FORM FINDING AND ANALYSIS OF SHELLS AND SLABS
BASED ON EQUILIBRIUM SOLUTIONS

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES of ETH ZURICH

(Dr. sc. ETH Zurich)

presented by

MARCO BAHR

Dipl.-Ing., Universität Leipzig

born on 18.12.1979

citizen of Germany

accepted on the recommendation of

Prof. Dr. Joseph Schwartz (examiner)
Prof. Dr. Aurelio Muttoni (co-examiner)

2017
Contents

Notations iv

Summary vii

Zusammenfassung ix

1 Introduction 1

2 Theoretical basis 7

2.1 Theory of plasticity 7

2.2 Material properties 8

2.2.1 Concrete 8

2.2.2 Reinforcement 8

3 Continuous Curved Stress Fields 10

3.1 Initial stress state 10

3.2 External loads 13

3.3 Continuous curved stress field 13

3.4 Discussion 16

4 Spatial Strut and Tie Networks 17

4.1 Initial Network 18

4.2 Generation of strut and tie networks 18

4.3 Control over the generation of strut and tie networks 20

4.4 Solutions of the system of equations 28

4.5 Lateral displacement of loads 29

4.6 Section Forces of a Grid Shell from an Equivalent Strut and Tie Network 30

4.7 Section Forces of a Beam Grillage from Equivalent Strut and Tie Networks 31

4.8 Discussion 33
8 Examples

8.1 Form finding of grid shell structure ........................................... 81

8.2 Form finding of a shell ............................................................. 87

8.3 Analysis of a slab ....................................................................... 101
  8.3.1 Torsionless load transfer ....................................................... 102
  8.3.2 Radial load transfer ............................................................... 111
  8.3.3 Assessment of the presented load transfer strategies ............ 118

9 Discussion / Outlook .................................................................. 122

References ..................................................................................... 127
Notations

Capital Roman letters

\( B \)  
intersection points at the yield surface

\( E \)  
stress field edge

\( F \)  
boundary force of a stress field

\( F_{eq} \)  
magnitude of the member force of the strut and tie network

\( F_{ini} \)  
magnitude of the initial member force

\( L \)  
linear function

\( M \)  
bending moment

\( M_B \)  
bending moment

\( M_D \)  
drilling moment

\( M_Z \)  
in-plane bending moment

\( N \)  
axial force

\( N_V \)  
in-plane shear force

\( S \)  
stress field

\( V \)  
transverse shear force

Small Roman letters

\( a_s \)  
reinforcement per unit length

\( e \)  
eccentricity value equals the length of the displacement vector

\( f_c \)  
compressive strength of concrete

\( f_{ct} \)  
tensile strength of concrete

\( f_y \)  
yield stress of reinforcement

\( h \)  
depth

\( h_{core} \)  
depth of the core element

\( h_x, h_y \)  
heights of inclined section planes

\( h_{x,core}, h_{y,core} \)  
heights of the inclined planes of the core element

\( l_d \)  
function of displacement vector lengths

\( l_m \)  
increase of the length between the tangent vectors of the initial surface and the continuous curved stress field

\( m_n, m_t \)  
transformed distributed bending moments

\( m_x, m_y \)  
distributed bending moments

\( m_{nt}, m_{tn} \)  
transformed distributed drilling moments

\( m_{xy}, m_{yx} \)  
distributed drilling moments

\( n_n, n_t \)  
transformed normal stress resultants

\( n_x, n_y \)  
normal stress resultants

\( n_{nt}, n_{tn} \)  
transformed in-plane shear stress resultants

\( n_{xy}, n_{yx} \)  
in-plane shear stress resultants

\( s_1, s_2 \)  
lever arms of the cover layer stress resultants
\( v_0 \) principal transverse shear stress resultant
\( v_{dx}, v_{dy} \) transverse shear stress resultant in direction of the unit displacement vector \( \vec{d} \)
\( v_{tr}, v_t \) transformed transverse shear stress resultants
\( v_x, v_y \) transverse shear stress resultants
\( s_i \) are length
\( u, v \) parameters of a vector function describing a surface
\( x_1, x_2 \) depths of the cover elements

**Small Greek letters**

\( \varepsilon \) strain
\( \sigma \) normal stress or related to stress fields a boundary stress
\( \sigma_1, \sigma_2 \) principal stresses
\( \sigma_i \) function of the magnitudes of the stress resultants on the initial surface
\( \sigma_m \) function of the magnitudes of the stress resultants of the curved stress field
\( \sigma_j, \sigma_k \) stress resultants
\( \tau \) transverse shear stress
\( \tau_c \) concrete shear strength
\( \tau_0 \) principal transverse shear stress
\( \varphi \) transformation angle
\( \varphi_0 \) transformation angle into the direction of principal transverse shear transfer

**Vectors with capital letters**

\( \vec{F} \) force vector or related to stress fields a boundary force vector
\( \vec{L} \) vector function defining a line in the stress space
\( \vec{P} \) position vector of a point
\( \vec{Q} \) external load vector or related to stress fields an edge load vector
\( \vec{R} \) reaction force vector

**Vectors with small letters**

\( \vec{d} \) unit displacement vector
\( \vec{e} \) unit vector
\( \vec{e}_{mx}, \vec{e}_{my} \) unit vector in direction of positive distributed bending moments
\( \vec{e}_{mxy}, \vec{e}_{myx} \) unit vector in direction of positive distributed drilling moments
\( \vec{e}_N \) unit vector in direction of positive normal forces
\( \vec{e}_{NV} \) unit vector in direction of positive in-plane shear forces
\( \vec{e}_{nx}, \vec{e}_{ny} \) unit vectors in direction of positive normal stress resultants
\( \vec{e}_x, \vec{e}_y, \vec{e}_z \) unit vectors in direction of the local x,y,z-axes
\( \vec{e}_{x0}, \vec{e}_{y0} \) unit vectors in direction of the local x,y-axes rotated through the angle \( \varphi_0 \)
\( \vec{i} \) vector function describing the initial surface
\( \vec{m} \) vector function of the shape of the continuous curved stress field
\( \vec{t}_i \) unit tangent vector function at the initial surface
\( \vec{t}_m \) unit tangent vector function at the continuous curved stress field
\( \vec{q} \) function of load vectors

Matrices and tensors

\( Q \) transformation matrix
\( T \) stress tensor
\( T_{\text{core}} \) stress tensor of the core element

Superscripts

\( I \) refers to the first, loaded strut and tie network or set of discrete curved stress fields
\( II \) refers to the second, not loaded strut and tie network or set of discrete curved stress fields
\( b \) refers to boundary stresses

Symbols

\( \angle(\vec{a},\vec{b}) \) angle between the vectors \( \vec{a} \) and \( \vec{b} \)
\( \perp \) perpendicular
Summary

Experimental and digital design methods for shells largely focus on statically optimal forms. In contrast, the type of curved surfaces in architecture has changed over the past decades: from being statically dominated to an increasingly architecturally determined formal language. Thus, a method is required that integrates the structure’s ability to bear flexural loads in the form finding process in order to illustrate the potential as well as the limits for deviations from the statically optimal form. As such a method would already meet the requirements for the analysis of structures, its scope can accordingly be extended.

Curved stress fields are an extension of thrust lines into two dimensions. Analogous to thrust lines, the curved stress fields do not necessarily define a shape, but describe the internal forces of the considered surface structure. In this thesis, two approaches are taken to determine of curved stress fields: a continuous and a discrete one.

In the continuous approach, a differential equation is developed that defines a curved membrane stress state which is in equilibrium with the external loads and is based on an initial surface with a tangential initial stress state. Although the corresponding differential equation has been established, the practical applicability is limited, due to the mathematical issues with the exact solution of differential equations of higher order.

Instead of approximating the solution of the differential equation numerically, a completely independent discrete approach to curved stress fields is developed. To allow an intuitive control over the load transfer, the discrete curved stress fields are generated based on spatial strut and tie networks. The shape of the strut and tie network is, in turn, obtained through a displacement of the nodes of the initial network. In contrast to alternative methods, the displacement of the nodes is completely free. The possibilities to control the generation of the spatial strut and tie networks are extensively discussed and a wide range of examples is shown.

To each network node a nodal zone is assigned, which is divided into triangular linear stress fields. The computation of the stress field boundary forces is based on the member forces of the strut and tie network and on the statical conditions among the stress fields within the nodal zone. A deduction of the internal forces of a shell and a slab from discrete curved stress fields is presented.

Curved stress fields define a membrane stress state, that is eccentric with respect to the middle surface of the structure. The admissible eccentricity thereof is limited by the structural resistance. A method to determine and illustrate the upper and lower limits of eccentricity is presented. To obtain such bounds of eccentricity that are directly linked to the curved stress fields, the boundary stresses of an infinitesimal element with inclined section planes are investigated.

The complete procedure, from the generation of spatial strut and tie networks to the determination of the bounds of eccentricity of the discrete curved stress fields, is implemented in a software application. The application is integrated in the 3D design software Rhinoceros© and is written in the programming language Python™. Complex computations are sourced out to the mathematical computation program Mathematica©. In three examples, the procedures proposed for (1) form
finding of a grid shell structure, (2) form finding of a shell and (3) the analysis of a slab, are investigated and explained in detail.

In the present thesis, a complete method for the determination of discrete curved stress fields has been developed. By the use of strut and tie networks, the method provides a highly intuitive control over the load transfer of the considered structures. For the practical application, the discrete approach of curved stress fields still requires a further optimization to provide competitive results.
Zusammenfassung


Der kontinuierliche Ansatz besteht darin, dass ein gekrümmter Membranspannungszustand, der im Gleichgewicht mit den äußeren Lasten ist, mit Hilfe einer Differentialgleichung beschrieben wird. Obwohl die entsprechende Differentialgleichung hergeleitet werden kann, so ist doch ihre praktische Anwendbarkeit, aufgrund der für die exakte Lösung von Differentialgleichungen höherer Ordnung bestehenden mathematischen Probleme, eingeschränkt.


Gekrümmte Spannungsfelder definieren einen in Bezug auf die Mittelebene des Tragwerks exzentrischen, ebenen Spannungszustand. Die zulässige Exzentrizität desselben ist durch den Tragwiderstand beschränkt. Um dies zu veranschaulichen, wird eine Methode zur Bestimmung der oberen und unteren Grenzen der Exzentrizität gezeigt. Für die Ermittlung des zulässigen Exzentrizitätsbereichs, der sich unmittelbar auf die gekrümmten Spannungsfelder bezieht, werden
die Randspannungen an einem infinitesimalen Element mit geneigten Querschnittsflächen untersucht.


In der vorliegenden Arbeit wurde eine vollständige Methode für die Bestimmung von diskreten gekrümmten Spannungsfeldern entwickelt. Die Methode bietet durch die zwischengeschalteten räumlichen Stabwerke eine intuitive Steuerung des Lastabtrags der untersuchten Tragwerke. Für eine praktische Anwendung der diskreten gekrümmten Spannungsfelder ist eine weitere Optimierung erforderlich, um realitätsnahe Ergebnisse zu erhalten.
1 Introduction

Motivation

Due to increasing use of digital design methods over past decades, the type of curved surfaces used in architecture has changed dramatically: from being statically dominated to an increasingly architecturally determined formal language [25]. As a result, the technical investment to realize these free-form structures has at times assumed considerable dimensions. Techniques applied in statically motivated form finding of structural surfaces are changing from experimental to digital methods, as the effort to generate digital models is considerably smaller. Irrespective of advances, which have been made in the field, research has been mainly focused on statically optimal forms. Hence, a method is required which is able to unify the architectural freedom of design with the engineer’s pursuit of efficient structures.

Review of form finding approaches

Form finding has been subject to experimental models for a long time. Besides very basic approaches like the hanging chain, which has, among others, been already used by Giovanni Poleni [52, p.192], more sophisticated models have mainly been introduced by pioneers like Antoni Gaudi [54, 20], Heinz Isler [45, pp.41], Frei Otto [12] and Sergio Musmeci [40]. With the distribution of computers, these models have been progressively replaced by digital approaches [24]. In the following, a brief summary of the most common digital form finding methods is given.

The transient stiffness method uses an iterative approach to find the shape of tension structures, which are exposed to large deformations. To determine the extensions of members of the structure a linear stress-strain relation is assumed. As deformations of a tensile structure are usually rather large, their effect on load transfer is taken into account, which requires an iterative process. The process is continued until the increment of the deformations has almost vanished and the static equilibrium is approximated [2, 26, pp.41]. Based on the analogy of catenary and arch, this method can be included in a form finding approach of compression-only forms [33, 5]. Tensile structures are mostly exposed to large deflections. The consideration of a relation of stresses and strains by this form finding approach thus appears imperative. Conversely, deformations of structural surfaces which are only subjected to compression, are in general rather small. Thus, the adaption of these approaches to compression-only structures, while keeping the relations of stresses and strains, may thus induce an unnecessary oversophistication.

The dynamic relaxation method is, in contrast to the former, a general technique for solving non-linear systems of equations [33]. It has originally been introduced to structural analysis by Otter [43] for the investigation of pre-stressed concrete nuclear vessels. The structural surface is modeled as a network of springs with the masses lumped in the joints [26, pp.61]. Out-of-balance forces cause an oscillation of the system about the equilibrium position. Damping brings the system to rest and leads to the aspired shape [26, p.61]. A computational implementation of this pseudo-dynamic approach, which enables form finding of networks subjected to compression and tension, has been developed by Kilian [23].
In contrast to the former methods, which presume a linear relation between stresses and strains, the force density method generates pure equilibrium solutions. It has originally been developed for cable networks, but allows also the consideration of compression members [51]. The non-linear equilibrium equations at the nodes of a network, involving geometrical parameters and member forces of the network, are transformed into linear equations by defining ratios of force to length for each member. The resulting system of linear equations can be directly solved [51, 26, pp.51]. The force density method is theoretically based on the lower bound theorem of the theory of plasticity and the assumption of a rigid-plastic material behavior, albeit authors in general do not refer to this fact. An adaption of the method to continuous membranes has been introduced as stress density method [30]. Maurin and Motro [31] propose the application of the stress density method for form finding of shells.

While the transient stiffness method and the dynamic relaxation method are based on iterative processes, the force density method allows a direct determination of the equilibrated shape of a structural surface. This is gained by presuming a rigid-plastic material behavior, which is acceptable for reinforced concrete structures. Although the first two methods allow influencing the generation process by introducing pre-stressing forces in the network, the force density method enables a more direct control by the choice of force-to-length ratios. Thus, the approach of the force density method appears more suitable for a form finding strategy, which allows the designer to consciously control the generation of the aspired form.

In the analysis and form finding of masonry vaults an approach also relying on the lower bound theorem of the theory of plasticity has been developed. As finite element analysis has been found not suitable [8, 6], an approach based on equilibrium solutions has been suggested [8, 19]. Block and Ochsendorf [7] proposed to extend the thrust line analysis into the third dimension, the so-called thrust network analysis. The method is based on a rigid-plastic material behavior of the masonry and creates equilibrium solutions for compression-only networks subjected to gravitational loads. In a first step, only the projection of the equilibrated network on a horizontal plane is considered. The relations among the projections of member forces of the network are determined by applying Maxwell’s definition of reciprocal figures. The non-linear equilibrium conditions at the nodes of a network, involving geometrical parameters and member forces, are transformed into linear equations, as the projected lengths and the forces of the projected members have been defined. A global scaling factor for the magnitudes of the member forces is used to manipulate the depth of the solution, such that it fits into a defined solution space.

The strategies of the force density method and thrust network analysis for transforming the system of non-linear equilibrium equations to a directly solvable linear one are basically identical. However, a more intuitive control of the generation of the equilibrated network is gained in thrust network analysis by decoupling the choice of member lengths and member forces compared to the force density method. This decoupling requires an intermediate step. Thus, in thrust network analysis at first the projection of the aspired network onto a horizontal plane is investigated.

Methods, which are claimed to contribute to the design process in terms of architecture or structural engineering must provide a certain freedom of design. The form finding methods, which have
been briefly discussed above, are all focused on the determination of statically optimal forms: forms of structures, which are only subjected to membrane forces, but not to bending. Results obtained by these methods may be satisfying to engineers, as these represent the most efficient solutions. From an architects perspective they may limit the freedom of design, as the effect of deviations from the statically optimal shape cannot be estimated. To consider the architectural freedom of design and the engineer’s pursuit of efficient structures, the method proposed in this thesis allows deviations from the statically optimal form within limits, which are defined by the bending resistance of the structure.

**Review of limit analysis of reinforced concrete shells and slabs**

A form finding method, which allows deviations from the statically optimal form while simultaneously assuring that the bending resistance of the structure is not exceeded, must already consider everything necessary for the analysis and design of a structure. Accordingly, the scope of the proposed method is not limited to form finding, but also includes the analysis and design of structural surfaces made of reinforced concrete.

Shell structures typically allow huge spans with only a few centimeters depth. This high degree of efficiency is a result of a funicular shape, which assures that under the common load conditions no bending but only membrane forces will occur. Accordingly, these kinds of structures are also often referred to as membrane shells. If considering the load bearing behavior only, slabs form the complete opposite to membrane shells. Membrane action is usually ignored for slabs, such that transverse loads are transferred by bending only. Since the type of curved surfaces used in architecture is changing from statically dominated membrane shells to increasingly formally determined shapes, this stark difference between slabs and shells is progressively vanishing. Membrane shells and slabs can be regarded as extreme cases, whereas shells, which are deviating from the statically optimal form and transfer loads by a combination of membrane action and bending, represent a mixture of these. Accordingly, the whole range from slabs to membrane shells is included in the scope of the proposed method.

Out of the existing form finding approaches those based on the lower bound theorem of the theory of plasticity are here considered most suitable for the development of a form finding method, which allows a direct control of the shape generation. These approaches in limit analysis of reinforced concrete shells and slabs are briefly summarized below.

Limit analysis of reinforced concrete shells "[...] is still in a relatively early stage of development" [49, p.491]. Research has been done concerning yield criteria [17, 50, 48][18, pp.270], but is either fundamental or limited to cylindrical shells or shells of revolution. In Save et al. [49, pp.491] and Calladine [9, pp.656], a number of upper-bound and lower-bound approaches for different special types of shells have been summarized.

In contrast, there has been extensive research in the field of limit analysis of reinforced concrete slabs. The most common methods are the yield-line theory and the strip method [29, p.81]. The yield-line theory, which has been developed by Johansen [22], determines upper bounds to the
ultimate load based on an investigation of the kinematic relations of a failure mechanism. Nevertheless, upper-bound methods are incompatible with the discussed form finding approaches as these focus predominantly on the analysis of structures. Concerning lower bound approaches in the limit analysis of reinforced concrete slabs a first significant method has been developed by Wolfensberger [56]. He proposed a division of the applied distributed loads into three components, which are then assumed to be transferred by bending moments about two orthogonal axes and by the torsional moment. By splitting a slab up into three imaginary slabs, each representing a single load bearing effect, its loading can easily be determined, especially when assuming constant values for the three load components. The strip method has been introduced by Hillerborg [16] and simplifies the description of the load bearing behavior compared to Wolfensberger [56], as the load component, which is assumed to be transferred by torsional moments, is set to zero. The slab is reduced to a torsionless grillage with infinitesimal spacing. Hillerborg states that the division of the applied distributed load into its components is basically arbitrary for every point of the slab [16, p.14]. Thus, the determination of the loading of the slab is easily done, once the division of the applied distributed load into its components is chosen. The strip method by Hillerborg is only applicable to slabs subjected to distributed loads. Morley [36] proposed an extension to the strip method to also consider single loads and column supports. Although the strip method became applicable to the most common slab configurations by this extension, its weak point, the disregard of the effects of torsion, remained. An approach, which solves this issue, was proposed by Meyboom [34]. Instead of a division into two sets of strips, the slab is split into a jigsaw of rectangular or trapezoidal segments derived from a yield line pattern. In each segment, the interaction of shear and normal stresses is investigated in detail based on a Sandwich model. The developed method is applicable to distributed and single loads and has been verified in a series of large scale tests [35].

