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New Characterization of Primes and its Consequences

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Abstract

A new characterization of a prime number as a sum of only two neighbouring natural numbers is established. The composite numbers are sums of two and also more neighbouring natural numbers. Another characterization as a difference between squares of two natural neighbouring natural numbers is discussed. A primeness criterion is derived as well as a method for decomposition of composite number.

1 Introduction

Since Euclid a prime number p is characterised by the fact, that it can be divided only by itself or 1. Thus the set of positive integers apart from 1 can be grouped into two subsets: The subset of prime numbers and the subset of composite numbers which are multiplications of prime numbers. Only 2 is the even prime number and all other prime numbers are odd.

Two new characterizations by means of neighbouring natural numbers (NNNs) are given in section 2 and a consequential primeness condition discussed in section 3. Section 4 shortly discusses the use of the same condition for the decomposition of composite numbers. Section 5 uses the possibility of expressing composite numbers as infinite series and section 6 makes use of the decomposition of odd natural numbers into left, middle and right numbers simplifying the methods.

2 A New Characterization

Characterization 1. A prime number other than 2 is a sum of two *and no more than* two NNNs.

Composite odd numbers are expressed as a sum of two NNNs but also as a sum of three or more NNNs. As a consequence of this characterization a condition for primeness and a method for decomposition of composites are given in sections 3 and 4.

It is clear that composite numbers other than $2^i, i \geq 2$ can be expressed as a sum of three, four or more NNNs. For odd composite numbers dividing by the smallest factor m leads to a sum of m NNNs. The composite numbers can be expressed as sums of m NNNs (1).

$$\frac{m^2 + m}{2} + mk \quad m \geq 2, k = 0, 1, 2, \dots \quad (1)$$

i.e.

$m = 2 :$	$3 + 2k$
$m = 3 :$	$6 + 3k$
$m = 4 :$	$10 + 4k$
$m = 5 :$	$15 + 5k$
$m = 6 :$	$21 + 6k$
$m = 7 :$	$28 + 7k$
$m = 8 :$	$36 + 8k$
$m = 9 :$	$45 + 9k$
$m = 10 :$	$55 + 10k$

Composite numbers which can be expressed as a sum of only m NNNs are given as follows:

only 2 :	all odd prime numbers
only 3 :	$3 \times 2^k, k \geq 1$
only 4 :	$2 \times (\text{odd prime numbers} \geq 5)$
only 5 :	$5 \times 2^k, k \geq 2$
only 6 :	none
only 7 :	$7 \times 2^k, k \geq 2$
only 8 :	$4 \times (\text{odd prime numbers} \geq 11)$
only 9 :	none
only 10 :	none
only 11 :	$11 \times 2^k, k \geq 3$
only 12 :	none
only 13 :	$13 \times 2^k, k \geq 3$
only 14 :	none
only 15 :	none
only 16 :	$8 \times (\text{odd prime numbers} \geq 17)$

This can be summarized as follows:

1. only 2 NNNs : all prime numbers
2. only p NNNs : $p \times 2^k, k \geq r$ such that 2^r is just $\geq \frac{p+1}{2}$ and p is prime
3. only 2^r NNNs : $2^{r-1} \times (\text{odd prime number} \geq \text{next prime number to } 2^r)$
4. all other composite numbers : none

Remark 1. For every composite number of m summands there is a lower limit either odd or even which is given by (1) with $k = 0$.

Remark 2. Only the composite numbers 2^i ($i = 2, 3, 4, \dots$) cannot be expressed as a sum of NNNs. The reason is that all the prime factors are 2's.

Characterization 2. A prime number other than 2 is the difference between squares of two NNNs and not between two squares of non-NNNs.

It is clear that this is directly related to characterization 1.

Remark 3. All odd numbers greater than 3 are a difference between squares of two NNNs.

Remark 4. If 0 is considered along with the natural numbers, then the odd composite numbers are also differences between two non-NNNs.

3 Condition of Primeness

An algebraic primeness condition can be derived from characterization 1. Adding the NNNs yields the odd composite number x . The chain is $a + (a+1) + \dots + b$

$$x = \frac{a+b}{2}(b-a+1) \quad (2)$$

where $b-a > 1$ and $a > 0$. Let $a+b = c$ and $2a-1 = d$, then¹

$$\begin{aligned} c^2 - cd &= 2x \\ c &= \frac{d + \sqrt{d^2 + 8x}}{2} \end{aligned} \quad (3)$$

A real solution is obtained if d can be found such that $d^2 + 8x$ is square. This can always be obtained by choosing $d = x - 2$, then $d^2 + 8x = (x+2)^2$ and therefore $c = x$ and $a = \frac{x-1}{2}$, i.e. all odd numbers - prime numbers and composite numbers - are sums of two NNNs. Taking the condition $b-a > 1$ into account means that d can be found such that $d < x-2$ for $d^2 + 8x$ to be square. Actually to check the condition of primeness, it is sufficient to check for $d < \frac{x}{3}$. The reason is clear when considering composite numbers as infinite series.

