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Towards single-valued polylogarithms in two variables for the seven-point remainder function in multi-Regge kinematics

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Abstract

We investigate single-valued polylogarithms in two complex variables, which are relevant for the seven-point remainder function in $\mathcal{N}=4$ super-Yang–Mills theory in the multi-Regge regime. After constructing these two-dimensional polylogarithms, we determine the leading logarithmic approximation of the seven-point remainder function up to and including five loops.

1. Introduction

One of the most interesting results in the study of scattering amplitudes in planar $\mathcal{N}=4$ super-Yang–Mills theory is that certain amplitudes can be constructed to high loop orders solely from understanding the space of functions describing the amplitude, its symmetries as well as its

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limiting behavior in special kinematic regimes. Indeed, following this program of bootstrapping the six-point amplitude, the remainder function is by now known up to four loops [1–4] and it is very likely that this program can be continued to higher orders.

Much less, however, is known about the seven-point amplitude. The MHV remainder function in general kinematics has been calculated up to two loops [5], while the symbol is known up to three loops [6,7]. It is therefore a sensible idea to consider specific kinematic configurations, hoping to obtain higher-order results in those special settings which can then be used as constraints on a potential ansatz for the full seven-point remainder function.

The special kinematic configuration considered in this paper is the multi-Regge limit. This limit has been studied in the seven-point case before: the Mandelstam regions have been classified [8,9], and the remainder function has been calculated in the most interesting Mandelstam region in the leading logarithmic approximation (LLA) [10]. Furthermore, the seven-point remainder function has been studied at strong coupling [11,12] as well as from the perspective of the symbol for two [13] and three loops [14].

In this paper, we follow the path of understanding for the six-point remainder function in the multi-Regge limit: while first expressed in Fourier–Mellin space up to next-to-leading logarithmic order (NLLA) [15–17], the identification of the relevant space of functions paved the way for an efficient evaluation of the remainder function in momentum space [18–20]. In the six-point case, these functions are single-valued harmonic polylogarithms (SVHPLs) which we will briefly review below.

Similarly, in ref. [10], the seven-point MHV remainder function was written down in Fourier–Mellin space. Due to the complicated nature of the integral, it was so far only evaluated up to two loops. In this paper we therefore set out to identify a suitable two-variable generalization of SVHPLs constituting the relevant space of functions in the seven-point case. We construct those functions from their differential behavior, which allows us to obtain results up to five loops.

The paper is organized as follows. In section 2 we review the six-point case and the construction of the SVHPLs describing the remainder function. In subsection 3.1 we then move on to seven gluons and highlight the differences to the six-point case and the two-dimensional generalization of HPLs in subsection 3.2 before constructing the two-dimensional analogue of SVHPLs in subsection 3.3. Using these functions, we obtain expressions for the remainder function in LLA up to five loops in subsection 3.4, before concluding in section 4.

2. Six-point remainder function in multi-Regge kinematics

2.1. Starting point/setup

The six-point remainder function $R_6^{\rm MHV}$ in $\mathcal{N}=4$ super-Yang–Mills theory[1] describes the discrepancy between the full amplitude and the BDS ansatz [21],

$$A_6^{\rm MHV} = A_{\rm BDS}\, e^{R_6^{\rm MHV}}\,. \tag{2.1}$$

While the calculation of the six-point remainder function in general kinematics requires a multitude of different techniques, its determination in the so-called multi-Regge kinematics is simpler. The multi-Regge limit refers to the kinematical regime of the scattering amplitude in which the

[1] For simplicity of presentation, we confine ourselves to the MHV remainder function; the six-point remainder function for NMHV is described and evaluated in refs. [18,3,4,20].

rapidities of the outgoing particles are strongly ordered. In terms of external momenta k_i, the multi-Regge limit can be most easily described using dual variables $x_{ii+1} := x_i - x_{i+1} := k_i$. Expressed in terms of dual conformal cross ratios u_i, the multi-Regge behavior reads:

$$u_1 := \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \to 1, \quad u_2 := \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \to 0, \quad u_3 := \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} \to 0, \tag{2.2}$$

where the reduced cross ratios

$$\tilde{u}_2 := \frac{u_2}{1-u_1} =: \frac{1}{|1+w|^2}, \quad \tilde{u}_3 := \frac{u_3}{1-u_1} =: \frac{|w|^2}{|1+w|^2} \tag{2.3}$$

are kept finite. As visible from eq. (2.3), six-point multi-Regge kinematics is completely determined by a complex parameter w and the large cross ratio u_1.

While the naïve limit eq. (2.2) yields a vanishing remainder function, a non-trivial result can be obtained by an analytic continuation to the so-called Mandelstam region [15,16], which is implemented by a clockwise continuation of the large cross ratio u_1,

$$u_1 \to e^{-2i\pi} u_1 \tag{2.4}$$

before taking the limit (2.2). For this setup, the six-point remainder function in multi-Regge kinematics can be written as a Fourier–Mellin integral [16,22,17]:

$$e^{R_6^{\mathrm{MHV}}+i\pi\delta}|_{\mathrm{MRK}} = \cos\pi\omega_{ab} + i\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}|w|^{2i\nu}\Phi_{\mathrm{reg}}^{\mathrm{MHV}}(\nu,n)$$
$$\times \exp\left[-\omega(\nu,n)\left(\log(1-u_1)+i\pi+\frac{1}{2}\log\frac{|w|^2}{|1+w|^4}\right)\right]. \tag{2.5}$$

In the above equation, the first term originates from a Regge pole exchange, while the second term comes from the exchange of a two-Reggeon bound state, which gives rise to a Regge-cut contribution. The so-called impact factor $\Phi_{\mathrm{reg}}^{\mathrm{MHV}}(\nu,n)$ and the BFKL eigenvalue $\omega(\nu,n)$ appearing in the latter have an expansion in powers of the loop-counting parameter $a = \frac{g^2 N_c}{8\pi^2}$:

$$\omega(\nu,n) = -a(E_{\nu,n} + aE_{\nu,n}^{(1)} + a^2 E_{\nu,n}^{(2)}) + \mathcal{O}(a^4),$$
$$\Phi_{\mathrm{reg}}^{\mathrm{MHV}} = 1 + a\,\Phi^{(1),\mathrm{MHV}} + a^2\,\Phi^{(2),\mathrm{MHV}} + \mathcal{O}(a^3)\,. \tag{2.6}$$

Physically, the BFKL eigenvalue $\omega(\nu,n)$ describes the evolution of the two-Reggeon bound state, while the impact factor describes the coupling of the two-Reggeon bound state to the physical gluons. While the first orders of these quantities were determined by direct calculation [16,22,17,18,2], a general solution based on the Wilson-loop OPE [23–26] has been identified in ref. [27].

In the six-point (and seven-point) calculations below, we will only need the lowest-order term of the BFKL eigenvalue, which reads

$$E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi\left(1+i\nu+\frac{|n|}{2}\right) + \psi\left(1-i\nu+\frac{|n|}{2}\right) - 2\psi(1), \tag{2.7}$$

where $\psi(x)$ is the digamma function. The two other quantities appearing in eq. (2.5), the phase δ and the Regge-pole contribution ω_{ab}, are related to the cusp anomalous dimension $\gamma_K(a)$ via

$$\omega_{ab} = \frac{1}{8}\gamma_K(a)\log|w|^2 \quad \text{and} \quad \delta = \frac{1}{8}\gamma_K(a)\log\frac{|w|^2}{|1+w|^4} \tag{2.8}$$

and are thus known to all orders [28].

