\max}| < i \leq |V \setminus F|$ we have $u_i \in K^1_{\max} \setminus F$, and finally, for all $|V \setminus F| < i \leq n$ we have $u_i \in F$. We draw the $d$-th coordinates step-by-step in the order given by the enumeration. The $d$-th coordinate of $u_k$ is written as $x_{kd}$, since the more consistent $x_{uk}$ might be somewhat hard to read. Together with $x_{kd}$, we draw $Y_{2kj}$ for all $1 \leq j < k$ (where $Y^2_{kj}$ stands in for $Y^2_{ukuj}$). Then, for all vertex pairs $(u_i, u_j)$ with $i, j \leq k$, we have complete information; we add all edges fulfilling (EIC) between such pairs. Therefore, the $k$-th step can be described as

- Draw $x_{kd}$ independently and uniformly from $X_d := [0, 1]$.
- For all $1 \leq j < k$, draw $Y^2_{kj}$ independently according to (2.9).
- For all $1 \leq j < k$, add the edge between $u_k$ and $u_j$ if (EIC) is satisfied.

The definition of the final three phases reflects the way we chose to enumerate the vertices:

Phase 4  This phase consists of steps 1 to $|V \setminus K^1_{\max}|$, i.e. it encompasses the following actions: Draw $x_{ud}$ and $Y^2_{uv}$ for all $u \neq v \in V \setminus K^1_{\max}$. Then, for all pairs $(u, v) \in (V \setminus K^1_{\max} \times V \setminus K^1_{\max})$, add the edge between $u$ and $v$ if (EIC) is satisfied. This results in a graph we call $G_2$.

Phase 5  This phase consists of steps $|V \setminus K^1_{\max}| + 1$ to $|V \setminus F|$, i.e. it encompasses the following actions: Draw $x_{ud}$ and $Y^2_{uv}$ for all $u \neq v \in K^1_{\max} \setminus F$. Then, for all pairs $(u, v) \in (V \setminus F \times V \setminus F)$, add the edge between $u$ and $v$ if (EIC) is satisfied. This results in a graph we call $G_3$.

Phase 6  This phase consists of steps $|V \setminus F| + 1$ to $n$, i.e. it encompasses the following actions: Draw $x_{ud}$ and $Y^2_{uv}$ for all $u \neq v \in K^1_{\max} \setminus F$. Then, for all pairs $(u, v) \in (V \times V)$, add the edge between $u$ and $v$ if (EIC) is satisfied. This results in a graph we call $G_4$, i.e. the “actual” graph induced by (EIC).

Our goal is to show that the sixth and last phase destroys all “small” cuts in the giant of $G_3$ (Lemma 2.5) while adding few vertices to it (Lemma 2.8). For $1 \leq i \leq 4$, we let $K^i_{\max}$ denote the connected component that contains $K^1_{\max}$. For proving that “small” cuts are destroyed, we will lower-bound the probability that an edge is present using

$$Y^2_{uv} < p^L_{uv}(|x_{ud} - x_{vd}|).$$ (LB2)
2.3. Generation Procedure

Algorithm 1 Sample a MCD model graph

\textbf{procedure} \textsc{SampleMCD}(d, α, β, n, w, δ)

// Phase 1
Draw \(x_{u_{1d}}, Y_{u_{1d}}\) for all \(u, v \in V\).
Add edges between pairs satisfying \(LB1 \rightarrow G_1\)
\(K^1_{\text{max}} \leftarrow\) giant of \(G_1\) \(|K^1_{\text{max}}| \geq s_{\text{max}}n\ \text{w.h.p}\)

// Phase 2
Choose \(f < \min\{\delta/12, s_{\text{max}}/12\}\)
\(F' \leftarrow 0\)
\textbf{for all} \(u \in V\) with \(w_u < B'\) \textbf{do}
\hspace{1em} Add \(u\) to \(F'\) with probability \(4f/s_{\text{max}}\)
\textbf{end for}