The review of the methods for limit analysis of reinforced concrete shells has illustrated that there is a need for a general method, which goes beyond the investigation of single types of shells. The developmental stage of methods for limit analysis of reinforced concrete slabs contrasts that of shells. Ultimate loads can be determined with a satisfactory accuracy using the method proposed by Meyboom [34]. However, this approach cannot be applied as intuitively as the strip method [16, 36], which allows the designer to easily understand and influence the load transfer of the slab. The method proposed in this thesis aims to be as illustrative as the strip method, while considering the effects of torsion in a slab.

Basic idea of this thesis

The reviews of form finding approaches and methods for limit analysis of reinforced concrete shells and slabs have clearly illustrated, that these topics are usually not connected to each other. This thesis is going to demonstrate how these apparently diverging subjects can be combined in a single approach. Although the overall aim is to provide a form finding method that allows conscious deviations from the statically optimal form, the first step is the determination of such statically optimal forms. However, these forms are not considered as the definite shape of the aspired structure. Instead, the
curved membrane stress state corresponding to the statically optimal shape is understood as a curved stress field, which represents the internal forces of the structure. The approach is comparable to thrust line analysis. A thrust line is the center line of the resultant of the internal forces of an arch structure (fig. 1.1a). Normal force, shear force and bending moment can be deduced from the resultant of the internal forces, which is acting along the thrust line. While normal and shear forces yield from the break down of the resultant of internal forces, the bending moment is determined by the normal component of the resultant force and its eccentricity towards the center line of the arch structure (fig. 1.1b). The internal forces of a shell can be derived analogously from a curved stress field. Within this thesis, two approaches for the determination of curved stress fields are presented: continuous curved stress fields in chapter 3 and discrete curved stress fields in chapter 5.

A thrust line may possess an eccentricity towards the center line of the corresponding arch structure. Besides the resultant of the internal forces, which is acting along the thrust line, this eccentricity is the decisive factor for the magnitude of the bending moment. Based on the magnitude of the resultant of the internal forces and the bending resistance of the considered structure, an upper and a lower bound for the eccentricity of the thrust line can be determined at every point of the structure. The bounds of eccentricity of all points create an envelop of structural resistance. As long as the thrust line lies within this envelop, the structure is capable of bearing the applied loads. Figure 1.2 shows the envelop of an arch structure with constant bending resistance over its length. In chapter 6, the determination of the bounds of eccentricity forming such an envelop for a shell or slab is investigated.
The combination of curved stress fields with the corresponding bounds of eccentricity allows the claimed extension of form finding beyond statically optimal forms. However, as the consequences of deviations from the statically optimal form are always present, the focus on structural issues is not lost. As a curved stress field is a representation of a structure’s internal forces and the bounds of eccentricity of the structure’s resistance, their combined application can also be used to analyze and design shells and slabs.
2 Theoretical basis

In this chapter, a brief summary of the fundamentals of the theory of plasticity is given. Furthermore, assumptions about the material behavior of concrete and reinforcement are made.

2.1 Theory of plasticity

The historical development of the theory and its application to reinforced concrete structures has been extensively summarized by Nielsen [42, pp.ix]. Methods based on the theory of plasticity presume either an elasto-plastic or a rigid-plastic material behavior. The focus of this thesis is laid on rigid-plastic material behavior. A rigid-plastic material model states that there is no strain until a yield stress is reached. Once the material is subjected to the yield stress, the loading cannot be increased any further and the strain may assume arbitrary values [42, p.1]. This strongly idealized material behavior is an admissible simplification, if the plastic strains significantly exceed the elastic strains [39, p.4]. A material model, which ignores strains occurring for stresses below the yield stress, is, of course, not suitable for the determination of any kind of deformations. However, especially for the determination of collapse loads of highly statically indeterminate structures, the theory of plasticity renders much better results than the theory of elasticity [46, p.v]. Two theorems exist in the theory of plasticity, which are fundamental for the determination of collapse loads [42, pp.8]. They have been independently formulated by Gvozdev [14] and Drucker et al. [10].

The Lower Bound Theorem

"A load system \([Q_S]\), based on a statically admissible stress field which nowhere violates the yield criterion is a lower bound to the collapse load \([Q_R]\)." [39, p.9]

A stress field is statically admissible, if it is in equilibrium with the external loads and meets the statical boundary conditions [39, p.9]. The yield criterion is not violated, if the stresses of the assumed stress field nowhere exceed the yield stress.

The Upper Bound Theorem

"A load system \([Q_K]\), which is in equilibrium with a kinematically admissible displacement field (velocity field) forming a mechanism is an upper bound to the collapse load \([Q_R]\)." [39, p.13]

To every yield criterion an associated flow rule exists, which specifies a direction of the corresponding strain rate for every yield point on the yield surface. A displacement field is kinematically admissible, if the assumed displacement corresponds to this flow rule. The displacement field also has to meet the geometrical boundary conditions [39, p.13][42, p.10]. A mechanism describes the dependencies among the kinematical displacements of a system. A valid mechanism possesses one degree of freedom, such that no independent kinematical displacements can occur within the system [39, p.13].
Statical discontinuities

If assuming a rigid-plastic material behavior, stresses below the yield stress do not cause strains and arbitrary strains result from the yield stress [42, p.1]. As stresses are thus virtually independent from strains, statical discontinuities may occur. A statical discontinuity is a sudden change in the magnitude of stresses along a so-called discontinuity line. Only stresses, which are tangential to the discontinuity line, may be subject to sudden changes. Stresses, which are normal to the discontinuity line, must be equal on both sides of the line to meet equilibrium (fig. 2.1) [42, p.13].

\[
\sigma_I^t \neq \sigma_{II}^t \quad \sigma_I^n = \sigma_{II}^n \quad \tau_{nt}^I = \tau_{nt}^{II}
\]

![Statical discontinuity](image)

**fig. 2.1** Statical discontinuity

### 2.2 Material properties

#### 2.2.1 Concrete

Regarding the transfer of normal stresses, concrete is idealized as a rigid-plastic material with a compressive yield stress \( f_c \) and no tensile strength (fig. 2.2). This assumption corresponds to the modified Coulomb failure criterion with zero tensile strength. Conversely, a certain tensile strength of the concrete must be provided for the transfer of shear, as only slabs and shells without transverse reinforcement are investigated. Thus, tensile stresses up to the reduced value of the actual tensile strength of concrete \( f_{ct} \) [29, p.35] are allowed for the transfer of shear (fig. 2.3).

#### 2.2.2 Reinforcement

The reinforcement steel is idealized as a rigid-plastic material with yield stresses \( f_y \) and \(-f_y\) (fig. 2.4). Reinforcement bars are assumed to resist only longitudinal forces. Instead of considering single reinforcement bars, the reinforcement is assumed to be continuously distributed, while keeping the reinforcement percentage.
**fig. 2.2** Assumed material behavior of concrete for the transfer of membrane stresses, Modified Coulomb failure criterion with no tensile strength:

a) Stress-strain diagram, b) Planar stress state

**fig. 2.3** Assumed material behavior of concrete for the transfer of transverse shear, Modified Coulomb failure criterion with a reduced tensile strength:

a) Stress plane, b) Planar stress state

**fig. 2.4** Assumed material behavior of reinforcement steel, stress-strain diagram
3 Continuous Curved Stress Fields

A continuous curved stress field is a curved membrane stress state, which is in equilibrium with the applied external loads. The membrane theory of shells describes such curved membrane stress states [13]. It presumes that the geometry of the membrane is given. The stresses are uniquely defined by the constitutive differential equation, if the boundary conditions are such that the equation is solvable [55]. The membrane stress in the membrane theory of shells is not defined according to a specific material behavior, but only relies on the equilibrium conditions.

If neither the geometry nor the stresses are known, one of these must be chosen to obtain a solution. Williams [55] used this fact to determine the form of vaults and sails under a presumed stress function. The stress function used in his approach is derived from the Biot-Savard law, which originates from electromagnetic theory. In terms of the theory of plasticity, the approach proposed by Williams [55] renders a lower bound to the collapse load. Following Williams’ path and based on the lower bound theorem a general approach for the determination of continuous curved stress fields is proposed in this chapter.

Curved stress fields are a method to determine and illustrate internal forces of slabs and shells. Reinforced concrete shells and slabs possess a quasi-infinite statical indeterminacy. Based on the lower bound theorem of the theory of plasticity, this fact can be utilized to consciously control the load transfer of shells and slabs. The constitutive differential equation of the membrane theory defines a relation between the stress magnitudes and the shape of a curved membrane stress state. Thus, for every function defining stress magnitudes, a shape is found, provided that the boundary conditions are such that the equation is solvable. In contrast to Williams’ approach [55], the stress function shall not be derived from the Biot-Savard law, but shall be used to consciously control the load transfer. Similar to the generation of compression-only networks as proposed by Block [7], an initial surface is introduced as an intermediate step. In contrast to Block [7], the initial surface does not have to be a projection of the aspired curved stress field. On this initial surface of defined geometry an initial stress state is chosen. This initial stress state does not have to be in equilibrium, neither with the external loads nor in itself. The continuous curved stress field is finally generated by deforming the initial surface, such that a curved membrane stress state is obtained, which is in equilibrium with the external loads.

An earlier stage of the method to generate continuous curved stress fields has already been published by the author [3]. Since then, the proposed method has been refined and improved.

3.1 Initial stress state

Although a continuous curved stress field is not a real structure, but only describes a curved membrane stress state in equilibrium with its external loads, its load bearing behavior is comparable to a membrane shell. To be able to transfer the applied loads by their form, horizontal thrust is needed. Imagine the horizontal projection of a shell which is only subjected to vertical loads. As the vertical loads vanish in the horizontal projection of the stress state of the shell, only a stress
state, which is completely caused by the horizontal thrust remains. This remaining stress state is regarded as the initial stress state, which will be used to influence the form and in turn the load bearing behavior of the continuous curved stress field. Returning to the example of the horizontal projection of a shell, the projected stresses are transformed into principal normal stresses. Illustrating the principal normal stresses graphically means to assign tangent vectors to every point of the horizontal projection of the shell. Each tangent vector represents one of the two principal stresses of the membrane stress state. The two resulting vector fields, each representing the principal normal stresses in every point of the projection of the shell, create streamlines, which are in the considered context known as trajectories. If the magnitudes of the principal stresses are manipulated while maintaining equilibrium in the projection plane, the form of the shell will accordingly change. An increase of the magnitudes by a global factor would result in a decrease of the overall rise of the shell and vice versa. Let us assume that there was a single trajectory of the horizontal projection of the membrane stress state of the shell, along which the principal stresses could be changed without affecting the horizontal equilibrium. An increase of the principal stresses along this particular trajectory would reduce the rise of the corresponding shell strip with infinitesimal width, if it could move independently from the rest of the shell. As this would of course violate the geometrical integrity of the shell, the infinitesimal strip will instead "attract" more loads which will result in a partial unloading of the area surrounding the considered strip, which in turn would effect the geometry of the shell etc. This example is intended to illustrate the duality of geometry and stresses, described by the constitutive differential equation of membrane theory, and how it can be affected by defining an initial stress state.

The discussion of the exemplary infinitesimal shell strip was based on several assumptions concerning the initial stress state and its relation to the continuous curved stress state. Each of these is now investigated and adapted to suit a general approach. The initial stress state was assumed to be the projection of the membrane stress state on a horizontal plane and was described by a continuous curved stress field. This assumption will be adequate for a wide range of cases. It is however not applicable to surfaces, which cannot be bijectively mapped by any of their planar projections, like hyperboloids. Furthermore, especially for the analysis of curved structures, it may be more useful to use the given geometry as initial surface rather than a planar surface. The relation between a curved surface and its projection has been described by a displacement vector field with all vectors being normal to the projection plane. This presumption results in a field of parallel vectors with varying magnitudes. The limitation of the vector orientation is omitted in favor of a differentiable function describing the directions of the displacement vector field. Thereby it is possible to map any possible shape of a continuous curved stress field bijectively.

Omitting the projective relation between the initial surface and the continuous curved stress field also leads to looser requirements for the definition of the initial stress state. An initial stress state, which is a projection of the continuous curved stress field, must be in equilibrium in itself and with the tangential components of the external loads. In contrast, an initial stress state, which is assigned to an initial surface that can be deformed without these constraints, must not meet equilibrium in general. This fact means a significant simplification for the definition of the initial stress state.
In the example discussed above, the initial stress state has been visualized by the trajectories that are formed by the principal normal stresses. As the principal normal stresses are perpendicular to each other by definition, the trajectories must accordingly also be orthogonal. When transforming the initial stress state to a continuous curved stress field, the direction of principal normal stresses does in general change, such that no benefit is gained from defining principal normal stresses on the initial surface. In contrast, it may even complicate the definition of an initial stress state. Thus, instead of principal normal stresses, stress resultants in arbitrary directions are used. Each stress resultant defines a certain relation between normal and in-plane shear stresses as illustrated in fig. 3.1. Similar to the discussed graphical interpretation of principal normal stresses by a tangent vector field for each of the two sets of principal normal stresses, a set of stress resultants is defined by a tangent vector field. The vector function, which assigns the tangential vectors to each point of the initial surface, must be differentiable. As the principal normal stresses form two overlaying tangent vector fields, which describe the membrane stress state, multiple tangent vector fields representing sets of stress resultants can be overlaid. At least two tangent vector fields of stress resultants are needed to completely describe the membrane stress, while the tangent vectors of both fields are required to be linearly independent in any point of the initial surface. The number of tangent vector fields of stress resultants must at least be two, but is basically not limited to an upper value.

In summary, the initial surface, which defines the geometry of the initial stress state, may be an arbitrarily shaped surface as the initial stress state does not have to meet equilibrium. It is used as an initial step towards the generation of the continuous curved stress field. The arrangement and number of boundaries of the initial surface must be identical to the aspired continuous curved stress field. This means, a continuous curved stress field, which possesses e.g. a closed curve as cross section, can only be generated from an initial surface of the same type. An initial surface is mathematically defined as the map of a compact, simply connected subset of $\mathbb{R}^2$ into $\mathbb{R}^3$ by a differentiable, injective function $\vec{f}$.

The actual initial stress state is defined by assigning tangential stress resultants to every point of the initial surface. Every stress resultant can be split up into a unit tangent vector, which specifies its direction, and a scalar factor that defines its magnitude. Accordingly, every tangent vector field of
stress resultants shall be described by a differentiable vector function \( \vec{t} \) for the unit tangent vectors and a differentiable scalar function \( \sigma_i \) of the magnitudes of the stress resultants. The constraint, that the tangent vectors of the vector fields representing stress resultants are required to be linearly independent, is expressed for two vector fields \( j \) and \( k \) by:

\[
\left| \vec{t}_{ij}(u_1, v_1) \cdot \vec{t}_{ik}(u_1, v_1) \right| \neq 1 \quad (j \neq k) \quad (3.1)
\]

with \( u_1 \) and \( v_1 \) being typical parameters of the vector function \( \vec{t} \), describing a typical point on the initial surface. The internal stress state does not have to be in equilibrium, neither in itself nor regarding the external loads. Its only function is to define an initial step, which can be transformed into an equilibrated continuous curved stress field by displacement of the points of the initial surface. Consequently, the external loads do not have to be considered at this stage, even if they possess tangential components.

A differentiable tangent vector field of stress resultants forms streamlines, a set of curves which is tangential to the vector field in every point. Every streamline curve can be reparametrized by its arc length \( s_i \). This fact shall be used to reparametrize the vector function \( \vec{t} \), which maps the initial surface and is so far defined as a function of the arbitrary parameters \( u \) and \( v \). A valid reparametrization is obtained by two tangent vector fields \( \vec{t}_{i1} \) and \( \vec{t}_{i2} \). The index \( i \) refers to the initial surface or the initial stress state. The arc lengths of any additional set of streamline curves described by a vector field \( \vec{t}_{ij} \) can be expressed by the arc lengths of the vector fields \( \vec{t}_{i1} \) and \( \vec{t}_{i2} \), such that the arc length \( s_{ij} \) is a function of \( s_{i1} \) and \( s_{i2} \). The tangent vector field \( \vec{t}_{ij} \) can be expressed by:

\[
\vec{t}_{ij} = \frac{\partial \vec{t}}{\partial s_{ij}} \quad (3.2)
\]

It must be noted, that the first derivative of a curve with respect to its arc length always results in a tangent vector of unit speed.

### 3.2 External loads

The external loads are defined by a continuous vector function \( \vec{q} \), which assigns a load vector of arbitrary direction to every point of the initial surface. Accordingly, no single loads can be considered.

### 3.3 Continuous curved stress field

The continuous curved stress field is generated by transforming the initial surface and the associated initial stress state into an equilibrated state with the external loads by displacement of the points of the initial surface. It is assumed that there is a differentiable, bijective vector function \( \vec{d} \), which describes the directions of displacements of the single points, and the differentiable scalar
function $l_d$ expressing its length, such that:

$$l_d \vec{d} = \vec{m} - \vec{i}$$  \hspace{1cm} (3.3)

with $\vec{m}$ as a differentiable vector function defining the shape of the equilibrated continuous curved stress field. The equilibrium condition at a typical point of the continuous curved stress field yields:

$$\sum_{j} \frac{\partial (\sigma_{mj} \vec{t}_{mj})}{\partial s_{ij}} + \vec{q} = \vec{0}$$  \hspace{1cm} (3.4)

with $\sigma_{mj}$ and $\vec{t}_{mj}$ representing stress magnitude and tangent vector function of the stress resultants of the continuous curved stress field.

As the stress magnitudes $\sigma_m$ and the tangent vectors $\vec{t}_m$ of the stress resultants of the continuous curved stress field are unknown, they will be derived from the defined initial stress state. Returning to the introductory example of section 3.1, the initial stress state was assumed to be the projection of the curved membrane stress state, which is described by a continuous curved stress field. Figure 3.3a illustrates the relations of a point on the initial surface, defined by $\vec{i}$, and a point on the continuous curved stress field, described by $\vec{m}$, under the assumption that the initial surface is a horizontal projection of the continuous curved stress field. Determining the map of the initial surface tangential vector $\vec{t}_i$ on the continuous curved stress field yields the vector $l_m \vec{t}_m$. The tangential vectors $\vec{t}_i$ and $\vec{t}_m$ are both unit vectors, such that the scalar factor $l_m$ describes the increase of the vector length, which occurs due to the geometrical constraints. The factor $l_m$ also describes the ratio of the stress magnitudes $\sigma_m$ to $\sigma_i$. In case of projection, the stress magnitudes change with the vector lengths of the associated tangential vectors. This correlation shall be extended to the general case in which, not only the length $l_d$ of the interconnecting vector between points on the initial surface and the continuous curved stress field changes, but also its direction described the unit vector $\vec{d}$. The geometrical relations of the general case are illustrated in fig. 3.3b. For the relation between stress resultants assigned to the initial surface $\sigma_i \vec{t}_i$ and the continuous curved...
stress field, $\sigma_m \vec{t}_m$ then applies to an arbitrary set of stress resultants denoted by the index $j$:

$$\sigma_{mj} \vec{t}_m = \sigma_{ij} \left( \vec{t}_{ij} + \frac{\partial (l_d \vec{d})}{\partial s_{ij}} \right)$$

(3.5)

The first derivative of eq. 3.5 with respect to the corresponding arc length on the initial surface $s_{ij}$ yields:

$$\frac{\partial (\sigma_{mj} \vec{t}_m)}{\partial s_{ij}} = \frac{\partial \sigma_{ij}}{\partial s_{ij}} \left( \vec{t}_{ij} + \frac{\partial (l_d \vec{d})}{\partial s_{ij}} \right) + \sigma_{ij} \left( \frac{\partial \vec{t}_{ij}}{\partial s_{ij}} + \frac{\partial^2 (l_d \vec{d})}{\partial s_{ij}^2} \right)$$

(3.6)

The right side of eq. 3.6 is substituted for the derivative of the stress resultant on the continuous curved stress field $\sigma_{mj} \vec{t}_{mj}$ with respect to the arc length $s_{ij}$ in eq. 3.4. The equilibrium condition of the continuous curved stress field results in:

$$\sum_j^n \left[ \frac{\partial \sigma_{ij}}{\partial s_{ij}} \left( \vec{t}_{ij} + \frac{\partial (l_d \vec{d})}{\partial s_{ij}} \right) + \sigma_{ij} \left( \frac{\partial \vec{t}_{ij}}{\partial s_{ij}} + \frac{\partial^2 (l_d \vec{d})}{\partial s_{ij}^2} \right) \right] + \vec{q} = \vec{0}$$

(3.7)

The differential equation 3.7 describes the transformation of the initial stress state into a continuous curved stress field, which is in equilibrium with the external loads $\vec{q}$.
3.4 Discussion

In this chapter, a differential equation (eq. 3.7) has been developed, which describes the relation between an initial stress state on an arbitrarily shaped initial surface and a curved membrane stress state, which is in equilibrium with the external loads.