Finally, the term $\log(1-u_1)$ in the integrand of eq. (2.5) is large because of the behavior of the cross ratio u_1 in the multi-Regge limit eq. (2.2). This suggests to organize the remainder function in powers of $\log(1-u_1)$ at each loop order:

$$R_6^{\mathrm{MHV}}|_{\mathrm{MRK}} = 2\pi i \sum_{\ell=2}^{\infty}\sum_{n=0}^{\ell-1} a^\ell \log^n(1-u_1)\Big[g_n^{(\ell)}(w,w^*) + 2\pi i\, h_n^{(\ell)}(w,w^*)\Big]. \tag{2.9}$$

In the above equation, all terms with $n=\ell-1$ are referred to as the leading logarithmic approximation (LLA) and terms with $n=\ell-1-k$ belong to (Next-to)^k-LLA. Since the imaginary and real parts $g_n^{(\ell)}$ and $h_n^{(\ell)}$ are not independent [18], it is sufficient to calculate all imaginary parts $g_n^{(\ell)}$ in order to determine the full remainder function. For more details on the six-point remainder function in the multi-Regge limit we refer the reader to ref. [20] and continue with the description of the relevant functions for the evaluation of eq. (2.5).

2.2. *Single-valued harmonic polylogarithms in one variable*

Before describing the functions governing the integral eq. (2.5), let us introduce harmonic polylogarithms (or HPLs for short) [29], which are defined as iterated integrals

$$H_{a_1,a_2,\ldots,a_n}(z) = \int_0^z \mathrm{d}t\, f_{a_1} H_{a_2,\ldots,a_n}(t)\,, \tag{2.10}$$

where the integration weights f_a are given as

$$f_{-1} = \frac{1}{1+t}, \quad f_0 = \frac{1}{t}, \quad \text{and} \quad f_1 = \frac{1}{1-t}. \tag{2.11}$$

In eq. (2.10), the length of the index vector $\vec{a} = \{a_1,\ldots,a_n\}$ is called the *weight* of a HPL, while z is referred to as the *argument*. For the six-point remainder function, only the latter two integration weights in eq. (2.11) will appear. Corresponding to the weights f_0 and f_1 we introduce two letters x_0 and x_1 which will be used as non-commutative bookkeeping variables in generating functions for polylogarithms below.

From their definition eq. (2.10) it is clear that HPLs satisfy the differential equation

$$\frac{\partial}{\partial z} H_{a_1,a_2,\ldots,a_n} = f_{a_1}(z) H_{a_2,\ldots,a_n}(z). \tag{2.12}$$

Harmonic polylogarithms of low weight can be conveniently expressed in terms of logarithms and dilogarithms, for example

$$H_{\underbrace{0,\ldots,0}_{w}}(z) = \frac{1}{w!}\log^w(z), \quad H_{\underbrace{1,\ldots,1}_{w}}(z) = \frac{1}{w!}(-\log(1-z))^w, \quad H_{\underbrace{0,\ldots,0}_{(w-1)},1}(z) = \mathrm{Li}_w(z)\,. \tag{2.13}$$

Furthermore, harmonic polylogarithms satisfy a scaling identity

$$H_{k\cdot a_1,\ldots,k\cdot a_n}(k\cdot z) = H_{a_1,\ldots,a_n}(z) \quad \text{for} \quad k\neq 0,\ z\neq 0\,, \tag{2.14}$$

which is valid whenever $a_n \neq 0$.

Given that the usual logarithm has a branch cut which is canonically chosen to lie along the negative real axis, its iterated and integrated versions have branch cuts, as well. However, it is possible to obtain single-valued harmonic polylogarithms (SVHPLs) by linearly combining products of the form $H_{s_1}(z)H_{s_2}(\bar{z})$ in a way that all branch cuts cancel. The combinations of HPLs leading to SVHPLs are unique and can be determined from demanding triviality of the monodromies around singular points of HPLs [30].

The lowest-weight SVHPLs[2] read

$$\begin{aligned}
\mathcal{L}_0(z) &= H_0(z) + H_0(\bar{z}) \\
\mathcal{L}_1(z) &= H_1(z) + H_1(\bar{z}) \\
\mathcal{L}_{0,0}(z) &= H_{0,0}(z) + H_{0,0}(\bar{z}) + H_0(z)H_0(\bar{z}) \\
\mathcal{L}_{1,0}(z) &= H_{1,0}(z) + H_{0,1}(\bar{z}) + H_1(z)H_0(\bar{z}) \\
\mathcal{L}_{1,0,1}(z) &= H_{1,0,1}(z) + H_{1,0,1}(\bar{z}) + H_{1,0}(z)H_1(\bar{z}) + H_1(z)H_{1,0}(\bar{z}) \\
&\vdots
\end{aligned} \tag{2.15}$$

While up to weight three the expressions follow an obvious pattern, ζ-values make an appearance starting at weight four, for example

$$\begin{aligned}
\mathcal{L}_{1,0,1,0}(z) = {} & H_{1,0,1,0}(z) + H_{0,1,0,1}(\bar{z}) + H_{1,0,1}(z)H_0(\bar{z}) + H_1(z)H_{0,1,0}(\bar{z}) \\
& + H_{1,0}(z)H_{01}(\bar{z}) - 4\zeta_3 H_1(\bar{z}).
\end{aligned} \tag{2.16}$$

An elaborate introduction to SVHPLs in the context of the six-point remainder function in MRK in which the method for solving the single-valuedness condition is carefully explained can be found in section 3 of ref. [18].

In the remainder of this subsection, let us collect several properties of SVHPLs which will be useful below: Two SVHPLs labeled by words s_1 and s_2 satisfy the shuffle relation

$$\mathcal{L}_{s_1}(z)\,\mathcal{L}_{s_2}(z) = \sum_{s\in s_1 \sqcup\!\sqcup\, s_2} \mathcal{L}_s(z)\,, \tag{2.17}$$

where the shuffle $s_1 \sqcup\!\sqcup\, s_2$ refers to all permutations of $s_1 \cup s_2$ which leave the order of elements in s_1 and s_2 unaltered. The generating functional for the SVHPLs,

$$\begin{aligned}
\mathcal{L}^{\{0,1\}}(z) = \sum_{s\in X(\{x_0,x_1\})} \mathcal{L}_s(z)s &= 1 + \mathcal{L}_0(z)\,x_0 + \mathcal{L}_1(z)\,x_1 + \mathcal{L}_{0,0}(z)\,x_0x_0 \\
&\quad + \mathcal{L}_{0,1}(z)\,x_0x_1 + \dots\,,
\end{aligned} \tag{2.18}$$

where $X(\{x_0, x_1\})$ are all words[3] in the alphabet $\{x_0, x_1\}$. This generating functional satisfies the differential equations

$$\frac{\partial}{\partial z}\mathcal{L}^{\{0,1\}}(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)\mathcal{L}^{\{0,1\}}(z), \qquad \frac{\partial}{\partial \bar{z}}\mathcal{L}^{\{0,1\}}(z) = \mathcal{L}^{\{0,1\}}(z)\left(\frac{y_0}{\bar{z}} + \frac{y_1}{1-\bar{z}}\right), \tag{2.19}$$

[2] As will be clear from the examples given in eq. (2.15), the $\mathcal{L}_s(z)$ are functions of both z and $\bar{z}$. For simplicity, however, we will denote these functions as $\mathcal{L}_s(z)$.

[3] For convenience, SVHPLs will be labeled by the indices of the letters rather than by the letters themselves.

where $\{y_0, y_1\}$ is an additional alphabet, which appears in the construction of SVHPLs in ref. [30] and is related by the single-valuedness condition to the alphabet $\{x_0, x_1\}$ mentioned above. Solving this condition order by order, one finds

$$\begin{aligned} y_0 &= x_0 \quad \text{and} \\ y_1 &= x_1 - \zeta_3(2x_0x_0x_1x_1 - 4x_0x_1x_0x_1 + 2x_0x_1x_1x_1 + 4x_1x_0x_1x_0 + \cdots) + \cdots , \end{aligned} \tag{2.20}$$

and can thus find the analogue of eq. (2.15) for $\mathcal{L}_s$ for an arbitrary label s constructed from the alphabet $\{x_0, x_1\}$. Note, however, that from eq. (2.19) the $\mathcal{L}_s(z)$ satisfy the same differential equation in z as the corresponding HPL $H_s(z)$, cf. eq. (2.12).