// Phase 3
\(F \leftarrow F' \cap K^1_{\text{max}}\)
Enumerate vertices from 1 to \(n\) such that
\[
\{u_1, \ldots, u_{|V\setminus K^1_{\text{max}}|}\} = V\setminus K^1_{\text{max}},
\{u_{|V\setminus K^1_{\text{max}}|+1}, \ldots, u_{|V\setminus F|}\} = K^1_{\text{max}} \setminus F,
\{u_{|V\setminus F|+1}, \ldots, u_n\} = F
\]
(Shorthand \(x_{kd}\) and \(Y_{kj}^2\) for \(x_{u_{kd}}\) and \(Y_{u_{kj}}^2\), respectively)

// Phase 4 (results in \(G_2\))
\textbf{for all} 1 \(\leq k \leq |V\setminus K^1_{\text{max}}|\) \textbf{do}
\hspace{1em} \textsc{Step}(k)
\textbf{end for}

// Phase 5 (results in \(G_3\))
\textbf{for all} \(|V\setminus K^1_{\text{max}}| < k \leq |V\setminus F|\) \textbf{do}
\hspace{1em} \textsc{Step}(k)
\textbf{end for}

// Phase 6 (results in \(G_4\))
\textbf{for all} \(|V\setminus F| < k \leq n\) \textbf{do}
\hspace{1em} \textsc{Step}(k)
\textbf{end for}
\textbf{end procedure}

\textbf{procedure} \textsc{Step}(k)
\hspace{1em} Draw \(x_{kd}\) uniformly from [0, 1].
\hspace{1em} \textbf{for all} 1 \(\leq j < k\) \textbf{do}
\hspace{2em} Draw \(Y_{kj}^2\) independently according to the CDF given in (2.9).
\hspace{2em} Add edge between \(u_k\) and \(u_j\) if (EIC) is satisfied.
\textbf{end for}
\textbf{end procedure}
2.4 Proof

With all these tools laid out, we are ready to tackle the “core” of the proof. We start by stating the following crucial lemma.

**Lemma 2.5** There is a constant \( \eta > 0 \) such that with high probability, \( K_{\text{max}}^3 \) has no \((\delta, \eta)\)-cut in \( G_A \). (More precisely, the subgraph of \( G_A \) induced by \( K_{\text{max}}^3 \) has no \((\delta, \eta)\)-cut.)

In order to prove this, we need the following auxiliary Lemma 2.6.

**Lemma 2.6** Consider the set of \( d \)-th vertex coordinates \( \{ x_{ud} \mid u \in V \} \). It is a set of \( n \) random variables drawn uniformly and independently from \( X_d = [0, 1] \). We divide \( X_d \) into \( M := \lceil n/l \rceil \) subintervals of equal length, where \( 0 < l < 1 \) is a constant. For all \( 0 \leq j < M \), we define the \( j \)-th subinterval as \( I_j := [j/M, (j+1)/M] \). Then, for every \( 0 < \delta < 1 \), and every \( l \), there is a constant \( r(\delta, l) > 0 \) having the following property: With probability \( 1 - e^{-\Theta(n)} \), there is no set \( S \) of \( n \) subintervals such that there are at least \( \delta n / 2 \) vertices \( u \) with \( x_{ud} \in S \).

**Proof** Fix a set \( S \) of \( n \) subintervals. The sum of the lengths of all subintervals in \( S \) is \( n/r \). Therefore, for every vertex \( u \), the probability of \( u \) getting placed in \( S \) (i.e. having \( x_{ud} \in S \)) is at least \( r \mu / 2 \), at most \( 2 r \mu \) (by virtue of \( r \mu / 2 \leq n/r \leq 2 r \mu \)), and for \( \mu \), the expected number of vertices placed in \( S \), we have \( 2 r \mu \geq \mu \geq r \mu / 2 \).