Example 1. $x = 105$

- $d = 1$

$$\left. \begin{aligned} c &= \frac{29+1}{2} = 15 \\ a &= 1 \end{aligned} \right\} \rightarrow b = 14$$

then $x = 1 + \dots + 14$

¹It is clear that if c is odd, then it is also a factor of x . If c is even, then $\frac{c}{2}$ is a factor of x .

- $d = 11$

$$\left. \begin{array}{l} c = \frac{11 + 31}{2} = 21 \\ a = 6 \end{array} \right\} \rightarrow b = 15$$

then $x = 6 + \dots + 15$

- $d = 23$

$$\left. \begin{array}{l} c = \frac{37 + 23}{2} = 30 \\ a = 12 \end{array} \right\} \rightarrow b = 18$$

then $x = 12 + \dots + 18$

- $d = 29$

$$\left. \begin{array}{l} c = \frac{29 + 41}{2} = 35 \\ a = 15 \end{array} \right\} \rightarrow b = 20$$

then $x = 15 + \dots + 20$

- $d = 37$

$$\left. \begin{array}{l} c = \frac{37 + 47}{2} = 42 \\ a = 19 \end{array} \right\} \rightarrow b = 23$$

then $x = 19 + \dots + 23$

- $d = 67$

$$\left. \begin{array}{l} c = \frac{67 + 73}{2} = 70 \\ a = 34 \end{array} \right\} \rightarrow b = 36$$

then $x = 34 + \dots + 36$

Remark 5. In example 1 three NNNs sums correspond to the three prime factors and another three NNNs sums are of double those lengths. This is under the condition that a is always positive.

4 Decomposition of composite numbers

Using expression (3) composite numbers can be decomposed when allowing for negative d , e.g. for $x = 105$

$$c = \frac{\pm 1 + \sqrt{(\pm 1)^2 + 8x}}{2} = \frac{30}{2}, \frac{28}{2}.$$

Getting rid of the factor 2 yields 15 and 7. Repeating that for 15 yields 3 and 5, then $105 = 3 \times 5 \times 7$.

5 Composite Numbers as Infinite Arithmetic Series

In [2] the composite numbers are expressed as

$$x = n^2 + 2nk \quad (4)$$

where $n \geq 3$ is odd and $k = 0, 1, 2, 3, \dots$. The rest of the odd numbers ≥ 3 are primes. Since the arithmetic series with composite n are included in those where n is prime, n can be restricted to be prime. Solving (4) for n yields

$$n = -k + \sqrt{k^2 + x}. \quad (5)$$

If k can be found such that $k^2 + x$ is square then x is composite, otherwise x is prime.

A trivial solution is given by $k = \frac{x-1}{2}$ which does not differentiate between prime numbers and composite numbers. For x to be composite find $k < \frac{x}{6}$ such that $k^2 + x$ is square, otherwise x is prime. Also the same procedure is used to more easily decompose composite numbers.

Example 2. $x = 105$ as above

$$\begin{aligned} x = 105 &\rightarrow k = \pm 4 \\ n &= \pm 4 + 11 = 15, 7 \end{aligned}$$

Doing the same for 15 gives $k = \pm 1 \rightarrow 3, 5$. There are other solutions: $k = 8$ and $k = 16$.

Remark 6. The valid solutions for (3) are given by $d < x - 2$ and the valid solutions for (5) are given by $k < \frac{x-1}{2}$. $d = x - 2$ and $d = \frac{x-1}{2}$ are the trivial solutions for (3) and (5) respectively.

Remark 7. Assuming $x = p_1 p_2$, then $k = \frac{p_1 - p_2}{2}$ is a solution of (5), and $d = \frac{2p_1 - 4p_2}{2}$ and $d = \frac{4p_1 - 2p_2}{2}$ are solutions of (3). In general there are more possibilities for (3) than there are for (5).

6 Left, Middle and Right Numbers [3]

The natural numbers $3 + 2k$ can be decomposed into three subsets:

- Left : $7 + 6k$ 7, 13, 19, 25, 31, 37, 43, 49, 55, 61, ...
 Middle : $3 + 6k$
 Right : $5 + 6k$ 5, 11, 17, 23, 29, 35, 41, 47, 53, 59, ...

The same can be done with even numbers. The reason for this decomposition is the Goldbach Conjecture where the following rules apply:

1. a left even number is a sum of two right prime numbers or $3 +$ one left prime number
2. a right even number is a sum of two left prime numbers or $3 +$ one right prime number
3. a middle even number is a sum of one left and one right prime number.