2.3. Calculation of the six-point remainder function

As pointed out at the end of subsection 2.1, the problem of calculating the remainder function $R_6^{\mathrm{MHV}}|_{\mathrm{MRK}}$ via eq. (2.5) boils down to evaluating the real part of the sum over the integral, which will yield the functions $g_n^{(\ell)}$. The crucial ingredients here are the loop expansions of the impact factor $\Phi_{\mathrm{reg}}^{\mathrm{MHV}}(\nu, n)$ and the BFKL eigenvalue $\omega(\nu, n)$ in eq. (2.6).

To calculate $g_n^{(\ell)}$ one would then expand the integral to the desired loop and logarithmic order, close the contour at infinity and sum up the residues. This, however, becomes cumbersome already beyond the lowest loop order.

As discussed in ref. [18], the functions $g_n^{(\ell)}$ can be expressed in terms of SVHPLs. This opens a natural and simpler way for the calculation of eq. (2.5): one starts from an ansatz in SVHPLs and compares the series expansions in (w, w^*) of both the ansatz and the integral. Following this approach the remainder function was calculated up to five loops, as well as for higher loop orders in LLA and NLLA [18,2,19].

A more direct evaluation of the remainder function eq. (2.5) was developed in ref. [20]. The key insight, first used in refs. [31,19], is that the leading term of any SVHPL is simply given by the harmonic polylogarithm with the same index structure

$$\mathcal{L}_s(w, w^*) = H_s(w) + \ldots , \tag{2.21}$$

as exemplified in eq. (2.15). Importantly, the term $H_w(w)$ is the only term in the expansion of the SVHPL which does not depend on w^*. Comparing with the dispersion relation (2.5), we see that the leading terms are encoded in the residues at $\nu = -\frac{in}{2}$ as only for those poles the residues will have no contribution from w^*.

Since the remainder function can be expressed in terms of SVHPLs exclusively, a viable approach consists of simply determining the leading terms, which will be a linear combination of HPLs and obtaining the full result by simply promoting HPLs to SVHPLs via

$$H_s(w) \to \mathcal{L}_s(w, w^*) . \tag{2.22}$$

In performing the above replacement, the contributions from the omitted residues are restored automatically.

In ref. [20], we started from this observation and identified recursion relations between different integrals which hold on the locus of the poles $\nu = -\frac{in}{2}$, but which by using eq. (2.22) lift to relations of the full result. Employing these relations, we reduced all integrals to a set of trivial basis integrals. This allowed us to efficiently evaluate the remainder function up to very high loop- and logarithmic orders and to prove Pennington's formula [32] for the six-point remainder function in LLA.

Given the success of this approach for the six-point remainder function, it is natural to ask, whether a similar formalism can be established for seven points. This idea is going to be discussed in the next section.

3. Seven-point remainder function in multi-Regge kinematics

3.1. From six to seven gluons

We now move on to the seven-point MHV remainder function in multi-Regge kinematics, which is defined similarly to the six-point case,

$$A_7^{\mathrm{MHV}} = A_{\mathrm{BDS}} e^{R_7^{\mathrm{MHV}}}. \tag{3.1}$$

The kinematics in this case is governed by seven conformal cross ratios,

$$\begin{aligned} u_{1,1} &= \frac{x_{37}^2 x_{46}^2}{x_{47}^2 x_{36}^2}, & u_{2,1} &= \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}, & u_{3,1} &= \frac{x_{35}^2 x_{26}^2}{x_{25}^2 x_{36}^2}, \\ u_{1,2} &= \frac{x_{14}^2 x_{57}^2}{x_{15}^2 x_{47}^2}, & u_{2,2} &= \frac{x_{16}^2 x_{25}^2}{x_{15}^2 x_{26}^2}, & u_{3,2} &= \frac{x_{36}^2 x_{27}^2}{x_{26}^2 x_{37}^2}, \\ \tilde{u} &= \frac{x_{13}^2 x_{47}^2}{x_{37}^2 x_{14}^2}. \end{aligned} \tag{3.2}$$

In the multi-Regge limit the cross ratios $u_{1,s}$, $s = 1, 2$, and $\tilde{u}$ approach 1, while all other cross ratios tend to zero. Due to a conformal Gram relation, only six of the above cross ratios are independent. It is, however, advantageous for what follows to keep all seven cross ratios explicitly. The remaining kinematic freedom in the multi-Regge limit is again given by the reduced cross ratios

$$\tilde{u}_{a,s} := \frac{u_{a,s}}{1 - u_{1,s}}, \tag{3.3}$$

which are finite in the multi-Regge limit and which we again parameterize by two complex variables w_1, w_2 defined as

$$\tilde{u}_{2,1} =: \frac{1}{|1+w_1|^2}, \quad \tilde{u}_{3,1} =: \frac{|w_1|^2}{|1+w_1|^2}, \quad \tilde{u}_{2,2} =: \frac{1}{|1+w_2|^2}, \quad \tilde{u}_{3,2} =: \frac{|w_2|^2}{|1+w_2|^2}. \tag{3.4}$$

A key difference to the six-point case is that several interesting Mandelstam regions exist, in which Regge cuts appear. However, as is shown in [8,9], the seven-point remainder function can be written as a linear combination of three elementary building blocks in every Regge region. These building blocks are usually called the *short cuts* which describe a Regge cut in the s_{45}- and s_{56}-channel, respectively, as well as the *long cut* which describes a Regge cut which spans the s_{456}-channel, see [8,9] for details and Fig. 1 for a pictorial representation. As it turns out, the short cuts are fully determined by the BFKL eigenvalue and impact factor of the six-point amplitude. Therefore, we focus on the long cut in which a new ingredient appears. To study the long cut, one can take the analytic continuation[4]

[4] Subtleties in the choice of path that arise due to the conformal Gram relation are, for example, discussed in ref. [12], but do not play a role here.

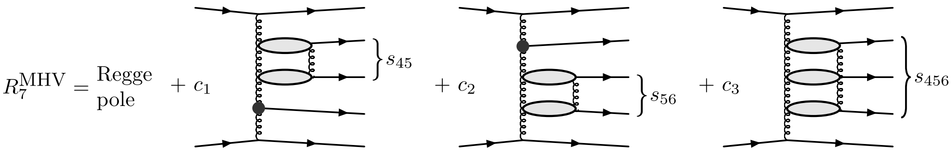

Fig. 1. In the seven-point case, the remainder function in every Regge region can be written as a linear combination of the Regge pole contribution, the *short cuts* in the s_{45}- and s_{56}-channel and the *long cut* in the combined s_{456}-channel. See [8,9] for details.