Since the \( x_{ud} \) are drawn independently, we can use a Chernoff inequality to bound the probability that at least \( \delta n / 2 \) vertices are placed in \( S \). In particular, we use the “strong” Chernoff bound given above (Theorem 2.2), which tells us that for a sum of i.i.d indicator random variables, called \( X \), and for any \( \epsilon > 0 \) we have

\[
\Pr[X \geq (1 + \epsilon)E[X]] \leq \left( \frac{e}{1 + \epsilon} \right)^{(1+\epsilon)E[X]} \quad (2.15)
\]

Note that in our case \( \delta / 2 \cdot n \geq \delta / 2 \cdot 1 / 27 \mu = \frac{\delta}{47} \mu \), so we set \( 1 + \epsilon = \frac{\delta}{47} \mu \).

Therefore, the Chernoff bound states that

\[
\Pr \left[ \# \{ u \mid x_{ud} \in S \} \geq \frac{\delta}{2} n \right] \leq \left( \frac{4 e r \mu / \delta}{\delta} \right)^{r l n / 2} = \left( \frac{4 e r \mu / \delta}{\delta} \right)^{l / 16} \cdot 2^{n / l} \quad (2.16)
\]

If we choose \( r \) small enough such that the term in square brackets is smaller than \( 1/2 \), this upper bound is equal to \( e^{-r' 2 n / l} \), where \( r' \) is a constant larger than \( \log 2 \).

As a final step, we take the union bound over all possible choices of \( S \). There are at most \( 2^M \leq 2^{2 n / l} \) ways to choose \( S \), so the probability that for at least one such \( S \) there are at least \( \delta n / 2 \) vertices in \( S \) is upper-bounded by \( e^{(\log 2) 2 n / l} e^{-r' 2 n / l} = e^{-\Theta(n)} \). \( \square \)
Note that restricting the set of vertices considered for Lemma 2.6 just makes the statement weaker; the lemma implies that for any \( V' \subseteq V \), w.h.p. there is no set \( S \) of \( rn \) subintervals such that there are at least \( \delta n / 2 \) vertices \( u \in V' \) with \( x_{ud} \in S \). We will make use of this for the following corollary that will help us lower-bound edge probabilities in the proofs of lemmas 2.5 and 2.8.

**Corollary 2.7** There is a constant \( P > 0 \) such that with high probability, the following holds for each step \( k \) of Algorithm 1, where \( \delta n / 2 < k \leq n \):

\[
\mathbb{P} \left[ \exists v \in A \text{ s.t. } u_k \sim v \right] \geq P.
\]

**Proof** Choose the constant \( l \) in Lemma 2.6 to be the same as in (2.7). With that choice of \( l \), for any two vertices which are in the same subinterval, the probability of the edge between those two vertices is at least \( c L / 2 \). This is independent of the \( Y_{uv} \) and \( x_{ui} \) for \( 0 \leq i < d \), and can be explained in the following way: For any \( u \neq v \in V \), being in the same subintervals implies \( |x_{ud} - x_{vd}| \leq l / n \), so in particular \( p_{uv}^L(\|x_u - x_v\|_{\min}) \geq c_L \). By Inequality (2.10), this means that

\[
\mathbb{P} \left[ Y_{uv}^2 < p_{uv}^L | |x_{ud} - x_{vd}| \leq l / n \right] \geq c_L / 2
\]

as a probability over \( Y_{uv} \). This is a result very similar to Inequality (2.7).

