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Holographic Minimal Models and Orbifold Conformal Field Theories

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MAXIMILIAN KELM

MSc ETH in Physics, ETH Zurich

born on

15 April, 1987

citizen of

Germany

accepted on the recommendation of

Prof. Dr. M. R. Gaberdiel, examiner

Prof. Dr. N. Beisert, co-examiner

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Abstract

This doctoral thesis is concerned with two-dimensional conformal field theories (CFTs) that appear in the holographic dualities with higher spin theories on three-dimensional anti-de-Sitter (AdS) space. The latter are generalisations of gravity which include a tower of massless particles of spin larger than 2. The dual CFTs can be formulated as coset models with $\mathcal{W}$ algebra symmetry. $\mathcal{W}$ algebras are higher-spin generalisations of the Virasoro algebra describing conformal symmetry.

We will give an introduction to both sides of the original higher-spin AdS_3/ CFT_2 duality and explain the statement of the duality as well as the most important pieces of evidence in its favour. In the process, the quantum $\mathcal{W}_\infty$ algebra describing the CFT side will be constructed by imposing algebraic consistency conditions. These will turn out to determine its structure uniquely up to two free parameters, one of which is the central charge of the CFT.

This analysis will then be carried over to a similar duality involving only even spin fields. As in the previous case, the resulting $\mathcal{W}$ algebra is fully determined up to two parameters by algebraic consistency. We will analyse the relationship of this algebra to known constructions of CFTs, namely the coset construction and the Drinfel'd-Sokolov reduction of the bulk symmetry algebra. This will provide an interpretation as the asymptotic symmetry algebra of the quantised higher spin gravity theory.

The second part of the thesis will aim at finding relations between an $\mathcal{N} = 2$ supersymmetric version of the CFTs studied before and CFTs that can be constructed as orbifolds of free theories. This might be a first step towards embedding the dual higher spin theories into string theory. First we will show that the CFT admits a description as an orbifold of free bosons and fermions by the unitary group $\mathrm{U}(N)$ in a certain limit. This continuous orbifold will then be shown to possess an extension which is given by an orbifold under the symmetric group S_{N+1}. The structure of both orbifolds will be studied in detail.

Zusammenfassung

Diese Doktorarbeit befasst sich mit zweidimensionalen konformen Feldtheorien (CFTs), welche in den holographischen Dualitäten mit Higher-Spin-Theorien im dreidimensionalen Anti-de-Sitter-Raum (AdS) vorkommen. Letztere sind verallgemeinerte Theorien der Gravitation, die masselose Teilchen mit Spin grösser als 2 beinhalten. Die dualen CFTs können als Nebenklassen-Modelle mit $\mathcal{W}$-Algebren-Symmetrie formuliert werden. $\mathcal{W}$-Algebren sind Higher-Spin-Verallgemeinerungen der Virasoro-Algebra, welche die konforme Symmetrie beschreibt.

Nach einer kurzen Einführung in beide Seiten der ursprünglichen $\mathrm{AdS}_3/\mathrm{CFT}_2$-Dualität mit höheren Spins werden wir die Aussage der Dualität sowie die wichtigsten Argumente für ihre Richtigkeit vorstellen. In diesem Zusammenhang werden wir auch die Quanten-$\mathcal{W}_\infty$-Algebra mit Hilfe von algebraischen Konsistenzbedingungen konstruieren, welche die Algebra bis auf zwei Parameter eindeutig bestimmen werden, wovon einer die zentrale Ladung der CFT ist.

Diese Analyse wird dann auch auf eine ähnliche Dualität angewendet werden, in der nur Felder mit geradem Spin vorkommen. Wie im zuvor behandelten Fall wird die so erzeugte $\mathcal{W}$-Algebra durch ihre algebraische Konsistenz bis auf zwei Parameter eindeutig bestimmt. Wir werden die Beziehung dieser Algebra zu mehreren bekannten CFT-Konstruktionen analysieren, und zwar zur Nebenklassen-Konstruktion und zur Drinfeld-Sokolov-Reduktion der Gravitationssymmetriealgebra. Dies erlaubt es uns, sie als asymptotische Symmetriealgebra der quantisierten Higher-Spin-Gravitationstheorie zu interpretieren.

Der zweite Teil dieser Arbeit beschäftigt sich mit der Beziehung zwischen einer $\mathcal{N} = 2$ supersymmetrischen Version der vorher untersuchten CFTs und solchen CFTs, die man mit Hilfe von Orbifold-Konstruktionen aus freien Theorien erhalten kann. Dies könnte einen ersten Schritt in Richtung einer Einbettung der dualen Higher-Spin-Theorien in die Stringtheorie darstellen. Dabei wird die CFT zunächst in einem gewissen Limes als Orbifold einer freien Theorie bezüglich der unitären Gruppe $\mathrm{U}(N)$ interpretiert werden. In einem zweiten Schritt wird gezeigt werden, dass dieser kontinuierliche Orbifold eine Erweiterung besitzt, welche sich als Orbifold unter der symmetrischen Gruppe S_{N+1} beschreiben lässt. Wir werden die Struktur beider Orbifold-Theorien eingehend studieren.

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Contents

Chapter 1

Introduction

1.1 Why higher spin holography?

Since the early days of quantum mechanics and at the latest with the advent of quantum field theory, it has been a major goal of theoretical physicists to find a quantum theory of general relativity. While the original version of the theory suffers from non-renormalisability, it might possess an ultraviolet completion which overcomes this problem. String theory has been considered a strong candidate for such a theory ever since its (re)discovery as a theory of gravity in the 1970s. However, as of 2016, the puzzle of quantum gravity is still not resolved, nor is string theory even understood to a satisfying degree. One major reason is that string theory is inherently mathematically difficult. Indeed, as was discovered in the 1990s, it is not only a theory of one-dimensional strings propagating in 10- or 26-dimensional spacetime, but also a theory of solitonic higher-dimensional objects called D-branes; and moreover, much of the mathematics needed to describe the theory have to be developed along the way.

Maldacena's discovery, in 1997, of the AdS/CFT correspondence [130, 170] sparked once again an enormous interest in string theory. He showed that superstring theory on an $\mathrm{AdS}_5 \times S^5$ background is dual to a supersymmetric cousin of quantum chromodynamics (QCD), $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory living on the four-dimensional boundary of AdS_5. The latter is a conformal field theory (CFT), i.e. a field theory which is scale-invariant and whose symmetry group is therefore an extension of the Poincaré group of ordinary relativistic field theories. Although the original statement of the duality was formulated in the 't Hooft limit where the number of colours becomes large, it is believed that it may also hold away from that limit. This duality is of utmost importance for various reasons:

First, the duality shows that there is an intimate relationship between two kinds of theories which had seemed incompatible for a long time, namely theories of gravity on the one hand and (conformal) quantum field theories on the other hand. The idea that gauge theories might be related to string theory in a particular limit of infinitely many colours was first brought up by 't Hooft [158] and made explicit by Maldacena's duality. Due to the quantum nature of the field theory side, there is now even new hope that the duality might give us a hint on how to quantise gravity.

Second, the AdS/CFT duality in its original version is a strong–weak duality: in the (planar) 't Hooft limit, it relates the strong coupling regime of the quantum field theory to the low energy regime of string theory, which can be described by an effective supergravity theory. Conversely, if the duality is to hold at all values of the coupling, the (stringy) strong curvature regime of the bulk theory is mapped to the perturbative regime of the boundary theory. This gives us perturbative computational access to originally non-perturbative problems. For instance, exploiting the similarity between $\mathcal{N} = 4$ SYM and QCD, one can use gravity methods to compute a lower bound on the ratio of shear viscosity and entropy density of quark-gluon plasma [138, 123], a quantity which used to be out of computational reach. The agreement of the outcome with experiment (see, e.g., [2, 129, 104, 89]) is both astonishing and encouraging.

Third, as more and more examples of dual theories are being found, the AdS/CFT correspondence has become one of the most important playgrounds for theoretical physicists, relating gravity theories to condensed matter systems or to fluid dynamics. While many of these so-called 'dualities' are only approximative in nature (as is the case with AdS/QCD mentioned above), the abundance of recurring structures in different areas all across physics clearly seems to indicate that there may still be much left to discover.

Finally, the AdS/CFT correspondence also serves as an explicit incarnation of 't Hooft's and Susskind's holographic principle [159, 155] (see also [26] and references therein), which, inspired by the Bekenstein-Hawking entropy formula for black holes, claims that the number of degrees of freedom of a gravitational theory scales like an area rather than a volume. This suggests to think of the information contained in the gravitational theory as stored on a holographic screen. If these concepts are to be taken seriously, any sensible quantum theory of gravity should be founded on this principle and formulated in such a way that the principle emerges from it as an obvious consequence, very much in the same way in which Einstein's equivalence principle is a consequence of his formulation of general relativity. The AdS/CFT correspondence already provides us with an explicit and suggestive dictionary that helps us translate gravity into a lower-dimensional non-gravitational theory.

However, one fundamental shortcoming of most of the holographic dualities found so far is that there has been little success in proving them, not least because of the strong–weak nature and the poor understanding we have of both sides beyond the perturbative regime. It was with the discovery of holographic dualities involving higher spin theories on anti-de-Sitter space that progress in this direction could be made.

Vasiliev's higher spin theories [165] are gauge theories of infinitely many interacting massless particles of arbitrarily high spin. This is a considerable extension of gravity, which is a gauge theory containing the spin-2 graviton, and should be described by gauging some infinite-dimensional extension of the Poincaré group. For a long time it was thought that such a theory could only be consistently defined if the fields of spin higher than 2 were either free or massive. This view was supported by numerous no-go theorems such as the Coleman-Mandula theorem [44], which however relies on S-matrix arguments that are not applicable to AdS space (note, however, the generalisation [131, 132] to the higher spin case). Indeed, higher spin theories rely heavily on a non-vanishing cosmological constant, which serves both as a coupling constant and as an infrared cutoff. Other no-go theorems can similarly be circumvented (see [16] for a review). In spacetime dimensions larger than 3, the theory requires the presence of an infinite tower of particles of ever increasing spin $s = 2, 3, 4, \ldots$, or only even spins in a minimal version. In particular, it contains the graviton as the spin-2 case, and particles of half-integer spins $s = 3/2, 5/2, \ldots$ (where $s = 3/2$ corresponds to the gravitino) in its supersymmetric versions. The construction of such a theory is not only an interesting academic exercise, but also provides a novel view on extended theories of gravity. Higher spin theories share some features with string theory such as non-locality and the presence of higher spin particles, without sharing all of its complications; after all, string theory contains an infinity of massive particles at each spin. Since the spin spectrum is reminiscent of the leading Regge trajectory of string theory, which consists of the lightest particle at each spin, it has been suggested that it might be possible to embed higher spin theory into string theory in the limit of tensionless strings, in which the particles become massless (see, e.g., [146] and references therein). This is another interesting limit of string theory beside the low-energy supergravity limit and might therefore elucidate string theory from a new perspective. Accordingly, string theory at finite string tension would then be a broken phase of an extended higher spin gauge theory, where the gauge symmetry is spontaneously broken by a gigantic Higgs mechanism termed *La grande bouffe* in [20].

Bearing this putative connection with string theory in mind, one might expect to find a CFT dual of higher spin theories as proposed by [154, 171, 133, 151], at least if the AdS/CFT correspondence is to hold in all regions of string theory parameter space. And indeed, the minimal version of Vasiliev theory on AdS_4 was found to be dual to the singlet sector of the 3-dimensional free or critical $\mathrm{O}(N)$ vector model in the large N limit [120, 152, 92, 93] (see [94] for a review). Here N is the number of colours. An important feature of these CFTs is that their number of degrees of freedom scales with N, as opposed to the Yang-Mills theories in the original version of the duality, where it scales like N^2. This is reflected in the bulk by the reduced spectrum of the higher spin theory with respect to full-fledged string theory. Even more importantly, this duality can be formulated as a relation between two weakly coupled theories, which opens up the possibility to perform perturbative checks of the duality and improve our understanding of the underlying mechanisms.

Later a lower-dimensional version of this duality was proposed [72, 81, 79, 73], relating higher spin theory on AdS_3 to a family of 2-dimensional coset minimal model CFTs (see [74] for a review), the simplest and most well-known exponents of which are the Virasoro minimal models such as the Ising model (cf. [39]). Conformal field theories in 2 dimensions differ from their higher-dimensional counterparts by the fact that their symmetry group (the conformal group or an extension thereof) is infinite-dimensional. This leads to an infinity of conserved quantities, which restricts the dynamics of the theory considerably and can in some cases even enable us to solve the theory exactly. On the bulk side of the duality these constraints are reflected by the absence of propagating degrees of freedom: as the Weyl tensor vanishes identically in three dimensions, gravitational waves cannot exist and the solutions to Einstein's equations and their higher spin counterparts are fully determined by the Ricci tensor and its higher spin generalisations. The only degrees of freedom are therefore located at the boundary. In spite of these heavy restrictions, the theory does describe gravity and even contains black holes [13, 12] which exhibit all the features one would expect from a 4-dimensional perspective, notably a horizon, entropy, temperature, and Hawking radiation. A further interesting aspect of the higher-spin dualities (both in 3 and 4 dimensions) is that they do not necessarily involve supersymmetry, thereby showing that supersymmetry is not an essential ingredient of the AdS/CFT correspondence. The simplifications described above have helped tremendously in studying the duality over the past years, and many variations of it have emerged: for instance, a proposal for a duality restricted to even spins was made in [3, 88]. This duality will lie at the heart of chapter 3. An $\mathcal{N} = 2$ extended supersymmetric generalisation was found in [45] and will be briefly reviewed in chapter 4, as it will be the starting point for chapters 5 and 6. Later, dualities with $\mathcal{N} = 1$ [46, 15] and (large) $\mathcal{N} = 4$ supersymmetry [75, 48, 38, 85, 14, 49] as well as matrix-extended versions [105, 75, 48, 38] (see, however, [37]) of the standard $\mathcal{N} = 2$ duality were identified, such that there now exists a plethora of higher-spin dualities on AdS_3 and it has become apparent that they are far from merely being an interesting but irrelevant coincident.

In parallel, the higher-spin/CFT dualities also helped improve our understanding of the relationship between higher spins and string theory. In [41] a generalisation of the aforementioned higher spin theory on AdS_4 that was dual to the $\mathrm{O}(N)$ vector model was argued to be a small-coupling limit of string theory on an $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background.

On the lower dimensional front it was argued in [76] that a certain limit of the coset theory appearing in the large $\mathcal{N} = 4$ duality of [75] could be described as an orbifold of a CFT of free particles by the unitary group $\mathrm{U}(N)$, i.e. a theory whose Hilbert space is restricted to those states that are invariant under the group action. It was further shown that this continuous orbifold can be embedded into a larger theory described by a discrete orbifold with respect to the symmetric group S_{N+1}. The latter theory in turn is believed to be dual to a system of D1- and D5-branes in string theory on $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$ (see [52] for a review). This cascade of identifications and embeddings suggests that string theory might be an extension of higher spin theory. Upon moving away from the limit in which the CFT is described by the continuous orbifold, the $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$ background should turn into $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$, which indeed exhibits the same large $\mathcal{N} = 4$ symmetry as the CFT and its higher spin dual. It is therefore suspected that the duality should continue to hold and become stringy in this regime. Exploiting this relationship, the expected symmetries of string theory could then be studied in some detail and constructed as an extension of higher spin symmetries in [77, 78].

A similar orbifold analysis in the case of the $\mathcal{N} = 2$ duality was carried out in [83, 84] and has been incorporated into this thesis as chapters 5 and 6.

1.2 Objectives and structure of the thesis

This thesis studies two aspects of higher spin holography in detail: the structure of $\mathcal{W}$ algebras and the orbifold constructions related to the $\mathcal{N} = 2$ version of the duality.

The goal of the first part is to understand the quantisation of the asymptotic symmetry algebras of (classical) higher spin theories, which is a first step towards a quantum formulation of the duality. In order to achieve this, we make a general ansatz for a quantum $\mathcal{W}$ algebra and solve for the structure constants by imposing the Jacobi identity. This is done in two cases: for the $\mathcal{W}_\infty$ algebra underlying the original duality involving fields of all integer spins greater than 1, and for the restriction to even spin fields described by the algebra $\mathcal{W}^{\mathrm{e}}_\infty$. In both cases we find that the algebra is uniquely determined up to the central charge and one additional free parameter, which can be identified with the coupling constant of the higher spin theory. In the first case, this yields a unique quantisation of the dual CFT for any value of the coupling parameter, as could be shown in [73] on the basis of our results. In the even spin case, however, there is a twofold ambiguity in the quantisation, which will be analysed in depth.

In the second part, the CFTs that appear in the $\mathcal{N} = 2$ version of the duality will be studied and related to orbifold constructions similar to what was done in [83, 84] for the $\mathcal{N} = 4$ case. In a first step, the large level limit of these Kazama-Suzuki coset models will be interpreted as a $\mathrm{U}(N)$ orbifold of a free theory. The untwisted sector of the orbifold will be shown to contain the CFT states that correspond to the perturbative sector of the holographically dual theory in the bulk. The twisted sectors, on the other hand, correspond to states that have been linked to non-perturbative conical defect solutions in the bulk. In a second step, this orbifold will then be shown to possess an extension which can be described as an orbifold by the symmetric group S_{N+1}. In analogy to the $\mathcal{N} = 4$ case, one may hope to use this relationship to find a stringy extension of $\mathcal{N} = 2$ higher spin theories in the future and understand the symmetries of string theory on that particular $\mathcal{N} = 2$ background.

The thesis is structured as follows:

Chapter 2 gives a brief introduction to the most important topics covered in this thesis. Notably, the standard bosonic higher spin theory on AdS_3 and its symmetry algebra $\mathfrak{hs}[\mu]$ are introduced in section 2.1. Section 2.2 introduces minimal model CFTs, $\mathcal{W}$ algebras and the most important ways to construct them. In particular, section 2.2.4 contains original material based on unpublished work in collaboration with Constantin Candu, Matthias Gaberdiel and Carl Vollenweider on the explicit construction of the algebra $\mathcal{W}_\infty[\mu]$. Section 2.3 then presents the main ideas and the most important arguments in favour of the proposed duality linking the higher spin theory from section 2.1 to the minimal models from section 2.2.

Chapter 3 is based on the paper [36] with Constantin Candu, Matthias Gaberdiel and Carl Vollenweider. Its main objective is to construct the algebra $\mathcal{W}^{\mathrm{e}}_\infty$ underlying the CFT dual to the even spin bulk theory as proposed in [3, 88]. This analysis is carried out in section 3.1.1 in analogy to the bosonic construction from section 2.2.4. The exact relation to the even higher spin algebra of the bulk theory is also found. Moreover, we present some hitherto unpublished comments on the relation between $\mathcal{W}^{\mathrm{e}}_\infty$ and the bosonic algebra $\mathcal{W}_\infty$ in 3.1.3. The remainder of chapter 3 discusses the different construction methods of $\mathcal{W}^e_\infty$, either by means of the Drinfel'd-Sokolov reduction (section 3.2), an introduction to which can be found in 2.2.3, or by a variety of coset constructions (section 3.3).

Chapter 4 serves as a short interlude to explain some of the theoretical concepts needed for the subsequent chapters, namely the simplest version of the $\mathcal{N} = 2$ Kazama-Suzuki cosets and the orbifold construction of conformal field theory.

The material in chapter 5 is joint work with Matthias Gaberdiel and has been published in [83]. It shows that the perturbative sector of the Kazama-Suzuki models which are the CFT duals of $\mathcal{N} = 2$ higher spin theory on AdS_3 can be rewritten as the untwisted sector of a continuous orbifold of free bosons and fermions in the large-level limit of the coset. This is the main result of section 5.1. The other, non-perturbative states of the coset are shown to correspond to the various twisted sectors in section 5.2, whose ground states are explicitly matched with certain coset states. We also identify two BPS states in each twisted sector from general orbifold considerations and find their counterparts on the coset side.

Chapter 6 presents the results of a follow-up project in collaboration with Matthias Gaberdiel [84], in which we study the relation of the Kazama-Suzuki coset from the previous chapter to the symmetric orbifold of free bosons and fermions. The symmetric orbifold is an extension of the continuous orbifold and might possess a string dual in analogy to the $\mathcal{N} = 4$ case. We decompose the untwisted sector in terms of coset representations in section 6.1 and find the structure of that sector to be given by multi-particle powers of a fundamental building block. A similar analysis is carried out for the twisted sectors in section 6.2, where each twisted sector now has its own building blocks.

We close with conclusions and an outlook in chapter 7. Four appendices contain the more technical material from chapters 2, 3, 5, and 6.

Chapter 2

Higher spin holography on AdS_3

The first signs of $\mathrm{AdS}_3/\mathrm{CFT}_2$ dualities emerged long before Maldacena's seminal paper [130] brought holography into the focus of attention. Already in 1986, Brown and Henneaux demonstrated [31] that the asymptotic symmetries of 3-dimensional Einstein gravity form a Poisson algebra isomorphic to two copies of the Virasoro algebra, and therefore describe a conformal field theory. This CFT has a finite central charge given by

$$c = \frac{3\ell}{2G} \,, \tag{2.1}$$

where ℓ is the curvature radius of AdS space and G is Newton's constant. It nevertheless took more than two decades until the full bearing of this discovery could be appreciated. After higher spin theories had been developed in the 1990s and the first examples for the AdS/CFT correspondence had been found, the possible existence of CFT duals to higher spin theories was soon brought up [171, 151]. A first explicit proposal for a higher spin version of an $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality was made by Klebanov & Polyakov [120], and Sezgin & Sundell [152]. As in the case of Maldacena's original $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [130], the duality was established in a limit in which the number of colours (and degrees of freedom) in the field theory becomes large. It is therefore a quasi-classical limit in which the (classical) gravity theories can be trusted. One distinguishing feature of those two kinds of dualities is however that gauge fields in super Yang-Mills theory transform in the adjoint representation of $\mathrm{SU}(N)$, and therefore their number scales like N^2 for large colour number N. On the other hand, the $\mathrm{O}(N)$ vector models whose singlet sector was proposed to be dual to 4d minimal higher spin theory are constructed out of N bosons (or fermions) transforming in the fundamental representation of $\mathrm{O}(N)$. Hence the number of degrees of freedom only scales with N, which amounts to a massive reduction in complexity. In the light of this duality the findings of

Brown and Henneaux could then be seen from a new perspective and interpreted as a potential sign for a lower-dimensional version of higher spin holography. The asymptotic symmetries of specific higher spin theories were computed in [106, 33] and found to be described by two copies of classical $\mathcal{W}_N$ algebras, which are non-linear extensions of the Virasoro algebra, to which they reduce in the special case $N = 2$. The central charge of these algebras was found to be (2.1) as in the pure gravity case. The analysis was found to work at the quantum level as well [80]. Since the quantum versions of the $\mathcal{W}_N$ algebras are the symmetry algebras of known minimal model CFTs [11] described by the family of coset theories

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} , \tag{2.2}$$

this paved the way towards the proposal by Gaberdiel & Gopakumar [72] of a duality between these $\mathcal{W}_N$ minimal models in a specific large N, k limit and higher spin theories on AdS$_3$. The limit has to be taken in such a way that the 't Hooft coupling

$$\lambda = \frac{N}{N+k} \tag{2.3}$$

is kept fixed. Just like their higher-dimensional counterparts, these CFTs are vector-like, i.e. the central charge scales like N. On the gravity side the parameter N corresponds to the maximal spin of the theory. In the $N = 2$ case underlying the original results of Brown & Henneaux, where the bulk theory is just spin-2 Einstein gravity and the boundary CFTs are the Virasoro minimal models, the central charge of the CFT is small and we are therefore in a quantum regime of the duality. An attempt towards a fully quantum formulation of the duality has been made in [39].

In this chapter we will provide the basics of the Chern-Simons formulation of 3d higher spin gravity in section 2.1 and of $\mathcal{W}$ algebras and minimal model CFTs in section 2.2. In section 2.3 we shall formulate the proposed duality and provide a quick overview over the various checks that have been performed in its support.

2.1 Higher spin theories in three dimensions

The idea of finding a theory of particles with spin greater than 1 goes back to Fierz and Pauli [60], who generalised the Dirac equation and found relativistic equations of motion for massive particles of arbitrary spin.

This was later turned into a gauge theory of massless non-interacting higher spin fields by Fronsdal [69] (see also [56] for the generalisation to half-integer spins and, e.g., [110, 126] for pedagogical introductions), who found the linear equation of motion for a spin-s particle $\varphi_{\mu_1 \cdots \mu_s}$ (a totally symmetric tensor of rank s

generalising the metric $g_{\mu\nu}$) propagating on a Minkowski background,

$$F_{\mu_1\cdots\mu_s} \equiv \Box\varphi_{\mu_1\cdots\mu_s} - s\,\partial_{(\mu_1}(\partial\cdot\varphi)_{\mu_2\cdots\mu_s)} + \frac{s(s-1)}{2}\partial_{(\mu_1}\partial_{\mu_2}\varphi'_{\mu_3\cdots\mu_s)} = 0\ . \quad (2.4)$$

Here parentheses around a set of indices denote their symmetrisation with weight 1 (i.e. the symmetrisation defines a projector), and we have made use of the definitions

$$(\partial\cdot\varphi)_{\mu_2\cdots\mu_s} = \partial_\mu\varphi^\mu{}_{\mu_2\cdots\mu_s}\ ,\quad \varphi'_{\mu_3\cdots\mu_s} = \varphi^\mu{}_{\mu\mu_3\cdots\mu_s}\ . \quad (2.5)$$

We also require $\varphi_{\mu_1\cdots\mu_s}$ to satisfy the double-trace condition

$$\varphi''_{\mu_1\cdots\mu_{s-4}} = 0 \quad (2.6)$$

if $s \geq 4$. $F_{\mu_1\cdots\mu_s}$ is called the Fronsdal tensor and generalises the linearised Ricci tensor to symmetric tensors of higher rank. The Fronsdal equation (2.4) is invariant under the gauge transformation

$$\delta\varphi_{\mu_1\cdots\mu_s} = s\,\partial_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}\ , \quad (2.7)$$

where ξ is a traceless rank $s-1$ tensor:

$$\xi' = 0\ . \quad (2.8)$$

This condition does of course only make sense if $s \geq 3$. For $s = 1$, equation (2.4) reduces to the Maxwell equation in vacuum,

$$\Box A_\mu - \partial_\mu(\partial\cdot A) = 0\ , \quad (2.9)$$

and for $s = 2$ it becomes the linearised vacuum Einstein equation:

$$\Box h_{\mu\nu} - 2\,\partial_{(\mu}(\partial\cdot h)_{\nu)} + \partial_\mu\partial_\nu h' = 0\ . \quad (2.10)$$

Furthermore, equation (2.7) reduces to a linearised diffeomorphism in that case. In the context of the AdS/CFT correspondence, it is relevant to generalise these backgrounds to curved spaces with constant curvature and a cosmological constant Λ, which are called de Sitter (dS) space in the case of positive cosmological constant and anti-de-Sitter (AdS) space for $\Lambda < 0$. In the case of an (A)dS$_d$ background the Fronsdal equation becomes [70, 57]

$$\hat{F}_{\mu_1\cdots\mu_s} + \frac{2\Lambda}{(d-1)(d-2)}\Big[\big(s^2 + (d-6)s - 2(d-3)\big)\varphi_{\mu_1\cdots\mu_s} + s(s-1)g_{(\mu_1\mu_2}\varphi'_{\mu_3\cdots\mu_s)}\Big] = 0\ , \quad (2.11)$$

where $\hat{F}$ is the Fronsdal tensor F with all partial derivatives replaced by covariant ones and contractions carried out with the (A)dS background metric $g_{\mu\nu}$.

Although it was long thought that higher spin theories could not be interacting, Vasiliev found a fully interacting (non-linear) theory in 4-dimensional dS and AdS space [162, 163, 164]. As it turned out, the presence of one higher spin field requires the presence of an infinite tower of them. These results were later extended to higher dimensions as well as to dimension 3 [21, 141, 142, 165]. In this dimension the Weyl tensor vanishes identically, and neither Einstein gravity nor higher spin gravity have any propagating degrees of freedom in the bulk. They are completely determined by their boundary dynamics and can be formulated as (topological) Chern-Simons theories. A further particularity of dimension 3 is that certain values of the coupling constant allow for a truncation of the theory to finitely many higher spin fields. In the following we shall study higher spin gravity on AdS_3 in more detail.

2.1.1 Chern-Simons gravity in $d = 2 + 1$

According to general relativity, gravity can be described as the effect of a warped spacetime on freely moving particles and their back-reaction on the structure of spacetime. This dynamics in absence of matter or radiation is encoded in the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0 \ , \tag{2.12}$$

where $g_{\mu\nu}$ is the metric tensor describing the geometry of spacetime, $R_{\mu\nu}$ the Ricci tensor and R the Ricci scalar which describe curvature, and Λ the cosmological constant. These equations are valid in any number of spacetime dimensions, but we will restrict to $d = 2 + 1$ in the following. They can be derived from the Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-\det g}\, \mathrm{d}^3 x \ . \tag{2.13}$$

There is an alternative formulation which makes use of a local vector frame (or *dreibein*) instead of the metric formulation. Let us define a local frame of 1-forms by e^a_μ, where $\mu = 0, 1, 2$ is the spacetime index and $a = 0, 1, 2$ labels the basis elements. We choose this frame to be orthonormal, which means

$$e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab} \quad \text{ or } \quad e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu} \ , \tag{2.14}$$

where η_{ab} is the Minkowski metric with signature $(-++)$. We can therefore express the metric in terms of the dreibein. If we want to stick to that language, we also need to express the Christoffel symbols and the curvature tensor in terms of differential forms. The analogue of the Christoffel symbols is given by the *connection 1-form* ω_μ^{ab} defined by the covariant derivative of a vector field $V^a = V^\mu e_\mu^a$:

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu^{ab} V_b \ . \tag{2.15}$$

The indices $a, b, c, \ldots$ are raised and lowered using the Minkowski metric. By requiring the connection to be compatible with the metric and torsion-free, we obtain an explicit expression for the connection 1-form in terms of the dreibein:

$$\omega_\mu^{ab} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}^{b]} - e^{\nu[a}e^{b]\rho}e_{\mu c}\partial_\nu e_\rho^c \ , \tag{2.16}$$

where square brackets denote anti-symmetrisation. The curvature tensor can be cast into the *curvature 2-form*

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \ . \tag{2.17}$$

In 3 dimensions, it will prove useful to define the Hodge dual of the differential forms introduced above with respect to the dreibein indices:

$$\begin{aligned} \omega^a &= \frac{1}{2}\varepsilon^a{}_{bc}\omega^{bc} \ , \\ R^a &= \frac{1}{2}\varepsilon^a{}_{bc}R^{bc} = \mathrm{d}\omega^a + \frac{1}{2}\varepsilon^a{}_{bc}\omega^b \wedge \omega^c \ . \end{aligned} \tag{2.18}$$

In this formalism, the Einstein-Hilbert action (2.13) becomes

$$S_{\mathrm{EH}} = \frac{1}{8\pi G}\int \left(e^a \wedge R_a - \frac{\Lambda}{6}\varepsilon_{abc}e^a \wedge e^b \wedge e^c\right) \ . \tag{2.19}$$

We now claim that up to boundary terms this can be written as a Chern-Simons theory [1, 169], i.e. a topological field theory with action

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi}\int \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right) \tag{2.20}$$

and equation of motion

$$F \equiv \mathrm{d}A + A \wedge A = 0 \ . \tag{2.21}$$

Here the *gauge connection* A is a 1-form which takes values in some Lie algebra $\mathfrak{g}$ and F is the curvature. (2.21) is therefore simply a flatness condition. As a special case of interest to us let us define $\mathfrak{g}$ to be generated by J_a, P_b for $a, b = 0, 1, 2$, satisfying

$$[J_a, J_b] = \varepsilon_{ab}{}^c J_c \ , \quad [J_a, P_b] = \varepsilon_{ab}{}^c P_c \ , \quad [P_a, P_b] = -\Lambda\,\varepsilon_{ab}{}^c J_c \ . \tag{2.22}$$

The J_a generate spacetime rotations and P_b generate translations, and the resulting Lie algebra is the algebra of isometries of a maximally symmetric Lorentzian manifold. Depending on the sign of the cosmological constant, this is either flat space, de Sitter space, or anti-de-Sitter space:

$$\mathfrak{g} = \begin{cases} \mathfrak{so}(3,1) \cong \mathrm{isom}(\mathrm{dS}_3) & \text{for } \Lambda > 0 \ , \\ \mathfrak{iso}(2,1) \cong \mathrm{isom}(\mathbb{R}^{2,1}) & \text{for } \Lambda = 0 \ , \\ \mathfrak{so}(2,2) \cong \mathrm{isom}(\mathrm{AdS}_3) & \text{for } \Lambda < 0 \ . \end{cases} \tag{2.23}$$

The trace defines a non-degenerate symmetric real bilinear form on $\mathfrak{g}$ by

$$\mathrm{Tr}(J_a J_b) = \mathrm{Tr}(P_a P_b) = 0 \ , \quad \mathrm{Tr}(J_a P_b) = \eta_{ab} \ . \tag{2.24}$$

We can now define the gauge connection by

$$A = (e^a_\mu P_a + \omega^b_\mu J_b)\, \mathrm{d}x^\mu \ . \tag{2.25}$$

From now on we will restrict ourselves to the AdS case, which is the one relevant for the holographic duality, and define $\Lambda = -\frac{1}{\ell^2}$, where $\ell > 0$ is the AdS radius. Then the terms in the Chern-Simons action (2.20) read

$$\int \mathrm{Tr}(A \wedge \mathrm{d}A) = 2 \int e^a \wedge \mathrm{d}\omega_a + \text{ boundary terms} \ , \tag{2.26}$$

$$\int \mathrm{Tr}(A \wedge A \wedge A) = \int \left(\frac{3}{2} \varepsilon_{abc} e^a \wedge \omega^b \wedge \omega^c - \frac{\Lambda}{2} \varepsilon_{abc} e^a \wedge e^b \wedge e^c \right) \ . \tag{2.27}$$

Setting the Chern-Simons level to the value

$$k = \frac{\ell}{4G} \ , \tag{2.28}$$

the full Chern-Simons action then becomes

$$\begin{aligned} S_{\mathrm{CS}}[A] &= \frac{\ell}{8\pi G} \int \left(e^a \wedge R_a - \frac{\Lambda}{6} \varepsilon_{abc} e^a \wedge e^b \wedge e^c \right) + \text{ boundary terms} \\ &= S_{\mathrm{EH}} + \text{ boundary terms} \ . \end{aligned} \tag{2.29}$$

Now observe that our Lie algebra $\mathfrak{g}$ is not simple. Indeed, we can write it as a direct sum

$$\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R})_+ \oplus \mathfrak{sl}(2,\mathbb{R})_- \ , \tag{2.30}$$

where the $\pm$ just serve as an index to label the two copies of $\mathfrak{sl}(2,\mathbb{R})$. This notation enables us to define the generators of $\mathfrak{sl}(2,\mathbb{R})_\pm$ by

$$J^\pm_a = \frac{1}{2}(J_a \pm \ell P_a) \tag{2.31}$$

satisfying the expected commutation relations

$$[J_a^{\pm}, J_b^{\pm}] = \varepsilon_{ab}{}^{c} J_c^{\pm} \ , \quad [J_a^{\pm}, J_b^{\mp}] = 0 \ . \tag{2.32}$$

The symmetric bilinear form in turn splits as

$$\mathrm{Tr}(J_a^{\pm} J_b^{\pm}) = \pm\frac{\ell}{2}\eta_{ab} \ , \quad \mathrm{Tr}(J_a^{+} J_b^{-}) = 0 \ . \tag{2.33}$$

Furthermore, we can define chiral gauge connections

$$A_\mu^{\pm} = \left(\omega_\mu^a \pm \frac{e_\mu^a}{\ell}\right) J_a^{\pm} \ , \tag{2.34}$$

leading to the identity

$$A_\mu = A_\mu^{+} + A_\mu^{-} \ . \tag{2.35}$$

So the Chern-Simons action splits into two equivalent and independent parts

$$S_{\mathrm{CS}}[A] = S_{\mathrm{CS}}[A^{+}] - S_{\mathrm{CS}}[A^{-}] \ , \tag{2.36}$$

and it will be sufficient to study one chiral copy. Here we have assumed that $A_a^{\pm}$ are two identical sets of generators with the same bilinear form. The relative sign between the original bilinear forms in (2.33) then leads to the sign in (2.36).

2.1.2 Higher spin algebras

It seems natural to generalise the analysis of the previous section to algebras of higher rank. Taking $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ leads to a Chern-Simons gravity theory which contains a spin-3 particle in addition to the spin-2 graviton of $\mathfrak{sl}(2,\mathbb{R})$. This is so because the restriction of the adjoint representation to the principally embedded[1] $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sl}(3,\mathbb{R})$ decomposes into a spin-2 and a spin-3 multiplet:

$$\mathfrak{sl}(3,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R}) \oplus D_3 \ . \tag{2.37}$$

Similarly, $\mathfrak{sl}(N,\mathbb{R})$ has particles of spin $2, 3, \ldots, N$:

$$\mathfrak{sl}(N,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R}) \oplus \bigoplus_{s=3}^{N} D_s \ , \tag{2.38}$$

where D_s is the $(2s-1)$-dimensional irreducible spin-s representation of $\mathfrak{sl}(2,\mathbb{R})$. This therefore suggests that the family of $\mathfrak{sl}(N,\mathbb{R})$ Chern-Simons theories leads to

[1] The principal $\mathfrak{sl}(2)$ embedding can be defined by the property that each representation D_s in the decomposition (2.38) intersects the Cartan subalgebra of $\mathfrak{sl}(N)$ non-trivially (cf. Thm. 5.2 in [122]). While this is not the standard definition of the principal embedding, it is the easiest one to generalise to infinite-dimensional Lie algebras.

the 3d version of higher spin gravity. Indeed, it was shown in [33] that the linearised version of these theories leads to the Fronsdal theory sketched in section 2.1.[2]

A generalisation of $\mathfrak{sl}(N,\mathbb{R})$ to non-integer N can be constructed [63, 19] (see also [74]) and will be called the higher-spin algebra $\mathfrak{hs}[\mu]$, where $\mu \in \mathbb{R}$ is a free parameter, and setting μ to a positive integer N leads to an algebra that admits a truncation to $\mathfrak{sl}(N,\mathbb{R})$. The first step is to construct an associative algebra

$$B[\mu] := \frac{\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))}{\langle C_2 - \frac{1}{4}(\mu^2-1)\mathbb{1}\rangle} \tag{2.39}$$

defined as the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))$ of $\mathfrak{sl}(2,\mathbb{R})$ and the two-sided ideal generated by the difference of the quadratic Casimir and a constant. Choosing the generators of $\mathfrak{sl}(2,\mathbb{R})$ as $J_0, J_\pm$ satisfying

$$[J_\pm, J_0] = \pm J_\pm \ , \quad [J_+, J_-] = 2J_0 \ , \tag{2.40}$$

the quadratic Casimir is given by

$$C_2 = J_0^2 - \frac{1}{2}(J_+J_- + J_-J_+) \ , \tag{2.41}$$

which is greater or equal than $-\frac{1}{4}$ for unitary highest weight representations of $\mathfrak{sl}(2,\mathbb{R})$. This algebra is spanned by $2s-1$ modes of spin s for all integers $s \geq 1$. We will denote them by V^s_m, where m ranges from $-s+1$ to $s-1$. They can be defined by

$$V^s_m = (-1)^{s-1-m}\,\frac{(s-1+m)!}{(2s-2)!}\,\mathrm{ad}(J_-)^{s-1-m}\,J_+^{s-1} \ . \tag{2.42}$$

Explicitly, the lowest-lying fields read

$$\begin{aligned}
& & & V^3_2 = J_+^2 \ , \\
& & V^2_1 = J_+ \ , \quad & V^3_1 = J_0J_+ + \tfrac{1}{2}J_+ \ , \\
V^1_0 = \mathbb{1} \ , \quad & & V^2_0 = J_0 \ , \quad & V^3_0 = \tfrac{1}{3}(J_-J_+ + 2J_0^2 + J_0) \cong J_0^2 - \tfrac{1}{12}(\mu^2-1) \ , \\
& & V^2_{-1} = J_- \ , \quad & V^3_{-1} = J_-J_0 + \tfrac{1}{2}J_- \ , \\
& & & V^3_{-2} = J_-^2 \ ,
\end{aligned} \tag{2.43}$$

where we have replaced $C_2 \cong \frac{1}{4}(\mu^2-1)$ in the expressions for V^3_0, V^3_{-1}, and V^3_{-2}. The associative product $\star$ on $B[\mu]$ is inherited from the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))$ and called the *lone star product*. A symmetric bilinear form can be defined using the trace given by the coefficient of V^1_0:

$$\mathrm{Tr}(V^s_m) = \delta^{s1}\delta_{m0} \quad \text{and linear extension.} \tag{2.44}$$

[2]This is also our reason for calling s the *spin*. Strictly speaking there is no spin in 3 dimensions. What we therefore mean by spin s in this context is a field that can be mapped to a symmetric rank-s tensor in Fronsdal's (or Vasiliev's) 3d higher spin theory.

The resulting bilinear form is non-degenerate for generic μ, but it degenerates for $\mu = N$ integer [161] (see also [62]):

$$\mathrm{Tr}(V^s_m \star A) = 0 \qquad \text{for all } s > |N|,\ A \in B[\mu = N]\ . \tag{2.45}$$

Therefore the modes V^s_m with $s > |N|$ span a two-sided ideal χ_N of $B[N]$,

$$\chi_N = \mathrm{span}\{V^s_m \colon s > |N|\}\ , \tag{2.46}$$

the quotient by which is isomorphic to the algebra of $|N| \times |N|$ matrices:

$$B[N]/\chi_N \cong \mathbb{R}^{|N|\times|N|}\ . \tag{2.47}$$

Having studied $B[\mu]$ enables us to define the higher-spin algebra for any μ by turning $B[\mu]$ into a Lie algebra and removing the centre spanned by the identity:

$$\mathfrak{hs}[\mu] = \frac{\mathrm{Lie}(B[\mu])}{\mathbb{R}\cdot\mathbb{1}}\ . \tag{2.48}$$

The V^2_m modes span an $\mathfrak{sl}(2)$ subalgebra, which is the principal embedding of $\mathfrak{sl}(2)$ into $\mathfrak{hs}[\mu]$. Their commutation relations with the other generators read

$$[V^2_n, V^s_m] = ((s-1)n - m)\, V^s_{n+m}\ . \tag{2.49}$$

V^s_m therefore has spin s, and $\mathfrak{hs}[\mu]$ decomposes into $\mathfrak{sl}(2)$ modules as follows:

$$\mathfrak{hs}[\mu] \to \mathfrak{sl}(2,\mathbb{R}) \oplus \bigoplus_{s=3}^{\infty} D_s\ . \tag{2.50}$$

A closed formula for general commutators exists as well and can be found in [139].

Therefore the Chern-Simons theory based on $\mathfrak{hs}[\mu]$ describes a theory involving one massless particle for each spin $s = 2, 3, \dots$. A special feature of higher-spin gravity in 3 dimensions is the possibility to truncate the theory to spins $s = 2, \dots, N$. As discussed earlier, the bilinear symmetric form degenerates for $\mu = N$, and the quotient by the corresponding ideal now becomes

$$\mathfrak{hs}[N]/\chi_N \cong \frac{\mathfrak{gl}(N,\mathbb{R})}{\mathbb{R}\cdot\mathbb{1}} \cong \mathfrak{sl}(N,\mathbb{R})\ , \tag{2.51}$$

giving us back the spin N gravity introduced in the previous section.

2.2 $\mathcal{W}$ algebras

After this brief introduction to higher spin gravity in three dimensions we shall turn to describe the two-dimensional conformal field theories that have been argued to be their holographic duals. Conformal field theories (CFTs) are quantum field theories that are scale invariant and whose symmetry group is therefore an extension of the Poincaré group, called the conformal group. In 2 dimensions this group becomes infinite-dimensional and its Lie algebra, the (centrally extended) Virasoro algebra,[3] is generated by the modes L_n ($n \in \mathbb{Z}$) of the chiral stress-energy tensor $L(z)$ satisfying the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\,\delta_{m,-n} \ . \tag{2.52}$$

In the language of operator product expansions (OPEs) of the stress-energy tensor

$$L(z) = \sum_n z^{-n-2} L_n \tag{2.53}$$

these commutators take the form

$$L(z)L(w) \sim \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w} \ . \tag{2.54}$$

Here we have only written down the singular terms of the OPE, which we have indicated by the $\sim$ sign, as only these are relevant for the commutators. The regular terms can, however, be completely reconstructed from the singular ones.

The infinite-dimensional symmetry algebra leads to infinitely many conserved charges, which in turn lead to relations between different correlators and can in some cases even allow to solve the theory explicitly. Of special importance in the study of CFTs are *Virasoro-primary* (or simply *primary*) fields Φ_h of conformal dimension h, which are defined by their OPE with the stress tensor taking the form

$$L(z)\Phi_h(w,\bar{w}) \sim \frac{h\,\Phi_h(w,\bar{w})}{(z-w)^2} + \frac{\partial_w \Phi_h(w,\bar{w})}{z-w} \ . \tag{2.55}$$

A *quasi-primary* field is one whose OPE with L agrees with (2.55) up to the third order pole, but may in addition have non-vanishing poles at order 4 or higher (the stress tensor itself is such an example). In the following we shall assume some familiarity with 2-dimensional conformal field theories, but self-contained introductions to the topic can be found, e.g., in [90, 24].

In the context of higher spin holography, the CFTs of interest to us have even larger symmetry algebras which contain the Virasoro algebra as a subalgebra. These extended conformal algebras are called $\mathcal{W}$ algebras (see [28] for a review). A *quantum $\mathcal{W}$ algebra* is a meromorphic conformal field theory (i.e., one in which OPEs have a pole structure like the one in equation (2.54)) spanned by a set of

[3]The actual symmetry algebra is a direct sum of two chiral copies of the Virasoro algebra, but we will mostly consider only one chiral version for simplicity.

Virasoro-primary conserved currents W^i of integer or half-integer conformal dimension s_i, their derivatives and their normal-ordered products. $\mathcal{W}$ algebras arise, for instance, as the symmetry algebras of coset theories and in the context of asymptotic symmetries of $(2+1)$-dimensional theories of gravity. Before turning to the explicit constructions of $\mathcal{W}$ algebras, let us give a simple example of such an algebra.

2.2.1 An example: $\mathcal{W}_3$

The simplest non-trivial example of a $\mathcal{W}$ algebra is the algebra $\mathcal{W}_3$ found by Zamolodchikov in [172]. It is linearly generated by the stress-energy tensor L satisfying the Virasoro algebra, a primary field W of conformal dimension 3, and the quasi-primary field

$$\Lambda(z) = LL(z) - \frac{3}{10}\partial^2 L(z) \tag{2.56}$$

of conformal dimension (or spin) 4. Here, LL denotes the normal-ordered product of L with itself, defined by the constant (i.e. leading regular) term in the OPE $L \star L$. The OPE of W with itself is uniquely fixed (up to an overall normalisation constant) by associativity constraints, and reads

$$\begin{aligned} W(z)W(w) \sim \frac{c/3}{(z-w)^6} + \frac{2L(w)}{(z-w)^4} + \frac{\partial L(w)}{(z-w)^3} + \frac{2\beta\Lambda(w) + \frac{3}{10}\partial^2 L(w)}{(z-w)^2} \\ + \frac{\beta\partial\Lambda(w) + \frac{1}{15}\partial^3 L(w)}{(z-w)} \end{aligned} \tag{2.57}$$

with $\beta = 16/(22+5c)$. In the following, we will write this in the more compact form

$$W \star W \sim \frac{c}{3} I \ , \tag{2.58}$$

which includes primary fields only. All other structure constants that are missing in this notation can be derived from these using conformal invariance, i.e. the associativity constraints that involve L:

$$L \star (L \star W) = (L \star L) \star W \ , \qquad L \star (W \star W) = (L \star W) \star W \ . \tag{2.59}$$

Note that the normal-ordered product fields LW and WW of spin 5 and 6, respectively, cannot appear in the OPE above, for different reasons: WW has too high a conformal dimension to appear in the singular part of $W \star W$, whereas LW cannot appear because $W \star W$ contains only quasi-primary fields of even conformal dimension. One can see this by using the bosonic property $W(z)W(w) = W(w)W(z)$, which implies that only fields with an odd number of derivatives can be present in odd poles of the OPE. This excludes quasi-primaries of odd conformal dimension.

In the following we shall study several ways to construct $\mathcal{W}$ algebras: the GKO coset construction, the Drinfel'd-Sokolov reduction and the bootstrap approach.

