Horizontal α-Harmonic Maps

Author(s):
Da Lio, Francesca; Rivière, Tristan

Publication Date:
2017

Permanent Link:
https://doi.org/10.3929/ethz-a-010817740

Horizontal α-Harmonic Maps

Francesca Da Lio* Tristan Riviere*

January 16, 2017

Abstract

Given a C^1 planes distribution P_T on all $\mathbb{R}^m$ we consider *horizontal α-harmonic maps*, $\alpha \geq 1/2$, with respect to such a distribution. These are maps $u \in H^\alpha(\mathbb{R}^k, \mathbb{R}^m)$ satisfying $P_T \nabla u = \nabla u$ and $P_T(u)(-\Delta)^\alpha u = 0$ in $\mathcal{D}'(\mathbb{R}^k)$. If the distribution of planes is integrable then we recover the classical case of α-harmonic maps with values into a manifold. In this paper we shall focus our attention to the case $\alpha = 1/2$ in dimension 1 and $\alpha = 2$ in dimension 2 and we investigate the regularity of the *horizontal α-harmonic maps*. In both cases we show that such maps satisfy a Schrödinger type system with an antisymmetric potential, that permits us to apply the previous results obtained by the authors in respectively in [14] and [6]. Finally we study the regularity of *variational α-harmonic* maps which are critical points of $\|(-\Delta)^{\alpha/2}u\|^2_{L^2}$ under the constraint to be tangent (horizontal) to a given planes distribution. We produce a convexification of this variational problem which permits to write it's Euler Lagrange equations.

Key words. Horizontal harmonic map, horizontal fractional harmonic map, sub-riemannian geometry, Schrödinger-type PDEs, conservation laws, commutators.

MSC 2010. 58E20, 35R11, 53C17, 35B65, 35S99, 49Q05.

Contents

1 Introduction **2**

2 Regularity of Horizontal Harmonic Maps in 2-D **9**

*Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland.

3 3-Commutators, Antisymmetry and Conservation Laws in 1-D **10**

3.1 A Regularity Result for Solutions to Linear Pseudo-Differential Equations involving Projections . . . 10

3.2 Multiplying 3-Commutators . . . 11

3.3 Conservation Laws for Fractional Schrödinger type PDEs with Antisymmetric Potentials. . . . 15

4 Regularity of Horizontal $1/2-$Harmonic Maps in 1-D **20**

4.1 Proof of Theorem 1.1. . . . 20

4.2 Proof of Proposition 1.1. . . . 20

5 Variational Harmonic Maps and $1/2-$Harmonic Maps into Plane Distributions. **21**

5.1 Variational Harmonic Maps into Plane Distributions . . . 21

5.1.1 The 1-D Case . . . 21

5.2 2-Dimensional Variational Harmonic Maps into Plane Distributions. . . . 22

5.3 Variational $1/2-$Harmonic Maps into Plane Distributions . . . 23

5.4 Reformulation of the Euler-Lagrange Equation . . . 30

A Appendix **31**

A.1 Integrable Distributions . . . 31

A.2 Rewriting the Commutators . . . 32

1 Introduction

The functions defined on a domain $U \subset \mathbb{R}^k$ and which are critical points to the Dirichlet Energy

$$E(u) = \frac{1}{2} \int_U |\nabla u|^2 \, dx^k$$

where dx^k denotes the Lebesgue measure in $\mathbb{R}^k$, satisfy the linear Laplace equation

$$-\Delta u = 0 \quad \text{in } \mathcal{D}'(U)$$

whose solutions are known to be real analytic in any dimension. The result extends of course to maps taking values into flat euclidean spaces $\mathbb{R}^m$. There has been a lot of geometric motivations for studying critical points of the *Dirichlet Energy* among maps forced to take values into a given oriented closed (compact without boundary) sub-manifold $\mathcal{N}^n$, that is within the Sobolev Space defined by

$$W^{1,2}(U, \mathcal{N}^n) := \left\{ u \in W^{1,2}(U, \mathbb{R}^m) \ ; \ u(x) \in \mathcal{N}^n \ \text{ for a. e. } x \in U \right\} .$$

The C^2 regularity on $\mathcal{N}^n$ is usually assumed in order to ensure at least the Frechet differentiability of E within $W^{1,2}(U,\mathcal{N}^n)$. If ν defines the unit normal multi-vector to the sub-manifold (under the regularity assumption on $\mathcal{N}^n$, we have that ν is C^1), critical points of E satisfy the following Euler Lagrange equation

$$\nu(u) \wedge \Delta u = 0 \qquad \text{in } \mathcal{D}'(U). \tag{1}$$

This equation makes sense at the distributional level since the composition of the C^1 tensor field ν with u is in $W^{1,2}$ and $\Delta u \in H^{-1}(U)$. If u is smooth equation (1) means that $\Delta u(x)$ is perpendicular to $T_{u(x)}\mathcal{N}^n$ for every $x \in U$ and generalizes the equation of geodesics in $\mathcal{N}^n$ for $k = 1$ to arbitrary k. Equation (1) is called *harmonic map equation* and is often presented in the following equivalent form (see [15])

$$-\Delta u = \sum_{j=1}^{k} A(u)(\partial_{x_j} u, \partial_{x_j} u) \tag{2}$$

where $A(z)(X,Y)$ is second fundamental form of $\mathcal{N}^n \hookrightarrow \mathbb{R}^m$ at the point $z \in \mathcal{N}^n$ along the pair of vectors $X, Y \in T_z\mathcal{N}^n$.

In [5], the authors initiated the analysis of $1/2-$*harmonic maps* into $\mathcal{N}^n$, in connection with the problem of free boundary minimal discs. These maps are critical points of the fractional energy on $\mathbb{R}^k$

$$E^{1/2}(u) := \int_{\mathbb{R}^k} |(-\Delta)^{1/4} u|^2 \, dx^k \tag{3}$$

within

$$H^{1/2}(\mathbb{R}^k, \mathcal{N}^n) := \left\{ u \in H^{1/2}(\mathbb{R}^k, \mathbb{R}^m) \ ; \ u(x) \in \mathcal{N}^n \ \text{ for a. e. } x \in \mathbb{R}^k \right\}.$$

The corresponding Euler-Lagrange equation is given by

$$\nu(u) \wedge (-\Delta)^{1/2} u \qquad \text{in } \mathcal{D}'(\mathbb{R}^k). \tag{4}$$

In [5, 6] various regularity results were established for weak solutions to (4) in the critical dimension 1. The proof of these results where using the existence of special structures in some reformulation of (4) called *3 commutators* which are bilinear pseudo-differential operators satisfying some *integrability by compensation* properties. These results could also be obtained by transforming the a-priori non-local PDE (4) into a local one by performing *ad-hoc* extensions and reflections (see [16] or [12]). More general *non-local non-linear elliptic problems* of the form

$$\nu(u) \wedge (-\Delta)^{\alpha} u \qquad \text{in } \mathcal{D}'(\mathbb{R}^k). \tag{5}$$

and further generalization have been studied in [17, 2, 7, ?]. Observe that, by introducing the field of orthogonal projection $m \times m$ matrices $P_T(z)$ onto the tangent spaces $T_z\mathcal{N}^n$ all the above equations can be rewritten in the form

$$P_T(u)\,(-\Delta)^\alpha u = 0 \tag{6}$$

where

$$\forall\, z \in \mathcal{N}^n \quad \forall\, Z \in \mathbb{R}^m \qquad P_T(z)\,Z := \sum_{i=1}^m P_T^{ij}(z)\,Z_j \quad \text{where } Z = \sum_{j=1}^m Z_j\,\varepsilon_j$$

and $(\varepsilon_j)_{j=1\cdots m}$ the canonical basis of $\mathbb{R}^m$.

The main purpose of the present work is to release the assumption that the field of orthogonal projection P_T is *integrable* and associated to a sub-manifold $\mathcal{N}^n$ and to consider the equation (6) for a general field of orthogonal projections P_T defined on the whole of $\mathbb{R}^m$ and for *horizontal maps* u satisfying $P_T(u)\nabla u = \nabla u$.

Let $P_T \in C^1(\mathbb{R}^m, M_m(\mathbb{R}))$ and $P_N \in C^1(\mathbb{R}^m, M_m(\mathbb{R}))$ such that

$$\left\{\begin{array}{l} P_T \circ P_T = P_T \quad P_N \circ P_N = P_N \\[2mm] P_T + P_N = I_m \\[2mm] \forall\, z \in \mathbb{R}^m \quad \forall\, U, V \in T_z\mathbb{R}^m \qquad < P_T U, P_N V >= 0 \\[2mm] \|\partial_z P_T\|_{L^\infty(\mathbb{R}^m)} < +\infty \end{array}\right. \tag{7}$$

where $< \cdot, \cdot >$ denotes the standard scalar product in $\mathbb{R}^m$. In other words P_T is a C^1 map into the orthogonal projections of $\mathbb{R}^m$. For such a distribution of projections P_T we denote by

$$n := \text{rank}(P_T).$$

Such a distribution identifies naturally with the distribution of $n-$planes given by the images of P_T (or the Kernel of P_T) and conversely, any C^1 distribution of $n-$dimensional planes defines uniquely P_T satisfying (7).

For any $\alpha \geq 1/2$ and for $k \geq 1$

$$\mathfrak{H}^\alpha(\mathbb{R}^k) := \left\{u \in H^\alpha(\mathbb{R}^k, \mathbb{R}^m) \quad ; \quad P_N(u)\,\nabla u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k)\right\}$$

Observe that this definition makes sense since we have respectively $P_T \circ u \in H^\alpha(S^1, M_m(\mathbb{R})$ and $\nabla u \in H^{\alpha-1}(\mathbb{R}^k, \mathbb{R}^m)$. We sometimes extend this definitions to other domains such as S^1 or a general riemannian surface Σ for $\alpha = 1$....etc. In the case $\alpha > k/2$ then $\mathfrak{H}^\alpha(\mathbb{R}^k)$ is a Finsler manifold (see Definition 3.8 in [20]).

Definition 1.1. *Given a C^1 plane distribution P_T in $\mathbb{R}^m$ satisfying (7), a map u in the space $\mathfrak{H}^\alpha(\mathbb{R}^k)$ is called* **horizontal α-harmonic** *with respect to P_T if*

$$\forall\, i = 1 \cdots m \qquad \sum_{j=1}^{m} P_T^{ij}(u)(-\Delta)^\alpha u_j = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^k) \tag{8}$$

and we shall use the following notation

$$P_T(u)\,(-\Delta)^\alpha u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^k).$$

□

Example: *Horizontal Harmonic Maps* in $\mathbb{C}^3$ for the distribution P_T given by

$$P_T(z)\,Z := Z - |z|^{-2}\ \left[Z\cdot(z_1,z_2,z_3)\ (z_1,z_2,z_3) + Z\cdot(iz_1,iz_2,iz_3)\ (iz_1,iz_2,iz_3)\right] \tag{9}$$

are given for instance by the conformal parametrization of *Special horizontal Surfaces* in S^5 (see [9]).

Remark 1.1. *In [10] the authors define a horizontal harmonic map a function which is in the same time horizontal with respect to the plane distribution and harmonic. In [10] it is proved in particular that in the case when P_T is issued from a riemannian submersion, also called* Carnot-Caratheodory space*, (which is the case of the previous example (9)) the normal projection of the tension field of any horizontal map in $\mathfrak{H}^1$ is necessary zero. Therefore the horizontal harmonic maps according to the Definition 1.1 are both horizontal and harmonic and the two definitions coincide. It would be interesting to inquire if such a result holds for $1/2-$harmonic maps.* □

When the plane distribution P_T is *integrable* that is to say when

$$\forall\ X, Y \in C^1(\mathbb{R}^m, \mathbb{R}^m) \quad P_N[P_T\,X, P_T\,Y] \equiv 0 \tag{10}$$

where $[\cdot,\cdot]$ denotes the Lie Bracket of vector-fields, using Fröbenius theorem the plane distribution corresponds to the tangent plane distribution of a $n-$dimensional *foliation* $\mathcal{F}$, (see e.g [11]). A smooth map u in $\mathfrak{H}^\alpha(\mathbb{R}^m)$ takes values everywhere into a *leaf* of $\mathcal{F}$ that we denote $\mathcal{N}^n$ and we are back to the classical theory of harmonic maps into manifolds. Observe that our definition includes the case of α-harmonic maps with values into a sub-manifold of the euclidean space and horizontal with respect to a planes distribution in this sub-manifold. Indeed it is sufficient to add to such a distribution the projection to the sub-manifold and extend the all to a tubular neighborhood of the sub-manifold.

In the present work we shall mostly focus our attention to the case $\alpha = 1/2$ in critical dimension 1 and $\alpha = 1$ in critical dimension 2. We establish below that harmonic maps from $\mathbb{R}^m$ into plane distributions satisfy an elliptic Schrödinger type system with an antisymmetric potential $\Omega \in L^2(\mathbb{R}^k, \mathbb{R}^k \otimes so(m))$ of the form

$$-\Delta u = \Omega(P_T) \cdot \nabla u. \tag{11}$$

Hence, following the analysis in [14] we deduce in two dimension the local existence on a disc D^2 of $A(P_T) \in L^\infty \cap W^{1,2}(D^2, Gl_m(\mathbb{R}))$ and $B(P_T) \in W^{1,2}(D^2, M_m(\mathbb{R}))$ such that

$$\operatorname{div}\left(A(P_T)\, \nabla u\right) = \nabla B(P_T)\, \nabla^\perp u \tag{12}$$

from which the regularity of u can be deduced using Wente's *Integrability by compensation* because of the estimate

$$\|\nabla B\, \nabla^\perp u\|_{H^{-1}(D^2)} \leq\, C\, \|\nabla B\|_{L^2}\, \|\nabla u\|_{L^2}. \tag{13}$$

One of the main contribution of our present work is to produce conservation laws corresponding to (12) but for general *horizontal* $1/2-$*harmonic maps* : locally, modulo some smoother terms coming from the application of non-local operators on cut-off functions, we construct $A(P_T) \in L^\infty \cap H^{1/2}(\mathbb{R}, GL_m(\mathbb{R}))$ and $B(P_T) \in H^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$ such that

$$(-\Delta)^{1/4}(A(P_T)\, v) = \mathcal{J}(B(P_T), v) + \text{cut-off}, \tag{14}$$

where $v := (P_T\,(-\Delta)^{1/4}v, \mathcal{R}\,(P_N(-\Delta)^{1/4}v))$ and $\mathcal{R}$ denotes the Riesz operator and $\mathcal{J}$ is a bilinear pseudo-differential operator satisfying

$$\|\mathcal{J}(B, v)\|_{H^{-1/2}(\mathbb{R})} \leq C\, \|(-\Delta)^{1/4}B\|_{L^2(\mathbb{R})}\, \|v\|_{L^2(\mathbb{R})}. \tag{15}$$

These facts imply the following theorem which is one of the main result of the present work.