$$\tilde{u} \to e^{-2\pi i}\tilde{u} \tag{3.5}$$

of the remainder function, before going to the multi-Regge limit. In this Mandelstam region, the short cuts do not contribute and the remainder function is fully determined by the long cut. In LLA, the remainder function in this region was stated in ref. [9] and reads

$$\begin{aligned} R_7^{\text{MHV}} = 1 + i\pi \sum_{\ell=2}^{\infty}\sum_{k=0}^{\ell-1} \frac{a^\ell}{k!(\ell-1-k)!} \log^k(1-u_{1,1}) \\ \times \log^{\ell-1-k}(1-u_{1,2})\, \mathcal{I}_7\left[E_{\nu,n}^k\, E_{\mu,m}^{\ell-k-1}\right], \end{aligned} \tag{3.6}$$

where we define

$$\begin{aligned} \mathcal{I}_7\left[\mathcal{F}(\nu,n,\mu,m)\right] := \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \int_{-\infty}^{+\infty}\frac{d\nu}{2\pi}\int_{-\infty}^{+\infty}\frac{d\mu}{2\pi}\, w_1^{i\nu+n/2}(w_1^*)^{i\nu-n/2} \\ \times w_2^{i\mu+m/2}(w_2^*)^{i\mu-m/2} C(\nu,n,\mu,m)\,\mathcal{F}(\nu,n,\mu,m), \end{aligned} \tag{3.7}$$

and where

$$C(\nu,n,\mu,m) = (-1)^{n+m}\frac{\Gamma\left(-i\nu-\frac{n}{2}\right)\Gamma\left(i\mu+\frac{m}{2}\right)\Gamma\left(i(\nu-\mu)+\frac{m-n}{2}\right)}{\Gamma\left(1+i\nu-\frac{n}{2}\right)\Gamma\left(1-i\mu+\frac{m}{2}\right)\Gamma\left(1-i(\nu-\mu)+\frac{m-n}{2}\right)} \tag{3.8}$$

is the so-called central emission vertex. Comparing expression (3.7) with the corresponding equation for the six-point case (2.5), we see that this is a new ingredient. Like in the six-point case there is again a nice physical interpretation of all the terms in eqs. (3.7) and (3.6), with the central emission vertex $C(\nu,n,\mu,m)$ describing the emission of a physical gluon from a bound state of two reggeized gluons and the BFKL eigenvalue $E_{\nu,n}$ describing the evolution of the bound state of two reggeized gluons, of which we have two because of the appearance of the central emission vertex. The impact factor which describes the coupling of the bound state to the physical gluons does not appear in eq. (3.7) since it is trivial in LLA. We can represent eq. (3.6) graphically as shown in Fig. 2. Note that eq. (3.6) has a similar form to eq. (2.9), only with two distinct large logarithms. Furthermore, as always in LLA, the real part vanishes.

Let us now study the symmetry properties of eq. (3.7). Using the expressions for the LLA BFKL eigenvalue eq. (2.7) and the central emission vertex eq. (3.8) we see that the remainder function is invariant under exchange of

$$\nu \leftrightarrow -\mu,\; n \leftrightarrow -m, \tag{3.9}$$

which corresponds to a swap

$$w_1 \leftrightarrow \frac{1}{w_2}, \tag{3.10}$$

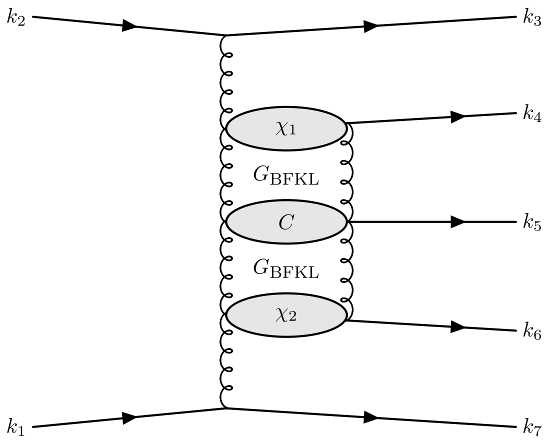

Fig. 2. Pictorial representation of eqs. (3.6) and (3.7), with C being the central emission vertex, G_{BFKL} describing the evolution of the two-Reggeon bound state and χ_1, χ_2 being the building blocks of the impact factor.

as well as under exchange of

$$n \leftrightarrow -n,\ m \leftrightarrow -m \tag{3.11}$$

which, in turn, corresponds to

$$w_1 \leftrightarrow w_1^*,\ w_2 \leftrightarrow w_2^*. \tag{3.12}$$

These symmetry properties will be very useful when evaluating the remainder function later on.

To evaluate eq. (3.7), one first closes the contours of the two integrals at infinity and then sums up the residues. A convenient choice is to close the contour of the μ-integration in the upper half-plane and the contour of the ν-integration in the lower half-plane. This corresponds to the choice $w_1 < 1$ and $w_2 > 1$, which is compatible with the symmetry $w_1 \leftrightarrow \frac{1}{w_2}$. In [10], this calculation was carried out for the integrals appearing at two loops, with the result

$$\mathcal{I}_7\left[E_{\nu,n}\right] = \frac{1}{2}\left(\log\left|1+\frac{w_1}{1+\frac{1}{w_2}}\right|^2 + \log\left|1+\frac{1+\frac{1}{w_2}}{w_1}\right|^2\right), \tag{3.13}$$

which takes the form of the six-point two-loop result with a rescaled variable, $w \to \frac{w_1}{1+\frac{1}{w_2}}$. From this result, we can also immediately obtain $\mathcal{I}_7\left[E_{\mu,m}\right]$ by making use of the symmetry (3.9) discussed before. Indeed, a special class of integrals is obtained when only one of the two energy eigenvalues $E_{\nu,n}$ or $E_{\mu,m}$ appears in the integrand, for definiteness let us choose $E_{\nu,n}$. In this case, the μ-integration can be carried out explicitly and results in a rescaling of the parameters (w_1, w_1^*). This reduces the integral to a six-point integral, that is, upon replacing

$$w \to \frac{w_1}{1+\frac{1}{w_2}} \tag{3.14}$$

one can effectively obtain the simple two-loop solution eq. (3.13) from the corresponding two-loop LLA integral of eq. (2.5). Starting from three loops, integrals with both types of energy eigenvalues $E_{\nu,n}$ and $E_{\mu,m}$ appear causing a more complicated result. We therefore have to resort to other means of solving the integral. As in the six-point case, a sensible starting point is trying to understand the relevant functions describing the remainder function.

3.2. Harmonic polylogarithms in two variables

Contrary to the situation in the six-point scenario, where the kinematics is determined by one complex parameter w, in the seven-point case we need to find SVHPLs in two complex variables w_1 and w_2 (cf. eq. (3.13)). In the discussion to follow we will use variables y and z, which will be related to w_1 and w_2 later on.

3.2.1. Two new letters

Harmonic Polylogarithms depending on two complex parameters – or two-dimensional harmonic polylogarithms (2dHPLs) for short[5] – have been constructed in ref. [33]. The implementation relies on introducing two new integration weights accompanying the weights f_a defined in eq. (2.11),

$$f_z = \frac{1}{t+z} \quad \text{and} \quad f_{1-z} = \frac{1}{1-t-z} \,. \tag{3.15}$$

Similar to the six-point scenario, where the function f_{-1} does not appear, the function f_z is not needed for the seven-point remainder function in MRK. In accordance, we will introduce the additional letter x_{1-z} only.

Up to weight three, all 2dHPLs can be expressed in terms of generalized polylogarithms. For the simplest cases the relations read:

$$\begin{aligned} H_{1-z}(y) &= H_1\Big(\frac{y}{1-z}\Big) = -\log\Big(1-\frac{y}{1-z}\Big), \\ H_{1,1-z}(y) &= \frac{1}{2}\log^2(1-y) - \log(1-y)\log(1-z) + \mathrm{Li}_2\Big(\frac{z}{1-y}\Big) - \mathrm{Li}_2(z)\,, \end{aligned} \tag{3.16}$$

where a complete set of relations leading to expressions for all possible labels up to weight three is presented in appendix A.2 of ref. [33].

While the labels y and z appear to be on unequal footing in the above formulæ, this is actually a choice of notation only: there are numerous relations between different representations of 2dHPLs. In particular, it is always possible to switch y and z in label and argument of a 2dHPL, for example:

$$H_{0,1,1-y}(z) = H_1(z)H_{0,1}(y) - H_1(y)H_{0,1}(z) - H_{0,1,1}(y) + H_{0,1,1}(z) + H_{0,1,1-z}(y)\,. \tag{3.17}$$

This type of relation, which can be easily derived for every label by reverting to the integral representation of 2dHPls, will be a crucial ingredient in fixing ζ-terms in 2dSVHPLs below.