2.2.2 WZW models and the GKO construction

The coset construction [95, 96] (see also [28]) is a way to construct a conformal field theory starting from a Wess-Zumino-Witten (WZW) model. The WZW model based on the semi-simple compact Lie group G at level k is defined by an action principle

$$S = \frac{|k|}{16\pi}\int_{\partial M} \mathrm{d}^2x\,\mathrm{Tr}(\partial^\mu g^{-1}\partial_\mu g) \\ + \frac{k}{24\pi i}\int_M \mathrm{d}^3x\,\varepsilon_{\mu\nu\rho}\,\mathrm{Tr}\left(\tilde{g}^{-1}(\partial^\mu\tilde{g})\tilde{g}^{-1}(\partial^\nu\tilde{g})\tilde{g}^{-1}(\partial^\rho\tilde{g})\right)\ , \quad (2.60)$$

where M is some 3-dimensional manifold with 2-dimensional boundary ∂M, g is a G-valued field on ∂M with extension $\tilde{g}$ to M, and $\varepsilon_{\mu\nu\rho}$ is the Levi-Civita tensor. Moreover, the trace is defined in the fundamental representation and the level k needs to be an integer for the Euclidean path integral of this theory to be well-defined. The equations of motion derived from varying the action define conserved currents $J(z)$ and $\bar{J}(\bar{z})$ on ∂M:

$$\partial_{\bar{z}}J = 0\ , \quad \partial_z\bar{J} = 0\ , \quad (2.61)$$

where

$$J(z) = -k(\partial_z g)g^{-1}\ , \quad \bar{J}(\bar{z}) = kg^{-1}\partial_{\bar{z}}g\ . \quad (2.62)$$

We will only consider the holomorphic currents $J(z)$ from now on, but the analysis is analogous for the anti-holomorphic ones. These currents take values in the Lie algebra $\mathfrak{g}$ of G, i.e. they can be expanded in a hermitian basis t_a of $\mathfrak{g}$,

$$J(z) = J^a(z)t_a\ , \quad (2.63)$$

with

$$[t_a, t_b] = i f_{ab}{}^c t_c \quad \text{and} \quad \mathrm{Tr}(t_a t_b) = d_{ab}\ . \quad (2.64)$$

Here d_{ab} is the Killing metric of $\mathfrak{g}$, which is used to raise and lower indices. The currents $J^a(z)$ then satisfy the OPEs

$$J^a(z)J^b(w) \sim \frac{kd^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(z)}{z-w}\ , \quad (2.65)$$

where d^{ab} is now the inverse Killing metric. Expressing the currents in terms of their Laurent modes,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}\ , \quad (2.66)$$

the mode algebra satisfies the commutation relations

$$[J^a_m, J^b_n] = i f^{ab}{}_c\, J^c_{m+n} + k\, m\, d^{ab}\, \delta_{m,-n} \ . \tag{2.67}$$

These define the affine Kac-Moody algebra $\mathfrak{g}_k$. We may then define the stress-energy tensor L_G of the WZW model by the so-called Sugawara construction

$$L_G(z) = \frac{1}{2(k+h^\vee)}\, d_{ab}(J^a J^b)(z) \ , \tag{2.68}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. For example, in the case of $\mathfrak{g} = \mathfrak{su}(N)$ we have $h^\vee = N$, and for $\mathfrak{so}(N)$ its value is $h^\vee = N - 2$. One can check that L_G satisfies the Virasoro algebra with central charge

$$c_G = \frac{k \dim G}{k + h^\vee} \ . \tag{2.69}$$

Moreover, the Kac-Moody currents are primary of conformal dimension 1:

$$L_G(z) J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \tag{2.70}$$

or

$$[L_m, J^a_n] = -n J^a_{m+n} \ . \tag{2.71}$$

Each integrable highest-weight module of $\mathfrak{g}_k$ with highest weight μ corresponds to a Virasoro-primary field of conformal dimension

$$h(\mu) = \frac{(\mu\,,\mu + 2\rho)}{2(k+h^\vee)} \ , \tag{2.72}$$

where $(\cdot\,,\cdot)$ denotes the Killing form and ρ is the Weyl vector of $\mathfrak{g}$. This can also be written in terms of the quadratic Casimir of $\mathfrak{g}$,

$$C_2^{\mathfrak{g}}(\mu) = \frac{1}{2}(\mu\,,\mu + 2\rho) \ . \tag{2.73}$$

The formula for the conformal dimension then simply becomes

$$h(\mu) = \frac{C_2^{\mathfrak{g}}(\mu)}{k + h^\vee} \ . \tag{2.74}$$

Starting from a Wess-Zumino-Witten model based on the group G, we can gauge this theory by a subgroup[4] $H < G$ and denote the resulting theory by

$$G/H \quad \text{or} \quad \frac{\mathfrak{g}_k}{\mathfrak{h}_{k'}} \ , \tag{2.75}$$

[4] H need not be a *normal* subgroup of G.

where $\mathfrak{h}$ is the Lie algebra of H, and its level k' is determined by the details of the embedding $\mathfrak{h} \subset \mathfrak{g}$. The coset algebra is defined as the algebra of all normal-ordered products of numerator currents and their derivatives which commute with the currents of the denominator theory. One can show that this satisfies the definition of a $\mathcal{W}$ algebra. Suppose L_G and L_H are the Sugawara stress tensors of the numerator and denominator theory, respectively, and J^a are the Kac-Moody currents of the denominator (which are also contained in the numerator); then the stress tensors act on these as

$$
\begin{aligned}
[(L_G)_m, J^a_n] &= -nJ^a_{m+n} \ , \\
[(L_H)_m, J^a_n] &= -nJ^a_{m+n} \ ,
\end{aligned}
\tag{2.76}
$$

and therefore

$$
L_{G/H} \equiv L_G - L_H \tag{2.77}
$$

commutes with the denominator currents and is a coset field. On the other hand, on any field in the numerator theory that commutes with all denominator currents, $L_{G/H}$ acts exactly like L_G, so we can interpret it as the coset stress tensor. It follows that the central charge is given by the difference of central charges

$$
c_{G/H} = c_G - c_H \ . \tag{2.78}
$$

An irreducible highest-weight representation μ of $\mathfrak{g}_k$ naturally decomposes into irreducible highest-weight representations ν of $\mathfrak{h}_{k'}$ in the following manner:

$$
\mu \to \bigoplus_{\nu} (\mu;\nu) \otimes \nu \ . \tag{2.79}
$$

The multiplicity spaces $(\mu;\nu)$ are then the representations of the coset, containing all states in μ which are highest weight with respect to $\mathfrak{h}_{k'}$. Due to the form of the stress tensor (2.77) their conformal dimensions are given by

$$
h(\mu;\nu) = h(\mu) - h(\nu) + n \ , \tag{2.80}
$$

where n is the level at which ν appears inside μ. Note that by taking the trace of the identity (2.79), we get the character branching

$$
\chi_\mu(q,z) = \sum_{\nu} b^\nu_\mu(q) \chi_\nu(q,z') \ . \tag{2.81}
$$

The branching function

$$
b^\nu_\mu(q) \equiv \mathrm{Tr}_{(\mu;\nu)}\, q^{L_0 - \frac{c}{24}} \tag{2.82}
$$

is therefore the character of the coset representation $(\mu;\nu)$, where the trace is taken over all states in $(\mu;\nu)$. The simplest case of such a representation is the one where both μ and ν are the 'trivial' representations whose highest weight state is the vacuum. The resulting 'vacuum representation' $(0;0)$ is then precisely the module generated by the coset algebra modes acting on the vacuum. By the operator–state correspondence of conformal field theory, this is in one-to-one correspondence to the coset algebra itself and the vacuum character therefore counts the fields in the coset algebra (including descendants).

As an example for the coset construction, let us generalise the $\mathcal{W}_3$ algebra studied earlier to the family $\mathcal{W}_N$ for any integer $N \geq 2$ (for $N = 2$, this is just the Virasoro algebra). These are generated by the stress tensor L and one primary field W^s for each spin $s = 3, \ldots, N$, as well as their normal-ordered products. For certain discrete values $c \leq N - 1$ of the central charge[5] they arise as the symmetry algebras of the coset $\mathcal{W}_{N,k}$ minimal models [11] defined by the diagonal cosets

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \tag{2.83}$$

with central charge

$$\begin{aligned} c_{N,k} &= \frac{k(N^2-1)}{N+k} + \frac{(N^2-1)}{N+1} - \frac{(k+1)(N^2-1)}{N+k+1} \\ &= (N-1)\left[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\right] \leq N-1 \ . \end{aligned} \tag{2.84}$$

For the holographic duality we will take both N and k large, but keep the ratio λ as defined in equation (2.3) constant. The central charge will then scale like

$$c_{N,k} \sim N(1-\lambda^2) \ , \tag{2.85}$$

hence the theory behaves like a vector model (rather than a matrix model) in this limit. Again, for the specific value $N = 2$, we get the Virasoro algebra and the Virasoro minimal models with central charges [68]

$$c_m = 1 - \frac{6}{m(m+1)} \ , \quad \text{where } m = k+2 \ . \tag{2.86}$$

For instance, the $m = 3$ case describes the critical Ising model.

The representations of the coset (2.83) are labelled by triples $(\mu, \rho; \nu)$, where μ, ρ and ν are integrable highest weights of $\mathfrak{su}(N)_k$, $\mathfrak{su}(N)_1$ and $\mathfrak{su}(N)_{k+1}$, respectively. Here, integrable means that the sum of the Dynkin labels must not exceed the level (k, 1 or $k+1$ in our case). It therefore follows that ρ has to be either trivial or a representation with Dynkin labels $\rho = [0^{n-1}, 1, 0^{N-n-1}]$ for some $1 \leq n \leq N-1$. Actually, ρ is uniquely determined by μ and ν from the condition that $\mu + \rho - \nu$

[5]For $c > N - 1$, the $\mathcal{W}_N$ algebras appear as the symmetry algebras of Toda theories.

must lie in the root lattice. We will therefore often drop it from our notation. Note that representations that are related by an outer automorphism of the affine $\mathfrak{su}(N)$ are isomorphic. The group of outer isomorphisms is in this case $\mathbb{Z}_N$, which acts on the highest weights by cyclically shifting their Dynkin labels. It is generated by

$$J: \quad [\Lambda_0;\Lambda_1,\ldots,\Lambda_{N-1}] \mapsto [\Lambda_{N-1};\Lambda_0,\ldots,\Lambda_{N-2}] \ . \tag{2.87}$$

Here Λ_0 is the affine Dynkin label, which is defined by the condition

$$k=\sum_{i=0}^{N-1}\Lambda_i \qquad \text{for } \mathfrak{su}(N)_k. \tag{2.88}$$

Therefore we have an identification of irreducible highest-weight representations

$$(J\mu, J\rho; J\nu) \cong (\mu,\rho;\nu) \ . \tag{2.89}$$

The simplest coset representations are the *minimal* representations

$$(\mathrm{f};0)\ , \quad (\bar{\mathrm{f}};0)\ , \quad (0;\mathrm{f})\ , \quad (0;\bar{\mathrm{f}})\ , \tag{2.90}$$

where the fundamental and anti-fundamental representations of $\mathfrak{su}(N)$ are given by

$$\mathrm{f}=[1,0,\ldots,0]\ , \qquad \bar{\mathrm{f}}=[0,\ldots,0,1]\ . \tag{2.91}$$

In the four representations (2.90), ρ is given by $\bar{\mathrm{f}}$, f, f and $\bar{\mathrm{f}}$, respectively. The conformal dimensions can be computed using equations (2.74) and (2.80), where the general formula for the quadratic Casimir of an $\mathfrak{su}(N)$ highest-weight representation $\Lambda=[\Lambda_1,\ldots,\Lambda_{N-1}]$,

$$C^{(N)}(\Lambda)=\sum_{i<j}\Lambda_i\Lambda_j\,\frac{i(N-j)}{N}+\frac{1}{2}\sum_j\Lambda_j^2\,\frac{j(N-j)}{N}+\sum_j\Lambda_j\,\frac{j(N-j)}{2}\ , \tag{2.92}$$

can be useful. The conformal dimensions of the minimal representations can then be computed to yield

$$\begin{aligned} h(\mathrm{f},\bar{\mathrm{f}};0)=h(\bar{\mathrm{f}},\mathrm{f};0)&=\frac{N^2-1}{2N(N+k)}+\frac{N^2-1}{2N(N+1)}=\frac{N-1}{2N}\left(1+\frac{N+1}{N+k}\right)\ ,\\ h(0,\mathrm{f};\mathrm{f})=h(0,\bar{\mathrm{f}};\bar{\mathrm{f}})&=\frac{N^2-1}{2N(N+1)}-\frac{N^2-1}{2N(N+k+1)}=\frac{(N-1)k}{2N(N+k+1)}\ . \end{aligned} \tag{2.93}$$

These representations are characterised by the property that they have the smallest number of states at low level beside the vacuum representation.

The algebras $\mathcal{W}_N$ are actually a special case of an even larger family of algebras called $\mathcal{W}_\infty[\mu]$, where μ is a real parameter. These are generated by the stress-energy tensor L and infinitely many primary conserved currents W^s for each $s \geq 3$. For $\mu = N$ integer, these acquire an ideal and the corresponding quotient gives us back $\mathcal{W}_N$. This is of course reminiscent of the higher spin algebras $\mathfrak{hs}[\mu]$, and indeed we will see that there exists an intimate relationship between these two families of algebras. The $\mathcal{W}_{N,k}$ algebras thus constructed will be studied in more detail in section 2.2.4.

2.2.3 The quantum Drinfel'd-Sokolov reduction

Another, purely algebraic method of constructing quantum $\mathcal{W}$ algebras is the quantum Drinfel'd-Sokolov reduction, the classical version of which was first introduced in [54]. Here we will follow the approach of [58] (see also [28] and references therein). Our starting point is once more an affine Lie algebra $\mathfrak{g}_k$, which we can use to define a WZW model as in the last section, and an affine subalgebra $\mathfrak{h}_{k'}$ by which we gauge (or 'reduce') that model.

In the cases we are interested in it is sufficient to consider a special case of $\mathfrak{h}_{k'}$ (but more general constructions are possible in principle). This special choice is induced by the principally embedded $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{g}$, under which the adjoint representation of $\mathfrak{g}$ decomposes into spin s_i modules of $\mathfrak{sl}(2)$ similarly as in equation (2.38),

$$\mathfrak{g} \to \bigoplus_i D_{s_i} \ , \tag{2.94}$$

where $s_i - 1$ are the exponents of $\mathfrak{g}$ and D_{s_i} is the $(2s_i - 1)$-dimensional irreducible representation of $\mathfrak{sl}(2)$. In particular, $\mathfrak{sl}(2)$ is the spin-2 module D_2 spanned by L_1, L_0 and L_{-1}. In general, we may take a basis T^i_m of D_{s_i} for $m = -s_i + 1, \ldots, s_i - 1$ with

$$[L_m, T^i_n] = \big((s_i - 1)m - n\big)T^i_{m+n} \ , \tag{2.95}$$

and $L_m = T^1_m$. We can then define $\mathfrak{h}$ as the nilpotent 'upper triangular' subalgebra generated by the T^i_m with $m > 0$ (i.e. the weight vectors e_α corresponding to positive roots α), and $\mathfrak{h}_{k'}$ as its affinisation. A $\mathfrak{g}$-valued current with central charge k can be expanded in this basis as follows:

$$J(z) = \sum_i \sum_{m=-s_i+1}^{s_i-1} J^m_i(z) T^i_m \ . \tag{2.96}$$

Furthermore we may define a one-dimensional representation χ on $\mathfrak{h}_{k'}$ by

$$\chi\big((J^m_i)_n\big) = \begin{cases} 1 \ , & \text{for } (i,m,n) = (1,1,-1) \ , \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{2.97}$$

where n is the affine mode number. Now we impose the first-class constraints $h \sim \chi(h)$ for all $h \in \mathfrak{h}_{k'}$, thereby setting $J_1^1(z)$ to 1 and all other $J_i^m(z)$ with $m > 0$ and $i \neq 1$ to 0 (hence the name 'reduction'). In the quantum case this is done using the method of BRST cohomology. We therefore introduce b and c ghosts (of ghost numbers -1 and $+1$, respectively) and construct the BRST charge Q of the constraints. Q then acts on the space of all normal-ordered products of currents in $\mathfrak{g}_k$, ghosts, and their derivatives (the so-called Hecke algebra). As usual in the BRST procedure, the ghost number introduces a $\mathbb{Z}$-grading on the Hecke algebra, and Q carries ghost number 1. Since Q is also nilpotent, $Q^2 = 0$, Q indeed induces a BRST cohomology on the Hecke algebra. The 0th cohomology then defines a $\mathcal{W}$ algebra

$$\mathcal{W}_{\mathrm{DS}}[\mathfrak{g}, k] \equiv H_Q^0(\mathfrak{g}_k, \mathfrak{h}_{k'}, \chi) = \frac{Q\text{-closed fields of ghost number } 0}{Q\text{-exact fields of ghost number } 0} \ , \tag{2.98}$$

which is called the quantum Drinfel'd-Sokolov reduction of $\mathfrak{g}_k$. It is generated by a Virasoro generator and primary fields W^i of conformal dimension s_i. One can explicitly construct the Virasoro generator and obtain the central charge

$$c = \mathrm{rank}\, \mathfrak{g} - 12|\alpha_+\rho + \alpha_-\rho^\vee|^2 \ , \tag{2.99}$$

where ρ and $\rho^\vee$ are, respectively, the Weyl vector and the dual Weyl vector of $\mathfrak{g}$, and we have used the definitions

$$\alpha_\pm = \pm(k + h^\vee)^{\mp\frac{1}{2}} \ . \tag{2.100}$$

The irreducible highest-weight representations of $\mathcal{W}_{\mathrm{DS}}[\mathfrak{g}, k]$ are characterised by highest weights

$$\Lambda = \alpha_+\Lambda_+ + \alpha_-\Lambda_- \ , \tag{2.101}$$

where Λ_+ and Λ_- are, respectively, dominant integral weights and dual dominant integral weights of $\mathfrak{g}$. The conformal dimension of such a representation is then given by

$$h(\Lambda) = \frac{1}{2}\big(\Lambda\, , \Lambda + 2(\alpha_+\rho + \alpha_-\rho^\vee)\big) \ . \tag{2.102}$$

It is interesting to note that the Drinfel'd-Sokolov reduction of a simply-laced Lie algebra $\mathfrak{g}$ at level k_{DS} coincides with the $\mathcal{W}$ algebra of the diagonal coset [28]

$$\frac{\mathfrak{g}_k \oplus \mathfrak{g}_1}{\mathfrak{g}_{k+1}} \ , \tag{2.103}$$

where the levels are related algebraically by

$$\frac{1}{k + h^\vee} = \frac{1}{k_{\mathrm{DS}} + h^\vee} - 1 \ . \tag{2.104}$$

With this relation the central charges (2.84) and (2.99) agree, as one may see using $\rho = \rho^\vee$ for $\mathfrak{g}$ simply laced and the Freudenthal-de Vries strange formula

$$\frac{|\rho|^2}{2h^\vee} = \frac{\dim \mathfrak{g}}{24} \ . \tag{2.105}$$

In particular, this relates the Drinfel'd-Sokolov reduction of $\mathfrak{sl}(N)$ to the $\mathcal{W}_{N,k}$ coset models (2.83).

2.2.4 Bootstrap construction of $\mathcal{W}_\infty[\mu]$

The family $\mathcal{W}_N$ of $\mathcal{W}$ algebras we have seen above can be generalised to non-integer values of N, leading to a family $\mathcal{W}_\infty[\mu]$. These algebras are generated by infinitely many primary fields W^s with $s = 3, 4, 5, \dots$. Their classical versions $\mathcal{W}_\infty^{\mathrm{cl}}$ [61, 117, 118] are Poisson algebras that were obtained in [116, 81, 32] as the asymptotic symmetry algebras of $\mathfrak{hs}[\mu]$ higher spin theories and we are therefore interested in their quantisation. However, upon quantising them by replacing Poisson brackets by commutators, the structure constants receive quantum corrections due to the non-linear terms which need to be turned into normal-ordered products. In principle $\mathcal{W}_\infty[\mu]$ should be obtained from $\mathfrak{hs}[\mu]$ by a generalised Drinfel'd-Sokolov reduction, since $\mathcal{W}_N$ can be obtained from $\mathfrak{sl}(N)$ in this way; and it should reduce to $\mathcal{W}_N$ for $\mu = N$ in the same way that $\mathfrak{hs}[\mu]$ reduces to $\mathfrak{sl}(N)$. However, it is not quite clear how to perform a Drinfel'd-Sokolov reduction on infinite-dimensional Lie algebras.

In the following, we will therefore pursue a different path and construct these algebras explicitly by studying the most general $\mathcal{W}$ algebra generated by primary fields W^s of spin $s = 3, 4, 5, \dots$. This section is based on a collaboration with Constantin Candu, Matthias Gaberdiel and Carl Vollenweider. The results were not published at first, but mentioned in [73] and rederived in [140], before the original work was finally fully explained in [166].

The most general ansatz[6] for the OPEs of these primaries reads

$$\begin{aligned}
W^3 \star W^3 &\sim c_{33}^4 W^4 + n_3 I \\
W^3 \star W^4 &\sim c_{34}^5 W^5 + c_{34}^3 W^3 \\
W^3 \star W^5 &\sim c_{35}^6 W^6 + a_{35}^6 A^6 + c_{35}^4 W^4 \\
W^4 \star W^4 &\sim c_{44}^6 W^6 + a_{44}^6 A^6 + c_{44}^4 W^4 + n_4 I \\
W^3 \star W^6 &\sim a_{36}^{8,1} A^{8,1} + c_{36}^7 W^7 + a_{36}^7 A^7 + c_{36}^5 W^5 + c_{36}^3 W^3
\end{aligned}$$

[6] Actually, an even more general ansatz would include more fields on the right-hand side, which are however excluded by parity symmetry, as will be explained shortly. We have checked explicitly that the additional structure constants in such an extended ansatz are forced to vanish by Jacobi identities.

$$
\begin{aligned}
W^4 \star W^5 &\sim a_{45}^{8,1} A^{8,1} + c_{45}^{7} W^7 + a_{45}^{7} A^7 + c_{45}^{5} W^5 + c_{45}^{3} W^3 \\
W^3 \star W^7 &\sim a_{37}^{9,1} A^{9,1} + c_{37}^{8} W^8 + a_{37}^{8,2} A^{8,2} + a_{37}^{8,3} A^{8,3} + a_{37}^{8,4} A^{8,4} \\
&\quad + c_{37}^{6} W^6 + a_{37}^{6} A^6 + c_{37}^{4} W^4 \\
W^4 \star W^6 &\sim a_{46}^{9,1} A^{9,1} + c_{46}^{8} W^8 + a_{46}^{8,2} A^{8,2} + a_{46}^{8,3} A^{8,3} + a_{46}^{8,4} A^{8,4} \\
&\quad + c_{46}^{6} W^6 + a_{46}^{6} A^6 + c_{46}^{4} W^4 \\
W^5 \star W^5 &\sim c_{55}^{8} W^8 + a_{55}^{8,2} A^{8,2} + a_{55}^{8,3} A^{8,3} + a_{55}^{8,4} A^{8,4} \\
&\quad + c_{55}^{6} W^6 + a_{55}^{6} A^6 + c_{55}^{4} W^4 + n_5 I \ .
\end{aligned}
\tag{2.106}
$$

The composite fields A^s are defined in terms of normal-ordered products of elementary primaries and corrections to make the fields primary:[7]

$$
\begin{aligned}
A^6 &= (W^3)^2 + \dots \\
A^7 &= W^3 W^4 + \dots \\
A^{8,1} &= \tfrac{4}{7} W^{3\prime} W^4 - \tfrac{3}{7} W^3 W^{4\prime} + \dots \\
A^{8,2} &= \tfrac{3(c+48)}{13c+516} W^{3\prime\prime} W^3 - \tfrac{7c+228}{2(13c+516)} (W^{3\prime})^2 + \dots \\
A^{8,3} &= W^3 W^5 + \dots \\
A^{8,4} &= (W^4)^2 + \dots \\
A^{9,1} &= \tfrac{5}{8} W^{3\prime} W^5 - \tfrac{3}{8} W^3 W^{5\prime} + \dots \ .
\end{aligned}
\tag{2.107}
$$

Notice that the ansatz (2.106) already incorporates three facts which seem to be true for $\mathcal{W}_\infty[\mu]$: First, as already observed and explained in the discussion of the $\mathcal{W}_3$ algebra in section 2.2.1, no primaries of odd conformal dimension can appear in an OPE of a field W^s with itself.

Second, we have used that in any CFT the identity I can only appear in OPEs of two primary fields of the same conformal dimension.

Third, the parity map which sends

$$
L \mapsto L \ , \quad W^s \mapsto (-1)^s W^s \tag{2.108}
$$

and respects the normal-ordered product is actually a Lie algebra automorphism. Therefore the right-hand side of an OPE $W^{s_1} \star W^{s_2}$ contains only fields[8] with parity charge $(-1)^{s_1+s_2}$. This explains why the OPEs $W^3 \star W^6$ and $W^4 \star W^5$ contain only the odd composite primary of spin 8, $A^{8,1}$, whereas $W^3 \star W^7$, $W^4 \star W^6$ and $W^5 \star W^5$ contain only the even ones. Also, the even composite primaries of spin 9 cannot appear in the singular part of the OPEs (2.106), and are therefore not explicitly listed here. Although we have checked that parity is enforced by Jacobi

[7] The full expressions can be found in appendix A.1.

[8] This is true for all fields appearing in the OPE, not just primary ones.

identities up to the level to which we have constructed $\mathcal{W}_\infty[\mu]$, an actual proof is missing as of yet.

We now follow a bootstrap approach to determine all unknown structure constants in the ansatz (2.106). By imposing Jacobi identities on the commutators of primary fields (as already done for some finitely generated $\mathcal{W}$ algebras in [113, 22, 109])

$$[[W^{s_1}_m, W^{s_2}_n], W^{s_3}_l] + [[W^{s_2}_n, W^{s_3}_l], W^{s_1}_m] + [[W^{s_3}_l, W^{s_1}_m], W^{s_2}_n] = 0 \ , \tag{2.109}$$

we obtain constraints on the structure constants which determine all of them uniquely, except for those that encode a field redefinition freedom, and a free parameter $\gamma = c^4_{33}$. The free structure constant γ is related to the parameter μ of the algebra; and different values of γ^2 lead, in general, to non-isomorphic algebras (since multiplying W^4 by a minus sign also sends $\gamma \mapsto -\gamma$, only γ^2 is a real parameter of the algebra). As for field redefinitions, each elementary primary W^s can be rescaled by any non-zero value without affecting its OPE with the stress-energy tensor. So for each W^s, there is one normalisation constant which we may fix at will and which is not determined by Jacobi identities. For $s = 3, 4$ we choose this normalisation constant to be n_s. Since n_6, n_7 and n_8 do not make an appearance in (2.106) yet, we choose c^6_{35}, c^7_{36} and c^8_{37} as normalisation constants for W^6, W^7 and W^8, respectively. For $s = 5$, it turns out that c^5_{34} is a good choice, whereas choosing n_5 would lead to solutions with several branches. In addition to this rescaling freedom, we can redefine an elementary primary W^s by adding a multiple of a composite primary of the same conformal dimension and parity:

$$\begin{aligned}
&W^6 \mapsto \hat{W}^6 \equiv W^6 + \alpha A^6 \ , \\
&W^7 \mapsto \hat{W}^7 \equiv W^7 + \beta A^7 \ , \\
&W^8 \mapsto \hat{W}^8 \equiv W^8 + \gamma_2 A^{8,2} + \gamma_3 A^{8,3} + \gamma_4 A^{8,4} \ .
\end{aligned} \tag{2.110}$$

Hence for each of the composite fields on the right-hand side of (2.110) we have an undetermined structure constant which we may set to zero if we wish. We choose these structure constants to be a^6_{35}, a^7_{36}, $a^{8,2}_{37}$, $a^{8,3}_{37}$, and $a^{8,4}_{37}$. For example, we can set $\hat{a}^6_{35} = 0$ by choosing $\alpha = a^6_{35}/c^6_{35}$. We would like to emphasise that this can be done only once for each composite field. For instance, once we have set $\hat{a}^6_{35}$ to zero, other structure constants such as $\hat{a}^6_{44}$ will be forced to take a specific value which is in general different from zero. Also note that we do not have this additional freedom for composite fields whose spin and parity do not agree. Indeed, the field $A^{8,1}$ can never appear in the same OPEs as W^8, and similarly for $A^{9,1}$ and W^9, which therefore makes it impossible to absorb these fields.

We can now turn to solving the Jacobi identities. Rather than doing this by hand, we have used OPE language and the Mathematica packages `OPEdefs` and

`OPEconf` by Thielemans (see [156, 157] for a documentation),[9] following the analysis of [109]. Writing $[AB]_n(z)$ for the nth order pole of the OPE of A and B, i.e.

$$A(z)B(w) = \sum_{n \leq h_A + h_B} \frac{[AB]_n(w)}{(z-w)^n} \ , \tag{2.111}$$

the Jacobi identity may be re-expressed by the following OPE identity derived from crossing symmetry of the fields [150] (see also [157]):

$$[A[BC]_p]_q = (-1)^{|A||B|}[B[AC]_q]_p + \sum_{l=1}^{\infty} \binom{q-1}{l-1} [[AB]_l C]_{p+q-l} \ , \tag{2.112}$$

where $|A|$ and $|B|$ are the fermion numbers of A and B, which vanish for the bosonic algebra $\mathcal{W}_\infty[\mu]$. The $q > 0$ terms of this identity are equivalent to the Jacobi identity and are implemented in the `OPEJacobi` function of the `OPEdefs` package. This enables us to solve the Jacobi identities order by order. Any Jacobi identity $[W^{s_1}, W^{s_2}, W^{s_3}]$ requires OPEs $W^{t_1} \star W^{t_2}$ up to

$$t_1 + t_2 \leq s_1 + s_2 + s_3 - 2 \ . \tag{2.113}$$

On the other hand, it turns out that the set of Jacobi identities with $s_1+s_2+s_3 \leq s$ for some integer s determines all structure constants appearing in the OPEs with $t_1+t_2 \leq s-2$, unless they are free in one of the senses explained above. Proceeding in this way, we notice that the first Jacobi identity, $[W^3, W^3, W^3]$, is trivially satisfied by the ansatz (2.106). The remaining Jacobi identities $[W^{s_1}, W^{s_2}, W^{s_3}]$ with $s_1 + s_2 + s_3 \leq 12$ are then $[W^3, W^3, W^4]$, $[W^3, W^4, W^4]$, $[W^3, W^3, W^5]$, $[W^3, W^3, W^6]$, $[W^3, W^4, W^5]$, and $[W^4, W^4, W^4]$. They lead to 33 independent equations. We started with 45 structure constants in (2.106), of which one, γ, is a free parameter and 11 are unfixed field redefinition constants. Therefore all 33 remaining (non-free) structure constants can be solved for, and the solutions can be found in appendix A. This leads to a 1-parameter family $\mathcal{W}_\infty(\gamma)$ of non-isomorphic algebras for each value of the central charge c.

2.2.5 Minimal representation and the triality relations

In the last section we found a 1-parameter family of $\mathcal{W}_\infty$ algebras, where the free parameter was the structure constant $\gamma = c_{33}^4$. We would like to relate γ to the parameter μ appearing in the family $\mathcal{W}_\infty[\mu]$. Following [73], this is most easily done by analysing the algebra as acting on a specific representation, the simplest examples being the minimal representations (2.90). As already mentioned in the

[9] We thank K. Thielemans for providing us with the packages.

discussion of section 2.2.2, these are characterised by having the fewest states at low level, aside from the vacuum representation. These representations consist of a highest weight state ϕ_{min} with conformal dimension h, its derivatives and its $\mathcal{W}$ descendants. They therefore possess the character

$$\chi_{\text{min}}(q) = \frac{q^h}{1-q} \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{1-q^n} = q^h \left(1 + q + 2q^2 + \ldots\right) . \tag{2.114}$$

Since the representation possesses only a single state at level 1, all states of the form $W^s_{-1}\phi_{\text{min}}$ must be proportional to $L_{-1}\phi_{\text{min}}$. Similarly, all states of the form $W^s_{-2}\phi_{\text{min}}$ must be a linear combination of two linearly independent states, $L_{-2}\phi_{\text{min}}$ and $L^2_{-1}\phi_{\text{min}}$. This gives us an infinity of null relations, from which one can deduce a relation between the conformal dimension h_{min} of the minimal representation and the parameter γ. Now we know that for $\mu = N$, our CFT reduces to the $\mathcal{W}_N$ minimal models, whose minimal representations $(\text{f};0)$ and $(0;\text{f})$ are well-known. We can then simply plug their μ-dependent conformal dimensions (where we analytically continue N to μ) into the formula derived above and obtain the triality relation

$$\gamma^2 = \frac{144\,(c+2)(\mu-3)\big(c(\mu+3)+2(4\mu+3)(\mu-1)\big)\,n_3^2}{c(5c+22)(\mu-2)\big(c(\mu+2)+(3\mu+2)(\mu-1)\big)\,n_4} . \tag{2.115}$$

This equation is cubic in μ and therefore has three roots $\mu_{1,2,3}$ for any given value of γ^2 and c, which satisfy

$$\mathcal{W}_\infty[\mu_1] \cong \mathcal{W}_\infty[\mu_2] \cong \mathcal{W}_\infty[\mu_3] . \tag{2.116}$$

In particular, setting $\mu = N$ and $c = c_{N,k}$ (formally allowing N and k to take any real value), the three roots read

$$\mu_1 = N , \quad \mu_2 = \frac{N}{N+k} , \quad \mu_3 = -\frac{N}{N+k+1} . \tag{2.117}$$

This triality relation shows that the symmetry algebra $\mathcal{W}_\infty[\mu = N]$ of the $\mathcal{W}_{N,k}$ coset model is equivalent to the algebra $\mathcal{W}_\infty[\mu = \lambda]$, where λ is given by the ratio

$$\lambda = \frac{N}{N+k} . \tag{2.118}$$

This result will be of utter importance for the duality introduced in the following section. It is also a conformation and generalisation of the *level–rank duality* proposed in [125, 7]. The latter claims the equivalence of the cosets

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \cong \frac{\mathfrak{su}(M)_l \oplus \mathfrak{su}(M)_1}{\mathfrak{su}(M)_{l+1}} , \tag{2.119}$$

where the positive integers N and M and the real numbers k and l are related by

$$k = \frac{N}{M} - N \ , \quad l = \frac{M}{N} - M \ . \tag{2.120}$$

Indeed, both pairs of parameters lead to the same central charge $c_{N,k} = c_{M,l}$, and solving for M yields

$$M = \frac{N}{N+k} \ , \tag{2.121}$$

so this is just a special case of the triality stated above.

2.3 $\mathcal{W}_{N,k}$ minimal model holography

We shall now formulate the holographic duality proposed in [72], thereby mostly following the review [74]. Based on the asymptotic symmetry calculations of [106, 33], it was claimed that higher spin gravity described by the $\mathfrak{hs}[\lambda]$ Chern-Simons theory on AdS_3 is holographically dual to the $\mathcal{W}_{N,k}$ minimal model CFTs (2.83) in the large N and k 't Hooft limit where

$$\lambda = \frac{N}{N+k} \ , \qquad 0 \le \lambda \le 1 \tag{2.122}$$

is kept fixed and corresponds to the gravity coupling parameter. In order to match the gravity spectrum with the one of the CFT, it was originally proposed in [72] to couple the gravity theory to two complex scalars with mass

$$M^2 = \lambda^2 - 1 \ . \tag{2.123}$$

However, as we will see in more detail below, it was later noted [74] that one of them corresponds to a non-perturbative degree of freedom and should be removed from the (perturbative) formulation of the duality. Therefore only one of the scalars was retained.

As already alluded to at the end of section 2.2.4, the triality relation (2.117) has an interesting consequence for the duality at finite central charge: by virtue of the triality, the algebras $\mathcal{W}_\infty[\mu = \lambda]$ and $\mathcal{W}_\infty[\mu = N]$, where λ is given by (2.122) and both algebras are taken at the same value of $c = c_{N,k}$, are in fact isomorphic. But $\mathcal{W}_\infty[\mu = N] \cong \mathcal{W}_N$ is the symmetry algebra of the coset (2.83) at *finite* N, which truncates at spin N. This enables us to conjecture the nature of the holographic duality away from the 't Hooft limit, and in particular in the quantum regime at finite N and k, where $c = c_{N,k}$ is now also finite. Since the classical algebra $\mathcal{W}_\infty^{\mathrm{cl}}[\mu]$ possesses a unique quantisation at any value of μ as shown in section 2.2.4, the $\mathfrak{hs}[\lambda]$ theory should then be dual to the $\mathcal{W}_{N,k}$ minimal models even at the quantum level, where once more $\lambda = N/(N+k)$.

The holographic conjecture has passed several non-trivial tests, which we are going to summarise in the following:

Asymptotic symmetries: It was shown by Brown and Henneaux [31] that the asymptotic symmetries of $\mathfrak{sl}(2)$ Chern-Simons gravity on an AdS_3 background as described in section 2.1.1 are given by two copies of the Virasoro algebra with central charge

$$c = \frac{3\ell}{2G} \ , \tag{2.124}$$

where ℓ is the AdS radius and G is the 3-dimensional Newton constant. This result was generalised to higher spin gravity in [106, 33] and [81, 32] (see also [116] for earlier work), where it was shown that the chiral asymptotic symmetry algebra of $\mathfrak{hs}[\mu]$ Chern-Simons gravity on AdS_3 is the classical $\mathcal{W}_\infty^{\mathrm{cl}}[\mu]$ algebra, which coincides with the quantum $\mathcal{W}_\infty[\mu]$ algebra in the semiclassical limit $c \to \infty$. The central charge of the boundary CFT is given by (2.124) as before.

Algebraically, one can in principle obtain $\mathcal{W}_\infty[\mu]$ from $\mathfrak{hs}[\mu]$ by a generalisation of the quantum Drinfel'd-Sokolov reduction to infinite-dimensional Lie algebras. The inverse operation has been described by Bowcock and Watts in [30] and consists in taking $c \to \infty$ and restricting to wedge modes $|m| < s$. We therefore call the original algebra the 'wedge algebra' of the asymptotic symmetry algebra. Note that for finite c, the wedge algebra need not be a subalgebra. Indeed, it was shown in [81] that the wedge algebra of $\mathcal{W}_\infty^{\mathrm{cl}}[\mu]$ is $\mathfrak{hs}[\mu]$, but it is not a subalgebra for $\mu \neq \pm 1$ and finite c due to the non-linearities which are suppressed by powers of $1/c$.

Spectrum: If the duality between the higher spin theory on AdS_3 and the minimal models is to hold, their partition functions need to agree. In order to evaluate the perturbative partition function of the gravity theory, it was shown in [80] (based on the techniques from [51]) that the 1-loop partition function on thermal AdS_3 receives the contribution

$$Z_{(s)}^{\text{1-loop}} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \tag{2.125}$$

from the spin s field. The total gauge contribution to the perturbative partition function is then the product of all spins $s \geq 2$,

$$Z_{\mathrm{hs}}^{\text{1-loop}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \equiv |\tilde{M}(q)|^2 \ , \tag{2.126}$$

where we have defined the modified MacMahon function

$$\tilde{M}(q) = \prod_{n=2}^{\infty} \frac{1}{(1-q^n)^{n-1}} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty} \frac{1}{1-q^n} \ . \tag{2.127}$$

This function corresponds exactly to the vacuum character of the CFT, $\chi_{(0;0)}$, as one can see by counting the states

$$W^{s_1}_{n_1} \cdots W^{s_m}_{n_m}\,\Omega \tag{2.128}$$

in the Poincaré-Birkhoff-Witt basis, where $s_1 \geq \cdots \geq s_m$ and $n_i \leq -s_i$, and Ω is the vacuum state.

The CFT also contains higher representations, which can be described by pairs $(\Lambda_+;\Lambda_-)$, where Λ_+ is a representation of the factor $\mathfrak{su}(N)_k$ in the numerator and Λ_- is a representation of the denominator $\mathfrak{su}(N)_{k+1}$. The representation of the $\mathfrak{su}(N)_1$ factor is then uniquely determined by these two as explained in section 2.2.2. The simplest representations of this kind are then the minimal representations

$$(\mathrm{f};0)\ ,\quad (0;\mathrm{f})\ , \tag{2.129}$$

as well as their conjugates $(\bar{\mathrm{f}};0)$ and $(0;\bar{\mathrm{f}})$. In the 't Hooft limit their conformal dimensions as computed in (2.93) become

$$h(\mathrm{f};0) = h(\bar{\mathrm{f}};0) \cong \frac{1}{2}(1+\lambda)\ ,\quad h(0;\mathrm{f}) = h(0;\bar{\mathrm{f}}) \cong \frac{1}{2}(1-\lambda)\ . \tag{2.130}$$

Making contact with the higher spin theory at $\lambda = \pm N$, one may be tempted to take the so-called semiclassical limit, where N is kept fixed while $c \to \infty$ (this corresponds formally to taking k to $-(N+1)$, or $\lambda = -N$). While this limit seems illegal from the point of view of the coset construction, it makes perfect sense from the perspective of the algebra $\mathcal{W}_\infty[\mu = N, c]$, which can be seen as an analytic continuation of $\mathcal{W}_{N,k}$. In this limit, (2.93) now becomes

$$h(\mathrm{f};0) = h(\bar{\mathrm{f}};0) \sim -\frac{1}{2}(N-1)\ ,\quad h(0;\mathrm{f}) = h(0;\bar{\mathrm{f}}) \sim -\frac{c}{2N^2}\ . \tag{2.131}$$

Only $h(\mathrm{f};0)$ remains finite, whereas $h(0;\mathrm{f})$ scales as $-c$. We therefore view $(\mathrm{f};0)$ as a perturbative state generating an infinity of higher states $(\Lambda_+;0)$. The coset representation $(0;\mathrm{f})$ and its tensor powers $(0;\Lambda_-)$, on the other hand, are non-perturbative in this regime and are believed to correspond to bound states of conical defect (or surplus) solutions with perturbative states in the bulk [40, 137].

We will therefore only keep the minimal representations $(\mathrm{f};0)$ and $(\bar{\mathrm{f}};0)$ in the perturbative spectrum. On the bulk side we can add a complex scalar to the higher spin theory (see [141, 142]), whose mass is fixed to be[10]

$$M^2 = \lambda^2 - 1 \ . \tag{2.132}$$

This mass is therefore related to the conformal dimension $\Delta = h + \bar{h} = 1 + \lambda$ of $(\mathrm{f};0)$ or $(\bar{\mathrm{f}};0)$ by

$$M^2 = \Delta(\Delta - 2) \ , \tag{2.133}$$

which corresponds precisely to the relation between mass and boundary conformal dimensions in the AdS/CFT dictionary.[11] The 1-loop partition function of this complex scalar in the bulk can be computed [91] to be

$$Z^{\text{1-loop}}_{\text{scalar}}(q) = \prod_{n,n'=0}^{\infty} \frac{1}{|1 - q^{h+n}\bar{q}^{h+n'}|^2} \ . \tag{2.134}$$

The perturbative CFT partition function, on the other hand, can be defined by

$$Z^{\text{CFT}}_{\text{pert}} = \sum_{\Lambda} |\chi_{(\Lambda;0)}|^2 \ , \tag{2.135}$$

where the sum is over all Young diagrams that have a finite number of boxes and antiboxes as $N, k \to \infty$. In the 't Hooft limit, these characters simplify considerably and the CFT partition function can be shown [79] to read

$$Z^{\text{CFT}}_{\text{pert}} = |\tilde{M}(q)|^2 \cdot \prod_{n,n'=0}^{\infty} \frac{1}{|1 - q^{h+n}\bar{q}^{h+n'}|^2} \tag{2.136}$$

in this limit. This corresponds exactly to the 1-loop partition function in the bulk, which is an important piece of evidence for the proposed duality.

Correlation functions: Having compared the symmetries and the spectrum of the two theories, it is a much more non-trivial check of the proposed duality to compare their correlation functions. For the 3-point functions between the perturbative complex scalar $\mathcal{O} = (\mathrm{f};0)$, its conjugate $\bar{\mathcal{O}}$ and a spin-s current $J^{(s)}$ in the 't Hooft limit,

$$\langle \mathcal{O}(z_1)\bar{\mathcal{O}}(z_2)J^{(s)}(z_3)\rangle \ , \tag{2.137}$$

[10] Although this mass square is negative for $\lambda < 1$, it satisfies the Breitenlohner-Freedman bound $M^2 \geq -1$ on AdS_3. Note that we have set the AdS radius to $\ell = 1$.

[11] Note that the conformal dimension $\Delta = h + \bar{h} = 1 - \lambda$ of the non-perturbative representation $(0;\mathrm{f})$ also solves this equation, which led to the original proposal with two complex scalars.

agreement with the corresponding boundary 3-point function of the higher spin theory was shown in [10] at arbitrary coupling λ (see also [42, 4] for earlier work).

The higher-point functions of the CFT seem to factorise for large N [136, 43], therefore reducing to the case of 2- and 3-point functions. In particular, the perturbative states of the form $(\Lambda; 0)$ form a closed subsector of the theory and behave like multi-particle states of the minimal representation $(\mathrm{f}; 0)$ at large N. This agrees nicely with the behaviour of the perturbative matter states in the bulk, which are also multi-particle states of the complex scalar.

Black hole entropy: Another non-trivial check of the holographic duality is the comparison of the higher spin version of BTZ black holes to the CFT (for a review of black holes in 3-dimensional higher spin gravity see [9]). In higher spin gravity diffeomorphism-invariant quantities such as geodesic length and the Ricci scalar are not invariant under general gauge transformations, and neither are the existence of a horizon or even the causal structure. It is therefore not clear a priori how to define black holes using these concepts. A way out has been found by generalising the BTZ solution [13, 12] of a black hole in ordinary 3d gravity, which is constructed as an orbifold of AdS_3. This solution looks locally like AdS_3, but has all the features of a black hole such as mass, charge, angular momentum, a horizon, entropy and temperature. Since the BTZ black hole is also a solution of higher spin gravity, a sensible and gauge invariant definition for a higher spin black hole was argued to be that the holonomies of the gauge connection around a contractible Euclidean time cycle should be equal to the ones of the BTZ black hole. With this definition, black holes dressed with additional higher spin charges could be constructed in [99, 124]. Any solution defined in this way exhibits a smooth horizon after a suitable gauge transformation [8], which is a necessary condition for a black hole to satisfy the first law of thermodynamics.

The partition function of the $\mathcal{W}_{N,k}$ minimal model CFT in the 't Hooft limit with insertion of the spin-3 chemical potential α is

$$Z_{\mathrm{CFT}}(\hat{\tau}, \alpha) = \mathrm{Tr}\left(\hat{q}^{L_0 - \frac{c}{24}} y^{W_0}\right) \ , \quad \hat{q} = e^{2\pi i \hat{\tau}} \ , \ y = e^{2\pi i \alpha} \ . \tag{2.138}$$

Here W_0 is the zero-mode of the spin 3 field, and the complex modular parameter $\hat{\tau}$ is proportional to the inverse of the temperature T,

$$\hat{\tau} \propto \frac{1}{T} \ . \tag{2.139}$$

This partition function was computed perturbatively up to the 6th power in y and in the high temperature ($\hat{\tau} \to 0$) limit in [82]. The result matches precisely the result from the bulk side as computed in [124]. Matching the partition function is tantamount to matching the entropy

$$S = (1 - \beta\partial_\beta) \log Z \ , \tag{2.140}$$

which therefore also agrees. The computation hence generalises the Cardy formula for the entropy of a CFT to a higher spin setup.

This agreement of the partition functions is a further noteworthy check of the duality, in particular because it takes place at a different point in parameter space than the previous checks and the bulk side contains a non-perturbative object (the black hole). Moreover, it also shows that the large-temperature limit of the bulk is dominated by black holes.

Chapter 3

Even spin holography

In the last chapter we have given an introduction to the minimal model holography relating a higher spin theory on 3-dimensional anti-de-Sitter space, based on $\mathfrak{hs}[\lambda]$ and coupled to a complex boson, to the $\mathcal{W}_{N,k}$ minimal models in the 't Hooft limit. In this chapter we will extend these results to the higher spin theory on AdS_3 that contains only gauge fields of even spin. This is the natural analogue of the Klebanov & Polyakov proposal, which involves the smallest (or minimal) higher spin theory on AdS_4. For the case of AdS_3, the gauge symmetry can be described by a Chern-Simons theory based on a suitable subalgebra of $\mathfrak{hs}[\mu]$, and it was argued in [3, 88] (see also [5]) that it should be dual to the $\mathrm{SO}(N)$ coset theories of the form

$$\frac{\mathfrak{so}(N)_k \oplus \mathfrak{so}(N)_1}{\mathfrak{so}(N)_{k+1}} \ .$$

While the (classical) asymptotic symmetry algebra of the bulk theory has not yet been determined explicitly, it is clear that it will be described by a classical $\mathcal{W}$ algebra that is generated by one field for every even spin $s = 2, 4, \dots$. One expects on general grounds that it will be non-linear, and hence the quantisation will exhibit the same subtleties as described above. As a consequence, it is actually simpler to approach this problem by constructing directly the most general *quantum* $\mathcal{W}$ algebra $\mathcal{W}^e_\infty[\mu]$ with this spin content. As in the case of $\mathcal{W}_\infty[\mu]$, one finds that the successive Jacobi identities fix the structure constants of all commutators in terms of a single parameter γ, as well as the central charge. For a suitable identification of γ and μ, we can then think of these algebras as the quantum Drinfel'd-Sokolov reduction of some subalgebra of $\mathfrak{hs}[\mu]$, which turns out to be the $\mathfrak{hs}^e[\mu]$ algebra of [88]. However, compared to the $\mathcal{W}_\infty[\mu]$ analysis of [73], there is an unexpected subtlety in that there are two natural ways in which one may identify γ and μ at finite c — the two identifications agree in the quasiclassical $c \to \infty$ limit, but differ in their $1/c$ corrections. This reflects the fact that $\mathfrak{hs}^e[\mu]$ truncates for $\mu = N$ to either $\mathfrak{sp}(N)$ (if N is even), or $\mathfrak{so}(N)$ (if N is odd), and that the Drinfel'd-Sokolov reduction of these non-simply-laced algebras are Langlands dual rather than equivalent (i.e. their Dynkin diagrams can be obtained from one another by exchanging short and long roots).

Since the quantum algebra $\mathcal{W}^e_\infty[\mu]$ is the most general $\mathcal{W}$ algebra with the given spin content, we can also identify the $\mathfrak{so}$ and $\mathfrak{sp}$ cosets (or rather their orbifolds) with these algebras. In this way we obtain again non-trivial identifications between quantum $\mathcal{W}^e_\infty$ algebras that explain and refine the holographic conjectures of [3] and [88], see equations (3.56) and (3.57) below. Furthermore, there are again non-trivial quantum equivalences between the algebras for different values of μ, which can be interpreted in terms of level–rank dualities of $\mathfrak{so}$ coset models that do not seem to have been noticed before, see equation (3.54).

This chapter is based on a joint paper with Constantin Candu, Matthias Gaberdiel and Carl Vollenweider [36]. It is organised as follows. In section 3.1 we construct the most general quantum $\mathcal{W}^e_\infty$ algebra, and explain how the different structure constants can be determined recursively from the Jacobi identities. We also consider various truncations to finitely generated algebras that have been studied in the literature before (see section 2.2), and explain that the wedge algebra of $\mathcal{W}^e_\infty$ is indeed the $\mathfrak{hs}^e[\mu]$ algebra of [88]. Section 3.2 is devoted towards identifying $\mathcal{W}^e_\infty[\mu]$ as a Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$. As in [73] the relation between the two algebras can be most easily analysed by studying some simple representations of the two algebras. It turns out that there is no canonical identification, but rather two separate choices that we denote by $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$, respectively; this nomenclature reflects the origin of this ambiguity, namely that $\mathfrak{hs}^e[\mu]$ truncates to either $C_n = \mathfrak{sp}(2n)$ or $B_n = \mathfrak{so}(2n+1)$, depending on whether $\mu = N$ is even or odd.