Theorem 1.1. *Let P_T be a C^1 distribution of planes (or projections) satisfying (7). Any map $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ satisfying*

$$P_T(u)\,(-\Delta)^{-1/2}u = 0 \qquad \textit{in } \mathcal{D}'(\mathbb{R}) \tag{16}$$

is in $\cap_{\delta<1} C^{0,\delta}_{loc}(\mathbb{R})$. □

Solutions to (16) are of special geometric interest because of the following proposition extending the well known fact in the integrable case which has been at the origin of the study of $1/2-$harmonic maps (see [6]).

Proposition 1.1. *An element in $\mathfrak{H}^{1/2}$ satisfying (16) has an harmonic extension $\tilde{u}$ in D^2 which is conformal and hence it is the boundary of a minimal disc whose exterior normal derivative $\partial_r \tilde{u}$ is orthogonal to the plane distribution given by P_T.* □

Example : We consider the field of projections corresponding to (9) but in $\mathbb{C}^2 \setminus \{0\}$ this time. That is

$$P_T(z)\, Z := Z - |z|^{-2}\ \left[Z \cdot (z_1, z_2)\ (z_1, z_2) + Z \cdot (iz_1, iz_2)\ (iz_1, iz_2)\right]. \tag{17}$$

Example of u satisfying (16) is given by solutions to the system

$$\left\{\begin{array}{l} \dfrac{\partial \tilde{u}}{\partial r} \in \ \mathrm{Span}\,\{u, i\, u\} \qquad \text{a. e.} \\[3mm] u \cdot \dfrac{\partial u}{\partial \theta} = 0 \qquad \text{a. e.} \\[3mm] i\, u \cdot \dfrac{\partial u}{\partial \theta} = 0 \qquad \text{a. e.} \end{array}\right. \tag{18}$$

where $\tilde{u}$ denotes the harmonic extension of u which happens to be conformal due to proposition 1.1 and define a minimal disc. An example of such a map is given by

$$u(\theta) := \frac{1}{\sqrt{2}}(e^{i\theta}, e^{-i\theta}) \quad \text{ where } \quad \tilde{u}(z, \overline{z}) = \frac{1}{\sqrt{2}}(z, \overline{z}). \tag{19}$$

Observe the solution in (19) is also an 1/2-harmonic into S^3 and it would be interesting to investigate whether this is the unique solution.[1] to (16) for P_T given by (17) modulo the composition with Möbius transformations of the form

$$e^{i\theta} \longrightarrow e^{i\sigma_0}\, \frac{e^{i\theta} - a}{1 - \overline{a}\, e^{i\theta}}$$

where $\sigma_0 \in \mathbb{R}$, $a \in \mathbb{C}$ and $|a| < 1$.

Despite the geometric relevance of equations (8) in the non-integrable case, it is however a-priori not the *Euler-Lagrange* equation of the variational problem consisting in finding the critical points of $\|(-\Delta)^{\alpha/2} u\|^2_{L^2}$ within $\mathfrak{H}^\alpha$ when P_T is not satisfying (10). This can be seen in the particular case where $\alpha = 1$ where the critical points to the *Dirichlet Energy* have been extensively studied in relation with the computation of *normal geodesics* in sub-riemannian geometry. We then introduce the following definition

Definition 1.2. *A map u in $\mathfrak{H}^\alpha$ is called* **variational $\alpha-$harmonic** *into the plane distribution P_T if it is a critical point of the $\|(-\Delta)^{\alpha/2} u\|^2_{L^2}$ within variations in $\mathfrak{H}^\alpha$ i.e. for any $u_t \in C^1((-1,1), \mathfrak{H}^\alpha)$ we have*

$$\frac{d}{dt} \|(-\Delta)^{\alpha/2} u_t\|^2_{L^2} \Big|_{t=0} = 0. \square$$

[1] A uniqueness result of that form can be obtained from [8] in the integrable case when

$$P_T(z)\, Z := Z - |z|^{-2}\ Z \cdot (z_1, z_2)\ (z_1, z_2)$$

Example of variational harmonic maps from S^1 into plane distribution is given by the sub-riemannian geodesics.

The last goal of the present work is to establish an *Euler Lagrange* equations characterizing "variational $\alpha-$harmonic into the plane distribution P_T" for $\alpha = 1$ and $\alpha = 1/2$ and to study the regularity of these solutions. This is done using a convexification of the variational problem following the spirit of the approach introduced by Strichartz in [19] for *normal geodesics* in sub-riemannian geometry. We prove in particular for the case $\alpha = 1/2$ that the smooth critical points of

$$\begin{aligned}\mathcal{L}^{1/2}(u,\xi) := &\int_{S^1} \frac{|(-\Delta)_0^{-1/4}(P_T(u)\xi)|^2}{2}\, d\theta \\ &- \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_T(u)\xi), (-\Delta)_0^{-1/4}\left(P_T(u)\frac{du}{d\theta}\right)\right\rangle\, d\theta \\ &- \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_N(u)\xi), (-\Delta)_0^{-1/4}\left(P_N(u)\frac{du}{d\theta}\right)\right\rangle\, d\theta\end{aligned} \tag{20}$$

in the co-dimension m Hilbert subspace of $H^{1/2}(S^1, \mathbb{R}^m) \times H^{-1/2}(S^1, \mathbb{R}^m)$ given by

$$\mathfrak{E} := \left\{ \begin{array}{c} (u,\xi) \in H^{1/2}(S^1,\mathbb{R}^m) \times H^{-1/2}(S^1,\mathbb{R}^m) \quad \text{s. t.} \\ \left(P_N(u), \frac{du}{d\theta}\right)_{H^{1/2},H^{-1/2}} = 0 \\ (-\Delta)_0^{-1/4}(P_T(u)\xi) \in L^2(S^1) \quad \text{and} \quad (-\Delta)_0^{-1/4}\left(P_T(u)\frac{du}{d\theta}\right) \in L^2(S^1) \end{array} \right\}$$

at the point where the constraint $\left(P_N(u), \frac{du}{d\theta}\right)_{H^{1/2},H^{-1/2}}$ is **non-degenerate** are "variational $1/2-$harmonic" into the plane distribution P_T in the sense of definition 1.2. It remains open the regularity of critical points of (20) or even of the 1/2 energy (3) in $\mathfrak{H}^{1/2}$ in the case when the constraint $\left(P_N(u), \frac{du}{d\theta}\right)_{H^{1/2},H^{-1/2}}$ is degenerate.

The paper is organized as follows. In Section 2 we prove the regularity of horizontal Harmonic maps in 2 dimension. In Section 3 we recall some commutators estimates, we find conservation laws associated to nonlocal Schrödinger type systems with antisymmetric potentials. In Section 4 we deduce Theorem 1.1 from the results obtained in Section 2 and Theorem 1.1. In Section 5 we find the Euler Lagrange equation associated to the Lagrangian (20) and we show that smooth critical points of (20) are actually variational harmonic and $1/2-$harmonic maps into a plane distribution P_T .

We finally mention that in the case of horozontal $1/2-$harmonic maps in dimension 1 we can consider as domain of definition indifferently either the real line $\mathbb{R}$ or the circle S^1.

Actually if $\Pi\colon S^1 \setminus \{-i\} \to \mathbb{R}$, $\Pi(\cos(\theta) + i\sin(\theta)) = \frac{\cos(\theta)}{1+\sin(\theta)}$ is the classical stereographic projection whose inverse is given by

$$\Pi^{-1}(x) = \frac{2x}{1+x^2} + i\left(-1 + \frac{2}{1+x^2}\right). \tag{21}$$

then the following relation between the 1/2 Laplacian in $\mathbb{R}$ and in S^1 holds:

Proposition 1.1 (Proposition 4.1, [4]). *Given $u : \mathbb{R} \to \mathbb{R}^m$ set $v := u \circ \Pi : S^1 \to \mathbb{R}^m$. Then $u \in L_{\frac{1}{2}}(\mathbb{R})$*[2] *if and only if $v \in L^1(S^1)$. In this case*

$$(-\Delta)^{\frac{1}{2}}_{S^1} v(e^{i\theta}) = \frac{((-\Delta)^{\frac{1}{2}}_{\mathbb{R}} u)(\Pi(e^{i\theta}))}{1+\sin\theta} \quad \textit{in } \mathcal{D}'(S^1 \setminus \{-i\}), \tag{22}$$

Observe that $1 + \sin(\theta) = |\Pi'(\theta)|$, and hence we have

$$\int_{S^1} (-\Delta)^{\frac{1}{2}} v(e^{i\theta})\, \varphi(e^{i\theta})\; d\theta = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u(x)\; \varphi \circ \Pi^{-1}(x)\; dx \quad \textit{for every } \varphi \in C^\infty_0(S^1 \setminus \{-i\}).$$

2 Regularity of Horizontal Harmonic Maps in 2-D

We prove the following theorem.

Theorem 2.1. *Let P_T be a C^1 map satisfying (7). Any map $u \in \mathfrak{H}^1(D^2)$ satisfying*

$$P_T(u)\,\Delta u = 0 \quad \textit{in } \mathcal{D}'(D^2) \tag{23}$$

is in $\cap_{\delta<1} C^{0,\delta}_{loc}(D^2)$. □

Proof of theorem 2.1. We have

$$\begin{aligned} -\Delta u &= \operatorname{div}(\nabla u) = \operatorname{div}(P_T(u)\,\nabla u) = P_T(u)\,(-\Delta u) + \nabla(P_T(u))\cdot\nabla u \\ &= \nabla(P_T(u))\cdot\nabla u = \nabla(P_T(u))\,P_T(u)\cdot\nabla u = -\nabla(P_N(u))\,P_T(u)\cdot\nabla u \end{aligned} \tag{24}$$

Observe that in one hand

$$\nabla(P_N(u))\,P_T(u) + P_N(u)\,\nabla P_T(u) = 0 \tag{25}$$

and in the other hand

$$(P_N(u)\,\nabla P_T(u))^t = \nabla P_T(u)\,P_N(u) \tag{26}$$

Hence combining (24), (25) and (26) together with the fact that $P_N(u)\nabla u \equiv 0$ we obtain

$$-\Delta u = \left[P_N(u)\,\nabla P_T(u) - (P_N(u)\,\nabla P_T(u))^t\right]\cdot\nabla u \tag{27}$$

Denote $\Omega := [P_N(u)\,\nabla P_T(u) - (P_N(u)\,\nabla P_T(u))^t]$. We have $\Omega \in L^2(D^2, so(m)\otimes\mathbb{R}^2)$. We can then apply the main result in [14] and deduce theorem 2.1. □

[2] We recall that $L_{\frac{1}{2}}(\mathbb{R}) := \left\{u \in L^1_{loc}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1+x^2}dx < \infty\right\}$

3 3-Commutators, Antisymmetry and Conservation Laws in 1-D

3.1 A Regularity Result for Solutions to Linear Pseudo-Differential Equations involving Projections

Denote $\mathcal{R}$ is the Riesz operator given by

$$\mathcal{R}\ :\ f=\sum_{n\in\mathbb{Z}} f_n\, e^{i\,n\,\theta}\ \longrightarrow\ \mathcal{R}f:=i\sum_{n\in\mathbb{Z}^*}\mathrm{sgn}(n)\, f_n\, e^{i\,n\,\theta}$$

The following lemma is a straightforward consequence of the classical Coifmann Rochberg and Weiss integrability by compensation (see [1]).

Theorem 3.1. *Let $m\in\mathbb{N}^*$, then there exists $\delta>0$ such that for any P_T, $P_N\in H^{1/2}(S^1,M_m(\mathbb{R}))$ satisfying*

$$\left\{\begin{array}{l} P_T\circ P_T=P_T\quad P_N\circ P_N=P_N\\[2mm] P_T+P_N=I_m\\[2mm] \text{for a. e. } e^{i\theta}\in S^1\quad \forall\, U,V\in\mathbb{R}^m\quad <P_T(\theta)U,P_N(\theta)V>=0\end{array}\right. \tag{28}$$

and

$$\int_{S^1}|(-\Delta)^{1/4}P_T|^2\ d\theta<\delta \tag{29}$$

then for any $p>1$ and for any $f\in L^p(S^1)$ with $\int_{S^1}f(\theta)\,d\theta=0$

$$(P_T+P_N\,\mathcal{R})\ f=0\quad\Longrightarrow\quad f=0 \tag{30}$$

We are going to extend the previous theorem to negative Sobolev Spaces.