In order to have a canonical representation, we will choose 2dHPLs with labels from $\{0, 1, 1-z\}$ for the argument y and 1dHPLs with labels from $\{0, 1\}$ for the argument z. Solving for shuffle relations by choosing a Lyndon basis [34] for the labels we will finally use

$$H_{\mathrm{Lyndon}(\{0,1,1-z\})}(y) \quad \text{and} \quad H_{\mathrm{Lyndon}(\{0,1\})}(z)\,. \tag{3.18}$$

As pointed out in ref. [13], this choice is actually a basis for the 2dHPLs appearing in the seven-point remainder function in MRK.

[5] In the following sections we will sometimes refer to the HPLs and SVHPLs discussed in subsection 2.2 as 1dHPLs and 1dSVHPLs, respectively.

3.2.2. Differential equations for 2dHPLs

With a second complex parameter entering the definition of 2dHPLs, an obvious question is the one about the differential behavior of those functions. While one could use relations like eq. (3.17) and thus trace back derivatives with respect to one variable appearing in the label of the polylogarithm to a derivative with respect to the argument, it is far more efficient to consider derivatives with respect to the labels of a 2dHPL directly. The necessary formulæ for taking those derivatives are listed and explained in Appendix A.

In terms of a generating function of 2dHPLs[6] with argument y

$$\begin{aligned} H^{\{0,1,1-z\}}(y) = & \sum_{s\in X(\{x_0,x_1,x_{1-z}\})} H_s(y)s \\ = & 1 + H_0(y)\,x_0 + H_1(y)\,x_1 + H_{1-z}(y)\,x_{1-z} \\ & + H_{0,0}(y)\,x_0x_0 + H_{0,1}(y)\,x_0x_1 + H_{0,1-z}(y)\,x_0x_{1-z} \\ & + H_{1,0}(y)\,x_1x_0 + H_{1,1}(y)\,x_1x_1 + H_{1,1-z}(y)\,x_1x_{1-z} \\ & + H_{1-z,0}(y)\,x_{1-z}x_0 + H_{1-z,1}(y)\,x_{1-z}x_1 + H_{1-z,1-z}(y)\,x_{1-z}x_{1-z} \\ & + \dots , \end{aligned} \tag{3.19}$$

the y-derivative can be written down immediately after considering the defining equation (2.10) together with the additional integration weight f_{1-z}:

$$\frac{\partial}{\partial y} H^{\{0,1,1-z\}}(y) = \left(\frac{x_0}{y} + \frac{x_1}{1-y} + \frac{x_{1-z}}{1-y-z}\right) H^{\{0,1,1-z\}}(y)\,. \tag{3.20}$$

Proceeding to the derivative with respect to z, it is no longer possible to write the derivative in multiplicative form as in eq. (3.20). Instead, one can describe the pattern of how letters and prefactors are attached to existing words depending on their particular letters. Writing

$$\frac{\partial}{\partial z} H^{\{0,1,1-z\}}(y) = \Xi\Big(H^{\{0,1,1-z\}}(y)\Big) \tag{3.21}$$

the operation Ξ acts as follows:

- for each sequence of letters $s = x_{1-z}\dots x_{1-z}$, promote

$$s \quad \rightarrow \quad \frac{x_0 s}{1-z} + \frac{x_1 s}{z} - \frac{s x_0}{1-z} - \frac{s x_1}{z} \tag{3.22a}$$

- for each sequence of letters $s = x_1\dots x_1$, promote

$$s \quad \rightarrow \quad \frac{x_{1-z}s}{z(1-z)} - \frac{s x_{1-z}}{z(1-z)} \tag{3.22b}$$

- to any complete word s, add a leading $1-z$:

$$s \quad \rightarrow \quad \frac{y}{(1-z)(1-y-z)}\, x_{1-z}s\,. \tag{3.22c}$$

[6] Again, instead of noting the full word s in the subscript of a 2dHPL, we just write the indices of the letters.

In order to extract the derivative of a particular 2dHPL with label s one expands both sides of eq. (3.21) and compares the coefficients of the word s.

In practice, aiming to find the z-derivative of $H_{0,1-z,1}(y)$ for example, one has to browse through all words of length two, which upon adding one letter following the rules eq. (3.22) will yield the word $x_0x_{1-z}x_1$. Starting from $H_{0,1}(y)x_0x_1$, one can insert a letter x_{1-z} between x_0 and x_1, making use of the first part of rule (3.22b). Taking $H_{0,1-z}(y)x_0x_{1-z}$, a letter x_1 can be appended to the right, which amounts to using the last part of rule (3.22a). Finally, the word $x_0x_{1-z}x_1$ can be reached by prepending x_0 to the word $x_{1-z}x_1$ accompanying $H_{1-z,1}(y)$, thus using the first term in rule (3.22a):

$$\frac{\partial}{\partial z}H_{0,1-z,1}(y)x_0x_{1-z}x_1 = \frac{H_{0,1}(y)}{z(1-z)}x_0x_{1-z}x_1 - \frac{H_{0,1-z}(y)}{z}x_0x_{1-z}x_1 + \frac{H_{1-z,1}(y)}{1-z}x_0x_{1-z}x_1 \,. \tag{3.23}$$

Another example, where one has to make use of rules (3.22a) and (3.22c) reads:

$$\frac{\partial}{\partial z}H_{1-z,0,0}(y)x_{1-z}x_0x_0 = \frac{y}{(1-z)(1-y-z)}H_{0,0}(y)\,x_{1-z}x_0x_0 - \frac{1}{1-z}H_{1-z,0}(y)x_{1-z}x_0x_0 \,. \tag{3.24}$$

3.3. Single-valued harmonic polylogarithms in two variables

The canonical way to identify single-valued versions of 2dHPLs would be to find a generalization of the single-valuedness condition formulated in ref. [30] for the alphabet $\{x_0, x_1\}$. However, although this generalization does most certainly exist, an explicit expression thereof is currently not known to us.[7] Therefore we continue on a different path: we postulate several constraints the single-valued versions of 2dHPLs should satisfy and later on argue that the functions thus constructed are indeed single-valued. In order to find those constraints, we take guidance by the properties of 1dHPLs reviewed in subsection 2.2:

- **Differential equations:** 1dSVHPLs satisfy the same differential equations as their 1dHPL counterpart (cf. eqs. (2.12) and (2.19)): therefore we require the generating functional of 2dSVHPLs with argument y

$$\begin{aligned}\mathcal{L}^{\{0,1,1-z\}}(y) &= \sum_{s\in X(\{x_0,x_1,x_{1-z}\})} \mathcal{L}_s(y)s \\ &= 1 + \mathcal{L}_0(y)\,x_0 + \mathcal{L}_1(y)\,x_1 + \mathcal{L}_{1-z}(y)\,x_{1-z} \\ &\qquad + \mathcal{L}_{0,0}(y)\,x_0x_0 + \mathcal{L}_{0,1}(y)\,x_0x_1 + \mathcal{L}_{0,1-z}(y)\,x_0x_{1-z} \\ &\qquad + \mathcal{L}_{1,0}(y)\,x_1x_0 + \mathcal{L}_{1,1}(y)\,x_1x_1 + \mathcal{L}_{1,1-z}(y)\,x_1x_{1-z} \\ &\qquad + \mathcal{L}_{1-z,0}(y)\,x_{1-z}x_0 + \mathcal{L}_{1-z,1}(y)\,x_{1-z}x_1 + \mathcal{L}_{1-z,1-z}(y)\,x_{1-z}x_{1-z} \\ &\qquad + \ldots\,,\end{aligned} \tag{3.25}$$

  to satisfy

$$\frac{\partial}{\partial y}\mathcal{L}^{\{0,1,1-z\}}(y) = \left(\frac{x_0}{y} + \frac{x_1}{1-y} + \frac{x_{1-z}}{1-y-z}\right)\mathcal{L}^{\{0,1,1-z\}}(y) \tag{3.26}$$

7 As pointed out in the conclusions, the paper [35] provides an explicit construction of those functions.

$$\frac{\partial}{\partial z}\mathcal{L}^{\{0,1,1-z\}}(y) = \Xi\Big(\mathcal{L}^{\{0,1,1-z\}}(y)\Big). \tag{3.27}$$

where the operation Ξ has been defined in eq. (3.22).