In section 3.3 we apply these results to the actual higher spin holography. In particular, we show that the (subalgebras of the) $\mathfrak{so}$ cosets fit into this framework, and hence deduce the precise relation between $\mathcal{WB}_\infty[\mu]$ or $\mathcal{WC}_\infty[\mu]$ and the $\mathfrak{so}$ coset algebras at finite N and k. We comment on the fact that the matching of the partition functions requires that we consider a non-diagonal modular invariant with respect to the orbifold subalgebra of the $\mathfrak{so}$ cosets (see section 4.6). We also explain that the non-trivial identifications among the $\mathcal{W}^e_\infty$ algebras imply a level–rank duality for the $\mathfrak{so}$ cosets themselves, and that also the cosets based on $\mathfrak{sp}(2n)$ and $\mathfrak{osp}(1|2n)$ can be brought into the fold. Finally, we show that, as in the case of the $\mathfrak{su}(N)$ cosets, only one of the two real scalars in the bulk theory should be thought of as being perturbative. Some of the more technical material has been collected in appendix B.

3.1 The even spin algebra

3.1.1 Construction

In this section we analyse the most general $\mathcal{W}_\infty$ algebra $\mathcal{W}_\infty^{\mathrm{e}}$ that is generated by the stress energy tensor L and one Virasoro-primary field W^s for each even spin $s = 4, 6, \ldots$. As we shall see, the construction allows for one free parameter in addition to the central charge.

The strategy of our analysis is as follows. First we make the most general ansatz for the OPEs of the generating fields W^s with each other. In a second step we then impose the constraints that come from solving the various Jacobi identities. Actually, instead of working directly in terms of modes and Jacobi identities, it is more convenient to do this analysis on the level of the OPEs. Then the relevant condition is that the OPEs are associative, as explained in 2.2.4.

3.1.1.1 Ansatz for OPEs

We know on general grounds that the conformal symmetry, i.e. the associativity of the OPEs involving the stress energy tensor L, fixes the coefficients of the Virasoro-descendant fields in the OPEs in terms of the Virasoro-primary fields. In order to make the most general ansatz we therefore only have to introduce free parameters for the coupling to the Virasoro-primary fields. Thus we need to know how many Virasoro-primary fields the algebra $\mathcal{W}_\infty^{\mathrm{e}}$ contains. This can be determined by decomposing the vacuum character of $\mathcal{W}_\infty^{\mathrm{e}}$

$$\chi_\infty(q) = \mathrm{Tr}_0 q^{L_0} = \prod_{s\in 2\mathbb{N}} \prod_{n=s}^{\infty} \frac{1}{1-q^n} = \chi_0(q) + \sum_{h=4}^{\infty} d(h)\chi_h(q) , \tag{3.1}$$

in terms of the Virasoro characters corresponding to the vacuum representation $\chi_0(q)$, and to a highest-weight representation with conformal dimension h

$$\chi_0(q) = \prod_{n=2}^{\infty} \frac{1}{1-q^n} , \qquad \chi_h(q) = q^h \prod_{n=1}^{\infty} \frac{1}{1-q^n} . \tag{3.2}$$

Note that since we are working at a generic central charge, there are no Virasoro null-vectors. The coefficients $d(h)$ in (3.1) are then the number of Virasoro-primary fields of conformal dimension h. Their generating function equals

$$P(q) = \sum_{h=4}^{\infty} d(h)q^h = (1-q)(\chi_{\mathrm{hs}}(q)-1) = q^4+q^6+2q^8+3q^{10}+q^{11}+6q^{12}+\cdots , \tag{3.3}$$

where $\chi_{\mathrm{hs}}(q) = \chi_\infty(q)/\chi_0(q)$ denotes the contribution of the higher-spin fields to the character χ_∞.

The most general ansatz for the OPEs is then

$$
\begin{aligned}
W^4 \star W^4 &\sim c_{44}^6 W^6 + c_{44}^4 W^4 + n_4 I \ , \\
W^4 \star W^6 &\sim c_{46}^8 W^8 + a_{46}^8 A^8 + c_{46}^6 W^6 + c_{46}^4 W^4 \ , \\
W^4 \star W^8 &\sim a_{48}^{11} A^{11} + c_{48}^{10} W^{10} + a_{48}^{10,1} A^{10,1} + a_{48}^{10,2} A^{10,2} + c_{48}^8 W^8 + a_{48}^8 A^8 \\
&\qquad + c_{48}^6 W^6 + c_{48}^4 W^4 \ , \\
W^6 \star W^6 &\sim c_{66}^{10} W^{10} + a_{66}^{10,1} A^{10,1} + a_{66}^{10,2} A^{10,2} + c_{66}^8 W^8 + a_{66}^8 A^8 + c_{66}^6 W^6 \\
&\qquad + c_{66}^4 W^4 + n_6 I \ ,
\end{aligned}
\tag{3.4}
$$

where we have only written out the contributions of the Virasoro primaries to the singular part of the OPEs. (As mentioned before, the conformal symmetry fixes the contributions of their Virasoro descendants uniquely.) Furthermore, A^8, $A^{10,1}$, $A^{10,2}$, A^{11} are the composite primary fields at level 8, 10, 11, respectively, as predicted by (3.3). They are of the form

$$
\begin{aligned}
A^8 &= \left(W^4\right)^2 + \cdots \ , \qquad & A^{10,1} &= W^{4\prime\prime} W^4 - \frac{9(48+c)}{8(64+c)} W^{4\prime} W^{4\prime} + \cdots \ , \\
A^{10,2} &= W^4 W^6 + \cdots \ , \qquad & A^{11} &= W^4 W^{6\prime} - \frac{3}{2} W^{4\prime} W^6 + \cdots \ .
\end{aligned}
\tag{3.5}
$$

Here the ellipses denote Virasoro descendants that have to be added in order to make these fields primary. Note that the construction of the even algebra is simplified with respect to $\mathcal{W}_\infty$ by the fact that we do not need to worry about parity symmetry; indeed, all fields are parity even by construction.

3.1.1.2 Structure constants

Next we want to determine the structure constants appearing in (3.4) by requiring the associativity of the multiple OPEs $W^{s_1} \star W^{s_2} \star W^{s_3}$. Note that in this calculation, we need to work with the full OPEs, rather than just their singular parts. The full OPE is in principle uniquely determined by its singular part, but the actual calculation is somewhat tedious. To do these computations efficiently we have therefore once more used the Mathematica packages `OPEdefs` and `OPEconf` of Thielemans.

More explicitly, we start by defining the OPE $W^4 \star W^4$ by the first line of (3.4), which does not contain any composite fields. We can then use this ansatz to compute the composite field A^8, and thus make an ansatz for the OPE $W^4 \star W^6$. At this step, we can already check the associativity of $W^4 \star W^4 \star W^4$, using the built-in function `OPEJacobi`.

The next step consists in computing the composite fields made from W^4 and W^6, i.e. the remaining composite fields in (3.5). Then we can make an ansatz for the remaining OPEs in (3.4), and check the associativity of $W^4 \star W^4 \star W^6$.

It should now be clear how we continue: in each step we first compute all the composite primary fields made of products of fundamental fields whose OPE we have already determined. This then allows us to make an ansatz for the next 'level' of OPEs. Then we can check the associativity of those triple products where all intermediate OPEs are known. Proceeding in this manner, we have computed the constraints arising from the associativity of the OPEs $W^{s_1} \star W^{s_2} \star W^{s_3}$ up to the total level $s_1 + s_2 + s_3 \leq 16$. The resulting relations are (for the sake of brevity we only give the explicit expressions up to total spin $s_1 + s_2 + s_3 \leq 14$ that can be calculated from the OPEs given explicitly in (3.4))

$$
\begin{aligned}
n_4 &= \tfrac{c(c-1)(c+24)(5c+22)}{12(2c-1)(7c+68)^2}\left(c_{46}^6\right)^2 - \tfrac{7c(c-1)(5c+22)}{72(2c-1)(7c+68)} c_{46}^6 c_{44}^4 + \tfrac{c(5c+22)}{72(c+24)}\left(c_{44}^4\right)^2 \ , \\
c_{44}^6 c_{46}^4 &= -\tfrac{8(c-1)(c+24)(5c+22)\left(c^2-172c+196\right)}{(2c-1)^2(7c+68)^3}\left(c_{46}^6\right)^2 \\
&\quad + \tfrac{28(c-1)(5c+22)\left(c^2-172c+196\right)}{3(2c-1)^2(7c+68)^2} c_{44}^4 c_{46}^6 + \tfrac{4(c-1)(5c+22)(11c+656)}{9(c+24)(2c-1)(7c+68)}\left(c_{44}^4\right)^2 \ , \\
c_{46}^8 a_{48}^{11} &= \left(\tfrac{888}{65c+2580} - \tfrac{40}{7c+68}\right) c_{46}^6 - \tfrac{2(13c+918)}{65c+2580} c_{44}^6 a_{46}^8 + \tfrac{224}{15(c+24)} c_{44}^4 \ , \\
c_{48}^8 &= \tfrac{192-31c}{26c+1032} c_{44}^6 a_{46}^8 + \tfrac{8(c(33c+1087)+11760)}{(7c+68)(13c+516)} c_{46}^6 - 2c_{44}^4 \ , \\
c_{44}^6 c_{46}^8 a_{48}^8 &= \tfrac{192-31c}{26c+1032}\left(c_{44}^6\right)^2\left(a_{46}^8\right)^2 - \tfrac{4\left(165c^3+10763c^2+140036c+38568\right)}{3(c+24)(c+31)(55c-6)} c_{44}^4 c_{44}^6 a_{46}^8 \\
&\quad - \tfrac{896(3c+46)(5c+3)((c-172)c+196)}{(c+31)(2c-1)(7c+68)^2(55c-6)}\left(c_{46}^6\right)^2 \\
&\quad + \tfrac{3136(3c+46)(5c+3)((c-172)c+196)}{3(c+24)(c+31)(2c-1)(7c+68)(55c-6)} c_{44}^4 c_{46}^6 \\
&\quad + \tfrac{8(33c^2+1087c+11760)}{(7c+68)(13c+516)} c_{44}^6 a_{46}^8 c_{46}^6 + \tfrac{448(3c+46)(5c+3)(11c+656)}{9(c+24)^2(c+31)(55c-6)}\left(c_{44}^4\right)^2 \ , \\
c_{46}^8 c_{48}^6 &= -\tfrac{35(c+50)(2c-1)(7c+68)}{3(c+24)(c+31)(55c-6)} c_{44}^4 c_{44}^6 a_{46}^8 + \tfrac{8\left(25c^3+615c^2-88272c+102332\right)}{3(c+24)(c+31)(55c-6)} c_{44}^4 c_{46}^6 \\
&\quad + \tfrac{16\left(425c^4+15145c^3+233766c^2+6507708c-7565544\right)}{(c+31)(7c+68)(13c+516)(55c-6)}\left(c_{46}^6\right)^2 \\
&\quad + \tfrac{7840(c+50)(2c-1)(7c+68)}{9(c+24)^2(c+31)(55c-6)}\left(c_{44}^4\right)^2 + \tfrac{604-4c}{13c+516} c_{44}^6 c_{46}^6 a_{46}^8 \ , \\
c_{44}^6 c_{46}^8 c_{48}^4 &= -\tfrac{(c-1)(c+24)(5c+22)\left(65c^4+8637c^3+364470c^2+2897944c+36384\right)}{2(2c-1)(3c+46)(5c+3)(7c+68)^2(13c+516)} c_{44}^6 a_{46}^8\left(c_{46}^6\right)^2 \\
&\quad + \tfrac{7(c-1)(5c+22)\left(65c^4+8637c^3+364470c^2+2897944c+36384\right)}{12(2c-1)(3c+46)(5c+3)(7c+68)(13c+516)} c_{44}^6 a_{46}^8 c_{44}^4 c_{46}^6 \\
&\quad - \tfrac{5(c-1)(c+50)(5c+22)\left(715c^4+90933c^3+2851076c^2+21154896c+6967008\right)}{12(c+24)(c+31)(3c+46)(5c+3)(13c+516)(55c-6)} \\
&\quad \times \left(c_{44}^4\right)^2 c_{44}^6 a_{46}^8 \\
&\quad - \tfrac{32(c-151)(c-1)(c+24)(5c+22)\left(c^2-172c+196\right)}{(2c-1)^2(7c+68)^3(13c+516)}\left(c_{46}^6\right)^3 \\
&\quad - \tfrac{56(c-1)(5c+22)\left(c^2-172c+196\right)\left(20c^3+24807c^2+765640c-185172\right)}{3(c+31)(2c-1)^2(7c+68)^2(13c+516)(55c-6)} c_{44}^4\left(c_{46}^6\right)^2 \\
&\quad + \tfrac{4(c-1)(5c+22)\left(5605c^4-408494c^3-70820464c^2-1703657536c+1312613664\right)}{9(c+24)(c+31)(2c-1)(7c+68)(13c+516)(55c-6)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left(c_{44}^{4}\right)^{2} c_{46}^{6} \\
&\quad + \tfrac{140(c-1)(c+50)(5c+22)(11c+656)}{27(c+24)^{2}(c+31)(55c-6)} \left(c_{44}^{4}\right)^{3} \ , \\
c_{44}^{6} c_{66}^{10} &= \tfrac{3}{4} c_{46}^{8} c_{48}^{10} \ , \\
\left(c_{44}^{6}\right)^{2} a_{66}^{10,1} &= \tfrac{3}{4} c_{44}^{6} c_{46}^{8} a_{48}^{10,1} + \tfrac{5(c+64)(c+76)(5c+22)(11c+232)}{3(c+24)(c+31)(17c+944)(55c-6)} c_{44}^{6} a_{46}^{8} c_{44}^{4} \\
&\quad + \tfrac{1120(c+64)(11c+656)(47c-614)}{9(c+24)^{2}(c+31)(17c+944)(55c-6)} \left(c_{44}^{4}\right)^{2} \\
&\quad + \tfrac{7840(c+64)(47c-614)\left(c^{2}-172c+196\right)}{3(c+24)(c+31)(2c-1)(7c+68)(17c+944)(55c-6)} c_{46}^{6} c_{44}^{4} \\
&\quad - \tfrac{2240(c+64)(47c-614)\left(c^{2}-172c+196\right)}{(c+31)(2c-1)(7c+68)^{2}(17c+944)(55c-6)} \left(c_{46}^{6}\right)^{2} \ , \\
c_{44}^{6} a_{66}^{10,2} &= \tfrac{6(c+64)(13c+248)}{(13c+516)(17c+944)} c_{44}^{6} a_{46}^{8} + \tfrac{192(c+64)(81c+1274)}{(7c+68)(13c+516)(17c+944)} c_{46}^{6} \\
&\quad - \tfrac{224(c+64)}{(c+24)(17c+944)} c_{44}^{4} + \tfrac{3}{4} c_{46}^{8} a_{48}^{10,2} \ , \\
c_{44}^{6} c_{66}^{8} &= \tfrac{4(4c+61)}{7c+68} c_{46}^{8} c_{46}^{6} - \tfrac{(11c+656)}{6(c+24)} c_{46}^{8} c_{44}^{4} \ , \\
\left(c_{44}^{6}\right)^{2} a_{66}^{8} &= -\tfrac{11c+656}{6(c+24)} c_{44}^{4} c_{44}^{6} a_{46}^{8} + \tfrac{784((c-172)c+196)}{3(c+24)(2c-1)(7c+68)} c_{44}^{4} c_{46}^{6} \\
&\quad - \tfrac{224((c-172)c+196)}{(2c-1)(7c+68)^{2}} \left(c_{46}^{6}\right)^{2} + \tfrac{4(4c+61)}{7c+68} c_{46}^{6} c_{44}^{6} a_{46}^{8} + \tfrac{112(11c+656)}{9(c+24)^{2}} \left(c_{44}^{4}\right)^{2} \ , \\
c_{44}^{6} c_{66}^{6} &= \tfrac{20\left(92c^{5}+2389c^{4}+39632c^{3}+4060c^{2}-212032c+193984\right)}{(2c-1)^{2}(7c+68)^{3}} \left(c_{46}^{6}\right)^{2} \\
&\quad + \tfrac{10\left(28c^{5}-5425c^{4}-525974c^{3}+387728c^{2}+3726976c-3870208\right)}{9(c+24)(2c-1)^{2}(7c+68)^{2}} c_{46}^{6} c_{44}^{4} \\
&\quad - \tfrac{20\left(13c^{4}-1637c^{3}-113622c^{2}+32168c+859328\right)}{27(c+24)^{2}(2c-1)(7c+68)} \left(c_{44}^{4}\right)^{2} \ , \\
\left(c_{44}^{6}\right)^{2} c_{66}^{4} &= -\tfrac{8(c-1)(c+24)(5c+22)((c-172)c+196)}{(1-2c)^{2}(7c+68)^{3}} \left(c_{46}^{6}\right)^{3} \\
&\quad + \tfrac{28(c-1)(5c+22)((c-172)c+196)}{3(1-2c)^{2}(7c+68)^{2}} c_{44}^{4} \left(c_{46}^{6}\right)^{2} \\
&\quad + \tfrac{4(c-1)(5c+22)(11c+656)}{9(c+24)(2c-1)(7c+68)} \left(c_{44}^{4}\right)^{2} c_{46}^{6} \ , \\
\left(c_{44}^{6}\right)^{2} n_{6} &= -\tfrac{2(c-1)^{2}c(c+24)^{2}(5c+22)^{2}((c-172)c+196)}{3(2c-1)^{3}(7c+68)^{5}} \left(c_{46}^{6}\right)^{4} \\
&\quad + \tfrac{(c-1)c(5c+22)^{2}(11c+656)}{162(c+24)^{2}(2c-1)(7c+68)} \left(c_{44}^{4}\right)^{4} \\
&\quad + \tfrac{14(c-1)^{2}c(c+24)(5c+22)^{2}((c-172)c+196)}{9(2c-1)^{3}(7c+68)^{4}} c_{44}^{4} \left(c_{46}^{6}\right)^{3} \\
&\quad - \tfrac{(c-1)c(5c+22)^{2}(c(c(17c-13105)+25330)-12092)}{54(2c-1)^{3}(7c+68)^{3}} \left(c_{44}^{4}\right)^{2} \left(c_{46}^{6}\right)^{2} \\
&\quad - \tfrac{7(c-1)c(5c+22)^{2}(c(8c+1161)-1244)}{162(1-2c)^{2}(c+24)(7c+68)^{2}} \left(c_{44}^{4}\right)^{3} c_{46}^{6} \ .
\end{aligned}
\tag{3.6}
$$

Let us comment on the implications of these results. Of the 23 structure constants that appear in (3.4), 8 remain undetermined by the above relations; for example, a convenient choice for the free structure constants is n_4, n_6, c_{46}^{8}, c_{48}^{10}, as well as a_{46}^{8}, $a_{48}^{10,1}$, $a_{48}^{10,2}$ and c_{44}^{4}. The first 4 of these just account for the freedom to normalise the fields W^4, W^6, W^8 and W^{10}, respectively. The appearance of a_{46}^{8}, $a_{48}^{10,1}$, and

$a_{48}^{10,2}$ reflects the freedom to redefine W^8 and W^{10} by adding to them composite fields of the same spin as explained for the case of $\mathcal{W}_\infty$ in section 2.2.4. Note that this freedom implies that the relations (3.6) must satisfy interesting consistency conditions. For example, if we redefine $\hat{W}^8$ in this manner, the structure constant a_{48}^{11} in the OPE $W^4 \star \hat{W}^8$ becomes $\hat{a}_{48}^{11} = a_{48}^{11} + \frac{a_{46}^8}{c_{46}^8}\frac{2(13c+918)}{65c+2580}c_{44}^6$, which then satisfies indeed the third equation of (3.6) with $\hat{a}_{46}^8 = 0$. For example, we can set $\hat{a}_{46}^8 = 0$ by redefining $W^8 \mapsto \hat{W}^8 \equiv W^8 + a_{46}^8/c_{46}^8 A^8$, and similarly in the other two cases.

Thus, at least up to the level to which we have analysed the Jacobi identities and up to field redefinitions, all structure constants are completely fixed in terms of c and the single fundamental structure constant c_{44}^4. Note that for a given choice of n_4, the sign of c_{44}^4 is again not determined since n_4 only fixes the normalisation of W^4 up to a sign. It seems reasonable to believe that this structure will continue, i.e. that all remaining structure constants are also uniquely fixed (up to field redefinitions) in terms of the central charge c and

$$\gamma = \left(c_{44}^4\right)^2 \ . \tag{3.7}$$

The situation is then analogous to what was found for $\mathcal{W}_\infty[\mu]$ before: the resulting algebra depends on one free parameter (in addition to the central charge c), and whenever we want to emphasise this dependence, we shall denote it by $\mathcal{W}_\infty^e(\gamma)$.

In a next step we want to relate $\mathcal{W}_\infty^e(\gamma)$ to the Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$. Before doing so, we can however already perform some simple consistency checks on the above analysis.

3.1.2 Truncations

Since our ansatz is completely general, it should also reproduce the various finite even $\mathcal{W}$ algebras that have been constructed in the literature before [113, 22]. More specifically, we can study for which values of γ, $\mathcal{W}_\infty^e$ develops an ideal such that the resulting quotient algebra becomes a finitely generated $\mathcal{W}$ algebra.

3.1.2.1 The algebra $\mathcal{W}(2,4)$

The simplest case is the so-called $\mathcal{W}(2,4)$ algebra, which is generated by a single Virasoro-primary field W^4 in addition to the stress-energy tensor. Thus we need to find the value of γ for which W^6, W^8, etc. lie in an ideal. Imposing c_{46}^4, c_{66}^4, c_{48}^4, and n_6 to vanish we obtain

$$\gamma = \frac{216(c+24)(c^2-172c+196)n_4}{c(2c-1)(7c+68)(5c+22)} \ . \tag{3.8}$$

The resulting quotient algebra is then in agreement with e.g. [113]. Note that A^8 does not lie in the ideal since the OPE of W^4 with A^8 contains terms proportional to W^4 which are non-vanishing for generic c. Thus we also need to require that $a_{46}^8 = 0$ (but c_{46}^8 need not be zero), which is automatically true by the above conditions. We have also analysed the consistency of the resulting algebra directly, i.e. repeating essentially the same calculation as in [113].

3.1.2.2 The algebras $\mathcal{W}(2,4,6)$

The next simplest case is the so-called $\mathcal{W}(2,4,6)$ algebra, which should appear from $\mathcal{W}_\infty^e$ upon dividing out the ideal generated by W^8, W^{10}, etc. This requires that we set c_{48}^4, c_{48}^6 and n_8 to zero. Furthermore, since the composite fields A^8, $A^{10,1}$, $A^{10,2}$ and A^{11} have a non-trivial image in the quotient (for generic c), we should expect that also a_{48}^8, $a_{48}^{10,1}$, $a_{48}^{10,2}$ and a_{48}^{11} vanish. Solving equations (3.6) together with these constraints then yields the two values for γ

$$\begin{aligned}\gamma = 2n_4\Big[&(18025c^6 + 1356090c^5 + 16727763c^4 - 537533674c^3 \\ &\qquad - 5470228116c^2 + 8831442312c - 300564000) \\ &\quad \pm (c-1)(5c+22)^2(11c+444)(13c+918)\sqrt{c^2-534c+729}\Big] \\ &\times \big[c(2c-1)(3c+46)(4c+143)(5c+3)(5c+22)(5c+44)\big]^{-1} \ . \end{aligned} \tag{3.9}$$

Up to a factor of $\frac{1}{2}$, this agrees with two of the four solutions found in [113]; incidentally, they are the ones which were claimed to be inconsistent in [121]. We have again also analysed the consistency of the resulting algebra directly, i.e. by working with an ansatz involving only W^4 and W^6.

As a matter of fact, there are two additional solutions that appear if we enlarge the ideal by also taking A^{11} to be part of it. Then we do not need to impose that $a_{48}^{11} = 0$, and the resulting algebras agree with the other two solutions[1] of [113], i.e. they are characterised by

$$\gamma = -\frac{4(5c^2+309c-14)^2\, n_4}{c(c-26)(5c+3)(5c+22)} \tag{3.10}$$

and

$$\gamma = \frac{216(10c^2+47c-82)^2\, n_4}{c(4c+21)(5c+22)(10c-7)} \ , \tag{3.11}$$

respectively. In the case of (3.10), the OPEs of W^4 and W^6 with A^{11} show that no additional field of dimension smaller than 11 needs to be included in the ideal. However, in the case of the algebra described by (3.11), the ideal also contains a certain linear combination of $A^{10,1}$ and $A^{10,2}$.

[1] Incidentally, there is a typo in [113]: the structure constant a_{46}^8 in the $\mathcal{W}(2,4,6)$ algebra satisfying (3.11) should be given by $(a_{46}^8)^2 = \frac{256(2c-1)(5c+3)^2(3c+46)^2(7c+68)n_6}{(31c-192)^2(c+11)(14c+11)(5c+22)(c+24)n_4}$.

3.1.3 Relation to $\mathcal{W}_\infty$

In section 2.2.4 we saw the construction of the larger algebra $\mathcal{W}_\infty$ and noticed that it possesses a parity automorphism multiplying all W^s with s odd by -1. The kernel of that map is an even spin subalgebra $\widetilde{\mathcal{W}}^e_\infty$ of $\mathcal{W}_\infty$, containing (a priori) the primary fields

$$W^4, W^6, A^6, W^8, A^{8,2}, A^{8,3}, A^{8,4}, \ \ldots \tag{3.12}$$

This algebra seems to be larger than $\mathcal{W}^e_\infty$, since it also contains even products of odd spin fields such as A^6, $A^{8,2}$ and $A^{8,3}$. However, one might be tempted to think that there may exist a suitable field redefinition such that these fields are completely absorbed by the elementary primaries W^s, as is possible in specific OPEs we have studied above. In that case the uniqueness of $\mathcal{W}^e_\infty$ would ensure that $\widetilde{\mathcal{W}}^e_\infty$ and $\mathcal{W}^e_\infty$ are isomorphic upon a suitable identification of γ, so $\mathcal{W}^e_\infty(\gamma')$ would be a subalgebra of $\mathcal{W}_\infty(\gamma)$ for some value of γ'. We are going to demonstrate that this is not the case for generic values of γ.

The idea of our proof is to restrict the algebra to a suitable truncation, where the number of structure constants is finite and completely under control. It is important to leave all free structure constants (except γ) unfixed for now, as they might be compatible with $\mathcal{W}^e_\infty$ only at specific values. Let us take the value $\mu = N$ at which $\mathcal{W}_\infty[\mu]$ possesses an ideal χ_N containing all elementary fields W^s for $s > N$. The restriction of that ideal to $\widetilde{\mathcal{W}}^e_\infty[N]$,

$$\tilde{\chi}_N = \chi_N \cap \widetilde{W}^e_\infty[N] \ , \tag{3.13}$$

is then an ideal of $\widetilde{W}^e_\infty[N]$ and can be factored out. Explicitly, we will choose $\mu = 6$, i.e. we factor out all W^s with $s > 6$. Therefore, if $\widetilde{\mathcal{W}}^e_\infty$ and $\mathcal{W}^e_\infty$ were isomorphic the quotient would have to be one of the $\mathcal{W}(2,4,6)$ algebras constructed in the previous section. By demanding the $\mathcal{W}_\infty$ structure constants c^4_{37} and c^6_{37} to vanish, we obtain the value[2] of c^4_{33}:

$$\left(c^4_{33}\right)^2 = \frac{243(c+2)(c+30)\, n_3^2}{c(2c+25)(5c+22)\, n_4} \ . \tag{3.14}$$

This agrees precisely with (2.115) for $\mu = 6$. Given this value of c^4_{33} we can then also determine the structure constant c^4_{44} of $\mathcal{W}_\infty[\mu = 6]$, which turns out to be

$$\left(c^4_{44}\right)^2 = \frac{\left(17c^2 + 1233c - 710\right)^2 n_4}{3c(c+2)(c+30)(2c+25)(5c+22)} \ . \tag{3.15}$$

This is not part of the (exhaustive) list of possible $\mathcal{W}(2,4,6)$ algebras found in [113] and characterised by (3.9), (3.10), and (3.11). More precisely, (3.15) agrees

[2] There is a second solution to these constraints, which might or might not correspond to some other ideal of $\mathcal{W}_\infty$.

with (3.9) (choosing the negative sign) to leading order as $c \to \infty$, but differs at the first subleading order. Similarly, for $\mu = 7$ the value of $\left(c_{44}^4\right)^2$ agrees with the other solution in (3.9) only to leading order in $1/c$.

As a further check, note that in $\mathcal{W}_\infty[\mu]$ it is not possible to simultaneously set a_{44}^6 and a_{46}^6 to 0 unless $\mu = \pm 1$. This is a proof by itself that A^6 cannot consistently be absorbed into W^6 by field redefinition. It thus becomes clear that we have truncated $\widetilde{\mathcal{W}}_\infty^e$ to an algebra of type $\mathcal{W}(2,4,6,6)$ above, where the second field of spin 6, A^6, is suppressed by $1/c$. That is why our structure constants reduce to the ones of a $\mathcal{W}(2,4,6)$ algebra in the large c limit, but differ at finite c.

We have therefore established that $\mathcal{W}_\infty^e$ is not a subalgebra of $\mathcal{W}_\infty$ for generic value of μ. This changes at $\mu = \pm 1$ however, where $\mathcal{W}_\infty[\mu]$ becomes the linear algebra[3] studied in [139], which is just the (centrally extended) continuation of $\mathfrak{hs}[1]$ beyond the wedge. $\mathcal{W}_\infty[1]$ has the c_{44}^4 structure constant

$$\left(c_{44}^4\right)^2 = \frac{216(3c+2)^2\, n_4}{c(c+2)(5c+22)}\,, \tag{3.16}$$

and since its even subalgebra has the correct field content[4] and our construction of $\mathcal{W}_\infty^e$ was unique, it corresponds to $\mathcal{W}_\infty^e$ at this value of c_{44}^4 (this has also been checked in [109]).

3.1.4 Identifying the wedge algebra

We expect from the analysis of [88] that $\mathcal{W}_\infty^e(\gamma)$ should arise as the Drinfel'd-Sokolov reduction of the even higher spin algebra. However, as was also explained in [88], it is not clear which higher spin algebra is relevant in this context, and two possibilities, $\mathfrak{hs}^e[\mu]$ and $\mathfrak{hso}[\mu]$, were proposed. Here, $\mathfrak{hs}^e[\mu]$ is spanned by the modes V_m^s from section 2.1.2 for which s is even, and $\mathfrak{hso}[\mu]$ by the V_m^s with $m+s$ even. In order to decide which of the two algebras is relevant, it is sufficient to determine the wedge algebra of $\mathcal{W}_\infty^e(\gamma)$, i.e. the algebra that is obtained by restricting the modes to the wedge $|m| < s$, and taking $c \to \infty$. (The reason for this is that restricting to the wedge algebra is in a sense the inverse of performing the Drinfel'd-Sokolov reduction, see [30, 81] for a discussion of this point.) As it turns out, the wedge algebra commutators of $\mathcal{W}_\infty^e(\gamma)$, as obtained from (3.4) together with (3.6), agree with the $\mathfrak{hs}^e[\mu]$ commutators (B.12) of appendix B.2

[3] Strictly speaking $\mathcal{W}_\infty[\pm 1]$ contains a linear *subalgebra* generated by L and quasi-primary fields $\widetilde{W}^s$ for $s = 3, 4, 5, \ldots$, without any composite fields.

[4] The argument is actually more subtle because the basis in which $\mathcal{W}_\infty[1]$ becomes linear is non-primary, whereas we assumed all W^s to be primary. To make contact with our construction of $\mathcal{W}_\infty^e$, we need to make them primary again, thereby reintroducing the even spin quasi-primaries that have previously been absorbed into $\widetilde{W}^s$.

provided we identify $c_{44}^4 = \sqrt{\gamma}$ with μ as

$$c_{44}^4 = \frac{12}{\sqrt{5}}(\mu^2 - 19) + \mathcal{O}(c^{-1}) \ . \tag{3.17}$$

Furthermore, we normalise our fields as

$$\begin{aligned} n_4 &= c(\mu^2-9)(\mu^2-4) \ , & n_6 &= c(\mu^2-25)(\mu^2-16)(\mu^2-9)(\mu^2-4) \ , \\ c_{46}^8 &= -8\sqrt{\tfrac{210}{143}} \ , & c_{48}^{10} &= -20\sqrt{\tfrac{6}{17}} \ , \end{aligned} \tag{3.18}$$

and take the field redefinition parameters to be $a_{46}^8 = a_{48}^{10,1} = a_{48}^{10,2} = 0$. Thus we conclude that the $\mathcal{W}_\infty^e(\gamma)$ algebra can be interpreted as the quantum Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$, where μ and γ are related as in (3.17); this will be further elaborated on in section 3.2.

We should mention in passing that $\mathfrak{hs}^e[\mu]$ and $\mathfrak{hso}[\mu']$ are not isomorphic (even allowing for some general relation between μ and μ'), since they possess different finite-dimensional quotient algebras, see [88]. Thus the above analysis also proves that the wedge algebra of $\mathcal{W}_\infty^e$ is *not* isomorphic to $\mathfrak{hso}[\mu]$ for any μ, and hence that $\mathcal{W}_\infty^e$ is *not* the quantum Drinfel'd-Sokolov reduction of $\mathfrak{hso}[\mu]$ for any μ.

3.1.5 Minimal representation

Our next aim is to determine the exact c dependence of (3.17). This can be done using the same trick as in [73] and [35], following the original analysis of [109]. The main ingredient in this analysis is a detailed understanding of the structure of the 'minimal representations' of $\mathcal{W}_\infty^e$. Recall that the duality of [88] suggests that $\mathcal{W}_\infty^e$ possesses two minimal representations whose character is of the form

$$\chi_{\min}(q) = \frac{q^h}{1-q} \prod_{s\in 2\mathbb{N}} \prod_{n=s}^{\infty} \frac{1}{1-q^n} \ , \tag{3.19}$$

and for which h is finite in the 't Hooft limit. It follows from this character formula that the corresponding representation has (infinitely) many low-lying null-vectors; this will allow us to calculate h as a function of c and γ.

Let us denote the primary field of the minimal representation by P^0. First, we need to make the most general ansatz for the OPEs $W^s \star P^0$. In order to do so we have to enumerate the number of Virasoro-primary states in the minimal $\mathcal{W}_\infty^e$ representation. Decomposing (3.19) in terms of irreducible Virasoro characters as

$$\chi_{\min}(q) = \sum_{n=0}^{\infty} d_{\min}(n)\, \chi_{h+n}(q) \ , \tag{3.20}$$

where $\chi_h(q)$ was defined in (3.2), $d_{\rm min}(n)$ equals then the multiplicity of the Virasoro primaries of conformal dimension $h+n$. It follows from (3.20) that the corresponding generating function is

$$\sum_{n=0}^{\infty} d_{\rm min}(n)\, q^n = \prod_{s=2}^{\infty}\prod_{n=2s}^{\infty} \frac{1}{1-q^n} = 1+q^4+q^5+2q^6+\cdots \ . \tag{3.21}$$

Then the most general ansatz for the OPEs $W^4 \star P^0$ and $W^6 \star P^0$ is

$$W^4 \star P^0 \sim w^4 P^0 \ , \qquad W^6 \star P^0 \sim w^6 P^0 + a^4 P^4 + a^5 P^5 \ , \tag{3.22}$$

where P^4 and P^5 are the primary fields of conformal dimension $h+4$ and $h+5$, respectively. Note that these fields are unique, as follows from (3.21); explicitly, they are of the form

$$P^4 = W^4 P^0 + \cdots \ , \qquad P^5 = \frac{h}{4+h} W^{4\prime} P^0 - \frac{4}{4+h} W^4 P^{0\prime} + \cdots \ , \tag{3.23}$$

where the ellipses stand for Virasoro descendants that are required to make these fields primary. As in [35], the condition that P^0 defines a representation of $\mathcal{W}^e_\infty$ is now equivalent to the constraint that all OPEs $W^{s_1}\star W^{s_2}\star P^0$ are associative. While we cannot test all of these conditions, imposing the associativity of $W^4 \star W^4 \star P^0$ implies already

$$\begin{aligned}
w^4 &= \frac{12h\big(c^2(9-2(h-1)h)+3c(h((49-12h)h-40)+2)-2h(h(12h+5)-14)\big)}{c(5c+22)(c(h-2)(2h-3)+h(4h-5))}\,\frac{n_4}{c_{44}^4} \ , \\
w^6 &= \frac{8(c-1)(5c+22)h\big(c(h+2)+15h^2-26h+8\big)(c(2h+3)+4h(12h-7))}{3c(c+24)(2c-1)(7c+68)(c(h-2)(2h-3)+h(4h-5))}\,\frac{n_4}{c_{44}^6} \ , \\
a^4 &= \frac{16(5c+22)(4h-9)\big(c(h+2)+15h^2-26h+8\big)}{(c+24)((c-7)h+c+3h^2+2)(2ch+c+2h(8h-5))}\,\frac{w^4}{c_{44}^6} \ , \\
a^5 &= \frac{20(5c+22)(h-4)(h-1)(c(2h+3)+4h(12h-7))}{(c+24)h((c-7)h+c+3h^2+2)(2ch+c+2h(8h-5))}\,\frac{w^4}{c_{44}^6} \ ,
\end{aligned} \tag{3.24}$$

up to a sign ambiguity of the self-coupling $c_{44}^4 = \pm\sqrt{\gamma}$. Furthermore, the conformal dimension h is determined by the equation

$$\gamma = \frac{144[c^2(-9-2h+2h^2)+3c(-2+40h-49h^2+12h^3)+2h(-14+5h+12h^2)]^2}{c(5c+22)[c(1+h)+2-7h+3h^2][(1+2h)c-10h+16h^2][(6-7h+2h^2)c-5h+4h^2]}\, n_4 \ . \tag{3.25}$$

Given γ and c, this is a sextic equation for h. We also note that our result is consistent with the one obtained in [109]. Moreover, we have checked that we arrive at the same result using commutators instead of OPEs; this calculation, which is analogous to the one performed in [73] for the algebra $\mathcal{W}_\infty[\mu]$, is presented in appendix B.1.

We should stress that the above constraints are necessary conditions for the minimal representation to exist, but do not prove that they are actually compatible with the full $\mathcal{W}_\infty^e$ structure. Furthermore, since we have only used the low-lying OPEs, our analysis actually holds for *any* algebra of type $\mathcal{W}(2,4,\dots)$ with no simple field of spin 5, and for any representation whose character coincides with (3.19) up to q^5, see also [109].

3.2 Drinfel'd-Sokolov reductions

As we have seen in section 3.1.4, the wedge algebra of $\mathcal{W}_\infty^e(\gamma)$ is $\mathfrak{hs}^e[\mu]$, where $\gamma = (c_{44}^4)^2$ is identified with a certain function of μ, see equation (3.17). Thus we should expect that the quantum $\mathcal{W}_\infty^e[\mu]$ algebras (where we now label $\mathcal{W}_\infty^e$ in terms of μ rather than γ) can be thought of as being the Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$. Actually, as we shall shortly see, the situation is a little bit more complicated. The subtlety we are about to encounter is related to the fact that $\mathfrak{hs}^e[\mu]$ is in some sense a non-simply-laced algebra.[5]

Since Drinfel'd-Sokolov reductions of infinite-dimensional Lie algebras are complicated, we shall first (as in [73]) consider the special cases when μ is a positive integer. Then $\mathfrak{hs}^e[\mu]$ can be reduced to finite-dimensional Lie algebras; indeed, as was already explained in [88], we have

$$\mathfrak{hs}^e[N]/\chi_N = \begin{cases} \mathfrak{so}(N) & \text{for } N \text{ odd} \\ \mathfrak{sp}(N) & \text{for } N \text{ even,} \end{cases} \tag{3.26}$$

where χ_N is the ideal of $\mathfrak{hs}^e[\mu]$ that appears for $\mu = N \in \mathbb{N}$. Note that in both cases, the resulting algebra is non-simply-laced, suggesting that $\mathfrak{hs}^e[\mu]$ should be thought of as being non-simply-laced itself.

As in [73] we should now expect that the quantum Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$ agrees, for $\mu = N$, with the quantum Drinfel'd-Sokolov reduction of $B_n = \mathfrak{so}(2n+1)$ or $C_n = \mathfrak{sp}(2n)$, respectively. The representation theory of these $\mathcal{WB}_n$ and $\mathcal{WC}_n$ algebras is well known, and thus, at least for these integer values of μ, we can compare the conformal dimension of the corresponding minimal representations with what was determined above, see equation (3.25). This will allow us to deduce an exact relation between γ and $\mu = N$ (for all values of the central charge). Analytically continuing the resulting expression to non-integer μ should then lead to the precise relation between γ and μ, for all values of μ.

[5] Note, however, the situation is also complicated by the fact that $\mathfrak{hs}^e[\mu]$ is infinite dimensional, and the construction of [28] only applies to finite-dimensional Lie algebras. On the other hand, given that things worked nicely [73] for the infinite-dimensional algebra $\mathfrak{hs}[\mu]$, we suspect that the infinite-dimensionality of $\mathfrak{hs}^e[\mu]$ is not the origin of the subtlety.

3.2.1 The B_n series approach

According to [28], the Drinfel'd-Sokolov reduction of $\mathfrak{so}(2n+1)$, which we shall denote by $\mathcal{WB}_n$, is an algebra of type $\mathcal{W}(2,4,\ldots,2n)$ with central charge

$$c_B = n - 12|\alpha_+\rho_B + \alpha_-\rho_B^\vee|^2 \tag{3.27}$$

and spectrum

$$h_\Lambda = \tfrac{1}{2}(\Lambda, \Lambda + 2\alpha_+\rho_B + 2\alpha_-\rho_B^\vee) \ , \qquad \Lambda \in \alpha_+ P_+ + \alpha_- P_+^\vee \ . \tag{3.28}$$

Here ρ_B and $\rho_B^\vee$ are the $\mathfrak{so}(2n+1)$ Weyl vector and covector, respectively, and P_+ and $P_+^\vee$ are the lattices of $\mathfrak{so}(2n+1)$ dominant weights and coweights, respectively. We work with the convention that the long roots have length squared equal to 2, and $\alpha_\pm$ are defined as in (2.100) by

$$\alpha_- = -\sqrt{k_B + 2n - 1} \ , \qquad \alpha_+ = \frac{1}{\sqrt{k_B + 2n - 1}} \ , \tag{3.29}$$

so that $\alpha_+\alpha_- = -1$. Furthermore k_B is the level that appears in the Drinfel'd-Sokolov reduction. Note that the dual Coxeter number of $\mathfrak{so}(2n+1)$ equals $g_B = 2n-1$. Plugging in the expressions for $\alpha_\pm$ into (3.27), the central charge of $\mathcal{WB}_n$ takes the form

$$\begin{aligned} c_B(\mu, k_B) &= -\frac{n[k_B(2n+1) + 4n^2 - 2n][2k_B(n+1) + 4n^2 - 3]}{k_B + 2n - 1} \\ &= \frac{(1-\mu)(k_B\mu + \mu^2 - 3\mu + 2)[k_B(1+\mu) + \mu^2 - 2\mu - 2]}{2(k_B + \mu - 2)} \ , \end{aligned} \tag{3.30}$$

where in the second line we have replaced $n = \frac{\mu-1}{2}$. The minimal representations of $\mathcal{WB}_n$ arise for $\Lambda = \Lambda_+ = \alpha_+\mathrm{f}$, and $\Lambda = \Lambda_- = \alpha_-\mathrm{f}^\vee$, where f is the highest weight of the fundamental $\mathfrak{so}(2n+1)$ representation, and $\mathrm{f}^\vee$ the corresponding coweight. The conformal dimensions of these two representations are

$$h_+ = h_{\Lambda_+} = -\frac{n(k_B + 2n - 2)}{k_B + 2n - 1} \ , \qquad h_- = h_{\Lambda_-} = k_B\left(n + \tfrac{1}{2}\right) + n(2n-1) \ . \tag{3.31}$$

They are both solutions of equation (3.25), provided $\gamma = \gamma_B(\mu, k_B)$ with γ_B equal to

$$\begin{aligned} \gamma_B = 144\,(&240 - 420k_B + 210k_B^2 - 30k_B^3 + 1188\mu - 1520k_B\mu + 773k_B^2\mu \\ &- 190k_B^3\mu + 19k_B^4\mu - 2138\mu^2 + 2237k_B\mu^2 - 818k_B^2\mu^2 + 107k_B^3\mu^2 + 264\mu^3 \\ &+ 614k_B\mu^3 - 703k_B^2\mu^3 + 220k_B^3\mu^3 - 20k_B^4\mu^3 + 1107\mu^4 - 1516k_B\mu^4 \\ &+ 615k_B^2\mu^4 - 75k_B^3\mu^4 - 644\mu^5 + 462k_B\mu^5 - 51k_B^2\mu^5 - 12k_B^3\mu^5 + k_B^4\mu^5 \end{aligned}$$

$$+ 67\mu^6 + 43k_B\mu^6 - 36k_B^2\mu^6 + 4k_B^3\mu^6 + 39\mu^7 - 36k_B\mu^7 + 6k_B^2\mu^7 - 12\mu^8$$
$$+ 4k_B\mu^8 + \mu^9)^2\, n_4 \,/\, \Big[c_B(\mu-3)(3k_B + k_B\mu - 6 + \mu^2)$$
$$\times (8 - 2k_B - 5\mu + k_B\mu + \mu^2)(1 - k_B - 4\mu + k_B\mu + \mu^2)$$
$$\times (1 - 3\mu + k_B\mu + \mu^2)(k_B + k_B\mu - 4 - 2\mu + \mu^2)$$
$$\times (2k_B - 2 - \mu + k_B\mu + \mu^2)(108 - 54k_B - 74\mu + 25k_B\mu - 5k_B^2\mu - 20\mu^2$$
$$+ 5k_B\mu^2 + 55\mu^3 - 30k_B\mu^3 + 5k_B^2\mu^3 - 30\mu^4 + 10k_B\mu^4 + 5\mu^5)\Big] \,, \qquad (3.32)$$

where we have again replaced $n = \frac{\mu-1}{2}$. For each μ, we therefore obtain a family of $\mathcal{W}_\infty^e$ algebras that depend on k_B; these algebras will be denoted by $\mathcal{WB}_\infty[\mu]$ (where we suppress the explicit k_B dependence). Note that, for fixed μ, these algebras really depend on k_B, rather than just on c_B: for a fixed c and μ, there are always two solutions $k_B^{(i)}$, $i = 1, 2$, for $c = c(\mu, k_B^{(i)})$, see (3.30). However, in general the corresponding γ values do not agree, $\gamma_B(\mu, k_B^{(1)}) \neq \gamma_B(\mu, k_B^{(2)})$, and hence the two solutions for k_B do not lead to isomorphic $\mathcal{W}_\infty^e$ algebras. This is different than what happened for $\mathcal{W}_\infty$ in [73], and closely related to the fact that $\mathfrak{hs}^e[\mu]$ is non-simply-laced, see below.

By construction, the algebras $\mathcal{WB}_\infty[\mu]$ truncate, for $\mu = 2n+1$, to $\mathcal{WB}_n$. (Note that also $\mathcal{WB}_n$ depends actually on the level k_B, and not just on c.) We have also checked that, for $n = 2$, $\gamma_B(2n+1, k_B)$ agrees with the γ given in equation (3.8) at $c = c_B(2n+1, k_B)$. Similarly, for $n = 3$, $\gamma_B(2n+1, k_B)$ agrees with the γ of equation (3.9) at $c = c_B(2n+1, k_B)$. (For $n = 3$, the two algebras corresponding to the two different solutions for k_B correspond to the choice of the branch cut in the square root of equation (3.9).)

3.2.2 The C_n series approach

The analysis for the Drinfel'd-Sokolov reduction of $\mathfrak{sp}(2n)$, which we shall denote by $\mathcal{WC}_n$, is essentially identical. Also $\mathcal{WC}_n$ is an algebra of type $\mathcal{W}(2, 4, \ldots, 2n)$, and its central charge equals

$$c_C = n - 12|\alpha_+\rho_C + \alpha_-\rho_C^\vee|^2 \,, \qquad (3.33)$$

where now ρ_C and $\rho_C^\vee$ are the Weyl vector and covector of $\mathfrak{sp}(2n)$, respectively. The spectrum is described by the analogue of equation (3.28), where[6]

$$\alpha_- = -\sqrt{k_C + 2n + 2}\,, \qquad \alpha_+ = \frac{1}{\sqrt{k_C + 2n + 2}}\,. \qquad (3.34)$$

[6]In our conventions, the *short* roots of $C_n = \mathfrak{sp}(2n)$ have length squared equal to 2.

Expressed in terms of n and k_C, the central charge then takes the form

$$\begin{aligned} c_C(\mu, k_C) &= -\frac{n[k_C(2n+1)+4n^2+4n)][k_C(2n-1)+4n^2-3]}{k_C+2n+2} \\ &= -\frac{\mu[(k_C+2)\mu+k_C+\mu^2][k_C(\mu-1)+\mu^2-3]}{2(k_C+\mu+2)} \ , \end{aligned} \tag{3.35}$$

where we have, in the second line, replaced $n = \frac{\mu}{2}$. The conformal dimensions of the minimal representations are now

$$h_+ = h_{\Lambda_+} = \frac{k_C(1-2n)-4n^2+3}{2k_C+4n+4} \ , \qquad h_- = h_{\Lambda_-} = n(k_C+2n+1) \ , \tag{3.36}$$

and they are both solutions of equation (3.25) provided $\gamma = \gamma_C(\mu, k_C)$ equals

$$\begin{aligned} \gamma_C = 144(&-224-520k_C-340k_C^2-68k_C^3-888\mu-1064k_C\mu-161k_C^2\mu \\ &+114k_C^3\mu+19k_C^4\mu-372\mu^2+687k_C\mu^2+946k_C^2\mu^2+227k_C^3\mu^2+730\mu^3 \\ &+1390k_C\mu^3+377k_C^2\mu^3-100k_C^3\mu^3-20k_C^4\mu^3+553\mu^4+134k_C\mu^4 \\ &-315k_C^2\mu^4-85k_C^3\mu^4-34\mu^5-326k_C\mu^5-129k_C^2\mu^5+4k_C^3\mu^5+k_C^4\mu^5 \\ &-111\mu^6-83k_C\mu^6+12k_C^2\mu^6+4k_C^3\mu^6-19\mu^7+12k_C\mu^7+6k_C^2\mu^7+4\mu^8 \\ &+4k_C\mu^8+\mu^9)^2 n_4 \Big/ \Big[c_C(\mu-2)(k_C\mu-5-k_C+\mu^2)(k_C\mu-1+\mu+\mu^2) \\ &\times(k_C\mu-3k_C-5-2\mu+\mu^2)(k_C\mu+4+3k_C+4\mu+\mu^2) \\ &\times(k_C\mu-2+k_C+2\mu+\mu^2)(k_C\mu+4+2k_C+3\mu+\mu^2) \\ &\times(-88-44k_C-44\mu-15k_C\mu-5k_C^2\mu-30\mu^2-25k_C\mu^2-15\mu^3 \\ &\quad+10k_C\mu^3+5k_C^2\mu^3+10\mu^4+10k_C\mu^4+5\mu^5)\Big] \ , \end{aligned} \tag{3.37}$$

where we have again replaced $n = \frac{\mu}{2}$. For each μ, we therefore obtain a family of $\mathcal{W}_\infty^e$ algebras that depend on k_C; these algebras will be denoted by $\mathcal{WC}_\infty[\mu]$ (where we suppress as before the explicit k_C dependence). Again, these algebras actually depend on k_C, rather than just c_C. By construction, $\mathcal{WC}_\infty[\mu]$ has the property that it truncates to $\mathcal{WC}_n$ for $\mu = 2n$. We have also checked that, for $n = 2$, $\gamma_C(2n, k_C)$ agrees with the γ of equation (3.8) at $c = c_C(2n, k_C)$. Similarly, for $n = 3$, $\gamma_C(2n, k_C)$ agrees with the γ of equation (3.9) at $c = c_C(2n, k_C)$, where again the two solutions for k_C correspond to the two signs in front of the square root in equation (3.9).