Theorem 3.2. *Let $m\in\mathbb{N}^*$, then there exists $\delta>0$ such that for any P_T, $P_N\in H^{1/2}(S^1,M_m(\mathbb{R}))$ satisfying*

$$\left\{\begin{array}{l} P_T\circ P_T=P_T\quad P_N\circ P_N=P_N\\[2mm] P_T+P_N=I_m\\[2mm] \text{for a. e. } e^{i\theta}\in S^1\quad \forall\, U,V\in\mathbb{R}^m\quad <P_T(\theta)U,P_N(\theta)V>=0\end{array}\right. \tag{31}$$

and

$$\int_{S^1}|(-\Delta)^{1/4}P_T|^2\ d\theta<\delta \tag{32}$$

then for any $f\in H^{-1/2}(S^1)$ with $<1,f>_{H^{1/2},H^{-1/2}}=0$

$$(P_T+P_N\,\mathcal{R})\ f=0\quad\Longrightarrow\quad f=0 \tag{33}$$

As we will see in the next subsections, the results on commutators in [1] does not apply to the negative Sobolev Spaces and we are going to make use of *integrability by compensation* results for the so called *3-commutators* introduced in [5] combined with Gauge theoretic arguments exploiting the antisymmetry of some terms in the spirit of [6].

The uniqueness result 3.2 under small energy assumptions implies the following regularity result.

Theorem 3.3. *Let $m \in \mathbb{N}^*$ and P_T, $P_N \in H^{1/2}(S^1, M_m(\mathbb{R}))$ satisfying*

$$\left\{\begin{array}{l} P_T \circ P_T = P_T \quad P_N \circ P_N = P_N \\ P_T + P_N = I_m \\ \text{for a. e. } e^{i\theta} \in S^1 \quad \forall U, V \in \mathbb{R}^m \quad < P_T(\theta)U, P_N(\theta)V >= 0 \end{array}\right. \tag{34}$$

then for any $f \in H^{-1/2}(S^1)$ with $< 1, f >_{H^{1/2}, H^{-1/2}} = 0$ and satisfying

$$(P_T + P_N \mathcal{R})\ f = 0 \tag{35}$$

we have $f \in L^p(S^1)$ for any $p < +\infty$. □

3.2 Multiplying 3-Commutators

In this section we recall regularity properties of some commutators we have introduced in [5, 6], called 3-commutators, and establish almost stability properties of 3-commutators under multiplication.

We introduce the following *commutators*:

$$T(Q, v) := (-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4}v + (-\Delta)^{1/4}Qv \tag{36}$$

and

$$S(Q, v) := (-\Delta)^{1/4}[Qv] - \mathcal{R}(Q\mathcal{R}(-\Delta)^{1/4}v) + \mathcal{R}((-\Delta)^{1/4}Q\mathcal{R}v) \tag{37}$$

$$F(Q, v) := \mathcal{R}[Q]\mathcal{R}[v] - Qv. \tag{38}$$

$$\Lambda(Q, v) := Qv + \mathcal{R}[Q\mathcal{R}[v]]\,. \tag{39}$$

In [5, 6] the authors obtained the following estimates.

Theorem 3.1. *Let $v \in L^2(\mathbb{R})$, $Q \in \dot{H}^{1/2}(\mathbb{R})$. Then $T(Q, v), S(Q, v) \in H^{-1/2}(\mathbb{R})$ and*

$$\|T(Q, v)\|_{H^{-1/2}(\mathbb{R})} \leq C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}(\mathbb{R})}\,; \tag{40}$$

$$\|S(Q, v)\|_{H^{-1/2}(\mathbb{R})} \leq C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}(\mathbb{R})}\,. \tag{41}$$

Actually in [3] we improve the estimates on the operators T, S.

Theorem 3.2. *Let Let $v \in L^2(\mathbb{R})$, $Q \in \dot{H}^{1/2}(\mathbb{R})$. Then $T(Q,v), S(Q,v) \in \mathcal{H}^1(\mathbb{R})$ and*

$$\|T(Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\,. \tag{42}$$

$$\|S(Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\,. \quad \Box \tag{43}$$

Theorem 3.1 is actually consequence of the following estimates for the dual operators of T and S.

Theorem 3.3. *Let $u, Q \in \dot{H}^{1/2}(\mathbb{R})$, denote*

$$T^*(Q,u) = (-\Delta)^{1/4}(Q(-\Delta)^{1/4}u) - (-\Delta)^{1/2}(Qu) + (-\Delta)^{1/4}(((-\Delta)^{1/4}Q)u)\,.$$

then $T^*(Q,u) \in \mathcal{H}^1(\mathbb{R})$ and

$$\|T^*(Q,u)\|_{\mathcal{H}^1(\mathbb{R})} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|u\|_{\dot{H}^{1/2}(\mathbb{R})}\,.\Box \tag{44}$$

Theorem 3.4. *Let $u, Q \in \dot{H}^{1/2}(\mathbb{R})$, denote*

$$S^*(Q,u) = (-\Delta)^{1/4}(Q(-\Delta)^{1/4}u) - \nabla(Q\mathcal{R}u) + \mathcal{R}(-\Delta)^{1/4}((-\Delta)^{1/4}Q\mathcal{R}u)\,.$$

Then $S^*(Q,u) \in \mathcal{H}^1(\mathbb{R})$ and

$$\|S^*(Q,u)\|_{\mathcal{H}^1(\mathbb{R})} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|u\|_{\dot{H}^{1/2}(\mathbb{R})}\,.\Box \tag{45}$$

Finally we have

Theorem 3.5. *Let $P, Q \in \dot{H}^{1/2}(\mathbb{R})$, denote*

$$\bar{T}(P,Q) = (-\Delta)^{1/4}P\mathcal{R}[(-\Delta)^{1/4}Q] + (-\Delta)^{1/4}[\mathcal{R}(-\Delta)^{1/4}[P]Q] - \nabla[PQ].$$

Then $\bar{T}(P,Q) \in \mathcal{H}^1(\mathbb{R})$ and

$$\|\bar{T}(P,Q)\|_{\mathcal{H}^1(\mathbb{R})} \le C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|P\|_{\dot{H}^{1/2}(\mathbb{R})}\,.\Box \tag{46}$$

Theorem 3.6. *For $f, v \in L^2$ it holds*

$$\|F(f,v)\|_{H^{-1/2}(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}\|v\|_{L^{2,\infty}(\mathbb{R})}\,. \tag{47}$$

and

$$\|F(f,v)\|_{\mathcal{H}^1(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\,. \tag{48}$$

Theorem 3.7. *For $Q \in \dot{H}^{1/2}(\mathbb{R})$, $v \in L^2(\mathbb{R})$ it holds*

$$\|\Lambda(Q,v)\|_{L^{2,1}(\mathbb{R})} \le C\|Q\|_{H^{1/2}(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\,. \tag{49}$$

Actually the estimate (49) is a consequence of the Coifman-Rochberg- Weiss estimate [1].

Next we prove a sort of stability of of the operators T, S, F with respect to the multiplication by a function $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Roughly speaking if we multiply them by a function $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we get a decomposition into the sum of a function in the Hardy Space and a term which is the product of function in $L^{2,1}$ by one in L^2.

Theorem 3.8. [Multiplication of F by $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$] *Let $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $f, v \in L^2(\mathbb{R})$. Then*

$$PF(f,v) = \underbrace{F(P\mathcal{R}[f], \mathcal{R}[v])}_{\in \mathcal{H}^1(\mathbb{R})} - \underbrace{\Lambda(P,f)}_{\in L^{2,1}}\, v \tag{50}$$

Proof of Theorem 3.8. We have

$$\begin{aligned} PF(f,v) &= P\mathcal{R}[f]\,\mathcal{R}[v] - Pfv \\ &= P\mathcal{R}[f]\,\mathcal{R}[v] + \mathcal{R}[P\mathcal{R}[f]]v - \mathcal{R}[P\mathcal{R}[f]]v - Pfv \\ &= F(P\mathcal{R}[f], \mathcal{R}[v]) - \Lambda(P,f)v. \end{aligned}$$

The conclusion follows from Theorem 3.6 and Theorem 3.7. □

Theorem 3.9. [Multiplication of T by $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$]
Let $P, Q \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v \in L^2(\mathbb{R})$. Then

$$PT(Q,v) = J_T(P,Q,v) + \mathcal{A}_T(P,Q)v, \tag{51}$$

where

$$\mathcal{A}_T(P,Q) = (-\Delta)^{-1/4}T^*(P,Q) = P(-\Delta)^{1/4}[Q] + (-\Delta)^{1/4}[P]Q - (-\Delta)^{1/4}[PQ] \in L^{2,1}$$

with

$$\|\mathcal{A}_T(P,Q)\|_{L^{2,1}} \le C\|(-\Delta)^{1/4}[P]\|_{L^2}\|(-\Delta)^{1/4}[Q]\|_{L^2}, \tag{52}$$

and

$$J_T(P,Q,v) := T(PQ,v) - T(P,Qv) \in \mathcal{H}^1(\mathbb{R})$$

with

$$\|J_T(P,Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \le C\left(\|(-\Delta)^{1/4}[P]\|_{L^2} + \|(-\Delta)^{1/4}[Q]\|_{L^2}\right)\|v\|_{L^2}. \tag{53}$$

Proof of Theorem 3.9. We have

$$\begin{aligned} PT(Q,v) &= P(-\Delta)^{1/4}[Qv] - PQ(-\Delta)^{1/4}[v] + P(-\Delta)^{1/4}[Q]v \\ &= \{P(-\Delta)^{1/4}[Q] - (-\Delta)^{1/4}[PQ] + (-\Delta)^{1/4}[P]Q\}v \\ &+ (-\Delta)^{1/4}[PQv] - PQ(-\Delta)^{1/4}v + (-\Delta)^{1/4}[PQ]v \\ &- \left((-\Delta)^{1/4}[PQv] + P(-\Delta)^{1/4}(Qv) - (-\Delta)^{1/4}[P]Qv\right) \\ &= (-\Delta)^{-1/4}[T^*(P,Q)]v + T(PQ,v) - T(P,Qv). \end{aligned}$$

Finally the estimates (52), (65) follow from Theorem 3.3 and Theorem 3.2. □

Now we consider the operator S. We first observe that given $Q \in H^{1/2}$ and $v \in L^2$ we have the following decomposition

$$\begin{aligned} \mathcal{R}[S(Q,v)] &= \tilde{S}(Q,v) - \mathcal{R}(-\Delta)^{1/4}[Q]v - (-\Delta)^{1/4}Q\mathcal{R}v \\ &= \tilde{S}(Q,v) + F(\mathcal{R}(-\Delta)^{1/4}[Q],v). \end{aligned} \tag{54}$$

where

$$\tilde{S}(Q,v) = \mathcal{R}(-\Delta)^{1/4}[Qv] + Q\mathcal{R}(-\Delta)^{1/4}[v] + \mathcal{R}(-\Delta)^{1/4}[Q]v.$$

From Theorems 3.2 and 3.6 it follows that $\tilde{S}(Q,v) \in \mathcal{H}^1$ and

$$\|\tilde{S}(Q,v)\|_{\mathcal{H}^1} \le \|v\|_{L^2}\|Q\|_{H^{1/2}}.$$

Theorem 3.10. [Multiplication of $\mathcal{R}S$ by a rotation $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$] *Let $P, Q \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v \in L^2(\mathbb{R})$. Then*

$$P\mathcal{R}[S(Q,v)] = \mathcal{A}_S(P,Q)v + J_S(P,Q,v) \tag{55}$$

where

$$\mathcal{A}_S(P,Q) := (-\Delta)^{-1/4}[\bar{T}(P,Q)] + \Lambda(P,\mathcal{R}(-\Delta)^{1/4}[Q]) \in L^{2,1}.$$

with

$$\|\mathcal{A}_S(P,Q)\|_{L^{2,1}} \le C\|(-\Delta)^{1/4}[P]\|_{L^2}\|(-\Delta)^{1/4}[Q]\|_{L^2},$$

and

$$J_S(P,Q,v) := \tilde{S}(PQ,v) - \tilde{S}(P,Qv) + F(\mathcal{R}[P(-\Delta)^{1/4}[Q]],v) \in \mathcal{H}^1(\mathbb{R})$$

with

$$\|J_S(P,Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \le C\left(\|(-\Delta)^{1/4}[P]\|_{L^2} + \|(-\Delta)^{1/4}[Q]\|_{L^2}\right)\|v\|_{L^2}.$$

Sketch of Proof. Let $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then

$$\begin{aligned} P\mathcal{R}[S(Q,v)] &= \tilde{S}(PQ,v) - \tilde{S}(P,Qv) \\ &+ \left\{P\mathcal{R}(-\Delta)^{1/4}Q + \mathcal{R}(-\Delta)^{1/4}[P]Q - \mathcal{R}(-\Delta)^{1/4}[PQ]\right\}v \\ &- P[\mathcal{R}(-\Delta)^{1/4}[Q]v - (-\Delta)^{1/4}Q\mathcal{R}v]. \end{aligned} \tag{56}$$

Next we estimate the term $P[\mathcal{R}(-\Delta)^{1/4}[Q]v - (-\Delta)^{1/4}Q\mathcal{R}v]$

$$\begin{aligned} P[\mathcal{R}(-\Delta)^{1/4}[Q]v + (-\Delta)^{1/4}Q\mathcal{R}v] &= \underbrace{\{P\mathcal{R}(-\Delta)^{1/4}[Q] - \mathcal{R}[P(-\Delta)^{1/4}[Q]]\}}_{\in L^{2,1}}v \\ &+ \underbrace{\mathcal{R}[P(-\Delta)^{1/4}[Q]]\,v + P(-\Delta)^{1/4}Q\,\mathcal{R}[v]}_{\in \mathcal{H}^1}. \end{aligned} \tag{57}$$

Therefore we can write

$$
\begin{aligned}
P\mathcal{R}[S(Q,v)] &= \mathcal{A}_S(P,Q)v \qquad (58)\\
&+ \underbrace{\tilde{S}(PQ,v) - \tilde{S}(P,Qv) + F(\mathcal{R}[P(-\Delta)^{1/4}[Q]],v)}_{\in\mathcal{H}^1}
\end{aligned}
$$

where

$$
\mathcal{A}_S(P,Q) := (-\Delta)^{-1/4}[\bar{T}(P,Q)] + \Lambda(P,\mathcal{R}(-\Delta)^{1/4}[Q]). \quad \Box
$$