Fix a \( \delta n / 2 < k \leq n \). Apply Lemma 2.6 to \( V_k \); this yields that with probability \( 1 - e^{\Theta(n)} \), there is no set \( S \) of \( rn \) subintervals such that there are at least \( \delta n / 2 \) vertices \( v \in V_k \) with \( x_{ud} \in S \). In particular, this implies that for every subset \( A \subseteq V_k \) of size at least \( \delta n / 2 \), there are more than \( rn \) subintervals \( I_j \) that contain at least one vertex of \( A \) (i.e. there is at least one \( v \in A \) such that \( x_{vd} \in I_j \)). Thus, with probability at least \( rn / M = rn / \lceil n / l \rceil \), the position \( x_{uv} \) is in a subinterval that contains at least one vertex of \( A \). If \( x_{ud} \) is in the same subinterval as \( x_{vd} \) for some \( v \in A \), we have \( \mathbb{P}[u \sim v] \geq c_L / 2 \) (see 2.18), and thus

\[
\mathbb{P} \left[ \exists v \in A \text{ s.t. } u_k \sim v \right] \geq rn / \lceil n / l \rceil \cdot c_L / 2 \geq P,
\]

for some constant \( P > 0 \).

The probability that Lemma 2.6 fails for a particular \( k \) is upper-bounded by \( e^{\Theta(n)} \); by a union-bound argument, the probability that it fails for at least one \( n \geq k > \delta n / 2 \) is upper-bounded by \( ne^{-\Theta(n)} = e^{-\Theta(n)} \).

We are now equipped to complete the proof of Lemma 2.5:

**Proof (of Lemma 2.5)** At the end of Phase 5, consider bipartitions of \( K_3 \) into two sets \( C_1 \) and \( C_2 \), both containing at least \( \delta n \) vertices. We would like to upper-bound the number of bipartitions that have less than \( \eta' n \) cross-edges.
(i.e. edges between $C_1$ and $C_2$), for a particular constant $\eta'$. Let $\mu := P \cdot fn/2$, where $P$ is the same constant as in Corollary 2.7. We apply Lemma 2.4 to $K^{3}_{\text{max}}$ with an $\epsilon > 0$ such that $\log(1 + \epsilon) < P fn/16 = \mu / (8n)$. The lemma provides an $\eta' > 0$ such that there are at most $(1 + \epsilon)^n$ bipartitions of $K^{3}_{\text{max}}$ into two subsets $C_1$ and $C_2$ with at most $\eta'n$ cross-edges at the end of Phase 5.

In particular, this is an upper bound for the number of bipartitions into two sets which both have size at least $\delta n$, with at most $\eta'n$ cross-edges.

Furthermore, a partition of $K^{3}_{\text{max}}$ also induces a bipartition of $F$. Since $F$ has at least $fn$ vertices w.h.p, we have for at least one $i \in \{1, 2\}$ that $|F \cap C_i| \geq fn/2$; w.l.o.g. this holds for $i = 1$. As we require $f < \delta/12$ (and $F$ has at most $6fn$ vertices w.h.p), we have $|C_2 \setminus F| \geq \delta n/2$. We can therefore apply Corollary 2.7 to obtain that in Phase 6, each vertex in $F \cap C_1$ receives at least one edge to $C_2 \setminus F$ with probability at least $P = \Theta(1)$.

Importantly, for any pair of vertices $v,w \in F \cap C_1$ the events “$v$ has at least one edge to $C_2 \setminus F$” and “$w$ has at least one edge to $C_2 \setminus F$” are independent (because we implicitly condition on the positions of the vertices in $C_2 \setminus F$), so we may apply a Chernoff bound. In particular, note that $\mu = P \cdot fn/2$ is a lower bound on the expected number of vertices in $F \cap C_1$ which have at least one edge to $C_2 \setminus F$ (and as such also a lower bound on the number of edges from $C_1$ to $C_2$). Insert this into Theorem 2.1ii) with $\epsilon = 1/2$, yielding that with probability at least $1 - e^{-\frac{1}{2}\mu}$ there are at least $\mu/2$ such edges.