3.2.3 Langlands duality

Naively, one would have expected that the two quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$ should be equivalent, but this is not actually the case: if we fix μ and c, and determine $k_B^{(i)}$, $i = 1, 2$, and $k_C^{(j)}$, $j = 1, 2$, by the requirement that

$$c = c_B(\mu, k_B^{(i)}) = c_C(\mu, k_C^{(j)}) \ , \tag{3.38}$$

then none of the four different algebras we obtain from $\mathcal{WB}_\infty[\mu]$ for $k_B = k_B^{(i)}$ and $\mathcal{WC}_\infty[\mu]$ for $k_C = k_C^{(j)}$ are equivalent to one another. Thus there is not a 'unique' quantisation of $\mathcal{W}^e_\infty[\mu]$!

The two constructions are, however, closely related to one another since we have the identifications

$$\begin{aligned} c_B(\mu+1, k_B) &= c_C(\mu, k_C) \\ \gamma_B(\mu+1, k_B) &= \gamma_C(\mu, k_C) \end{aligned} \qquad \text{when} \quad (k_B+\mu-1)(k_C+\mu+2) = 1 \ . \tag{3.39}$$

This relation is the 'analytic continuation' of the Langlands duality that relates $B_n = \mathfrak{so}(2n+1)$ and $C_n = \mathfrak{sp}(2n)$. Indeed, the Dynkin diagrams of B_n and C_n are obtained from one another upon reversing the arrows, i.e. upon exchanging the roles of the long and the short roots. Correspondingly, the root system of one algebra can be identified with the coroot system of the other (provided we scale the roots and coroots appropriately — this is the reason for our non-standard normalisation convention for the roots of C_n). It is then manifest from the above formulae that the central charge and spectrum is the same provided we also exchange the roles of α_+ and α_-. In terms of the levels k_B and k_C, this is then equivalent to the requirement that $(k_B+\mu-1)(k_C+\mu+2) = 1$ for $\mu = 2n$. Thus we can think of $\mathcal{WB}_\infty[\mu+1]$ and $\mathcal{WC}_\infty[\mu]$ to be related by Langlands duality for all μ.

The ambiguity in the definition of the quantum algebra associated with $\mathcal{W}^e_\infty[\mu]$ therefore simply reflects that Langlands duality acts non-trivially on $\mathfrak{hs}^e[\mu]$, i.e. that $\mathfrak{hs}^e[\mu]$ is non-simply-laced. This is to be contrasted with the case of $\mathcal{W}_\infty[\mu]$ where the two solutions of k for a given μ and c actually gave rise to equivalent $\mathcal{W}_\infty$ algebras, see equation (2.9) of [73], reflecting the fact that $\mathfrak{hs}[\mu]$ can be thought of as being 'simply-laced'.

3.2.4 Classical limit

In the semiclassical limit of large levels, the two quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$ actually become equivalent. More concretely, if we choose the normalisation of n_4 as in section 3.1.4, we have in the semiclassical limit

$$c_B \sim -\tfrac{1}{2}\mu(\mu^2-1)k_B + \mathcal{O}(k_B^0) \ , \qquad c_C \sim -\tfrac{1}{2}\mu(\mu^2-1)k_C + \mathcal{O}(k_C^0) \ , \tag{3.40}$$

$$\gamma_B \sim \tfrac{144}{5}(\mu^2-19)^2 + \mathcal{O}(k_B^{-1}) \ , \qquad \gamma_C \sim \tfrac{144}{5}(\mu^2-19)^2 + \mathcal{O}(k_C^{-1}) \ . \tag{3.41}$$

In particular, the central charges agree, and the parameter γ is of the form predicted by equation (3.17), recalling that $\gamma = (c^4_{44})^2$. Thus both quantum algebras $\mathcal{WB}_\infty[\mu]$

and $\mathcal{WC}_\infty[\mu]$ define consistent quantisations of the classical Poisson algebra, and both can be thought of as Drinfel'd-Sokolov reductions of $\mathfrak{hs}^e[\mu]$. However, as mentioned before, the $\mathcal{O}(c^{-1})$ corrections in equation (3.41) are different, reflecting the non-trivial action of Langlands duality as described by equation (3.39).

3.2.5 Self-dualities

While the parameters μ and k are well-suited for characterising the classical limits of the algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$, they do not directly parametrise the inequivalent $\mathcal{W}^e_\infty$ algebras. (The following discussion is directly parallel to the analogous analysis for the case of $\mathcal{W}_\infty[\mu]$ in [73].) Indeed, as was stressed in section 3.1.1.2, the parameters distinguishing between different $\mathcal{W}^e_\infty$ algebras are c and γ. It follows from equation (3.32) that there are 12 different combinations $(\mu_i, k_B^{(i)})$ that give rise to the *same* quantum algebra $\mathcal{WB}_\infty[\mu]$, and likewise for $\mathcal{WC}_\infty[\mu]$, see equation (3.37). Six of these identifications can be written down simply, while the other six require cubic roots; the simple identifications for $\mathcal{WB}_\infty[\mu]$ relate (μ, k_B) to

$$\begin{aligned}
\big(\mu_2, k_B^{(2)}\big) &= \big(\mu^2 + \mu(k_B - 2) + k_B - 1\ ,\ 3 - \tfrac{1}{\mu + k_B - 2} - \mu_2\big) \\
\big(\mu_3, k_B^{(3)}\big) &= \big(\mu^2 + \mu(k_B - 4) - k_B + 4\ ,\ \tfrac{1}{\mu + k_B - 3} + 3 - \mu_3\big) \\
\big(\mu_4, k_B^{(4)}\big) &= \big(\tfrac{\mu(\mu + k_B - 3)}{\mu + k_B - 2}\ ,\ -\mu - k_B + 5 - \mu_4\big) \\
\big(\mu_5, k_B^{(5)}\big) &= \big(\tfrac{2}{\mu + k - 3} + \mu + 1\ ,\ \tfrac{1}{\mu + k_B - 2} + 2 - \mu_5\big) \\
\big(\mu_6, k_B^{(6)}\big) &= \big(-\tfrac{k_B}{\mu + k_B - 2} - \mu + 2\ ,\ 2 - \tfrac{1}{\mu + k_B - 3} - \mu_6\big)\ .
\end{aligned} \tag{3.42}$$

Note that all of these identifications are generated by the two primitive transformations $(\mu, k_B) \mapsto (\mu_2, k_B^{(2)})$ and $(\mu, k_B) \mapsto (\mu_3, k_B^{(3)})$. Similarly, for $\mathcal{WC}_\infty[\mu]$ the simple identifications relate (μ, k_C) to

$$\begin{aligned}
\big(\mu_2, k_C^{(2)}\big) &= \big(\mu^2 + \mu(k_C + 2) + k_C\ ,\ \tfrac{1}{\mu + k_C + 1} - 1 - \mu_2\big) \\
\big(\mu_3, k_C^{(3)}\big) &= \big(\mu^2 + \mu k_C - k_C - 3\ ,\ -\tfrac{1}{\mu + k_C + 2} - 1 - \mu_3\big) \\
\big(\mu_4, k_C^{(4)}\big) &= \big(-\tfrac{\mu(\mu + k_C + 1)}{\mu + k_C + 2}\ ,\ -\mu - k_C - 3 - \mu_4\big) \\
\big(\mu_5, k_C^{(5)}\big) &= \big(-\tfrac{2}{\mu + k_C + 1} + \mu - 1\ ,\ \tfrac{1}{\mu + k_C + 2} - 2 - \mu_5\big) \\
\big(\mu_6, k_C^{(6)}\big) &= \big(-\tfrac{k_C}{\mu + k_C + 2} - \mu\ ,\ -\tfrac{1}{\mu + k_C + 1} - 2 - \mu_6\big)\ .
\end{aligned} \tag{3.43}$$

Again, all of these identifications are generated by the two primitive transformations $(\mu, k_C) \mapsto (\mu_2, k_C^{(2)})$ and $(\mu, k_C) \mapsto (\mu_3, k_C^{(3)})$.

3.3 The coset constructions

It was proposed in [3, 88] that the higher spin theory on AdS_3 based on the even spin algebra — from what we have said above, it is now clear that the relevant algebra is in fact $\mathfrak{hs}^e[\lambda]$ — should be dual to the 't Hooft limit of the $\mathfrak{so}(2n)$ cosets

$$\mathcal{WD}_{n,k} = \frac{\mathfrak{so}(2n)_k \oplus \mathfrak{so}(2n)_1}{\mathfrak{so}(2n)_{k+1}} \ , \tag{3.44}$$

where the 't Hooft limit consists of taking $n, k \to \infty$ while keeping the parameter

$$\lambda = \frac{2n}{2n+k-2} \quad \text{fixed.} \tag{3.45}$$

This therefore suggests that the corresponding quantum $\mathcal{W}^e_\infty$ algebras should be isomorphic. Given that there are two different quantisations of the Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$ (see section 3), there should therefore be two identifications, relating $\mathcal{WD}_{n,k}$ to either $\mathcal{WB}_\infty[\lambda]$ or $\mathcal{WC}_\infty[\lambda]$. In this section we want to explain in detail these different relations. As in the case of $\mathcal{W}_\infty[\mu]$ studied in [73], the (correctly adjusted) correspondences will actually turn out to hold even at finite n and k.

3.3.1 The D_n cosets

In a first step we need to understand the structure of the $\mathcal{W}$ algebra underlying the cosets (3.44). By the usual formula we find that its central charge equals

$$c_{\mathfrak{so}}(2n,k) = n\left[1 - \frac{(2n-2)(2n-1)}{(k+2n-2)(k+2n-1)}\right] \ . \tag{3.46}$$

In order to determine the spin spectrum of the $\mathcal{W}$ algebra we can use that D_n is simply laced, and hence that (3.44) is isomorphic [28] to the Drinfel'd-Sokolov reduction of D_n, which we denote by $\mathcal{WD}_n$; this algebra is of type $\mathcal{W}(2,4,\ldots,2n-2,n)$. In the 't Hooft limit, i.e. for $n \to \infty$, the spin spectrum of $\mathcal{WD}_n$ involves all even spins (with multiplicity one), and hence becomes a $\mathcal{W}^e_\infty$ algebra, but for finite n, this is not the case because of the additional spin n generator, which we shall denote by V. However, as was already explained in [108, 23], $\mathcal{WD}_n$ possesses an outer $\mathbb{Z}_2$ automorphism σ — this is actually the automorphism that is inherited from the spin-flip automorphism of $\mathfrak{so}(2n)$ — under which the generators of spin $2,4,\ldots,2n-2$ are invariant, while the spin n generator V is odd. Then, the 'orbifold' subalgebra $\mathcal{WD}^\sigma_n$, i.e. the σ-invariant subalgebra of $\mathcal{WD}_n$, has the right spin content. It is generated, in addition to the σ-invariant generators of $\mathcal{WD}_n$ of spin $2, 4, \ldots, 2n-2$, by the normal-ordered product of spin $2n$ VV, as well as its higher derivatives that are schematically of the form $V\partial^{2l}V$, see [23].[7]

These arguments imply that we can generate $\mathcal{WD}_n^\sigma$ by (a subset of) the fields contained in $\mathcal{W}^e_\infty$. Hence, $\mathcal{WD}_n^\sigma$ is a quotient of $\mathcal{W}^e_\infty$ and we can characterise it again in terms of the central charge c, and the parameter $\gamma = (c_{44}^4)^2$. As before, a convenient method to compute γ is by comparing the conformal dimension of the 'minimal' representations. Since $\mathcal{WD}_n^\sigma$ is a subalgebra of $\mathcal{WD}_n$, each representation of $\mathcal{WD}_n$ defines also a representation of $\mathcal{WD}_n^\sigma$. In particular, the 'minimal' representations of $\mathcal{WD}_n$ that are labelled by $(v;0)$ and $(0;v)$ — see [88] for our conventions — are also minimal for $\mathcal{WD}_n^\sigma$, and their conformal dimensions equal

$$h(v;0) = \frac{1}{2}\left[1 + \frac{2n-1}{k+2n-2}\right] , \qquad h(0;v) = \frac{1}{2}\left[1 - \frac{2n-1}{k+2n-1}\right] . \tag{3.47}$$

Both solve equation (3.25) for $\gamma = \gamma_{\mathfrak{so}}(N,k)$, where $N = 2n$ and

$$\begin{aligned}\gamma_{\mathfrak{so}} = 144\big(&-224 + 744k - 860k^2 + 408k^3 - 68k^4 + (376 - 1336k \\ &+ 1267k^2 - 386k^3 + 19k^4)N + (-124 + 857k - 599k^2 + 76k^3)N^2 \\ &+ (-52 - 252k + 94k^2)N^3 + (24 + 36k)N^4\big)^2 n_4 \,/\, \Big[c_{\mathfrak{so}}(2+k)(N-1) \\ &\times (2k - 4 + N)(k - 5 + 2N)(3k - 4 + 2N)(2k - 2 + 3N)(3k - 5 + 4N) \\ &\times (88 - 132k + 44k^2 - 132N + 73kN + 5k^2N + 44N^2 + 10kN^2)\Big] . \end{aligned} \tag{3.48}$$

It is interesting that also $h = n$ solves equation (3.25) for $\gamma = \gamma_{\mathfrak{so}}(2n, k)$, thus implying that also the field V generates a minimal representation of $\mathcal{WD}_n^\sigma$.

3.3.2 The B_n cosets

A closely related family of cosets is obtained from (3.44) by considering instead the odd $\mathfrak{so}$ algebras, i.e.

$$\mathcal{WB}(0,n)^{(0)} = \frac{\mathfrak{so}(2n+1)_k \oplus \mathfrak{so}(2n+1)_1}{\mathfrak{so}(2n+1)_{k+1}} . \tag{3.49}$$

These $\mathcal{W}$ algebras can be identified with the bosonic subalgebra of the Drinfel'd-Sokolov reduction of the superalgebras $\mathfrak{osp}(1|2n)$ or $B(0,n)$, see [28]. The latter is a $\mathcal{W}$ algebra of type $\mathcal{W}(2, 4, \ldots, 2n, n+\frac{1}{2})$, and we shall denote it by $\mathcal{WB}(0,n)$.

[7] The counting of the quasiprimary higher spin states is essentially equivalent to the counting of the higher-spin fields of a theory of a single real boson, see e.g. [120]. Note that, as is also explained in [23], the resulting algebra is neither freely generated nor infinitely generated, i.e. there are relations between the $\mathcal{W}^e_\infty$ type generators that effectively reduce these generators to a finite set.

Since the field of conformal weight $n+\frac{1}{2}$ is fermionic — we shall denote it by S in the following — the bosonic subalgebra does not include S, but contains instead the normal-ordered products $S\partial^{2l+1}S$ with $l=0,1,\ldots$ — since $SS=0$ we now always have an odd number of derivatives.[8] Thus the generating fields include, in addition to the bosonic generating fields of $\mathcal{WB}(0,n)$ of spin $2,4,\ldots,2n$, fields of spin $2n+2, 2n+4,\ldots$; in particular, $\mathcal{WB}(0,n)^{(0)}$ is therefore again a quotient of $\mathcal{W}^e_\infty$, and can be characterised in terms of γ and c. The analysis is essentially identical to what was done for the $\mathfrak{so}(2n)$ case above — indeed, the central charge, as well as the conformal dimensions of the minimal representations are obtained from (3.46) and (3.47) upon replacing $2n\mapsto 2n+1$, and thus γ is simply $\gamma=\gamma_{\mathfrak{so}}(N,k)$, where $N=2n+1$ and $\gamma_{\mathfrak{so}}$ was already defined in (3.48). Thus these two families of cosets are naturally analytic continuations of one another.

As an additional consistency check we note that the algebra $\mathcal{WB}(0,1)^{(0)}$ is of type $\mathcal{W}(2,4,6)$, see [27], and its structure constants are explicitly known [107]. In section 3.1.2.2 we have reproduced this algebra as a quotient of $\mathcal{W}^e_\infty$. The corresponding value of γ, given in equation (3.11), agrees indeed with $\gamma_{\mathfrak{so}(3)}$.

3.3.3 Level–rank duality

The expressions (3.46) and (3.48) are invariant under the transformation

$$N\mapsto N\ , \qquad k\mapsto -2N-k+3\ . \tag{3.50}$$

For even $N=2n$ this is a consequence of the Langlands self-duality of D_n, which in turn follows from the fact that D_n is simply laced, implying that the Drinfel'd-Sokolov reduction has the symmetry $\alpha_\pm\mapsto-\alpha_\mp$. As a result, $\mathcal{WD}_n$ actually only depends on c, rather than directly on k. This is reflected in the fact that $\gamma_{\mathfrak{so}}$ can be written as an unambiguous function of N and c as

$$\begin{aligned}\gamma_{\mathfrak{so}} = 72\big(&2c^2(N^2-2N-18)+3c(6N^3-49N^2+80N-8)\\ &+2N(6N^2+5N-28)\big)^2 n_4\Big/\Big[(5c+22)c(c(N^2-7N+12)+2N^2-5N)\\ &\times(c(N+1)+4N^2-5N)(2c(N+2)+3N^2-14N+8)\Big]\ .\end{aligned} \tag{3.51}$$

Note that, in the large c limit, $\mathcal{WD}^\sigma_n$ becomes a classical Poisson algebra, which can be identified with the σ-invariant classical Drinfel'd-Sokolov reduction of D_n. In fact, taking n_4 as in equation (3.18), it follows from equation (3.51) that the corresponding γ parameter equals

$$\gamma_{\mathfrak{so}}=\tfrac{144}{5}(\mu^2-19)^2+\mathcal{O}(c^{-1})\ , \qquad \text{where} \quad \mu=2n-1\ . \tag{3.52}$$

[8] The counting of the quasiprimary higher spin fields is in this case analogous to that of counting the higher spin fields of a single free fermion.

Note that this ties in with the fact that the wedge algebra of $\mathcal{WD}_n^\sigma$ is the σ-invariant subalgebra of $\mathfrak{so}(2n)$, which in turn equals $\mathfrak{so}(2n-1)$. This explains why (3.52) agrees with (3.41) for $\mu = 2n-1$.

Next we observe that equation (3.51) is a polynomial equation of order 6 in N, with coefficients that are functions of γ and c, and hence there is a six-fold ambiguity in the definition of N. If we parametrise $c = c_{\mathfrak{so}}(N,k)$, then the algebra associated with (N,k) is equivalent to the one associated with

$$
\begin{aligned}
(N_2, k_2) &= \Big(\frac{k+2N-3}{k+N-2}\,,\ \frac{k}{k+N-2}\Big)\,, \\
(N_3, k_3) &= \Big(\frac{k}{k+N-1}\,,\ \frac{2N+k-3}{k+N-1}\Big)\,,
\end{aligned}
\tag{3.53}
$$

while the other three solutions involve cubic roots. Obviously, we can also replace $k \mapsto -2N-k+3$ without modifying the algebra, see (3.50), and thus, expressed in terms of N and k, there are 12 different pairs (N_i, k_i) that define the same algebra. We should also mention that the third solution above is obtained by applying the map $(N,k) \mapsto (N_2,k_2)$ twice. This fundamental transformation has a nice interpretation in terms of a level–rank type duality rather similar to the one appearing for $\mathfrak{su}(N)$ in section 2.2.5:

$$
\left(\frac{\mathfrak{so}(N)_k \oplus \mathfrak{so}(N)_1}{\mathfrak{so}(N)_{k+1}}\right)^\sigma \cong \left(\frac{\mathfrak{so}(M)_l \oplus \mathfrak{so}(M)_1}{\mathfrak{so}(M)_{l+1}}\right)^\sigma\,, \tag{3.54}
$$

where

$$
k = \frac{N-1}{M-1} - N + 2\,, \qquad l = \frac{M-1}{N-1} - M + 2\,, \tag{3.55}
$$

and the superscript σ means that we take the σ-invariant subalgebra if N or M are even integers. Obviously, as a true level–rank duality, this only makes sense if M and N are positive integers. As far as we are aware, this level–rank duality has not been noticed before.

3.3.4 Holography

With these preparations we can now return to the main topic of this section, the precise relation between the σ-even subalgebra of the $\mathfrak{so}(2n)$ cosets of equation (3.44), and the quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$. As we have explained before, all three algebras are in general (quotients of) $\mathcal{W}_\infty^e$ algebras, and hence are uniquely characterised in terms of γ and c. By comparing the relations (3.30) and (3.32) for $\mathcal{WB}_\infty[\mu]$ with (3.46) and (3.48) for the $\mathfrak{so}(2n)$ cosets, we conclude that we have the identification

$$
\mathcal{WD}_{n,k}^\sigma \cong \mathcal{WB}_\infty[\lambda_B]\,, \quad \text{with} \quad \lambda_B = \frac{2n-2}{k+2n-2}\,, \quad k_B = k+2n+1-\lambda_B\,. \tag{3.56}
$$

Similarly, for the case of $\mathcal{WC}_\infty[\mu]$ we find instead from (3.35) and (3.37) that

$$\mathcal{WD}^\sigma_{n,k} \cong \mathcal{WC}_\infty[\lambda_C] \ , \quad \text{with} \quad \lambda_C = \frac{2n}{k+2n-2} \ , \quad k_C = k+2n-3-\lambda_C \ . \tag{3.57}$$

Obviously, using the self-duality relations of the various algebras, see equations (3.42), (3.43) and (3.53), there are also other versions of these identifications, but the above is what is relevant in the context of minimal model holography: the above analysis shows that the (σ-even subalgebra of the) $\mathfrak{so}$ cosets[9] are equivalent to the quantum Drinfel'd-Sokolov reduction of the $\mathfrak{hs}^e[\lambda_{B/C}]$ algebras with $\lambda_{B/C}$ given above. Note that λ_C agrees exactly with λ given in (3.45) above, see also [88], while for λ_B the difference is immaterial in the 't Hooft limit. These statements are now true even at finite n and k, hence giving the correct quantum version of the even spin holography conjecture.

3.3.5 The semiclassical behaviour of the scalar fields

With our detailed understanding of the symmetry algebras at finite c, we can now also address the question of whether the duals of the two minimal coset fields of [3, 88] should be thought of as being perturbative or non-perturbative excitations of the higher spin bulk theory. As in the case studied in [73], this issue can be decided by studying the behaviour of their conformal dimensions in the semiclassical limit, i.e. for $c \to \infty$.

Let us consider then the $\mathcal{WD}^\sigma_n$ coset at fixed n. If c takes one of the actual minimal model values, $c = c_{\mathfrak{so}}(2n, k)$ with $k \in \mathbb{N}$, see equation (3.46), the algebra has the two minimal representations $(v; 0)$ and $(0; v)$, whose conformal dimensions are given in equation (3.47). Written in terms of n and c (rather than n and k), they take the form

$$h_\pm(n, c) = \frac{1}{2}\left(1 + \frac{n - c \pm \sqrt{(c-n)\,(c-(3-4n)^2 n)}}{4(n-1)n}\right) \ , \tag{3.58}$$

where $h(v; 0) = h_+(n, c)$ and $h(0; v) = h_-(n, c)$. Since we know that the algebra $\mathcal{WD}^\sigma_n$ depends only on c (rather than k), it is then clear that (3.58) are the conformal weights of minimal representations for any value of c.

We are interested in the semiclassical limit, which consists of taking $c \to \infty$ at fixed n. There is obviously an ambiguity in how precisely c is analytically continued, but taking c, say, along the positive real axis to infinity, we read off

[9] For $n \in \mathbb{N} + \frac{1}{2}$, the left hand side of equations (3.56) and (3.57) should be understood as the chiral algebra of the cosets (3.49). Coset interpretations exist also when n is a negative half-integer, see section 3.3.7.

from (3.58) that

$$h(v;0) = h_+(n,c) = \frac{1-\mu}{2} + \mathcal{O}(c^{-1}) \ , \tag{3.59}$$

$$h(0;v) = h_-(n,c) = \frac{c}{\mu^2-1} + \mathcal{O}(1) \ , \tag{3.60}$$

where $\mu = 2n-1$, see equation (3.52). In this limit $h(v;0)$ remains finite, while $h(0;v)$ is proportional to c. Thus we conclude that only the coset representation $(v;0)$ corresponds to a perturbative scalar of the higher spin theory based on $\mathfrak{hs}^e[\mu]$, while $(0;v)$ describes a non-perturbative excitation. This is directly analogous to what happened in [73].

3.3.6 The full orbifold spectrum

Now that we have understood the relation between the symmetries in the duality conjecture of [3, 88] we can come back to the comparison of the partition functions that was performed in [88]. It was shown there that the spectrum of the charge conjugate modular invariant of the $\mathcal{WD}_{n,k}$ algebra coincides, in the 't Hooft limit, with the bulk 1-loop partition function of a suitable higher spin theory on thermal AdS_3.

As we have seen above, at finite n and c, the relevant symmetry algebra is actually not $\mathcal{WD}_{n,k}$, but only the σ-invariant subalgebra $\mathcal{WD}^\sigma_{n,k}$. Every representation of $\mathcal{WD}_{n,k}$ defines also a representation of $\mathcal{WD}^\sigma_{n,k}$, and hence the charge conjugation (or A-type) modular invariant of the $\mathcal{WD}_{n,k}$ algebra also defines a consistent partition function with respect to $\mathcal{WD}^\sigma_{n,k}$. However, from the latter point of view, it is not the charge conjugation modular invariant, but rather of what one may call 'D-type'.

It is then natural to ask whether the charge conjugation (A-type) modular invariant of $\mathcal{WD}^\sigma_{n,k}$ also has a bulk interpretation. We shall not attempt to answer this question here, but we shall only show that it leads to a *different* partition function in the 't Hooft limit. Thus, if the charge-conjugation modular invariant of $\mathcal{WD}^\sigma_{n,k}$ also has a consistent AdS_3 dual, this must be a different theory than the one considered in [3, 88].

In the charge conjugation (A-type) modular invariant of the $\mathcal{WD}^\sigma_{n,k}$ algebra, every untwisted representation of $\mathcal{WD}^\sigma_{n,k}$ appears once. Obviously, not all representations of $\mathcal{WD}^\sigma_{n,k}$ arise as subrepresentations of untwisted $\mathcal{WD}_{n,k}$ representations. In particular, each σ-twisted representation of $\mathcal{WD}_{n,k}$ (for which V is half-integer moded) also leads to an untwisted representation of $\mathcal{WD}^\sigma_{n,k}$. Since σ is inherited from the outer automorphism of $\mathfrak{so}(2n)$, these twisted representations of $\mathcal{WD}_{n,k}$

can be described via the cosets

$$\frac{\mathfrak{so}(2n)^{(2)}_k \oplus \mathfrak{so}(2n)^{(2)}_1}{\mathfrak{so}(2n)^{(2)}_{k+1}} \ , \tag{3.61}$$

where $\mathfrak{so}(2n)^{(2)}_k$ is the twisted affine algebra, see e.g. [97] for an introduction. The representations of $\mathfrak{so}(2n)^{(2)}_k$ are labelled by $\mathfrak{so}(2n-1)$ dominant highest weights Ξ, satisfying certain integrability conditions, and the corresponding conformal dimensions equal

$$h_{\mathfrak{so}(2n)^{(2)}_k}(\Xi) = \frac{C(\Xi)}{2(k+2n-2)} + \frac{k(2n-1)}{16(k+2n-2)} \ , \tag{3.62}$$

where C is the Casimir of $\mathfrak{so}(2n-1)$. The conformal dimension of the representations of (3.61) can then be obtained from (3.62) by the usual coset formula. In particular, the twisted vacuum, where we take Ξ to be the vacuum representation ($\Xi = 0$) of $\mathfrak{so}(2n-1)$ for all 3 factors in equation (3.61), has conformal dimension

$$\frac{1}{16}\left[1 - \frac{(2n-1)(2n-2)}{(k+2n-2)(k+2n-1)}\right] \ . \tag{3.63}$$

This state does not appear in the 1-loop bulk higher spin calculation of [88], and thus the dual of the charge conjugation modular invariant of $\mathcal{WD}^{\sigma}_{n,k}$ must be a different bulk theory than the one considered in [88].

3.3.7 Other minimal models

Let us close this discussion with a comment about other minimal models one may consider. As we have seen in sections 3.3.1 and 3.3.2, the dual of the even higher spin theories on AdS can be identified with the cosets of *either* the $\mathfrak{so}$(even) or the $\mathfrak{so}$(odd) algebras. It is then natural to ask how the cosets of the $\mathfrak{sp}$ algebras fit into this picture. Using the field counting techniques of [34] (see also [46]) one can show that the cosets[10]

$$\frac{\mathfrak{sp}(2n)_k \oplus \mathfrak{sp}(2n)_{-1}}{\mathfrak{sp}(2n)_{k-1}} \tag{3.64}$$

possess a $\mathcal{W}^{\rm e}_\infty$ symmetry in the 't Hooft limit. The essential points of this calculation are (i) that $\mathfrak{sp}(2n)_{-1}$ has a free field realisation in terms of n $\beta\gamma$-systems; and (ii) that the coset vacuum character can be computed by counting $\mathfrak{sp}(2n)$ invariant products of $\beta\gamma$-fields and their derivatives, using standard arguments of classical invariant theory.

[10] In our conventions, the *short* roots of $\mathfrak{sp}(2n)$ have length squared equal to 2.

It is then natural to ask what $\mathcal{W}_\infty^e$ algebras the cosets (3.64) lead to when analytically continued in n and k. The answer can be schematically formulated as

$$\frac{\mathfrak{sp}(2n)_k \oplus \mathfrak{sp}(2n)_{-1}}{\mathfrak{sp}(2n)_{k-1}} \cong \left(\frac{\mathfrak{so}(-2n)_{-k} \oplus \mathfrak{so}(-2n)_1}{\mathfrak{so}(-2n)_{-k+1}}\right)^\sigma , \tag{3.65}$$

where both cosets stand for the corresponding $\mathcal{W}_\infty^e$ algebras (or their quotients), and the equality means that both the analytically continued central charge and the self-coupling γ agree.

Incidentally, there is an independent check for our claim that the cosets (3.64) are quotients of $\mathcal{W}_\infty^e$. For $n = 1$, the coset (3.64) is known to be of type $\mathcal{W}(2, 4, 6)$, see [25],[11] and its structure constants have been computed explicitly in [55], coinciding with the solution given in equation (3.10) of section 3.1.2.2. We also note that the corresponding value of γ agrees indeed with $\gamma_{\mathfrak{so}(-2)}$, as required by (3.65).

The above arguments apply similarly for the cosets

$$\frac{\mathfrak{osp}(1|2n)_k \oplus \mathfrak{osp}(1|2n)_{-1}}{\mathfrak{osp}(1|2n)_{k-1}} , \tag{3.66}$$

for which the emergence of a $\mathcal{W}_\infty^e$ symmetry in the 't Hooft limit can be proven using analogous methods, in particular, noting that $\mathfrak{osp}(1|2n)_{-1}$ has a free field realisation in terms of a single Majorana fermion and n $\beta\gamma$-systems. In this case, the analogue of (3.65) is

$$\frac{\mathfrak{osp}(1|2n)_k \oplus \mathfrak{osp}(1|2n)_{-1}}{\mathfrak{osp}(1|2n)_{k-1}} \cong \frac{\mathfrak{so}(-2n+1)_{-k} \oplus \mathfrak{so}(-2n+1)_1}{\mathfrak{so}(-2n+1)_{-k+1}} . \tag{3.67}$$

[11] We thank the authors of [46] for drawing our attention to this reference.

Chapter 4

The supersymmetric duality and orbifold constructions

4.1 The $\mathcal{N} = 2$ supersymmetric duality

Having discussed holographic dualities involving the $\mathfrak{su}(N)$ and $\mathfrak{so}(N)$ minimal models, we would now like to turn to the supersymmetric case. In particular, we will be concerned with the special case of $\mathcal{N} = 2$ supersymmetry for the rest of this thesis, which lies halfway between the bosonic cases we have studied in the previous sections and the maximal $\mathcal{N} = 4$ supersymmetry. This is an important step towards understanding the $\mathcal{N} = 4$ holographic duality [75] and its relation to string theory [76, 77, 78]. In the following we shall give a brief introduction to the $\mathcal{N} = 2$ generalisation of the original bosonic duality that has first been proposed in [45]. Parts of this overview are directly taken from our paper [83].

4.1.1 Kazama-Suzuki models

Let us first describe the family of conformal field theories which have been related in [45] to the $\mathcal{N} = 2$ supersymmetric Vasiliev theory on AdS_3 described in [141]. These are $\mathcal{N} = 2$ superconformal field theories (SCFTs), which means that their vacuum sector contains an extension of the conformal algebra, the $\mathcal{N} = 2$ *superconformal algebra* (SCA). It consists of the stress-energy tensor L of spin 2 generating a Virasoro subalgebra, two supercharges $G^{\pm}$ of spin $\frac{3}{2}$ and a U(1) current J of spin 1. They satisfy the OPEs

$$
\begin{aligned}
&L \star L \sim \tfrac{c}{2} I \ , && L \star J \sim J \ , && L \star G^{\pm} \sim \tfrac{3}{2} G^{\pm} \ , \\
&J \star J \sim \tfrac{c}{3} I \ , && J \star G^{\pm} \sim \pm G^{\pm} \ ,
\end{aligned}
$$

$$G^+ \star G^- \sim 2\,J + \tfrac{2c}{3} I \ , \qquad G^\pm \star G^\pm \sim 0 \ . \tag{4.1}$$

The $\mathcal{N}=2$ SCFTs obtained by a coset construction have been classified by Kazama and Suzuki [114, 115]. The simplest examples of such cosets are of the form

$$\frac{\mathfrak{su}(N+1)^{(1)}_{N+k+1}}{\mathfrak{su}(N)^{(1)}_{N+k+1} \oplus \mathfrak{u}(1)^{(1)}_{\kappa}} \cong \frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)_\kappa} \ . \tag{4.2}$$

Here the first description is in terms of $\mathcal{N}=1$ superalgebras, where the superscript '(1)' means that we have supersymmetrised the Kac-Moody algebra such that for each (spin-1) current of the algebra we also have a Majorana fermion of spin $\frac{1}{2}$. The second expression is in terms of bosonic affine algebras, where we have used that the fermions decouple and become free after a suitable change of basis of the superalgebras. Out of the N^2+2N free fermions of the numerator, only $2N$ survive the coset construction. These can be represented as the lowest states in the vector representation of $\mathfrak{so}(2N)_1$. Since we only consider the NS sector of the theory, the $\mathfrak{so}(2N)_1$ factor in the coset only stands for the vacuum and vector representations. Furthermore, the level of the $\mathfrak{u}(1)$ factor equals $\kappa = N(N+1)(N+k+1)$, and the central charge of the coset is

$$c = (N-1) + \frac{Nk(N+2)}{N+k+1} - \frac{(N^2-1)(k+1)}{N+k+1} = \frac{3Nk}{N+k+1} \ . \tag{4.3}$$

These cosets have manifest $\mathcal{N}=1$ supersymmetry, but Kazama and Suzuki showed [114, 115] that they actually even exhibit $\mathcal{N}=2$ supersymmetry. In order to understand the coset construction in detail, we need to explain how the denominator algebra sits inside the numerator. At the level of Lie groups, the denominator $\mathrm{SU}(N)\times\mathrm{U}(1)$ is 'embedded' into $\mathrm{SU}(N+1)$ via the (N-to-one) mapping

$$(v,w) \mapsto \begin{pmatrix} \bar{w}v & 0 \\ 0 & w^N \end{pmatrix} \ , \tag{4.4}$$

where $w \in \mathrm{U}(1)$ is a phase, while $v \in \mathrm{SU}(N)$ is an $N \times N$ matrix. Similarly, the 'embedding' into $\mathrm{SO}(N,N)$ (whose complexified Lie algebra agrees with the complexification of $\mathfrak{so}(2N)$) is defined by

$$(v,w) \mapsto \begin{pmatrix} \bar{w}^{N+1}v & 0 \\ 0 & w^{N+1}\bar{v} \end{pmatrix} \ . \tag{4.5}$$

At the level of Lie algebras the embedding works accordingly. Our conventions are chosen so that the free fermions and bosons have $\mathrm{U}(1)$ charge $\pm(N+1)$. An explicit construction of the spin-1 and spin-2 currents can be found in appendix D (see also [34] and [37]).

The $s\mathcal{W}_\infty$ algebra of this coset has been studied in detail in [35] (see also [6]). It is generated by the SCA and one $\mathcal{N}=2$ multiplet $W^{(s)}$ for each integer $s \geq 2$, where $W^{(s)}$ consists of the Virasoro-primary fields

$$W^{s0} \ , \quad W^{s\,\pm} = G^{\pm}_{-\frac{1}{2}} W^{s0} \ , \quad W^{s1} = \tfrac{1}{4}(G^{+}_{-\frac{1}{2}} G^{-}_{-\frac{1}{2}} - G^{-}_{-\frac{1}{2}} G^{+}_{-\frac{1}{2}}) W^{s0} \tag{4.6}$$

of conformal dimensions s, $s+\frac{1}{2}$ and $s+1$, respectively. In terms of Virasoro-primary fields (or quasi-primary in the case of L itself), the algebra is therefore generated by one field of spin 1 and two fields for each integer or half-integer spin $s > 1$. The fields in (4.6) then satisfy the following OPEs with the SCA:

$$\begin{aligned}
&L \star W^{s0} \sim s\, W^{s0} \ , \quad L \star W^{s\pm} \sim \left(s + \tfrac{1}{2}\right) W^{s\pm} \ , \quad L \star W^{s1} \sim (s+1)\, W^{s1} \ , \\
&J \star W^{s0} \sim 0 \ , \qquad J \star W^{s\pm} \sim \pm W^{s\pm} \ , \qquad J \star W^{s1} \sim s\, W^{s0} \ , \\
&G^{\pm} \star W^{s0} \sim \mp W^{s\pm} \ , \quad G^{\pm} \star W^{s1} \sim (s + \tfrac{1}{2}) W^{s\pm} \ , \\
&G^{\pm} \star W^{s\pm} \sim 0 \ , \qquad G^{\pm} \star W^{s\mp} \sim 2\, W^{s1} + 2s\, W^{s0} \ .
\end{aligned}$$

It was shown in [35] that any $\mathcal{N}=2$ $s\mathcal{W}_\infty$ algebra with the field content mentioned above is fixed by the Jacobi identities up to two free parameters: the central charge c and the self-coupling of the supermultiplet $W^{(2)}$, $\gamma = (c_{22}^{2})^2$. The wedge algebra is

$$\mathfrak{shs}[\mu] \oplus \mathbb{R} = \frac{\mathcal{U}\left(\mathfrak{osp}(1|2)\right)}{\langle C^{\mathfrak{osp}} - \frac{1}{4}\mu(\mu-1)\mathbb{1}\rangle} \ . \tag{4.7}$$

This construction is analogous to the bosonic higher-spin algebra defined by (2.39) and (2.48), but this time it is based on the universal enveloping algebra of $\mathfrak{osp}(1|2)$, owing to the fact that $\mathcal{N}=2$ supergravity on AdS$_3$ can be described as a Chern-Simons theory based on $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ [1]. By definition we have $\mathfrak{shs}[\mu] = \mathfrak{shs}[1-\mu]$. Furthermore, for $-\mu = N \in \mathbb{N}$, $\mathfrak{shs}[\mu]$ acquires a maximal ideal χ_N and truncates to

$$\mathfrak{shs}[-N]/\chi_N \cong \mathfrak{sl}(N+1|N) \ . \tag{4.8}$$

For the identification of the wedge algebra, one can find a relation between γ and μ using the minimal representation technique introduced in section 2.2.5 and finds

$$\gamma = \frac{8(c+3)^2(c-2c\mu-3\mu^2)^2\, n_2}{c(c-1)(c-3+6\mu)(\mu+1)(c-3\mu)(2c+3\mu-c\mu)} \ , \tag{4.9}$$

where n_2 is the central term in the OPE of W^{20} with itself. We see that the right-hand side of equation (4.9) is quartic in μ, which leads to a quadrality relation (as opposed to the triality found in the original bosonic $\mathcal{W}_\infty$ algebra). This quadrality leads to the isomorphism of $\mathcal{N}=2$ $s\mathcal{W}_\infty$ algebras

$$s\mathcal{W}_\infty[\mu_1] \cong s\mathcal{W}_\infty[\mu_2] \cong s\mathcal{W}_\infty[\mu_3] \cong s\mathcal{W}_\infty[\mu_4] \ , \tag{4.10}$$

where the different values of μ are related by

$$\mu_1 = \mu \ , \quad \mu_2 = \frac{c(1-\mu)}{c+3\mu} \ , \quad \mu_3 = \frac{c+3\mu}{3(\mu-1)} \ , \quad \mu_4 = -\frac{c}{3\mu} \ . \tag{4.11}$$

The representations of the coset are labelled by $(\Lambda_+;\Lambda_-,\ell)$, where Λ_+ is an integrable weight of $\mathfrak{su}(N+1)_k$ and Λ_- an integrable weight of $\mathfrak{su}(N)_{k+1}$, while ℓ denotes the $\mathfrak{u}(1)$ charge. The selection rule is

$$\frac{|\Lambda_+|}{N+1} - \frac{|\Lambda_-|}{N} - \frac{\ell}{N(N+1)} \in \mathbb{Z} \ , \tag{4.12}$$

where $|\Lambda| = \sum_j j\Lambda_j$, and we have the field identification

$$(\Lambda_+;\Lambda_-,\ell) \ \cong \ \big(J^{(N+1)}\Lambda_+; J^{(N)}\Lambda_-, \ell-(k+N+1)\big) \ , \tag{4.13}$$

where J denotes the outer automorphism of equation (2.87), i.e., it maps (for the case of $\mathfrak{su}(N+1)$)

$$\Lambda = [\Lambda_0;\Lambda_1,\ldots,\Lambda_N] \ \mapsto \ J^{(N+1)}\Lambda = [\Lambda_N;\Lambda_0,\Lambda_1,\ldots,\Lambda_{N-1}] \ . \tag{4.14}$$

Since the field identification acts simultaneously on a weight in $\mathfrak{su}(N+1)$ and $\mathfrak{su}(N)$, it has order $N(N+1)$; this then ties together with the fact that the $\mathfrak{u}(1)$ charge ℓ is defined modulo $\kappa = N(N+1)(N+k+1)$.
The conformal dimension of the representation $(\Lambda_+;\Lambda_-,\ell)$ equals

$$h(\Lambda_+;\Lambda_-,\ell) = \frac{C^{(N+1)}(\Lambda_+)}{N+k+1} - \frac{C^{(N)}(\Lambda_-)}{N+k+1} - \frac{\ell^2}{2N(N+1)(N+k+1)} + n \ , \tag{4.15}$$

where n is a half-integer, describing the level at which (Λ_-,ℓ) appears in the representation Λ_+, and $C^{(N)}(\Lambda)$ is the quadratic Casimir of the $\mathfrak{su}(N)$ weight Λ. Finally, the U(1) charge (with respect to the U(1) generator of the superconformal $\mathcal{N}=2$ algebra) equals

$$q(\Lambda_+;\Lambda_-,\ell) = \frac{\ell}{N+k+1} + s \ , \tag{4.16}$$

where $s \in \mathbb{Z}$ denotes the charge contribution of the descendants. For example, the representation

$$(\mathrm{f};0,N): \qquad h = \frac{N}{2(N+k+1)} \ , \quad q = \frac{N}{N+k+1} \ , \tag{4.17}$$

where f denotes the fundamental representation of $\mathfrak{su}(N+1)$, describes a chiral primary, as does

$$h(0;\mathrm{f},-(N+1)) = \frac{1}{2} - \frac{(N^2-1)}{2N(N+k+1)} - \frac{(N+1)}{2N(N+k+1)} = \frac{k}{2(N+k+1)} \ , \tag{4.18}$$

for which the U(1) charge equals

$$q(0;\mathrm{f},-(N+1)) = \frac{-(N+1)}{N+k+1} + 1 = \frac{k}{N+k+1} \ . \tag{4.19}$$

Here the additional terms in (4.18) and (4.19) appear because for $(0;\mathrm{f},-(N+1))$ the representation of the denominator arises only at the first excited level.

4.1.2 $\mathcal{N}=2$ holography

In close analogy to the bosonic cases, the Kazama-Suzuki coset models (4.2) were shown to be dual to a supersymmetric higher spin theory based on $\mathfrak{shs}[\lambda]$ [141, 142] defined in (4.7) in the large N, k 't Hooft limit where

$$\lambda = \frac{N}{N+k+1} \tag{4.20}$$

is kept constant [45, 34]. This theory is coupled to two $\mathcal{N}=2$ matter multiplets, each one consisting of two complex scalars of masses

$$(M^B_\lambda)^2 = -1+\lambda^2 \ , \quad (M^B_{1-\lambda})^2 = -1+(1-\lambda)^2 \ , \tag{4.21}$$

and two Dirac fermions of mass

$$(M^F)^2 = \left(\lambda - \frac{1}{2}\right)^2 \ . \tag{4.22}$$

These are then dual to the perturbative coset representations whose lowest exponents are the minimal representations

$$(\mathrm{f};0,N) \ , \quad (\bar{\mathrm{f}};0,-N) \ , \quad (0;\mathrm{f},-(N+1)) \ , \quad (0;\bar{\mathrm{f}},N+1) \tag{4.23}$$

of conformal dimensions

$$\begin{aligned} h(\mathrm{f};0,N) = h(\bar{\mathrm{f}};0,N) &= \frac{N}{2(N+k+1)} \cong \frac{\lambda}{2} \ , \\ h(0;\mathrm{f},-(N+1)) = h(0;\bar{\mathrm{f}},N+1) &= \frac{k}{2(N+k+1)} \cong \frac{1-\lambda}{2} \ . \end{aligned} \tag{4.24}$$

This duality has undergone several non-trivial checks: First, the asymptotic symmetries of $\mathfrak{shs}[\lambda]$ (for the case of $\mathfrak{sl}(N+1|N)$, this had already been done in [111]) were shown to agree with the coset algebra [105, 101]; it was also shown in [35] that the structure constants of $s\mathcal{W}_\infty[\mu]$ reduce to those of $\mathfrak{shs}[\mu]$ on the wedge and in the limit $c \to \infty$. Furthermore, the perturbative partition functions of the two theories were shown to agree in the 't Hooft limit in [34], and three-point functions were matched in [47].

4.2 Orbifolds

The orbifold construction is a procedure to obtain a new conformal field theory out of some original CFT by making it invariant under a group G acting on the target space manifold (see [90, 24] for pedagogical introductions). 2-dimensional CFTs such as the ones found in string theory usually live on Riemann surfaces describing the world sheet, the most important example beyond tree level being the torus. In the AdS/CFT case that is of interest to us, the CFT lives on the boundary of (thermal) AdS_3, which is also a torus. We will therefore assume our CFT to be defined on a torus with modular parameter τ. For the sake of simplicity, we will also assume the group G to be finite, but we will see examples of (compact) continuous groups acting on target space in chapter 5.

The Hilbert space of the *untwisted sector* consists of all states of the original theory that are invariant under the group action. The projector onto this sector can be written as

$$P^{(\mathrm{U})} = \frac{1}{|G|} \sum_{g \in G} g \ . \tag{4.25}$$

It is straightforward to see that $(P^{(\mathrm{U})})^2 = P^{(\mathrm{U})}$ and that $g \cdot (P^{(\mathrm{U})}\,\xi) = P^{(\mathrm{U})}\,\xi$ for any state ξ and group element g, which shows that $(P^{(\mathrm{U})}\,\xi)$ is indeed in the untwisted sector. The untwisted partition function is then given by the trace over the Hilbert space of the original CFT with insertion of the projector $P^{(\mathrm{U})}$:

$$Z^{(\mathrm{U})}(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}} \left(P^{(\mathrm{U})}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \ , \quad q = e^{2\pi i \tau} \ . \tag{4.26}$$

In order to make the theory modular invariant, however, one needs to include *twisted sectors* as well. The fields $X(z)$ in this sector are defined on the torus, but pick up a twist $h \in G$ when going around the space cycle:

$$X(z+1) = hX(z) \ . \tag{4.27}$$

These fields should also be invariant under some group action g; indeed, acting on (4.27) with some $g \in G$ and assuming invariance, we obtain

$$ghX(z) = gX(z+1) = X(z+1) = hX(z) = hgX(z) \ . \tag{4.28}$$

Hence we can only demand invariance under all $g \in G$ which commute with h. These form the centraliser subgroup $C^h < G$. So the partition function of the h-twisted sector reads

$$Z^{(h)}(\tau, \bar{\tau}) = \mathrm{Tr}_{\mathcal{H}_h} \left(P^{(h)}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \ , \tag{4.29}$$

where $P^{(h)}$ is now the projector onto the C^h-invariant subspace of the h-twisted space $\mathcal{H}_h$, i.e.

$$P^{(h)} = \frac{1}{|C^h|} \sum_{g \in C^h} g \ . \tag{4.30}$$

Note that for any field X in the h-twisted sector and $g \in G$, gX is in the h'-twisted sector for $h' = ghg^{-1}$ since

$$(gX)(z+1) = ghX(z) = ghg^{-1}(gX)(z) \ . \tag{4.31}$$

Moreover, the centralisers of h and h' are isomorphic: if a commutes with h, then $a' = gag^{-1}$ commutes with h'. It follows that for any h' that is conjugate to h, the h'-twisted sector and the h-twisted sector are in fact equivalent and we need to include only one of them into our theory. The twisted sectors therefore have to be labelled by conjugacy classes $[h]$ rather than elements of G. The full orbifold partition function then reads

$$Z_{\mathrm{orb}}(\tau, \bar{\tau}) = \sum_{[h] \subset G} \frac{1}{|C^{[h]}|} \sum_{g \in C^h} \mathrm{Tr}_{\mathcal{H}_h} \left(g\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \ . \tag{4.32}$$

The untwisted sector corresponds to the conjugacy class $[e]$ consisting only of the neutral element $e \in G$. If G is an abelian group, the centraliser of any element is the whole group, and each conjugacy class consists of only one element. The partition function (4.32) therefore simplifies to

$$Z_{\mathrm{orb}}(\tau, \bar{\tau}) = \frac{1}{|G|} \sum_{g,h \in G} \mathrm{Tr}_{\mathcal{H}_h} \left(g\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \tag{4.33}$$

in that case.