Remark 3.1. We mention without entering into the details that in 2-D the Jacobian $J(a,b) = \nabla(a)\,\nabla^{\perp}(b)$ satisfies a stability property enjoyed by the operators (36), (37), (38) with respect to the multiplication by $P \in W^{1,2}(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$ as well. More precisely we may define the following two zero order pseudo-differential operators: $\mathrm{Grad}(X) := \nabla\mathrm{div}\Delta^{-1}(X)$, $\mathrm{Rot}(Y) = \nabla^{\perp}\mathrm{curl}\Delta^{-1}(Y)$. If $a,b \in W^{1,2}(\mathbb{R}^2)$ and $P \in W^{1,2}(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$ then

$$
\begin{aligned}
J(a,b) &= \nabla(a)\,\nabla^{\perp}(b) \qquad (59)\\
&= \mathrm{Grad}(\nabla(a))\,\mathrm{Rot}(\nabla^{\perp}(b)) - \mathrm{Rot}(\nabla(a))\,\mathrm{Grad}(\nabla^{\perp}(b));
\end{aligned}
$$

and

$$
\begin{aligned}
P\,J(a,b) &= P\,\nabla(a)\,\nabla^{\perp}(b) \qquad (60)\\
&= \underbrace{[P\mathrm{Grad}(\nabla(a)) - \mathrm{Grad}(P\nabla(a))]}_{\in L^{2,1}(\mathbb{R}^2)}\,\mathrm{Rot}(\nabla^{\perp}(b))\\
&+ \underbrace{\mathrm{Grad}(P\nabla(a))\,\mathrm{Rot}(\nabla^{\perp}(b)) - \mathrm{Rot}(P\nabla(a))\,\mathrm{Grad}(\nabla^{\perp}(b))}_{\in\mathcal{H}^1(\mathbb{R}^2)}.
\end{aligned}
$$

3.3 Conservation Laws for Fractional Schrödinger type PDEs with Antisymmetric Potentials.

The aim of this part is to construct conservation laws for fractional Schrödinger type PDEs with antisymmetric potentials. More precisely we are going to consider a nonlocal system of the form

$$
(-\Delta)^{1/4}v = \Omega_0 v + \Omega_1 v + \mathcal{Z}(Q,v) + g(x) \qquad (61)
$$

where $v \in L^2(\mathbb{R})$, $Q \in H^{1/2}(\mathbb{R})$, $\mathcal{Z}\colon H^{1/2}(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathcal{H}^1(\mathbb{R})$ is a linear combination of the operators (38), (36) and (37) introduced in the previous section, $\Omega_0 \in L^2(\mathbb{R}, so(m))$, $\Omega_1 \in L^{2,1}(\mathbb{R})$, $g(x)$ is a tempered distribution.

Theorem 3.11. *Let $v \in L^2(\mathbb{R},\mathbb{R}^m)$ be a solution of (61), where $\Omega_0 \in L^2(\mathbb{R}, so(m))$, $\Omega_1 \in L^{2,1}(\mathbb{R})$, $\mathcal{Z}$ is a linear combination of the operators (38), (36) and (37), $\mathcal{Z}(Q,v) \in \mathcal{H}^1$ for every $Q \in H^{1/2}$, $v \in L^2$ with*

$$\|\mathcal{Z}(Q,v))\|_{\mathcal{H}^1} \quad \leq \quad C\|Q\|_{H^{1/2}}\|v\|_{L^2}$$

There exists $\varepsilon_0 > 0$ such that if $(\|\Omega_0\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{H^{1/2}}) < \varepsilon_0$, then there exist $A = A(\Omega_0,\Omega_1,Q) \in H^{1/2}(\mathbb{R}, GL_m(\mathbb{R}))$ and an operator $B = B(\Omega_0,\Omega_1,Q) \in H^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$ such that

$$\|A\|_{H^{1/2}} + \|B\|_{H^{1/2}} \leq C(\|\Omega_0\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{H^{1/2}}) \tag{62}$$

$$dist(\{A, A^{-1}\}, SO(m)) \leq C(\|\Omega_0\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{H^{1/2}}) \tag{63}$$

and

$$(-\Delta)^{1/4}[Av] = \mathcal{J}(B,v) + Ag, \tag{64}$$

where $\mathcal{J}$ is a linear operator in B, v, $\mathcal{J}(B,v) \in \mathcal{H}^1(\mathbb{R})$ and

$$\|\mathcal{J}(B,v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C\|B\|_{H^{1/2}}\|v\|_{L^2}\,. \tag{65}$$

Proof of Theorem 3.11. We first observe that since the operator $\mathcal{Z}(Q,v))$ is a linear combination of the operators F, S and T, it satisfies the following stability property: if $Q, P \in \dot{H}^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $v \in L^2$ then

$$P\mathcal{Z}(Q,v) = \mathcal{A}_{\mathcal{Z}}(P,Q)v + J_{\mathcal{Z}}(P,Q,v), \tag{66}$$

where

$$\|\mathcal{A}_{\mathcal{Z}}(P,Q)\|_{L^{2,1}} \leq C\|(-\Delta)^{1/4}[P]\|_{L^2}\|(-\Delta)^{1/4}[Q]\|_{L^2},$$

and

$$\|J_{\mathcal{Z}}(P,Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C\left(\|(-\Delta)^{1/4}[P]\|_{L^2} + \|(-\Delta)^{1/4}[Q]\|_{L^2}\right)\|v\|_{L^2}.$$

Step 1: From Theorem 1.2 in [6] there exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\|\Omega_0\|_{L^2} < \varepsilon_0$, then there exists $P = P(\Omega_0) \in \dot{H}^{1/2}(\mathbb{R}, SO(m))$ such that

$$\left\{\begin{array}{ll} (i) & P^{-1}(-\Delta)^{1/4}P - (-\Delta)^{1/4}P^{-1}P = 2\,\Omega_0\,; \\ & \\ (ii) & \|(-\Delta)^{1/4}P\|_{L^2} \leq C\|\Omega_0\|_{L^2}\,. \end{array}\right. \tag{67}$$

Moreover

$$\begin{aligned} P\Omega_0P^{-1} - P^{-1}(-\Delta)^{1/4}P &= -\frac{\left(P(-\Delta)^{1/4}[P^{-1}] - (-\Delta)^{1/4}[P]P^{-1}\right)}{2} \\ &= -(-\Delta)^{-1/4}(T^*(P^{-1},P)) \in L^{2,1}. \end{aligned}$$

Step 2: Estimate of $(-\Delta)^{1/4}[Pv]$.

$$
\begin{aligned}
(-\Delta)^{1/4}[Pv] &= (-\Delta)^{1/4}[Pv] - P(-\Delta)^{1/4}[v] + (-\Delta)^{1/4}[P]v \\
&+ P(-\Delta)^{1/4}[v] - (-\Delta)^{1/4}[P]v \\
&= T(P,v) + P\{\Omega_0 v + \Omega_1 v + \mathcal{Z}(P,v) + g(x)\} - (-\Delta)^{1/4}[P]v \\
&= T(P,v) + J_{\mathcal{Z}}(P,Q,v) + \mathcal{A}_{\mathcal{Z}}(P,Q)v + P\Omega_1 P^{-1}(Pv) \\
&+ [P\Omega_0 P^{-1} - (-\Delta)^{1/4}[P]P^{-1}](Pv) + Pg \\
&= \varpi(\Omega_0,\Omega_1,Q)(Pv) + J_{T,\mathcal{Z}}(P,v) + Pg
\end{aligned} \tag{68}
$$

where

$$
\varpi(\Omega_0,\Omega_1,Q) = P\Omega_1 P^{-1} + \mathcal{A}_{\mathcal{Z}}(P,Q) + [P\Omega_0 P^{-1} - (-\Delta)^{1/4}[P]P^{-1}] \in L^{2,1}
$$

with

$$
\|\varpi\|_{L^{2,1}} \le C(\|(-\Delta)^{1/4}[Q]\|_{L^2} + \|\Omega_0\|_{L^2} + \|\Omega_1\|_{L^{2,1}}),
$$

and $J_{T,\mathcal{Z}}(P,Q,v) = T(Q,v) + J_{\mathcal{Z}}(P,Q,v) \in \mathcal{H}^1(\mathbb{R})$ with

$$
\|J_{T,\mathcal{Z}}(P,Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \le C\left(\|(-\Delta)^{1/4}[P]\|_{L^2} + \|(-\Delta)^{1/4}[Q]\|_{L^2}\right)\|v\|_{L^2}.
$$

Moreover the operator $\mathcal{J}$ is linear and it has the following property: if $M,P,Q \in \dot{H}^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $v \in L^2$ then it holds the following decomposition:

$$
M J_{T,\mathcal{Z}}(P,Q,v) = \omega(M,P,Q,)v + \mathcal{G}(M,P,Q,v), \tag{69}
$$

with $\omega(M,P,Q,v) \in L^{2,1}(\mathbb{R})$ and $\mathcal{G}(M,P,Q,v) \in \mathcal{G}^1(\mathbb{R})$. This decomposition follows from the fact that $J_{T,\mathcal{Z}}(P,Q,v)$ is a linear combination of the operators F, T and S.

Step 3: Given $\mathcal{E} \in W^{1/2,(2,1)} \cap L^\infty$ (that we will chose later in a suitable way), from the above computations it follows that

$$
\begin{aligned}
(-\Delta)^{1/4}[(Id+\mathcal{E})Pv] &= T((Id+\mathcal{E}),Pv) \\
&+ (Id+\mathcal{E})(-\Delta)^{1/4}[Pv] - (-\Delta)^{1/4}[(Id+\mathcal{E})]Pv \\
&= T((Id+\mathcal{E}),Pv) + (Id+\mathcal{E})\{\varpi(\Omega_0,\Omega_1,Q)(Qv) + J_{T,\mathcal{Z}}(P,Q,v)\} \\
&- (-\Delta)^{1/4}[\mathcal{E}]Pv \\
&= T((Id+\mathcal{E}),Pv) + \mathcal{G}((Id+\mathcal{E}),P,Q,v) + (Id+\mathcal{E})Pg \\
&+ [(Id+\mathcal{E})\varpi(\Omega_0,\Omega_1,Q) + \omega(Id+\mathcal{E}),P,Q)P^{-1}]Pv - (-\Delta)^{1/4}[\mathcal{E}]Pv.
\end{aligned} \tag{70}
$$

Set $\tilde{\omega}((Id+\mathcal{E}),\Omega_0,\Omega_1,Q) := [(Id+\mathcal{E})\varpi(\Omega_0,\Omega_1,Q)+\omega((Id+\mathcal{E}),P,Q)P^{-1}]$. We have $\tilde{\omega}$ is linear with respect to $\mathcal{E}$, $\tilde{\omega}((Id+\mathcal{E}),\Omega_0,\Omega_1,Q)\in L^{2,1}$ and

$$\|\tilde{\omega}((Id+\mathcal{E}),\Omega_0,\Omega_1,Q)\|_{L^{2,1}} \le C\|\mathcal{E}\|_{L^\infty}(\|(-\Delta)^{1/4}[Q]\|_{L^2}+\|\Omega_0\|_{L^2}+\|\Omega_1\|_{L^{2,1}})\,.$$

We choose $\mathcal{E}$ to be a solution in $W^{1/2,(2,1)}\cap L^\infty$

$$(-\Delta)^{1/4}[\mathcal{E}] = \tilde{\omega}(Id+\mathcal{E},\Omega_0,\Omega_1,Q). \tag{71}$$

Such a solution satisfies

$$\|\mathcal{E}\|_{L^\infty}\le C\|(-\Delta)^{1/4}[\mathcal{E}]\|_{L^{2,1}}\le C(\|(-\Delta)^{1/4}[Q]\|_{L^2}+\|\Omega_0\|_{L^2}+\|\Omega_1\|_{L^{2,1}}).$$

By combining (70) and (71) it follows

$$(-\Delta)^{1/4}[(Id+\mathcal{E})Pv] = \mathcal{J}((Id+\mathcal{E}),P,Q,v)+(Id+\mathcal{E})Pg, \tag{72}$$

where

$$\mathcal{J}((Id+\mathcal{E}),\Omega_0,\Omega_1,Q,v) = T((Id+\mathcal{E}),Pv)+\mathcal{G}((Id+\mathcal{E}),P,Q,v)$$

$\mathcal{J}((Id+\mathcal{E}),\Omega_0,\Omega_1,Q,v)\in\mathcal{H}^1(\mathbb{R})$ with

$$\|\mathcal{J}((Id+\mathcal{E}),\Omega_0,\Omega_1,Q,v)\|_{\mathcal{H}^1(\mathbb{R})}\le C(\|(-\Delta)^{1/4}[Q]\|_{L^2}+\|\Omega_0\|_{L^2}+\|\Omega_1\|_{L^{2,1}})\|v\|_{L^2}.$$

We set $A=A(\Omega_0,\Omega_1,Q)=(Id+\mathcal{E})P$ and $B=B(\Omega_0,\Omega_1,Q)=((Id+\mathcal{E}),P,Q)$, where P satisfies (67) and $\mathcal{E}$ is a solution of (71). It is evident that $(Id+\mathcal{E})$ dependson Ω_0,Ω_1,Q.