The developed differential equation (eq. 3.7) does not describe a fundamentally new relation, but represents the constitutive differential equation of the membrane theory [11, p.167] in a different form. However, the actual achievement is, that this constitutive differential equation has now been implemented in a form finding method for the geometry of continuous curved membrane stress states. The basic idea, to use the membrane theory for form finding, has already been applied by Williams [55]. But in contrast to Williams’ approach [55], the method developed in this thesis puts the same idea in a more general context, as, theoretically curved stress fields of any shape can be generated.

Furthermore, with the introduction of an initial stress state in the form finding process, defined by tangent vector fields of stress resultants, a step towards an intuitively and consciously controlled load transfer has been made. A comparable procedure has been proposed by Block [7] for the determination of compression-only networks. In the form finding approach proposed by Block [7] an initial stress state as a projection of the actual stress state is defined. This procedure has been extended here to apply to general problems of form finding of continuous curved stress fields by removing the constraints, which occur in cases with a projective relation between initial and equilibrated stress state.

The user’s ability to strongly influence the load transfer does simultaneously form a disadvantage. Besides the differential equation, which describes the geometry of a continuous curved stress field based on a user-defined initial stress state, several boundary conditions are necessary to obtain a solution. Green and Zerna wrote about this problem: "In general it is not possible to satisfy all boundary conditions by using membrane theory, [...]" [13, p.407]. Green and Zerna [13], of course, approach the problem with different prerequisites, as they presume a given geometry of a membrane shell and aim to determine the corresponding stress function. Nonetheless, irrespective of a predefined geometry or stress function, as the constitutive equations of membrane theory and the method proposed in this thesis originate from the same principles, their statement does still apply. Accordingly, only for a few configurations of the initial stress state boundary conditions and thus solutions of the differential equation (eq. 3.7) can be found.

Besides very basic cases, the differential equation (eq. 3.7) cannot be solved explicitly. Numerical techniques, such as e.g. finite difference method, are in general necessary to find solutions to more advanced problems. Thus, the exact shape of a continuous curved stress field could in most cases only be approximated. Compared to a form finding method, which approximates the problem already from the beginning by a discrete approach, a continuous approach features some disadvantages. Especially the requirement of differentiability of the initial stress function and the fact that no abrupt changes of the external loads can be considered, limit the applicability of a continuous approach. Thus, the investigation of continuous curved stress fields shall not be continued. Instead, the developed procedure for the determination of continuous curved stress fields will be used for an approach to the generation of discrete curved stress fields.
4 Spatial Strut and Tie Networks

In the last chapter a method for the generation of continuous curved stress fields has been developed. Due to the mathematical problems related to continuous solutions in general, a further development towards an applicability of the method is omitted in favor of a discrete approach. The proposed discrete approach consists of two parts. At first, a spatial strut and tie network is generated, which is in equilibrium with the applied loads concentrated in the network nodes. Based on this network, a pattern of triangular stress fields is then developed, which is regarded as an approximation to continuous curved stress fields. This chapter deals with the generation of spatial strut and tie networks, while the next will explain their transformation into triangular stress fields.

The force density method [51] is a form finding method, which is purely based on the equilibrium conditions of the nodes of a pin-jointed network. Continuous curved stress fields also describe pure equilibrium solutions. The force density method, thus, provides a starting point for the development of a discretization of the method of continuous curved stress fields. The equilibrium of a node in a pin-jointed network is described by a non-linear vector equation, involving geometrical parameters and the forces of the concerned network members. Accordingly, the equilibrium conditions of all nodes of a network form a system of non-linear equations. Schek [51] proposed to transform the non-linear equations to linear ones by defining force to length ratios of each member. These user-defined ratios are used to influence the form of the equilibrated network.

Although choosing ratios of member forces to member lengths provides a good approach for transforming the equilibrium conditions of network nodes to linear equations, a statically founded relation does not exist between these two parameters. Thus, the choice of these parameters shall be decoupled here to gain a more intuitive way to influence the form of an equilibrated network. Block [7] achieved a decoupling of member force and member length by introducing an intermediate step in the method. As already discussed in chapter 3, first the horizontal projection of the equilibrated network is investigated [7]. A projected network is designed, including the projected member lengths. The projected member forces are determined only based on the equilibrium conditions of the projected system. Projected member lengths and projected member forces are, thus, defined individually. The form finding approach of spatial strut and tie networks shall not be limited to cases, which can be projected on a plane, to assure a general applicability of the method. Thus, the projected network used by Block [7] is interpreted more generally as an initial network, which is not limited to the geometrical constraints of projection.

The procedure, which has been developed for continuous curved stress fields, will basically be kept and adapted to the necessities of a discrete approach. In contrast to the continuous approach of chapter 3, not an initial surface, but an initial network is used to define an initial stress state. The set up of the arbitrarily shaped initial system defines the arrangement of nodes and members in the network and thereby provides a means to influence load transfer. Further control over the form of the spatial strut and tie network and its load bearing behavior is gained through the possibility to define the axial forces of the initial members. In general, the initial network does not have to meet equilibrium (see sec. 8.1). The spatial strut and tie network, which is in equilibrium with
The external loads, is obtained by displacing the nodes of the initial system. Thus, neither node positions nor lengths of the members are in general fixed to their initial value. The members of the initial network and the spatial strut and tie networks are presumed to be only subjected to axial loads and pin-jointed in the nodes.

4.1 Initial Network

The shape and the arrangement of members and nodes of the initial network may be arbitrary. In general, the initial network does not have to meet equilibrium conditions, considering neither the applied external loads nor solely initial member forces. Thus, the magnitude of initial member forces is also arbitrary and external loads are not included in the initial system. An initial member force is denoted by the scalar $F_{ini}$.

Supported nodes may be placed everywhere in the system. Just as every other node of the initial network, the position of supported nodes is not predefined to be fixed to their initial position. The reaction force of a support is represented by the vector $\vec{R}$.

External loads can only be applied at the nodes of the initial network. Thus, distributed loads must be summarized to single loads. External loads may act in any direction. An external load is represented by the vector $\vec{Q}$.

4.2 Generation of strut and tie networks

The equilibrated strut and tie network is obtained by displacing the nodes of the initial network in a way that every node is in equilibrium with the applied external load. The displacement of an initial node is expressed by the multiplication of the eccentricity value $e$ with the unit vector $\vec{d}$, which describes the displacement direction. If the initial system is a projection of the discrete curved stress field, the ratios of real to projected member length and of real to projected member force will match. This phenomenon has already been discussed in detail in section 3.3. As in the continuous approach (see ch. 3), this match of ratios is also presumed for the general case, so that the ratio of initial to equilibrated member length is assumed to equal the ratio of initial to equilibrated member force. The proposed definition of the relation between member forces of the initial and the strut and tie network, illustrated in fig. 4.1, forms the basis of the generation of discrete curved stress fields. Equation 4.1 results from this basic assumption.

$$F_{eq,AB} = F_{ini,AB} \frac{\left\| \vec{P}_B - \vec{P}_A \right\|}{\left\| \vec{P}_B' - \vec{P}_A' \right\|}$$  \hspace{1cm} (4.1)

The geometrical relation of an initial network node to its counterparts on the strut and tie network is defined by:

$$\vec{P}_i = \vec{P}_i' + e_i \vec{d}_i$$  \hspace{1cm} (4.2)
At each node of the initial system an equilibrium condition is formulated. The nodal equilibrium condition states that the sum of all member forces and the applied external load must result in zero. Equation 4.3 shows the equilibrium condition for an exemplary node A.

\[
\sum_{i \in I_A} \left( \frac{\vec{F}_{eq,i} \cdot (\vec{P}'_i - \vec{P}'_A)}{\| \vec{P}'_i - \vec{P}'_A \|} \right) + \vec{Q}_A = \vec{0} \quad (4.3)
\]

with \( I_A = \{ \text{indexes of neighboring nodes around} \; \vec{P}_A \} \)

At an exemplary supported node C a vector of reaction forces is added to the equilibrium condition, such that applies:

\[
\sum_{i \in I_C} \left( \frac{\vec{F}_{eq,i} \cdot (\vec{P}'_i - \vec{P}'_C)}{\| \vec{P}'_i - \vec{P}'_C \|} \right) + \vec{Q}_C + \vec{R}_C = \vec{0} \quad (4.4)
\]

with \( I_C = \{ \text{indexes of neighboring nodes around} \; \vec{P}_C \} \)

The relation between an arbitrarily shaped initial network and the corresponding strut and tie network results when eqs. 4.1 and 4.2 are substituted for the strut and tie member force and the position vectors of the network nodes in eqs. 4.3 and 4.4. This yields for a typical node \( \vec{P}_A \):

\[
\sum_{i \in I_A} \left( \frac{\vec{F}_{ini,i} \cdot (\vec{P}'_i + e_i \vec{d}_i) - (\vec{P}'_A + e_A \vec{d}_A)}{\| \vec{P}'_i - \vec{P}'_A \|} \right) + \vec{Q}_A = \vec{0} \quad (4.5)
\]

with \( I_A = \{ \text{indexes of neighboring nodes around} \; \vec{P}_A \} \)
and for a typical supported node $P_C$:

$$\sum_{i \in I_C} \left( F_{ini,Ci} \left( \frac{\vec{P}_i + e_i \vec{d} \cdot \vec{P}_i}{\| \vec{P}_i - \vec{P}_C \|} \right) - (\vec{P}_C + e_C \vec{d} \cdot \vec{P}_C) \right) + \bar{Q}_C + \bar{R}_C = \bar{0}$$  \hspace{1cm} (4.6)

with $I_C = \{ \text{indexes of neighboring nodes around } P_C \}$

The equilibrium conditions of all nodes of the initial network form a system of non-linear equations.

### 4.3 Control over the generation of strut and tie networks

The scalar unknowns in the system of equations describe magnitudes of initial member forces, components of displacement vectors of nodes and components of reaction forces. When using the proposed approach, the scalar unknowns outnumber the scalar equations. Control over the generation of strut and tie networks is gained by assigning values to the additional unknowns. For all arrangements of initial networks, the number of additional unknowns matches the sum of the number of members and the number of reaction force components. The set of unknowns, which are considered to be additional, is not predetermined. Hence, besides the actual assignment of values, the selection of a particular set of additional unknowns is also decisive for the result of the generation process.

Components of displacement vectors of nodes can be directly set. Assigning a value to one of the components results in limiting the displacement of a node to a plane. In case of two predetermined components, a direction of displacement is defined. The same effect is gained by defining relations between the components of displacement of a node, which additionally allows limitations independent of the axes of the global coordinate system. For some nodes of a system, such as supported nodes, the choice of all components of displacement is in general useful.

The value of member forces of the equilibrated strut and tie network may theoretically also be set directly, but this will cause mathematical problems for solving the overall equations system. Hence, the choice of the initial member forces is preferred instead. Although the definition of initial member forces allows no direct control over the member forces of the equilibrated strut and tie network, a sufficient control over the generation process is, however, provided. As the scale of chosen initial member forces remains, in general, unchanged, dependencies among members, like a classification into primary and secondary elements, can be defined. Besides, a direct influence on the resulting system is also gained by defining the distribution of compression and tension forces in the initial system, as the sign of member forces remains unchanged (see eq. 4.1).

Just as the components of node displacement, the components of reaction forces can be set directly. This specification can be a powerful means to control the load transfer. Especially in cases for which an uniform load transfer to all or a group of supports is aspired, a set-up can be used, in which a single component of the reaction force at the supports is set to a specific value or a relation among the involved reaction force components is defined. Besides the control over load transfer,
The solvability of the equation system describing the strut and tie network is, besides mathematical issues which are discussed in section 4.4, affected by the specified values of the additional unknowns. Only if these do not violate the equilibrium conditions of the aspired strut and tie network, a solution for the equation system is found. Such violations arise e.g. from an inadmissible definition of reaction force components or the limitation of the displacement component parallel to the applied loads for a group of neighboring nodes, in a way that they lie on a common plane. Depending on the arrangement of the network, further reasons for potential violations of the equilibrium conditions may occur.

For the further discussion of solution strategies, it is presumed that the supported nodes of the considered networks are fixed to their initial positions. The remaining number of additional unknowns then always matches the number of members in the network. To obtain optimal control over the generation of strut and tie networks, the choice of the set of additional unknowns should
only allow one solution for the corresponding equation system. Unique solutions for the strut and tie networks require uniquely defined initial member forces.

The most direct strategy is to set the magnitudes of all initial member forces. This approach will always enable an equilibrated solution of the network as long as the magnitudes of at least two initial member forces per node are different from zero. Even initial configurations, which violate the equilibrium conditions, will always result in an equilibrated system, as equilibrium is found by displacement of the nodes. Thus, this strategy is suitable especially to systems with only a few supported nodes, as shown in fig. 4.1a.
An alternative strategy to define all initial member forces in order to obtain a unique solution is to assume a projective relationship between the initial and the strut and tie network, which limits the displacements of all nodes to one and the same direction. This is obtained by either setting two components of the displacement vectors of all nodes to zero or by defining two relations between the components of the displacement vectors. Possible sets of initial member forces, which can be considered as additional unknowns besides the defined displacement direction, are those member forces, that must be set to uniquely determine a force diagram of the initial network. Figure 4.5 shows an example of a strut and tie network generated by assuming a projective relation. To uniquely define the force diagram (fig. 4.5c), the magnitude of an initial member force of one segment out of each of the curves A to E (fig. 4.5b) could be chosen, while the irrelevant initial member forces along curves F, 1 and 6 are assumed to be zero. With the initial member forces determined by a force diagram, a unique solution for the strut and tie network is obtained.

Projection in the broadest sense includes all kinds of linear limitation of node displacement. The conditions for networks or sets of nodes within networks, which have been limited in their displacement to lines of different directions, are comparable to the ones for the actual projection. Only those initial member forces that are necessary for the unique definition of the force diagram corresponding to the initial network may be set additionally. This condition does still apply to arrangements with curved initial networks (fig. 4.6). In contrast to planar initial networks, auxiliary external loads at the nodes must be introduced to meet the equilibrium conditions at the initial network.

**fig. 4.5**  
a) Strut and tie network generated by using a projective relation, each node is subjected to a single load of 5kN; b) plan view of the initial system; c) force diagram of the initial member forces

**fig. 4.6**  
Strut and tie network generated by using a projective relation based on a curved initial network, each node is subjected to a single load of 5kN
Figures 4.3 and 4.5 show different solutions of one and the same problem by selecting the values of different sets of additional unknowns. Both systems are subjected to the same vertical loads and the magnitudes assigned to the initial member forces, which are included in the set of additional unknowns, are identical in both examples. A projective relation allows the best control over the geometry of the resulting strut and tie network, as fig. 4.5a illustrates. Conversely, the choice of all initial member forces may potentially lead to deviations from the originally defined geometry as the difference between the planar initial system and the horizontal projection of the strut and tie network (dashed lines) in fig. 4.3 indicates. Concerning the efficiency of load transfer, the choice of all initial member forces leads to the better result for the investigated system. Especially in the back-most part of the system, the potential of bi-axial load transfer is used considerably better than in the projected system. This results from the possible free displacement of all nodes, so that the most suitable form of the system is reached for the assumed initial member forces.

Beside the two discussed strategies to select the set of additional unknowns, it is also possible to use a combination of both instead. Figure 4.4 illustrates a solution generated by using such a combination. To assure that the horizontal projection of the unsupported edge of the strut and tie network does not deviate from the initial system, the horizontal displacement of the corresponding nodes has been fixed. In contrast, the remaining nodes of the system have not been limited in their displacement. Instead, the magnitudes of the initial forces of the members between them have been set. The limitation concerning the choice of initial member forces as additional member forces does, in this case, only apply to the horizontally fixed nodes. By the combination of a projective relation along the unsupported edge and the choice of all member forces in the remaining system, the advantages of both strategies have been unified in the solution: control over the decisive geometrical parameters and an effective load transfer. An alternative approach is to fix only one horizontal component of node displacement, which allows a greater number of initial member forces to be specified directly (see figs. 4.7 and 4.8). The constraint for the definition of initial member force, formulated above for the projection strategy, applies analogously to the fixed

![fig. 4.7](image) Strut and tie network with fixed horizontal components of displacement in lateral direction, each node is subjected to a single load of 5kN
direction of displacement. Accordingly, the initial member forces must meet the nodal equilibrium condition within the initial network in the fixed direction of displacement, while the equilibrium conditions in the free directions of displacement may be violated in the initial network, as equilibrium can be found through displacement of the nodes. From the above discussed systems with limited horizontal displacement of all or a group of nodes, a general condition for the additional choice of initial member forces can be concluded. The initial member forces must meet the nodal equilibrium conditions in the fixed directions of displacement and the set of specified initial member forces must not be redundant for the determination of all initial member forces of the network.

The so far discussed sets of additional unknowns always lead to unique solutions as all initial member forces are directly or indirectly defined uniquely. If using alternative sets of additional unknowns, the equation systems possess infinite or no solutions. Besides desirable solutions for the strut and tie network, the solution sets also contain results that define strut and tie networks of an irregular shape which can hardly be practically used. Related to the desirable solutions, two types of solution sets are distinguished, the first, which only contains one desirable solution, and a second with infinite desirable solutions (see fig. 4.9). The type of solution set which is obtained depends on the choice of additional unknowns. If the chosen additional unknowns limit the rise of the strut and tie network directly through a component of displacement or indirectly through an initial member force, a solution set with only one desirable solution is obtained.

Regarding the discussed strategy with projection between the initial and the strut and tie network (see fig. 4.5), a solution set with infinite solutions is obtained, if e.g. all three components of displacement of a group of nodes are defined instead of the specified initial member forces. Although the choice of the group of nodes, at which the third component of displacement is defined is arbitrary, the best results are obtained, if the group of nodes consists of one node per curves A to E (fig. 4.10). As the rise of the strut and tie network is specified through the vertical component of displacement, the solution set to the corresponding equation system contains only one desirable solution.