Finally, let $\eta := \min\{\eta', \mu/(2n)\}$. According to the first part, there are at most $(1 + \epsilon)^n$ bipartitions of $K^{3}_{\text{max}}$ into two sets such that both have size at least $\delta n$, and have at most $\eta n \leq \eta'n$ cross-edges. According to the second part, any such bipartition has at least $\mu/2 \geq \eta n$ cross-edges after Phase 6 with probability not less than $1 - e^{-\frac{1}{2}\mu}$. Therefore, if we take a union bound over all such bipartitions, we get that the probability that there is a $(\delta, \eta)$-cut (i.e. a bipartition with less than $\eta n$ cross-edges) in the subgraph of $G_4$ induced by $K^{3}_{\text{max}}$ is upper-bounded by $(1 + \epsilon)^n e^{-\frac{1}{2}\mu} = e^{-\Theta(n)}$. \hfill \Box

What remains is to show that going through Phase 6 does not add too many vertices to the giant.

**Lemma 2.8** There is a constant $f(\delta) > 0$ such that we have that with high probability

$$|K^{4}_{\text{max}}| \leq |K^{3}_{\text{max}}| + 3\delta n. \quad (2.20)$$

Note that trivially $K^{3}_{\text{max}} \subseteq K^{4}_{\text{max}}$.

We will do this by showing that for an appropriate $f$, after Phase 5 there are few vertices outside of $K^{3}_{\text{max}}$ that are in large components, or in components which have at least one “heavy” vertex. The proof will be concluded by demonstrating that $F$ has few edges to vertices which fall in neither of these categories.
2.4. Proof

Let us begin with showing that few vertices are in large components which are not the giant:

**Lemma 2.9** There is a constant $s_t$ such that with high probability, the graph $G_3$ has less than $\delta n$ vertices which are in a component that a) is not the giant, and b) has size at least $s_t$.

**Proof** Choose $s_t$ to be larger than $4/(s_{\text{max}} P)$, where $P$ is the same as in Corollary 2.7. Consider Phase 5 of Algorithm 1, which is used to generate $G_3$. It consists of steps $|V \setminus K_{\text{max}}^1| + 1$ through $|V \setminus F|$. Recall that

$$\left\{ u_k \in V \mid |V \setminus K_{\text{max}}^1| < k \leq |V \setminus F| \right\} = K_{\text{max}}^1 \setminus F. \quad (2.21)$$

Let $A_k$ be the set of vertices in $G_2$ which have the following property: At the end of step $k - 1$, they are in a component of size at least $s_t$ but not in the giant. For each step $k$ of Phase 5, we define the indicator random variable $Z_k$ which is $1$ if and only if $A_k + 1 \neq A_k$, i.e. if and only if step $k$ adds an edge from $u_k$ to at least one vertex in $A_k$ (thus attaching it to the giant). We apply Corollary 2.7 yielding that with high probability, at each step $k$ with $|A_k| \geq \delta n$, vertex $u_k$ receives an edge to $A_k$ with probability at least $P$, i.e.

$$\mathbb{P} [Z_k \mid |A_k| \geq \delta n] \geq P; \quad (2.22)$$

we call this a “hit”. Additionally, we define indicator random variables $Z'_k$ as follows:

$$Z'_k := \begin{cases} Z_k, & \text{if } |A_k| \geq \delta n \\ 0, & \text{otherwise}. \end{cases} \quad (2.23)$$

Here, $B_P$ is a indicator random variable with success probability $P$. Note that while the $Z'_k$ are not completely independent, we have $\mathbb{P}[Z'_k = 1 \mid A_k] \geq P$ for all $k$ and all $A_k \subseteq V_k$, i.e. not depending on the outcome of the previous steps. Therefore, we may lower-bound the sum over the $Z'_k$ by a sum over $|K_{\text{max}}^1 \setminus F|$ independent instances of $B_P$. We get

$$\mathbb{E} \left( \sum_{k=|V \setminus K_{\text{max}}^1|+1}^{\mid V \setminus F \mid} Z'_k \right) \geq \mathbb{E} \left( \sum_{k=|V \setminus K_{\text{max}}^1|+1}^{\mid V \setminus F \mid} B_P \right) > P \cdot \frac{s_{\text{max}} n}{2} =: \mu. \quad (2.24)$$