We get

$$(-\Delta)^{1/4}[Av] = \mathcal{J}(B,v)+Ag \tag{73}$$

We observe that by construction we have

$$\begin{array}{rcl}\|A\|_{H^{1/2}}+\|B\|_{H^{1/2}} &\le& C(\|\Omega_0\|_{L^2}+\|\Omega_0\|_{L^{2,1}}+\|Q\|_{H^{1/2}})\\ dist(A,SO(m)) &\le& \|A-P\|_{L^\infty}\le C\|\mathcal{E}\|_{L^\infty}\le C(\|\Omega_0\|_{L^2}+\|\Omega_0\|_{L^{2,1}}+\|Q\|_{H^{1/2}})\\ \|\mathcal{J}(B,v)\|_{\mathcal{H}^1(\mathbb{R})} &\le& C\|B\|_{H^{1/2}}\|v\|_{L^2}.\end{array}$$

We can conclude the proof. □

Theorem 3.4. *Let $m\in\mathbb{N}^*$, then there exists $\delta>0$ such that for any P_T, $P_N\in H^{1/2}(\mathbb{R},M_m(\mathbb{R}))$ satisfying*

$$\left\{\begin{array}{l} P_T\circ P_T=P_T \quad P_N\circ P_N=P_N\\ P_T+P_N=I_m\\ \textit{for a. e. } x\in\mathbb{R}\quad \forall U,V\in\mathbb{R}^m \quad <P_T(x)U,P_N(x)V>=0\end{array}\right. \tag{74}$$

and

$$\int_{\mathbb{R}}|(-\Delta)^{1/4}P_T|^2\,d\theta<\delta \tag{75}$$

then for any $f\in H^{-1/2}(\mathbb{R})$

$$(P_T+P_N\,\mathcal{R})\,f=0 \quad\Longrightarrow\quad f=0 \tag{76}$$

Proof of Theorem 3.4.

We first set $f := (-\Delta)^{1/2}u$. From (76) it follows that

$$\left\{\begin{array}{l} P_T(-\Delta)^{1/2}u = 0 \\ \\ P_N\mathcal{R}(-\Delta)^{1/2}u = 0 \end{array}\right. \tag{77}$$

The set $v = (P_T(-\Delta)^{1/4}u, P_N\mathcal{R}(-\Delta)^{1/4}u)^t$. In [6] it has been proved that v satisfies a non-local Schrödinger type system of the form (61) with $g \equiv 0$ $\Omega_0 = \Omega_0(P_T) \in L^2(\mathbb{R}, so(\mathbb{R}^m))$ $\Omega_1 = \Omega_1(P_T) \in L^{2,1}$, $\mathcal{Z}(P_T, v)$ is a linear operator in P_T, v, $\mathcal{Z}(P_T, v) \in \mathcal{H}^1$ with

$$\begin{array}{rcl} \|\Omega_0\|_{L^2} &=& \|\Omega_0(P_T)\|_{L^2} \le C\|P_T\|_{H^{1/2}} \\ \|\Omega_1\|_{L^{2,1}} &=& \|\Omega_1(P_T)\|_{L^{2,1}} \le C\|P_T\|_{H^{1/2}} \\ \|\mathcal{Z}(P_T, v))\|_{\mathcal{H}^1} &\le& C\|P_T\|_{H^{1/2}}\|v\|_{L^2} \end{array}$$

(see appendix A.2). From Theorem 3.11 it follows that if δ is small enough then there exist $A = A(P_T)$ and $B = B(P_T)$ such that

$$(-\Delta)^{1/4}[Av] = \mathcal{J}(B, v) \tag{78}$$

and

$$\begin{array}{rcl} \|A\|_{H^{1/2}} + \|B\|_{H^{1/2}} &\le& C\|P_T\|_{H^{1/2}} \\ dist(A, SO(m)) &\le& \le C\|P_T\|_{H^{1/2}} \\ \|\mathcal{J}(B, v)\|_{\mathcal{H}^1(\mathbb{R})} &\le& C\|B\|_{H^{1/2}}\|v\|_{L^2}. \end{array} \tag{79}$$

From (78) and (79) it follows that

$$\begin{array}{rcl} \|v\|_{L^2} &=& \|A^{-1}Av\|_{L^2} \le C\|A^{-1}\|_{L^\infty}\|Av\|_{L^2} \\ &\le& C\|(-\Delta)^{-1/4}\mathcal{J}(B, v)\|_{L^{2,1}} \le C\|B\|_{H^{1/2}}\|v\|_{L^2} \\ &\le& C\|P_T\|_{H^{1/2}}\|v\|_{L^2} \le C\varepsilon_0\|v\|_{L^2}. \end{array} \tag{80}$$

Again if δ is small enough then (80) yields $v \equiv 0$.

Claim: $v \equiv 0 \quad \Rightarrow f = 0$.

Proof of the Claim.

If δ is small enough then $v \equiv 0$ implies that $P_T(-\Delta)^{1/4}u = 0$ and $P_N\mathcal{R}(-\Delta)^{1/4}u = 0$. Now observe that

$$\mathcal{R}[P_N(-\Delta)^{1/4}u] = \underbrace{\mathcal{R}[P_N(-\Delta)^{1/4}u] - \underbrace{P_N\mathcal{R}(-\Delta)^{1/4}u}_{=0}}_{\in L^{2,1}} \tag{81}$$

Therefore from Theorem 3.6 and the fact that $P_T(-\Delta)^{1/4}u = 0$ it follows that

$$\begin{aligned} \|(-\Delta)^{1/4}u\|_{L^2} &= \|P_N(-\Delta)^{1/4}u]\|_{L^2} = \|\mathcal{R}[P_N(-\Delta)^{1/4}u]\|_{L^2} \\ &\le C\|P_T\|_{H^{1/2}}\|v\|_{L^2} \le C\delta\|(-\Delta)^{1/4}u\|_{L^2}. \end{aligned} \tag{82}$$

Therefore if $C\delta < 1$ then (82) implies $(-\Delta)^{1/4}u \equiv 0$ and therefore $f = 0$.

We can conclude the proof of the claim and of the Theorem 3.4. □

4 Regularity of Horizontal $1/2-$Harmonic Maps in 1-D

4.1 Proof of Theorem 1.1.

The proof of Theorem 1.1 follows by combining Theorem 3.4 and localization arguments used in [6]. □

4.2 Proof of Proposition 1.1.

A first proof when $P_T \in C^2(\mathbb{R}^m)$. In that case we have that $u \in C^{1,\alpha}(S^1)$. Denote $\tilde{u}$ the harmonic extension of u. It is well known that the *Hopf differential* of $\tilde{u}$

$$|\partial_{x_1}\tilde{u}|^2 - |\partial_{x_2}\tilde{u}|^2 - \,2\,i\,\langle\partial_{x_1}\tilde{u}, \partial_{x_2}\tilde{u}\rangle = f(z)$$

is holomorphic. Considering on $S^1 = \partial D^2$

$$2\,\langle\partial_r\tilde{u}, \partial_\theta\tilde{u}\rangle = -\,\sin 2\theta\,\left(|\partial_{x_1}\tilde{u}|^2 - |\partial_{x_2}\tilde{u}|^2\right) - \,\cos 2\,\theta\,\left(-2\,\langle\partial_{x_1}\tilde{u}, \partial_{x_2}\tilde{u}\rangle\right) = -\Im\left(z^2\,f(z)\right)$$

Since $0 = P_T(u)\,(-\Delta)^{1/2}u = P_T(u)\,\partial_r\tilde{u}$ and $0 = P_N(u)\,\partial_\theta u = P_N(u)\,\partial_\theta\tilde{u}$ on ∂D^2 we have that

$$\Im\left(z^2\,f(z)\right) = 0 \quad \text{ on } \partial D^2$$

hence the holomorphic function $z^2\,f(z)$ is equal to a real constant. Since $f(z)$ cannot have a pole at the origin we have that $z^2 f(z)$ is identically equal to zero and hence $\tilde{u}$ is conformal. □

5 Variational Harmonic Maps and $1/2-$Harmonic Maps into Plane Distributions.

5.1 Variational Harmonic Maps into Plane Distributions

5.1.1 The 1-D Case

In this subsection we consider the well known case of critical points of the Dirichlet energy within the space

$$\mathfrak{H}^1(S^1) := \left\{ u \in H^1(S^1, \mathbb{R}^m) \quad ; \quad P_N(u)\frac{du}{d\theta} = 0 \quad \text{in } \mathcal{D}'(S^1) \right\}$$

We introduce the following Lagrangian defined on the Hilbert Space $H^1(S^1, \mathbb{R}^m) \times L^2(S^1, \mathbb{R}^m)$

$$\mathcal{L}^1(u, \xi) := \int_{S^1} \frac{\langle \xi, P_T(u)\xi \rangle}{2}\, d\theta - \int_{S^1} \xi \cdot \frac{du}{d\theta}\, d\theta \tag{83}$$

A point (u, ξ) is a critical point to $\mathcal{L}$ if and only if for any $(w, \eta) \in H^1(S^1, \mathbb{R}^m) \times L^2(S^1, \mathbb{R}^m)$ we have

$$\int_{S^1} \langle \eta, P_T(u)\, \xi \rangle\, d\theta - \int_{S^1} \eta \cdot \frac{du}{d\theta}\, d\theta + \int_{S^1} \frac{\langle \xi, d_w P_T(u)\xi \rangle}{2}\, d\theta - \int_{S^1} \xi \cdot \frac{dw}{d\theta}\, d\theta = 0 \tag{84}$$

This is equivalent to

$$\left\{ \begin{array}{l} \dfrac{du}{d\theta} = P_T(u)\, \xi \\[3mm] \dfrac{d\xi_k}{d\theta} = -\dfrac{1}{2} \langle \xi, \partial_{z_k} P_T(u)\xi \rangle \qquad \forall\, k = 1 \cdots m \end{array} \right. \tag{85}$$

This implies first that $d\xi/d\theta \in L^1(S^1)$ which gives that $\xi \in C^0(S^1)$. Hence we deduce that $(u, \xi) \in C^1(S^1) \times C^1(S^1)$. This is the case of normal geodesics in sub-riemannian geometric.

Assume that P_T is <u>integrable</u>, i.e. satisfies (10), then taking the θ derivative of the first equation of (85) gives

$$P_T(u)\, \frac{d^2u}{d\theta^2} = P_T(u)\, d_{\frac{du}{d\theta}} P_T(u)\, \xi + P_T(u)\, \frac{d\xi}{d\theta} \tag{86}$$

We have using the second equation of (85)

$$\begin{array}{l} \left\langle \varepsilon_l, P_T(u)\, \dfrac{d\xi}{d\theta} \right\rangle = \displaystyle\sum_{k=1}^m P_T^{lk}\, \frac{d\xi_k}{d\theta} = -\frac{1}{2} \sum_{i,j,k=1}^m P_T^{lk}\, \partial_{z_k} P_T^{ij}\, \xi_i\, \xi_j \\[3mm] = -\frac{1}{2} \sum_{i,j,k,s=1}^m P_T^{lk}\, \partial_{z_k} P_T^{is}\, P_T^{sj}\, \xi_i\, \xi_j - \frac{1}{2} \sum_{i,j,k,s=1}^m P_T^{lk}\, P_T^{is}\, \partial_{z_k} P_T^{sj}\, \xi_i\, \xi_j \\[3mm] = -\, \langle \xi, d_{P_T \varepsilon_l} P_T\ P_T \xi \rangle \end{array} \tag{87}$$

Combining the previous with Lemma A.1 gives

$$\left\langle \varepsilon_l, P_T(u)\,\frac{d\xi}{d\theta}\right\rangle = -\left\langle \xi, d_{P_T\xi}P_T\;P_T\varepsilon_l\right\rangle$$

Using the symmetry of the matrix $d_{P_T\xi}P_T$ and P_T we have

$$\left\langle \varepsilon_l, P_T(u)\,\frac{d\xi}{d\theta}\right\rangle = -\left\langle d_{P_T\xi}P_T\,\xi, P_T\varepsilon_l\right\rangle = -\left\langle \varepsilon_l, P_T\,d_{\frac{du}{d\theta}}P_T\,\xi\right\rangle$$

so in other words we have proved in the integrable case

$$P_T(u)\,\frac{d\xi}{d\theta} = -P_T(u)\,d_{\frac{du}{d\theta}}P_T(u)\,\xi. \tag{88}$$

Combining (86) and (88) we obtain

$$P_T(u)\,\frac{d^2u}{d\theta^2} = 0.$$

which is the well known harmonic map equation (16) for $\alpha = 1$.

5.2 2-Dimensional Variational Harmonic Maps into Plane Distributions.

Following the 1-dimensional case one can introduce for pairs $(u,\xi) \in W^{1,2}(D^2,\mathbb{R}^m) \times L^2(D^2,\mathbb{R}^2\otimes\mathbb{R}^m)$

$$\mathcal{L}(u,\xi) := \int_{D^2} \frac{\sum_{i=1}^2 \left\langle \xi^l, P_T(u)\xi^l\right\rangle}{2} - \sum_{i=1}^2 \xi^l\cdot\partial_{x_l}u\;dx^2$$

A pair (u,ξ) is a critical point of $\mathcal{L}$ if and only if

$$\begin{cases} \dfrac{\partial u}{\partial x_l} = P_T(u)\,\xi^l \\[2ex] \operatorname{div}\xi_k = -\dfrac{1}{2}\displaystyle\sum_{l=1}^2 \left\langle \xi^l, \partial_{z_k}P_T(u)\,\xi^l\right\rangle \qquad \forall\, k = 1\cdots m \end{cases} \tag{89}$$

where

$$\operatorname{div}\xi_k = \partial_{x_1}\xi_k^1 + \partial_{x_2}\xi_k^2.$$

Similarly as (87) we have

$$\left\langle \varepsilon_i, P_T(u)\operatorname{div}\xi\right\rangle = -\sum_{l=1}^2 \left\langle \xi^l, \partial_{P_T\varepsilon_i}P_T(u)\,P_T(u)\,\xi^l\right\rangle \qquad \forall\, k = 1\cdots m$$

Hence

$$\begin{aligned}\operatorname{div}(P_T(u)\xi)_i &= \sum_{l=1}^{2}\sum_{j=1}^{m} d_{\partial_{x_l}u}P_T^{ij}(u)\,\xi_j^l - \sum_{l=1}^{2}\sum_{j,k=1}^{m}\xi_j^l\ d_{P_T\varepsilon_i}P_T^{jk}(u)\,(P_T(u)\xi^l)_k\\ &= \sum_{l=1}^{2}\sum_{j,k=1}^{m}\xi_j^l\,\partial_{z_k}P_T^{ij}(u)\,(P_T(u)\xi^l)_k - \sum_{l=1}^{2}\sum_{j,k,s=1}^{m}\xi_j^l\ P_T^{is}\,\partial_{z_s}P_T^{jk}(u)\,(P_T(u)\xi^l)_k\\ &= \sum_{l=1}^{2}\sum_{j,k,s=1}^{m}\xi_j^l\,P_T^{ks}(u)\,\partial_{z_s}P_T^{ij}(u)\,(P_T(u)\xi^l)_k - \sum_{l=1}^{2}\sum_{j,k,s=1}^{m}\xi_j^l\ P_T^{is}(u)\,\partial_{z_s}P_T^{kj}(u)\,(P_T(u)\xi^l)_k\end{aligned}$$

Denote

$$\Omega_l^{ik} := \sum_{j,s=1}^{m}\xi_j^l\,P_T^{is}\,\partial_{z_s}P_T^{kj}(u) - \xi_j^l\ P_T^{ks}\,\partial_{z_s}P_T^{ij}(u)$$

We have by definition

$$\forall\, l=1,2\quad \forall\, i,k\in\{1\cdots m\}\qquad \Omega_l^{ik} = -\,\Omega_l^{ki}$$

Moreover $\Omega\in L^2$ and u satisfies the following system

$$-\,-\Delta u = \Omega\cdot\nabla u\quad \text{ in } D^2$$

Hence, using [14], we have that $\nabla u\in\cap_{p<2}W^{1,p}_{loc}(D^2)$.