The sets of additional unknowns discussed so far did only involve components of displacement

![fig. 4.8](image.png) Strut and tie network with fixed horizontal component of displacement in longitudinal direction, each node is subjected to a single load of 5kN
and initial member forces. If including components of reaction forces in the set of additional unknowns, the resulting equation systems then possess infinite or no solutions, as the initial member forces cannot be defined uniquely due to the limited number of additional unknowns. However, as
the use of reaction force components gives easy control over the load transfer, it is worth discussing. In the strategy with a projective relation between initial and strut and tie network (fig. 4.5), initial member forces have been chosen as the remaining additional unknowns. For an identical arrangement of nodes and members, as used in this example and analogous systems, the initial member forces may e.g. be replaced by a relation among the vertical reaction force components of the supported nodes at the one end of the curves A to E. As the relation among the reaction force components of these five supports reduces the number of remaining undefined additional unknowns only by four, the set of chosen additional unknowns can be completed by any initial member force or a third component of displacement of any node. In both ways it is observed that only one solution out of the solution set can be considered as desirable, while the other obtained strut and tie networks are of an irregular shape (fig. 4.9a). If the set of additional unknowns consists of the components of displacement, which are necessary to define the projective relation between initial and strut and tie network, and of the relations among reaction force components or actual values for reaction force components, the solution sets contain an infinite number of desirable shapes for the strut and tie network with different rises, as none of the chosen additional unknowns limits the rise of the strut and tie network directly through a vertical component of displacement or indirectly through an initial member force (fig. 4.9b). The possibilities for controlling or at least influencing the actually obtained solution out of the solution set resulting from the computation of the equation system, is discussed in section 4.4.

Applying the above discussed strategies for the choice of the set of additional unknowns allows to generate complex equilibrated networks. Figure 4.11a illustrates a strut and tie network with closed curves as cross section, which is subjected to a combination of dead and wind loads. It was generated by setting the z-components of displacement of all nodes to zero. This results in a limitation of the displacement of the nodes to horizontal planes. Additionally, the initial member forces of the horizontal rings were chosen. Beside the topmost ring, which was defined to be in

**fig. 4.10** Strut and tie network with fixed horizontal displacement at all nodes and a defined vertical component of displacement at one node per curves A to E, each node is subjected to a single load of 5kN
tension, compression member forces were assumed for the other rings.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig4.11.png}
\caption{Initial (a) and strut and tie (b) network generated by choosing a combination of displacement components and initial member forces, the initial member forces specified in a) have each been set for all members of a ring, the nodes are subjected to single loads with a vertical component of 5kN and a horizontal component of 1.7kN.}
\end{figure}

4.4 Solutions of the system of equations

Two types of strategies for the choice of the additional unknowns have been discussed, strategies that lead to unique solutions and strategies from which infinite solutions result. The distinction between the two types is done based on the unique definition of the initial member forces. This unique definition leads to a transformation of the non-linear equation system to a linear one. Strictly speaking the equation systems remain non-linear, if the set of additional unknowns consists of a combination of initial member forces and components of displacement. However, if a unique definition of all initial member forces can be deduced from the set of additional unknowns, a linear equation system is achieved through an intermediate step in these cases. The strategies, in which all initial member forces are defined directly or indirectly, follow the same approach as the force density method [51]. The only difference is that the choice of member forces and member
lengths is decoupled and done on an initial network as also proposed by Block [7]. The resulting system of linear equations is always solvable for all practically relevant cases.

The non-linearity of the equation system remains, if not all magnitudes of the initial member forces are defined. A solution of the system of non-linear equations can only be approximated using e.g. the Newton-Raphson method. To approach a solution by the Newton-Raphson method, an initial guess for the values of the unknowns must be made, which means that for all components of displacement of nodes, initial member forces and reaction force components, which are not part of the set of additional unknowns, an initial value for the numerical search for the solution must be set. The necessary quality of the initial guess to find a solution of a non-linear equation system depends on the individual arrangement of the network. In general, the required quality increases with the complexity of the network. Furthermore, the actually obtained solution out of the solution set is controlled by the initial guess. Regarding the solution sets obtained from non-linear equation systems a distinction depending on the number of desirable solutions was made in section 4.3. For the solution sets containing only one desirable solution it is observed, that even with an imprecise initial guess for the values of the unknowns, the desirable solution is obtained. Accordingly, strategies for the choice of additional unknowns, that result in this type of solution set, possess an applicability which is comparable to strategies that lead to unique solutions. A different situation arises for strategies that result in an infinite number of desirable solutions. The influence on the actual obtained solution out of the solution set is limited and does, in general, not allow to determine a strut and tie network with a specific rise. Nonetheless, depending on the quality of the initial guess of the remaining unknowns, the actually obtained solution lies within an acceptable range around the initial guess.

4.5 Lateral displacement of loads

A lateral displacement of loads may occur, unless the displacement of loaded nodes has been defined to be tangential to the line of action of the loads. This lateral displacement causes a change of the originally applied load situation, which is, of course, not admissible. If a generated strut and tie network is used for the determination of internal forces or analysis of a structural surface, a lateral displacement of loaded points must, thus, be avoided.

A different situation arises from a strut and tie network applied for form-finding. Most external loads are dependent on the form of the structure. As loads are dependent on the final form of a structure, magnitudes and points of application of loads will adapt to a changing form of the structure. Thus, the final form of a structure can only be found in an iterative process and a lateral displacement of the initially assumed loads is compensated by iteratively adapting the loads to the evolving structure.
4.6 Section Forces of a Grid Shell from an Equivalent Strut and Tie Network

As already shown in fig. 1.1, the section forces of an arch structure can be deduced from a thrust line. The two-dimensional pendant of an arch is a grid shell, just as is a strut and tie network for a thrust line. Accordingly, the section forces of a grid shell can be deduced from an equivalent strut and tie network. A strut and tie network is equivalent if it possesses identical supports and loading as the grid shell. Figure 4.12 shows a grid shell and a possible equivalent strut and tie network.

![Grid shell structure and equivalent strut and tie Network](image)

\[ N_i = F_{eq,i} \cdot \cos \alpha_i \]  
\[ V_i = F_{eq,i} \cdot \sin \alpha_i \]  
\[ M_i = F_{eq,i} \cdot \cos \alpha_i \cdot e_i = N_i \cdot e_i \]
with $\alpha_i$ as the angles between the strut and tie member forces $F_{eq,i}$ and the normal forces $N_i$. The number of equivalent strut and tie networks, that express the section forces of a grid shell, is unlimited, as the approach of strut and tie networks is based on the lower bound theorem of the theory of plasticity. Torsion and transverse bending cannot be considered by this approach.

### 4.7 Section Forces of a Beam Grillage from Equivalent Strut and Tie Networks

A beam differs from an arch in its load bearing behavior, which is dominated by bending instead of axial force. The deduction of section forces from a thrust line always yields solutions that may show bending, but in which axial forces are an inherent and rather dominant component. If a thrust line is now combined with a tie, as shown in fig. 4.14, a different situation arises. The force couple formed by the thrust line and the tie allows to describe a loading bearing behavior only consisting of bending and shear without axial forces (fig. 4.14). The relation of internal forces of a beam and the force couple described by the thrust line and the tie has already been shown by Muttoni [38, pp.234].

![fig. 4.14](image)

**fig. 4.14**  Simple beam under constant load with a tied thrust line describing its section forces

An analogous model to the tied thrust line in two dimensions is obtained by combining two strut and tie networks with an identical arrangement of nodes and members, as shown figs. 4.15 and 4.16a. Within this thesis these will be referred to as combined strut and tie networks. The loads are not necessarily borne by only one of these two strut and tie networks, as loads might also be applied to both of them (fig. 4.16b). The number of possible solutions is unlimited, as for single strut and tie networks applies.

The relation of the section forces of the beam grillage members and the members of the equivalent combined strut and tie networks, illustrated in fig. 4.17, is expressed by:

$$N_i = F_{eq,i}^I \cdot \cos \alpha_i^I + F_{eq,i}^{II} \cdot \cos \alpha_i^{II}$$  \hspace{1cm} (4.10)

$$V_i = F_{eq,i}^I \cdot \sin \alpha_i^I + F_{eq,i}^{II} \cdot \sin \alpha_i^{II}$$  \hspace{1cm} (4.11)

$$M_i = F_{eq,i}^I \cdot \cos \alpha_i^I \cdot e^I + F_{eq,i}^{II} \cdot \cos \alpha_i^{II} \cdot e^{II}$$  \hspace{1cm} (4.12)

With $F_{eq,i}^I \cdot \cos \alpha_i^I = -F_{eq,i}^{II} \cdot \cos \alpha_i^{II}$, a load bearing behavior with only bending and shear is described, which is usually predominant in beam grillages. The values $\alpha_i^I$ and $\alpha_i^{II}$ describe the angles between the normal forces $N_i$ and the member forces $F_{eq,i}^I$ and $F_{eq,i}^{II}$, respectively.
fig. 4.15  a) Axes of a beam grillage, b) Equivalent combined strut and tie networks

fig. 4.16  a) Equivalent combined strut and tie networks with loads applied to the network in tension; b) Equivalent combined strut and tie networks with loads applied to both

fig. 4.17  a) Node of a beam grillage; b) Corresponding nodes of equivalent combined strut and tie networks, in which the loads are only applied to one of the networks
4.8 Discussion

For the generation of spatial strut and tie networks, a method has been developed to determine equilibrated shapes of pin-jointed networks. Although spatial strut and tie networks are considered a first step towards discrete curved stress fields, the method is, nonetheless, a generally applicable tool, as sections 4.6, 4.7 and the form finding example in section 8.1 show. The form of an strut and tie network is obtained by a transformation of a kinematic initial system into an equilibrium state for a particular load case.

A determination of the form of equilibrated pin-jointed networks solely using the equilibrium conditions of the nodes has already been proposed by Schek [51] with the force density method. Although this method is a powerful tool to determine shapes of pin-jointed network structures, it does not allow for the same high level of control of the shape, as is gained through an intermediate step using an initial network. Thrust network analysis [7] on the other hand, uses horizontal projection as an intermediate step for the generation of a compression-only network. Based on Maxwell’s definition of reciprocal figures, the projected network is developed simultaneously with the force diagram of the projected member forces, such that equilibrium is assured. The necessary additional unknowns for the determination of the equilibrated network can, thus, be intuitively chosen by the user. This approach for the intermediate step, however, limits the application of thrust network analysis [7] to equilibrated networks, which can be projected on a horizontal plane. The method of strut and tie networks, in contrast, provides an applicability to any shape of networks, while still allowing an intuitive control over the generation process. It must be acknowledged that, compared to thrust network analysis [7], the proposed method is less intuitive, owing to the wider range of applications. However, provided a rough idea of the aspired form and some experience in the relationship of form and forces, the proposed method represents an efficient tool to create equilibrated pin-jointed networks.

The Newton-Raphson method has been proposed to solve the non-linear equation systems. It approximates the solution based on an initial guess for the unknowns. With this method, a solution can be found for every choice of additional unknowns and specified values, that do not violate equilibrium conditions of the aspired strut and tie network. However, if other strategies for the choice of additional unknowns are used, as those discussed in section 4.3, the actual benefit from the solutions found reduces remarkably, as the necessary quality of the initial guess for the unknowns increases. The increase of the necessary quality of the initial guess goes along with an increase of complexity of the investigated networks. For simple arrangements of nodes and members, a rather imprecise initial guess is sufficient to find a solution. In contrast, for systems with a high number of nodes and members, solutions are only found with an initial guess that is close to the actual solution.
5 Discrete Curved Stress Fields

The generation of strut and tie networks was the first step in the discrete approach to describe the stresses of shells and slabs. The development of discrete curved stress fields forms the second step. Although strut and tie networks provide an intuitive control over the load distribution in a shell or slab, the stresses within the structure cannot be directly concluded. To transform the member forces of a strut and tie network into bi-axial stresses, they are, first, converted into one-axial stress fields, which form nodal zones in the nodes of the strut and tie network. Such nodal zones are widely used in the analysis of planar concrete structures (see [39]). Depending on the number of adjoined members in the node and geometrical constraints, the nodal zone is subdivided into triangular stress fields. If the size of the nodal zones is maximized, a pattern of triangular stress fields throughout the strut and tie network evolves. These patterns of triangular stress fields approximate continuous curved stress fields, as discussed in chapter 3.

The discrete curved stress fields represent the curved equivalent to stress fields in planar systems (see [15], [39]). Due to the kinked shape of discrete curved stress fields, the number of constraints which must be considered is significantly higher than in the planar case. Thus, not only constant but also linear stress fields are investigated for their applicability. Strut and tie networks may be used as a form finding tool, but do, in general, only represent an equilibrium solution under the given statical constraints. As this also applies to curved stress fields, the internal stresses of shells or slabs may be deduced from these. The fundamental principle and necessary constraints are investigated and discussed.

5.1 Constant Stress Fields

A constant stress field is a triangular field with a constant planar bi-axial stress state. It is widely used in the rigid-plastic analysis of planar reinforced concrete structures subjected to in-plane loads [15, 39]. The constant stress state yields constant boundary stresses at the edges of the stress field, which can be summarized in a resultant boundary force applied in the middle of the edge (see fig. 5.1). The boundary forces must meet the following equilibrium conditions:

\[ \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \]  \hspace{1cm} (5.1)
\[ \vec{F}_1 \times (\vec{P}_C - \vec{P}_A) + \vec{F}_2 \times (\vec{P}_B - \vec{P}_A) = 0 \]  \hspace{1cm} (5.2)

Equations 5.1 and 5.2 assure the equilibrium of forces the torque equilibrium. Equation 5.2 simultaneously expresses the equality of shear stress at the corners of the stress field [15]. As the stress field is planar, the vector eqs. 5.1 and 5.2 reduce to three scalar equations in total.
5.2 Linear Stress Fields

A triangular linear stress field is defined by three stress states in each of its corner points and corresponding linear functions [15]. The stress states are expressed by the stress tensors $T^A$, $T^B$ and $T^C$ (fig. 5.2) with the corresponding linear functions $L_A$, $L_B$ and $L_C$ (fig. 5.3). The stress tensor of the stress field is then defined by:

$$T = T^A L_A + T^B L_B + T^C L_C$$

(5.3)
The linear stress state within the stress field yields linear boundary stresses at the edges, which vary in intensity and direction. The boundary stresses can be summarized by two resultant boundary forces per edge. To meet force equilibrium, the following condition must apply for the boundary forces:

\[
\vec{F}_{11} + \vec{F}_{12} + \vec{F}_{21} + \vec{F}_{22} + \vec{F}_{31} + \vec{F}_{32} = \vec{0} \tag{5.4}
\]

The equality of shear stress at the corner points [15] is expressed by:

\[
\vec{F}_{11} \times (\vec{P}_B - \vec{P}_A) + \vec{F}_{12} \times (\vec{P}_C - \vec{P}_A) = 0 \tag{5.5}
\]

\[
\vec{F}_{12} \times (\vec{P}_C - \vec{P}_B) + \vec{F}_{21} \times (\vec{P}_A - \vec{P}_B) = 0 \tag{5.6}
\]

\[
\vec{F}_{22} \times (\vec{P}_A - \vec{P}_C) + \vec{F}_{31} \times (\vec{P}_B - \vec{P}_C) = 0 \tag{5.7}
\]

The eqs. 5.5 to 5.7 already include the condition of torque equilibrium, which results from the sum of these three equations [15].

### 5.3 Nodal Zones of Strut and Tie Networks

A member force can be easily transformed to one-axial constant stress fields. By assigning a strip of a surface to a member, its force can be transformed to stresses. Depending on the position of the member within the strip, one or two constant one-axial stress fields are necessary to meet equilibrium (fig. 5.4). A single constant stress field is sufficient, if the member is positioned in the middle of the strip (fig. 5.4a). As this constraint cannot be met for the final nodal zones, a member force will be transformed to two constant one-axial stress fields (fig. 5.4b).
fig. 5.4 Transformation of a member force to: a) a single constant one-axial stress field, b) two constant one-axial stress fields

To enable an adaption of the nodal zones to a double curved surface, the two constant uniaxial stress fields per member cannot be on a common plane. Equilibrium conditions require that the boundary in-between the two stress fields does not match the member axis (fig. 5.5).

fig. 5.5 Member of a strut and tie network with equivalent constant uniaxial stress fields

At the nodes of the strut and tie network, the two constant one-axial stress fields for each member form a nodal zone with two edges per member (fig. 5.6).
fig. 5.6  Nodal zone formed by the constant stress fields of the members of the strut and tie network at a single node.

By maximizing the widths of the stress fields, the stress fields themselves vanish with only the nodal zones remaining (fig. 5.7).

fig. 5.7  Spatial strut and tie network with maximized nodal zones.
5.4 Division of Nodal Zones into Stress Fields

Division of the nodal zone into stress fields means to divide the nodal zone geometrically into triangles. As the edges of the nodal zone are, in general, not on a plane, kinks between these triangles evolve. At the edges between stress fields an additional force is needed to meet equilibrium between the boundary forces of the neighboring stress fields (fig. 5.8). This additional force is provided by the applied external loads. The single load applied to the node of the strut and tie network is distributed along the edges of the corresponding stress fields. From the equilibrium conditions at a typical edge between linear stress fields results:

\[ \vec{F}_{21A} + \vec{F}_{21B} + \vec{Q}_{21} = \vec{0} \]  
(5.8)

\[ \vec{F}_{22A} + \vec{F}_{22B} + \vec{Q}_{22} = \vec{0} \]  
(5.9)

The outside edges of the nodal zone may also be involved, as long as they form an edge between stress fields and are, accordingly, not part of an unsupported boundary of the set of curved stress fields.

The direction of the edge loads must be limited to the direction of the corresponding single load of the strut and tie network, to avoid a transfer of stresses parallel to the stress fields. As a cause of the discretized geometry and the equilibrium conditions at the edges of the stress fields, edge loads may show an orientation opposite to the one of the single load applied to the corresponding network node. The equilibrium conditions related to the single load and the edge load resultants within the nodal zone and the constraint of identical direction demand that:

\[ \sum_{i \in I} \vec{Q}_i = \vec{0} \]  
\[ \text{with } I = \{ \text{indexes of all edge load resultants of the nodal zone} \} \]  
(5.10)

\[ \vec{Q}_i = \frac{\vec{Q}_i \cdot \vec{Q}}{\|\vec{Q}\|^2} \vec{Q} \]  
(5.11)
with \( \vec{Q} \) as the vector of single load applied to the network node. If no edge loads are applied along the outside nodal zone edges, the adjacent stress fields must be tangential to the corresponding member axis of the strut and tie network.

Planar systems of constant stress fields are kinematic. Solutions can only be found, if the geometry of the stress fields is specifically adapted to the applied loads, in a way that an unstable state of equilibrium evolves [15]. A system of constant triangular stress fields is only statically determinate, if the system consists of exactly three stress fields, in which each of the stress fields possesses a common edge with the other two (fig. 5.9). As this special case cannot be expanded to be useable for non-triangular nodal zones, it has no meaning for further consideration.

![fig. 5.9 Stable system of constant triangular stress fields](image)

The fact that planar systems of constant stress fields are kinematic also applies if kinks between the stress fields are introduced to adapt the stress fields to a curved geometry. Due to a curved shape and respective geometrical constraints, the complexity to adapt the geometry to create a system in an unstable state of equilibrium increases significantly. Systems of linear stress fields form stable systems, due to their two boundary forces per edge. Accordingly, solutions can be found for every arbitrary pre-defined geometry of the system of stress fields, such that, besides geometrical constraints, no further conditions must be taken into account for the division of a nodal zone into linear triangular stress fields.

### 5.5 Deduction of the Section Forces of a Shell from an Equivalent Discrete Curved Stress Field

Analogously to the deduction of section forces of a grid shell from an equivalent strut and tie network, as shown in section 4.6, the stresses of a shell can also be deduced from a statically equivalent set of curved stress fields (fig. 5.10). As constant stress fields can be regarded as a special case of linear stress fields, the following deduction only considers linear stress fields.
First of all, the shell needs to be reduced to its middle surface, which in turn is approximated by triangles. The corner points of these triangles and the corner points of the stress fields of the equivalent set of discrete curved stress fields are connected by multiples of the unit vector $\vec{d}$ (fig. 5.11). If the equivalent curved stress fields represent the loading of the shell, then the triangles,
which approximate the middle surface of the shell, are combined stress, shear and moment fields.
Regarding the boundary forces of the curved stress fields, the transformation of these creates in-plane forces, transverse shear forces and moments at the corresponding application points of the triangular fields of the shell. The transformation rules can be deduced using the equilibrium conditions. If equilibrium is met for the boundary forces, the curved stress field and the combined stress, shear and moment field are also in equilibrium. Related to the shear field, all necessary conditions are met. For stress and moment fields, in contrast, also the condition of equality of shear stress at the corner points must be met. While the compliance of the transformation with the equilibrium conditions is obviously given, the condition of equality of shear stress needs a more detailed investigation.