- **Limiting behavior:** In the limit $z \to 0$, the alphabet will shrink since $x_{1-z} \to x_1$. Thus we demand to recover the corresponding 1dSVHPLs in the limit $(1-z) \to 1$.
- **Switching variables:** Consistency with the relations switching the variable in the label and the argument. This means we require all relations like eq. (3.17) to hold upon replacing $H \to \mathcal{L}$.

In short, we require 2dSVHPLs to inherit the properties of 2dHPLs and in addition we demand a consistent reduction to 1dSVHPLs in the limit $z \to 0$. We will explicitly show in the following that those constraints are indeed sufficient to pin down 2dSVHPLs in two variables to at least weight four.

In practice, we can gain some experience regarding the structure of the 2dSVHPLs by studying a simple ad hoc construction: We start from the 2dHPLs as given by Gehrmann and Remiddi and reviewed in subsection 3.2. Since these functions can be expressed in terms of generalized polylogarithms up to weight three, we can promote each 1dHPL separately to its single-valued version using relations like eq. (2.15).

This is most easily explained using an example: a candidate for a single-valued 2dHPL can be obtained via

$$\begin{aligned} H_{1,1-z}(y) &= \frac{1}{2}\log^2(1-y) - \log(1-y)\log(1-z) + \mathrm{Li}_2\left(\frac{z}{1-y}\right) - \mathrm{Li}_2(z) \\ &= H_{1,1}(y) - H_1(y)H_1(z) + H_{0,1}\left(\frac{z}{1-y}\right) - H_{0,1}(z) \\ &\to \mathcal{L}_{1,1}(y) - \mathcal{L}_1(y)\mathcal{L}_1(z) + \mathcal{L}_{0,1}\left(\frac{z}{1-y}\right) - \mathcal{L}_{0,1}(z). \end{aligned} \tag{3.28}$$

Making use of the scaling relation

$$H_{a_1,a_2,\ldots,a_n}\left(\frac{y}{1-z}\right) = H_{(1-z)a_1,(1-z)a_2,\ldots,(1-z)a_n}(y), \tag{3.29}$$

where $a_i \in \{0,1\}$ and which holds whenever the last index is not 0 (see the discussion around eq. (2.14)), as well as

$$H_{\underbrace{0,\ldots,0}_{n}}\left(\frac{y}{1-z}\right) = \frac{1}{n!}\left(H_0(y) + H_1(z)\right)^n, \tag{3.30}$$

one can express the above candidate for a single-valued polylogarithm in terms of 2dHPLs:

$$\begin{aligned} \mathcal{L}_{1,1-z}(y) = {} & H_{1,1-z}(y) + H_1(y)H_{1-\bar{z}}(\bar{y}) + H_{1-\bar{z},1}(\bar{y}) + \mathcal{L}_0(z)\Big(H_{1-\bar{z}}(\bar{y}) - H_1(\bar{y})\Big) \\ & - \mathcal{L}_1(z)H_1(\bar{y}). \end{aligned} \tag{3.31}$$

Note that in eq. (3.31) the variable z in the label is complex-conjugated whenever the argument of the HPL is $\bar{y}$. Following this ad hoc approach we find functions up to weight three which perfectly match the first orders of the integral eq. (3.6). Furthermore, note that we did not express 1dSVHPLs of z in terms of usual HPLs, as this relates to a feature of 2dSVHPLs to be elaborated on below: if expressed in the basis eq. (3.18), 2dSVHPLs split into a *canonical* part as well as a part in which 1dSVHPLs of argument z are multiplied by 2dHPLs of arguments y and $\bar{y}$ (with labels possibly containing $1-z$ and $1-\bar{z}$). By *canonical* we refer to the pattern

$$\mathcal{L}_{a_1,a_2,\dots,a_n}(y)\big|_{\text{can.}} = \sum_{k=0}^{n} H_{a_1,\dots,a_k}(y) H_{\bar{a}_n,\bar{a}_{n-1},\dots,\bar{a}_{k+1}}(\bar{y}) \tag{3.32}$$

that is already present for 1dSVHPLs (see eq. (2.15)). Quite naturally, the additional terms one finds beyond the canonical part are exactly those needed to preserve the derivative rule eq. (3.27).

Since our ad hoc construction of 2dSVHPLs only works up to weight three, we would now like to turn these observations into a construction of higher-weight 2dSVHPLs by the following algorithm: We start from a known 2dSVHPL and add a letter to the left by integrating in y, thus making use of the differential equation (3.26). This fixes the 2dSVHPL of higher weight up to a function of $\bar{y}$, z and $\bar{z}$.

Based on the assumptions that HPLs of argument z only appear in single-valued combinations we make the most general ansatz of terms

$$\mathcal{L}_{a_1,\dots,a_n}(z) H_{b_1,\dots,b_m}(\bar{y}), \tag{3.33}$$

where $a_i \in \{0, 1\}$ and $b_i \in \{0, 1, 1-\bar{z}\}$, compatible with the overall weight. Demanding that this ansatz satisfies relations eq. (3.27) for differentiation in z then fixes the ansatz completely.

To clarify the procedure, let us consider an example. Starting from the obvious weight one expressions

$$\begin{aligned}
\mathcal{L}_0(y) &= H_0(y) + H_0(\bar{y}),\\
\mathcal{L}_1(y) &= H_1(y) + H_1(\bar{y}),\\
\mathcal{L}_{1-z}(y) &= H_{1-z}(y) + H_{1-\bar{z}}(\bar{y}),
\end{aligned} \tag{3.34}$$

we can, for example, write down an ansatz for $\mathcal{L}_{1,1-z}(y)$ as

$$\begin{aligned}
\mathcal{L}_{1,1-z}(y) = {} & H_{1,1-z}(y) + H_{1-\bar{z},1}(\bar{y}) + H_1(y) H_{1-\bar{z}}(\bar{y})\\
& + \mathcal{L}_0(z)\,(c_1 H_0(\bar{y}) + c_2 H_1(\bar{y}) + c_3 H_{1-\bar{z}}(\bar{y}))\\
& + \mathcal{L}_1(z)\,(c_4 H_0(\bar{y}) + c_5 H_1(\bar{y}) + c_6 H_{1-\bar{z}}(\bar{y}))\,,
\end{aligned} \tag{3.35}$$

which by differentiation with respect to z is then fixed to give eq. (3.31). Carrying this out for all functions up to weight three, we can compare the results of this algorithm with our ad hoc construction and find a perfect matching, as we should. Additional complications start from weight four where ζ-values are going to appear. As the 2dSVHPLs at weight three do not contain any zetas, those terms cannot be fixed by the y- and z-derivative and we have to add a term

$$\zeta_3\Big(c_1 H_1(\bar{y}) + c_2 H_{1-\bar{z}}(\bar{y})\Big) \tag{3.36}$$

to the ansatz of every 2dSVHPLs at weight four. Note that the two terms above are the only ones consistent with the reduction $z \to 0$, as well as vanishing in the limit $\bar{y} \to 0$. Demanding a consistent reduction in the limit $z \to 0$ and consistency with the relations exchanging argument and label fixes most of the ζ-terms, but not all. However, we can impose an additional constraint: as shown in [33], a 2dHPL evaluated at $y = 1 - z$ can be written as a combination of 1dHPLs of argument z. Similarly, setting $y = 1-z$, $\bar{y} = 1-\bar{z}$ in our ansatz, we expect to obtain a combination of 1dSVHPLs of weight four. As it turns out, this constraint is strong enough to fix all coefficients. As an example containing a ζ-value, we find

$$\begin{aligned}
\mathcal{L}_{0,0,1,1-z}(y) = {} & \mathcal{L}_{0,0,1,1-z}(y)\big|_{\text{can.}} + \mathcal{L}_{0,0,1}(z) H_{1-\bar{z}}(\bar{y})\\
& + \mathcal{L}_{0,1}(z)\Big(H_0(y) H_{1-\bar{z}}(\bar{y}) + H_{1-\bar{z},0}(\bar{y})\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{L}_1(z)\Big(-H_{0,0}(y)H_1(\bar{y})-H_0(y)H_{1,0}(\bar{y})-H_{1,0,0}(\bar{y})\Big)\\
&+\mathcal{L}_0(z)\Big(-H_{0,0}(y)H_1(\bar{y})+H_{0,0}(y)H_{1-\bar{z}}(\bar{y})-H_0(y)H_{1,0}(\bar{y})\\
&\qquad+H_0(y)H_{1-\bar{z},0}(\bar{y})-H_{1,0,0}(\bar{y})+H_{1-\bar{z},0,0}(\bar{y})\Big)-2\zeta_3H_{1-\bar{z}}(\bar{y}).
\end{aligned}
\tag{3.37}
$$