5.3 Variational $1/2-$Harmonic Maps into Plane Distributions

On $H^s(S^1)$ ($s\in\mathbb{R}$ arbitrary) we define the following operator for any $\alpha\in[0,1]$

$$(-\Delta)_0^{-\alpha/2}\qquad f\in H^s(S^1)\longrightarrow v=(-\Delta)_0^{-\alpha/2}f\in H_0^{s+\alpha}(S^1)$$

where v satisfies

$$(-\Delta)^{\alpha/2}v = f - \frac{1}{2\pi}\int_{S^1} f$$

and is given explicitly by

$$v := \sum_{n\in\mathbb{Z}^*} f_n|n|^{-\alpha}\ e^{in\theta}\quad \text{where}\quad f=\sum_{n\in\mathbb{Z}} f_n\,e^{i\,n\,\theta}.$$

Observe that v satisfies

$$\int_{S^1} v(\theta)\ d\theta = 0.$$

Hence this gives in particular that

$$(-\Delta)_0^{-\alpha/2}\circ(-\Delta)_0^{-\beta/2}f = (-\Delta)_0^{-(\alpha+\beta)/2}\,f \tag{90}$$

We have also for any $f \in H^{-\alpha}(S^1,\mathbb{R}^m)$ and $g \in L^2(S^1,\mathbb{R}^m)$

$$\int_{S^1} (-\Delta)_0^{-\alpha/2} f(\theta)\ g(\theta)\ d\theta = (f, (-\Delta)_0^{-\alpha/2}\, g)_{H^{-\alpha},H^{\alpha}} \tag{91}$$

We introduce the following Lagrangian defined on the sub-manifold of the Hilbert Space $H^{1/2}(S^1,\mathbb{R}^m) \times H^{-1/2}(S^1,\mathbb{R}^m)$ given by

$$\mathfrak{E} := \left\{ \begin{array}{c} (u,\xi) \in H^{1/2}(S^1,\mathbb{R}^m) \times H^{-1/2}(S^1,\mathbb{R}^m) \quad \text{s. t.} \\ \\ \left(P_N(u), \dfrac{du}{d\theta}\right)_{H^{1/2},H^{-1/2}} = 0 \\ \\ (-\Delta)_0^{-1/4}(P_T(u)\xi) \in L^2(S^1) \quad \text{ and } \quad (-\Delta)_0^{-1/4}\left(P_T(u)\dfrac{du}{d\theta}\right) \in L^2(S^1) \end{array} \right\}$$

Let

$$\begin{aligned} \mathcal{L}^{1/2}(u,\xi) := \int_{S^1} \frac{|(-\Delta)_0^{-1/4}(P_T(u)\xi)|^2}{2}\ d\theta - \\ \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_T(u)\xi), (-\Delta)_0^{-1/4}\left(P_T(u)\frac{du}{d\theta}\right)\right\rangle\ d\theta \\ -\int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_N(u)\xi), (-\Delta)_0^{-1/4}\left(P_N(u)\frac{du}{d\theta}\right)\right\rangle\ d\theta \end{aligned}$$

Observe that if $u \in \mathfrak{H}^{1/2}$ we have $(u, du/d\theta) \in \mathfrak{E}$ and

$$\mathcal{L}^{1/2}\left(u, \frac{du}{d\theta}\right) = -\frac{1}{2}\int_{S^1} |(-\Delta)^{1/4}u|^2\ d\theta.$$

Assume now (u,ξ) is a critical point of $\mathcal{L}^{1/2}$ in $\mathfrak{E}$. Hence for any choice of $(w,\eta) \in C^\infty(S^1,\mathbb{R}^m) \times C^\infty(S^1,\mathbb{R}^m)$ where w satisfies the constraint

$$\int_{S^1} P_N(u)\,\frac{dw}{d\theta}\,d\theta + \left\langle d_w P_N(u)\,,\,\frac{du}{d\theta}\right\rangle_{H^{1/2},H^{-1/2}} = 0 \tag{92}$$

we have respectively

$$\begin{aligned} \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_T(u)\xi), (-\Delta)_0^{-1/4}(P_T(u)\eta)\right\rangle\ d\theta \\ -\int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_T(u)\eta), (-\Delta)_0^{-1/4}\left(P_T(u)\frac{du}{d\theta}\right)\right\rangle\ d\theta \\ -\int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_N(u)\eta), (-\Delta)_0^{-1/4}\left(P_N(u)\frac{du}{d\theta}\right)\right\rangle\ d\theta = 0 \end{aligned} \tag{93}$$

and

$$
\begin{aligned}
&\int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_T(u)\xi), (-\Delta)_0^{-1/4}(d_w P_T(u)\xi)\right\rangle \, d\theta \\
&\quad - \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(d_w P_T(u)\xi), (-\Delta)_0^{-1/4}\left(P_T(u)\frac{du}{d\theta}\right)\right\rangle \, d\theta \\
&\quad - \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_T(u)\xi), (-\Delta)_0^{-1/4}\left(d_w P_T(u)\frac{du}{d\theta}\right)\right\rangle \, d\theta \\
&\quad - \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(d_w P_N(u)\xi), (-\Delta)_0^{-1/4}\left(P_N(u)\frac{du}{d\theta}\right)\right\rangle \, d\theta \\
&\quad - \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_N(u)\xi), (-\Delta)_0^{-1/4}\left(d_w P_N(u)\frac{du}{d\theta}\right)\right\rangle \, d\theta \\
&\quad - \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_T(u)\xi), (-\Delta)_0^{-1/4}\left(P_T(u)\frac{dw}{d\theta}\right)\right\rangle \, d\theta \\
&\quad - \int_{S^1} \left\langle (-\Delta)_0^{-1/4}(P_N(u)\xi), (-\Delta)_0^{-1/4}\left(P_N(u)\frac{dw}{d\theta}\right)\right\rangle \, d\theta = 0
\end{aligned}
\tag{94}
$$

The first equation (94) implies using (90), (91) and the symmetry of the matrices P_T and P_N

$$
P_T(u)\,(-\Delta_0)^{-1/2}\left(P_T(u)\,\xi - P_T(u)\frac{du}{d\theta}\right) - P_N(u)\,(-\Delta_0)^{-1/2}\left(P_N(u)\frac{du}{d\theta}\right) = 0 \tag{95}
$$

This implies

$$
\left\{
\begin{array}{l}
P_T(u)\,(-\Delta_0)^{-1/2}\left(P_T(u)\,\xi - P_T(u)\dfrac{du}{d\theta}\right) = 0 \\[2ex]
P_N(u)\,(-\Delta_0)^{-1/2}\left(P_N(u)\dfrac{du}{d\theta}\right) = 0
\end{array}
\right.
\tag{96}
$$

Multiplying the second equation by $du/d\theta$ and integrating by parts gives

$$
\int_{S^1} \left|(-\Delta)_0^{-1/4}\left(P_N(u)\frac{du}{d\theta}\right)\right|^2 \, d\theta = 0
$$

this gives

$$
P_N(u)\frac{du}{d\theta} \equiv Cte \tag{97}
$$

Since the membership of u to $\mathfrak{E}$ imposes

$$
\left(P_N(u), \frac{du}{d\theta}\right)_{H^{1/2},H^{-1/2}} = 0
$$

Hence we have

$$P_N(u)\frac{du}{d\theta} \equiv 0 \tag{98}$$

or in other words $u \in \mathcal{H}$. Multiplying now the first equation by $\xi - du/d\theta$ and integrating by parts gives

$$P_T(u)\,\xi - \frac{du}{d\theta} \equiv Cte \tag{99}$$

With these informations at hand (94) becomes

$$\begin{aligned}
&-\int_{S^1} \left\langle \frac{dw}{d\theta}, P_T(u)\,(-\Delta)_0^{-1/2}\,(P_T(u)\,\xi) + P_N(u)\,(-\Delta)_0^{-1/2}\,(P_N(u)\,\xi)\right\rangle \\
&+\int_{S^1} \left\langle d_w P_T(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle\, d\theta \\
&-\int_{S^1} \left\langle d_w P_T(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_T(u)\,\xi)\right\rangle\, d\theta = 0
\end{aligned} \tag{100}$$

Combining (92) and (100) and assuming u is a non degenerate point of the constraint

$$\left(P_N(u), \frac{du}{d\theta}\right)_{H^{1/2},H^{-1/2}} = 0$$

we obtain the existence of $\lambda = (\lambda_1 \cdots \lambda_m) \in \mathbb{R}^m$ such that for any $k = 1 \cdots m$

$$\begin{aligned}
&\frac{d}{d\theta}\left(P_T(u)\,(-\Delta)_0^{-1/2}\,(P_T(u)\,\xi) + P_N(u)\,(-\Delta)_0^{-1/2}\,(P_N(u)\,\xi)\right)^k = \\
&\quad -\left\langle \partial_{z_k} P_T(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle + \left\langle \partial_{z_k} P_T(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_T(u)\,\xi)\right\rangle \\
&\quad +\left\langle \lambda, \partial_{z_k} P_T(u)\,\frac{du}{d\theta} - \partial_{\frac{du}{d\theta}} P_T(u)\,\varepsilon_k\right\rangle
\end{aligned} \tag{101}$$

Assume that P_T is <u>integrable</u>, i.e. satisfies (10). Taking the multiplication of (101) with $P_T(u)$ gives

$$
\begin{aligned}
&\langle \varepsilon_i, P_T(u)\,\mathcal{R}\,P_T(u)\,\xi\rangle \\
&+\left\langle \varepsilon_i, P_T(u)\,\frac{dP_T(u)}{d\theta}\,(-\Delta)_0^{-1/2}\,(P_T(u)\,\xi) + P_T(u)\,\frac{dP_N(u)}{d\theta}\,(-\Delta)_0^{-1/2}\,(P_N(u)\,\xi)\right\rangle \\
&= -\left\langle \partial_{P_T(u)\,\varepsilon_i} P_T(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle \qquad (102)\\
&\quad+\left\langle \partial_{P_T(u)\,\varepsilon_i} P_T(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_T(u)\,\xi)\right\rangle \\
&\quad+\left\langle \lambda, \partial_{P_T(u)\,\varepsilon_i} P_T(u)\,\frac{du}{d\theta} - \partial_{\frac{du}{d\theta}} P_T(u)\,P_T(u)\,\varepsilon_i\right\rangle
\end{aligned}
$$

where $\mathcal{R}$ is the Riesz operator given by

$$
\mathcal{R}\ :\ f = \sum_{n\in\mathbb{Z}} f_n\, e^{i\,n\,\theta}\ \longrightarrow\ \mathcal{R}f := i\sum_{n\in\mathbb{Z}^*}\operatorname{sgn}(n)\, f_n\, e^{i\,n\,\theta}
$$

Since $P_N(u)\frac{du}{d\theta} = 0$, on can use lemma A.1 in order to infer

$$
\left\langle \lambda, \partial_{P_T(u)\,\varepsilon_i} P_T(u)\,\frac{du}{d\theta} - \partial_{\frac{du}{d\theta}} P_T(u)\,P_T(u)\,\varepsilon_i\right\rangle = 0 \qquad (103)
$$

moreover, using again lemma A.1, the symmetry of the matrices $dP_N(u)/d\theta$ and $P_T(u)$, we obtain

$$
\begin{aligned}
&\left\langle \partial_{P_T(u)\,\varepsilon_i} P_T(u)\,\tfrac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle = \left\langle \partial_{\frac{du}{d\theta}} P_T(u)\,P_T(u)\,\varepsilon_i, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle \\
&= -\left\langle \partial_{\frac{du}{d\theta}} P_N(u)\,P_T(u)\,\varepsilon_i, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle \\
&= -\left\langle \frac{dP_N(u)}{d\theta}\,P_T(u)\,\varepsilon_i, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle \\
&= -\left\langle P_T(u)\,\varepsilon_i, \frac{dP_N(u)}{d\theta}\,(-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle \\
&= -\left\langle \varepsilon_i, P_T(u)\frac{dP_N(u)}{d\theta}\,(-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle \qquad (104)
\end{aligned}
$$

and similarly we have

$$\left\langle\partial_{P_T(u)\,\varepsilon_i} P_T(u)\,\tfrac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_T(u)\,\xi)\right\rangle = \left\langle\partial_{\frac{du}{d\theta}} P_T(u)\,P_T(u)\,\varepsilon_i, (-\Delta_0)^{-1/2}(P_T(u)\,\xi)\right\rangle$$

$$= \left\langle\varepsilon_i, P_T(u)\frac{dP_T(u)}{d\theta}\,(-\Delta_0)^{-1/2}(P_T(u)\,\xi)\right\rangle \tag{105}$$

Combining (102)...(105) we obtain

$$0 = P_T(u)\,\mathcal{R}\,P_T(u)\,\xi = P_T(u)\,\mathcal{R}\,\tfrac{du}{d\theta} = P_T(u)\,(-\Delta)^{1/2}u \tag{106}$$

which is exactly the $1/2-$harmonic map equation.