The vector equations 5.5 to 5.7 result from the condition of the equality of shear stress. Splitting eq. 5.5 up into its components yields:

\begin{align*}
F_{32,y} (P_{2,z} - P_{1,z}) - F_{32,z} (P_{2,y} - P_{1,y}) + F_{11,y} (P_{3,z} - P_{1,z}) - F_{11,z} (P_{3,y} - P_{1,y}) &= 0 \\ F_{32,z} (P_{2,x} - P_{1,x}) - F_{32,x} (P_{2,z} - P_{1,z}) + F_{11,z} (P_{3,x} - P_{1,x}) - F_{11,x} (P_{3,z} - P_{1,z}) &= 0 \\ F_{32,x} (P_{2,y} - P_{1,y}) - F_{32,y} (P_{2,x} - P_{1,x}) + F_{11,x} (P_{3,y} - P_{1,y}) - F_{11,y} (P_{3,x} - P_{1,x}) &= 0
\end{align*}

The scalar notation of eqs. 5.6 and 5.7 can be written analogously. If projecting a valid stress field \( S_1 \) and its boundary forces on the x-y-plane (fig. 5.12), the projection still complies with the condition of the equality of shear stress, as the left side of eqs. 5.12 and 5.13 becomes zero and eq. 5.14 remains unchanged. If moving the corner points of the stress field \( S_1 \) along the z-axis by arbitrary values, a new stress field \( S_2 \) is obtained. The boundary forces of \( S_2 \) are projections of the boundary forces of stress field \( S_1 \) on the plane described by stress field \( S_2 \). To assure a feasible projection it is presumed that the following applies for the displacement vectors \( \vec{v}_1 \), \( \vec{v}_2 \) and \( \vec{v}_3 \), which describe the displacement of the corner points of \( S_1 \) to its image of \( S_2 \):

\[ -\infty < \| \vec{v}_1 \|, \| \vec{v}_2 \|, \| \vec{v}_3 \| < \infty \] (5.15)

The projections of \( S_1 \) and \( S_2 \) on the x-y-plane are identical. The eqs. 5.12 to 5.14 describe the torque equilibrium of the two boundary forces next to a common stress field corner with respect to the midpoint of the stress field. The moment vectors for each of the boundary forces are normal to the plane of the respective stress field. Accordingly, the respective moment vectors of the projected stress field on the x-y-plane represent the z-components of the moment vectors of the stress fields \( S_1 \) and \( S_2 \). If the condition of the equality of shear stress is met for the projected stress field, the sum of the moment vectors of the two boundary forces next to a common stress field corner results in \( \vec{0} \). The z-component of the respective vector sum of the stress field \( S_2 \), and thereby the vector sum itself, is also \( \vec{0} \). Thus, the described transformation of a valid stress field \( S_1 \) yields, in turn, a valid stress field \( S_2 \).

The orientation of the coordinate system, on which this deduction is based on, is in general arbitrary. Thus, the described facts are not only valid for displacements along the global z-axis but for any direction. It has been shown that the image stress field \( S_2 \) of a valid stress field \( S_1 \) is in turn a valid stress field, as it has been created through displacement of the corner points along paral-
Transformation of a linear stress field $S_1$ to a linear stress field $S_2$

de displacement vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ with its boundary forces being projections of the original boundary forces on the new stress field’s plane. The displacement vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ must not be perpendicular to the normal vectors of one of the stress field planes. The displacement vector in an arbitrary point of the stress field is expressed by:

$$\vec{v} = L_1 \vec{v}_1 + L_2 \vec{v}_2 + L_3 \vec{v}_3$$  \hspace{1cm} (5.16)

with $L_1$, $L_2$ and $L_3$ as linear functions describing the influence of the corresponding displacement vector (see fig. 5.3).

If displacing a linear stress field by a linear vector function $\vec{v}$ (see eq. 5.16), also a transverse shear stress field and a moment field are necessary to express the original situation. A moment field, which is defined by a linear stress field with a linearly changing eccentricity, is determined by a quadratic function. A quadratic moment field can be represented by two linear stress fields with a linearly changing distance. If the stress fields, used to describe the moment field, can be found by...
displacement of the corner points of the original stress field, the moment field is also valid.

The boundary forces of a single discrete curved stress field can be referred to a triangular field on the middle surface of a shell (fig. 5.11). Consequently, this will create a stress, shear and moment field, which, in turn, meets the equilibrium conditions and the condition of the equality of shear stress. The boundary forces and moments of the field on the middle surface of the shell are determined by:

\[
N_{mn} = \vec{F}_{mn} \cdot \vec{e}_{Nm} \tag{5.17}
\]

\[
N_{Vmn} = \vec{F}_{mn} \cdot \vec{e}_{NVm} \tag{5.18}
\]

\[
V_{mn} = \vec{F}_{mn} \cdot \vec{e}_z \tag{5.19}
\]

\[
M_{Bmn} = \left( \vec{F}_{mn} \times (e \vec{d}) \right) \cdot \vec{e}_{NVm} \tag{5.20}
\]

\[
M_{Dmn} = \left( \vec{F}_{mn} \times (e \vec{d}) \right) \cdot (-\vec{e}_{Nm}) \tag{5.21}
\]

\[
M_{Zmn} = \left( \vec{F}_{mn} \times (e \vec{d}) \right) \cdot \vec{e}_z \tag{5.22}
\]

with indexes \( m \) and \( n \) denoting the position of the boundary force or moment and the unit vectors:

\[
\vec{e}_{Nm} = \begin{pmatrix} e_{Nmx} \\ e_{Nmy} \\ 0 \end{pmatrix}, \quad \vec{e}_{NVmn} = \vec{e}_z \times \vec{e}_{Nm} = \begin{pmatrix} -e_{Nmy} \\ e_{Nmx} \\ 0 \end{pmatrix} \tag{5.23}
\]

In contrast to the common definition of internal forces and moments, an in-plane bending moment vector \( M_{Zmn} \) is defined, which occurs if the displacement vector \( \vec{d} \) is not parallel to the \( z \)-axis of the local coordinate system of the middle surface stress field. This specialty is caused by an incompatibility of the definitions of the local coordinate system and the orientation of the displacement vector \( \vec{d} \) and is discussed in detail in section 6.1.

\[\text{fig. 5.13} \quad \text{A slab and the equivalent combined sets of curved stress fields}\]
5.6 Deduction of the Section Forces of a Slab from Equivalent Discrete Curved Stress Fields

In section 4.7, the deduction of internal forces of a beam grillage from a set of two combined strut and tie networks has been discussed, while in section 5.5, the conditions for the transformation of stress fields to combined stress, shear and moment fields are deduced. The combination of both approaches leads to the deduction of the section forces of a slab from two combined sets of curved stress fields (fig. 5.13) which are based on two combined strut and tie networks as shown in figs. 4.16 and 4.17. The boundary forces and moments of a triangular field on the middle surface of a slab are deduced from the boundary forces of the corresponding two stress fields (fig. 5.14). The principle of parallel displacement vectors, which connect the corner points of the stress fields to the fields on the middle surface (see sec. 5.5), now has to be applied to the stress fields of both sets of curved stress fields to obtain valid fields on the middle surface of the slab.

*Fig. 5.14* Transformation of two linear stress fields to a combined stress, shear and moment field on the middle surface of a slab
The boundary forces and moments of the combined stress, shear and moment field on the middle surface of the slab can be determined by:

\[ N_{mn} = \left( \vec{F}_I^{mn} + \vec{F}_II^{mn} \right) \cdot \vec{e}_{Nm} \]  
\[ N_{Vmn} = \left( \vec{F}_I^{mn} + \vec{F}_II^{mn} \right) \cdot \vec{e}_{NVm} \]  
\[ V_{mn} = \left( \vec{F}_I^{mn} + \vec{F}_II^{mn} \right) \cdot \vec{e}_z \]  
\[ M_{Bmn} = \left( \vec{F}_I^{mn} \times (eI^{d}) + \vec{F}_II^{mn} \times (eII^{d}) \right) \cdot \vec{e}_{NVm} \]  
\[ M_{Dmn} = \left( \vec{F}_I^{mn} \times (eI^{d}) + \vec{F}_II^{mn} \times (eII^{d}) \right) \cdot (-\vec{e}_{Nm}) \]  
\[ M_{Zmn} = \left( \vec{F}_I^{mn} \times (eI^{d}) + \vec{F}_II^{mn} \times (eII^{d}) \right) \cdot \vec{e}_z \]

The superscripts \(^I\) and \(^II\) refer to the respective set of curved stress fields.

5.7 Discussion

The approach of discrete curved stress fields allows a transformation of the member forces of strut and tie networks into bi-axial stress states. Thereby the intuitive control over the load transfer, which is provided by strut and tie networks, can be used for two-dimensional structures subjected to lateral loads. The inner stresses of an actual structure can then be deduced from the determined sets of discrete curved stress fields.

This two-step procedure, which consists of strut and tie networks and their conversion to curved stress fields, requires specific geometrical constraints, which may partly result in edge loads of opposite orientation compared to the actually applied loads. A different procedure to determine the geometry of the pattern of triangular stress fields may, of course, lead to solutions avoiding edge loads of opposite orientation. However, as the intuitive control over the load transfer by the use of strut and tie networks would be lost, this partly unrealistic approach to the distribution of the external loads over a nodal zone is taken into account.

For the transformation of the member forces of strut and tie networks into bi-axial stress states, two types of triangular stress fields have been discussed, constant and linear stress fields. In the software application, described in section 7.2, standardized concepts to divide the middle surface into nodal zones and the nodal zones into stress fields are used, which are mainly based on geometrical constraints. As systems of constant stress fields require an adaptation of the stress field geometry to the actual applied loads, due to their statical overdeterminacy, linear stress fields are used instead. Concepts for the transformation of strut and tie networks into sets of constant stress fields may be subject to further research.

For the generation of strut and tie networks, the free displacement of nodes is a fundamental principle, such that in general, also a displacement of nodes laterally to the applied external loads is allowed (see ch. 4.5). Sets of curved stress fields are meant to represent the internal stresses of corresponding shell or slab structures. As a displacement of nodes laterally to the applied external load would change the actual load case, curved stress fields may solely be based on those strut and tie networks that only allow a displacement of nodes parallel to the applied external loads.
6 Bounds of Eccentricity

The method of curved stress fields is a holistic description of internal forces of shells and slabs. It avoids the separate consideration of the single internal forces, as usually applied [53, p.508]. The possibility to illustrate the internal forces in a single figure is its main benefit. Thus, yield criteria, which require a break down of the forces of a curved stress field to the usually applied internal forces, would undermine the underlying concept of curved stress fields. The method presented in this chapter is intended to be a description of the load bearing capacity of shells and slabs retaining the basic idea of curved stress fields.

An illustration of the load bearing capacity of a structure, which is equivalent to the visualization of the internal forces, must accordingly also be represented by an eccentric membrane stress state. Based on the load bearing capacity of the structure and the same underlying membrane stress state as defined by the corresponding curved stress field, two eccentricities can be determined. If the considered curved stress field had exactly one of these two eccentricities towards its reference surface, the structure would be loaded up to its ultimate limit state. Thus, the two eccentricity values and the membrane stress state of the corresponding curved stress field together represent the load bearing capacity of the structure, are referred to as the bounds of eccentricity. These bounds of eccentricity represent, strictly speaking, merely lower bounds to the ultimate strengths [29]. The influence of buckling is not considered for the determination of the bounds of eccentricity.

6.1 Element with inclined section planes

The determination of structural resistance requires the consideration of the structure not only by its middle surface but with its actual depth. In sections 5.5 and 5.6, the stress state defined by the eccentric curved stress field has been referred to the respective triangular field on the middle surface of the structure. These triangular fields are extended by the depth of the structure to determine the load bearing capacity.

fig. 6.1 Comparison of (a) an element with orthogonal edges and (b) an element with inclined section planes

The moment vectors, resulting from the eccentricity of the curved stress fields towards the counterparts on the middle surface, are all tangential to the same plane. The displacement vector \( \vec{d} \), which is identical in every point of the middle surface as discussed in section 5.5, is the normal vector.
to this plane of moments. The bending and drilling moment vectors, acting at the section planes of an orthogonal element of the structure as shown in fig. 6.1a, are tangential to the middle plane of the element. Unless the normal vector to the middle plane $\vec{z}$ and the displacement vector $\vec{d}$ are identical, the moment vectors, which result from the eccentricity of the curved stress fields, cannot be represented by the bending and drilling moment vectors only. An approach to solve this problem without the introduction of an additional moment vector, is to redefine the underlying element for the determination of the structural resistance, so that it provides bending and drilling moment vectors that are also perpendicular to the displacement vector $\vec{d}$. This redefinition of the moment vectors demands a correspondent adjustment of the section planes of the element. The section planes must be tangential to the displacement vector $\vec{d}$ to enable a correct transformation of the redefined moment vectors to stresses. The inclined section planes of the element are accordingly spanned by a tangential vector to the middle plane and the displacement vector $\vec{d}$.

The direction of normal stresses and forces for the element with inclined section planes is adopted from the element with orthogonal edges. The same applies for in-plane shear stresses and forces. The directions of the bending and drilling moment vectors at the inclined section planes are found by determining the cross product of the normal or in-plane shear stress vectors and the displacement vector $\vec{d}$. Thus, the bending moment vectors are defined to be tangential to the section planes and perpendicular to the displacement vector $\vec{d}$. The drilling moment vectors are normal to the corresponding section planes and are also perpendicular to the displacement vector $\vec{d}$. The unit vectors describing the direction of bending and drilling moments are defined by:

$$\vec{e}_{mx} = \frac{\vec{d} \times \vec{e}_x}{||\vec{d} \times \vec{e}_x||} \quad \text{and} \quad \vec{e}_{my} = \frac{\vec{d} \times \vec{e}_y}{||\vec{d} \times \vec{e}_y||}$$

(6.1)

$$\vec{e}_{myx} = \frac{\vec{d} \times \vec{e}_y}{||\vec{d} \times \vec{e}_y||} \quad \text{and} \quad \vec{e}_{mxy} = \frac{\vec{d} \times \vec{e}_x}{||\vec{d} \times \vec{e}_x||}$$

(6.2)

$\vec{e}_x$ and $\vec{e}_y$ denote unit vectors in direction of the $x$- and $y$-axis, respectively. The stress state in the structural element is defined by the stress tensor:

$$\mathbf{T} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \quad \text{with} \quad \tau_{xy} = \tau_{yx}, \ \tau_{xz} = \tau_{zx} \quad \text{and} \quad \tau_{yz} = \tau_{zy}$$

(6.3)

The boundary stresses at an inclined section plane are determined by the multiplication of the stress tensor with the respective normal vector of the section plane. The normal vectors of the section planes are determined by:

$$\vec{e}_{nx} = \frac{\vec{e}_y \times \vec{d}}{||\vec{e}_y \times \vec{d}||} \quad \text{and} \quad \vec{e}_{ny} = \frac{\vec{d} \times \vec{e}_x}{||\vec{d} \times \vec{e}_x||}$$

(6.4)
For a section plane facing positive $x$-values results:

$$
\mathbf{T} \cdot \vec{e}_{nx} = \begin{pmatrix}
\sigma_{x} + e_{nx,y} \tau_{xy} + e_{nx,z} \tau_{xz} \\
\sigma_{y} + e_{ny,x} \tau_{xy} + e_{ny,z} \tau_{yz} \\
e_{nx,x} \tau_{xz} + e_{nx,y} \tau_{yz}
\end{pmatrix}
$$

(6.5)

with $e_{nx,x}$, $e_{nx,y}$ and $e_{nx,z}$ as the $x$, $y$ and $z$ components of the normal vector $\vec{e}_{nx}$. As the eccentricity vector $\vec{d}$ is in general not normal to the $x$-$y$ plane, the three unit vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{d}$ define a skew coordinate system. The transformation from the local Cartesian coordinate system to this skew coordinate system is expressed by the transformation matrix:

$$
\mathbf{Q} = \begin{pmatrix}
1 & 0 & -\frac{\cos \angle(\vec{e}_z, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \\
0 & 1 & -\frac{\cos \angle(\vec{e}_x, \vec{d})}{\cos \angle(\vec{e}_x, \vec{d})} \\
0 & 0 & 1
\end{pmatrix}
$$

(6.6)

By applying the transformation matrix $\mathbf{Q}$, the vector of boundary stresses is transformed into components in direction of the stress resultants. To distinguish between the internal stresses included in the stress tensor and the boundary stresses, the latter are denoted by the superscript $^b$. The transformed boundary stress vector at a section facing positive $x$-values in the $x$-$y$-$d$ coordinate system is defined by:

$$
\begin{pmatrix}
\sigma_x^b \\
\tau_{xy}^b \\
\tau_{dx}^b
\end{pmatrix} = \mathbf{Q} \cdot (\mathbf{T} \cdot \vec{e}_{nx})
$$

(6.7)

The boundary stresses at the other section planes are found analogously using the respective normal vectors. The section planes of the infinitesimal element are parallelograms. The sectional areas of the skew element are parallelograms. Their heights are determined by:

$$
h_x = h \frac{\sin \angle(\vec{e}_y, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \quad \text{and} \quad h_y = h \frac{\sin \angle(\vec{e}_x, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})}
$$

(6.8)

The stress resultants applied at the middle plane are found by integration of the boundary stresses with respect to the depth of the element. Including the relation expressed by eq. 6.8, the stress resultants are defined by:

$$
n_x = \frac{\sin \angle(\vec{e}_y, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \int_{-h/2}^{h/2} \sigma_x^b \, dz \quad \text{and} \quad n_y = \frac{\sin \angle(\vec{e}_x, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \int_{-h/2}^{h/2} \sigma_y^b \, dz
$$

(6.9)

$$
n_{yx} = \frac{\sin \angle(\vec{e}_y, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \int_{-h/2}^{h/2} \tau_{xy}^b \, dz \quad \text{and} \quad n_{xy} = \frac{\sin \angle(\vec{e}_x, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \int_{-h/2}^{h/2} \tau_{xy}^b \, dz
$$

(6.10)

$$
v_{dx} = \frac{\sin \angle(\vec{e}_y, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \int_{-h/2}^{h/2} \tau_{dx}^b \, dz \quad \text{and} \quad v_{dy} = \frac{\sin \angle(\vec{e}_x, \vec{d})}{\cos \angle(\vec{e}_z, \vec{d})} \int_{-h/2}^{h/2} \tau_{dy}^b \, dz
$$

(6.11)
The cross product of the eccentricity vector described by \( e \vec{d} \) with the transformed boundary stress vector (eq. 6.7) describes a moment field applied at a section plane facing positive \( x \)-values:

\[
(e \vec{d}) \times \begin{pmatrix}
\sigma_x^b \\
\tau_{yx}^b \\
\tau_{dy}^b
\end{pmatrix} = \alpha_x^b e (\vec{d} \times \vec{e}_x) + \tau_{yx}^b e (\vec{d} \times \vec{e}_y) + \tau_{dy}^b e (\vec{d} \times \vec{d})
\] (6.12)

The cross product of \( \vec{d} \) with itself in the last term of eq. 6.12 yields \( \vec{0} \). The remaining two components each define a moment field in direction of one of the unit moment vectors \( \vec{e}_{mx} \) and \( \vec{e}_{my} \).