Chapter 5

The continuous orbifold of $\mathcal{N} = 2$ minimal models

In this chapter we will study the $\mathcal{N} = 2$ Kazama-Suzuki cosets

$$\frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)_{N(N+1)(N+k+1)}} \tag{5.1}$$

that were introduced in section 4.1.1 in the limit where the level k is taken to infinity, while N is large but fixed. It will turn out that the limit theory has an interpretation as a $\mathrm{U}(N)$ orbifold of $2N$ free fermions and bosons that both transform as $\mathbf{N} \oplus \bar{\mathbf{N}}$ under $\mathrm{U}(N)$.[1] This is the natural generalisation of the bosonic analysis of [87], where it was shown that the cosets

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \tag{5.2}$$

admit a description in terms of an orbifold of $N-1$ free bosons by the Lie group $\mathrm{SU}(N)$. In each of these cases the limit is taken in the spirit of [145] (rather than say [144]), see also [64, 67] for other instances where this kind of construction has been considered. Note that this orbifold picture is the natural 2d analogue of the $\mathrm{U}(N)$ (or $\mathrm{O}(N)$) singlet sector of a theory of free bosons or fermions in 3d that appears in the duality with higher spin theories on AdS_4 [120, 152].

We shall consider the usual charge conjugation modular invariant of the coset (5.1). In particular, we shall see in section 5.1 that the part of the CFT spectrum that corresponds to the perturbative higher spin degrees of freedom

$$\mathcal{H}_{\text{pert}} = \bigoplus_{\Lambda} \mathcal{H}_{(0;\Lambda)} \otimes \overline{\mathcal{H}}_{(0;\Lambda^*)} \tag{5.3}$$

[1] The idea that the limit theory has such an interpretation was already mentioned in [143], following on from the analysis of [65], where this was shown explicitly for $N = 1$.

can be identified, for $k\to\infty$, with the subspace of the free field theory of $2N$ bosons and fermions that are singlets with respect to $\mathrm{U}(N)$, i.e., with the untwisted sector of the continuous orbifold. The remaining coset primaries, i.e., those of the form $(\Lambda_+;\Lambda_-)$ with $\Lambda_+\neq 0$, can then be interpreted in terms of the various twisted sectors of the continuous orbifold, as shall be shown in section 5.2. In fact, as has been mentioned in section 4.2 for usual orbifolds, the untwisted sector is not modular invariant by itself, and the twisted sectors are required in order to restore modular invariance. For the case at hand where we have supersymmetry, the identification of the different coset primaries with the twisted sectors can be worked out in detail, and a number of non-trivial consistency checks can be performed. In particular, we have compared the conformal dimension of the twisted sector ground states with that calculated from the coset viewpoint; we have also determined the fermionic excitation spectrum directly from the coset perspective.

The material in this chapter is based on a collaboration with Matthias Gaberdiel and has been published in [83] (see also [66] for related work). A similar analysis for the case of another class of Kazama-Suzuki cosets, given by

$$\frac{\mathfrak{so}(2N+2)^{(1)}_{2N+k}}{\mathfrak{so}(2N)^{(1)}_{2N+k}\oplus\mathfrak{so}(2)^{(1)}_{\kappa}}\ , \tag{5.4}$$

was later carried out in [59].

5.1 The untwisted sector of the continuous orbifold

We are interested in taking the $k\to\infty$ limit of the cosets

$$\frac{\mathfrak{su}(N+1)^{(1)}_{k+N+1}}{\mathfrak{su}(N)^{(1)}_{k+N+1}\oplus\mathfrak{u}(1)^{(1)}_{\kappa}}\ \cong\ \frac{\mathfrak{su}(N+1)_k\oplus\mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1}\oplus\mathfrak{u}(1)_\kappa}\ , \tag{5.5}$$

which were introduced in section 4.1. For the case $N=1$ with $c=3$, this was worked out in some detail in [65], where it was shown that the resulting theory can be interpreted in terms of a continuous $\mathrm{U}(1)$ orbifold. Here we want to extend the discussion to general N. The idea that the limit theory may be interpreted in terms of a $\mathrm{U}(N)$ orbifold was already sketched in [143]; in the following, we shall pursue a somewhat different approach and be much more explicit.

The discussion of [34] as well as the analogous analysis in [87] suggests that the underlying free theory consists of $2N$ free bosons and free fermions that transform as

$$\mathbf{N}_{-(N+1)}\oplus\bar{\mathbf{N}}_{N+1} \tag{5.6}$$

with respect to $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ in the denominator. The relevant orbifold group is therefore $\mathrm{SU}(N) \times \mathrm{U}(1)$, or equivalently $\mathrm{U}(N)$,[2] where the group acts simultaneously on both left- and right-movers.

One reason in favour of this idea is that the central charge approximates in this limit

$$c = \frac{3Nk}{N+k+1} \cong 3N \ , \tag{5.7}$$

in agreement with a description in terms of $2N$ free bosons and fermions. Furthermore, the ground states of the representations $(0; \mathrm{f}, -(N+1))$ and $(0; \bar{\mathrm{f}}, (N+1))$ can be identified with the $\mathbf{N}+\bar{\mathbf{N}}$ free fermions since their conformal dimension and $\mathfrak{u}(1)$ charge become in this limit

$$h(0; \mathrm{f}, -(N+1)) = h(0; \bar{\mathrm{f}}, (N+1)) = \frac{1}{2} \ , \tag{5.8}$$

as well as

$$q(0; \mathrm{f}, -(N+1)) = +1 \ , \qquad q(0; \bar{\mathrm{f}}, (N+1)) = -1 \ . \tag{5.9}$$

Each of these representations has two $\mathcal{N}=2$ descendants with $h=1$, which can in turn be identified with the free bosons. For the actual coset partition function, left- and right-movers are grouped together, i.e., $(0; \mathrm{f}, -(N+1))$ for the left-movers appears together with $(0; \bar{\mathrm{f}}, (N+1))$ for the right-movers, etc., and this is precisely what the $\mathrm{U}(N)$ singlet condition achieves.

Concretely, we therefore claim that the untwisted sector of the $\mathrm{U}(N)$ orbifold of $2N$ free bosons and fermions, transforming as in (5.6), corresponds to the subsector of the coset theory

$$\mathcal{H}_0 = \bigoplus_{\Lambda, u} \mathcal{H}_{(0;\Lambda,u)} \otimes \bar{\mathcal{H}}_{(0;\Lambda^*,-u)} \tag{5.10}$$

in the limit $k \to \infty$. Here the sum runs over all representations Λ that appear in finite tensor powers of the fundamental or anti-fundamental representation of $\mathfrak{su}(N)$ — in the limit $k \to \infty$, the k-dependent bound on the integrable $\mathfrak{su}(N)_{k+1}$ representations disappears — and Λ^* denotes the representation conjugate to Λ. Furthermore, u must satisfy the selection rule that $(N+1)|\Lambda| + u = 0 \bmod N(N+1)$.

In the following we will give strong evidence in favour of this claim by showing that the partition functions agree. In section 5.2 we shall then also explain how the twisted sectors of the continuous orbifold can be understood from the coset viewpoint.

[2]The discrete subgroup of $\mathrm{SU}(N) \times \mathrm{U}(1)$ that needs to be factored out to obtain $\mathrm{U}(N)$ acts trivially.

5.1.1 The partition function from the coset

We want to show that the spectrum of the untwisted sector of the $\mathrm{U}(N)$ orbifold coincides with equation (5.10) by comparing partition functions. In order to do so, we need to understand the character of the coset representations $(0;\Lambda,u)$ in the limit $k\to\infty$. For large k, the character of an affine representation Λ of $\mathfrak{su}(N)_k$ is given by

$$\mathrm{ch}^{N,k}_{\Lambda}(v;q)=\frac{q^{h^{N,k}_{\Lambda}}\left[\mathrm{ch}^{N}_{\Lambda}(v)+\mathcal{O}(q^{k-\sum_i\Lambda_i+1})\right]}{\prod\limits_{n=1}^{\infty}\left[(1-q^n)^{N-1}\prod\limits_{i\neq j}(1-v_i\bar{v}_jq^n)\right]}\ . \tag{5.11}$$

Here v_i are the eigenvalues of $v\in\mathrm{SU}(N)$, $\mathrm{ch}^N_{\Lambda}(v)$ is the character of Λ restricted to the zero-mode subalgebra $\mathfrak{su}(N)$, Λ_i are the Dynkin labels of Λ, and we define

$$h^{N,k}_{\Lambda}=\frac{C^{(N)}(\Lambda)}{N+k}\ , \tag{5.12}$$

where $C^{(N)}(\Lambda)$ is, as before, the quadratic Casimir of Λ. For example, the vacuum character $\mathrm{ch}^{N+1,k}_0(v,w;q)$ of $\mathfrak{su}(N+1)_k$ with $v\in\mathrm{SU}(N)$ and $w\in\mathrm{U}(1)$ embedded into $\mathrm{SU}(N+1)$ as in (4.4) equals

$$\mathrm{ch}^{N+1,k}_0=\frac{1+\mathcal{O}(q^{k+1})}{\prod\limits_{n=1}^{\infty}\Big[(1-q^n)^N\prod\limits_{i\neq j}(1-v_i\bar{v}_jq^n)\prod\limits_{i=1}^{N}\left[(1-\bar{w}^{N+1}v_iq^n)(1-w^{N+1}\bar{v}_iq^n)\right]\Big]}\ . \tag{5.13}$$

Moreover, the representations of the $\mathfrak{so}(2N)_1$ factor in the numerator are the vacuum and vector representation, as well as either of the two spinor representations. In terms of the free fermions (that are equivalent to $\mathfrak{so}(2N)_1$), the former two correspond to the NS sector, while the latter are accounted for in terms of the R sector. In the following we shall concentrate on the NS sector[3] for which the contribution of the $2N$ free fermions equals

$$\theta(v,w;q)=\prod_{n=1}^{\infty}\prod_{i=1}^{N}(1+\bar{w}^{N+1}v_iq^{n-\frac{1}{2}})(1+w^{N+1}\bar{v}_iq^{n-\frac{1}{2}})\ . \tag{5.14}$$

The characters of the denominator, on the other hand, are given in that limit by

$$\mathrm{ch}^{N,k+1}_{\Lambda,u}(v,w;q)=\frac{q^{h^{N,k+1}_{\Lambda}+\frac{u^2}{2\kappa}}\left(w^u+\mathcal{O}(q^{\frac{\kappa}{2}-|u|})\right)\left(\mathrm{ch}^N_{\Lambda}(v)+\mathcal{O}(q^{k-\sum_i\Lambda_i+2})\right)}{\prod_{n=1}^{\infty}\left[(1-q^n)^N\prod_{i\neq j}(1-v_i\bar{v}_jq^n)\right]}\ . \tag{5.15}$$

[3] In the duality to the higher spin theory on AdS_3 only the NS-NS sector plays a role since the conformal dimension of the RR sector states is proportional to the central charge, see the discussion in [88].

The coset character associated to $(0;\Lambda,u)$ is then given by the branching function $b^{N,k}_{0;\Lambda,u}(q)$, which is defined by

$$\mathrm{ch}^{N+1,k}_0(v,w;q)\,\theta(v,w;q)=\sum_{\Lambda,u} b^{N,k}_{0;\Lambda,u}(q)\,\mathrm{ch}^{N,k+1}_{\Lambda,u}(v,w;q)\ . \tag{5.16}$$

Combining the explicit expressions given above, the branching functions take the form (see also [34])

$$b^{N,k}_{0;\Lambda,u}(q)=q^{-h^{N,k+1}_{\Lambda}-\frac{u^2}{2\kappa}}\Big[a^N_{0;\Lambda,u}(q)+\mathcal{O}(q^{k-\sum_i\Lambda_i+2})+\mathcal{O}(q^{\frac{\kappa}{2}-|u|})\Big]\,, \tag{5.17}$$

where $a^N_{0;\Lambda,u}(q)$ is the multiplicity of $w^u\mathrm{ch}^N_\Lambda(v)$ in

$$\sum_{\Lambda,u} a^N_{0;\Lambda,u}(q)\,w^u\mathrm{ch}^N_\Lambda(v)=\prod_{n=1}^{\infty}\prod_{i=1}^{N}\frac{(1+\bar{w}^{N+1}v_iq^{n-\frac{1}{2}})(1+w^{N+1}\bar{v}_iq^{n-\frac{1}{2}})}{(1-\bar{w}^{N+1}v_iq^{n})(1-w^{N+1}\bar{v}_iq^{n})}\ . \tag{5.18}$$

It therefore follows that the partition function $\mathcal{Z}_0$ of (5.10) equals for $k\to\infty$

$$\mathcal{Z}_0=\lim_{k\to\infty}(q\bar{q})^{-\frac{c}{24}}\sum_{\Lambda,u}|b^{N,k}_{0;\Lambda,u}(q)|^2=(q\bar{q})^{-\frac{N}{8}}\sum_{\Lambda,u}|a^N_{0;\Lambda,u}(q)|^2\ , \tag{5.19}$$

where we sum over all finite Young diagrams Λ of at least $N-1$ rows, and u must be of the form $u=(N+1)(-|\Lambda|+nN)$ with $n\in\mathbb{Z}$. In the second equality, we have used that since Λ and u are finite (and do not grow with k), the prefactor in equation (5.17), $h^{N,k+1}_{\Lambda}+\frac{u^2}{2\kappa}$, vanishes in the limit, and the higher-order terms in the bracket become irrelevant.

5.1.2 Comparison with the untwisted orbifold sector

We shall now compare this result to the U(N) orbifold of $2N$ free fermions and bosons that transform as $\mathbf{N}\oplus\bar{\mathbf{N}}$ of U(N), cf., equation (5.6). Labelling again the elements of U(N) in terms of SU(N) $\times$ U(1) via the 'embedding'

$$\imath\colon(v,w)\mapsto w^{-(N+1)}\cdot v=\bar{w}^{(N+1)}\cdot v\ , \tag{5.20}$$

the partition function with the insertion of these group elements takes the form

$$\imath(v,w)\cdot\mathcal{Z}_{\mathrm{free}}=(q\bar{q})^{-\frac{N}{8}}\prod_{n=1}^{\infty}\prod_{i=1}^{N}\frac{|1+\bar{w}^{(N+1)}v_iq^{n-\frac{1}{2}}|^2|1+w^{N+1}\bar{v}_iq^{n-\frac{1}{2}}|^2}{|1-\bar{w}^{(N+1)}v_iq^{n}|^2|1-w^{N+1}\bar{v}_iq^{n}|^2}\,, \tag{5.21}$$

where we have used that the central charge equals $c=3N$. The untwisted sector of this orbifold theory consists of the states that are U(N) invariant. Put differently, the untwisted sector is therefore the multiplicity space of the trivial representation of $\mathrm{U}(N)$ acting on the free theory with partition function $\mathcal{Z}_{\text{free}}$. Since (5.21) is, up to the prefactor, just the charge-conjugate square of the coset numerator character (5.18), this amounts to finding the trivial representation in

$$(0;\Lambda_1,u_1)\otimes(0;\Lambda_2,u_2) \tag{5.22}$$

for some representations (Λ_i,u_i) $(i=1,2)$ of $\mathfrak{su}(N)\oplus\mathfrak{u}(1)$, where the first factor corresponds to the left-movers and the second one to the right-movers. This tensor product contains the trivial representation if and only if $\Lambda_1=\Lambda_2^*$ and $u_1=-u_2$, where Λ_2^* is the representation conjugate to Λ_2, and it always does so with multiplicity one. Thus we conclude that the partition function of the untwisted sector equals

$$\mathcal{Z}_{\mathrm{U}}=(q\bar{q})^{-\frac{N}{8}}\sum_{\Lambda,u}|a^N_{0;\Lambda,u}(q)|^2\,, \tag{5.23}$$

matching precisely (5.19). This yields convincing evidence that the coset subsector of states $(0;\Lambda,u)$ can indeed be described by the untwisted sector of the $\mathrm{U}(N)$ orbifold introduced above.

5.2 Twisted sectors of the continuous orbifold

The remaining states, i.e., those with $\Lambda_+\neq 0$, should then arise from the twisted sector of the continuous orbifold. In the following we shall be able to make this correspondence rather concrete. The main reason why we can be much more explicit (see equation (5.28) below) than in the corresponding bosonic analysis of [87] is that the $\mathcal{N}=2$ superconformal symmetry is quite restrictive and in particular implies that the ground state energy of the twisted sectors is linear in the twist.

To begin with, let us briefly review the basic logic of the continuous orbifold approach of [87]. As was explained there, continuous compact groups (such as $\mathrm{U}(N)$) behave in many respects like finite groups, and one may therefore believe that an orbifold by a continuous compact group can be constructed essentially as in the familiar finite case. In particular, the untwisted sector just consists of the invariant states of the original theory, while the twisted sectors are labelled by the conjugacy classes of the orbifold group. Finally, in each such twisted sector, only the states that are invariant with respect to the centraliser of the twist element survive.

For the case of $\mathrm{U}(N)$, the conjugacy classes are labelled by the elements in the Cartan torus $\mathrm{U}(1)^N$ modulo the action of the Weyl group, i.e., the permutation group S_N. Furthermore, the centraliser of a generic element of the Cartan torus is again just the Cartan torus itself, i.e., the orbifold projection in the twisted sector will just guarantee that the partition function is invariant under the T-transformation, $\tau \mapsto \tau + 1$.

Let us parametrise the elements of the Cartan torus by the diagonal matrices

$$\mathrm{diag}(e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_N}) \ , \qquad -\tfrac{1}{2} < \alpha_i \leq \tfrac{1}{2} \quad (i = 1, \ldots, N) \ . \tag{5.24}$$

Since the Weyl group permutes these entries, the conjugacy classes (and thus the twisted sectors) can actually be labelled by

$$\alpha = [\alpha_1, \ldots, \alpha_N] \ , \tag{5.25}$$

where now, in addition, $\alpha_i \leq \alpha_j$ for $i < j$. In this section, we will argue that the ground state of the sector with twist α can be identified, in the limit $k \to \infty$, with the coset representative

$$\Big(\Lambda_+(\alpha); \Lambda_-(\alpha), u(\alpha)\Big) \ , \tag{5.26}$$

where $m \in \{0, \ldots, N\}$ is chosen such that

$$\alpha_i \leq 0 \text{ for } i \leq m \qquad \text{and} \qquad \alpha_i \geq 0 \text{ for } i > m \ , \tag{5.27}$$

and we define

$$\begin{aligned}\Lambda_+(\alpha) = [k(\alpha_2 - \alpha_1), \ldots, k(\alpha_m - \alpha_{m-1}), -k\alpha_m,& \\ k\alpha_{m+1}, k(\alpha_{m+2} - \alpha_{m+1}), \ldots, k(\alpha_N - \alpha_{N-1})] \ ,&\end{aligned} \tag{5.28}$$

$$\Lambda_-(\alpha) = [k(\alpha_2 - \alpha_1), \ldots, k(\alpha_N - \alpha_{N-1})] \ , \qquad u(\alpha) = k\sum_{i=1}^{N} \alpha_i \ , \tag{5.29}$$

where each entry of the weights is projected onto the integer part (and we also adjust $u(\alpha)$ correspondingly). These weights are then allowed at level k since we have

$$\sum_{j=1}^{N+1} \big[\Lambda_+(\alpha)\big]_j = \sum_{j=1}^{N} \big[\Lambda_-(\alpha)\big]_j = k(\alpha_N - \alpha_1) \leq k \ . \tag{5.30}$$

One also easily checks that (5.26) satisfies the selection rule (4.12). Conversely, for every coset primary $(\Lambda_+; \Lambda_-, u)$, we can write, after a suitable field redefinition if necessary, $\Lambda_+ \equiv \Lambda_+(\alpha)$ for some α of the form (5.25) with $-\frac{1}{2} < \alpha_i \leq \frac{1}{2}$ and $\alpha_i \leq \alpha_j$ for $i < j$; indeed, the corresponding α may be taken to be

$$\alpha = \frac{1}{k}\left[-\sum_{i=1}^{m} \Lambda_i, -\sum_{i=2}^{m} \Lambda_i, \ldots, -\Lambda_m, \Lambda_{m+1}, \sum_{i=m+1}^{m+2} \Lambda_i, \ldots, \sum_{i=m+1}^{N} \Lambda_i\right] , \tag{5.31}$$

where we choose m such that

$$\sum_{i=1}^{m} \Lambda_i < \frac{k}{2} \ , \qquad \text{and} \qquad \sum_{i=m+1}^{N} \Lambda_i \leq \frac{k}{2} \ . \tag{5.32}$$

We will give three main pieces of evidence for this identification: we will show in section 5.2.1 that the conformal dimension of the coset primary (5.26) agrees with the ground state energy of the α-twisted state; we will confirm that the fermionic excitation spectrum of the coset primary has the expected form (see section 5.2.2); and we shall show in section 5.2.3 that the twisted sector has BPS descendants precisely as suggested by the orbifold picture.

5.2.1 Conformal dimension

In the α-twisted sector the free fermions and bosons are simultaneously twisted (as they transform in the same representation of $\mathrm{U}(N)$, see equation (5.6) above). As a consequence, the ground state energy of the α-twisted sector should simply be

$$h(\alpha) = \frac{1}{2}\sum_{i=1}^{N} |\alpha_i| \ . \tag{5.33}$$

(We have outlined the calculation of the twisted sector ground state energy in appendix C.1, see in particular equation (C.14).) We therefore need to show that the conformal dimension of (5.26) agrees with (5.33).

In order to determine the conformal dimension of (5.26), we use (4.15) and note that the quadratic Casimir of a weight Λ of $\mathfrak{su}(N)$ is given by

$$C^{(N)}(\Lambda) = \sum_{i<j} \Lambda_i \Lambda_j \, \frac{i(N-j)}{N} + \frac{1}{2}\sum_j \Lambda_j^2 \, \frac{j(N-j)}{N} + \sum_j \Lambda_j \, \frac{j(N-j)}{2} \ . \tag{5.34}$$

The key step of the computation is to calculate the difference of the Casimirs, which turns out to equal

$$\Delta C = C^{(N+1)}\big(\Lambda_+(\alpha)\big) - C^{(N)}\big(\Lambda_-(\alpha)\big) = \frac{\left(k\sum_{i=1}^{N}\alpha_i\right)^2}{2N(N+1)} + \frac{k}{2}\left[-\sum_{i=1}^{m}\alpha_i + \sum_{i=m+1}^{N}\alpha_i\right] . \tag{5.35}$$

It then follows that the conformal dimension is indeed given by

$$\begin{aligned} h\Big(\Lambda_+(\alpha);\Lambda_-(\alpha),u(\alpha)\Big) &= \frac{\Delta C}{N+k+1} - \frac{u(\alpha)^2}{2N(N+1)(N+k+1)} \\ &= \frac{k}{2(N+k+1)}\Big(-\sum_{i=1}^{m}\alpha_i + \sum_{i=m+1}^{N}\alpha_i\Big) \cong \frac{1}{2}\sum_{i=1}^{N}|\alpha_i| \end{aligned} \tag{5.36}$$

in the limit $k\to\infty$. Here we have used that the excitation number n in (4.15) vanishes because the representation $\Lambda_-(\alpha)$ appears in the branching of $\Lambda_+(\alpha)$ from $\mathfrak{su}(N+1)$ to $\mathfrak{su}(N)$, as follows from the discussion in appendix C.2.

We should also mention that the U(1) charge of the coset primary equals

$$q\Big(\Lambda_+(\alpha);\Lambda_-(\alpha),u(\alpha)\Big) = \frac{u(\alpha)}{N+k+1} \cong \sum_{i=1}^{N} \alpha_i \ , \tag{5.37}$$

which also agrees with what one expects based on the twisted sector analysis. Note that the ground state is a chiral primary if all twists are positive, and an anti-chiral primary if all twists are negative; we shall come back to a more detailed analysis of the BPS states in the twisted sectors in section 5.2.3.

5.2.2 The fermionic excitation spectrum

We can test the above correspondence further by calculating the actual excitation spectrum of the fermions in the twisted sector. Recall that the free fermions correspond to the coset primaries $(0;\mathrm{f},-(N+1))$ and $(0;\bar{\mathrm{f}},(N+1))$, respectively. We can therefore determine the 'twist' of these fermions by evaluating the change in conformal dimension upon fusion with these fields. As a by-product of this analysis we will also be able to show that the above coset primaries are indeed ground states.

More specifically, suppose that $(\Lambda_+;\Lambda_-,u)$ is the (ground) state of a twisted sector. Then we consider the fusion products

$$(\Lambda_+;\Lambda_-,u)\otimes\big(0;\mathrm{f},-(N+1)\big) = \bigoplus_{l=0}^{N-1}\big(\Lambda_+;\Lambda_-^{-(l)},u-(N+1)\big) \ , \tag{5.38}$$

where $\Lambda^{-(l)}$ with $l=0,\ldots,N-1$ denotes the N different weights that appear in the tensor product $\Lambda\otimes\mathrm{f}$. Similarly we define

$$(\Lambda_+;\Lambda_-,u)\otimes\big(0;\bar{\mathrm{f}},(N+1)\big) = \bigoplus_{l=0}^{N-1}\big(\Lambda_+;\Lambda_-^{+(l)},u+(N+1)\big) \ , \tag{5.39}$$

where $\Lambda^{+(l)}$ labels the weights that appear in $\Lambda\otimes\bar{\mathrm{f}}$; a closed formula for both cases is given by

$$\Lambda_j^{\epsilon(l)} = \begin{cases} \Lambda_j+\epsilon & j=l \\ \Lambda_j-\epsilon & j=l+1 \\ \Lambda_j & \text{otherwise.} \end{cases} \tag{5.40}$$

Here $\epsilon=\pm$, and we have assumed that all $\Lambda_j\neq 0$ so that all N fusion channels $\Lambda^{\epsilon(l)}$ are indeed allowed. (We will comment on the situation when this is not the case at the end of this subsection.)

Now the 'twist' of the fermionic excitations of the twisted sector state $(\Lambda_+;\Lambda_-,$ $u)$ can be determined by calculating the difference of conformal dimension of the coset primaries that appear in (5.38) and (5.39), relative to the original state. Indeed, generically, there will be N different such twists, corresponding to the N different fusion channels in (5.40), and this ties in with the fact that there are N fundamental fermions (as well as their conjugates). One cross-check of our analysis will be that the twists of the fermions and their conjugates will be opposite, and this will indeed turn out to be the case.

In order to calculate this difference of conformal dimension we note that it follows from (4.15) that

$$\begin{aligned}\delta h^{(l)} &\equiv h\Big(\Lambda_+;\Lambda_-^{\epsilon(l)},u+\epsilon(N+1)\Big)-h(\Lambda_+;\Lambda_-,u)\\ &=\frac{1}{N+k+1}\Big(C^{(N)}(\Lambda_-)-C^{(N)}(\Lambda_-^{\epsilon(l)})\Big)\\ &\qquad-\frac{1}{2N(N+1)(N+k+1)}\Big(2\epsilon u(N+1)+(N+1)^2\Big)+n\ .\end{aligned}\tag{5.41}$$

The difference of Casimir operators turns out to equal

$$\begin{aligned}\delta C^{(l)}&=C^{(N)}(\Lambda_-)-C^{(N)}(\Lambda_-^{\epsilon(l)})\\ &=-\frac{\epsilon}{N}\sum_{i=1}^{N-1}i\,\Lambda_i+\epsilon\sum_{j=l+1}^{N-1}\Lambda_j+\frac{1}{2N}\big(\epsilon N^2-2l\epsilon N-\epsilon N+1-N\big)\ ,\end{aligned}\tag{5.42}$$

where Λ_j are the Dynkin labels of Λ_-. Thus we find that

$$\begin{aligned}\delta h^{(l)}=n+\frac{1}{N+k+1}\Big[-\frac{\epsilon}{N}\Big(\sum_{i=1}^{N-1}i\,\Lambda_i+u\Big)+\epsilon\sum_{j=l+1}^{N-1}\Lambda_j\Big]\\ +\frac{1}{2(N+k+1)}(\epsilon N-2l\epsilon-(2+\epsilon))\ .\end{aligned}\tag{5.43}$$

In the limit $k\to\infty$, the second line can be ignored (since none of the terms in the numerator can depend on k), and hence we get approximately

$$\delta h^{(l)}\cong n+\frac{\epsilon}{N+k+1}\Big[\sum_{j=l+1}^{N-1}\Lambda_j-\frac{1}{N}\Big(\sum_{i=1}^{N-1}i\,\Lambda_i+u\Big)\Big]\ .\tag{5.44}$$

Applying this formula to the state (5.26) and using (5.28) yields then

$$\delta h^{(l)}\cong n-\epsilon\,\alpha_{l+1}\ ,\tag{5.45}$$

where α_{l+1} denotes the different components of the twist in (5.25). For the free fermions,[4] the selection rule of the $\mathfrak{so}(2N)_1$ factor implies that $n=\frac{1}{2}$. Thus, the

[4] Technically, this means we have to consider the so-called 'even' fusion of the associated coset fields, see [134, 135], as well as [71]. In order to analyse the bosonic descendants (that sit in the same $\mathcal{N}=2$ representation), we then have to consider the 'odd' fusion rules.

excitations of the fermions are shifted away from the untwisted NS value $\delta h = \frac{1}{2}$ by the twist α_{l+1}. Furthermore, this twist is opposite for the fermions and the anti-fermions, i.e., it is proportional to ϵ. This then agrees precisely with what should be the case for the α-twisted sector.

It is worth stressing that the derivation of (5.44) was completely general, and did, in particular, not assume any specific properties of the state $(\Lambda_+; \Lambda_-, u)$. Thus we can use it to read off the twist of *any* coset state, which therefore equals

$$\alpha_j \cong -\frac{1}{N+k+1}\Big[\sum_{i=j}^{N-1}\Lambda_j - \frac{1}{N}\Big(\sum_{i=1}^{N-1} i\,\Lambda_i + u\Big)\Big] , \tag{5.46}$$

where the Λ_j are, as before, the Dynkin labels of Λ_-. Note that finite excitations only change the Λ_i and u by a finite amount, which can be neglected in the limit $k \to \infty$. We therefore conclude that finitely excited states live in the same twisted sector as the corresponding ground state. Again, this is what should be the case for the α-twisted sector.

Finally, we comment on the special situation for which some of the $\Lambda_j = 0$. In that case, there are actually fewer fermionic excitations since some of the l in (5.40) are not allowed. This phenomenon also has a very natural interpretation from the continuous orbifold perspective: because of equation (5.28), $\Lambda_j = 0$ implies that $\alpha_{j+1} = \alpha_j$. Then the centraliser of the corresponding element of the Cartan torus (5.24) is *bigger* than just the Cartan torus itself, since it includes, in particular, the SU(2) subgroup that rotates the two twists α_j and α_{j+1} into one another. This means that actually fewer fermionic excitations survive the orbifold projection in the twisted sector, in perfect agreement with the fact that we also have fewer coset descendants. The analysis works similarly if more than one $\Lambda_j = 0$, etc.

It remains to show that the coset states (5.26) actually correspond to the *ground states* of the α-twisted sector. For the fermionic excitations with $n = \frac{1}{2}$ this is obvious from the above (given that, by construction, each $|\alpha_j| \leq \frac{1}{2}$). The argument for the bosonic descendants (for which $n = 0$ is possible) requires more work and is spelled out in appendix C.3.

5.2.3 BPS descendants

Finally, we want to analyse the BPS descendants of the twisted sector ground states. For the case with $\mathcal{N} = 4$ superconformal symmetry, it is well known from the analysis of the symmetric orbifold, see e.g. [127], that each twisted sector of the symmetric orbifold contains a BPS descendant that is obtained from the twisted sector ground state upon applying all fermionic generators whose mode number is less than 1/2. For the case at hand, i.e., the situation with $\mathcal{N} = 2$ superconformal symmetry, we expect that each twisted sector should contain *two* BPS states, one chiral primary that is obtained by applying all $q=+1$ fermionic modes whose mode number is less than $1/2$ to the twisted sector ground state; and one anti-chiral primary that is obtained by applying all $q=-1$ fermionic modes whose mode number is less than $1/2$. Actually, as we shall see, this expectation is borne out; quite surprisingly, the relevant chiral and anti-chiral states remain BPS even at finite N and k.

To be more specific, let us consider the twisted sector ground state defined in equation (5.26). In order to obtain the chiral primary descendant we have to apply the fermionic modes associated to $(0;\mathrm{f},-(N+1))$ whose mode numbers are less than $1/2$. Thus we should consider the descendant where we add a box to each of the first m rows, i.e., the coset primary

$$\Big(\Lambda_+(\alpha);\Lambda_-(\alpha)^{(\mathrm{BPS})},u(\alpha)^{(\mathrm{BPS})}\Big)\ , \tag{5.47}$$

where, for $m\geq 1$,

$$\Lambda_-(\alpha)^{(\mathrm{BPS})}=\Big[k(\alpha_2-\alpha_1),\ldots,k(\alpha_{m+1}-\alpha_m)+1,\ldots,k(\alpha_N-\alpha_{N-1})\Big]\ , \tag{5.48}$$

and

$$u(\alpha)^{(\mathrm{BPS})}=k\sum_{i=1}^{N}\alpha_i-m(N+1)\ . \tag{5.49}$$

We now claim that this defines a chiral primary operator, even for finite N and k.[5] Similarly, the anti-chiral primary is obtained by applying the fermionic modes associated to $(0;\bar{\mathrm{f}},(N+1))$ whose mode numbers are less than $1/2$, i.e., by removing a box in each of the rows $m+1,\ldots,N$. The corresponding anti-chiral primary is then

$$\Big(\Lambda_+(\alpha);\Lambda_-(\alpha)^{(\overline{\mathrm{BPS}})},u(\alpha)^{(\overline{\mathrm{BPS}})}\Big)\ , \tag{5.50}$$

where, for $m<N$,

$$\Lambda_-(\alpha)^{(\overline{\mathrm{BPS}})}=\Big[k(\alpha_2-\alpha_1),\ldots,k(\alpha_{m+1}-\alpha_m)+1,\ldots,k(\alpha_N-\alpha_{N-1})\Big]=\Lambda_-(\alpha)^{(\mathrm{BPS})}, \tag{5.51}$$

but now

$$u(\alpha)^{(\overline{\mathrm{BPS}})}=k\sum_{i=1}^{N}\alpha_i+(N-m)(N+1)\ . \tag{5.52}$$

Note that both states satisfy the selection rule (4.12) because

$$|\Lambda_+(\alpha)|=-k\sum_{i=1}^{N}\alpha_i+(N+1)k\alpha_N \tag{5.53}$$

[5] One way to see this is to note that, up to a field identification, this coset primary satisfies $\Lambda_-=P\Lambda_+$, where P is the restriction to the first $N-1$ Dynkin labels. We thank Stefan Fredenhagen for pointing this out to us.

and

$$|\Lambda_-(\alpha)^{(\mathrm{BPS})}| = |\Lambda_-(\alpha)^{(\overline{\mathrm{BPS}})}| = -k\sum_{i=1}^{N}\alpha_i + Nk\alpha_N + m\ . \tag{5.54}$$

To show that these states are indeed chiral and anti-chiral primaries, we again first compute the difference of the Casimirs; using the result from (5.35) we obtain

$$\begin{aligned}\Delta C &= C^{(N+1)}\big(\Lambda_+(\alpha)\big) - C^{(N)}\big(\Lambda_-(\alpha)^{(\mathrm{BPS})}\big)\\ &= C^{(N+1)}\big(\Lambda_+(\alpha)\big) - C^{(N)}\big(\Lambda_-(\alpha)\big) + k\sum_{i=1}^{m}\alpha_i - \frac{mk}{N}\sum_{i=1}^{N}\alpha_i - \frac{N+1}{2N}m(N-m)\\ &= \frac{\big[u(\alpha)^{(\mathrm{BPS})}\big]^2}{2N(N+1)} + \frac{1}{2}u(\alpha)^{(\mathrm{BPS})}\ . \end{aligned}\tag{5.55}$$

Equations (4.15) and (4.16) then directly lead to

$$\begin{aligned}h\Big(\Lambda_+(\alpha);\Lambda_-(\alpha)^{(\mathrm{BPS})},u(\alpha)^{(\mathrm{BPS})}\Big) &= \frac{u(\alpha)^{(\mathrm{BPS})}}{2(N+k+1)} + \frac{m}{2} \\ &= \frac{1}{2}\,q\Big(\Lambda_+(\alpha);\Lambda_-(\alpha)^{(\mathrm{BPS})},u(\alpha)^{(\mathrm{BPS})}\Big)\ ,\end{aligned}\tag{5.56}$$

so these states are indeed chiral primary. Similarly, using

$$u(\alpha)^{(\mathrm{BPS})} = u(\alpha)^{(\overline{\mathrm{BPS}})} - N(N+1) \tag{5.57}$$

we compute

$$\begin{aligned}h\Big(\Lambda_+(\alpha);\Lambda_-(\alpha)^{(\overline{\mathrm{BPS}})},u(\alpha)^{(\overline{\mathrm{BPS}})}\Big) &= -\frac{u(\alpha)^{(\overline{\mathrm{BPS}})}}{2(N+k+1)} + \frac{N-m}{2} \\ &= -\frac{1}{2}\,q\Big(\Lambda_+(\alpha);\Lambda_-(\alpha)^{(\overline{\mathrm{BPS}})},u(\alpha)^{(\overline{\mathrm{BPS}})}\Big)\ ,\end{aligned}\tag{5.58}$$

and thus these states are anti-chiral primary as claimed.

Note that for $m = 0$, all twists are non-negative, and so by (5.36) and (5.37) already the ground state is chiral primary. Similarly, the ground state with $m = N$ is anti-chiral primary since all twists are non-positive.

Chapter 6

The symmetric orbifold of $\mathcal{N} = 2$ minimal models

Having shown in chapter 5 how to relate the Kazama-Suzuki coset models in the large k limit to a continuous orbifold of $2N$ free bosons and fermions, let us now take this analysis one step further and compare the coset to a much bigger theory, namely the *symmetric* orbifold of $\mathbb{T}^2$. The motivation for studying this construction is its possible relation to string theory on AdS_3.

In one dimension higher the so-called ABJ triality [41] suggests a relation between a higher spin theory on AdS_4, a 3d CFT and a string theory in a certain limit. For the case of AdS_3 a somewhat different proposal was made in [76], where the $\mathcal{N} = 4$ superconformal generalisation [75] of the original bosonic minimal model holography of chapter 2 was shown to define a subtheory of the symmetric orbifold of $\mathbb{T}^4$, which is believed to be dual to string theory on an $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ background, see [52] for a review. More precisely, the CFT duals of the $\mathcal{N} = 4$ higher spin theories on AdS_3 are described by the so-called Wolf space cosets (see [149, 147, 153, 98, 160, 148] for some early literature on this subject). In the limit where the level k tends to infinity, these cosets simplify in a way completely analogous to the discussion of chapter 5, viz. they become a theory of $4(N+1)$ free bosons and fermions subject to a $\mathrm{U}(N)$ singlet constraint. They then form a natural subsector of the untwisted sector of the symmetric orbifold where the same free theory is only subjected to a singlet constraint under the permutation group $S_{N+1} \subset \mathrm{U}(N)$. The relations between these theories and the possible embedding of $\mathcal{N} = 4$ higher spin theory into string theory are sketched diagrammatically in figure 6.1.

It is obviously tempting to believe that this sort of relation is not just restricted to the maximally supersymmetric setting, but that the less supersymmetric higher-

$$
\begin{tikzcd}
\text{Wolf space cosets} \arrow[r, leftrightarrow, "\text{dual}"] \arrow[d, "\text{large level}"] & \begin{array}{c}\mathcal{N}=4\text{ higher spin}\\ \text{theory on AdS}_3\end{array} \arrow[d, hook, dotted, "?"] \\
\begin{array}{c}(\mathbb{T}^4)^{N+1}/\mathrm{U}(N)\\ \text{orbifold}\end{array} \arrow[d, hook] & \begin{array}{c}\text{String theory}\\ \text{on AdS}_3\times S^3\times S^3\times S^1\end{array} \arrow[d, "\text{large radius}"] \\
\begin{array}{c}(\mathbb{T}^4)^{N+1}/S_{N+1}\\ \text{orbifold}\end{array} \arrow[r, leftrightarrow, "\text{dual}"] & \begin{array}{c}\text{String theory}\\ \text{on AdS}_3\times S^3\times\mathbb{T}^4\end{array}
\end{tikzcd}
$$

Figure 6.1: A schematic overview of $\mathcal{N}=4$ higher spin and stringy dualities. The duality between the Wolf space cosets and $\mathcal{N}=4$ higher spin theory is to be restricted to the perturbative sector, and the orbifold theories need to be restricted to the untwisted sector for the embedding to make sense. Upon identification of the large $\mathcal{N}=4$ symmetries, a large level of the Wolf space cosets corresponds to a large radius of one of the S^3 factors in string theory on $\mathrm{AdS}_3\times S^3\times S^3\times S^1$. The latter then looks like $\mathrm{AdS}_3\times S^3\times\mathbb{T}^4$ when restricting to states without winding and momentum.

spin/CFT dualities may also be related naturally to string theory. This is quite plausible following the general philosophy of [102], see also [100, 17, 18] for subsequent work. A candidate for a possible stringy extension of $\mathcal{N}=2$ higher spins has, however, yet to be found.

This chapter is based on the paper [84] with Matthias Gaberdiel. It is organised as follows. In section 6.1 we define the symmetric orbifold in question, and explain how the large level limit of the relevant KS models describe a sub-sector of this theory. In particular, we study the embedding in detail for the untwisted sector, where we can give very concrete decompositions in terms of the representations of the $\mathcal{N}=2$ $s\mathcal{W}_\infty$ algebra. Section 6.2 is devoted to understanding how the twisted sector states of the symmetric orbifold can be similarly described in terms of these representations; we study in detail the (2)-cycle, as well as the $(2)^2$-cycle twisted sector, for which we give detailed decomposition formulae; we also explain how the structure of a general twisted sector can be understood in similar terms. Finally, we undertake (in section 6.2.4) first steps towards characterising the higher spin representations that are relevant for the description of the twisted sector, generalising the discussion of [78] to the $\mathcal{N}=2$ case. Some aspects of sections 6.2.4 and 6.2.5 are based on an initial collaboration with Shouvik Datta. A self-contained description of the low-lying bosonic generators of the $s\mathcal{W}_\infty$ algebra in terms of the KS cosets can be found in appendix D.

6.1 The untwisted sector of the symmetric orbifold

In the last chapter the coset

$$\frac{\mathfrak{su}(N+1)^{(1)}_{N+k+1}}{\mathfrak{su}(N)^{(1)}_{N+k+1}\oplus\mathfrak{u}(1)^{(1)}_{\kappa}} \cong \frac{\mathfrak{su}(N+1)_k\oplus\mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1}\oplus\mathfrak{u}(1)_\kappa} \tag{6.1}$$

was shown to agree with an orbifold theory of N free complex bosons and fermions by the continuous orbifold group $\mathrm{U}(N)$ in the limit where the level $k\to\infty$. A similar approach was applied to the $\mathcal{N}=4$ Wolf space cosets in [76], where it was shown that the corresponding coset algebra is a natural subalgebra of the chiral algebra of the symmetric orbifold; in turn the symmetric orbifold is believed to be dual to string theory on AdS_3, thus exhibiting how the higher spin theory is embedded into string theory. In this chapter we want to analyse how the $\mathcal{N}=2$ cosets (6.1) can be related to an $\mathcal{N}=2$ symmetric orbifold. This should be a first step towards understanding the string theory interpretation of the corresponding $\mathcal{N}=2$ higher spin theory.

Recall that the continuous orbifold describes the theory of N free complex bosons and fermions transforming in the fundamental (and anti-fundamental) representation of $\mathrm{U}(N)$. Thus it can be represented as the continuous orbifold $(\mathbb{T}^2)^N/\mathrm{U}(N)$.[1] The untwisted sector consists of the states that are invariant under the action of $\mathrm{U}(N)$. The full orbifold theory includes also a twisted sector for each conjugacy class of $\mathrm{U}(N)$.

As in the $\mathcal{N}=4$ case one may then consider, instead of the $\mathrm{U}(N)$ action, the permutation action of $S_{N+1}\subset\mathrm{U}(N)$. To explain this, it is natural to start with a theory of $N+1$ free bosons and fermions, on which S_{N+1} acts by permutations. This action is not irreducible since the sum of all bosons (or fermions) is invariant under the permutation action,

$$N+1\cong N\oplus 1\ . \tag{6.2}$$

Here and in the following, normal font is used to denote representations of S_{N+1}, while bold font is reserved for representations of $\mathrm{U}(N)$. The N-dimensional repre-

[1] Strictly speaking the relevant orbifold is $(\mathbb{R}^2)^N/\mathrm{U}(N)$, since the $\mathrm{U}(N)$ action is not compatible with discrete momenta. However, we shall usually refer to it as the torus orbifold since the zero momentum sector (which is what we shall be considering) is independent of the radius of the torus.

sentation on the right hand side is irreducible and is called the standard representation of S_{N+1}. In a suitable basis this representation acts on only N copies of $\mathbb{T}^2$, so the orbifold of $N+1$ copies decomposes in fact as

$$(\mathbb{T}^2)^{N+1}/S_{N+1} \cong (\mathbb{T}^2)^N/S_{N+1} \oplus \mathbb{T}^2 \ . \tag{6.3}$$

The free torus which transforms as a singlet under S_{N+1} is not of much interest to us and we will often drop it; in the following we shall therefore mainly concentrate on the non-trivial part of the symmetric orbifold. This will be the symmetric orbifold theory which will be related to the KS models.

In order to see the relation to the KS models we recall that the standard representation ρ of S_{N+1} acting on the N tori maps permutations to unitary (actually even orthogonal) $N\times N$ matrices. Thus we can view $\rho(S_{N+1})$ as a finite subgroup of $\mathrm{U}(N)$, and since the standard representation is faithful, that subgroup is isomorphic to S_{N+1}. Furthermore, as discussed in [76], the fundamental (and anti-fundamental) representation of $\mathrm{U}(N)$ branches down to the standard representation of S_{N+1}. Thus the $\mathrm{U}(N)$-invariant states of the free theory form a consistent subsector of the S_{N+1} invariant states, and hence the untwisted sector of the continuous orbifold is a subsector of the untwisted sector of the symmetric orbifold.

In the rest of this section we shall analyse the untwisted sector of the symmetric orbifold from the viewpoint of the continuous orbifold. The twisted sectors of the symmetric orbifold will be discussed in the following section.