The expressions for all other Lyndon basis elements up to weight four can be found in the file attached to the arXiv submission of this paper.

Going on to weight five, our constraints do not seem to be strong enough to fix all coefficients, which is due to the growth of both the number of Lyndon basis elements and the larger number of terms appearing in the ansatz for the ζ-terms, i.e. the analogue of eq. (3.36) at weight five. We are only able to fully fix those 2dSVHPLs at weight five whose label contains two different indices only.[8] The expressions for the weight-five 2dSVHPLs can also be found in the file attached to the arXiv submission, but note that those still contain fudge factors. It would be interesting to see if there are additional constraints which allow to completely fix the functions at weight five as well.

Up to weight three it is obvious from our ad hoc construction that the resulting functions are single-valued: they are composed from single-valued components by definition. For higher weights we can only argue that this is indeed the case: Starting from our ansatz and fixing all fudge coefficients does not only reproduce the 2dSVHPLs constructed naïvely, but yields functions, which including their ζ-parts perfectly match the analytical properties of the integral eq. (3.6). While this does not prove single-valuedness, the perfect matching with the explicit calculation of the integral strongly supports our conjecture.

3.4. Matching the results

Now that we have constructed a suitable class of functions, we want to generate expressions for the seven-point remainder function eq. (3.6) beyond two loops. We do this by simply writing down an ansatz with the correct weight and matching the series expansion of the ansatz to data generated from calculating residues of eq. (3.6). This also allows us to identify the variables in argument and label and we find that

$$
(y,z)\to\left(-w_1,-\frac{1}{w_2}\right)
\tag{3.38}
$$

is the correct choice. In the following we will use the abbreviation $x:=\frac{1}{w_2}$.

This leads to the following results at two loops,

$$
\begin{aligned}
\mathcal{I}_7\left[E_{\nu,n}\right]&=\frac{1}{2}\mathcal{L}_1(-x)\mathcal{L}_{1+x}(-w_1)+\frac{1}{2}\mathcal{L}_{0,1+x}(-w_1)+\frac{1}{2}\mathcal{L}_{1+x,0}(-w_1)\\
&\quad+\mathcal{L}_{1+x,1+x}(-w_1),\\
\mathcal{I}_7\left[E_{\mu,m}\right]&=-\frac{1}{2}\mathcal{L}_1(-w_1)\mathcal{L}_0(-x)-\frac{1}{2}\mathcal{L}_1(-w_1)\mathcal{L}_1(-x)+\frac{1}{2}\mathcal{L}_0(-x)\mathcal{L}_{1+x}(-w_1)\\
&\quad+\mathcal{L}_1(-x)\mathcal{L}_{1+x}(-w_1)-\frac{1}{2}\mathcal{L}_{1,1+x}(-w_1)-\frac{1}{2}\mathcal{L}_{1+x,1}(-w_1)\\
&\quad+\mathcal{L}_{1+x,1+x}(-w_1)+\frac{1}{2}\mathcal{L}_{0,1}(-x)+\frac{1}{2}\mathcal{L}_{1,0}(-x)+\mathcal{L}_{1,1}(-x).
\end{aligned}
$$

[8] This is not surprising, as those 2dSVHPLs can be constructed from 1dSVHPLs using the rescaling identity.

Note again that $\mathcal{I}_7\left[E_{\mu,m}\right]$ can be obtained from $\mathcal{I}_7\left[E_{\nu,n}\right]$ by using the symmetry (3.9) as well as the relations (3.17). At three loops we find

$$\begin{aligned}
\mathcal{I}_7\left[E_{\nu,n}^2\right] = & \frac{1}{2}\mathcal{L}_1(-x)\mathcal{L}_{0,1+x}(-w_1) + \frac{1}{4}\mathcal{L}_{1,1}(-x)\mathcal{L}_{1+x}(-w_1) + \frac{1}{4}\mathcal{L}_1(-x)\mathcal{L}_{1+x,0}(-w_1) \\
& + \mathcal{L}_1(-x)\mathcal{L}_{1+x,1+x}(-w_1) + \frac{1}{4}\mathcal{L}_{0,0,1+x}(-w_1) + \frac{1}{2}\mathcal{L}_{0,1+x,0}(-w_1) \\
& + \mathcal{L}_{0,1+x,1+x}(-w_1) + \frac{1}{4}\mathcal{L}_{1+x,0,0}(-w_1) + \mathcal{L}_{1+x,0,1+x}(-w_1) \\
& + \mathcal{L}_{1+x,1+x,0}(-w_1) + 2\mathcal{L}_{1+x,1+x,1+x}(-w_1)
\end{aligned}$$

as well as

$$\begin{aligned}
& \mathcal{I}_7\left[E_{\nu,n}\, E_{\mu,m}\right] \\
& = -\frac{1}{4}\mathcal{L}_{0,1}(-w_1)\mathcal{L}_0(-x) - \frac{1}{4}\mathcal{L}_{0,1}(-w_1)\mathcal{L}_1(-x) - \frac{3}{4}\mathcal{L}_{0,1}(-w_1)\mathcal{L}_{1+x}(-w_1) \\
& \quad - \frac{1}{4}\mathcal{L}_1(-w_1)\mathcal{L}_{0,1}(-x) + \frac{1}{2}\mathcal{L}_{0,1}(-x)\mathcal{L}_{1+x}(-w_1) + \frac{1}{4}\mathcal{L}_0(-x)\mathcal{L}_{0,1+x}(-w_1) \\
& \quad + \frac{1}{2}\mathcal{L}_1(-x)\mathcal{L}_{0,1+x}(-w_1) - \frac{1}{4}\mathcal{L}_{1,0}(-w_1)\mathcal{L}_0(-x) - \frac{1}{4}\mathcal{L}_{1,0}(-w_1)\mathcal{L}_1(-x) \\
& \quad - \frac{3}{4}\mathcal{L}_{1,0}(-w_1)\mathcal{L}_{1+x}(-w_1) + \frac{1}{4}\mathcal{L}_{1,0}(-x)\mathcal{L}_{1+x}(-w_1) - \frac{1}{2}\mathcal{L}_{1,1}(-w_1)\mathcal{L}_0(-x) \\
& \quad - \frac{1}{2}\mathcal{L}_{1,1}(-w_1)\mathcal{L}_1(-x) - \frac{1}{4}\mathcal{L}_1(-w_1)\mathcal{L}_{1,1}(-x) + \mathcal{L}_{1,1}(-x)\mathcal{L}_{1+x}(-w_1) \\
& \quad - \frac{1}{4}\mathcal{L}_1(-x)\mathcal{L}_{1,1+x}(-w_1) + \frac{1}{4}\mathcal{L}_0(-x)\mathcal{L}_{1+x,0}(-w_1) + \frac{1}{2}\mathcal{L}_1(-x)\mathcal{L}_{1+x,0}(-w_1) \\
& \quad + \frac{1}{2}\mathcal{L}_0(-x)\mathcal{L}_{1+x,1+x}(-w_1) + \frac{3}{2}\mathcal{L}_1(-x)\mathcal{L}_{1+x,1+x}(-w_1) + \frac{1}{2}\mathcal{L}_{0,1,1+x}(-w_1) \\
& \quad + \frac{1}{2}\mathcal{L}_{0,1+x,1}(-w_1) + \frac{1}{2}\mathcal{L}_{0,1+x,1+x}(-w_1) + \frac{1}{4}\mathcal{L}_{1,0,1+x}(-w_1) - \frac{1}{2}\mathcal{L}_{1,1,1+x}(-w_1) \\
& \quad + \frac{1}{2}\mathcal{L}_{1,1+x,0}(-w_1) - \frac{1}{2}\mathcal{L}_{1,1+x,1+x}(-w_1) + \frac{1}{4}\mathcal{L}_{1+x,0,1}(-w_1) + \mathcal{L}_{1+x,0,1+x}(-w_1) \\
& \quad + \frac{1}{2}\mathcal{L}_{1+x,1,0}(-w_1) - \frac{1}{2}\mathcal{L}_{1+x,1,1}(-w_1) + \frac{1}{2}\mathcal{L}_{1+x,1+x,0}(-w_1) - \frac{1}{2}\mathcal{L}_{1+x,1+x,1}(-w_1) \\
& \quad + 2\mathcal{L}_{1+x,1+x,1+x}(-w_1).
\end{aligned}$$