Table 1: Addition - Simple Algebra

+	L	M	R
L	R	L	M
M	L	M	R
R	M	R	L

Table 2: Multiplication - Simple Algebra

×	L	M	R
L	L	M	R
M	M	M	M
R	R	M	L

6.1 Left and Right prime numbers

The left composite numbers are given by:

$$x = n^2 + 6nk, \quad n = 5, 7, 11, 13, \dots \quad (6)$$

The right composites are given by

$$x = n^2 + \left\{ \begin{matrix} 4n \\ 2n \end{matrix} \right\} + 6nk, \quad (7)$$

where $4n$ is chosen for a left n and $2n$ for a right n respectively. It is to be noticed, that the series of right composite numbers begins later in comparison to the series of left composite numbers.

The left primes are: 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, ...

The right primes are: 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, ...

There are 11 left and 12 right prime numbers under 100. Since the prime numbers begin earlier and the composite numbers begin later than their left counterparts, it is to be expected that the number of right prime numbers is higher than the number of left prime numbers. The limits of this difference are

$$\{0, \pi_{\sqrt{x}} + 1\} \quad (8)$$

where $\pi_{\sqrt{x}}$ is the number of prime numbers less than or equal to \sqrt{x} , which coincides with the number of arithmetic series starting under x . The higher limit is obtained with $x = 60, \dots$, while the lower limit is obtained with $x = 40, 80, 230, \dots$

According to [4] there is a turning point where the number of left prime numbers becomes larger than the number of right prime numbers (including the prime number 2). This point is 608981813029. That means that the number of left prime numbers surpasses the number of right prime numbers at this value or a little earlier. The reason for this is that the overlapping of composite numbers in the different arithmetic series begins earlier with the left numbers. At infinity the number of left and right primes should be equal so that there must be other turning points.

6.2 Using Left & Right Primes in the Primeness Criteria

Given are the two criteria:

square $k^2 + x$ $k < \frac{x}{6}$ for x to be composite

square $d^2 + 8x$ d is odd, and $d < \frac{x}{3}$

For left x : It is easy to see from (6) that, assuming no factor 5, $k < \frac{x}{14}$ for x to be composite and $k = 0, 3, 6, 9, \dots$. Using (6) and choosing

$$d = -n + 6k \quad (9)$$

results in

$$d^2 + 8x = (3n + 6k)^2 \quad (10)$$

and $d = 1, 7, 13, 19, 25, \dots$ and $d = 5, 11, 17, 23, 29, \dots$.

For right x : Similarly from (7), $k < \frac{x}{14}$ and $k = 1, 2, 4, 5, 7, 8, \dots$. The result for d depends on whether n is a right or left number:

1. n is a right number:

$$\begin{aligned} x &= n^2 + 2n + 6nk \wedge n = 5 + 6\bar{k} \\ \Rightarrow d &= -3 - 6\bar{k} + 6k \end{aligned}$$

which results in

$$d^2 + 8x = (17 + 18\bar{k} + 6k)^2, \bar{k} \in \mathbb{N}_0^+.$$

2. n is a left number:

$$\begin{aligned} x &= n^2 + 4n + 6nk \wedge n = 7 + 6\bar{k} \\ \Rightarrow d &= -3 - 6\bar{k} + 6k \end{aligned}$$

which results in

$$d^2 + 8x = (25 + 18\bar{k} + 6k)^2, \bar{k} \in \mathbb{N}_0^+.$$

Hence for right composite numbers $d = 3, 9, 15, 21, 27, \dots$ and for left composite numbers $d = 1, 7, 13, 19, 25, \dots$ and $d = 5, 11, 17, 23, 29, \dots$.

A further simplification in choosing k or d can be obtained by using the characteristics of squares: There is no square which ends on 2, 3, 7 or 8 on the right.

Conclusions

In this paper a new characterization of primes as a sum of only two NNNs is given with its consequences for the condition primeness and decomposition of composites. Also the simplification using left and right numbers is discussed.

References

- [1] Carter Bays and Richard H. Hudson. Details of the first region of integers x with $\pi_{3,2}(x) < \pi_{3,1}(x)$. *Math. Comp.*, 32:571–576, 1978.
- [2] Mohamed Mansour. Special semi-groups in elementary number theory. *International Journal of Pure & Applied Mathematics*, 57(3):419–433, 2009.
- [3] Mohamed Mansour. On the road to goldbach, 10 2014. URL <http://www.emeriten.ethz.ch/Doku2014/Emer-Mansour-14.pdf>.
- [4] Hans Riesel. *Prime Numbers and Computer Methods for Factorization*. Number 1 in 57. Birkhuser Basel, 1 edition, 1994.