6.1.1 Perturbative decomposition of the untwisted sector

The untwisted sector of the symmetric orbifold by S_{N+1} contributes to the partition function as

$$Z^{(\mathrm{U})}(q,\bar{q},y,\bar{y}) = |\mathcal{Z}_{\mathrm{vac}}(q,y)|^2 + \sum_R |\mathcal{Z}_R^{(\mathrm{U})}(q,y)|^2 \ , \tag{6.4}$$

where $\mathcal{Z}_{\mathrm{vac}}$ denotes the vacuum character, and R labels the non-trivial irreducible representations of S_{N+1}. In order to avoid having to write repeatedly $N+1$, we now change notation and replace the N from (6.1), (6.2) and (6.3) by $\tilde{N}$, and define $N \equiv \tilde{N}+1$; in any case, we shall always be considering the large N (and hence large $\tilde{N}$ limit) for which this distinction is immaterial. In their analysis [53], Dijkgraaf, Moore, Verlinde and Verlinde computed the partition function of the symmetric orbifold X^N/S_N in the R-R sector with insertion of $(-1)^{F+\bar{F}}$, which reads

$$\sum_{N=0}^{\infty} p^N \tilde{Z}_{\mathrm{R}}(S^N(X)) = \prod_{m=1}^{\infty} \prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}} \frac{1}{\left(1 - p^m q^{\frac{\Delta}{m}} \bar{q}^{\frac{\bar{\Delta}}{m}} y^{\ell} \bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}} \ . \tag{6.5}$$

Here we have indicated by the tilde that we have inserted a factor of $(-1)^{F+\bar{F}}$, and $c(\varDelta,\bar{\varDelta},\ell,\bar{\ell})$ are the expansion coefficients of the R-R partition function (with insertion of $(-1)^{F+\bar{F}}$) of the base manifold X,

$$\tilde{Z}_{\mathrm{R}}(X)=\sum_{\varDelta,\bar{\varDelta},\ell,\bar{\ell}} c(\varDelta,\bar{\varDelta},\ell,\bar{\ell})\, q^{\varDelta}\bar{q}^{\bar{\varDelta}}y^{\ell}\bar{y}^{\bar{\ell}}\ . \tag{6.6}$$

In our case, $X=\mathbb{T}^2$ and the partition function factorises into its chiral parts, with $c(\varDelta,\bar{\varDelta},\ell,\bar{\ell})=c(\varDelta,\ell)c(\bar{\varDelta},\bar{\ell})$. The chiral partition function reads (as in [76] we will be ignoring the momentum and winding states)

$$\begin{aligned}\tilde{Z}_{\mathrm{R}}^{(\mathrm{chiral})}(\mathbb{T}^2)&=i\frac{\vartheta_1(z,\tau)}{\eta^3(\tau)}=-(y^{\frac{1}{2}}-y^{-\frac{1}{2}})\prod_{n=1}^{\infty}\frac{(1-yq^n)(1-y^{-1}q^n)}{(1-q^n)^2}\\&=-y^{\frac{1}{2}}+y^{-\frac{1}{2}}+q(y^{\frac{3}{2}}-3y^{\frac{1}{2}}+3y^{-\frac{1}{2}}-y^{-\frac{3}{2}})\\&\quad+3\,q^2(y^{\frac{3}{2}}-3y^{\frac{1}{2}}+3y^{-\frac{1}{2}}-y^{-\frac{3}{2}})\\&\quad+q^3(-y^{\frac{5}{2}}+9y^{\frac{3}{2}}-22y^{\frac{1}{2}}+22y^{-\frac{1}{2}}-9y^{-\frac{3}{2}}+y^{-\frac{5}{2}})+\mathcal{O}(q^4)\ ,\end{aligned}\tag{6.7}$$

where

$$\vartheta_1(z,\tau)=i(y^{\frac{1}{2}}-y^{-\frac{1}{2}})q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1-y\,q^n)(1-y^{-1}\,q^n)\ . \tag{6.8}$$

In our analysis we will only be concerned with the NS-NS sector. The partition function in that sector can be obtained from (6.5) by spectral flow

$$y\to y\,q^{\frac{1}{2}}\ ,\qquad \bar{y}\to\bar{y}\,\bar{q}^{\frac{1}{2}}\ ,\qquad p\to p\,q^{\frac{1}{8}}\,\bar{q}^{\frac{1}{8}}\,y^{\frac{1}{2}}\,\bar{y}^{\frac{1}{2}}\ . \tag{6.9}$$

This leads to an overall factor of $(q\bar{q})^{-\frac{N}{8}}=(q\bar{q})^{-\frac{c}{24}}$, which we will suppress throughout this chapter for better readability. (Effectively, this is equivalent to multiplying the right-hand side of the last replacement in (6.9) by an additional factor of $(q\bar{q})^{\frac{1}{8}}$.) We then obtain the symmetric orbifold generating function in the NS-NS sector (without a $(-1)^{F+\bar{F}}$ insertion)

$$\begin{aligned}&\sum_{N=0}^{\infty}p^N Z(S^N(X))\\&\quad=\prod_{m=1}^{\infty}\prod_{\substack{\varDelta,\bar{\varDelta}\\ \ell,\bar{\ell}}}\frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p^m q^{\frac{\varDelta}{m}+\frac{\ell}{2}+\frac{m}{4}}\bar{q}^{\frac{\bar{\varDelta}}{m}+\frac{\bar{\ell}}{2}+\frac{m}{4}}y^{\ell+\frac{m}{2}}\bar{y}^{\bar{\ell}+\frac{m}{2}}\right)^{c(\varDelta,\bar{\varDelta},\ell,\bar{\ell})}}\ .\end{aligned}\tag{6.10}$$

Now the generating function of the untwisted sector corresponds to the $m=1$ factor of (6.10),

$$\sum_{N=0}^{\infty}p^N Z^{(\mathrm{U})}(S^N(X))=\prod_{\substack{\varDelta,\bar{\varDelta}\\ \ell,\bar{\ell}}}\frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p\,q^{\varDelta+\frac{\ell}{2}+\frac{1}{4}}\bar{q}^{\bar{\varDelta}+\frac{\bar{\ell}}{2}+\frac{1}{4}}y^{\ell+\frac{1}{2}}\bar{y}^{\bar{\ell}+\frac{1}{2}}\right)^{c(\varDelta,\bar{\varDelta},\ell,\bar{\ell})}}\ , \tag{6.11}$$

and the chiral vacuum character (the partition function of the $\mathcal{W}$ algebra) of the orbifold $(\mathbb{T}^2)^{\tilde{N}+1}/S_{\tilde{N}+1}$ can be found from (6.11) by setting $\bar{\Delta}=0, \bar{\ell}=-\frac{1}{2}$ and taking N large enough so that the coefficients stabilise; it is given by

$$\begin{aligned}\mathcal{Z}'_{\text{vac}} &= 1+q^{\frac{1}{2}}(y+y^{-1})+4q+6q^{\frac{3}{2}}(y+y^{-1})+4q^2(y^2+6+y^{-2})\\ &\quad+q^{\frac{5}{2}}(y^3+37y+37y^{-1}+y^{-3})+7q^3(4y^2+17+4y^{-2})+\mathcal{O}(q^{\frac{7}{2}})\ . \qquad (6.12)\end{aligned}$$

In order to obtain the vacuum character of the orbifold $(\mathbb{T}^2)^{\tilde{N}}/S_{\tilde{N}+1}\equiv(\mathbb{T}^2)^{N-1}/S_N$, we have to divide this by the chiral partition function of $\mathbb{T}^2$, which means we neglect the torus that transforms as a singlet under $S_N\equiv S_{\tilde{N}+1}$ and corresponds to the trivial factor in the permutation representation of $S_{\tilde{N}+1}$, see equation (6.2). Since this torus partition function is given by

$$Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2)=\prod_{n=1}^{\infty}\frac{(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2})}{(1-q^n)^2}\ , \qquad (6.13)$$

where we have once more suppressed the prefactor $q^{-\frac{1}{8}}$, we obtain the modified vacuum character

$$\begin{aligned}\mathcal{Z}_{\text{vac}}(q,y) &= \frac{\mathcal{Z}'_{\text{vac}}}{Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2)}\\ &= 1+q+2q^{\frac{3}{2}}(y+y^{-1})+q^2(y^2+8+y^{-2})+10q^{\frac{5}{2}}(y+y^{-1})\\ &\quad+q^3(5y^2+32+5y^{-2})+q^{\frac{7}{2}}(2y^3+47y+47y^{-1}+2y^{-3})\\ &\quad+q^4(y^4+37y^2+142+37y^{-2}+y^{-4})+\mathcal{O}(q^{\frac{9}{2}})\ . \qquad (6.14)\end{aligned}$$

This vacuum character counts the chiral states that transform trivially under S_N, and hence includes, in particular, the character of the $\mathcal{N}=2$ coset $s\mathcal{W}_\infty$ algebra (in the limit $k\to\infty$). Thus the vacuum sector should decompose into the coset characters as

$$\mathcal{Z}_{\text{vac}}(q,y)=\sum_{\Lambda}n(\Lambda)\,\chi_{(0;\Lambda)}(q,y)\ . \qquad (6.15)$$

Indeed, by comparing both sides of the equation order by order in q, we find explicitly

$$\begin{aligned}\mathcal{Z}_{\text{vac}}(q,y) &= \chi_{(0;0)}(y,q)+\chi_{(0;[2,0,\ldots,0])}(q,y)+\chi_{(0;[0,0,\ldots,0,2])}(q,y)\\ &\quad+\chi_{(0;[3,0,\ldots,0,0])}(q,y)+\chi_{(0;[0,0,0,\ldots,0,3])}(q,y)\\ &\quad+\chi_{(0;[2,0,\ldots,0,1])}(q,y)+\chi_{(0;[1,0,0,\ldots,0,2])}(q,y)\\ &\quad+2\cdot\chi_{(0;[4,0,\ldots,0,0])}(q,y)+2\cdot\chi_{(0;[0,0,0,\ldots,0,4])}(q,y)\\ &\quad+\chi_{(0;[0,2,0,\ldots 0,0])}(q,y)+\chi_{(0;[0,0,\ldots 0,2,0])}(q,y)\\ &\quad+\chi_{(0;[3,0,\ldots,0,1])}(q,y)+\chi_{(0;[1,0,0,\ldots,0,3])}(q,y)\end{aligned}$$

$$
\begin{aligned}
&+ 2 \cdot \chi_{(0;[2,0,0,\ldots,0,2])}(q,y) \\
&+ \chi_{(0;[2,1,0,\ldots,0,1])}(q,y) + \chi_{(0;[1,0,\ldots,0,1,2])}(q,y) \\
&+ \chi_{(0;[0,2,0,\ldots,0,1])}(q,y) + \chi_{(0;[1,0,\ldots,0,2,0])}(q,y) \\
&+ 3 \cdot \chi_{(0;[3,0,\ldots,0,2])}(q,y) + 3 \cdot \chi_{(0;[2,0,\ldots,0,3])}(q,y) \\
&+ \chi_{(0;[1,1,0,\ldots,0,2])}(q,y) + \chi_{(0;[2,0,\ldots,0,1,1])}(q,y) \\
&+ \chi_{(0;[3,1,0,\ldots,0])}(q,y) + \chi_{(0;[0,\ldots,0,1,3])}(q,y) \\
&+ 2 \cdot \chi_{(0;[4,0,\ldots,0,1])}(q,y) + 2 \cdot \chi_{(0;[1,0,\ldots,0,4])}(q,y) \\
&+ \chi_{(0;[2,1,0,\ldots,0,1,0])}(q,y) + \chi_{(0;[0,1,0,\ldots,0,1,2])}(q,y) \\
&+ \chi_{(0;[1,1,0,\ldots,0,1,1])}(q,y) + \mathcal{O}(q^{\frac{9}{2}}) \ . \qquad (6.16)
\end{aligned}
$$

As in [76], this is precisely of the form (6.15), with $n(\Lambda)$ denoting the multiplicity of the S_N singlet representation in the $\mathrm{U}(N-1)$ representation Λ, where we think of Λ as a $S_{\tilde{N}+1} \equiv S_N$ representation using the embedding $S_{\tilde{N}+1} \subset \mathrm{U}(\tilde{N})$.[2]

Furthermore, as in [77], we can identify the single-particle generators that generate this extended $\mathcal{W}$ algebra; if we had not divided out by the diagonal $\mathbb{T}^2$, the generating function of the single-particle generators would have been (see [77])

$$
\begin{aligned}
\sum_{s,l} \tilde{d}(s,l) q^s y^l &= (1-q) \left[Z_{\mathrm{NS}}^{(\mathrm{chiral})}(\mathbb{T}^2)(q,y) - 1 \right] \\
&= (1-q) \left[\prod_{n=1}^{\infty} \frac{(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2})}{(1-q^n)^2} - 1 \right] , \qquad (6.17)
\end{aligned}
$$

where the factor of $(1-q)$ removes the derivatives, and $\tilde{d}(s,l)$ are the number of single-particle generators of spin s and charge l. Dividing out by the diagonal torus removes just the contribution coming from the two free fermions and bosons; thus the actual generating function equals

$$
\begin{aligned}
\sum_{s,l} d(s,l) q^s y^l &= (1-q) \Big[Z_{\mathrm{NS}}^{(\mathrm{chiral})}(\mathbb{T}^2)(q,y) - \Big(1 + \frac{q^{\frac{1}{2}}(y+y^{-1})}{(1-q)} + 2\frac{q^1}{(1-q)} \Big) \Big] \\
&= q + 2q^{\frac{3}{2}}(y+y^{-1}) + q^2(6+y^2+y^{-2}) + 6q^{\frac{5}{2}}(y+y^{-1}) + \cdots \ . \qquad (6.18)
\end{aligned}
$$

These single-particle generators generate then the $\mathcal{W}$ algebra in the sense that

$$
\mathcal{Z}_{\mathrm{vac}}(q,y) = \prod_{s,l} \prod_{n=0}^{\infty} \frac{1}{(1-y^l q^{s+n})^{(-1)^{2s} d(s,l)}} \ . \qquad (6.19)
$$

[2] We thank Marco Baggio for helping us compute these multiplicities.

They should sit in wedge representations of the $\mathcal{N}=2$ $s\mathcal{W}_\infty$ algebra, and one finds, analogously to [77], that we have the decomposition

$$\sum_{s,l} d(s,l) q^s y^l = (1-q) \sum_{m,n=0}^{\infty}{}' \chi^{(\text{wedge})}_{(0;[m,0,0,\ldots,0,0,n])}(q,y) \ , \tag{6.20}$$

where the prime indicates that the terms with $(m,n)=(0,0),(1,0),(0,1)$ are not included in the sum. Note that the term with $m=n=1$ accounts precisely for the generators of the original $s\mathcal{W}_\infty$ algebra. We have checked these identities up to order q^{15}, and it should be straightforward to prove them using the techniques of [77].

We can similarly extract the characters corresponding to the second sum in (6.4). For example, the representation that contains, among others, the coset states

$$(0;\mathrm{f}) \ , \qquad (0;\bar{\mathrm{f}}) \ , \tag{6.21}$$

is associated to R being the standard representation of S_N. The corresponding character $\mathcal{Z}_1$ is obtained from the coefficient of $\bar{q}(\bar{y}+\bar{y}^{-1})$ in $Z^{(\mathrm{U})}$, from which one has to subtract the contribution from $|\mathcal{Z}'_{\text{vac}}|^2$ and then divide by the torus partition function again. This character turns out to be given by

$$\begin{aligned}
\mathcal{Z}_1 &= \frac{\mathcal{Z}'_{\text{vac}}(Z^{(\text{chiral})}_{\text{NS}}(\mathbb{T}^2)-1)}{Z^{(\text{chiral})}_{\text{NS}}(\mathbb{T}^2)} = \mathcal{Z}'_{\text{vac}} - \mathcal{Z}_{\text{vac}} \\
&= q^{\frac{1}{2}}(y+y^{-1}) + 3q + 4q^{\frac{3}{2}}(y+y^{-1}) + q^2(3y^2+16+3y^{-2}) \\
&\quad + q^{\frac{5}{2}}(y^3+27y+27y^{-1}+y^{-3}) + q^3(23y^2+87+23y^{-2}) \\
&\quad + 5q^{\frac{7}{2}}(2y^3+29y+29y^{-1}+2y^{-3}) \\
&\quad + q^4(3y^4+141y^2+433+141y^{-2}+3y^{-4}) + \mathcal{O}(q^{\frac{9}{2}}) \ .
\end{aligned} \tag{6.22}$$

It can be decomposed into coset characters in the $k\to\infty$ limit according to

$$\begin{aligned}
\mathcal{Z}_1(q,y) = {} & \chi_{(0;[1,0,\ldots,0])}(q,y) + \chi_{(0;[0,\ldots,0,1])}(q,y) \\
& + \chi_{(0;[2,0,\ldots,0])}(q,y) + \chi_{(0;[0,\ldots,0,2])}(q,y) \\
& + \chi_{(0;[1,0,\ldots,0,1])}(q,y) \\
& + 2\cdot\chi_{(0;[3,0,\ldots,0])}(q,y) + 2\cdot\chi_{(0;[0,\ldots,0,3])}(q,y) \\
& + \chi_{(0;[1,1,0,\ldots,0])}(q,y) + \chi_{(0;[0,\ldots,0,1,1])}(q,y) \\
& + 2\cdot\chi_{(0;[2,0,\ldots,0,1])}(q,y) + 2\cdot\chi_{(0;[1,0,\ldots,0,2])}(q,y) \\
& + 3\cdot\chi_{(0;[4,0,\ldots,0])}(q,y) + 3\cdot\chi_{(0;[0,\ldots,0,4])}(q,y) \\
& + 2\cdot\chi_{(0;[2,1,0,\ldots,0])}(q,y) + 2\cdot\chi_{(0;[0,\ldots,0,1,2])}(q,y)
\end{aligned}$$

$$
\begin{aligned}
&+ \chi_{(0;[0,2,0,\ldots,0])}(q,y) + \chi_{(0;[0,\ldots,0,2,0])}(q,y) \\
&+ 4\cdot\chi_{(0;[3,0,\ldots,0,1])}(q,y) + 4\cdot\chi_{(0;[1,0,\ldots,0,3])}(q,y) \\
&+ 2\cdot\chi_{(0;[1,1,0,\ldots,0,1])}(q,y) + 2\cdot\chi_{(0;[1,0,\ldots,0,1,1])}(q,y) \\
&+ 5\cdot\chi_{(0;[2,0,\ldots,0,2])}(q,y) \\
&+ \chi_{(0;[2,0,\ldots,0,1,0])}(q,y) + \chi_{(0;[0,1,0\ldots,0,2])}(q,y) \\
&+ 4\cdot\chi_{(0;[3,1,0,\ldots,0])}(q,y) + 4\cdot\chi_{(0;[0,\ldots,0,1,3])}(q,y) \\
&+ \chi_{(0;[0,1,1,0,\ldots,0])}(q,y) + \chi_{(0;[0,\ldots,0,1,1,0])}(q,y) \\
&+ 7\cdot\chi_{(0;[4,0,\ldots,0,1])}(q,y) + 7\cdot\chi_{(0;[1,0,\ldots,0,4])}(q,y) \\
&+ 4\cdot\chi_{(0;[2,1,0,\ldots,0,1])}(q,y) + 4\cdot\chi_{(0;[1,0,\ldots,0,1,2])}(q,y) \\
&+ 3\cdot\chi_{(0;[0,2,0,\ldots,0,1])}(q,y) + 3\cdot\chi_{(0;[1,0,\ldots,0,2,0])}(q,y) \\
&+ 9\cdot\chi_{(0;[3,0,\ldots,0,2])}(q,y) + 9\cdot\chi_{(0;[2,0,\ldots,0,3])}(q,y) \\
&+ 2\cdot\chi_{(0;[3,0,\ldots,0,1,0])}(q,y) + 2\cdot\chi_{(0;[0,1,0,\ldots,0,3])}(q,y) \\
&+ 5\cdot\chi_{(0;[1,1,0,\ldots,0,2])}(q,y) + 5\cdot\chi_{(0;[2,0,\ldots,0,1,1])}(q,y) \\
&+ \chi_{(0;[1,1,0,\ldots,0,1,0])}(q,y) + \chi_{(0;[0,1,0,\ldots,0,1,1])}(q,y) \\
&+ \chi_{(0;[2,0,1,0\ldots,0,1])}(q,y) + \chi_{(0;[1,0,\ldots,0,1,0,2])}(q,y) \\
&+ 3\cdot\chi_{(0;[2,1,0,\ldots,0,1,0])}(q,y) + 3\cdot\chi_{(0;[0,1,0,\ldots,0,1,2])}(q,y) \\
&+ \chi_{(0;[1,0,1,0,\ldots,0,2])}(q,y) + \chi_{(0;[2,0,\ldots,0,1,0,1])}(q,y) \\
&+ 6\cdot\chi_{(0;[1,1,0,\ldots,0,1,1])}(q,y) + \mathcal{O}(q^{\frac{9}{2}})\ .
\end{aligned}
\tag{6.23}
$$

This time, the coefficients of the coset characters $\chi_{(0;\Lambda)}$ correspond precisely to the multiplicity of the $(N-1)$-dimensional standard representation of S_N inside Λ.[3] This is obviously in line with the fact that the $\tilde{N} = N-1$ boson and fermion fields (that give rise to the representations (6.21)) transform precisely in this representation of the permutation group.

6.1.2 The building blocks of the untwisted sector

Having identified the lowest two representations of S_N by explicitly evaluating the orbifold partition function order by order in q, we will now turn to a more systematic analysis of the untwisted sector. We will show that it organises itself in terms of multi-particle powers of the 'minimal representation' $\mathcal{Z}_1$, in parallel to what was observed in [78].

Let us first introduce the wedge character χ_1 pertaining to $\mathcal{Z}_1$ by stripping off the modes outside of the wedge,

$$
\mathcal{Z}_1 = \mathcal{Z}_{\text{vac}} \cdot \chi_1 \qquad \text{or} \qquad \chi_1 = Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2) - 1\ , \tag{6.24}
$$

[3] Once more we thank Marco Baggio for helping us compute these multiplicities.

where $\mathcal{Z}_{\text{vac}}$ is the vacuum character (that counts the modes outside the wedge); explicitly, we have

$$\begin{aligned}\chi_1(q,y) &= \sum_{\substack{(\Delta,\ell)\\ \neq(0,-\frac{1}{2})}} |c(\Delta,\ell)|\, q^{\Delta+\frac{\ell}{2}+\frac{1}{4}}\, y^{\ell+\frac{1}{2}} \\ &= q^{\frac{1}{2}}\left(y+y^{-1}\right)+3\,q+3\,q^{\frac{3}{2}}\left(y+y^{-1}\right)+q^2\left(y^2+9+y^{-2}\right)+\mathcal{O}(q^{\frac{5}{2}})\ .\end{aligned} \tag{6.25}$$

Then we claim that the full partition function of the untwisted sector for $N\to\infty$ can be written as

$$Z^{(\mathrm{U})}(q,\bar{q},y,\bar{y}) = |\mathcal{Z}_{\text{vac}}(q,y)|^2\left(1+\sum_{\Lambda}|\chi_\Lambda(q,y)|^2\right)\ , \tag{6.26}$$

where Λ runs over all Young diagrams, and $\chi_\Lambda(q,y)$ is the Λ-symmetrised power of $\chi_1(q,y)$ given by (see e.g. [128])

$$\chi_\Lambda(q,y) = \frac{1}{m!}\sum_{\rho\in S_m}\chi^\Lambda_m(\rho)\prod_{k=1}^{m}\mathcal{F}^{k-1}\chi_1\left(q^k,y^k\right)^{a_k(\rho)}\ . \tag{6.27}$$

Here $m=|\Lambda|$ is the number of boxes of Λ, $\chi^\Lambda_m(\rho)$ is the character of Λ seen as an S_m-representation, $a_k(\rho)$ is the number of k-cycles in the permutation ρ, and $\mathcal{F}$ is the involutive mapping that acts on a character or partition function by insertion of $(-1)^{F+\bar{F}}$. So denoting $\mathcal{F}\chi_1$ by $\tilde{\chi}_1$, the first few characters read

$$\begin{aligned}\chi_{\tiny\yng(1)}(q,y) &= \chi_1(q,y)\ ,\\ \chi_{\tiny\yng(2)}(q,y) &= \frac{1}{2}\left(\chi_1(q,y)^2+\tilde{\chi}_1(q^2,y^2)\right)\ ,\\ \chi_{\tiny\yng(1,1)}(q,y) &= \frac{1}{2}\left(\chi_1(q,y)^2-\tilde{\chi}_1(q^2,y^2)\right)\ ,\\ \chi_{\tiny\yng(3)}(q,y) &= \frac{1}{6}\left(\chi_1(q,y)^3+3\chi_1(q,y)\tilde{\chi}_1(q^2,y^2)+2\chi_1(q^3,y^3)\right)\ ,\\ \chi_{\tiny\yng(1,1,1)}(q,y) &= \frac{1}{6}\left(\chi_1(q,y)^3-3\chi_1(q,y)\tilde{\chi}_1(q^2,y^2)+2\chi_1(q^3,y^3)\right)\ ,\\ \chi_{\tiny\yng(2,1)}(q,y) &= \frac{1}{3}\left(\chi_1(q,y)^3-\chi_1(q^3,y^3)\right)\ .\end{aligned} \tag{6.28}$$

A proof of (6.26) will be given at the end of section 6.2.3. We have checked agreement of equations (6.11) and (6.26) for up to three boxes and up to order $\mathcal{O}(q^2)\mathcal{O}(\bar{q}^2)$, which is the lowest order to which the Young diagrams Λ with four boxes contribute.

6.2 Twisted sectors of the symmetric orbifold

The twisted sectors are labelled by conjugacy classes $[g]$ of S_N, and consist of those states which are invariant under C^g, the centraliser of g in S_N. The conjugacy classes of S_N can be labelled by cycle structures

$$(1)^{N_1}(2)^{N_2}(3)^{N_3}\cdots(m)^{N_m}\ , \qquad \text{where } \sum_{i=1}^{m} N_i = N\ . \tag{6.29}$$

The conjugacy class labelled by such a string consists of all elements of S_N that can be decomposed into N_2 2-cycles, N_3 3-cycles, etc. The centraliser of this conjugacy class is then

$$C^{(1)^{N_1}(2)^{N_2}\cdots(m)^{N_m}} \cong S_{N_1}\times(S_{N_2}\ltimes\mathbb{Z}_2^{N_2})\times\cdots\times(S_{N_m}\ltimes\mathbb{Z}_m^{N_m})\ . \tag{6.30}$$

The n free fermions and bosons corresponding to an n-cycle have twists of i/n, for $i=1,\ldots,n$, and the corresponding $\mathbb{Z}_n$ acts by the usual phases on them. On the other hand, the S_{N_n} factors in the semi-direct products permute the N_n different n-cycles among each other.

Since states are tensor products of left- and right-moving states, the action of the centraliser on these chiral states need not be trivial (only the combined action on left- and right-movers must be). The partition function of the $[g]$-twisted sector will thus have the structure

$$Z^{[g]} = \sum_R |\mathcal{Z}_R^{[g]}|^2\ , \tag{6.31}$$

where R labels the different irreducible representations of the centraliser $C^{[g]}$. We will see examples of this below.

6.2.1 The 2-cycle twisted sector

We will start our analysis of the twisted sector with the subsector corresponding to a 2-cycle twist, which is the simplest example. The partition function of the 2-cycle twisted sector in the ordinary symmetric orbifold can be obtained from the generating function; more specifically, the R-R sector expression can be extracted from the $m=1$ and $m=2$ factors of (6.5),

$$\begin{aligned}\sum_{N=0}^{\infty} p^N \tilde{Z}_{\mathrm{R}}^{(2)}(S^N\mathbb{T}^2) = p^2 &\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}{}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^{\ell}\bar{y}^{\bar{\ell}}\\ &\times\prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}}\frac{1}{(1-p\,q^{\Delta}\bar{q}^{\bar{\Delta}}y^{\ell}\bar{y}^{\bar{\ell}})^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\ ,\end{aligned} \tag{6.32}$$

where the prime at the sum indicates that $\Delta-\bar{\Delta}$ has to be even. Flowing to the NS-NS sector and considering the stabilising limit of large N we find for the partition function without $(-1)^{F+\bar{F}}$ insertion

$$\begin{aligned} Z^{(2)}(S^N\mathbb{T}^2) = {}& \sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}{}' |c(\Delta,\bar{\Delta},\ell,\bar{\ell})|\, q^{\frac{1}{2}(\Delta+\ell+1)}\bar{q}^{\frac{1}{2}(\bar{\Delta}+\bar{\ell}+1)}y^{\ell+1}\bar{y}^{\bar{\ell}+1} \\ & \times \prod_{\substack{(\Delta,\bar{\Delta},\ell,\bar{\ell}) \\ \neq(0,0,-\frac{1}{2},-\frac{1}{2})}} \frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}q^{\Delta+\frac{\ell}{2}+\frac{1}{4}}\bar{q}^{\bar{\Delta}+\frac{\bar{\ell}}{2}+\frac{1}{4}}y^{\ell+\frac{1}{2}}\bar{y}^{\bar{\ell}+\frac{1}{2}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}} \ . \end{aligned} \tag{6.33}$$

We then obtain the partition function we are interested in by dividing by the left- and right-moving torus partition function $Z_{\text{NS}}(\mathbb{T}^2)=|Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2)|^2$,

$$\begin{aligned} Z^{(2)}(q,\bar{q},y,\bar{y}) = \frac{q^{\frac{1}{4}}\bar{q}^{\frac{1}{4}}}{y^{\frac{1}{2}}\bar{y}^{\frac{1}{2}}}\Big[& 1+y\bar{y}+(y\bar{y}^2+3y+3\bar{y}+\bar{y}^{-1})\bar{q}^{\frac{1}{2}} \\ & +(y^2\bar{y}+y+\bar{y}+y^{-1})q^{\frac{1}{2}} \\ & +2(y^2\bar{y}^2+2y^2+5y\bar{y}+2\bar{y}^2+5+2y\bar{y}^{-1} \\ & \quad +2y^{-1}\bar{y}+y^{-1}\bar{y}^{-1})q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}}+\cdots\Big] \ . \end{aligned} \tag{6.34}$$

Since the centraliser of this sector (ignoring the $N-2$ sectors that are not affected by the twist — invariance with respect to this subgroup will just guarantee that the remaining factors give rise to a factor equal to the untwisted sector $Z^{(\text{U})}$ for large N) is simply $S_2\cong\mathbb{Z}_2$, there are two representations that contribute, namely

$$\begin{aligned} \mathcal{Z}_+(q,y) &= \mathcal{Z}_{\text{vac}}\cdot\sum_{\Delta\text{ even},\ell}|c(\Delta,\ell)|\,q^{\frac{1}{2}(\Delta+\ell+1)}y^{\ell+1} \\ &= y^{\frac{1}{2}}q^{\frac{1}{4}}+(y^{\frac{3}{2}}+3y^{-\frac{1}{2}})q^{\frac{3}{4}}+(10y^{\frac{1}{2}}+3y^{-\frac{3}{2}})q^{\frac{5}{4}} \\ &\quad +(12y^{\frac{3}{2}}+27y^{-\frac{1}{2}}+y^{-\frac{5}{2}})q^{\frac{7}{4}}+\mathcal{O}(q^{\frac{9}{4}}) \ , \end{aligned} \tag{6.35}$$

and

$$\begin{aligned} \mathcal{Z}_-(q,y) &= \mathcal{Z}_{\text{vac}}\cdot\sum_{\Delta\text{ odd},\ell}|c(\Delta,\ell)|\,q^{\frac{1}{2}(\Delta+\ell+1)}y^{\ell+1} \\ &= y^{-\frac{1}{2}}q^{\frac{1}{4}}+(3y^{\frac{1}{2}}+y^{-\frac{3}{2}})q^{\frac{3}{4}}+(3y^{\frac{3}{2}}+10y^{-\frac{1}{2}})q^{\frac{5}{4}} \\ &\quad +(y^{\frac{5}{2}}+27y^{\frac{1}{2}}+12y^{-\frac{3}{2}})q^{\frac{7}{4}}+\mathcal{O}(q^{\frac{9}{4}}) \ . \end{aligned} \tag{6.36}$$

Defining the wedge characters $\chi_\pm^{(2)}$ by

$$\mathcal{Z}_\pm=\mathcal{Z}_{\text{vac}}\cdot\chi_\pm^{(2)} \ , \tag{6.37}$$

the whole sector can then simply be written as

$$\begin{aligned} Z^{(2)} &= Z^{(\mathrm{U})} \cdot \left(|\chi_+^{(2)}|^2 + |\chi_-^{(2)}|^2\right) \\ &= |\mathcal{Z}_{\mathrm{vac}}|^2 \cdot \left(1 + \sum_{\Lambda} |\chi_\Lambda(q,y)|^2\right) \cdot \left(|\chi_+^{(2)}|^2 + |\chi_-^{(2)}|^2\right) \ . \end{aligned} \tag{6.38}$$

The two wedge characters $\chi_\pm$ have the same leading q behaviour, and their lowest terms are described by the coset representations [83]

$$([k/2,0,\ldots,0];[k/2,0,\ldots,0]) \qquad \text{and} \qquad ([k/2,0,\ldots,0];[k/2+1,0,\ldots,0]) \tag{6.39}$$

for large k, respectively, i.e., have twist $\xi = [-1/2,0,\ldots,0]$ in the continuous orbifold picture. One of these states can be obtained from the other by acting on it with a fermionic zero-mode. In fact, both $\chi_\pm$ can be written in terms of coset representations (for $k \to \infty$), and we have checked that up to order q^2 we have

$$\begin{aligned} \mathcal{Z}_+(q,y) = {} & \chi_{([k/2,0,\ldots,0];[k/2+1,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2-1,0,\ldots,0])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2+3,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2,1,0,\ldots,0])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2+1,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2,1,0,\ldots,0,1])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2-2,1,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+3,0,\ldots,0,1])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2-1,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+2,1,0,\ldots,0])}(q,y) \\ & + 2\cdot\chi_{([k/2,0,\ldots,0];[k/2-1,2,0,\ldots,0])}(q,y) + 2\cdot\chi_{([k/2,0,\ldots,0];[k/2+1,0,\ldots,0,2])}(q,y) \\ & + 2\cdot\chi_{([k/2,0,\ldots,0];[k/2-2,1,0,\ldots,0,1])}(q,y) + \mathcal{O}(q^{\frac{9}{4}}) \ , \\ \mathcal{Z}_-(q,y) = {} & \chi_{([k/2,0,\ldots,0];[k/2,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+2,0,\ldots,0])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2-1,1,0,\ldots,0])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2-1,1,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+1,1,0,\ldots,0])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2+2,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+1,1,0,\ldots,0,1])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2-2,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+4,0,\ldots,0])}(q,y) \\ & + 2\cdot\chi_{([k/2,0,\ldots,0];[k/2,0,\ldots,0,2])}(q,y) + 2\cdot\chi_{([k/2,0,\ldots,0];[k/2-2,2,0,\ldots,0])}(q,y) \\ & + \chi_{([k/2,0,\ldots,0];[k/2-3,1,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2-2,0,\ldots,0,1])}(q,y) \\ & + \mathcal{O}(q^{\frac{9}{4}}) \ . \end{aligned} \tag{6.40}$$

As in [76], we can understand the multiplicities in these decompositions systematically: $\mathcal{Z}_\pm$ contains all those coset representations

$$([k/2,0,\ldots,0];[k/2+l_0,\Lambda']) \tag{6.41}$$

for which $l_0 + \sum_i \Lambda'_i$ is odd or even, respectively.[4] This is due to the fact that $l_0 + \sum_i \Lambda'_i$ counts the number of twisted modes by which the ground state

$$(\Lambda_+; \Lambda_-) = ([k/2, 0, \ldots, 0]; [k/2, 0, \ldots, 0]) \tag{6.42}$$

has been excited. Each of these twisted modes has odd parity under the $\mathbb{Z}_2$ in the centraliser. In addition, each state has to be invariant under the S_{N-2} factor of the centraliser — the states that are not invariant are accounted for by the middle factor in (6.38). For the boxes in the first row of Λ_-, this is automatically true, so the overall multiplicity with which $(\Lambda_+; \Lambda_-)$ contributes to $\mathcal{Z}_\pm$ is determined by the multiplicity of the trivial S_{N-2} representation inside the $\mathrm{SU}(N-2)$ representation Λ'. Using the (by now) standard embedding $S_{N-2} \subset \mathrm{U}(N-3) \subset \mathrm{SU}(N-2)$, we obtain the decompositions

$$\begin{aligned} (\mathbf{N-2})_{\mathrm{SU}(N-2)} &\to (N-3)_{S_{N-2}} \oplus 1_{S_{N-2}} \,, \\ \overline{(\mathbf{N-2})}_{\mathrm{SU}(N-2)} &\to (N-3)_{S_{N-2}} \oplus 1_{S_{N-2}} \,. \end{aligned} \tag{6.43}$$

Hence states with $\Lambda' = \square$ or $\Lambda' = \overline{\square}$ have multiplicity 1. Moreover, the symmetric product of two boxes contains two S_{N-2} singlets, whereas the antisymmetric product contains none. This explains why states with $\Lambda' = \begin{smallmatrix}\square\\ \square\end{smallmatrix}$ do not appear in the decomposition, whereas states with $\Lambda' = \square\!\square$ appear with multiplicity 2. The tensor product of a box with an antibox, $\overline{\square} \otimes \square$, contains two singlets, but one of them corresponds to the $s\mathcal{W}_\infty$ generators and hence does not give rise to a new representation; the resulting multiplicity in the coset decomposition is therefore again 1.

6.2.2 The twisted sector with two 2-cycles

The next, slightly more complicated step is to study the sector whose twist corresponds to the conjugacy class of permutations which have two 2-cycles. This means that two of the free bosons and fermions are twisted, while all the others are untwisted. We are interested in this sector because it contains the operators corresponding to exactly marginal deformations of the theory, which should, in particular, allow us to study the behaviour upon switching on the string coupling constant, compare [86]. By the same reasoning as before, we can obtain the gen-

[4] Here we sum only over the first few Dynkin labels of Λ', such that anti-boxes and their tensor powers do not contribute to the $\mathbb{Z}_2$ parity. Actually, we should treat Λ' as a $\mathrm{U}(N-2)$ rather than $\mathrm{SU}(N-2)$ representation, since an anti-box of $\mathrm{U}(N-2)$ differs from $[0, \ldots, 0, 1]$ of $\mathrm{SU}(N-2)$ by its $\mathrm{U}(1)$ charge, which we have suppressed in our notation.

erating function of the partition function from (6.10)

$$\sum_{N=0}^{\infty} p^N Z^{(2)^2}(S^N\mathbb{T}^2) = \frac{p^4}{2}\Bigg[\Big(\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}{}' |c(\Delta,\bar{\Delta},\ell,\bar{\ell})| q^{\frac{1}{2}(\Delta+\ell+1)}\bar{q}^{\frac{1}{2}(\bar{\Delta}+\bar{\ell}+1)} y^{\ell+1}\bar{y}^{\bar{\ell}+1}\Big)^2 + \sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}{}' c(\Delta,\bar{\Delta},\ell,\bar{\ell}) q^{\Delta+\ell+1}\bar{q}^{\bar{\Delta}+\bar{\ell}+1} y^{2\ell+2}\bar{y}^{2\bar{\ell}+2}\Bigg] \times \prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}} \frac{1}{(1-pq^{\Delta+\frac{\ell}{2}+\frac{1}{4}}\bar{q}^{\bar{\Delta}+\frac{\bar{\ell}}{2}+\frac{1}{4}}(-y)^{\ell+\frac{1}{2}}(-\bar{y})^{\bar{\ell}+\frac{1}{2}})^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}} \,. \tag{6.44}$$

In the first term, a factor of $(-1)^{\ell+\bar{\ell}+1}$ has again been absorbed into the absolute value $|c(\Delta,\bar{\Delta},\ell,\bar{\ell})|$, whereas the second term contains a factor of $(-1)^{2(\ell+\bar{\ell}+1)} = 1$. As before, the partition function for our symmetric orbifold can be obtained by taking N large, and dividing by the partition function of the free $\mathbb{T}^2$ theory. We thus obtain

$$\begin{aligned} Z^{(2)^2} &= q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}}(1+y\bar{y}+y^{-1}\bar{y}^{-1}) + q^{\frac{1}{2}}\bar{q}^{1}\big(y\bar{y}+y^{-1}\bar{y}^{-1}+3(y+y^{-1})+4(\bar{y}+\bar{y}^{-1})\big) \\ &\quad + q^{1}\bar{q}^{\frac{1}{2}}\big(y\bar{y}+y^{-1}\bar{y}^{-1}+4(y+y^{-1})+3(\bar{y}+\bar{y}^{-1})\big) \\ &\quad + q^{1}\bar{q}^{1}\big(38+3(y^2\bar{y}^2+y^{-2}\bar{y}^{-2})+17(y+y^{-1})(\bar{y}+\bar{y}^{-1}) \\ &\qquad + 7(y^2+\bar{y}^2+\bar{y}^{-2}+y^{-2})\big) + \cdots \,. \end{aligned} \tag{6.45}$$

The centraliser of this sector is

$$C^{(2)^2} = S_{N-4}\times(S_2\ltimes\mathbb{Z}_2^2)\,. \tag{6.46}$$

Again, we can ignore the action of the S_{N-4} part — this will only ensure that the $N-4$ untwisted bosons and fermions from the directions that are unaffected by the twist reproduce again the contribution from the untwisted sector. The remaining group $S_2\ltimes\mathbb{Z}_2^2\cong D_8$ (the dihedral group of order 8) has five irreducible representations, four of dimension 1, and one of dimension 2. In order to describe them, we first note that the abelian $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup has 4 different one-dimensional representations that are characterised by the eigenvalues $(\pm\,,\pm)$ of the two non-trivial $\mathbb{Z}_2$ generators. In D_8, both $(+\,,+)$ and $(-\,,-)$ give rise to two one-dimensional representations each that differ by the sign under the exchange of S_2 — this accounts for the 4 one-dimensional representations. The two-dimensional representation of D_8 is spanned by the two states with mixed charges $(\pm\,,\mp)$ that are exchanged under the action of S_2.

The simplest way to describe the contribution of these representations to the twisted sector is in multi-particle form. It follows from the derivation from equation (6.44) that the $(2)^2$ sector has the partition function

$$Z^{(\mathrm{U})}\cdot\frac{1}{2}\Bigg[\Bigg(\Big|\sum_{\substack{\Delta\ \text{even},\\ \ell}}|c(\Delta,\ell)|\,q^{\frac{1}{2}(\Delta+\ell+1)}y^{\ell+1}\Big|^2+\Big|\sum_{\substack{\Delta\ \text{odd},\\ \ell}}|c(\Delta,\ell)|\,q^{\frac{1}{2}(\Delta+\ell+1)}y^{\ell+1}\Big|^2\Bigg)^2$$
$$+\Big|\sum_{\substack{\Delta\ \text{even},\\ \ell}}c(\Delta,\ell)q^{\Delta+\ell+1}y^{2\ell+2}\Big|^2+\Big|\sum_{\substack{\Delta\ \text{odd},\\ \ell}}c(\Delta,\ell)q^{\Delta+\ell+1}y^{2\ell+2}\Big|^2\Bigg]\ . \tag{6.47}$$

Since the wedge characters of the 2-cycle twisted sector, see equation (6.37), are given by

$$\chi_{\pm}^{(2)}=\sum_{\substack{\Delta\ \text{even/odd},\\ \ell}}|c(\Delta,\ell)|\,q^{\frac{1}{2}(\Delta+\ell+1)}y^{\ell+1}\ , \tag{6.48}$$

the above $(2)^2$-sector partition function can then be written as

$$Z^{(2)^2}=Z^{(\mathrm{U})}\cdot\Big[\big|(\chi_{+}^{(2)})_{\ydiagram{2}}\big|^2+\big|(\chi_{+}^{(2)})_{\ydiagram{1,1}}\big|^2+\big|(\chi_{-}^{(2)})_{\ydiagram{2}}\big|^2+\big|(\chi_{-}^{(2)})_{\ydiagram{1,1}}\big|^2+\big|\chi_{+}^{(2)}\chi_{-}^{(2)}\big|^2\Big]\ , \tag{6.49}$$

where

$$\begin{aligned}(\chi_{\pm}^{(2)})_{\ydiagram{2}/\ydiagram{1,1}}(q,y)&=\frac{1}{2}\big(\chi_{\pm}^{(2)}(q,y)^2\pm\tilde{\chi}_{\pm}(q^2,y^2)\big)\\&=\frac{1}{2}\Big[\Big(\sum_{\substack{\Delta\ \text{even/odd},\\ \ell}}|c(\Delta,\ell)|\,q^{\frac{1}{2}(\Delta+\ell+1)}y^{\ell+1}\Big)^2\\&\qquad\pm\sum_{\substack{\Delta\ \text{even/odd},\\ \ell}}c(\Delta,\ell)\,q^{\Delta+\ell+1}y^{2\ell+2}\Big]\ .\end{aligned} \tag{6.50}$$

Each of the terms in (6.49) corresponds to one of the five irreducible representations of D_8, and can be organised in terms of coset representations. In order to describe this in detail, let us start from the ground state that has the eigenvalues $(+\,,+)$ with respect to the two $\mathbb{Z}_2$ factors; it appears in the $(\chi_{-}^{(2)})_{\ydiagram{1,1}}$ sector,[5] is an element of the coset representation

$$\big(\Lambda_{+};\Lambda_{-}\big)=\big([0,k/2,0,\dots,0];[0,k/2,0,\dots,0]\big)\ , \tag{6.51}$$

and therefore has the twist $\xi=[-1/2,-1/2,0,\dots,0]$. All other states of the $(2)^2$-twisted sector can be obtained by adding boxes to Λ_{-} (while leaving Λ_{+} invariant), yielding

$$\Lambda_{-}=[l_1,k/2+l_2,\Lambda']\ , \tag{6.52}$$

[5]Our convention for the definition of $\chi_{\pm}^{(2)}$ follows [76], and is motivated by the fact that $\pm$ corresponds to even/odd in equation (6.48); this then leads to the somewhat strange (but inevitable) conclusion that the corresponding $\mathbb{Z}_2$ eigenvalue is $\mp$, see also equation (7.17) and (7.18) of [76].

where $l_1, l_2 \in \mathbb{Z}$, and Λ' denotes the remaining $N-4$ Dynkin labels. For example, $l_1 = 0$, $l_2 = 1$ contains the ground state transforming as $(-\,,-)$ with respect to the two $\mathbb{Z}_2$ factors — it appears in the sector $(\chi_+^{(2)})_{\square\square}$ — while $l_1 = 1$, $l_2 = 0$ contains the ground state with eigenvalues $(+\,,-)$, which appears in the sector $\chi_+^{(2)}\chi_-^{(2)}$. The other two dihedral representations only contribute at order q^1; in terms of coset representations we have the decompositions

$$
\begin{aligned}
\mathcal{Z}_{\mathrm{vac}} \cdot (\chi_+^{(2)})_{\square\square} = {} & \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0])} \\
& + \chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0,1])} \\
& + \chi_{([0,k/2,0,\ldots,0];[2,k/2+1,0,\ldots,0])} + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2-2,1,0,\ldots,0])} \\
& + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,1])} + 2\cdot\chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,1])} \\
& + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0,1])} \\
& + \mathcal{O}(q^2)\ , \\
\mathcal{Z}_{\mathrm{vac}} \cdot (\chi_+^{(2)})_{\genfrac{}{}{0pt}{}{\square}{\square}} = {} & \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0])} \\
& + \chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0,1])} \\
& + \chi_{([0,k/2,0,\ldots,0];[2,k/2+1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,0,\ldots,0])} \\
& + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0,1,0])} \\
& + 2\cdot\chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,0,1])} \\
& + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0,1])} \\
& + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2-2,1,0,\ldots,0])} + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,1])} \\
& + \mathcal{O}(q^2)\ , \\
\mathcal{Z}_{\mathrm{vac}} \cdot (\chi_-^{(2)})_{\square\square} = {} & \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0])} \\
& + \chi_{([0,k/2,0,\ldots,0];[0,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0,1])} \\
& + \chi_{([0,k/2,0,\ldots,0];[2,k/2+1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,0,\ldots,0])} \\
& + \chi_{([0,k/2,0,\ldots,0];[0,k/2-1,0,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0,1,0])} \\
& + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])} + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,1])} \\
& + 2\cdot\chi_{([0,k/2,0,\ldots,0];[0,k/2-1,1,0,\ldots,0,1])} \\
& + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0,1])} \\
& + \mathcal{O}(q^2)\ , \\
\mathcal{Z}_{\mathrm{vac}} \cdot (\chi_-^{(2)})_{\genfrac{}{}{0pt}{}{\square}{\square}} = {} & \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0])} \\
& + \chi_{([0,k/2,0,\ldots,0];[0,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0,1])} \\
& + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,0,\ldots,0])} + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])} \\
& + 2\cdot\chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,1])} + 2\cdot\chi_{([0,k/2,0,\ldots,0];[0,k/2-1,1,0,\ldots,1])}
\end{aligned}
$$

$$
\begin{aligned}
&+ \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0,1])} \\
&+ \mathcal{O}(q^2) \ , \\
\mathcal{Z}_{\mathrm{vac}} \cdot \chi_+^{(2)} \chi_-^{(2)} = &\ \chi_{([0,k/2,0,\ldots,0];[1,k/2,0,\ldots,0])} \\
&+ 2 \cdot \chi_{([0,k/2,0,\ldots,0];[1,k/2-1,1,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[1,k/2,0,\ldots,0,1])} \\
&+ \chi_{([0,k/2,0,\ldots,0];[1,k/2-1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[1,k/2+1,0,\ldots,0])} \\
&+ 2 \cdot \chi_{([0,k/2,0,\ldots,0];[1,k/2-2,1,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[1,k/2-1,0,\ldots,1])} \\
&+ 4 \cdot \chi_{([0,k/2,0,\ldots,0];[1,k/2-1,1,0,\ldots,0,1])} \\
&+ \chi_{([0,k/2,0,\ldots,0];[1,k/2-1,0,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[1,k/2,0,\ldots,0,1,0])} \\
&+ 2 \cdot \chi_{([0,k/2,0,\ldots,0];[1,k/2,1,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[1,k/2+1,0,\ldots,0,1])} \\
&+ 2 \cdot \chi_{([0,k/2,0,\ldots,0];[3,k/2,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[3,k/2-1,0,\ldots,0])} \\
&+ 2 \cdot \chi_{([0,k/2,0,\ldots,0];[3,k/2-2,0,\ldots,0])} + \mathcal{O}(q^2) \ .
\end{aligned} \tag{6.53}
$$

The systematics of the decompositions are analogous to the 2-cycle twist case, see the discussion following equation (6.42) above, but are somewhat more complicated. Each box appended to the first two rows of Λ_- of the ground state (6.51) has odd parity under one of the two $\mathbb{Z}_2$'s. As a consequence, the states that appear in the mixed sector $\chi_+^{(2)}\chi_-^{(2)}$ are precisely those that have an odd number of them, i.e., for which l_1 is odd. Conversely, the other four representations contain the states with l_1 even, but the selection rules among them are more subtle, and indeed, the same coset representation can appear in different D_8 decompositions. For example, the lowest state in the representation

$$
\Lambda_- = [2, k/2, 0, \ldots, 0] \tag{6.54}
$$

can be constructed as an excitation of the twisted sector ground state with a fermionic zero-mode and a bosonic $(-\frac{1}{2})$-mode involving the same twisted coordinate. Then the state has $(+\,,+)$ charge under $\mathbb{Z}_2^2$, and we can either symmetrise or anti-symmetrise it with respect to the S_2 factor. That is why this state appears both in $(\chi_-^{(2)})_{\square\square}$ and in $(\chi_-^{(2)})_{\substack{\square\\\square}}$. But we can also construct the lowest state of (6.54) by exciting the twisted sector ground state with a fermionic zero-mode from one twisted coordinate, and a bosonic $(-\frac{1}{2})$-mode from the other, and symmetrise with respect to S_2.[6] In this case the charge is $(-\,,-)$ under $\mathbb{Z}_2^2$, and the state is even under the S_2; thus the representation (6.54) also appears in the decomposition of $(\chi_+^{(2)})_{\substack{\square\\\square}}$.

[6]The antisymmetric combination is actually a supersymmetric descendant of the excitation by the two fermionic zero-modes and is therefore part of $([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0])$.