In fact the correspondence between critical points of $\mathcal{L}^{1/2}$ and critical points of the $1/2-$energy within $\mathfrak{H}^{1/2}$ goes beyond the very special case of integrable plane distributions. Precisely we have the following theorem.

Theorem 5.1. *Let (u,ξ) be a smooth critical point of $\mathcal{L}^{1/2}$ in $\mathfrak{E}$ then u is a critical point of*

$$E^{1/2}(u) = \int_{S^1} |(-\Delta)^{1/4}u|^2\ d\theta$$

within the space $\mathfrak{H}^{1/2}$ of horizontal $H^{1/2}-$maps. □

Proof of theorem 5.1. Since u is assumed to be smooth we can make use locally of an orthonormal frame $e_1\cdots e_n$ generating the plane distribution given by the Images of P_T. With this frame at hand we can introduce the *control* $\alpha_1(\theta),\cdots,\alpha_n(\theta)$ such that

$$\frac{du}{d\theta} = \sum_{i=1}^n \alpha_i(\theta)\,e_i(u(\theta)) \tag{107}$$

Classical considerations from control theory in sub-riemannian framework (see for instance [13]) asserts that an infinitesimal variation of an horizontal map satisfying (107) is given by w satisfying

$$\frac{dw}{d\theta} = \sum_{i=1}^n v_i(\theta)\ e_i(u(\theta)) + \sum_{i=1}^n \alpha_i(\theta)\ d_w e_i(u(\theta)). \tag{108}$$

where the $v_i(\theta)$ are arbitrary so that the constraint (92) is satisfied. Since $P_T\,P_N = 0$, we have

$$d_wP_T(u)\,P_N + P_T(u)\,d_wP_N(u) = 0$$

Hence this implies, using that $d_wP_T = -\,d_wP_N$,

$$P_T(u)\,d_wP_T(u)\,P_T(u) = -P_T(u)\,d_wP_N(u)\,P_T(u) = d_wP_T(u)\,P_N(u)\,P_T(u) = 0 \tag{109}$$

Hence

$$\begin{aligned} P_T(u)\, d_w P_T(u)\, \frac{du}{d\theta} &= P_T(u)\, d_w P_T(u)\, P_T(u)\, \frac{du}{d\theta} = 0 \\ \Longrightarrow \quad d_w P_T(u)\, \frac{du}{d\theta} &= P_N(u)\, d_w P_T(u)\, \frac{du}{d\theta}. \end{aligned} \tag{110}$$

Since

$$P_T := \sum_{i=1}^{n} e_i \otimes e_i$$

We have that

$$d_w P_T(u)\, \frac{du}{d\theta} = \sum_{i,j=1}^{n} \alpha_j\ d_w e_i \cdot e_j\ e_i + \sum_{i=1}^{n} \alpha_i\ d_w e_i. \tag{111}$$

Combining (108), (110) and (111) we obtain

$$d_w P_T(u)\, \frac{du}{d\theta} = \sum_{i=1}^{n} \alpha_i\ P_N(u)\, d_w e_i = P_N\, \frac{dw}{d\theta}. \tag{112}$$

Inserting this identity in (100)

$$\begin{aligned} &-\int_{S^1} \left\langle \frac{dw}{d\theta}, P_T(u)\, (-\Delta)_0^{-1/2}\, (P_T(u)\, \xi) + P_N(u)\, (-\Delta)_0^{-1/2}\, (P_N(u)\, \xi) \right\rangle\ d\theta \\ &+\int_{S^1} \left\langle P_N\, \frac{dw}{d\theta}, (-\Delta_0)^{-1/2} (P_N(u)\, \xi) \right\rangle\ d\theta \\ &-\int_{S^1} \left\langle P_N\, \frac{dw}{d\theta}, (-\Delta_0)^{-1/2} (P_T(u)\, \xi) \right\rangle\ d\theta = 0 \end{aligned} \tag{113}$$

which is equivalent to

$$\int_{S^1} \left\langle \frac{dw}{d\theta}, (-\Delta)_0^{-1/2} \left(\frac{du}{d\theta} \right) \right\rangle\ d\theta = 0 \tag{114}$$

Since this holds for any perturbation w of u in $\mathfrak{H}^{1/2}$, we have proved the theorem. □

5.4 Reformulation of the Euler-Lagrange Equation

Observe that (101) becomes

$$
\begin{aligned}
&((P_T(u)\,\mathcal{R}\,P_T(u) + P_N(u)\,\mathcal{R}\,P_N(u))\,\xi)^k \\
&+\left(\frac{dP_T(u)}{d\theta}\left((-\Delta)_0^{-1/2}\,(P_T(u)\,\xi)\right) + \frac{dP_N(u)}{d\theta}\,(-\Delta)_0^{-1/2}\,(P_N(u)\,\xi)\right)^k = \\
&\quad+\left\langle \partial_{z_k}P_N(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_N(u)\,\xi)\right\rangle + \left\langle \partial_{z_k}P_T(u)\,\frac{du}{d\theta}, (-\Delta_0)^{-1/2}(P_T(u)\,\xi)\right\rangle \\
&\quad+\left\langle \lambda, \partial_{z_k}P_T(u)\,\frac{du}{d\theta} - \partial_{\frac{du}{d\theta}}P_T(u)\,\varepsilon_k\right\rangle
\end{aligned}
\tag{115}
$$

This gives

$$
\begin{aligned}
&((P_T(u)\,\mathcal{R}\,P_T(u) + P_N(u)\,\mathcal{R}\,P_N(u))\,\xi)^k = \\
&\sum_{j=1}^m\left(\sum_{i=1}^m(\partial_{z_k}P_N^{ij} - \partial_{z_j}P_N^{ik})\,(-\Delta_0)^{-1/2}(P_N(u)\,\xi)^i + (\partial_{z_k}P_T^{ij} - \partial_{z_j}P_T^{ik})\,(-\Delta_0)^{-1/2}(P_T(u)\,\xi)^i\right)\frac{du^j}{d\theta} \\
&+\sum_{j=1}^m\left(\sum_{i=1}^m(\partial_{z_k}P_T^{ij} - \partial_{z_j}P_T^{ik})\,\lambda^i\right)\frac{du^j}{d\theta}
\end{aligned}
\tag{116}
$$

Denote

$$
\begin{aligned}
\omega^{kj} &:= \left(\sum_{i=1}^m(\partial_{z_k}P_N^{ij} - \partial_{z_j}P_N^{ik})\,(-\Delta_0)^{-1/2}(P_N(u)\,\xi)^i + (\partial_{z_k}P_T^{ij} - \partial_{z_j}P_T^{ik})\,((-\Delta_0)^{-1/2}(P_T(u)\,\xi)^i + \lambda^i)\right) \\
&= \left(\sum_{i=1}^m(\partial_{z_k}P_N^{ij} - \partial_{z_j}P_N^{ik})\,(-\Delta_0)^{-1/2}\xi^i + (\partial_{z_k}P_T^{ij} - \partial_{z_j}P_T^{ik})\,\lambda^i\right)
\end{aligned}
$$

Observe that ω is antisymmetric and the equation becomes

$$
(P_T\,\mathcal{R}\,P_T + P_N\,\mathcal{R}\,P_N)\,\xi = \omega\,P_T\,\xi \tag{117}
$$

Let

$$
v = \begin{pmatrix} P_T\,\xi \\ \mathcal{R}\,P_N\,\xi \end{pmatrix}
$$

Observe that

$$
(P_T\mathcal{R} + P_N)\,v = \begin{pmatrix} P_T\,\mathcal{R}\,P_T\,\xi \\ P_N\,\mathcal{R}\,P_N\,\xi \end{pmatrix} \tag{118}
$$

If one multiplies (117) by $P_T(u)$ one gets that $w := P_T(u)\xi$ satisfies

$$\left\{\begin{array}{l} P_T\,\mathcal{R}\,w = \Omega\,w \\ \\ P_N\,w = 0 \end{array}\right. \tag{119}$$

where $\Omega := P_T\,\omega\,P_T$ is <u>antisymmetric</u> which is a "deformation" of the $1/2-$harmonic equation

$$\left\{\begin{array}{l} P_T\,\mathcal{R}\,w = 0 \\ \\ P_N\,w = 0 \end{array}\right. \tag{120}$$

where $w := du/d\theta$.

A Appendix

A.1 Integrable Distributions

The goal of the present section is to establish the following elementary lemma which is well known.

Lemma A.1. *Let P_T be a C^1 plane distribution satisfying (7) and assume P_T is integrable, i.e. satisfies (10), then*

$$\forall\ X,Y\in C^1(\mathbb{R}^m,\mathbb{R}^m)\qquad \textit{we have}\quad d_{P_TX}P_T\ P_TY = d_{P_TY}P_T\ P_TX \tag{A.1}$$

or in other words

$$\forall\ i\,j\,k\in\{1\cdots m\}\qquad \sum_{s,t=1}^m \partial_{z_t}P_T^{is}\ P_T^{sk}\ P_T^{tj} = \sum_{s,t=1}^m \partial_{z_t}P_T^{is}\ P_T^{sj}\ P_T^{tk} \tag{A.2}$$

□

Proof of lemma A.1. Let $(\varepsilon_i)_{i=1\cdots m}$ be the canonical basis of $\mathbb{R}^m$. We have

$$[P_T\,\varepsilon_j, P_T\,\varepsilon_k] = \sum_{s,t=1}^m \left(P_T^{tj}\,\partial_{z_t}P_T^{sk} - P_T^{tk}\,\partial_{z_t}P_T^{sj}\right)\ \varepsilon_s$$

Equation (10) becomes

$$\forall\,i,j,k\qquad \sum_{s,t=1}^m(\delta^{is}-P_T^{is})\ \left(P_T^{tj}\,\partial_{z_t}P_T^{sk} - P_T^{tk}\,\partial_{z_t}P_T^{sj}\right) = 0$$

which gives

$$0=\sum_{t=1}^{m} P_T^{tj}\,\partial_{z_t}P_T^{ik}-P_T^{tk}\,\partial_{z_t}P_T^{ij}-\sum_{s,t=1}^{m} P_T^{tj}\,P_T^{is}\,\partial_{z_t}P_T^{sk}-P_T^{tk}\,P_T^{is}\,\partial_{z_t}P_T^{sj} \tag{A.3}$$

Using the fact that $P_T\circ P_T=P_T$ we have

$$-\sum_{s=1}^{m} P_T^{is}\,\partial_{z_t}P_T^{sk}=-\partial_{z_t}P_T^{ik}+\sum_{s=1}^{m}\partial_{z_t}P_T^{is}\,P_T^{sk} \tag{A.4}$$

Combining (A.3) and (A.4) gives then (A.2) and lemma A.1 is proved. □

A.2 Rewriting the Commutators

In this section we recall the explicit form of the matrices Ω_0,Ω_1 and of the operator $\mathcal{Z}$ introduced in (61) in the case of 1/2-harmonic maps.

We introduce some notations: for $Q\in H^{1/2}\cap L^\infty$ we denote by

$$(-\Delta)^{1/4}\{Q\},\quad \mathcal{R}\circ(-\Delta)^{1/4}\{Q\},\quad (-\Delta)^{1/4}\{Q\circ\mathcal{R}\}$$

the pseudo-differential operators given respectively by the laws :

$$v\mapsto(-\Delta)^{1/4}[Qv],\quad v\mapsto\mathcal{R}(-\Delta)^{1/4}[Qv],\quad v\mapsto(-\Delta)^{1/4}[Q\mathcal{R}[v]]$$

for $v\in L^2$.

Proposition A.1. *Let $u\in\dot H^{1/2}(\mathbb{R},\mathcal{N})$ be a weak 1/2-harmonic map. Then the following equation holds*

$$\begin{aligned}\Delta^{1/4}v=(-\Delta)^{1/4}\begin{pmatrix}P_T(-\Delta)^{1/4}u\\ \mathcal{R}P_N(-\Delta)^{1/4}u\end{pmatrix}&=\tilde\Omega+\Omega_1\begin{pmatrix}P_T(-\Delta)^{1/4}u\\ \mathcal{R}P_N(-\Delta)^{1/4}u\end{pmatrix}\\&+\Omega\begin{pmatrix}P_T(-\Delta)^{1/4}u\\ \mathcal{R}P_N(-\Delta)^{1/4}u\end{pmatrix},\end{aligned} \tag{A.5}$$

where $\Omega=\Omega(P_T)\in L^2(\mathbb{R},so(2m))$, $\Omega_1=\Omega_1(P_T)\in L^{2,1}$ with

$$\|\Omega\|_{L^2},\|\Omega_1\|_{L^{2,1}}\le C(\|P_T\|_{H^{1/2}}+\|P_T\|^2_{H^{1/2}}),$$

and

$$\tilde\Omega=(C-2D)\begin{pmatrix}P_T(-\Delta)^{1/4}u\\ \mathcal{R}P_N(-\Delta)^{1/4}u\end{pmatrix} \tag{A.6}$$

where the matrices C and D are $2\times 2m$ matrices whose components are made by pseudo-differential operators: for $j\in\{1,\dots,m\}$

$$
\begin{aligned}
c_{1j} &= (-\Delta)^{1/4}\{P_T\} - P_T(-\Delta)^{1/4} + (-\Delta)^{1/4}[P_T] \\
c_{1,j+m} &= (-\Delta)^{1/4}\{P_T\circ\mathcal{R}\} - P_T(-\Delta)^{1/4}\circ\mathcal{R} + (-\Delta)^{1/4}[P_T]\circ\mathcal{R} \\
c_{2,j} &= \mathcal{R}(-\Delta)^{1/4}\{P_N\} - P_N(-\Delta)^{1/4}\circ\mathcal{R} - (-\Delta)^{1/4}[P_N]\circ\mathcal{R} \\
c_{2,j+m} &= (-\Delta)^{1/4}\{P_N\} + P_N(-\Delta)^{1/4} - (-\Delta)^{1/4}[P_N]\,.
\end{aligned}
\tag{A.7}
$$

and

$$
\begin{aligned}
d_{1,j} &= 0 \\
d_{1,j+m} &= \mathcal{R}[\omega_1] + \omega_1\mathcal{R} \\
d_{2,j} &= (-\Delta)^{1/4}P_N + \mathcal{R}((-\Delta)^{1/4}P_N)\mathcal{R} \\
d_{2,j+m} &= [\mathcal{R}[\omega_2] + \omega_2\mathcal{R}] - ((-\Delta)^{1/4}P_N)\mathcal{R} - \mathcal{R}((-\Delta)^{1/4}P_N).
\end{aligned}
\tag{A.8}
$$

Moreover for $w\in L^2$,

$$
\|c_{ij}(w)\|_{\mathcal{H}^1(\mathbb{R})}, \|d_{ij}(w)\|_{\mathcal{H}^1(\mathbb{R})} \le C[\|P_T\|_{\dot H^{1/2}(\mathbb{R}))} + \|P_T\|^2_{\dot H^{1/2}(\mathbb{R}))}]\|w\|_{L^2}\,. \tag{A.9}
$$

In order to prove Proposition A.1 we recall the following Proposition (Proposition 1.1 in [6]).