For the last step, we use linearity of expectation and the fact that we require $f < s_{\text{max}}/12$ (and $F$ has at most $6fn$ vertices w.h.p), which implies $|K_{\text{max}}^1 \setminus F| > s_{\text{max}} n / 2$. Applying the Chernoff bound from Theorem 2.1ii) to $\sum B_P$ yields that w.h.p. $\sum B_P \geq \mu / 2$ and therefore also that w.h.p. $\sum Z'_k \geq \mu / 2$. Now, under the assumption of $\sum Z'_k \geq \mu / 2$, there are two possibilities: If $A_{|V \setminus F|+1}$ is smaller than $\delta n$, we are done (as this is exactly the set of vertices in components of size at least $s_t$ but not the giant in $G_3$). Otherwise, each case of $Z'_k = 1$ actually resulted from a “hit”, so at
least $s_t \cdot \mu / 2$ vertices were eliminated from $A$. However, by our choice of $s_t > 4 / (s_{\text{max}} P) = 2n / \mu$, this leads to a contradiction. Therefore, under the assumption of $\sum_k Z'_k \geq \mu / 2$ (and thus with high probability), the set $A_{|V\setminus F|+1}$ has less than $\delta n$ vertices as desired. □

Furthermore, most vertices in small components also have small weight:

**Lemma 2.10** There is a constant $B > 0$ such that there are at most $\delta n / s_t$ vertices with weight at least $B$ which are in a component of size at most $s_t$ in $G_3$. In particular, there are at most $\delta n / s_t$ vertices in $G_3$ which are in a component of size at most $s_t$ and share that component with a vertex with weight at least $B$. For convenience, we choose $B \geq B'$ (so that the weight of vertices in $F$ is guaranteed to be at most $B$).

**Proof** We use (PL2) to upper-bound the total number of vertices with weight at least $B$. In particular, we set $\eta = 1$ in (PL2) and $w$ to $\max\{B', (c_2 s_t / \delta)^{1/(\beta - 2)}\} =: B$. Inequality (PL2) then yields that there are at most $\delta n / s_t$ vertices with weight at least $B$. Furthermore, there are clearly no more than $\delta n$ vertices which are in a component of size at most $s_t$ and share that component with a vertex with weight at least $B$. □

In order to prove that few edges leave $F$, we need the additional following auxiliary result:

**Lemma 2.11** With probability $1 - n^{-\omega(1)}$, there is no vertex $u$ with weight at most $B$ with $\deg(u) > \log^2 n$ in $G_4$.

**Proof** (This proof follows a similar argument from the proof of Lemma 23 in [2].) Let $u$ be a vertex with weight $w_u \leq B$, and let $x_u$ be its (fixed) position in $\mathbb{T}^d$. Note that conditioned on $x_u$, $\deg(u)$ is a sum of independent Bernoulli random variables. This will enable us to use a Chernoff bound (Theorem 2.1iii) to bound the probability that $\deg(u)$ is large. As $w_u$ is bounded by a constant, there is a constant $U$ such that $\mathbb{E}[\deg(u)] \leq U$ according to Lemma 22 from [2]. Inserting this into the Chernoff bound with $t = \log^2 n$ yields

$$
\Pr[\deg(v) > \log^2 n] \leq 2^{-\log^2 n} = n^{-\omega(1)}. \quad (2.25)
$$

(Doing so is valid because for large enough $n$, we have $t > 2eU \geq 2e\mathbb{E}[\deg(u)]$.) Finally, we can take a union bound over all vertices with weight at most $B$. Since there are at most $n$ such vertices, the probability that at least one of them has degree larger than $\log^2 n$ is still $n \cdot n^{-\omega(1)} = n^{-\omega(1)}$. □

Now, we are equipped to prove

**Lemma 2.12** There is an $f > 0$ such that with high probability, there are at most $\delta n / s_t$ edges from $F$ to vertices that have weight at most $B$. 