6.2.3 Sectors of arbitrary twist

While the detailed description of the decompositions into $s\mathcal{W}_\infty$ characters becomes more and more cumbersome, some aspects of the twisted sector can be described quite generally. In particular, the partition function of any twisted sector can be written in 'multiparticle' form, generalising equation (6.49).[7] Let us first explain this for the twisted sectors $(2)^n$ corresponding to multiple 2-cycle twists. By the DMVV formula (6.5), the generating function for this part of the partition function in the R-R sector equals

$$\sum_{N=0}^{\infty} p^N Z_{\mathrm{R}}^{(2)^n}(S^N\mathbb{T}^2) = {\prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' \left.\frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p^2\, q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^{\ell}\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\right|_{p^{2n}} \\ \times p^{2n}\prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}}\frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p\, q^{\Delta}\bar{q}^{\bar{\Delta}}y^{\ell}\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\ . \tag{6.55}$$

We recognise the second factor as the untwisted partition function of S_{N-2n}, which is indistinguishable from the untwisted partition function of S_N as $N\to\infty$. The first factor, on the other hand, can be expressed in terms of sums of squares of all possible symmetrisations of the elementary characters $\chi_{\pm}^{(2)}$. To see this, let us write

$$\left.{\prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' \frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p^2\, q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^{\ell}\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\right|_{p^{2n}} = \\ = \exp\left.\left[-{\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})\log\left(1-(-1)^{\ell+\bar{\ell}+1}p^2\, q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^{\ell}\bar{y}^{\bar{\ell}}\right)\right]\right|_{p^{2n}} \\ = \exp\left.\left[{\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})\sum_{k=1}^{\infty}\frac{p^{2k}}{k}(-1)^{k(\ell+\bar{\ell}+1)}q^{\frac{k\Delta}{2}}\bar{q}^{\frac{k\bar{\Delta}}{2}}y^{k\ell}\bar{y}^{k\bar{\ell}}\right]\right|_{p^{2n}}\ .$$

Changing the order of summation and flowing to the NS-NS sector, this becomes

$$\exp\left.\left[\sum_{k=1}^{\infty}\frac{p^{2k}}{k}{\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})(-1)^{k(\ell+\bar{\ell}+1)}q^{\frac{k}{2}(\Delta+\ell+1)}\bar{q}^{\frac{k}{2}(\bar{\Delta}+\bar{\ell}+1)}y^{k(\ell+1)}\bar{y}^{k(\bar{\ell}+1)}\right]\right|_{p^{2n}}$$

[7] This observation was recently also made in [112].

$$= \exp\left[\sum_{k=1}^{\infty}\frac{p^{2k}}{k}\left(|\mathcal{F}^{k-1}\chi_+^{(2)}(q^k,y^k)|^2+|\mathcal{F}^{k-1}\chi_-^{(2)}(q^k,y^k)|^2\right)\right]\Bigg|_{p^{2n}}$$

$$= \sum_{m=0}^{n}\exp\left[\sum_{k=1}^{m}\frac{p^{2k}}{k}|\mathcal{F}^{k-1}\chi_+^{(2)}(q^k,y^k)|^2\right]\Bigg|_{p^{2m}}$$

$$\cdot \exp\left[\sum_{k=1}^{n-m}\frac{p^{2k}}{k}|\mathcal{F}^{k-1}\chi_-^{(2)}(q^k,y^k)|^2\right]\Bigg|_{p^{2(n-m)}} .$$

Next we note that

$$\begin{aligned}\exp\left[\sum_{j=1}^{m}\frac{p^{2j}}{j}|\mathcal{F}^{j-1}\chi(q^j,y^j)|^2\right]\Bigg|_{p^{2m}} &= \sum_{\substack{k_1,\ldots,k_m\geq 0\\ \sum_j jk_j=m}}\frac{1}{\prod_{i=1}^m i^{k_i}k_i!}\prod_{j=1}^{m}|\mathcal{F}^{j-1}\chi(q^j,y^j)|^{2k_j}\\ &= \frac{1}{m!}\sum_{\rho\in S_m}\prod_{j=1}^{m}|\mathcal{F}^{j-1}\chi(q^j,y^j)|^{2a_j(\rho)}\\ &= \sum_{\Lambda\in Y_m}|\chi_\Lambda(q,y)|^2 \ . \end{aligned} \tag{6.56}$$

In the second equality, we have used that $m!/\prod_{i=1}^m i^{k_i}k_i!$ is the number of elements in the conjugacy class $C_{k_1,\ldots,k_m}$ of S_m, which consist of k_i cycles of length i. On the other hand, the last equality follows from (6.27) and the column orthogonality of S_m characters,

$$\sum_{\Lambda\in Y_m}\left(\chi_m^\Lambda(\rho)\right)^2=\frac{m!}{|C_\rho|} \qquad \text{for any } \rho\in S_m \ . \tag{6.57}$$

Here the sum is over all Young diagrams of m boxes or all irreducible representations of S_m. It follows that

$$Z^{(2)^n}(S^N\mathbb{T}^2)=Z^{(\mathrm{U})}(S^{N-2n}\mathbb{T}^2)\cdot\sum_{k=0}^{n}\sum_{\Lambda_1\in Y_k}|(\chi_+^{(2)})_{\Lambda_1}(q,y)|^2\sum_{\Lambda_2\in Y_{n-k}}|(\chi_-^{(2)})_{\Lambda_2}(q,y)|^2 \ , \tag{6.58}$$

thus generalising (6.49) to the case $n>2$.

So far we have only dealt with multiple 2-cycles, but the analysis is fairly analogous for the twist $(m)^n$ consisting of n non-overlapping m-cycles. The analogue of equation (6.55) for $m\geq 2$ is now

$$\sum_{N=0}^{\infty}p^N Z_{\mathrm{R}}^{(m)^n}(S^N\mathbb{T}^2)=\prod_{\substack{\Delta,\bar\Delta,\ell,\bar\ell\\ m|(\Delta-\bar\Delta)}}\frac{1}{\left(1-(-1)^{\ell+\bar\ell+1}p^m\,q^{\frac{\Delta}{m}}\bar q^{\frac{\bar\Delta}{m}}y^\ell\bar y^{\bar\ell}\right)^{c(\Delta,\bar\Delta,\ell,\bar\ell)}}\Bigg|_{p^{mn}}$$

$$\times\, p^{mn} \prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}} \frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1} p\, q^{\Delta}\bar{q}^{\bar{\Delta}} y^{\ell}\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\ . \tag{6.59}$$

The analysis goes through essentially unmodified, and we find that we can express the partition function of this sector in terms of the elementary characters

$$\chi_i^{(m)}(q,y) = \sum_{\substack{\Delta,\ell \\ \Delta\equiv i\,(\mathrm{mod}\ m)}} |c(\Delta,\ell)|\, q^{\frac{\Delta}{m}+\frac{\ell}{2}+\frac{m}{4}}\, y^{\ell+\frac{m}{2}} \qquad \text{for } i=1,\ldots,m \tag{6.60}$$

as

$$Z^{(m)^n}(S^N\mathbb{T}^2) = Z^{(\mathrm{U})}(S^{N-mn}\mathbb{T}^2)\cdot \sum_{\substack{k_1,\ldots,k_m\geq 0 \\ \sum_j k_j=n}} \prod_{i=1}^{m} \sum_{\Lambda\in Y_{k_i}} |(\chi_i^{(m)})_{\Lambda}(q,y)|^2\ . \tag{6.61}$$

In particular, in the sector whose twist is just one cycle of length m, we have $n=1$ and thus

$$Z^{(m)}(S^N\mathbb{T}^2) = Z^{(\mathrm{U})}(S^{N-m}\mathbb{T}^2)\cdot \sum_{i=1}^{m} |\chi_i^{(m)}(q,y)|^2\ . \tag{6.62}$$

It remains to combine these statements to cover the general case of a twist with cycle structure $(1)^{N_1}(2)^{N_2}\cdots(n)^{N_n}$, i.e., N_i cycles of length i for $i=1,\ldots,n$, where $\sum_i iN_i = N$. By the DMVV formula (6.5), the R-R partition function factorises into n components pertaining to the different cycle lengths

$$Z_{\mathrm{R}}^{(1)^{N_1}\cdots(n)^{N_n}}(S^N\mathbb{T}^2) = \prod_{m=1}^{n} \prod_{\substack{\Delta,\bar{\Delta},\ell,\bar{\ell} \\ m|(\Delta-\bar{\Delta})}} \left.\frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1} p^m\, q^{\frac{\Delta}{m}}\bar{q}^{\frac{\bar{\Delta}}{m}} y^{\ell}\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\right|_{p^{mN_m}}\ , \tag{6.63}$$

and correspondingly for the NS-NS sector. Plugging in our results from above, we obtain

$$\begin{aligned} Z^{(1)^{N_1}\cdots(n)^{N_n}}(S^N\mathbb{T}^2) &= \prod_{m=1}^{n} \sum_{\substack{k_1,\ldots,k_m\geq 0 \\ \sum_j k_j=N_m}} \prod_{i=1}^{m} \sum_{\Lambda\in Y_{k_i}} |(\chi_i^{(m)})_{\Lambda}(q,y)|^2 \\ &= Z^{(\mathrm{U})}(S^{N_1}\mathbb{T}^2)\cdot \prod_{m=2}^{n} \sum_{\substack{k_1,\ldots,k_m\geq 0 \\ \sum_j k_j=N_m}} \prod_{i=1}^{m} \sum_{\Lambda\in Y_{k_i}} |(\chi_i^{(m)})_{\Lambda}(q,y)|^2\ . \end{aligned} \tag{6.64}$$

Thus we can think of the entire twisted sector as consisting of the 'multiparticle' contributions of the fundamental building blocks (6.60).

As was already alluded to before, essentially the same techniques also allow us to prove the identity equation (6.26) for the untwisted sector partition function. Since $\chi_1^{(1)}(q,y) = Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2)(q,y) = 1 + \chi_1(q,y)$ and

$$\mathcal{Z}'_{\text{vac}}(q,y) = \prod_{\substack{(\varDelta,\ell)\\ \neq(0,-\frac{1}{2})}} \frac{1}{\left(1 - q^{\varDelta+\frac{\ell}{2}+\frac{1}{4}}(-y)^{\ell+\frac{1}{2}}\right)^{c(\varDelta,\ell)}}\ , \tag{6.65}$$

we get, for $N\to\infty$,

$$\begin{aligned} Z^{(\text{U})}(S^N\mathbb{T}^2)(q,\bar{q},y,\bar{y}) &= \prod_{\substack{(\varDelta,\bar{\varDelta},\ell,\bar{\ell})\\ \neq(0,0,-\frac{1}{2},-\frac{1}{2})}} \frac{1}{\left(1 - q^{\varDelta+\frac{\ell}{2}+\frac{1}{4}}\bar{q}^{\bar{\varDelta}+\frac{\bar{\ell}}{2}+\frac{1}{4}}(-y)^{\ell+\frac{1}{2}}(-\bar{y})^{\bar{\ell}+\frac{1}{2}}\right)^{c(\varDelta,\bar{\varDelta},\ell,\bar{\ell})}} \\ &= |\mathcal{Z}'_{\text{vac}}(q,y)|^2 \cdot \exp\sum_{k=1}^{\infty}\frac{1}{k}|\mathcal{F}^{k-1}\chi_1(q^k,y^k)|^2 \\ &= |\mathcal{Z}'_{\text{vac}}(q,y)|^2 \sum_{m=0}^{\infty}\sum_{\varLambda\in Y_m}|\chi_\varLambda(q,y)|^2\ , \end{aligned} \tag{6.66}$$

which reproduces (6.26) upon dividing by $Z_{\text{NS}}(\mathbb{T}^2)$, see equation (6.14).

6.2.4 Twisted representations of the wedge algebra

Given the multiparticle structure of the entire twisted sector, see (6.64), it only remains to understand the structure of the building blocks $\chi_i^{(m)}$ (that account for the individual 'particles'). These wedge characters count states that sit in representations of the wedge subalgebra $\mathfrak{shs}[\mu]$ of $s\mathcal{W}_\infty[\mu]$. In this section we undertake first steps to understand the structure of these higher-spin representations. This should shed light on the 'particle' structure of the stringy extension of the higher spin theory; in [78] the relevant analysis was done for the bosonic toy model consisting of a single boson, here we explain the $\mathcal{N}=2$ generalisation.

As was explained at the beginning of this chapter, the m-cycle twisted sector is generated by complex fermions and bosons of twist $\xi_i = \frac{i}{m}$, where $i=1,\ldots,m$. Since the $s\mathcal{W}_\infty$ generators are neutral bilinears in the currents (and since their mode numbers continue to be integers or half-integers depending on the statistics), the contribution coming from the individual twisted (complex) bosons and fermions decouple from one another, and we can think of the representation as consisting of an m-fold tensor product of the individual twist-ξ_i contributions. Apart from one untwisted component corresponding to $i=m$ — this does not contribute to the wedge character — the other $(m-1)$ components all lead to representations whose wedge character is of the form (see also [78])

$$\chi_\xi(q,y) = q^h \prod_{n=1}^{\infty} \frac{(1+zyq^{n-\frac{1}{2}-\xi})(1+z^{-1}y^{-1}q^{n-\frac{1}{2}+\xi})}{(1-zq^{n-\xi})(1-z^{-1}q^{n-1+\xi})}\Bigg|_{z^p} . \tag{6.67}$$

Here we have assumed that $0<\xi<\frac{1}{2}$, and z keeps track of the twist, i.e., the terms with a given power of z^p pick up the same phase under the cyclic group $\mathbb{Z}_m$ in the centraliser. In the following, we shall concentrate on the z^0 case, for which the states transforms trivially under $\mathbb{Z}_m$. The q-expansion of this character is

$$\begin{aligned}\chi_\xi(q,y) = q^h\big(1 + yq^{\frac{1}{2}} + 2q + (3y+y^{-1})q^{\frac{3}{2}} \\ + (y^2+6)q^2 + (8y+3y^{-1})q^{\frac{5}{2}} + \ldots\big) .\end{aligned} \tag{6.68}$$

For $\xi < \frac{1}{2} < 1$ there is a similar answer where y is replaced by $y \mapsto y^{-1}$; the case $\xi = \frac{1}{2}$ is a bit special since there are then fermionic zero modes.

Each such representation has a single descendant at level 1/2, and is therefore a special case of what one may like to call a 'level-1/2 representation', compare the terminology of [78]. Thus we can learn about the structure of the twisted sector by studying general level-1/2 representations, and this is what we shall be doing in the following.

Suppose ϕ is the ground state of a level-1/2 representation. Let us assume for definiteness that ϕ is annihilated by $G^-_{-1/2}$ (rather than $G^+_{-1/2}$), i.e.,

$$G^-_{-1/2}\phi = 0 , \tag{6.69}$$

as well as by all the other negative-charge fermionic spin-s supercharges, i.e.,

$$W^{s-}_{-1/2}\phi = 0 \qquad \text{for } s = 2,3,\ldots . \tag{6.70}$$

(This is the situation that is relevant for (6.68); the conjugate solution arises for $\frac{1}{2} < \xi < 1$.) Here we have denoted the generators of the spin-s multiplet by (see e.g., [35])

$$W^{s0} , \qquad W^{s\pm} , \qquad W^{s1} \tag{6.71}$$

of spin s, $s+\frac{1}{2}$, and $s+1$, respectively. The corresponding modes then transform in a representation of the superconformal algebra

$$\begin{aligned}
[G^\pm_r, W^{s0}_n] &= \mp W^{s\pm}_{r+n} \\
\{G^\pm_r, W^{s\pm}_r\} &= 0 \\
\{G^\pm_r, W^{s\mp}_r\} &= \pm\big((2s-1)r - t\big)\, W^{s0}_{r+t} + 2W^{s1}_{r+t} \\
[G^\pm_r, W^{s1}_n] &= \big(sr - \tfrac{1}{2}\, n\big)\, W^{s\pm}_{r+n} .
\end{aligned} \tag{6.72}$$

Let us denote the eigenvalues of the zero-modes W_0^{s0} and W_0^{s1} on the ground state ϕ by w^{s0} and w^{s1}, respectively. Then it follows from (6.70) that

$$0 = G^+_{1/2}\, W^{s-}_{-1/2}\, \phi = \big(s w^{s0} + 2 w^{s1}\big)\, \phi \tag{6.73}$$

and hence

$$w^{s1} = -\frac{1}{2} s w^{s0} \ . \tag{6.74}$$

Note that for $s=1$ this reduces to the familiar chiral primary condition, namely that $h = -\frac{1}{2}q$, where $q = w^{10}$ is the U(1) charge with respect to the spin-1 field in the $\mathcal{N}=2$ supermultiplet, and $h = w^{11}$ is the conformal dimension.

The other condition that follows from the level-1/2 condition is that all the states generated by the $W^{s+}_{-1/2}$ modes from the ground state are proportional to $G^+_{-1/2}\phi$, i.e.,

$$W^{s+}_{-1/2}\phi = \alpha(s)\, G^+_{-1/2}\phi \ . \tag{6.75}$$

Applying $G^-_{1/2}$ to this relation and using the above commutation relations, we find that

$$\alpha(s) = -\frac{s w^{s0}}{2h} \ , \tag{6.76}$$

where we have used (6.74).

In order to obtain a relation between the different quantum numbers $\alpha(s)$, we finally apply the W_0^{20} mode to both sides of equation (6.75). For example, for the case where $s=2$ and using the $[W^{20}_m, W^{2+}_r]$ commutation relation, we conclude that

$$\alpha(3) = -\frac{8q_3(5\nu^2 - 8\sqrt{3}\nu\alpha(2) - 15(8\alpha(2)^2+3))}{9(\nu-5)} \ , \tag{6.77}$$

where $\nu = 2\mu - 1$ and q_3 is a normalisation constant of W^{30}. This determines $\alpha(3)$ as a function of $\alpha(2)$. Continuing in this manner, we obtain a recursion relation for all $\alpha(s)$. This shows that all higher quantum numbers w^{s0} and w^{s1} are recursively determined. Thus the assumption that there is a single state at level 1/2 implies that the most general level-1/2 representation is characterised by only two quantum numbers

$$h \equiv w^{11} \quad \text{and} \quad \alpha(2) \equiv -\frac{w^{20}}{h} \ . \tag{6.78}$$

6.2.5 A relation between the parameters

As in the bosonic analysis of [78], it seems that the actual ξ-twisted representation is a special type of level-1/2 representation, and has in fact one fewer state at level 3/2 than a generic level-1/2 representation.[8] One should therefore expect that it is characterised by a special relation between the two eigenvalues in (6.78). In order to work out what this relation should be, we can use that the ξ-twisted representation is described, in the coset language, by the large k limit of the coset representation $([\xi k, 0, \ldots, 0]; [\xi k, 0, \ldots, 0])$ [83]. In order to evaluate the eigenvalues of L and W^{20} on this coset state, we have worked out the form of the spin-2 fields in the coset; this is discussed, in some detail, in the appendix. With the notation of appendix D, in particular, (D.19), (D.21), (D.23) and (D.26), we find that in the (large c and $\nu = -1$) 't Hooft limit

$$\begin{aligned} L &= L_{\rm b} + L_{\rm f} + \frac{3}{2c} J^2 \ , \\ W^{20} &= \frac{1}{\sqrt{3}} \left(-L_{\rm b} + 2\, L_{\rm f}\right) \ . \end{aligned} \tag{6.79}$$

The mode expansions of the stress tensor of a single free boson and fermion are given by (the fermion has NS boundary conditions)

$$\begin{aligned} (L_{\rm b})_m &= \sum_{n\in\mathbb{Z}}^{\infty} :\bar{\alpha}_{m-n}\alpha_n: \ , \\ (L_{\rm f})_m &= \frac{1}{2} \sum_{r\in\mathbb{Z}+1/2}^{\infty} (2r-m) :\bar{\psi}_{m-r}\psi_r: \ . \end{aligned} \tag{6.80}$$

Here the bosonic and fermionic modes satisfy the usual commutation relations

$$\begin{aligned} &[\alpha_m, \alpha_n] = 0 = [\bar{\alpha}_m, \bar{\alpha}_n] \ , \qquad [\alpha_m, \bar{\alpha}_n] = m\delta_{m,-n} \ , \\ &\{\psi_r, \psi_s\} = 0 = \{\bar{\psi}_r, \bar{\psi}_s\} \ , \qquad \{\psi_r, \bar{\psi}_s\} = \delta_{r,-s} \ . \end{aligned} \tag{6.81}$$

In the ξ-twisted sector, the boson and fermion mode numbers get shifted, and the zero mode of the stress tensor picks up a normal-ordering contribution

$$\begin{aligned} (L_{\rm b})_0 &= \sum_{r\in\mathbb{Z}+\xi} :\bar{\alpha}_{-r}\alpha_r: \ + \ \frac{1}{2}\xi\,(1-\xi) \ , \\ (L_{\rm f})_0 &= \sum_{s\in\mathbb{Z}+\frac{1}{2}+\xi} s :\bar{\psi}_{-s}\psi_s: \ + \ \frac{\xi^2}{2} \ . \end{aligned} \tag{6.82}$$

For large c we then find for the eigenvalues of L_0 and W_0^{20}

$$h = \frac{\xi}{2} \ , \qquad w^{20} = \frac{1}{2\sqrt{3}}\,\xi(3\xi - 1) \ . \tag{6.83}$$

[8]This will be discussed in more detail in [50].

Eliminating ξ from the above equations yields

$$\alpha(2) = -\frac{w^{20}}{h} = -\frac{1}{\sqrt{3}}\,(3\xi - 1) = -\,\frac{6h-1}{\sqrt{3}}\ . \tag{6.84}$$

This is therefore the additional relation which characterises the special level-1/2 representations that arise in the twisted sector.

Chapter 7

Conclusions and outlook

Since it was first proposed in 2010, the duality between higher spin theories on AdS_3 and minimal model CFTs [72] has gone a long way towards a better understanding of the AdS/CFT correspondence on the one hand and the relation between higher spin theories and string theory on the other hand. It has become apparent over the last years that the original duality is not just an isolated and peculiar coincident, but, quite on the contrary, just one example of an impressive family of such dualities.

The aim of this thesis was twofold. The first aim was to study the quantum symmetries of the minimal model CFTs believed to be dual to 3d higher spin theories. In particular, it could be shown in section 2.2.4 that the classical $\mathcal{W}_\infty^{\mathrm{cl}}[\lambda]$ algebras which arise as the asymptotic extensions of the bulk symmetries admit a unique quantisation for any value of λ and central charge c. This analysis could then be carried over to the algebra $\mathcal{W}_\infty^{\mathrm{e}}$ restricted to even spins, which was the subject of chapter 3, where we also showed that the first few commutators of the wedge algebra of $\mathcal{W}_\infty^{\mathrm{e}}$ agree with those of the even spin subalgebra $\mathfrak{hs}^e[\mu]$ of $\mathfrak{hs}[\mu]$. This was an important confirmation of the proposal of [3, 88] that the dual higher spin theory on AdS_3 should be described in terms of a Chern-Simons theory based on $\mathfrak{hs}^e[\mu]$. Furthermore, given the usual relation between wedge algebras and Drinfel'd-Sokolov reductions, $\mathcal{W}_\infty^{\mathrm{e}}$ should be thought of as the quantum Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$. As we have explained, there are actually two different quantisations of the classical Drinfel'd-Sokolov reduction of $\mathfrak{hs}^e[\mu]$, which we called $\mathcal{WB}_\infty$ and $\mathcal{WC}_\infty$, respectively. We have argued that this ambiguity is closely related to the fact that $\mathfrak{hs}^e[\mu]$ is non-simply-laced. We also saw, in close analogy with [73], that only one of the 'scalar' excitations should be thought of as being perturbative, while the other should correspond to a non-perturbative classical solution.

The second goal of the thesis was to use the holographic duality in order to understand the relation between higher spins and strings from the CFT point of view. This was done in the $\mathcal{N} = 2$ setting, which lies halfway between the less complicated bosonic cases and the $\mathcal{N} = 4$ cases where the string theories are under better control.

As a first step, we gave evidence in chapter 5 that the large level limit of the SU(N) Kazama-Suzuki coset models could be understood as a U(N) orbifold theory of N free complex bosons and fermions. In particular, the subsector of the coset theory consisting of the states of the form $(0; \Lambda, u)$ — these are dual to the excitations of one complex scalar multiplet of the higher spin theory — corresponds to the untwisted sector of this orbifold, as followed from the comparison of the partition functions. We also identified the twisted sector ground states from the coset perspective, and showed that their conformal dimension, their excitation spectrum and their BPS descendants match the orbifold prediction. In particular, the BPS states are generated from the ground states by exciting them with all fermions or antifermions whose twist has the same sign. This analysis could also be carried over to Kazama-Suzuki models of SO-type [59] and to the $\mathcal{N} = 4$ Wolf space cosets [76] by other authors.

As a second step, we 'embedded' this theory into the symmetric orbifold of $\mathbb{T}^2$ in chapter 6 by showing that the latter is a (quite substantial) modular extension of the former. This is the $\mathcal{N} = 2$ analogue of the $\mathcal{N} = 4$ construction of [76] where the relevant symmetric orbifold is known to be dual to string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. It is therefore tempting to believe, in particular given the discussions of [102, 100, 17, 18], that also the symmetric orbifold of $\mathbb{T}^2$ should be dual to some string theory on AdS_3. This will be an interesting topic to study in the future.

A possible generalisation of our analysis could also be to study the $\mathcal{N} = 2$ Kazama-Suzuki cosets

$$\frac{\mathfrak{su}(N+M)^{(1)}_{k+N+M}}{\mathfrak{su}(N)^{(1)}_{k+N+M} \oplus \mathfrak{su}(M)^{(1)}_{k+N+M} \oplus \mathfrak{u}(1)^{(1)}_{\kappa}}\,, \tag{7.1}$$

of which our cosets were the special case with $M = 1$. These cosets can be viewed as a 'matrix-like' extension of the $M = 1$ case, similar to what was considered for the case of AdS_4 in [41]. Understanding their correct AdS_3 dual (see [49] for first steps in this direction) and a possible relation to a symmetric orbifold construction could lead to a more obvious connection with string theory, which might also have some common traits with the ABJ triality [41].

These generalisations will hopefully lead to a more detailed understanding of the exact relationship between higher spins and string theory. This should ultimately pave the way towards a new approach to string theory and lead to new insights about nature and consequences of that theory.

Appendix A

Explicit results on $\mathcal{W}_\infty$

A.1 Composite primary fields

The composite primary fields of $\mathcal{W}_\infty$ can be computed using the `OPEPPole` function in the Mathematica package `OPEconf` by Kris Thielemans [156]. In terms of the fields and OPEs introduced in section 2.2.4 they are explicitly given by

$$\begin{aligned}
A^6 &= W^3W^3 - \tfrac{(5c+76)\,c_{33}^4}{36(c+24)}W^{4\,\prime\prime} - \tfrac{22\,c_{33}^4}{3(c+24)}LW^4 - \tfrac{16(191c+22)\,n_3}{c(2c-1)(5c+22)(7c+68)}L(LL) \\
&\quad - \tfrac{6(67c^2+178c-752)\,n_3}{c(2c-1)(5c+22)(7c+68)}L''L - \tfrac{3(225c^2+1978c+776)\,n_3}{2c(2c-1)(5c+22)(7c+68)}L'L' \\
&\quad + \tfrac{(-5c^3-20c^2+476c+32)\,n_3}{2c(2c-1)(5c+22)(7c+68)}L^{(4)} \qquad (A.1)\\
A^7 &= W^3W^4 - \tfrac{94\,c_{34}^5}{11c+350}LW^5 - \tfrac{4(257c+83)\,c_{34}^3}{(c+23)(5c-4)(7c+114)}L(LW^3) \\
&\quad + \tfrac{(-437c^3-9089c^2-22454c+76152)\,c_{34}^3}{12(c+2)(c+23)(5c-4)(7c+114)}L''W^3 \\
&\quad + \tfrac{(-1565c^3-35801c^2-142046c-65208)\,c_{34}^3}{36(c+2)(c+23)(5c-4)(7c+114)}L'W^{3\,\prime} \\
&\quad + \tfrac{(-355c^3+329c^2+52214c+12072)\,c_{34}^3}{18(c+2)(c+23)(5c-4)(7c+114)}LW^{3\,\prime\prime} \\
&\quad - \tfrac{(c+19)\,c_{34}^5}{11c+350}W^{5\,\prime\prime} - \tfrac{(25c^4-930c^3-17157c^2+115358c+26904)\,c_{34}^3}{432(c+2)(c+23)(5c-4)(7c+114)}(W^3)^{(4)} \qquad (A.2)\\
A^{8,1} &= \tfrac{4}{7}W^{3\,\prime}W^4 - \tfrac{3}{7}W^3W^{4\,\prime} - \tfrac{3\,c_{34}^5}{7(c+7)}L'W^5 + \tfrac{6\,c_{34}^5}{35(c+7)}LW^{5\,\prime} \\
&\quad - \tfrac{6(127c+18)\,c_{34}^3}{7(c+2)(5c-4)(7c+114)}L'(LW^3) + \tfrac{4(127c+18)\,c_{34}^3}{7(c+2)(5c-4)(7c+114)}L(LW^{3\,\prime}) \\
&\quad + \tfrac{(323c^2+1578c-608)\,c_{34}^3}{42(c+2)(5c-4)(7c+114)}L'''W^3 + \tfrac{(151c^2+336c-796)\,c_{34}^3}{7(c+2)(5c-4)(7c+114)}L''W^{3\,\prime} \\
&\quad + \tfrac{(245c^2+396c+244)\,c_{34}^3}{14(c+2)(5c-4)(7c+114)}L'W^{3\,\prime\prime} + \tfrac{5(43c^2-261c-34)\,c_{34}^3}{42(c+2)(5c-4)(7c+114)}LW^{3\,\prime\prime\prime} \\
&\quad + \tfrac{(5c+32)\,c_{34}^5}{210(c+7)}W^{5\,\prime\prime\prime} + \tfrac{(5c^3-245c^2+616c+92)\,c_{34}^3}{280(c+2)(5c-4)(7c+114)}(W^3)^{(5)} \qquad (A.3)\\
A^{8,2} &= \tfrac{3(c+48)}{13c+516}W^{3\,\prime\prime}W^3 + \tfrac{(-7c-228)}{2(13c+516)}W^{3\,\prime}W^{3\,\prime} - \tfrac{108}{13c+516}L(W^3W^3)
\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{12(1927c-3543)\,c_{33}^4}{(c+31)(13c+516)(55c-6)} L(LW^4) - \tfrac{18(4c^2+211c-4083)\,c_{33}^4}{(c+31)(13c+516)(55c-6)} L''W^4 \\
&+ \tfrac{18(5c^2-218c-4218)\,c_{33}^4}{(c+31)(13c+516)(55c-6)} L'W^{4\,\prime} + \tfrac{(805c^2+18649c+28254)\,c_{33}^4}{(c+31)(13c+516)(55c-6)} LW^{4\,\prime\prime} \\
&+ \tfrac{144(1919c-642)\,n_3}{c(3c+46)(5c+3)(5c+22)(13c+516)} L(L(LL)) \\
&+ \tfrac{24(1861c^2+14814c+50184)\,n_3}{c(3c+46)(5c+3)(5c+22)(13c+516)} L''(LL) \\
&+ \tfrac{6(6895c^2+80424c-67212)\,n_3}{c(3c+46)(5c+3)(5c+22)(13c+516)} L'(L'L) \\
&+ \tfrac{(805c^3+29516c^2+197676c+169488)\,n_3}{2c(3c+46)(5c+3)(5c+22)(13c+516)} L^{(4)}L \\
&+ \tfrac{(935c^3+61940c^2+793908c+767376)\,n_3}{2c(3c+46)(5c+3)(5c+22)(13c+516)} L^{(3)}L' \\
&+ \tfrac{9(149c^3+6116c^2+77580c-85392)\,n_3}{4c(3c+46)(5c+3)(5c+22)(13c+516)} L''L'' \\
&+ \tfrac{(35c^3+1883c^2+31434c-36504)\,c_{33}^4}{16(c+31)(13c+516)(55c-6)} (W^4)^{(4)} \\
&+ \tfrac{(175c^4+15990c^3+178120c^2-721656c-19152)\,n_3}{240c(3c+46)(5c+3)(5c+22)(13c+516)} L^{(6)} \qquad\qquad (A.4)
\end{aligned}
$$

$$
\begin{aligned}
A^{8,3} = {}& W^3W^5 - \tfrac{122\,c_{35}^6}{13c+516} LW^6 - \tfrac{122\,a_{35}^6}{13c+516} L(W^3W^3) \\
&- \tfrac{(5c+114)\,a_{35}^6}{39c+1548} \left(W^{3\,\prime\prime}W^3 + W^{3\,\prime}W^{3\,\prime}\right) - \tfrac{(5c+114)\,c_{35}^6}{78c+3096} W^{6\,\prime\prime} \\
&+ \tfrac{4(671(55c^2+1699c-186)\,a_{35}^6c_{33}^4 - 3(6123c^2+245662c+104232)\,c_{35}^4)}{3(c+24)(c+31)(13c+516)(55c-6)} L(LW^4) \\
&+ \big(22(1375c^4+79875c^3+1288610c^2+4134612c-466488)\,a_{35}^6c_{33}^4 \\
&\quad - 3(22945c^4+1741388c^3+39639076c^2+260703456c-158539968) \\
&\qquad \times c_{35}^4\big) \\
&\quad \times [18(c+24)(c+31)(5c+22)(13c+516)(55c-6)]^{-1}\, L''W^4 \\
&+\big(176(1375c^4+79875c^3+1288610c^2+4134612c-466488)\,a_{35}^6c_{33}^4 \\
&\quad - 3(85475c^4+6323680c^3+138838124c^2+904228176c+441415296) \\
&\qquad \times c_{35}^4\big) \\
&\quad \times [72(c+24)(c+31)(5c+22)(13c+516)(55c-6)]^{-1}\, L'W^{4\,\prime} \\
&+ \big(2(114125c^4+5992175c^3+84458440c^2+258686332c-29233248) \\
&\qquad \times a_{35}^6c_{33}^4 \\
&\quad - 3(15925c^4+589200c^3-7046892c^2-214236368c-84029568)\,c_{35}^4\big) \\
&\quad \times [36(c+24)(c+31)(5c+22)(13c+516)(55c-6)]^{-1}\, LW^{4\,\prime\prime} \\
&+ \big(4(1375c^5+82625c^4+1448360c^3+6711832c^2+7802736c-932976) \\
&\qquad \times a_{35}^6c_{33}^4 \\
&\quad + 3(-325c^5+3740c^4+1053340c^3+11335552c^2-170758272c \\
&\qquad - 69449472)\,c_{35}^4\big) \\
&\quad \times [432(c+24)(c+31)(5c+22)(13c+516)(55c-6)]^{-1}\, (W^4)^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{1952(191c+22)\, a^6_{35} n_3}{c(2c-1)(5c+22)(7c+68)(13c+516)} L(L(LL)) \\
&+ \tfrac{4(14171c^2+76342c-132600)\, a^6_{35} n_3}{c(2c-1)(5c+22)(7c+68)(13c+516)} L''(LL) \\
&+ \tfrac{(56455c^2+712118c+182136)\, a^6_{35} n_3}{c(2c-1)(5c+22)(7c+68)(13c+516)} L'(L'L) \\
&+ \tfrac{2(680c^3+19829c^2+122c-233112)\, a^6_{35} n_3}{3c(2c-1)(5c+22)(7c+68)(13c+516)} L^{(4)} L \\
&+ \tfrac{(2915c^3+163162c^2+1041168c+56928)\, a^6_{35} n_3}{6c(2c-1)(5c+22)(7c+68)(13c+516)} L''' L' \\
&+ \tfrac{45(15c^3+372c^2+724c+912)\, a^6_{35} n_3}{2c(2c-1)(5c+22)(7c+68)(13c+516)} L'' L'' \\
&+ \tfrac{(225c^4+20670c^3+106475c^2-1346498c+35544)\, a^6_{35} n_3}{360c(2c-1)(5c+22)(7c+68)(13c+516)} L^{(6)} \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
A^{8,4} = {} & W^4 W^4 - \tfrac{134}{13c+516} \left(c^6_{44}\, LW^6 + a^6_{44}\, L(W^3 W^3) \right) \\
& - \tfrac{(7c+216)}{52c+2064} \left(c^6_{44}\, W^6{}'' + 2a^6_{44} \left(W^3{}'' W^3 + W^3{}' W^3{}' \right) \right) \\
& + \tfrac{4(737(55c^2+1699c-186)\, a^6_{44} c^4_{33} - 7(3497c^2+143471c+185244)\, c^4_{44})}{3(c+24)(c+31)(13c+516)(55c-6)} L(LW^4) \\
& + \big(11(385c^3 + 23773c^2 + 365682c - 40176)\, a^6_{44} c^4_{33} \\
& \qquad - 2(2977c^3 + 186817c^2 + 2459172c - 10551168)\, c^4_{44} \big) \\
& \quad \times [6(c+24)(c+31)(13c+516)(55c-6)]^{-1}\, L'' W^4 \\
& + \big(22(385c^3 + 23773c^2 + 365682c - 40176)\, a^6_{44} c^4_{33} \\
& \qquad - (10075c^3 + 695962c^2 + 12258696c + 20136384)\, c^4_{44} \big) \\
& \quad \times [6(c+24)(c+31)(13c+516)(55c-6)]^{-1}\, L' W^4{}' \\
& + \big((15565c^3 + 816867c^2 + 10328252c - 1136460)\, a^6_{44} c^4_{33} \\
& \qquad - (8320c^3 + 494391c^2 + 6176258c - 13466568)\, c^4_{44} \big) \\
& \quad \times [9(c+24)(c+31)(13c+516)(55c-6)]^{-1}\, L W^4{}'' \\
& + \big(2(385c^4 + 24543c^3 + 413228c^2 + 691188c - 80352)\, a^6_{44} c^4_{33} \\
& \qquad - (455c^4 + 21219c^3 - 111446c^2 - 9024216c + 14935104)\, c^4_{44} \big) \\
& \quad \times [144(c+24)(c+31)(13c+516)(55c-6)]\, (W^4)^{(4)} \\
& + 16 \big(134(2865c^3 + 45979c^2 + 31616c + 3036)\, a^6_{44} n_3 \\
& \qquad - 3(64922c^3 + 2721945c^2 + 5753619c - 134676)\, n_4 \big) \\
& \quad \times [c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(13c+516)]^{-1} \\
& \quad \times L(L(LL)) \\
& + 8 \big(3(43695c^4 + 1096227c^3 + 6433406c^2 - 1772840c - 3148608)\, a^6_{44} n_3 \\
& \qquad - (75478c^4 + 3555117c^3 + 20414771c^2 - 70417080c + 12489264)\, n_4 \big) \\
& \quad \times [c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(13c+516)]^{-1}\, L''(LL) \\
& + \big(3(386565c^4 + 13116359c^3 + 115756152c^2 + 85517140c + 12421104)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times a_{44}^6 n_3 \\
&\qquad - 4(253565c^4 + 14040162c^3 + 163680028c^2 \\
&\qquad\qquad\qquad + 233430552c + 1572768)\, n_4\big) \\
&\quad \times \left[c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(13c+516)\right]^{-1} L'(L'L) \\
&+ \big(3(19395c^5 + 934857c^4 + 10600112c^3 + 4580428c^2 \\
&\qquad\qquad\qquad - 128888208c - 76811904)\, a_{44}^6 n_3 \\
&\qquad + 2(-16250c^5 - 853312c^4 - 5725441c^3 + 114072166c^2 \\
&\qquad\qquad\qquad + 527210784c + 229314528) n_4\big) \\
&\quad \times \left[6c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(13c+516)\right]^{-1} L^{(4)} L \\
&+ \big(3(61215c^5 + 4070129c^4 + 66914380c^3 + 296660444c^2 \\
&\qquad\qquad\qquad + 103648384c - 30677952)\, a_{44}^6 n_3 \\
&\qquad - 4(45955c^5 + 2563084c^4 + 29966096c^3 + 24033944c^2 \\
&\qquad\qquad\qquad - 37436544c + 203840640)\, n_4\big) \\
&\quad \times \left[12c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(13c+516)\right]^{-1} L''' L' \\
&+ 3\big(45(315c^5 + 15369c^4 + 188088c^3 + 444268c^2 + 599568c + 238464) \\
&\qquad\qquad \times\, a_{44}^6 n_3 \\
&\qquad - 2(6916c^5 + 322963c^4 + 1583702c^3 - 10458112c^2 \\
&\qquad\qquad\qquad + 119173176c - 19257120)\, n_4\big) \\
&\quad \times \left[4c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(13c+516)\right]^{-1} L'' L'' \\
&+ \big(3(4725c^6 + 400785c^5 + 6619040c^4 + 4856722c^3 - 310076664c^2 \\
&\qquad\qquad\qquad - 183843960c + 1543392)\, a_{44}^6 n_3 \\
&\qquad - 2(4550c^6 + 246120c^5 + 1939005c^4 - 41511678c^3 \\
&\qquad\qquad\qquad - 628847552c^2 - 934493328c + 3244608)\, n_4\big) \\
&\quad \times \left[720c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)(13c+516)\right]^{-1} L^{(6)}
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
A^{9,1} = {} & \tfrac{5}{8} W^{3\,\prime} W^5 - \tfrac{3}{8} W^3 W^{5\,\prime} \\
& + \tfrac{5c+42}{48(7c+68)} \left(c_{35}^6\, W^{6\,\prime\prime\prime} + 2\, a_{35}^6\, W^{3\,\prime\prime\prime} W^3 + 6\, a_{35}^6\, W^{3\,\prime\prime} W^{3\,\prime}\right) \\
& + \tfrac{23}{12(7c+68)} \left(3c_{35}^6\, L' W^6 + c_{35}^6\, L W^{6\,\prime} + 3a_{35}^6\, L'(W^3 W^3) + 2a_{35}^6\, L(W^{3\,\prime} W^3)\right) \\
& + \tfrac{253(5c^2+17c-22)\, a_{35}^6 c_{33}^4 - 3(665c^2+6642c+1768)\, c_{35}^4}{18(c-1)(c+24)(5c+22)(7c+68)} \left(2\, L'(LW^4) - L(LW^{4\,\prime})\right) \\
& - \tfrac{22(75c^3+1115c^2+2594c-3784)\, a_{35}^6 c_{33}^4 - 3(861c^3+18290c^2+96648c+2176)\, c_{35}^4}{432(c-1)(c+24)(5c+22)(7c+68)} L''' W^4 \\
& - \tfrac{88(25c^3+295c^2+604c-924)\, a_{35}^6 c_{33}^4 - 3(1099c^3+18390c^2+55112c-192576)\, c_{35}^4}{192(c-1)(c+24)(5c+22)(7c+68)} L'' W^{4\,\prime}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4(215c^2+809c-1024)\, a_{35}^6 c_{33}^4 - 3(455c^2+4812c+3808) c_{35}^4}{576(c-1)(c+24)(7c+68)} L' W^{4\,\prime\prime} \\
&- \frac{16(235c^3+2589c^2+5052c-7876)\, a_{35}^6 c_{33}^4 + 3(-889c^3+2158c^2+107656c+27200)\, c_{35}^4}{864(c-1)(c+24)(5c+22)(7c+68)} L W^{4\,\prime\prime\prime} \\
&- \big(2(175c^4+765c^3-11112c^2-37876c+48048)\, a_{35}^6 c_{33}^4 \\
&\qquad - 3(35c^4-2054c^3-11552c^2+117056c+32640)\, c_{35}^4\big) \\
&\quad \times [17280(c-1)(c+24)(5c+22)(7c+68)]^{-1}\, (W^4)^{(5)} \\
&- \frac{(1111c-438)\, a_{35}^6 n_3}{6c(2c-1)(5c+22)(7c+68)} L'''(LL) - \frac{141(17c+14)\, a_{35}^6 n_3}{4c(2c-1)(5c+22)(7c+68)} L''(L'L) \\
&+ \frac{5(7c+114)\, a_{35}^6 n_3}{8c(2c-1)(5c+22)(7c+68)} L'(L'L') - \frac{(175c^2+348c-3684)\, a_{35}^6 n_3}{80c(2c-1)(5c+22)(7c+68)} L^{(5)} L \\
&- \frac{(95c^2+634c-752)\, a_{35}^6 n_3}{16c(2c-1)(5c+22)(7c+68)} L^{(4)} L' - \frac{75(3c^2+6c+8)\, a_{35}^6 n_3}{16c(2c-1)(5c+22)(7c+68)} L''' L'' \\
&- \frac{(135c^3+264c^2-7550c-156)\, a_{35}^6 n_3}{20160c(2c-1)(5c+22)(7c+68)} L^{(7)} \ .
\end{aligned}
\tag{A.7}
$$

A.2 Structure constants

The structure constants in the ansatz (2.106) for $\mathcal{W}_\infty$ are determined by Jacobi identities, which can be implemented using the function `OPEJacobi` in Thielemans' Mathematica package `OPEdefs` (we also used `OPEconf` in order to be able to work with primary fields only). The resulting expressions are collected below:[1]

$$
\begin{aligned}
c_{34}^3 &= \frac{c_{33}^4 n_4}{n_3} \ , \\
c_{35}^4 &= \frac{5(c+7)(5c+22)\, (c_{33}^4)^2 n_4}{(c+2)(7c+114)\, c_{34}^5 n_3} - \frac{60\, n_3}{c\, c_{34}^5} \ , \\
c_{44}^6 &= \frac{4\, c_{34}^5 c_{35}^6}{5\, c_{33}^4} \ , \\
a_{44}^6 &= \frac{4\, c_{34}^5 a_{35}^6}{5\, c_{33}^4} + \frac{30(5c+22)\, n_4}{(c+2)(7c+114)\, n_3} \ , \\
c_{44}^4 &= \frac{3(c+3)\, c_{33}^4 n_4}{(c+2)\, n_3} - \frac{288(c+10)\, n_3}{c(5c+22)\, c_{33}^4} \ , \\
a_{36}^{8,1} &= \frac{(3c+116)\, a_{35}^6 c_{33}^4}{3(c+24)\, c_{35}^6} + \frac{1920(2c-1)\, n_3}{c(c+24)(5c+22)\, c_{34}^5 c_{35}^6} - \frac{160(c+7)(2c-1)\, (c_{33}^4)^2 n_4}{(c+2)(c+24)(7c+114)\, c_{34}^5 c_{35}^6 n_3} \ , \\
c_{36}^5 &= -\frac{(7c-8)\, c_{33}^4 c_{34}^5 a_{35}^6}{6(c+24)\, c_{35}^6} - \frac{960(c+10)^2\, n_3}{c(c+24)(5c+22)\, c_{35}^6} + \frac{35(c+7)(7c^2+122c+688)\, (c_{33}^4)^2 n_4}{4(c+2)(c+24)(7c+114)\, c_{35}^6 n_3} \ , \\
c_{36}^3 &= -\frac{2(c-97)\, (c_{33}^4)^2 a_{35}^6 n_4}{9(c+24)\, c_{35}^6 n_3} - \frac{4(c+2)(c+23)(5c-4)(7c+114)\, a_{35}^6 n_3}{c(2c-1)(5c+22)(7c+68)\, c_{35}^6} \\
&\quad + \frac{5(c+7)(5c+22)(7c-8)\, (c_{33}^4)^3 (n_4)^2}{6(c+2)(c+24)(7c+114)\, c_{34}^5 c_{35}^6 (n_3)^2} - \frac{10(7c-8)\, c_{33}^4 n_4}{c(c+24)\, c_{34}^5 c_{35}^6} \ , \\
a_{45}^{8,1} &= \frac{240(c+7)\, c_{33}^4 n_4}{(c+2)(7c+114)\, c_{34}^5 n_3} - \frac{2880\, n_3}{c(5c+22)\, c_{33}^4 c_{34}^5} \ , \\
c_{45}^7 &= \frac{2\, c_{35}^6 c_{36}^7}{3\, c_{33}^4} \ , \\
a_{45}^7 &= \frac{4(c+13)\, a_{35}^6}{3(c+24)} + \frac{2\, c_{35}^6 a_{36}^7}{3\, c_{33}^4} - \frac{6720(c+10)\, n_3}{c(c+24)(5c+22)\, c_{33}^4 c_{34}^5} + \frac{560(c+7)(c+10)\, c_{33}^4 n_4}{(c+2)(c+24)(7c+114)\, c_{34}^5 n_3} \ ,
\end{aligned}
$$

[1] Incidentally, the expression for c_{46}^6 in [166] has a typo in one of the signs.