Substitution of the remaining cross product terms in eq. 6.12 with the unit moment vectors (eqs. 6.1 and 6.2) yields:

\[
(e \vec{d}) \times \begin{pmatrix}
\sigma_x^b \\
\tau_{yx}^b \\
\tau_{dy}^b
\end{pmatrix} = \sigma_x^b e ||\vec{d} \times \vec{e}_x|| \vec{e}_{mx} + \tau_{yx}^b e ||\vec{d} \times \vec{e}_y|| \vec{e}_{my}
\] (6.13)

which in turn can be transformed to:

\[
(e \vec{d}) \times \begin{pmatrix}
\sigma_x^b \\
\tau_{yx}^b \\
\tau_{dy}^b
\end{pmatrix} = \sigma_x^b e \left((\vec{d} \times \vec{e}_x) \cdot \vec{e}_{mx}\right) \vec{e}_{mx} + \tau_{yx}^b e \left((\vec{d} \times \vec{e}_y) \cdot \vec{e}_{my}\right) \vec{e}_{my}
\] (6.14)

The equations for the moment fields on the other section planes of the skew element can be deduced analogously. Integrating the functions of the moment fields over the depth of the element, while substituting the eccentricity value \( e \) with \( \left( z / \cos \angle(\vec{e}_z, \vec{d}) \right) \), provides the magnitudes of the boundary moments:

\[
m_x = \frac{\sin \angle(\vec{e}_y, \vec{d})}{\cos^2 \angle(\vec{e}_z, \vec{d})} \left((\vec{d} \times \vec{e}_x) \cdot \vec{e}_{mx}\right) \int_{-h/2}^{h/2} \sigma_x^b z \, dz
\] (6.15)

\[
m_y = \frac{\sin \angle(\vec{e}_x, \vec{d})}{\cos^2 \angle(\vec{e}_z, \vec{d})} \left((\vec{d} \times \vec{e}_y) \cdot \vec{e}_{my}\right) \int_{-h/2}^{h/2} \sigma_x^b z \, dz
\] (6.16)

\[
m_{yx} = \frac{\sin \angle(\vec{e}_y, \vec{d})}{\cos^2 \angle(\vec{e}_z, \vec{d})} \left((\vec{d} \times \vec{e}_x) \cdot \vec{e}_{my}\right) \int_{-h/2}^{h/2} \tau_{yx}^b z \, dz
\] (6.17)

\[
m_{xy} = \frac{\sin \angle(\vec{e}_x, \vec{d})}{\cos^2 \angle(\vec{e}_z, \vec{d})} \left((\vec{d} \times \vec{e}_y) \cdot \vec{e}_{mx}\right) \int_{-h/2}^{h/2} \tau_{yx}^b z \, dz
\] (6.18)

Note, that the values for \( m_{yx} \) and \( m_{xy} \) differ unless the angles \( \angle(\vec{e}_z, \vec{d}) \) and \( \angle(\vec{e}_y, \vec{d}) \) are identical.
6.1.1 Transformation of stress resultants

The unit vectors of the normal, in-plane shear and transverse shear stresses form a skew coordinate system. According to the definition made for transverse shear, its direction is parallel to the eccentricity vector \( \vec{d} \) and is, thus, independent from the direction of the membrane stresses. Therefore, the transformation of membrane stresses and the transformation of transverse shear stresses can be considered separately. When treated separately, the transformation rules turn out to be identical to orthogonal elements. The transformation of stress resultants at elements with inclined section planes is shown in figs. 6.2 and 6.3. The corresponding equations, which are, for example, covered in [29, p.46], are adapted to the notations of this thesis and are given below:

\[
\begin{align*}
n_n &= n_x \cos^2 \varphi + n_y \sin^2 \varphi + 2 n_{xy} \sin \varphi \cos \varphi \\
n_i &= n_x \sin^2 \varphi + n_y \cos^2 \varphi - 2 n_{xy} \sin \varphi \cos \varphi \\
n_{nt} &= (n_x - n_i) \sin \varphi \cos \varphi + n_{xy} (\cos^2 \varphi - \sin^2 \varphi) \\
v_n &= v_x \sin \varphi + v_y \cos \varphi \\
v_t &= -v_x \sin \varphi + v_y \cos \varphi
\end{align*}
\]

The unit vectors of bending and drilling moments define a planar skew coordinate system. Due to the definition made for moments at inclined section planes, the unit vectors of bending and drilling moments are not orthogonal to each other. As the moments are defined as cross products involving the eccentricity vector \( \vec{d} \), all moment vectors lie on the same plane to which the vector \( \vec{d} \) is normal. Based on the definitions made for the orientation of the unit moment vectors (eqs. 6.1 and 6.2) and considering that the unit moment vectors \( \vec{e}_{mx} \) and \( \vec{e}_{my} \) do not enclose a right angle, the transformation rules for moment vectors result in (see fig. 6.4):

\[
\begin{align*}
m_n &= m_x \cos \varphi + m_{xy} \sin \varphi \cos \varphi \\
m_i &= m_x \sin \varphi - m_{xy} \sin \varphi \cos \varphi \\
m_{nt} &= -m_{xy} \cos \varphi - m_x \sin \varphi \cos \varphi \\
m_{nt} &= m_{xy} \sin \varphi + m_x \cos \varphi \cos \varphi
\end{align*}
\]

with the angles \( \alpha_m \) defined by:

\[
\cos \alpha_{mnt} = -\vec{e}_{mn} \cdot \vec{e}_{nt} \\
\cos \alpha_{mxt} = -\vec{e}_{mx} \cdot \vec{e}_{nt} \\
\cos \alpha_{mnt} = \vec{e}_{mn} \cdot \vec{e}_{nt}
\]
6.1.2 Principal transverse shear

Transverse shear stresses in an orthogonal stress space can be converted to a principal shear stress [28]. This conversion is based on the transformation rules. Since the transformation rules of transverse shear are identical for orthogonal and skew elements, the determination of magnitude
and direction of the principal transverse shear stress is also identical, such that:

\[ v_0 = \sqrt{v_x^2 + v_y^2}, \quad \tan \varphi_0 = \frac{v_y}{v_x} \quad (6.29) \]

### 6.1.3 Stress fields and corresponding part of the structure

Just as the discussed alternative infinitesimal element, the part of the structure that corresponds to a curved stress field provides cross sections that are tangential to the displacement vector \( \vec{d} \). The part of the structure is idealized to possess a constant depth, such that the upper and lower face are triangular planes that are parallel to the triangular field of the middle surface.

The boundary forces and moments, that are applied at the triangular piece of the structure, can be directly derived from the boundary forces of the corresponding curved stress fields (fig. 6.5). Dependent on the type of structure, the stress state is represented by one or two sets of curved stress fields, as described in sections 5.5 and 5.6. The force equilibrium between the boundary forces of a stress field out of a set of curved stress fields, denoted by the superscript \( I \), and the corresponding boundary forces of the part of the structure yields:

\[ \vec{F}_m^I = N_{mn}^I \vec{e}_{Nm} + N_{Vmn}^I \vec{e}_{VNm} + V_{mn}^I \vec{d} \quad (6.30) \]

with \( \vec{e}_{Nm} \) and \( \vec{e}_{VNm} \) as the unit vectors in direction of the positive normal and in-plane shear force at the edge \( m \), which are defined with respect to the local coordinate system:

\[ \vec{e}_{Nm} = \begin{pmatrix} e_{Nmx} \\ e_{Nmy} \\ 0 \end{pmatrix}, \quad \vec{e}_{VNm} = \vec{e}_z \times \vec{e}_{Nm} = \begin{pmatrix} -e_{Nmy} \\ e_{Nmx} \\ 0 \end{pmatrix} \quad \vec{d} = \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \quad (6.31) \]

The indexes \( m \) and \( n \) denote the edge number and the application point respectively. Based on the unit vectors, the magnitudes of normal force \( N_{mn}^I \), in-plane shear force \( N_{Vmn}^I \) and transverse shear force \( V_{mn}^I \) can be determined by:

\[ N_{mn}^I = \frac{e_{Nmx}}{e_{Nmx}^2 + e_{Nmy}^2} F_{max}^I + \frac{e_{Nmy}}{e_{Nmx}^2 + e_{Nmy}^2} F_{may}^I - \frac{d_x}{d_z \left( e_{Nmx}^2 + e_{Nmy}^2 \right)} e_{Nmx} \begin{pmatrix} e_{Nmx} + e_{Nmy} \end{pmatrix} F_{mcz}^I \quad (6.32) \]

\[ N_{Vmn}^I = \frac{e_{Nmy}}{e_{Nmx}^2 + e_{Nmy}^2} F_{max}^I + \frac{e_{Nmx}}{e_{Nmx}^2 + e_{Nmy}^2} F_{may}^I + \frac{d_y}{d_z \left( e_{Nmx}^2 + e_{Nmy}^2 \right)} e_{Nmx} \begin{pmatrix} e_{Nmx} - e_{Nmy} \end{pmatrix} F_{mcz}^I \quad (6.33) \]

\[ V_{mn}^I = \frac{1}{d_z} F_{mcz}^I \quad (6.34) \]

The same applies analogously to the boundary forces derived from a curved stress field out of the set denoted by the superscript \( II \). The boundary forces of both curved stress fields sum up to:

\[ N_{mn} = N_{mn}^I + N_{mn}^{II} \quad (6.35) \]

\[ N_{Vmn} = N_{Vmn}^I + N_{Vmn}^{II} \quad (6.36) \]

\[ V_{mn} = V_{mn}^I + V_{mn}^{II} \quad (6.37) \]
Fig. 6.5  Triangular part of the structure with corresponding curved stress fields out of the curved stress field sets I and II and bounds of eccentricity
Based on the normal forces $N_{I\, mn}$, $N_{II\, mn}$ and the in-plane shear forces $N_{I\, V\, mn}$, $N_{II\, V\, mn}$ the bending moment $M_{B\, mn}$ and the drilling moment $M_{D\, mn}$ can be determined by:

$$M_{B\, mn} = (N_{I\, mn} e_I + N_{II\, mn} e_{II}) (\vec{d} \times \vec{e}_{Nm}) \cdot \vec{e}_{MB\, m}$$  \hspace{1cm} (6.38)

$$M_{D\, mn} = (N_{I\, V\, mn} e_I + N_{II\, V\, mn} e_{II}) (\vec{d} \times \vec{e}_{NV\, m}) \cdot \vec{e}_{MD\, m}$$  \hspace{1cm} (6.39)

with $e_I$ and $e_{II}$ as the eccentricity values of the respective curved stress field. As the curved stress fields are not parallel to the corresponding triangular part of the middle surface of the structure, the eccentricity values $e_I$ and $e_{II}$ are defined by linear functions dependent on the local coordinates $x$ and $y$. The moment unit vectors $\vec{e}_{MB\, m}$ and $\vec{e}_{MD\, m}$ are defined by:

$$\vec{e}_{MB\, m} = \frac{\vec{d} \times \vec{e}_{Nm}}{|\vec{d} \times \vec{e}_{Nm}|}$$  \hspace{1cm} (6.40)

$$\vec{e}_{MD\, m} = \frac{\vec{d} \times \vec{e}_{NV\, m}}{|\vec{d} \times \vec{e}_{NV\, m}|}$$  \hspace{1cm} (6.41)

As the transverse shear force $V_{mn}$ is defined to be parallel to the displacement vector $\vec{d}$, it has no impact on moments.

Based on the determined boundary forces, the stress state of the corner points of the stress and shear field on the middle surface are deduced. Hajdin [15] proposed a determination of the stress tensors of the corner points of a linear stress field for membrane stresses. This approach does not include transverse shear and the skew orientation of the transverse shear boundary forces complicates the use of stress tensors. Therefore, the stress resultants, defining the stress states at the corner points of the linear stress and shear field on the middle surface, are deduced using the transformation rules of stress resultants (see sec. 6.1.1).

![fig. 6.6 Linear stress field with virtual orthogonal edges at the corner point $P_A$](image)

The approach is based on the idea, that the boundary stresses of a stress field edge and a corresponding virtual orthogonal edge describe the stress state at the common corner point (fig. 6.6). At
a typical stress field corner point \( P_A \), the boundary stresses at the adjacent edges \( E_1 \) and \( E_3 \) and the corresponding virtual orthogonal edges \( E_{1\perp} \) and \( E_{3\perp} \) describe the same stress state with respect to the \( e_1-e_{1\perp}-z \) and the \( e_3-e_{3\perp}-z \) coordinate system, respectively. The transformation of the stress state from the \( e_3-e_{3\perp}-z \) to the \( e_1-e_{1\perp}-z \) coordinate system yields:

\[
\begin{align*}
  n_{1A} &= n_{3A} \cos^2 \varphi_{31} + n_{3\perp A} \sin^2 \varphi_{31} + 2 n_{V3A} \sin \varphi_{31} \cos \varphi_{31} \\
  n_{1\perp A} &= n_{3A} \sin^2 \varphi_{31} + n_{3\perp A} \cos^2 \varphi_{31} - 2 n_{V3A} \sin \varphi_{31} \cos \varphi_{31}
\end{align*}
\]  

(6.42)

(6.43)

with the signed angle \( \varphi_{31} \) defined by:

\[
\varphi_{31} = \arccos(\hat{e}_{3\perp} \cdot \hat{e}_{1\perp}) \left( \frac{\hat{e}_{3\perp} \times \hat{e}_{1\perp}}{|\hat{e}_{3\perp} \times \hat{e}_{1\perp}|} \right) \]  

(6.44)

Solving eqs. 6.42 and 6.43 for \( n_{1\perp A} \) yields:

\[
n_{1\perp A} = n_{1A} \left( \frac{1}{\sin^2 \varphi_{31}} - 1 \right) + n_{3A} \left( 2 - \frac{1}{\sin^2 \varphi_{31}} \right) - n_{V3A} \frac{2}{\tan \varphi_{31}}
\]  

(6.45)

The boundary stresses at stress field corner point \( P_A \) are determined by:

\[
\begin{align*}
  n_{1A} &= \frac{2N_{11}}{|P_A - \hat{P}_B|} \quad n_{V1A} = \frac{2N_{V11}}{|P_A - \hat{P}_B|} \quad n_{3A} = \frac{2N_{32}}{|P_A - \hat{P}_C|} \quad n_{V3A} = \frac{2N_{V32}}{|P_A - \hat{P}_C|}
\end{align*}
\]  

(6.46)

The stress state with respect to the local \( x \)- and \( y \)-axes can then be deduced from the stress state in the \( e_1-e_{1\perp}-z \) coordinate system

\[
\begin{align*}
  n_x &= n_{1A} \cos^2 \varphi_1 + n_{1\perp A} \sin^2 \varphi_1 + 2 n_{V1A} \sin \varphi_1 \cos \varphi_1 \\
  n_y &= n_{1A} \sin^2 \varphi_1 + n_{1\perp A} \cos^2 \varphi_1 - 2 n_{V1A} \sin \varphi_1 \cos \varphi_1 \\
  n_{xy} &= (n_{1A} - n_{1\perp A}) \sin \varphi_1 \cos \varphi_1 + n_{V1A} (\cos^2 \varphi_1 - \sin^2 \varphi_1)
\end{align*}
\]  

(6.47)

(6.48)

(6.49)

with the signed angle \( \varphi_1 \) defined by:

\[
\varphi_1 = \arccos(\hat{e}_{1\perp} \cdot \hat{e}_x) \left( \frac{\hat{e}_{1\perp} \times \hat{e}_x}{|\hat{e}_{1\perp} \times \hat{e}_x|} \right)
\]  

(6.50)

Based on the same approach involving eqs. 6.22 and 6.23, the transverse shear stress resultants result in:

\[
\begin{align*}
  v_{dx} &= v_{1A} \frac{\sin (\varphi_1 + \varphi_{31})}{\sin \varphi_{31}} - v_{3A} \frac{\sin \varphi_1}{\sin \varphi_{31}} \quad \text{and} \quad v_{dy} = v_{1A} \frac{\cos (\varphi_1 + \varphi_{31})}{\sin \varphi_{31}} - v_{3A} \frac{\cos \varphi_1}{\sin \varphi_{31}}
\end{align*}
\]  

(6.51)

with:

\[
\begin{align*}
  v_{1A} &= \frac{2V_{11}}{|P_A - \hat{P}_B|} \quad \text{and} \quad v_{3A} = \frac{2V_{32}}{|P_A - \hat{P}_C|}
\end{align*}
\]  

(6.52)

The quadratic moment field is expressed by the linear membrane stress states of each of the corresponding curved stress fields and the associated linear eccentricity functions. Accordingly, the membrane stress resultants must also be determined separately for each of the curved stress fields.
The partial stress resultants $n'_x$, $n'_y$, $n'_{xy}$, $n''_x$, $n''_y$ and $n''_{xy}$ are determined analogously with eqs. 6.47, 6.48, 6.49 and 6.45. Based on the partial stress resultants, the moments are defined by:

\[
\begin{align*}
    m_x &= (n'_x e^I + n''_x e^{II})(\vec{d} \times \vec{e}_x) \cdot \vec{e}_{mx} \\
    m_y &= (n'_y e^I + n''_y e^{II})(\vec{d} \times \vec{e}_y) \cdot \vec{e}_{my} \\
    m_{yx} &= (n'_{yx} e^I + n''_{yx} e^{II})(\vec{d} \times \vec{e}_x) \cdot \vec{e}_{mxy} \\
    m_{xy} &= (n'_{xy} e^I + n''_{xy} e^{II})(\vec{d} \times \vec{e}_x) \cdot \vec{e}_{mxy}
\end{align*}
\]

Note, that the values for $m_{xy}$ and $m_{yx}$ differ unless the angles enclosed by the vectors $\vec{d}$, $\vec{e}_x$ and $\vec{d}$, $\vec{e}_y$ are identical.

### 6.2 Sandwich model

Sandwich models are commonly used in the analysis and design of slabs [4], [42, pp.73], [28], [34], [21]. The cross section of the slab is hereby divided into two cover and a core layer. While the cover layers are supposed to be subjected to membrane stress states, which result from the division of moments into force couples, the transverse shear force is transferred by the core layer. When applying the Sandwich model to shells, also normal and in-plane shear forces must be considered in addition to moments and transverse shear. The stresses resulting from these forces are added to the membrane stress states of the cover layers, in a way that the core layer remains in pure shear.

