Conference Poster

PCE-based imprecise Sobol’ indices

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### Problem Statement & Context

A computational model is defined as a mapping:

\[ x \in \mathbb{D} \subseteq \mathbb{R}^N \rightarrow y = M(x) \in \mathbb{R} \]

- \( x \) is modelled by an imprecise random vector \( X \), which accounts for both aleatory uncertainty (natural variability) and epistemic uncertainty (lack of knowledge).
- The elements of \( X \) are assumed statistically independent.
- The computational model is considered as a black-box.

**Goal:** Sensitivity analysis – estimate the influence of each component \( X_i \in X \) on the random response \( Y = M(X) \).

### PCE-based Sobol' Indices

Considering a **probabilistic** input vector \( X \), then a Polynomial Chaos Expansion (PCE) meta-model surrogates \( M \):

\[ Y = M(X) \approx \sum_{\alpha \in \mathcal{A}} a_{\alpha} \psi_{\alpha}(X) \]

- \( \psi_{\alpha}(X) \): multivariate orthonormal polynomials with respect to \( X \).
- \( a_{\alpha} \): coefficients of the polynomials.
- \( \mathcal{A} \): set of \( \alpha \) indices determined by an appropriate truncation scheme.

**Sparse PCE:** obtained with least-angle regression (LARS).

Then, PCE-based Sobol’ indices read:

\[ S_{\alpha}(P) = \frac{\sum_{\alpha_{\mathcal{A}}} a_{\alpha}^2 / \sum_{\alpha_{\mathcal{A}}} a_{\alpha}^2}{\sum_{\alpha \in \mathcal{A}} a_{\alpha}} \]

- \( \mathcal{I}_{\alpha_{\mathcal{A}}} = \{ \alpha \in \mathcal{A} : \alpha_k > 0 \text{ for all } k \in \{1, \ldots, i\}, \alpha_k = 0 \text{ otherwise} \} \)
- Postprocessing of PCE coefficients \( \rightarrow \) cheap.
- Variance decomposition of probabilistic variability.

⇒ Extension to imprecise probabilities?

### Augmented PCE

**Definition:** Augmented input vector \( V = (C, \Theta) \) with \( C_i = F_{X_i}(X_i, \Theta_i) \) and hence \( C_i \sim \mathcal{U}(0,1) \). Then:

\[ W = M(F_X(C, \Theta)) \quad \text{def} \quad M^{aug}(V) \]

Consider \( \Theta \), as uniform distributions, PCE meta-model for \( W \) as a function of \( V \):

\[ W \approx M^{PCE}(V) = \sum_{\alpha \in \mathcal{A}} a_{\alpha} \psi_{\alpha}(V) \]

where \( \psi_{\alpha}(V) \) are multivariate orthonormal polynomials with respect to \( V \).

### PCE-based Imprecise Sobol’ Indices

**Reordering** to a PCE in terms of \( C \) (aleatory uncertainty):

\[ W(\Theta) = \sum_{\alpha \in \mathcal{A}} a_{\alpha} \psi_{\alpha}(C) \]

where \( a_{\alpha} \) is a combination of \( a_{\alpha} \) and \( \psi_{\alpha}(\Theta) \). Then, bounds of the Sobol’ indices:

\[ S_{\alpha}(P) = \min_{\Theta_{\mathcal{A}}} \frac{\sum_{\alpha_{\mathcal{A}}} a_{\alpha}^2 / \sum_{\alpha_{\mathcal{A}}} a_{\alpha}^2}{\sum_{\alpha \in \mathcal{A}} a_{\alpha}} \]

\[ S_{\alpha}(P) = \max_{\Theta_{\mathcal{A}}} \frac{\sum_{\alpha_{\mathcal{A}}} a_{\alpha}^2 / \sum_{\alpha_{\mathcal{A}}} a_{\alpha}^2}{\sum_{\alpha \in \mathcal{A}} a_{\alpha}} \]

⇒ Postprocessing of augmented PCE ⇒ cheap optimizations.


### Parametric P-box

**Definition:** CDF \( F_{X_i} \) (aleatory uncertainty) with interval-valued distribution parameters \( \Theta \) (epistemic uncertainty).

e.g. an imprecise Gaussian variable

\[ X \sim \mathcal{N}(\mu_x, \sigma_x), [\mu_x, \sigma_x] \]

- \( \Theta = \{\mu_x, \sigma_x\} \)


### Example: Simply Supported Truss

**Problem:** assess deflection \( u(x) \) of truss (Hurtado, 2013):

- Loads \( P_i \), \( i = 1, \ldots, 7 \) independent.
- \( \mu P_i \in [55, 100] \text{ kN}, \sigma P_i \in [13, 17] \text{ kN} \).

**Augmented PCE:** \( N = 100 \) LHS samples.

**Results:**

- Computation of first order indices
- High accuracy in estimates of Sobol’ indices

**Conclusions:**

- The augmented input space allows for a distinction between aleatory and epistemic uncertainty in \( X \).
- **Imprecise Sobol’ indices** allow for a distinction between aleatory and epistemic uncertainty in sensitivity analysis.
- **Augmented PCE** makes sensitivity analysis tractable for expensive-to-evaluate models with random input described by parametric p-boxes.

### Imprecise Sobol’ Indices

**Idea:** Separation of sources of uncertainty within Sobol’ indices:

- Aleatory uncertainty ⇒ value of conventional Sobol’ indices
- Epistemic uncertainty ⇒ interval-valued indices