$$
\begin{aligned}
c_{45}^{5} &= \frac{5(c+7)(17c+126)\,c_{33}^{4}n_4}{2(c+2)(7c+114)\,n_3} - \frac{720(c+10)\,n_3}{c(5c+22)\,c_{33}^{4}}\ ,\\
c_{45}^{3} &= \frac{5(c+7)(5c+22)\,(c_{33}^{4})^2(n_4)^2}{(c+2)(7c+114)\,c_{34}^{5}n_3^2} - \frac{60\,n_4}{c\,c_{34}^{5}}\ ,\\
a_{37}^{9,1} &= \frac{4(c+13)(11c+820)\,c_{33}^{4}c_{34}^{5}a_{35}^{6}}{5(c+24)(11c+350)\,c_{35}^{6}c_{36}^{7}} + \frac{2(11c+820)\,c_{34}^{5}a_{36}^{7}}{5(11c+350)\,c_{36}^{7}} + \frac{2304(73c^2+1149c-850)\,n_3}{c(c+24)(5c+22)(11c+350)\,c_{35}^{6}c_{36}^{7}}\\
&\quad - \frac{3(3797c^3+82090c^2+387832c-306880)\,(c_{33}^{4})^2n_4}{(c+2)(c+24)(7c+114)(11c+350)\,c_{35}^{6}c_{36}^{7}n_3}\ ,\\
c_{37}^{6} &= \frac{(1372c^4+57159c^3+985274c^2+8331408c+27861120)\,(c_{33}^{4})^2n_4}{(c+2)(c+24)(7c+114)(11c+350)\,c_{36}^{7}n_3}\\
&\quad - \frac{2(29c-60)(2(c+13)a_{35}^{6}c_{33}^{4}+(c+24)a_{36}^{7}c_{35}^{6})c_{34}^{5}}{5(c+24)(11c+350)\,c_{36}^{7}}\\
&\quad - \frac{18(1323c^3+52400c^2+759236c+3957600)\,n_3}{c(c+24)(5c+22)(11c+350)\,c_{36}^{7}}\ ,\\
a_{37}^{6} &= \frac{15(c+7)(2c-1)(5c+22)(7c+68)(167c^2+2186c-1392)\,(c_{33}^{4})^3(n_4)^2}{4(c+2)^2(c+23)(c+24)(5c-4)(7c+114)^2\,c_{34}^{5}c_{35}^{6}c_{36}^{7}(n_3)^2} - \frac{2(29c-60)\,c_{34}^{5}a_{35}^{6}a_{36}^{7}}{5(11c+350)\,c_{36}^{7}}\\
&\quad - \frac{4(c+13)(29c-60)\,c_{33}^{4}c_{34}^{5}(a_{35}^{6})^2}{5(c+24)(11c+350)\,c_{35}^{6}c_{36}^{7}} - \frac{18(1323c^3+52400c^2+759236c+3957600)\,a_{35}^{6}n_3}{c(c+24)(5c+22)(11c+350)\,c_{35}^{6}c_{36}^{7}}\\
&\quad + \frac{(6090c^6+353824c^5+8034345c^4+98606461c^3+679706700c^2+1790854612c-2216076240)}{(c+2)(c+23)(c+24)(5c-4)(7c+114)(11c+350)}\\
&\qquad \times \frac{(c_{33}^{4})^2a_{35}^{6}n_4}{c_{35}^{6}c_{36}^{7}n_3}\\
&\quad - \frac{(c+51)(2c-1)(5c+22)(7c+68)\,a_{36}^{7}c_{33}^{4}n_4}{2(c+2)(c+23)(5c-4)(7c+114)\,c_{36}^{7}n_3} - \frac{45(2c-1)(7c+68)(167c^2+2186c-1392)\,c_{33}^{4}n_4}{c(c+2)(c+23)(c+24)(5c-4)(7c+114)\,c_{34}^{5}c_{35}^{6}c_{36}^{7}}\ ,\\
c_{37}^{4} &= \frac{5(c+7)(5c+22)(6860c^5+233021c^4+2210045c^3+2684318c^2-18804472c+11668160)\,(c_{33}^{4})^4(n_4)^2}{12(c+2)^2(c+23)(c+24)(5c-4)(7c+114)(11c+350)\,c_{34}^{5}c_{35}^{6}c_{36}^{7}(n_3)^2}\\
&\quad - \frac{(5c+22)(457c^3-2188c^2-156865c+78820)(2(c+13)\,c_{33}^{4}a_{35}^{6}+(c+24)\,c_{35}^{6}a_{36}^{7})\,(c_{33}^{4})^2n_4}{18(c+2)(c+23)(c+24)(5c-4)(11c+350)\,c_{35}^{6}c_{36}^{7}n_3}\\
&\quad - \frac{(103700c^6+5037443c^5+82080149c^4+484133372c^3+156571028c^2-4060675888c+2813946240)}{c(c+2)(c+23)(c+24)(5c-4)(7c+114)(11c+350)}\\
&\qquad \times \frac{5(c_{33}^{4})^2n_4}{c_{34}^{5}c_{35}^{6}c_{36}^{7}}\\
&\quad + \frac{23040(c+10)^2(29c-60)\,(n_3)^2}{c^2(c+24)(5c+22)(11c+350)\,c_{34}^{5}c_{35}^{6}c_{36}^{7}}\\
&\quad - \frac{(11c^2+626c+32880)(2(c+13)a_{35}^{6}c_{33}^{4}+(c+24)a_{36}^{7}c_{35}^{6})\,n_3}{c(c+24)(11c+350)c_{35}^{6}c_{36}^{7}}\ ,\\
a_{46}^{9,1} &= \frac{117(3c+26)\,c_{33}^{4}n_4}{(c+2)(7c+114)\,c_{35}^{6}n_3} + \frac{6\,c_{34}^{5}a_{35}^{6}}{5\,c_{35}^{6}} - \frac{6912\,n_3}{c(5c+22)\,c_{33}^{4}c_{35}^{6}}\ ,\\
c_{46}^{8} &= \frac{4c_{36}^{7}c_{37}^{8}}{7c_{33}^{4}}\ ,\\
a_{46}^{8,2} &= \frac{640(c+7)(2c-1)(5c+22)(29c^2+533c-870)\,(c_{33}^{4})^2(n_4)^2}{21(c+2)^2(c+24)(5c-4)(7c+114)^2\,c_{34}^{5}c_{35}^{6}(n_3)^2}\\
&\quad - \frac{4(5c+22)((3c+116)(29c^2+533c-870)\,c_{33}^{4}a_{35}^{6}-(c+24)(9c^2+37c+1788)\,c_{35}^{6}a_{36}^{7})\,n_4}{63(c+2)(c+24)(5c-4)(7c+114)\,c_{35}^{6}n_3}\\
&\quad - \frac{2560(2c-1)(29c^2+533c-870)\,n_4}{7c(c+2)(c+24)(5c-4)(7c+114)\,c_{34}^{5}c_{35}^{6}} + \frac{4\,a_{37}^{8,2}c_{36}^{7}}{7\,c_{33}^{4}}\ ,\\
a_{46}^{8,3} &= \frac{15(8611c^3+301020c^2+3170988c+11305504)\,c_{33}^{4}n_4}{7(c+2)(c+24)(7c+114)(11c+350)\,c_{35}^{6}n_3} - \frac{11520(c+10)(169c+3370)\,n_3}{7c(c+24)(5c+22)(11c+350)\,c_{33}^{4}c_{35}^{6}}\\
&\quad - \frac{22(3c^2+314c+5024)\,c_{34}^{5}a_{35}^{6}}{7(c+24)(11c+350)\,c_{35}^{6}} + \frac{4(11c+68)\,c_{34}^{5}a_{36}^{7}}{7(11c+350)\,c_{33}^{4}} + \frac{4\,c_{36}^{7}a_{37}^{8,3}}{7\,c_{33}^{4}}\ ,
\end{aligned}
$$

$$
\begin{aligned}
a^{8,4}_{46} &= \frac{4\Big(c(5c+22)c^5_{34}\big(2(3c+116)\,(c^4_{33})^2 a^6_{35}+3(c+24)\,c^6_{35}(c^4_{33}a^7_{36}+c^7_{36}a^{8,4}_{37})\big)+11520(2c-1)\,c^4_{33}n_3\Big)}{21c(c+24)(5c+22)\,c^4_{33}c^5_{34}c^6_{35}} \\
&\quad - \frac{1280(c+7)(2c-1)\,(c^4_{33})^2 n_4}{7(c+2)(c+24)(7c+114)\,c^5_{34}c^6_{35}n_3}\ , \\
c^6_{46} &= \frac{(147c^3+4237c^2+46786c+181360)\,c^4_{33}n_4}{2(c+2)(c+24)(7c+114)\,n_3} - \frac{(17c-32)\,c^5_{34}a^6_{35}}{15(c+24)} - \frac{192(7c^2+195c+1628)\,n_3}{c(c+24)(5c+22)\,c^4_{33}}\ , \\
a^6_{46} &= \frac{120(c+7)(2c-1)(5c+22)(7c+68)\,(c^4_{33})^2(n_4)^2}{(c+2)^2(c+24)(7c+114)^2\,c^5_{34}c^6_{35}(n_3)^2} + \frac{(119c^3+2691c^2+35054c+187616)\,c^4_{33}a^6_{35}n_4}{2(c+2)(c+24)(7c+114)\,c^6_{35}n_3} \\
&\quad - \frac{1440(2c-1)(7c+68)\,n_4}{c(c+2)(c+24)(7c+114)\,c^5_{34}c^6_{35}} - \frac{(17c-32)\,c^5_{34}(a^6_{35})^2}{15(c+24)\,c^6_{35}} - \frac{192(7c^2+195c+1628)\,a^6_{35}n_3}{c(c+24)(5c+22)\,c^4_{33}c^6_{35}}\ , \\
c^4_{46} &= \frac{5(c+7)(5c+22)(7c^2+82c+288)\,(c^4_{33})^3(n_4)^2}{(c+2)^2(c+24)(7c+114)\,c^5_{34}c^6_{35}(n_3)^2} \\
&\quad - \frac{2(c-9)(5c+22)\,(c^4_{33})^2a^6_{35}n_4}{3(c+2)(c+24)\,c^6_{35}n_3} - \frac{60(113c^3+3100c^2+26724c+77632)\,c^4_{33}n_4}{c(c+2)(c+24)(7c+114)\,c^5_{34}c^6_{35}} \\
&\quad + \frac{46080(c+10)^2\,(n_3)^2}{c^2(c+24)(5c+22)\,c^4_{33}c^5_{34}c^6_{35}} + \frac{8(23c^3-2674c^2-40088c+33664)\,a^6_{35}n_3}{c(c+24)(2c-1)(7c+68)\,c^6_{35}}\ , \\
c^8_{55} &= \frac{10\,c^6_{35}c^7_{36}c^8_{37}}{21\,c^4_{33}c^5_{34}}\ , \\
a^{8,2}_{55} &= \frac{10(5c+22)(9c^2+37c+1788)(2(c+13)\,c^4_{33}a^6_{35}+(c+24)\,c^6_{35}a^7_{36})\,n_4}{189(c+2)(c+24)(5c-4)(7c+114)\,c^5_{34}n_3} \\
&\quad - \frac{400(193c^3-12430c^2-299960c+243744)n_4}{21c(c+2)(c+24)(5c-4)(7c+114)\,(c^5_{34})^2} \\
&\quad + \frac{100(c+7)(5c+22)\big(193c^3-12430c^2-299960c+243744\big)\,(c^4_{33})^2(n_4)^2}{63(c+2)^2(c+24)(5c-4)(7c+114)^2\,(c^5_{34})^2(n_3)^2} + \frac{10\,c^6_{35}c^7_{36}a^{8,2}_{37}}{21\,c^4_{33}c^5_{34}}\ , \\
a^{8,3}_{55} &= \frac{20(c+13)(11c+68)\,a^6_{35}}{21(c+24)(11c+350)} + \frac{10(11c+68)\,c^6_{35}a^7_{36}}{21(11c+350)\,c^4_{33}} + \frac{10\,c^6_{35}c^7_{36}a^{8,3}_{37}}{21\,c^4_{33}c^5_{34}} \\
&\quad - \frac{9600(c+10)(169c+3370)\,n_3}{7c(c+24)(5c+22)(11c+350)\,c^4_{33}c^5_{34}} + \frac{25(3343c^3+92550c^2+614104c+2418752)\,c^4_{33}n_4}{7(c+2)(c+24)(7c+114)(11c+350)\,c^5_{34}n_3}\ , \\
a^{8,4}_{55} &= \frac{20(c+13)\,c^4_{33}a^6_{35}}{21(c+24)c^5_{34}} + \frac{10\,c^6_{35}a^7_{36}}{21\,c^5_{34}} + \frac{10\,c^6_{35}c^7_{36}a^{8,4}_{37}}{21\,c^4_{33}c^5_{34}} + \frac{400(c+7)(11c+166)\,(c^4_{33})^2n_4}{7(c+2)(c+24)(7c+114)\,(c^5_{34})^2n_3} \\
&\quad - \frac{4800(11c+166)\,n_3}{7c(c+24)(5c+22)\,(c^5_{34})^2}\ , \\
c^6_{55} &= \frac{5(37c^2+425c+2202)\,c^4_{33}c^6_{35}n_4}{3(c+2)(7c+114)\,c^5_{34}n_3} - \frac{60(19c+218)\,c^6_{35}n_3}{c(5c+22)\,c^4_{33}c^5_{34}}\ , \\
a^6_{55} &= \frac{5(37c^2+425c+2202)\,c^4_{33}a^6_{35}n_4}{3(c+2)(7c+114)\,c^5_{34}n_3} - \frac{60(19c+218)\,a^6_{35}n_3}{c(5c+22)\,c^4_{33}c^5_{34}} - \frac{450(39c+178)\,n_4}{c(c+2)(7c+114)\,(c^5_{34})^2} \\
&\quad + \frac{75(c+7)(5c+22)(39c+178)\,(c^4_{33})^2(n_4)^2}{2(c+2)^2(7c+114)^2\,(c^5_{34})^2(n_3)^2}\ , \\
c^4_{55} &= \frac{25(c+7)^2(5c+22)(17c+126)\,(c^4_{33})^3(n_4)^2}{2(c+2)^2(7c+114)^2\,(c^5_{34})^2(n_3)^2} - \frac{150(c+7)(41c+366)\,c^4_{33}n_4}{c(c+2)(7c+114)\,(c^5_{34})^2} \\
&\quad + \frac{43200(c+10)\,(n_3)^2}{c^2(5c+22)\,c^4_{33}(c^5_{34})^2}\ , \\
n_5 &= \frac{5(c+7)(5c+22)(c^4_{33})^2(n_4)^2}{(c+2)(7c+114)\,(c^5_{34})^2n_3} - \frac{60\,n_3n_4}{c\,(c^5_{34})^2}\ .
\end{aligned}
\tag{A.8}
$$

A.3 $\mathcal{W}_\infty$ in terms of commutators

We are now going to describe the algebra $\mathcal{W}_\infty$ in terms of commutators between the lowest fields. We will work in the conventions of [81, 73], which differ from the ones we used in the previous analysis. In particular, the normalisation of the fields W^s for $s \geq 4$ is fixed by setting $c_{3\,s-1}^{s} = s$ (which also fixes $c_{33}^{4} = 4$), and the free parameter of the algebra is now given by a function of N_3 and N_4, which are defined as follows:

$$N_3 = \frac{6}{5c}\, n_3 \ , \qquad N_4 = \frac{6}{7c}\, n_4 \ . \tag{A.9}$$

The explicit structure constants in these conventions then read

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\, m(m^2-1)\,\delta_{0,m+n} \\
[L_m, W^3_n] &= (2m-n)W^3_{m+n} \\
[L_m, W^4_n] &= (3m-n)W^4_{m+n} \\
[L_m, W^5_n] &= (4m-n)W^5_{m+n} \\
[W^3_m, W^3_n] &= (m-n)\left(2\,W^4_{m+n} + \frac{40N_3}{5c+22}\,Q^4_{m+n}\right) \\
&\quad + \frac{N_3}{12}(m-n)\left(2m^2 - mn + 2n^2 - 8\right) L_{m+n} \\
&\quad + \frac{N_3 c}{144}\, m(m^2-1)(m^2-4)\,\delta_{0,m+n} \ , \\
[W^3_m, W^4_n] &= \frac{84N_4}{25(c+2)N_3} Q^{6,1}_{m+n} + (3m-2n)\left(W^5_{m+n} + \frac{1456N_4}{N_3(175c+2850)} Q^5_{m+n}\right) \\
&\quad + \frac{N_4}{15N_3}\left(-5m^3 + 5m^2 n + 17m - 51mn^2 + n^3 - 9n\right) W^3_{m+n} \ , \\
[W^3_m, W^5_n] &= \left(\frac{1792(c+7)N_4}{15(c+2)(7c+114)N_3} - \frac{160N_3}{15c+66}\right) Q^{7,1}_{m+n} \\
&\quad + (2m-n)\Bigg(2\,W^6_{m+n} + \frac{a^6_{35}}{3} A^6_{m+n} \\
&\qquad + \frac{16\big(56(5c^2+57c+154)N_4 - 25(7c^2+128c+228)N_3^2\big)}{15(c+2)(c+24)(7c+114)N_3} Q^{6,2}_{m+n}\Bigg) \\
&\quad - \left(-28m^3 + 21m^2 n + m\left(88 - 9n^2\right) + 2n\left(n^2 - 16\right)\right) \\
&\qquad \times \left(\frac{14(c+7)(5c+22)N_4}{225(c+2)(7c+114)N_3} - \frac{N_3}{36}\right) W^4_{m+n} \ , \\
[W^4_m, W^4_n] &= 3(m-n)\Bigg(W^6_{m+n} + \left(\frac{a^6_{35}}{6} + \frac{7(5c+22)N_4}{(c+2)(7c+114)N_3}\right) A^6_{m+n} \\
&\quad + \frac{8\left(49(c+3)(5c+22)N_4 - 175(c+2)(c+10)N_3^2\right)}{15(c+2)(c+24)(5c+22)N_3} Q^{6,2}_{m+n} \\
&\quad - \frac{7(19c-524)N_4}{54(2c-1)(7c+68)} Q^{6,3}_{m+n} + \frac{56(72c+13)N_4}{3(2c-1)(5c+22)(7c+68)} Q^{6,4}_{m+n}\Bigg) \\
&\quad + (m-n)\left(m^2 - mn + n^2 - 7\right)
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\left(\frac{7(c+3)N_4}{15(c+2)N_3}-\frac{5(c+10)N_3}{15c+66}\right)W^4_{m+n}+\frac{49N_4}{9(5c+22)}Q^4_{m+n}\right)\\
&+\frac{N_4}{360}(m-n)\big(3m^4-2m^3n+m^2\left(4n^2-39\right)-2mn\left(n^2-10\right)\\
&\quad+3(n^4-13n^2+36)\big)L_{m+n}\\
&+\frac{N_4c}{4320}\,m(m^2-1)(m^2-4)(m^2-9)\,\delta_{0,m+n}\ ,
\end{aligned}
\tag{A.10}
$$

where we have used the composite quasi-primary fields

$$
\begin{aligned}
Q^4 &= LL-\frac{3}{10}\,L''\ ,\\
Q^5 &= LW^3-\frac{3}{14}\,W^{3\,\prime\prime}\ ,\\
Q^{6,1} &= L'W^3-\frac{2}{3}LW^{3\,\prime}+\frac{1}{12}W^{3\,\prime\prime\prime}\ ,\\
Q^{6,2} &= LW^4-\frac{1}{6}W^{4\,\prime\prime}\ ,\\
Q^{6,3} &= L'L'-\frac{4}{5}L''L-\frac{1}{42}L^{(4)}\ ,\\
Q^{6,4} &= LQ^4-\frac{1}{6}Q^{4\,\prime\prime}\ ,\\
A^6 &= W^3W^3-\frac{5}{126}\left(14W^{4\,\prime\prime}+7\frac{40N_3}{5c+22}Q^{4\,\prime\prime}+\frac{3N_3}{4}L^{(4)}\right)-\frac{88}{3(c+24)}Q^{6,2}\\
&\quad+\frac{5(43c-844)N_3}{36(2c-1)(7c+68)}Q^{6,3}-\frac{40(191c+22)N_3}{3(2c-1)(5c+22)(7c+68)}Q^{6,4}\ ,\\
Q^{7,1} &= L'W^4-\frac{1}{2}LW^{4\,\prime}+\frac{1}{20}W^{4\,\prime\prime\prime}\ .
\end{aligned}
\tag{A.11}
$$

The primary field A^6 agrees with the one defined in equation (A.1). Note that L, Q^4, $Q^{6,3}$, and $Q^{6,4}$ are all descendants of $\mathbb{1}$ and can therefore only appear in commutators of two fields of the same conformal dimension.

Appendix B

The even spin algebra $\mathfrak{hs}^e[\mu]$

B.1 Minimal representations of $\mathfrak{hs}^e[\mu]$ using commutators

In section 3.1.5 we computed the structure constant c_{44}^4 in terms of the conformal dimension h of minimal representations. The calculation was carried out using OPEs. An alternative, but equivalent approach uses commutators rather than OPEs and shall be sketched in this appendix.
We will need the following commutators of $\mathcal{W}_\infty^e$:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}m(m^2-1)\delta_{0,m+n} \ , \\
[L_m, W_n^4] &= (3m-n)W_{m+n}^4 \\
[W_m^4, W_n^4] &= \frac{1}{2}(m-n)\left(c_{44}^6 W_{m+n}^6 + q_{44}^{6,1}Q_{m+n}^{6,1} + q_{44}^{6,2}Q_{m+n}^{6,2} + q_{44}^{6,3}Q_{m+n}^{6,3}\right) \\
&\quad + \frac{1}{36}\left(m^2 - mn + n^2 - 7\right)(m-n)\left(c_{44}^4 W_{m+n}^4 + q_{44}^4 Q_{m+n}^4\right) \\
&\quad + \big(3(m^4+n^4) - (2mn+39)(m^2+n^2) + 4m^2n^2 + 20mn + 108\big) \\
&\qquad \times \frac{1}{3360}(m-n)q_{44}^2 L_{m+n} \\
&\quad + \frac{1}{5040}m(m^2-1)(m^2-4)(m^2-9)n_4\delta_{0,m+n} \ ,
\end{aligned} \tag{B.1}$$

where the composite quasiprimary fields $Q^4, Q^{6,1}, Q^{6,2}$ and $Q^{6,3}$ are given by

$$\begin{aligned}
Q^4 &= LL - \frac{3L''}{10} \ , & Q^{6,1} &= LW^4 - \frac{W^{4\prime\prime}}{6} \ , \\
Q^{6,2} &= L'L' - \frac{4}{5}L''L - \frac{\partial^4 L}{42} \ , & Q^{6,3} &= L(LL) - \frac{1}{3}L'L' - \frac{19}{30}L''L - \frac{\partial^4 L}{36} \ .
\end{aligned} \tag{B.2}$$

Solving the Jacobi identity $[L_m,[W^4_n,W^4_l]]+\text{cycl.}=0$, we find that

$$q^2_{44}=\frac{8}{c}n_4\ ,\quad q^{6,1}_{44}=\frac{28}{3(c+24)}c^4_{44}\ ,\qquad q^{6,2}_{44}=-\frac{2(19c-524)}{3c(2c-1)(7c+68)}n_4\ ,$$
$$q^{6,3}_{44}=\frac{96(72c+13)}{c(2c-1)(5c+22)(7c+68)}n_4\ ,\qquad q^4_{44}=\frac{168}{c(5c+22)}n_4\ . \tag{B.3}$$

This fixes the structure constants of the Virasoro descendants in terms of their primaries. Similarly, by considering Jacobi identities of higher level, we can reobtain in this manner the relations between structure constants given in section 3.1.1.2.

Recall from section 3.1.5 that the defining property of a minimal representation is a character of the form

$$\frac{q^h}{1-q}\prod_{s\in 2\mathbb{N}}^{\infty}\prod_{n=s}^{\infty}\frac{1}{1-q^n}=q^h(1+q+2q^2+3q^3+\dots)\ , \tag{B.4}$$

where h is the conformal dimension of the highest weight state Φ.

Thus, at level 1 all the states must be proportional to $L_{-1}\Phi$, at level 2 they are linear combinations of, say, $L^2_{-1}\Phi$ and $L_{-2}\Phi$, and at level 3 of, for instance, $L_{-3}\Phi$, $L_{-2}L_{-1}\Phi$ and $L^3_{-1}\Phi$. Therefore, we can conclude that the representation must have null relations of the form

$$\mathcal{N}_{1W^4}=(W^4_{-1}-\frac{2w^4}{h}L_{-1})\Phi\ , \tag{B.5}$$
$$\mathcal{N}_{2W^4}=(W^4_{-2}+aL^2_{-1}+bL_{-2})\Phi\ , \tag{B.6}$$
$$\mathcal{N}_{3W^4}=(W^4_{-3}+dL_{-3}+eL_{-2}L_{-1}+fL^3_{-1})\Phi\ , \tag{B.7}$$

where w^4 is the eigenvalue of the zero mode of W^4 on Φ. The coefficient in front of L_{-1} in $\mathcal{N}_{1W^4}$ follows from the condition

$$L_1\mathcal{N}_{1W^4}=0\ . \tag{B.8}$$

Similarly, the coefficients a and b in $\mathcal{N}_{2W^4}$ can be determined from the conditions

$$L^2_1\mathcal{N}_{2W^4}=0\quad\text{and}\quad L_2\mathcal{N}_{2W^4}=0\ , \tag{B.9}$$

and d, e and f from $L_3\mathcal{N}_{3W^4}=0$, $L_2L_1\mathcal{N}_{3W^4}=0$ and $L^3_1\mathcal{N}_{3W^4}=0$. The result is

$$a=-\frac{(5c+16h)w^4}{h(2ch+c+2h(8h-5))}\ ,\qquad b=\frac{4(11-8h)w^4}{2ch+c+2h(8h-5)}\ ,$$
$$d=-\frac{6\big[c(h+3)(2h-3)+2h(h-2)(8h-21)-22\big]w^4}{\big[(c-7)h+c+3h^2+2\big]\big[2ch+c+2h(8h-5)\big]}\ ,$$
$$e=-\frac{12\big[c(6h(h-1)-2)+h(h(8h-15)+9)\big]w^4}{h\big[(c-7)h+c+3h^2+2\big]\big[2ch+c+2h(8h-5)\big]}\ ,$$

$$f = -\frac{(5c+22)(c-h)w^4}{h\big[(c-7)h+c+3h^2+2\big]\big[2ch+c+2h(8h-5)\big]} \ . \tag{B.10}$$

Finally, solving the slightly more involved null relations

$$W_1^4\mathcal{N}_{1W^4} = 0 \ , \qquad W_2^4\mathcal{N}_{2W^4} = 0 \ , \qquad W_3^4\mathcal{N}_{3W^4} = 0 \ , \tag{B.11}$$

and plugging in the structure constants (B.3) leads to the same expressions for w^4, w^6 and c_{44}^4 as those obtained in (3.24) and (3.25) by associativity. Here w^6 is the eigenvalue of the zero mode of W^6 on Φ.

B.2 Structure constants of $\mathfrak{hs}^e[\mu]$

The algebra $\mathfrak{hs}^e[\mu]$ is a subalgebra of $\mathfrak{hs}[\mu]$ and the structure constants of the latter are known explicitly, see [62]. We have rescaled the generators of this reference so that the first few commutation relations take the form

$$[L_m, W_n^s] = ((s-1)m-n)\, W_{m+n}^s \ , \tag{B.12}$$

$$\begin{aligned}[W_m^4, W_n^4] = &-\frac{20}{\sqrt{7}}P_6^{44}(m,n)W_{m+n}^6 + \frac{12}{\sqrt{5}}\left(\mu^2-19\right)P_4^{44}(m,n)W_{m+n}^4 \\ &+ 8\left(\mu^4-13\mu^2+36\right)P_2^{44}(m,n)L_{m+n} \ ,\end{aligned} \tag{B.13}$$

$$\begin{aligned}[W_m^4, W_n^6] = &-8\sqrt{\frac{210}{143}}P_8^{46}(m,n)W_{m+n}^8 + \frac{14}{\sqrt{5}}\left(\mu^2-49\right)P_6^{46}(m,n)W_{m+n}^6 \\ &- \frac{20}{\sqrt{7}}\left(\mu^4-41\mu^2+400\right)P_4^{46}(m,n)W_{m+n}^4 \ ,\end{aligned} \tag{B.14}$$

$$\begin{aligned}[W_m^6, W_n^6] = &-252\sqrt{\frac{5}{2431}}P_{10}^{66}(m,n)W_{m+n}^{10} \\ &+ 28\sqrt{\frac{6}{143}}\left(\mu^2-115\right)P_8^{66}(m,n)W_{m+n}^8 \\ &- \frac{40}{3\sqrt{7}}\left(\mu^2-88\right)\left(\mu^2-37\right)P_6^{66}(m,n)W_{m+n}^6 \\ &+ \frac{14}{\sqrt{5}}(\mu^2-49)(\mu^2-25)(\mu^2-16)P_4^{66}(m,n)W_{m+n}^4 \\ &+ 12(\mu^2-25)(\mu^2-16)(\mu^2-9)(\mu^2-4)P_2^{66}(m,n)L_{m+n} \ ,\end{aligned} \tag{B.15}$$

$$\begin{aligned}[W_m^4, W_n^8] = &-20\sqrt{\frac{6}{17}}P_{10}^{48}(m,n)W_{m+n}^{10} - \frac{72}{13\sqrt{5}}\left(277-3\mu^2\right)P_8^{48}(m,n)W_{m+n}^8 \\ &- 40\sqrt{\frac{210}{143}}(\mu^2-49)(\mu^2-36)P_6^{48}(m,n)W_{m+n}^6 \ ,\end{aligned} \tag{B.16}$$

where $P_{s''}^{ss'}(m,n)$ are the universal polynomials containing the mode dependence of the structure constants in a commutator of quasiprimary fields of a CFT. They are given by

$$P^{ss'}_{s''}(m,n) := \sum_{r=0}^{s+s'-s''-1} \binom{s+m-1}{s+s'-s''-r-1} \times \frac{(-1)^r (s-s'+s'')_{(r)} (s''+m+n)_{(r)}}{r!(2s'')_{(r)}} \ , \quad \text{(B.17)}$$

where we have introduced the Pochhammer symbols $x_{(r)} = \Gamma(x+r)/\Gamma(x)$. When m, n are restricted to the wedge, these universal polynomials are essentially the Clebsch-Gordan coefficients of $\mathfrak{sl}(2)$ [29]. The proportionality factors between the generators T^j_m of [62] and our generators W^s_m are explicitly

$$T^j_m = \sqrt{\frac{(j-m)!(j+m)!}{(2j)!}} W^{j+1}_m \ . \quad \text{(B.18)}$$

Appendix C

Aspects of the continuous orbifold twisted sector

C.1 Twisted sector ground state energies

In this appendix we collect together some formulae for the ground state energies of twisted fermions and bosons.

C.1.1 Complex free fermions

We begin with the case of free fermions twisted by α with $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. Let us consider a pair of complex fermions that pick up eigenvalues $e^{\pm 2\pi i \alpha}$ under the twist. The relevant twining character, i.e. the character with the insertion of the eigenvalues $e^{\pm 2\pi i\alpha}$, equals then in the NS-sector

$$\chi_\alpha(\tau) = \frac{\vartheta_3(\tau,\alpha)}{\eta(\tau)} \ , \tag{C.1}$$

where we use the definitions

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \ , \tag{C.2}$$

$$\vartheta_3(\tau,z) = \prod_{n=1}^{\infty} (1-q^n)\,(1+yq^{n-1/2})\,(1+y^{-1}q^{n-1/2}) \ , \tag{C.3}$$

as well as $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$. To obtain the ground state energy of the twisted sector we perform an S-modular transformation, using the transformation rules

$$\eta(-\tfrac{1}{\tau}) = (-i\tau)^{1/2}\,\eta(\tau) \tag{C.4}$$

$$\vartheta_3(-\tfrac{1}{\tau}, \tfrac{z}{\tau}) = (-i\tau)^{1/2}\, e^{i\pi z^2/\tau}\, \vartheta_3(\tau, z) \ , \tag{C.5}$$

to obtain for the α-twisted partition function

$$\chi_\alpha(-\tfrac{1}{\tau}) = e^{i\pi\alpha^2\tau}\, \frac{\vartheta_3(\tau, \tau\alpha)}{\eta(\tau)} \tag{C.6}$$

$$= q^{-\frac{1}{24}}\, e^{i\pi\alpha^2\tau} \prod_{n=1}^{\infty} (1 + e^{2\pi i\tau\alpha} q^{n-1/2})\, (1 + e^{-2\pi i\tau\alpha} q^{n-1/2}) \ . \tag{C.7}$$

Thus the ground state energy of the α-twisted sector equals

$$\Delta h_{\text{fer}} = \frac{1}{2}\alpha^2 \ . \tag{C.8}$$

C.1.2 Complex free bosons and susy case

The analysis for a pair of complex bosons is essentially identical. Now the relevant twining character equals

$$\chi_\alpha(\tau) = -2\sin(\pi\alpha)\, \frac{\eta(\tau)}{\vartheta_1(\tau, \alpha)} \ , \tag{C.9}$$

where $\vartheta_1(\tau, z)$ is defined by

$$\vartheta_1(\tau, z) = -2q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^n)\, (1 - yq^n)\, (1 - y^{-1} q^n) \ . \tag{C.10}$$

The modular transformation behaviour of $\vartheta_1(\tau, z)$ is

$$\vartheta_1(-\tfrac{1}{\tau}, \tfrac{z}{\tau}) = -i(i\tau)^{1/2} e^{i\pi z^2/\tau}\, \vartheta_1(\tau, z) \ , \tag{C.11}$$

and hence the twisted character equals

$$\chi_\alpha(-\tfrac{1}{\tau}) = i\frac{\sin(\pi\alpha)}{\sin(\pi\tau\alpha)}\, e^{-i\pi\alpha^2\tau}\, q^{-\frac{2}{24}} \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i\alpha\tau} q^n)\, (1 - e^{-2\pi i\alpha\tau} q^n)} \ . \tag{C.12}$$

For $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$ we read off from the leading $q \to 0$ behaviour that

$$\Delta h_{\text{bos}} = \frac{1}{2}|\alpha| - \frac{1}{2}\alpha^2 \ . \tag{C.13}$$

Note that for a supersymmetric theory, i.e., for a theory where both bosons and fermions are twisted by the same amount, the total ground state energy is then

$$\Delta h_{\text{tot}} = \Delta h_{\text{bos}} + \Delta h_{\text{fer}} = \frac{|\alpha|}{2} \ , \tag{C.14}$$

which is indeed linear in $|\alpha|$.

C.2 Branching rules

In this appendix we explain the branching rules of $\mathfrak{su}(N+1) \supset \mathfrak{su}(N)$. They were first derived by Weyl [167] in terms of $\mathfrak{u}(N)$ tensors (see, e.g., [168] and [119] for more modern and general treatments).

Let $\Lambda = [\Lambda_1, \ldots, \Lambda_N]$ be a highest weight of $\mathfrak{su}(N+1)$. The procedure can be divided into three steps:

1. Interpret Λ as a highest weight of $\mathfrak{u}(N+1)$ rather than $\mathfrak{su}(N+1)$.
2. Let r_i denote the number of boxes in the ith row of the Young diagram associated with Λ,
$$r_i = \sum_{j=i}^{N} \Lambda_j \ . \tag{C.15}$$
Then under the branching $\mathfrak{u}(N+1) \supset \mathfrak{u}(N)$ we have the decomposition
$$\Lambda \to \bigoplus_{\tilde{\Lambda}} \tilde{\Lambda} \ , \tag{C.16}$$
where $\tilde{\Lambda} = [\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_N]$ are highest weights of $\mathfrak{u}(N)$ whose rows $\tilde{r}_i$ satisfy
$$r_1 \geq \tilde{r}_1 \geq r_2 \geq \tilde{r}_2 \geq \cdots \geq \tilde{r}_N \geq 0 \ , \tag{C.17}$$
each $\tilde{\Lambda}$ appearing once.
3. In the end, each $\tilde{\Lambda}$ has to be restricted to $\mathfrak{su}(N)$ by removing the last Dynkin label.

Equation (C.17) means that from each row $i = 1, \ldots, N$, any number $a_i = 0, \ldots, \Lambda_i$ of boxes may be removed, such that the new number of boxes in the ith row becomes
$$\tilde{r}_i = r_i - a_i \ . \tag{C.18}$$
So the weights $\tilde{\Lambda}$ are labelled by the vectors $\mathbf{a} = (a_1, \ldots, a_N)$ and we write $\Lambda(\mathbf{a})$ for the restriction to $\mathfrak{su}(N)$ of the $\tilde{\Lambda}$ labelled by $\mathbf{a}$. The Dynkin labels of $\Lambda(\mathbf{a})$ are given by
$$\Lambda(\mathbf{a})_i = \tilde{r}_i - \tilde{r}_{i+1} = \Lambda_i - a_i + a_{i+1} \tag{C.19}$$
for $i = 1, \ldots, N-1$, and thus the branching rules may be written as
$$\Lambda \to \bigoplus_{\mathbf{a}} \Lambda(\mathbf{a}) \ . \tag{C.20}$$

C.3 The ground state analysis

In this appendix we shall show that the coset states (5.26) actually define twisted sector ground states. In particular, we need to show that $\delta h^{(l)}$ in (5.45) is non-negative for all $l = 0, \ldots, N-1$. Because the individual twists satisfy $|\alpha_i| \leq \frac{1}{2}$, only the representations with $n = 0$ have a chance of lowering the conformal dimension of the original state. The $\mathfrak{so}(2N)_1$ selection rule implies that $n = \frac{1}{2}$ for the actual fermionic excitations, but $n = 0$ can arise for the bosonic excitations (that come from the same multiplets). Thus we need to analyse (i) whether $n = 0$ is allowed in the fusion with $(0; \mathrm{f}, -(N+1))$ or $(0; \bar{\mathrm{f}}, (N+1))$; and (ii) if so, whether the relevant term in (5.44) is then positive.

The condition that $n = 0$ is possible simply means that $\Lambda_-(\alpha)^{\epsilon(l)}$ is contained in $\Lambda_+(\alpha)$ under the branching rules of $\mathfrak{su}(N+1) \supset \mathfrak{su}(N)$. In the notation of appendix C.2 the original coset state (5.26) corresponds to the choice $\Lambda_+(\alpha) \equiv \Lambda$, and $\Lambda_-(\alpha) \equiv \Lambda(\mathbf{a})$ with

$$\mathbf{a} = \mathbf{a}^{(m)} = (0, \ldots, 0, \Lambda_{m+1}, \ldots, \Lambda_N) \ . \tag{C.21}$$

Furthermore, generically the fusion with $(0; \mathrm{f}, -(N+1))$ or $(0; \bar{\mathrm{f}}, (N+1))$ leads to

$$\Lambda(\mathbf{a})^{\epsilon(l)} = \Lambda(\mathbf{a}') \, , \quad \text{where} \quad a'_j = \begin{cases} a_j & j \neq l+1 \\ a_{l+1} + \epsilon & j = l+1 \ . \end{cases} \tag{C.22}$$

However, this representation only appears in the above branching rules of the same $\Lambda_+(\alpha) \equiv \Lambda$ if all a'_j satisfy $0 \leq a'_j \leq \Lambda_j$. Thus we see that $n = 0$ is only allowed if

$$\begin{aligned} &\text{for } \epsilon = + \qquad a_{l+1} < \Lambda_{l+1} \quad \text{i.e.,} \ \ l \leq m \\ &\text{for } \epsilon = - \qquad 0 < a_{l+1} \quad \text{i.e.,} \ \ l \geq m+1 \ . \end{aligned} \tag{C.23}$$

(We are assuming here, for simplicity, that all $\Lambda_j \neq 0$.) But for these values of ϵ and l, it then follows from (5.27) that $-\epsilon\, \alpha_{l+1} \geq 0$. This therefore shows that $\delta h^{(l)}$ in (5.45) is indeed non-negative.

C.3.1 Other potential twisted sector ground states

It is also not hard to show that among the ‘light states’, i.e., those that have $n = 0$, the only twisted sector ground states are in fact those described in (5.26). The most general light states are of the form

$$\Big(\Lambda; \Lambda(\mathbf{a}), -|\Lambda| + (N+1) \sum_{j=1}^{N} a_j \Big) \, , \qquad |\Lambda| = \sum_{j=1}^{N} j\, \Lambda_j \ , \tag{C.24}$$

where $\mathbf{a} = (a_1, \ldots, a_N)$, and the a_i take the values $a_i = 0, \ldots, \Lambda_i$, $i = 1, \ldots, N$. We want to show that among these states, the only ones that are twisted sector ground states, i.e., annihilated by all positive fermionic and bosonic modes, are those for which $\mathbf{a}$ is of the form (C.21). In order to analyse this issue, we determine the analogue of (5.44), which now takes the form

$$\delta h^{(l)} \cong n + \frac{\epsilon}{N+k+1}\left((\Lambda_{l+1} - a_{l+1}) + \sum_{i=l+2}^{N} \Lambda_i - A \right), \qquad A = \sum_{i=1}^{N} a_i \ . \quad \text{(C.25)}$$

Using (C.22), we have again that $n = 0$ is only allowed for $\epsilon = +$ if $a_{l+1} < \Lambda_{l+1}$, and for $\epsilon = -$ if $a_{l+1} > 0$ — otherwise the representation $\Lambda(\mathbf{a}')$ does not appear in the branching rules of $\mathfrak{su}(N+1) \supset \mathfrak{su}(N)$. It follows that if $0 < a_j < \Lambda_j$, both values $\epsilon = \pm$ allow for $n = 0$ and thus one of the two $\delta h^{(l)}$ will be negative. So for a ground state, each a_j is either $a_j = 0$ or $a_j = \Lambda_j$.

As a last step, we show that in fact $\mathbf{a} = \mathbf{a}^{(m)}$ for some $m = 0, \ldots, N$. Requiring (C.25) to be non-negative for all l, we obtain the inequalities

$$\text{if } a_{l+1} = 0 \ : \qquad A \le \sum_{j=l+1}^{N} \Lambda_j \quad \text{(C.26)}$$

(recall that for $a_{l+1} = 0$, $n = 0$ occurs for $\epsilon = +$) and

$$\text{if } a_{l+1} = \Lambda_{l+1} \ : \qquad \sum_{j=l+2}^{N} \Lambda_j \le A \quad \text{(C.27)}$$

(since for $a_{l+1} = \Lambda_{l+1}$, $n = 0$ occurs for $\epsilon = -$).

The sequence of partial sums $P_r = \sum_{j=r}^{N} \Lambda_j$ is strictly decreasing, whereas A takes the same value in all of these inequalities. This implies that the a_j have to be chosen in such a way that $\mathbf{a} = (0, \ldots, 0, \Lambda_{m+1}, \ldots, \Lambda_N) = \mathbf{a}^{(m)}$. This completes the proof.

Appendix D

The $\mathcal{N} = 2$ coset analysis

In this appendix we explain in some detail the construction of the spin 1 and 2 currents of the coset

$$\frac{\mathfrak{su}(N+1)^{(1)}_{k+N+1}}{\mathfrak{su}(N)^{(1)}_{k+N+1} \oplus \mathfrak{u}(1)^{(1)}_{\kappa}} \tag{D.1}$$

with $\kappa = N(N+1)(N+k+1)$. We will closely follow the analysis of [85] in the $\mathcal{N} = 4$ case and [103]. The numerator consists of $N(N+2)$ bosonic currents $\mathcal{J}^A$ and free fermions ψ^A transforming in the adjoint representation of $\mathfrak{su}(N+1)$. Given a hermitian orthonormal basis t^A_{ij} of $\mathfrak{su}(N+1)$ satisfying

$$[t^A, t^B] = i f^{ABC} t^C \qquad \text{and} \qquad \mathrm{Tr}(t^A\, t^B) = \delta^{AB} \ , \tag{D.2}$$

which we order in such a way that t^a for $a = 1, \ldots, N^2 - 1$ form a hermitian orthonormal basis of $\mathfrak{su}(N)$, the numerator fields satisfy the commutation relations

$$\begin{aligned}
[\mathcal{J}^A_m, \mathcal{J}^B_n] &= i f^{ABC} \mathcal{J}^C_{m+n} + (k+N+1)\delta^{AB}\delta_{m,-n} \ , \\
[\mathcal{J}^A_m, \psi^B_r] &= i f^{ABC} \psi^C_{m+r} \ , \\
\{\psi^A_r, \psi^B_s\} &= \delta^{AB}\delta_{r,-s} \ .
\end{aligned} \tag{D.3}$$

Restricting the adjoint representation to the denominator subalgebra, it decomposes as

$$\mathfrak{su}(N+1) \to \mathfrak{su}(N) \oplus \mathfrak{u}(1) \oplus \mathbf{N} \oplus \bar{\mathbf{N}} \ . \tag{D.4}$$

We can decouple the currents from the fermions by defining

$$J^A = \mathcal{J}^A + \frac{i}{2} f^{ABC}(\psi^B \psi^C) \tag{D.5}$$

in the numerator or

$$\widetilde{J}^a = \mathcal{J}^a + \frac{i}{2} f^{abc}(\psi^b \psi^c) \tag{D.6}$$

in the denominator, where again lower-case indices from the beginning of the alphabet range from 1 to N^2-1 only. These currents and the fermion bilinears give rise to the bosonic coset

$$\frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)_\kappa} \ . \tag{D.7}$$

From the $N(N+2)$ fermions in the numerator we subtract the N^2 fermions in the denominator. The $2N$ surviving fermions can be defined by

$$\psi^i = t^A_{N+1,i}\psi^A \ , \qquad \bar{\psi}^i = t^A_{i,N+1}\psi^A \ , \tag{D.8}$$

satisfying

$$\begin{aligned} \{\psi^i_r, \bar{\psi}^j_s\} &= \delta^{ij}\delta_{r,-s} \ , \\ \{\psi^i_r, \psi^j_s\} = \{\bar{\psi}^i_r, \bar{\psi}^j_s\} &= 0 \ . \end{aligned} \tag{D.9}$$

The bosonic currents in the numerator can be split up in J^a for $a=1,\ldots,N^2-1$, J^i and $\bar{J}^i$, for $i=1,\ldots,N$, and K, where we define

$$J^i = t^A_{N+1,i}J^A \ , \qquad \bar{J}^i = t^A_{i,N+1}J^A \ , \qquad K = (N+1)\, t^A_{N+1,N+1}J^A \ . \tag{D.10}$$

Here, $\bar{J}^i$ and J^i correspond to the $\mathbf{N}$ and $\bar{\mathbf{N}}$ of $\mathfrak{su}(N)$, respectively, while K is the $\mathfrak{u}(1)$ current embedded into $\mathfrak{su}(N+1)$. The $\mathfrak{u}(1)$ embedding into $\mathfrak{so}(2N)$ can be written as

$$j = -(N+1)(\psi^i\bar{\psi}^i) \ . \tag{D.11}$$

The total $\mathfrak{u}(1)$ current is then equal to $K+j$. It will be useful to express the decoupled $\mathfrak{su}(N)_{k+1}$ currents in terms of the decoupled $\mathfrak{su}(N+1)_k$ currents:

$$\widetilde{J}^a = J^a + t^a_{ij}(\psi^i\bar{\psi}^j) \ , \tag{D.12}$$

where we have assumed, without loss of generality, that the matrices t^A for $A = N^2,\ldots,N(N+2)-1$ are of the form

$$t^A = \left(\begin{array}{c|c} 0_N & * \\ \hline * & 0 \end{array}\right) \ , \qquad A = N^2,\ldots,N(N+2)-1 \ , \tag{D.13}$$

and that $t^{N(N+2)}$ is diagonal. We also define the unique spin-1 primary of the coset, which is also the lowest field in the superconformal algebra, as

$$J = \frac{1}{N+k+1}\left(K - \frac{k}{N+1}\,j\right) \ . \tag{D.14}$$

Then the stress-energy tensor of the coset theory is given by the difference of the numerator and denominator Sugawara tensors:

$$\begin{aligned} L &= L_{\mathfrak{su}(N+1)} - L_{\mathfrak{su}(N)} - L_{\mathfrak{u}(1)} + L_{\text{free fermions}} \\ &= \frac{1}{2(N+k+1)} \Bigg((J^i \bar{J}^i) + (\bar{J}^i J^i) + k \left((\partial\psi^i \bar{\psi}^i) - (\psi^i \partial\bar{\psi}^i) \right) \\ &\qquad - 2\, t^a_{ij} \left(J^a (\psi^i \bar{\psi}^j) \right) - \frac{2}{N(N+1)} (Kj) \Bigg) \,, \end{aligned} \tag{D.15}$$

where we have used that

$$(J^A J^A) = (J^a J^a) + (J^i \bar{J}^i) + (\bar{J}^i J^i) + \frac{1}{N(N+1)} (KK) \,. \tag{D.16}$$

We can split up the stress-energy tensor into three mutually commuting stress-energy tensors given by

$$\begin{aligned} L_{\mathrm{b}} &= \frac{1}{2(N+k+1)} \left((J^i \bar{J}^i) + (\bar{J}^i J^i) - \frac{1}{N+k} (J^a J^a) - \frac{1}{Nk} (KK) \right) , \\ L_{\mathrm{f}} &= \frac{k}{2(N+k+1)} \Bigg((\partial\psi^i \bar{\psi}^i) - (\psi^i \partial\bar{\psi}^i) - \frac{2}{k} t^a_{ij} \left(J^a (\psi^i \bar{\psi}^j) \right) \\ &\qquad + \frac{1}{k(N+k)} (J^a J^a) - \frac{1}{N(N+1)^2} (jj) \Bigg) , \\ L_{(\mathrm{JJ})} &= \frac{N+k+1}{2Nk} (JJ) \,, \end{aligned} \tag{D.17}$$

with central charges

$$\begin{aligned} c_{\mathrm{b}} &= \frac{N(k-1)(N+2k+1)}{(N+k)(N+k+1)} \,, \\ c_{\mathrm{f}} &= \frac{k(N-1)(k+2N+1)}{(N+k)(N+k+1)} \,, \\ c_{(\mathrm{JJ})} &= 1 \,, \end{aligned} \tag{D.18}$$

such that the total stress-energy tensor reads

$$L = L_{\mathrm{b}} + L_{\mathrm{f}} + L_{(\mathrm{JJ})} \tag{D.19}$$

with total central charge

$$c = c_{\mathrm{b}} + c_{\mathrm{f}} + c_{(\mathrm{JJ})} = \frac{3Nk}{N+k+1} \,. \tag{D.20}$$

There is another elementary primary field of conformal dimension 2, which was called W^{20} in [35]. We can make an ansatz

$$W^{20} = \alpha\, L_{\mathrm{b}} + \beta\, L_{\mathrm{f}} + \gamma\, L_{(\mathrm{JJ})} \ . \tag{D.21}$$

From the analysis in [35], we know that W^{20} satisfies the OPE

$$\begin{aligned} W^{20}(z)W^{20}(w) \sim{}& \frac{n_2}{(z-w)^4} + c_{22}^2 \frac{W^{20}(w)}{(z-w)^2} + \frac{4\,n_2}{c-1}\frac{\big(L(w) - \frac{3}{2c}\,(JJ)(w)\big)}{(z-w)^2} \\ &+ \frac{c_{22}^2}{2}\frac{W^{20\,\prime}(w)}{z-w} + \frac{2\,n_2}{c-1}\frac{\big(L'(w) - \frac{3}{2c}\,(JJ)'(w)\big)}{z-w} \ . \end{aligned} \tag{D.22}$$

Demanding this as well as a vanishing central term in the OPE $L \star W^{20}$, we obtain

$$\begin{aligned} \alpha &= -\sqrt{\frac{2k(N-1)(2N+k+1)(N+k+1)\,n_2}{N(k-1)(N+2k+1)(3Nk-(N+k+1))}} \ , \\ \beta &= -\frac{N(k-1)(N+2k+1)}{k(N-1)(k+2N+1)}\,\alpha \\ &= \sqrt{\frac{2N(k-1)(N+2k+1)(N+k+1)\,n_2}{k(N-1)(2N+k+1)(3Nk-(N+k+1))}} \ , \\ \gamma &= 0 \ . \end{aligned} \tag{D.23}$$

This then also reproduces correctly the form of $(c_{22}^2)^2$ as predicted by equation (3.27) of [35]. For the normalisation of W^{20} we choose the convention

$$n_2 = -\frac{c}{6}(\nu+3)(\nu-3) \ , \tag{D.24}$$

where

$$\begin{aligned} \nu &= 2\mu - 1 = \frac{N-k-1}{N+k+1} \ , \\ c &= \frac{3Nk}{N+k+1} \ . \end{aligned} \tag{D.25}$$

In the $c \to \infty$ limit, the parameters then become

$$\alpha \to -\frac{\nu+3}{2\sqrt{3}} \ , \qquad \beta \to -\frac{\nu-3}{2\sqrt{3}} \ . \tag{D.26}$$

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