![Sandwich model](image-url)
6.2.1 Core layer

The core layer is assumed to be subjected to a pure constant transverse shear stress state. The corresponding stress tensor is defined by:

$$
T_{\text{core}} = \begin{pmatrix}
0 & 0 & \tau_{xz} \\
0 & 0 & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & 0
\end{pmatrix}
$$

with \(\tau_{xz} = \tau_{zx}\) and \(\tau_{yz} = \tau_{zy}\) \hspace{1cm} (6.57)

The multiplication of the stress tensor with the normal vector of the section planes provides the boundary stresses. For a section plane facing positive \(x\)-values results:

$$
T_{\text{core}} \cdot \vec{e}_{nx} = \begin{pmatrix}
\tau_{xz} e_{nx,z} \\
\tau_{yz} e_{nx,z} \\
\tau_{zx} e_{nx,z} + \tau_{zy} e_{nx,y}
\end{pmatrix}
$$

with \(e_{nx,x}, e_{nx,y}\) and \(e_{nx,z}\) as the \(x, y\) and \(z\) component of the normal vector \(\vec{e}_{nx}\). The normal vector \(\vec{e}_{nx}\) is defined by the cross product of the unit vector in direction of the local \(y\)-axis and the eccentricity vector \(\vec{d}\) (see eq. 6.4) with \(\vec{e}_{y} = (0, 1, 0)\) with respect to the local coordinate system. Thus, for the component of the normal vector in direction of the local \(y\)-axis results \(e_{nx,y} = 0\). The boundary stress vector at a section plane facing positive \(x\)-values simplifies to:

$$
T_{\text{core}} \cdot \vec{e}_{nx} = \begin{pmatrix}
\tau_{xz} e_{nx,z} \\
\tau_{yz} e_{nx,z} \\
\tau_{zx} e_{nx,z}
\end{pmatrix}
$$

Due to the inclined section planes, the boundary stress vector also possesses normal and in-plane shear stress components, besides the transverse shear component. The stress resultants in the skew coordinate system, defined by the unit vectors \(\vec{e}_{x}, \vec{e}_{y}\) and \(\vec{d}\), are obtained by applying the transformation matrix (see eq. 6.6) and multiplying with the height of the section plane:

$$
\begin{pmatrix}
n_{vx} \\
n_{vyx} \\
v_{dx}
\end{pmatrix} = Q \cdot (T_{\text{core}} \cdot \vec{e}_{nx}) h_{x,\text{core}}
$$

which provides in explicit notation:

$$
n_{vx} = h_{x,\text{core}} \left( \tau_{xz} e_{nx,z} - \tau_{zx} \frac{e_{nx,x} \cos \angle(\vec{e}_{x}, \vec{d})}{\cos \angle(\vec{e}_{z}, \vec{d})} \right) \hspace{1cm} (6.61)
$$

$$
n_{vyx} = h_{x,\text{core}} \left( \tau_{yz} e_{nx,z} - \tau_{zy} \frac{e_{nx,y} \cos \angle(\vec{e}_{y}, \vec{d})}{\cos \angle(\vec{e}_{z}, \vec{d})} \right) \hspace{1cm} (6.62)
$$

$$
v_{dx} = h_{x,\text{core}} \tau_{zx} \frac{e_{nx,z} \cos \angle(\vec{e}_{z}, \vec{d})}{\cos \angle(\vec{e}_{x}, \vec{d})} \hspace{1cm} (6.63)
$$
The boundary stress resultants at a section plane facing positive \( y \)-values are deduced analogously. The multiplication of the stress tensor with the normal vector of the section plane yields:

\[
T_{\text{core}} \cdot \vec{e}_{ny} = \begin{pmatrix}
\tau_{xz} \ e_{ny,z} \\
\tau_{yz} \ e_{ny,z} \\
\tau_{yz} \ e_{ny,y}
\end{pmatrix}
\]  \hspace{1cm} (6.64)

If the boundary stress vector is transformed to the \( x-y-d \) coordinate system and the result is multiplied with the height of the section plane \( h_y \), then a vector consisting of the three boundary stress resultants is obtained:

\[
\begin{pmatrix}
    n_{vxy} \\
    n_{vy} \\
    v_{dy}
\end{pmatrix} = Q \cdot (T_{\text{core}} \cdot \vec{e}_{ny}) \ h_{y,\text{core}}
\]  \hspace{1cm} (6.65)

which yields in explicit notation:

\[
n_{vxy} = h_{y,\text{core}} \left( \tau_{xz} \ e_{ny,z} - \tau_{yz} \ e_{ny,y} \frac{\cos \angle(\vec{e}_{y}, \vec{d})}{\cos \angle(\vec{e}_{z}, \vec{d})} \right) \]  \hspace{1cm} (6.66)

\[
n_{vy} = h_{y,\text{core}} \left( \tau_{yz} \ e_{ny,z} - \tau_{yz} \ e_{ny,y} \frac{\cos \angle(\vec{e}_{y}, \vec{d})}{\cos \angle(\vec{e}_{z}, \vec{d})} \right) \]  \hspace{1cm} (6.67)

\[
v_{dy} = h_{y,\text{core}} \tau_{yz} \ e_{ny,y} \frac{1}{\cos \angle(\vec{e}_{z}, \vec{d})}
\]  \hspace{1cm} (6.68)

Using a Sandwich model approach with cover layers of different depths, an eccentricity of the core layer towards the middle surface of the structure results. The bending moments caused may be ignored due to the very limited significance of the membrane stress resultants and their rather small eccentricity.

### 6.2.2 Cover layers

The depths of the cover layers of Sandwich models used for slabs can be derived from investigations of beams and slabs subjected to torsion [34, pp.25]. Marti [28] suggested to assign a thickness to the cover layers and to design the reinforcement, so that the concrete strength is not exceeded. Jäger [21, p.43], for example, proposed to set the depth of the cover layer to the double of the concrete cover. The main benefit from this assumption is that the resultant membrane stresses assigned to concrete and reinforcement act in the same plane.

In contrast to slabs, the stress state in shells is dominated by membrane stresses rather than bending. Sandwich models with two cover layers of equal and rather small depth, as they would result from the approaches described above, seem less appropriate. Instead, one of the cover elements shall provide a greater depth, while the depth of the other cover element is defined to be twice the concrete cover. As the focus for the determination of the bounds of eccentricity lays on structures with the capability to bear a non-negligible amount of flexural loads, the depth of cover elements is
limited to assure ductile behavior. The investigated Sandwich model types are described in detail in section 7.3 and are illustrated in fig. 7.12.

For the transformation of the internal forces of a structural element to the Sandwich model, the force equilibrium of the membrane stress resultants requires including the membrane stress resultants of the core element:

\[ n_x = n_{x1} + n_{x2} + n_{vx} \]  \hspace{1cm} (6.69)
\[ n_y = n_{y1} + n_{y2} + n_{vy} \]  \hspace{1cm} (6.70)
\[ n_{yx} = n_{yx1} + n_{yx2} + n_{vyx} \]  \hspace{1cm} (6.71)
\[ n_{xy} = n_{xy1} + n_{xy2} + n_{vxy} \]  \hspace{1cm} (6.72)

From the torque equilibrium results:

\[ m_x = n_{x1} s_1 + n_{x2} s_2 \]  \hspace{1cm} (6.73)
\[ m_y = n_{y1} s_1 + n_{y2} s_2 \]  \hspace{1cm} (6.74)
\[ m_{yx} = n_{yx1} s_1 + n_{yx2} s_2 \]  \hspace{1cm} (6.75)
\[ m_{xy} = n_{xy1} s_1 + n_{xy2} s_2 \]  \hspace{1cm} (6.76)

with \( s_1 \) and \( s_2 \) denoting the lever arms of the upper and lower cover layer, respectively. A possible eccentricity of the core layer and its membrane stress resultants, due to cover layers of different depths, is ignored.

### 6.3 Yield criteria for membrane elements

The cover layers of a Sandwich-model are subjected to a membrane stress state. The yield criteria for the membrane stress state have been subject to extensive investigations. Nielsen [41] developed seven yield criteria under the assumption of orthogonal reinforcement. These yield criteria have been refined by Müller [37] and extended to skew reinforcement by Marti [27, pp.76]. Below, the seven yield criteria for orthogonal reinforcement are shown:

\[ Y_1 : \quad n_{xy}^2 - (a_{sx} f_{yx} - n_x)(a_{sy} f_{yy} - n_y) = 0 \]
\[ Y_2 : \quad n_{xy}^2 - (h f_c - a_{sy} f_{yy} + n_y)(a_{sy} f_{yy} - n_y) = 0 \]
\[ Y_3 : \quad n_{xy}^2 - (a_{sx} f_{yx} - n_x)(h f_c - a_{sx} f_{yx} + n_x) = 0 \]
\[ Y_4 : \quad n_{xy}^2 - (h f_c/2)^2 = 0 \]  \hspace{1cm} (6.77)
\[ Y_5 : \quad n_{xy}^2 + (a_{sx} f'_{yx} + n_x)(h f_c + a_{sx} f'_{yx} + n_x) = 0 \]
\[ Y_6 : \quad n_{xy}^2 + (h f_c + a_{sy} f'_{yy} + n_y)(a_{sy} f'_{yy} + n_y) = 0 \]
\[ Y_7 : \quad n_{xy}^2 - (h f_c + a_{sy} f'_{yy} + n_y)(h f_c + a_{sy} f'_{yy} + n_y) = 0 \]

with \( a_{sx} \) and \( a_{sy} \) as the reinforcement per unit length, \( f_{yx} \) and \( f_{yy} \) as the reinforcement strength, while \( f'_{yx} \) and \( f'_{yy} \) denote yielding of reinforcement under compression, \( h \) as the depth of the considered cover layer and \( f_c \) as the compressive strength of concrete.
Figure 6.8 illustrates the closed yield surface defined by the seven yield criteria (eqs. 6.77). Each of the yield criteria defines two parts of the yield surface, for positive and negative values of the in-plane shear stress resultant $n_{xy}$. The scope for each yield criterion is defined by the intersections of the yield surface parts and results in:

- $Y_1 : a_{sx} f_{yx} + a_{sy} f_{yy} - h f_c \leq n_x + n_y, \quad n_x \leq a_{sx} f_{yx}, \quad n_y \leq a_{sy} f_{yy}$
- $Y_2 : a_{sx} f_{yx} - f_c h/2 \leq n_y \leq a_{sy} f_{yy}$
- $a_{sy} f_{yy} - a_{sx} f_{yx} - f_c h \leq n_x + n_y \leq a_{sx} f_{yx} + a_{sy} f_{yy} - f_c h$
- $Y_3 : a_{sx} f_{yx} - f_c h/2 \leq n_x \leq a_{sx} f_{yx}$
- $a_{sx} f_{yx} - a_{sy} f_{yy} - f_c h \leq n_x + n_y \leq a_{sx} f_{yx} + a_{sy} f_{yy} - f_c h$
- $Y_4 : -f_c h/2 - a_{sx} f_{yx} \leq n_x \leq a_{sx} f_{yx} - f_c h/2$
- $-f_c h/2 - a_{sy} f_{yy} \leq n_y \leq a_{sy} f_{yy} - f_c h/2$
- $Y_5 : -f_c h - a_{sx} f_{yx} \leq n_x \leq -f_c h/2 - a_{sx} f_{yx}$
- $-f_c h - a_{sy} f_{yy} - a_{sy} f_{yy} \leq n_x + n_y \leq a_{sy} f_{yy} - a_{sx} f_{yx} - f_c h$
- $Y_6 : -f_c h - a_{sy} f_{yy} \leq n_y \leq -f_c h/2 - a_{sy} f_{yy}$
- $-f_c h - a_{sy} f_{yy} - a_{sx} f_{yx} \leq n_x + n_y \leq a_{sx} f_{yx} - a_{sy} f_{yy} - f_c h$
- $Y_7 : n_x \geq -f_c h - a_{sx} f_{yx}, \quad n_y \geq -f_c h - a_{sy} f_{yy}, \quad n_x + n_y \leq -f_c h - a_{sx} f_{yx} - a_{sy} f_{yy}$

### 6.4 Yield criteria for elements subjected to transverse shear

Besides areas near supports, slabs and shells are in general only subjected to rather small transverse shear forces [29, p.113]. Thus, only structures without transverse reinforcement are considered. Failure mechanisms of members without transverse reinforcement have been discussed e.g. in [47] and [1]. There, the shear strengths are deduced from models with cracked shear zones. Since the focus in the present thesis lays on the investigation of eccentric membrane stresses throughout the complete structure, which is mainly not subjected to decisive transverse shear, the further
investigation is based on a comparatively simple model with uncracked shear zone. Provided an uncracked shear zone, transverse shear is transferred by two orthogonal compression and tension stress fields of equal stress magnitudes inclined at 45° to the middle surface [1].

The direction of principal transverse shear transfer is defined by the angle \( \phi_0 \) (see eq. 6.29) with respect to the \( x \)-axis. The unit vector in direction of the principal shear transfer \( \vec{e}_{x0} \) and a vector \( \vec{e}_{y0} \), which is orthogonal to it, are obtained by rotation of the unit vectors in direction of the \( x \) and \( y \)-axis \( \vec{e}_x \) and \( \vec{e}_y \) by the angle \( \phi_0 \) about the \( z \)-axis. If transforming the stress tensor of the core layer \( T_{\text{core}} \) to the \( x_0\text{-}y_0\text{-}z \) coordinate system, a stress tensor only containing the principal transverse shear stress \( \tau_0 \) results:

\[
Q_0^T \cdot T_{\text{core}} \cdot Q_0 = \begin{pmatrix}
0 & 0 & \tau_0 \\
0 & 0 & 0 \\
\tau_0 & 0 & 0
\end{pmatrix}
\]  \( (6.79) \)

The transformation matrix \( Q_0 \) from the \( x\text{-}y\text{-}z \) to the \( x_0\text{-}y_0\text{-}z \) coordinate system is defined by:

\[
Q_0 = \begin{pmatrix}
\cos \phi_0 & \sin \phi_0 & 0 \\
-\sin \phi_0 & \cos \phi_0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  \( (6.80) \)

The normal vector at an inclined section plane in the \( x_0\text{-}y_0\text{-}z \) coordinate system facing positive values of \( x_0 \) is defined by:

\[
\vec{e}_{n0} = \frac{\vec{e}_{y0} \times \vec{d}}{||\vec{e}_{y0} \times \vec{d}||} = \begin{pmatrix}
e_{n0,x}
0
e_{n0,z}
\end{pmatrix}
\]  \( (6.81) \)

and the boundary stresses in the \( x_0\text{-}y_0\text{-}z \) coordinate system at the inclined section plane corresponding to the normal vector \( \vec{e}_{n0} \) are determined by the vector:

\[
Q_0^T \cdot T_{\text{core}} \cdot Q_0 \cdot \vec{e}_{n0} = \begin{pmatrix}
\tau_0 e_{n0,z} \\
0 \\
\tau_0 e_{n0,x}
\end{pmatrix}
\]  \( (6.82) \)

The transformation of the boundary stress vector in the \( x_0\text{-}y_0\text{-}d \) coordinate system, using the transformation matrix given by:

\[
Q_{0d} = \begin{pmatrix}
1 & 0 & -\frac{\cos \angle(\vec{e}_{0,d})}{\cos \angle(\vec{e},d)} \\
0 & 1 & -\frac{\cos \angle(\vec{e}_{0,d})}{\cos \angle(\vec{e},d)} \\
0 & 0 & \frac{1}{\cos \angle(\vec{e},d)}
\end{pmatrix}
\]  \( (6.83) \)

and multiplication with the height of the considered inclined section plane of the core layer found
by:

$$h_{0,\text{core}} = h_{\text{core}} \frac{\sin \angle(\vec{e}_{y0}, \vec{d})}{\cos \angle(\vec{e}_{z}, \vec{d})} \quad (6.84)$$

yields the vector of the boundary stress resultants:

$$\begin{pmatrix} n_{vx0} \\ n_{vyx0} \\ v_{d0} \end{pmatrix} = Q_0 \cdot (Q_0 \cdot T_{\text{core}} \cdot Q_0^T \cdot \vec{e}_{n0}) \cdot h_{0,\text{core}} \quad (6.85)$$

Using eqs. 6.84 and 6.85, the principal shear stress resultant in direction of the eccentricity vector \( \vec{d} \) results in:

$$v_{d0} = \tau_0 h_{\text{core}} e_{n0,0} \frac{\sin \angle(\vec{e}_{y0}, \vec{d})}{\cos^2 \angle(\vec{e}_{z}, \vec{d})} \quad (6.86)$$

The minimal depth of an uncracked core element is, thus, determined by:

$$h_{\text{core}} \geq \frac{v_{d0} \cos^2 \angle(\vec{e}_{z}, \vec{d})}{\tau_c e_{n0,0} \sin \angle(\vec{e}_{y0}, \vec{d})} \quad (6.87)$$

with \( \tau_c \) as the shear strength of concrete. According to the applied model of an uncracked core layer, the concrete shear strength \( \tau_c \) is deduced from the two orthogonal compression and tension stress fields of equal magnitude inclined at 45° to the middle surface (see fig. 6.9). The magnitude of the two stress fields is therein limited by the tensile strength of concrete \( f_{ct} \). The concrete shear strength results in:

$$\tau_c = 2 \cos^2 45° f_{ct} = f_{ct} \quad (6.88)$$

![fig. 6.9](image)

Transverse shear transfer in an uncracked core element with inclined section planes

### 6.5 Determination of the bounds of eccentricity

The bounds of eccentricity illustrate the load bearing capacity of a structure at a particular point by an eccentric membrane stress state. The membrane stress state is given by the corresponding curved stress field. Based on the load bearing capacity, the two maximal eccentricities of this
membrane stress state towards the middle surface of the structure are determined.

The load bearing behavior of a slab is described by two combined sets of curved stress fields (see sec. 5.6). It is hereby allowed to apply the external loads to one or both of these sets. The bounds of eccentricity are always determined for only one set of curved stress fields. Hence, the external loads should, in this context, be only applied to the set of curved stress fields, for which the bounds of eccentricity are determined, to obtain an illustrative representation.

The load bearing capacity of the structure is investigated by means of a Sandwich model. For each of its cover layers, the yield criteria, as reproduced in section 6.3, form a closed yield surface (fig. 6.8). The membrane stress state of the set of curved stress fields for shells and the combined sets of curved stress fields for slabs, respectively, are transformed to functions of the membrane stress resultants of the cover layers dependent on the eccentricity value \( e \). For the top cover layer results:

\[
\begin{align*}
   n_{x1} &= (n_{x1}' + n_{x1}'') - n_{yx1} \frac{s_2}{s_1 + s_2} + (n_{y1}' + n_{y1}'') \cos \angle (\vec{e}_x, \vec{d}) \frac{s_2}{s_1 + s_2} \tag{6.89} \\
   n_{y1} &= (n_{y1}' + n_{y1}'') - n_{yx1} \frac{s_2}{s_1 + s_2} + (n_{xy1}' + n_{xy1}'') \cos \angle (\vec{e}_x, \vec{d}) \frac{s_2}{s_1 + s_2} \tag{6.90} \\
   n_{yx1} &= (n_{yx1}' + n_{yx1}'') - n_{yx1} \frac{s_2}{s_1 + s_2} + (n_{xy1}' + n_{xy1}'') \cos \angle (\vec{e}_x, \vec{d}) \frac{s_2}{s_1 + s_2} \tag{6.91}
\end{align*}
\]

with \( s_1 \) and \( s_2 \) as the lever arms of the stress resultants of the top and bottom cover layer, and superscripts \( ' \) and \( '' \) referring to the two sets of curved stress fields. The equations for a bottom cover layer are found analogously. The functions of the 3 membrane stress resultants of a cover layer define the line \( \bar{L}_1 \) in stress space (fig. 6.8). The intersections of the line \( \bar{L}_1 \) and the yield surface define the upper and lower bound of eccentricity denoted with \( B_1^+ \) and \( B_1^- \) in fig. 6.8. The vector function of \( \bar{L}_1 \) represents a typical stress state of a cover layer of both a shell and a slab.

For shells with \( n_{x1}' = n_{y1}' = n_{yx1}'' = 0 \) and planar slabs subjected to bending without membrane action with \( n_{x1}' = n_{y1}' = n_{xy1}'' \), \( n_{yx1}' = n_{yx1}'' \) and \( e'' = 0 \), the eqs. 6.89, 6.90 and 6.91 simplify and result, transformed into explicit notation, in:

\[
\begin{align*}
   n_{y1} &= \frac{n_y}{n_x} n_{x1} + \frac{n_{y1}' s_2}{n_{y1}' s_1 + s_2} n_{yx1} - \frac{s_2}{s_1 + s_2} n_{vy} \tag{6.92} \\
   n_{yx1} &= \frac{n_{yx1}}{n_x} n_{x1} + \frac{n_{yx1}' s_2}{n_{yx1}' s_1 + s_2} n_{yx1} - \frac{s_2}{s_1 + s_2} n_{vyx} \tag{6.93}
\end{align*}
\]

To assure an admissible stress state, the bounds of eccentricity determined for one of the cover elements must be verified considering the respective other cover element.