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2.4. Proof

Recall how $F$ was defined: We include each vertex with weight less than $B'$ independently with probability $4f/s_{\max}$ in a set $F'$. (The weight bound $B' \leq B$ was chosen such that there are at least $s_{\max}n/2$ vertices with weight less than $B'$ in $K_{\max}^1$.) Then, the set $F$ is given as $F' \cap K_{\max}^1$. Note that the generation of $F'$ is completely independent of sampling the actual graph, so we might as well draw $F'$ before generating even $G_1$. Using a Chernoff bound (Theorem 2.1i with $\epsilon = 1/2$), we can show that w.h.p we have $|F'| \leq (6fn/s_{\max}) =: f'n$. Let $S$ be the set of all vertices which have weight at most $B$; note that $F' \subseteq S$. We will show that for small enough $f$ (and thus $f'$), there are at most $\delta n/s_t$ edges from $F'$ to $S$ in $G_4$; this completes the argument, as $F = F' \cap K_{\max}^1 \subseteq F'$. The following argument actually works for any graph generated according to the distance model.

The result will follow directly from Theorem 2.3; in the following, we describe the choice of parameters for the theorem. The subgraph of $G_4$ induced by $S$ is defined by the collection of all vertex positions $x_u$, and all $Y_{uv}$, for $u \neq v \in S$. In order to keep the number of random variables for Theorem 2.3 low, we group some of the $Y_{uv}$ together: For $u \in S$, let $Y_u$ be the tuple that contains all $Y_{uv}$ such that $v < u$ (respective to some enumeration of $S$; it need not be the same one as in Algorithm 1). Note that with the $x_u$ and $Y_{uv}$, there is a grand total of at most $2n$ random variables, so we may set $m$ in Theorem 2.3 to $2n$. (In fact, $2n - 1$ would do, as one of the $Y_u$ is an empty tuple.)

For notational simplicity, let $Z$ be the tuple containing all $x_u$ and $Y_{uv}$ and let $\omega$ denote an outcome of $Z$ (or equivalently, it represents a particular realization of the subgraph induced by $S$). The function $g(\omega)$ is simply defined as the number of edges from $F'$ to $S$ in $\omega$. Note that $0 \leq g(\omega) \leq |F'| \cdot |S| \leq f'n^2$. This corresponds to $M$ in the theorem. We define the “bad” subset $B$ as follows:

$$B := \{\omega \in \Omega \mid \exists u \in S : \text{deg}(u) \geq \log^2 n\}. \quad (2.26)$$

Observe that for any $\omega, \omega' \in B$ which differ in at most two components, we have

$$|g(\omega) - g(\omega')| \leq 2\log^2 n =: c \quad (2.27)$$

since the outcome of every $x_u$ and $Y_{uv}$ affects at most $\log^2 n$ edges (because $\omega$ and $\omega'$ are both in $B$ and all edges a single coordinate affects have a vertex in common). This corresponds to $c$ in the theorem.

According to Lemma 2.11 (and using the fact that the weights of vertices in $S$ are bounded by $B$), the probability that there is at least one vertex $u \in S$ with $\text{deg}(u) > \log^2 n$ is bounded by $n^{-\omega(1)}$. Therefore, we can conclude that $P[B] = n^{-\omega(1)}$. Furthermore, applying lemma 22 from [2] then yields that there is a constant $U$ such that the expected degree of each vertex in $S$, and
2. Proof of non-separability

thus \( F' \) is at most \( U \). Since \( g \) is upper-bounded by the sum of degrees of vertices in \( F' \), we get that

\[
E[g(Z)] \leq E \left[ \sum_{u \in F'} \deg(u) \right] \leq U f'n. \tag{2.28}
\]

This allows us to set \( t := 2U f'n - E[g(Z)] \geq U f'n \) for Theorem 2.3.