Proposition A.2. *Let $u\in \dot H^{1/2}(\mathbb{R},\mathcal{N})$ be a weak $1/2$-harmonic map. Then the following equation holds*

$$
\begin{aligned}
\Delta^{1/4}v = (-\Delta)^{1/4}\begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix} &= \tilde\Omega + \Omega_1\begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix} \\
&+ \Omega\begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix},
\end{aligned}
\tag{A.10}
$$

where $\Omega=\Omega(P_T)\in L^2(\mathbb{R},so(2m))$, $\Omega_1=\Omega_1(P_T)\in L^{2,1}$ with

$$
\|\Omega\|_{L^2}, \|\Omega_1\|_{L^{2,1}} \le C(\|P_T\|_{H^{1/2}} + \|P_T\|^2_{H^{1/2}}),
$$

$$
\tilde\Omega = \begin{pmatrix} -2F(\omega_1(P_T),(P_N\Delta^{1/4}u)) + T(P_T,u) \\ -2F(\mathcal{R}((-\Delta)^{1/4}P_N),\mathcal{R}((-\Delta)^{1/4}u)) - 2F(\omega_2(P_T),P_N((-\Delta)^{1/4}u) + \mathcal{R}(S(P_N,u)) \end{pmatrix}
$$

$\omega_1(P_T),\omega_2(P_T)\in L^2$ and

$$
\|\omega_1(P_T)\|_{L^2},\ \|\omega_2(P_T)\|_{L^2} \le C(\|P_T\|_{H^{1/2}} + \|P_T\|^2_{H^{1/2}}).
$$

Proof of Proposition A.1. We next rewrite the matrix $\tilde{\Omega}$ as the product of a matrix of pseudodifferential operators times $v = (P_T(-\Delta)^{1/4}u, \mathcal{R}P_N(-\Delta)^{1/4}u)^t$.

Step 1. We rewrite $T(P_T, (-\Delta)^{1/4}u)$ and $\mathcal{R}(S(P_N, (-\Delta)^{1/4}u))$ in terms of $P_T(-\Delta)^{1/4}u$ and $\mathcal{R}P_N(-\Delta)^{1/4}u$. We observe that

$$(-\Delta)^{1/4}u = P_N(-\Delta)^{1/4}u - \mathcal{R}[\mathcal{R}P_N(-\Delta)^{1/4}u)].$$

By linearity we get

$$T(P_T, (-\Delta)^{1/4}u) = T_1(P_T, P_T(-\Delta)^{1/4}u) + T_2(P_T, \mathcal{R}P_N(-\Delta)^{1/4}u). \tag{A.11}$$

where for $Q \in H^{1/2}$, $v_1, v_2 \in L^2$ it holds

$$\begin{aligned} T_1(Q, v_1) \;&:=\; (-\Delta)^{1/4}[Qv_1] \\ &\quad -Q(-\Delta)^{1/4}[v_1] + (-\Delta)^{1/4}[Q]v_1. \end{aligned} \tag{A.12}$$

$$\begin{aligned} T_2(Q, v_2) \;&:=\; -(-\Delta)^{1/4}[Q\mathcal{R}[v_2]] \\ &\quad +Q(-\Delta)^{1/4}[\mathcal{R}v_2] - (-\Delta)^{1/4}[Q]\mathcal{R}v_2. \end{aligned} \tag{A.13}$$

Moreover

$$\mathcal{R}[S(P_N, (-\Delta)^{1/4}u)] = S_1(P_N, P_T(-\Delta)^{1/4}u) + S_2(P_N, \mathcal{R}P_N(-\Delta)^{1/4}u). \tag{A.14}$$

$$\begin{aligned} S_1(Q, v_1) \;&:=\; \mathcal{R}(-\Delta)^{1/4}[Qv_1] \\ &\quad Q(-\Delta)^{1/4}\mathcal{R}[v_1] - (-\Delta)^{1/4}[Q]\mathcal{R}[v_1]. \end{aligned} \tag{A.15}$$

$$\begin{aligned} S_2(Q, v_2) \;&:=\; (-\Delta)^{1/4}[Qv_2] \\ &\quad +Q(-\Delta)^{1/4}[v_2] - (-\Delta)^{1/4}[Q]v_2. \end{aligned} \tag{A.16}$$

We write

$$\begin{pmatrix} T(P_T, (-\Delta)^{1/4}u) \\ \mathcal{R}[S(P_N, (-\Delta)^{1/4}u)] \end{pmatrix} = C \begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix} \tag{A.17}$$

where the matrix C is a $2 \times 2m$ matrix whose components are made by pseudo-differential operators: for $j \in \{1, \dots, m\}$

$$\begin{aligned} c_{1j} &= (-\Delta)^{1/4}\{P_T\} - P_T(-\Delta)^{1/4} + (-\Delta)^{1/4}[P_T] \\ c_{1,j+m} &= (-\Delta)^{1/4}\{P_T \circ \mathcal{R}\} - P_T(-\Delta)^{1/4} \circ \mathcal{R} + (-\Delta)^{1/4}[P_T] \circ \mathcal{R} \\ c_{2,j} &= \mathcal{R}(-\Delta)^{1/4}\{P_N\} - P_N(-\Delta)^{1/4} \circ \mathcal{R} - (-\Delta)^{1/4}[P_N] \circ \mathcal{R} \\ c_{2,j+m} &= (-\Delta)^{1/4}\{P_N\} + P_N(-\Delta)^{1/4} - (-\Delta)^{1/4}[P_N]\,. \end{aligned} \tag{A.18}$$

From (40) and (41) it follows that for $v \in L^2$ the following estimate holds

$$\|c_{ij}(v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C \, \|P_T\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}.$$

Step 2. Now we rewrite the following matrix

$$\begin{pmatrix} F_1 \\ F_2 + F_3 \end{pmatrix} \tag{A.19}$$

where

$$\begin{aligned} F_1 := F(\omega_1, P_N \Delta^{1/4} u) &= -F(\omega, \mathcal{R}[\mathcal{R}[P_N \Delta^{1/4} u]]) \\ &= \mathcal{R}[\omega_1]\mathcal{R}[P_N(-\Delta)^{1/4} u] + \omega_1 \mathcal{R}[\mathcal{R}[P_N \Delta^{1/4} u]], \end{aligned} \tag{A.20}$$

$$\begin{aligned} F_2 &:= F(\omega_2, P_N((-\Delta)^{1/4} u) = -F(\omega_2, \mathcal{R}[\mathcal{R}[P_N \Delta^{1/4} u]]) \\ &= \mathcal{R}[\omega_2]\mathcal{R}[P_N(-\Delta)^{1/4} u] + \omega_2 \mathcal{R}[\mathcal{R}[P_N \Delta^{1/4} u]]. \end{aligned} \tag{A.21}$$

and

$$\begin{aligned} & F_3 := F(\mathcal{R}((-\Delta)^{1/4} P_N), \mathcal{R}((-\Delta)^{1/4} u)) \\ = \; & \underbrace{F(\mathcal{R}((-\Delta)^{1/4} P_N), \mathcal{R} P_T((-\Delta)^{1/4} u))}_{(1)} + \underbrace{F(\mathcal{R}((-\Delta)^{1/4} P_N), \mathcal{R} P_N((-\Delta)^{1/4} u))}_{(2)}. \end{aligned} \tag{A.22}$$

We have

$$(1) \;=\; (-\Delta)^{1/4} P_N P_T((-\Delta)^{1/4} u)) - \mathcal{R}((-\Delta)^{1/4} P_N)\mathcal{R} P_T((-\Delta)^{1/4} u)$$

$$(2) \;=\; -((-\Delta)^{1/4} P_N)\mathcal{R}(\mathcal{R} P_N((-\Delta)^{1/4} u))) - \mathcal{R}((-\Delta)^{1/4} P_N)\mathcal{R} P_N((-\Delta)^{1/4} u)).$$

By using (A.20), (A.21), (A.22) we can rewrite (A.19) in terms of a matrix of pseudo differential operators.

$$\begin{pmatrix} F_1 \\ F_2 + F_3 \end{pmatrix} = D \begin{pmatrix} P_T(-\Delta)^{1/4} u \\ \mathcal{R} P_N(-\Delta)^{1/4} u \end{pmatrix} \tag{A.23}$$

where C is a matrix of pseudo differential operators given by

$$\begin{aligned} d_{1,j} &= 0 \\ d_{1,j+m} &= \mathcal{R}[\omega_1] + \omega_1 \mathcal{R} \\ d_{2,j} &= (-\Delta)^{1/4} P_N + \mathcal{R}((-\Delta)^{1/4} P_N)\mathcal{R} \\ d_{2,j+m} &= [\mathcal{R}[\omega_2] + \omega_2 \mathcal{R}] - ((-\Delta)^{1/4} P_N)\mathcal{R} - \mathcal{R}((-\Delta)^{1/4} P_N). \end{aligned} \tag{A.24}$$

Moreover for $v \in L^2$,

$$\|d_{ij}(v)\|_{\mathcal{H}^1(\mathbb{R})} \le C \left[\|P_T\|_{\dot{H}^{1/2}(\mathbb{R}))} + \|P_T\|^2_{\dot{H}^{1/2}(\mathbb{R}))}\right] \|v\|_{L^2} \,. \tag{A.25}$$

By combining (A.17) and (A.23) we get

$$\tilde{\Omega} = (C - 2D) \begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix}. \tag{A.26}$$

We conclude the proof of Proposition A.1. □

References

[1] Coifman, R. R.; Rochberg, R.; Weiss, G. *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) 103 (1976), no. 3, 611-635.

[2] Da Lio, F. *Fractional Harmonic Maps into Manifolds in odd dimension* $n > 1$, Calculus of Variations and PDEs, Volume 48, Issue 3-4, (2013), 421-445.

[3] Da Lio, F. *Compactness and Bubbles Analysis for Half-Harmonic Maps into Spheres*, arXiv:1210.2653, Annal,es de l'Institut Henri Poincaré / Analyse non linéaire 32 (2015), pp. 201-224.

[4] Da Lio F., L. Martinazzi L, Rivière T, *Blow-up analysis of a nonlocal Liouville-type equation*, arXiv:1503.08701, APDE Vol.8, no 7 (2015), 1757-1805.

[5] Da Lio,F. & Rivière,T. *Three-term commutator estimates and the regularity of 1/2-harmonic maps into spheres,* Anal. PDE 4 (2011), no. 1, 149190.

[6] Da Lio, F.& Rivière,T. *Sub-criticality of non-local Schrödinger systems with antisymmetric potentials and applications to half-harmonic maps*, Advances in Mathematics 227, (2011), 1300-1348.

[7] Da Lio, F. & Schikorra, A. *(n,p)-harmonic maps: regularity for the sphere case*, arXiv:1202.1151v1, Advances in Calculus of Variations, DOI 10.1515/acv-2012-0107 .

[8] Fraser A. & Schoen, R. *Uniqueness theorems for free boundary minimal disks in space forms*, arXiv:1409.1632 .

[9] Haskins, M., *Special Lagrangian cones*, Amer. J. Math. 126 (2004), no. 4, 845-871.

[10] Jost, J.& Yang, Yi-Hu *Heat flow for horizontal harmonic maps into a class of Carnot-Caratheodory spaces.* Math. Res. Lett. 12 (2005), no. 4, 513529.

[11] Lang, S. *Fundamentals of differential geometry*, Springer-Verlag (2001).

[12] Millot,V. & Sire, Y. *On a Fractional Ginzburg-Landau Equation and* 1/2*-Harmonic Maps into Spheres*, Archive for Rational Mechanics and Analysis 215, Issue 1, (2015), pp 125-210.

[13] Montgomery, R. *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, 2002.

[14] Rivière, T. *Conservation laws for conformally invariant variational problems*, Invent. Math. 168 (2007), no. 1, 1–22.

[15] Rivière, T. *Conformally invariant Variational Problems*, book in preparation (2016).

[16] Scheven, A. *Partial regularity forstationary harmonic maps at a free boundary*, Math. Z. 253 (2006), , 135–157.

[17] Schikorra A. *Regularity of n/2 harmonic maps into spheres* , Journal of Differential Equations 252 (2012) 1862–1911.

[18] Schikorra A. *epsilon-regularity for systems involving non-local, antisymmetric operators*, Calc. Var. P.D.E. 54(4), (2015), 3531-3570.

[19] Strichartz, R. *Sub-Riemannian geometry*, J. Differential Geom. 24 (1986), no. 2, 221-263.

[20] Struwe, M. *Variational methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems.* Sriinger-Verlag, (2008).