As explained before, the remaining integral at three loops, $\mathcal{I}_7\left[E_{\mu,m}^2\right]$, can be obtained by symmetry. All further results up to five loops are too lengthy to be reproduced here and we refer the reader to the file accompanying the arXiv submission of this paper.

Before we close let us make one additional but important remark. Recall that the key property of the one-dimensional SVHPLs that allowed us to directly construct the full result from a small set of residues in the six-point was that the leading term of the 1dSVHPL was a HPL with the same index structure which only depends on w. However, comparing with the explicit expressions of our 2dSVHPLs, e.g. eq. (3.31), we see that a similar statement holds – the leading term of a given 2dSVHPL is the 2dHPL with the same index structure which only depends on y and z but not on the complex-conjugated variables. This means that we should be able to construct the full result solely from the residues at

$$\nu = -\frac{i\,n}{2}, \quad \text{and} \quad \mu = \frac{i\,m}{2}, \tag{3.39}$$

which leads to an expression in 2dHPLs, and then making the replacement

$$H_w(y) \to \mathcal{L}_w(y). \tag{3.40}$$

We have checked that this indeed reproduces the full result. This in turn means that it should be possible to follow a procedure similar to [20] to reconstruct the remainder function from a set of simple basis integrals.

It is because of this remark that we present the formula for the five-loop remainder function in terms of 2dSVHPLs in the attached file, even though we cannot fully fix the ζ-parts of all those functions yet. We have obtained the full result in 2dHPLs and checked that the prescription eq. (3.40) works for the ζ-free part. Furthermore, the integrals contributing to the five-loop remainder function which only contain one kind of energy eigenvalue,

$$\mathcal{I}_7\left[E^4_{\nu,n}\right], \mathcal{I}_7\left[E^4_{\mu,m}\right] \tag{3.41}$$

only contain 2dSVHPLs of weight five with two different indices, which we fully understand. Here, too, the prescription eq. (3.40) works. We are therefore convinced that the formula as written down in the file attached to the arXiv submission is correct.

4. Conclusions

Setting up an efficient approach for calculating the MHV remainder function for seven points in $\mathcal{N} = 4$ super-Yang–Mills requires the construction of single-valued harmonic polylogarithms in two variables, 2dSVHPLs. In this paper, we have started the investigation of their analytical properties and constructed those functions up to and including weight four. The analytical constraints we are using, however, are not strong enough for completely determining the single-valued version of 2dHPLs starting from weight five, since we cannot fix the coefficients of all terms proportional to zeta values.

Upon availability of expressions for 2dHPLs, it is possible to apply a similar concept as the one introduced in ref. [19]: by calculating a certain subset of the residues contributing to the seven-point MHV remainder function only, one can determine the leading term of the 2dSVHPLs, which later on can be promoted to the full single-valued expression.

Using this method, we have expressed the remainder function in terms of 2dSVHPLs up to five loops. It would be interesting to see whether there are further constraints on the 2dSVHPLs which would allow us to go beyond weight five.

While we provide an ad-hoc construction of the 2dSVHPLs, in a recent paper [35] an explicit construction of those functions is provided to arbitrary weight in a different language. Naturally, it would be interesting to compare our results to the more general approach in ref. [35].

Furthermore, the pattern we find is simple enough to suggest that it should be possible to identify a formula similar to the formula derived in ref. [32] for the LLA of the six-point remainder function.

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Appendix A. Derivatives of harmonic polylogarithms

Derivatives of HPLs can be most easily stated in general form using the language of Goncharov polylogarithms. Given their integral definition

$$G_{a_1,a_2,\ldots,a_n}(y) = \int_0^y \frac{\mathrm{d}t}{t-a_1} G_{a_2,\ldots,a_n}(t)\,, \tag{A.1}$$

and comparing with the integration weights f_0, f_1 and f_{1-z} leads to

$$G_{a_1,a_2,\ldots,a_n}(y) = (-1)^{(\#(1)+\#(1-z))} H_{a_1,a_2,\ldots,a_n}(y)\,, \tag{A.2}$$

that is, the relative sign is determined by the number of 1's and $(1-z)$'s appearing in the label. In terms of Goncharov polylogarithms, derivatives with respect to label and argument read [36]:

$$\begin{aligned}
\frac{\partial}{\partial y} G_{a_1,a_2,\ldots,a_n}(y) &= \frac{1}{y-a_1} G_{a_2,\ldots,a_n}(y) \\
\frac{\partial}{\partial a_i} G_{a_1,a_2\ldots,a_n}(y) &= \frac{1}{a_{i-1}-a_i} G_{a_1,\ldots,\hat{a}_{i-1},\ldots,a_n}(y) \;+\; \frac{1}{a_i-a_{i+1}} G_{a_1,\ldots,\hat{a}_{i+1},\ldots,a_n}(y) \\
&\quad - \frac{a_{i-1}-a_{i+1}}{(a_{i-1}-a_i)(a_i-a_{i+1})} G_{a_1,\ldots,\hat{a}_i,\ldots,a_n}(y) \\
\frac{\partial}{\partial a_n} G_{a_1,\ldots,a_n}(y) &= \frac{1}{a_{n-1}-a_n} G_{a_1,\ldots\hat{a}_{n-1},a_n}(y) - \frac{a_{n-1}}{(a_{n-1}-a_n)a_n} G_{a_1,\ldots,a_{n-1}}(y)\,,
\end{aligned} \tag{A.3}$$

where $\hat{a}$ denotes omission of the respective entry. Given the terms of the form $\frac{1}{a_{i-1}-a_i}$ in the derivatives, it is obvious that for neighboring elements of the same kind one will get divergent terms. However, by shifting the entries in the label by a small value ϵ one can safely determine the derivative and successively take $\epsilon \to 0$. For example one finds:

$$\begin{aligned}
\frac{\partial}{\partial z} H_{1-z,1-z,1-z}(y) &= -\frac{\partial}{\partial z} G_{1-z+\epsilon,1-z,1-z+\epsilon}(y)\Big|_{\epsilon\to 0} \\
&= -\frac{y\, G_{1-z,1-z}(y)}{(1-z)(1-y-z)} \\
&= -\frac{y\, H_{1-z,1-z}(y)}{(1-z)(1-y-z)}\,.
\end{aligned} \tag{A.4}$$

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