Putting all of this together, Theorem 2.3 yields

\[
P \left[ g(Z) - E[g(Z)] \geq t \right] \leq 2e^{-\frac{(Uf'n)^2}{8 \Delta^2 n \log^2 n}} + \left( \frac{2n \cdot f'n^2}{2 \log^2 n} + 1 \right) n^{-\omega(1)} = n^{-\omega(1)}. \tag{2.29}
\]

If we choose \( f \) small enough such that \( 2U f' < \delta/s_t \), we get the desired statement, since for such a choice of \( f' \), the theorem implies that w.h.p.

\[
g(Z) \leq E[g(Z)] + t = 2U f'n < \delta n/s_t.
\]

We now have all we need to complete the proof of Lemma 2.8

Proof (of Lemma 2.8) Each vertex in \( G_3 \) is an element of exactly one of the following sets:

1. \( K_3^{\text{max}} \).
2. The set of vertices in a component of size at least \( s_t \) but not in \( K_3^{\text{max}} \).
3. The set of vertices in a component of size less than \( s_t \) which contains at least one vertex with weight at least \( B \).
4. The set of vertices in a component of size less than \( s_t \) in which all vertices have weight less than \( B \). We call this set \( T \).

Going through Phase 6 will attach some vertices from sets 2–4 to \( K_3^{\text{max}} \); our goal is to show that w.h.p at most \( 3\delta n \) vertices are attached this way. Lemma 2.9 tells us that w.h.p. the second set contains at most \( \delta n \) vertices. Lemma 2.10 gives the same result for the third set. Therefore these two sets can contribute at most \( \delta n \) vertices each, and the remaining \( \delta n \) must come from \( T \) (set 4). However, according to Lemma 2.12, there are only at most \( \delta n/s_t \) edges from \( F \) to \( T \) w.h.p. (since vertices in \( T \) all have weight less than \( B \)). Each such edge can attach at most \( s_t \) vertices from \( T \) to \( K_3^{\text{max}} \), so w.h.p at most \( \delta n \) edges from \( T \) are attached, completing the argument. \( \square \)

Theorem 2.13 With high probability, \( K_3^{\text{max}} \) has no \((4\delta, \eta)\)-cut, where \( \eta \) is the same as in Lemma 2.8.

Proof Assume there is a \((4\delta, \eta)\)-cut of \( K_3^{\text{max}} \); such a cut induces a bipartition of vertices into two set \( C_1 \) and \( C_2 \) which are both of size at least \( 4\delta n \).

Consider the restriction of \( C_1 \) and \( C_2 \) to \( K_3^{\text{max}} \). Lemma 2.8 implies that for
We studied the separator properties of the MCD model for dimensions greater than one, and proved that this class of random graphs does not have small separators. This is a substantial difference to the one-dimensional case and to Euclidean GIRGs, which is mainly due to the independent lower bounds given as (LB1) and (LB2). This shows that the underlying geometry has a big impact on fundamental properties of the graph, and helps decide which geometries are suitable for modelling real-world networks; note that removing few edges is usually enough to disconnect these [4]. Future research could include investigating distance functions that interpolate between MCD and Euclidean/maximum norm. Under the MCD model, two vertices are likely connect if their positions are close along the first axis, OR the second axis, . . . , OR the $d$-th axis, whereas in Euclidean GIRGs, they have to be close along the first axis, AND the second axis, . . . , AND the $d$-th axis. Being able to easily mix and match these two would allow for formulating intuitive models, with connection criteria resembling boolean formulae (such as “Two people are likely to be friends if they both like football AND live close to each other, OR if both like the same particular online game”). Hopefully, the tools laid out in this thesis for analyzing the MCD model will prove useful in that endeavour as well.

Acknowledgements  I would like to thank my supervisor Johannes Lengler for suggesting a very interesting topic, and for his patient and helpful support.
Bibliography


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