The upper and lower bound of eccentricity are each functions dependent on the local coordinates \( x \) and \( y \). The functions are based on the linear membrane stress functions of linear stress fields and the seven yield criteria for membrane elements. Both are continuous, but only the linear membrane stress functions are differentiable. Accordingly, the functions of the bounds of eccentricity are continuous and non-differentiable. Therefore, the created surfaces representing the structural resistance for each curved stress field may possess kinks. In certain cases, a sudden switch of
the decisive cover element from the top to the bottom cover element, or vice versa, is possible. However, the function also remains continuous in these cases. Due to the discontinuity of the tangential stresses along the edges of the curved stress fields, the bounds of eccentricity do not define continuous surfaces throughout the structure, but do, in contrast, possess discontinuities along the stress field edges.

6.6 Discussion

The method of the bounds of eccentricity allows a graphical comparison of the loads of a structural surface, represented by a curved stress field, to the load bearing capacity of the structure. This graphical comparison allows a consideration of material strengths in the form finding process, such that a form does not have to match the developed curved stress field, but may differ from it as long as the curved stress field is enclosed by the two bounds of eccentricity. These bounds illustrate the resistance of the structure. Accordingly, the proposed method is also suited for the analysis of given structures. Besides the graphical illustration of the load bearing capacity of a structure and its application in form finding and analysis, the presented approach is also applicable to the design of reinforcement or the depth of a structure.

The approach of the bounds of eccentricity is based on a rather unconventional infinitesimal element with inclined section planes. Its main benefit is that the moments caused by the eccentric membrane stress field of the corresponding curved stress field can be directly transformed into stresses. Using an orthogonal element instead would mean that the moments caused by the corresponding eccentric curved stress field cannot be transformed into the moments acting at the element without introducing a third moment vector besides the bending and drilling moment vectors. This additional moment vector could be an in-plane bending moment vector parallel to the normal vector $\mathbf{z}$.

Provided linear stress fields, in-plane bending moments, as $M_{Zmn}$ in fig. 6.10, could be transformed into force couples of the normal forces added to the normal boundary forces $N_{mn}$ of the edge. The membrane stress state defined by the normal and tangential boundary forces $N_{mn}$ and $N_{Vmn}$, with $m$ and $n$ denoting the respective indexes of the application points, was overlayed by a membrane stress state caused by the transformed in-plane bending moments. Whether this additional stress state meets the conditions for linear stress fields (see eqs. 5.4 to 5.7) would have to be investigated. In contrast, the application of elements with inclined section planes preserves the direct link between the membrane and shear stress fields on the middle surface of the structure and the corresponding curved stress field.

Stability issues have not been considered in this approach of the bounds of eccentricity. Stability problems may only occur for slender shell constructions, which just provide a very limited bending resistance and must, thus, possess a statically optimized shape. As the focus of the present thesis lays on shell structures that provide a bending resistance which allows perceivable deviations from statically optimized shapes, it is feasible to ignore stability issues in the determination of the structural resistance.
fig. 6.10  Triangular part of the structure with section planes that are perpendicular to the middle plane and corresponding curved stress fields out of sets I and II
7 Implementation in an application software

The methodology presented in chapters 4, 5 and 6 is implemented in an application software to verify and test its abilities. The focus of the program lays on functionality and does only provide rudimentary user dialogs. The programmed application is written in Python™ [44] and is embedded in the 3D-Design-Software Rhinoceros© [32]. This approach allows an intuitive graphical input of the fundamental geometry and a graphical output of the determined results. Within the Python™ application, the user input is processed and data for the different equation systems is prepared. The actual computation is transferred to the software Mathematica© [57], which provides appropriate solvers. The obtained results are, in turn, processed in the Python™ application and illustrated in the model space of the 3D-Design-Software Rhinoceros©. The computation in Mathematica© is prepared by adding specific data to template files. The Mathematica© command line interface is used to launch the computation in Mathematica© and to return the obtained results, such that the process is completely controlled by the Python™ application.

7.1 Generation of strut and tie networks

The starting point of the generation of an equilibrated spatial strut and tie network is the definition of an initial network. The initial network determines the initial arrangement of nodes and members and is created by using the drawing tools provided by the 3D-Design-Software Rhinoceros© before the actual launch of the application (see fig. 7.1a). The application offers the possibility to densify this initial network drawn by the user (see fig. 7.1b). The automated densification of the initial network can adapt the position of the new nodes to a predefined surface.

The supports of the strut and tie networks are defined by so-called support zones, which are drawn by the user as surface elements using the tools provided by Rhinoceros© (see fig. 7.1c). For the generation of strut and tie networks, the definition of support points would be sufficient. Supports are, nonetheless, specified by support zones to already define the scope for arranging the curved stress fields, as described in section 7.2. The supports are assumed to provide three reaction force components. The external loads specified at the nodes may vary in magnitude and direction from node to node.

As discussed in section 4.3, the unknowns outnumber the equations in the system of equations describing the equilibrium conditions of the strut and tie network. The user gains control over the generation of the strut and tie networks by choosing the set of additional unknowns and the values of the additional unknowns. The application offers the user several options:

**Initial member forces** allows the definition of the initial member forces, which gives a rather intuitive control over the generation process.

**Displacement direction** defines constraints for the displacement of nodes, which allows to create a projective relation between the initial network and the equilibrated strut and tie network or parts of it.
fig. 7.1 Steps of the generation of a single strut and tie network; a) initial network; b) automatically densified initial network; c) support zones (gray squares) and external loads; d) initial values for the additional unknowns; e) initial values for the remaining unknowns as initial guess; f) the generated equilibrated strut and tie network
**Node displacement** allows the user to specify a displacement of nodes, which is useful if the aspired shape of the strut and tie network must pass specific points.

**Fix supports** sets the displacement of the support nodes to zero, which matches with the most common assumption and allows in combination with the choice of the initial member forces the most intuitive control over the generation process.

**Reaction forces** enables the user to specify values for reaction force components or to define relations among reaction force components of a group of supports, which is a powerful means for a direct control over load transfer.

If only initial member forces have been chosen as additional unknowns, besides fixed support positions, the equation system is transformed into a linear one. For all other chosen sets of additional unknowns, the equation system remains non-linear. The solution of non-linear systems is approximated using the Newton-Raphson method. As the success of the solver and the solution found in case of an infinite solution set are dependent on an initial guess for the remaining unknowns, the application offers the user the possibility to change and adapt the preset values.

The equation system, representing the equilibrium conditions of the strut and tie networks with the user-defined additional unknowns, and additional control commands are created by the Python™ application and written into a file. The actual computation in Mathematica® is launched via the command line interface. The appropriate solving method for the given equation system is automatically chosen by Mathematica®. The results of the computation are in turn handed back to the Python™ application via the command line interface.

With the displacements determined for the nodes of the strut and tie network, a graphical output of the equilibrated strut and tie network is generated using coloring of the single members to display compression (blue) or tension (red) member forces. The complete set of results, including the member forces of the equilibrated strut and tie network, is stored in a separate file.

In section 4.7, the combination of two strut and tie networks is discussed to describe the internal forces of a beam grillage. The external loads may be applied to both of the combined networks. Within the application, the loads are only applied to one network. The second strut and tie network, which is combined with the loaded network, is automatically generated based on the same initial network. The position of the support nodes is set to the same position as in the loaded network, and the initial member forces are derived from the member forces of the equilibrated loaded strut and tie network.

The application is a general tool that provides the possibility to generate any kind of equilibrated strut and tie networks, not only those that can be interpreted as grid shells (fig. 7.2). However, to enable the generation of an equilibrated structure by displacement of the nodes, the initial network must be kinematic.
7.2 Discrete curved stress field

The generation of curved stress fields is based on a corresponding strut and tie network. While the generation of strut and tie networks provides a high degree of control for the user through the definition of the additional unknowns, the generation of curved stress fields is mainly automated. The biggest challenge for this stage of the application is to create a geometry for the set of curved stress fields that complies statically with the corresponding strut and tie network and approximates the shape indicated by the corresponding strut and tie network as good as possible. To avoid an increase of complexity for the determination of the stress field geometry, linear stress fields are used instead of constant stress fields, that would require an adaptation of the stress field geometry to the actual applied loads, due to the overdeterminacy of the corresponding statical systems. After the geometry is defined, the boundary forces of the single linear stress fields are determined node-wise based on the member forces of the corresponding strut and tie network.

If a strut and tie network is a part of the generation of discrete curved stress fields, the corresponding initial network represents a discretization of the middle surface of the structure. Thus, the initial network must be defined based on the actual middle surface of the considered structure. It must be arranged in a way that the external loads of the network nodes are statically equivalent to the external load applied at the middle surface of the structure. Further, the displacement vectors of all network nodes must be parallel to each other to comply with the restriction discussed in section 5.6.
The geometry of the nodal zones including their stress fields is defined on the middle surface of the structure, which is specified by the user. The initial division of the middle surface into nodal zones is carried out by only considering geometrical considerations. The nodal zone boundary consists of two edges per member. These form a composed nodal zone edge. A composed nodal zone edge is defined by three corner points. The corner point in the middle is positioned in the midpoint of the corresponding initial network member. The other two corner points are placed in the center of the middle surface slices enclosed by a closed chain of network members. The position of the corner points of the nodal zones can at this stage of the application be influenced by so-called weighting factors. The weighting factors have been introduced to adapt the size of the nodal zone to the magnitudes of the member forces adjoining a network node. The definition of these weighting factors is either done automatically based on the sum of the member force magnitudes of the adjoining members or can be specified by the user.
The corner points of the nodal zones of the equilibrated strut and tie network are then derived from its counterparts on the middle surface. The eccentricity vectors between the corner points of the middle surface and of the curved stress fields must all be parallel, as discussed in section 5.6. The length of the eccentricity vectors is found by applying an interpolation based on the geometry of the strut and tie network. The division of the nodal zones on the middle surface and the curved stress fields is done individually based on the same principles to assure that corner points of the discrete curved stress fields are connected with their counterparts on the middle surface by multiples of the same vector.

The boundary forces applied at a composed nodal zone edge of an equilibrated strut and tie network must be in equilibrium with the corresponding member force. Equilibrium of forces is easily assured by applying the respective equilibrium conditions. To also meet torque equilibrium, the geometry of the composed nodal zone edge must, in general, be adapted. The number of admissible shapes of the composed nodal zone edge to meet torque equilibrium is infinite. Within the application, this is achieved by moving the middle corner point of the composed nodal zone edge from its initial position on the network member in direction of the eccentricity vector, while the two endpoints of the composed nodal zone edge remain in their position. Unless the nodal zone is planar, torque equilibrium is only obtained, if the composed nodal zone edge does not intersect the corresponding network member. As larger kinks in-between stress fields are presumed to result in stress peaks, an algorithm is used to smooth the indicated surface by moving the nodal zone corner points along the eccentricity vector.

Figures 7.3 to 7.8 illustrate the nodal zones for network nodes with three to eight adjacent members as used by the application software. The boundary forces of the composed nodal zone edges are directly derived from the member forces. The nodal zone boundaries are not subjected to edge loads. Accordingly, the member is tangent to the respective stress fields adjoining a composed nodal zone edge. The stress field edge between these two stress fields is parallel to the corresponding network member and its length is initially set to the quarter of the length of the
Composed nodal zone edges, with their midpoint being on or very close to the plane spanned by the end points and the external load vector, cause the boundary forces of the adjoined stress fields to tend to infinity. To avoid this phenomenon, the composed nodal zone edges are "folded" by changing the position of its midpoint.

A major criteria for the design of the nodal zone division was to keep the number of additional unknowns as small as possible, so that the definition of the strut and tie network is sufficient to control the load transfer. Also an automatic choice of the values of additional unknowns that might develop to much influence on the result compared to the users input should be avoided. The degree of statical indeterminacy of all kinds of divisions used is one. This value is a specialty of the defined divisions and no general property of nodal zones, which consist of triangular linear stress fields.

Along the stress field edges within the nodal zone, linear edge loads are applied. These are necessary to meet equilibrium conditions, unless the stress fields share a common plane. Along the composed nodal zone edges, no edge loads are applied, as the adjoining stress fields of neighboring nodal zones share a common plane, according to the definition made. The linear edge loads must be in equilibrium with the single external load of the corresponding node of the strut and tie network and parallel to it. The orientation of the edge loads would preferably be identical to the orientation of the corresponding single external load, but in order to find solutions for a wide range of arrangements of curved stress fields, this criterion, however, had to be dismissed. For
compatibility reasons, a linear edge load is represented by two resultants applied at the application points of the boundary forces of the linear stress fields.

The geometry of the discrete curved stress fields and the stress fields on the middle surface is generated by the Python™ application and the necessary parameters for the computation of the stress field boundary forces are added to prepared Mathematica® template files. The computation in Mathematica® is launched by the Python™ application via the command line interface.

Based on the geometry determined for the discrete curved stress fields, the stress field boundary forces and edge loads are computed. The stress field boundary forces are determined for the set of stress fields included in a single nodal zone, rather than considering all discrete stress fields of the structure simultaneously. As an initial step, the boundary forces of each composed nodal zone edge are determined based on the corresponding member forces using the equilibrium conditions. In a second step, the linear system of equations is prepared for computation. It consists of the force equilibrium condition within a single stress field (eq. 5.4), the three conditions for the equality of shear stress at the corner points of a stress field (eqs. 5.5, 5.6, 5.7) and the equilibrium conditions at the stress field edges involving the boundary forces of the neighboring stress fields and the respective edge load resultant. The stress field equations are expressed with respect to a local coordinate system. The vector equations for the equality of shear stress at the stress field corner points simplify accordingly to scalar equations. Combined with the force equilibrium condition of the stress field, five scalar equations per stress field result. Based on the equilibrium conditions at the stress field edges, another three equations result per application point of the boundary forces.
The equation system is finally complemented by the equilibrium conditions between the resultants of the linear edge loads and the single external load applied at the corresponding network node. As the edge loads and the single external load are defined to be parallel, the number of the respective equations reduces to one, assuring force equilibrium and another two for torque equilibrium. In the system of equations, the number of unknowns outnumber the equations by one. The linear system of equations is, thus, partly solved symbolically. The remaining unknown of the nodal zone is determined by a minimization of the absolute maximum of the principal stresses of the stress fields of the considered nodal zone. The minimization is executed in Mathematica©.

Besides the possibility described to influence the size of the nodal zones by so-called weighting factors, the size of the internal nodal zone is automatically varied. The variation is achieved by changing the lengths of the stress field edges in-between the stress fields, adjoining the composed nodal zone edges, using a global factor. This iterative variation aims to minimize the absolute maximum of the principal stresses of the stress fields of the considered nodal zone. For each of the variations, a complete run of the computation of the stress field boundary forces and edge loads is necessary. The boundary forces of the stress fields, that result from the computation, are returned from Mathematica© to the Python™ application via the command line interface. Based on the boundary forces of the curved stress fields, the boundary forces and moments of the stress fields on the middle surface could be determined, as described in sections 5.5 and 5.6. Since the determination of the bounds of eccentricity is based on an alternative definition of the stress field boundary forces, the output of the boundary forces as an intermediate result has not been implemented in the application.

**7.3 Bounds of eccentricity**

Bounds of eccentricity illustrate the maximal eccentricities of a corresponding curved stress field. Thus, the computer based determination of bounds of eccentricity is added to the application developed for the generation of curved stress fields (see sec. 7.2). The depth of the structure and the concrete covers, as well as the percentage of reinforcements, are defined by the user, while the orientation of the reinforcement is proposed by the application. The determination of the bounds of eccentricity is limited to orthogonal reinforcements. Based on the given discrete middle surface, a local orthogonal coordinate system is automatically determined for every node. Afterwards, it can be manually corrected at single nodes. Its tangential axes $x$ and $y$ are equal to the directions of reinforcement. In contrast to the general methodology, as described in chapter 6, the determination of the bounds of eccentricity by the application has for slabs been limited to cases with a planar middle surface, such that only the loaded curved stress fields, denoted by the superscript $I$, possess an eccentricity.

In an initial step, the boundary forces of the curved stress fields are referred to the middle surface stress fields. At the middle surface stress field, the boundary forces are split up into two membrane force components and a transverse shear force component. The two membrane force components are tangential to the middle surface stress field with one being normal and the other tangential to the corresponding stress field edge (see fig. 6.5). The transverse shear force component is parallel
to the eccentricity vector $\vec{d}$, as defined in section 6.1.3. Based on the boundary forces, membrane
and transverse shear stress states at the three corner points of the stress field on the middle sur-
face are deduced. Based on these three stress states and the corresponding linear functions (see
sec. 5.2), the stress states in every point of the linear stress field can be determined.

The transverse shear stress resultants $v_{dx}$ and $v_{dy}$ are transformed to the principal transverse shear
stress resultant $v_{d0}$. Based $v_{d0}$ a minimal depth of the core layer is determined. If the minimal
depth exceeds the actual depth of the core layer, zones of potential problems in the transfer of
transverse shear are indicated.

The actual determination of the bounds of eccentricity is done using Sandwich models. To con-
sider the different load bearing behavior of slabs and shells, three types of Sandwich models are
used (fig. 7.9). While the first model focuses on the application for slabs, the other two have been
developed for shells. Nonetheless, every type of structure is investigated by the application using
all three Sandwich models. Reinforcement is not considered as single bars, but as a continuous
reinforcement slab providing the strengths of the actual reinforcement in the orthogonal directions
$x$ and $y$. The vertical position of the reinforcement layers of the two directions is not distinguished.
The depths of the cover layers of the Sandwich model, shown in fig. 7.9a, corresponds to twice the
depth of the concrete cover. In the other two models the depth of the one cover layer is also twice
the depth of the concrete cover, while the depth of the other cover layer is varying. The depth
iteration starts with the depth $\max(a_{sx}, a_{sy}) f_y / f_c$ and increases in steps of $1\, \text{mm}$ to the maximal
depth of $0.5d$. The reinforcement is ignored in this cover layer, as the lever arm would otherwise
be dependent on the actual loading of concrete and reinforcement.

$$
\begin{align*}
\text{a)} & \quad \text{b)} & \quad \text{c)} \\
& & \\
\text{fig. 7.9 Types of sandwich models, a) type AA, b) type AB with } \max(a_{sx}, a_{sy}) f_y / f_c \leq x_1 \leq 0.5d, & c) \text{ type BA with } \max(a_{sx}, a_{sy}) f_y / f_c \leq x_2 \leq 0.5d
\end{align*}
$$

The lever arms of the stress resultants of the top and bottom cover layers, denoted by $s_1$ and $s_2$
respectively, are defined by:

$$
s_1 = h - x_1 / 2 \quad \text{and} \quad s_2 = h - x_2 / 2 \quad (7.1)
$$

The relations of the forces and moments of the Sandwich model cover layers and the infinitesimal
element are specified in eqs. 6.69 to 6.72 and 6.73 to 6.76. In case of cover layers of different
depths, the core layer possesses an eccentricity towards the middle surface. As the eccentricity
and the magnitudes of the membrane stress resultants of the core layer are rather small, compared
to the lever arms and membrane stress resultants of the cover layers, its impact on the bending
resistance is ignored.

76
For the determination of the bounds of eccentricity, non-linear equations must be solved. These are composed of eqs. 6.89, 6.90 and 6.91 or of the equivalent equations for the bottom cover element and each of the yield criteria of eqs. 6.77. The solution set consists of values for the eccentricity $e$. Equations 6.78, which define the scope of the seven yield criteria of the membrane element, are used to verify the eccentricity values within the solution sets. For the top and the bottom cover elements, a minimal and a maximal bound of eccentricity are determined. The respectively lower absolute values represent the actual bounds of eccentricity.

The described procedure has been implemented in an algorithm programed in Mathematica© [57]. The Python™ [44] application embedded in Rhinoceros© [32] provides this algorithm with the necessary data and launches it automatically. As the bounds of eccentricity are solved numerically, the shape of a surface representing the bounds of eccentricity of a curved stress field is only approximated. To minimize the calculation time, the bounds are only evaluated for the corner points of a triangular field of the middle surface. The bounds of eccentricities of all other points are interpolated linearly, such that the surface, representing the bounds of eccentricity of a corresponding curved stress field, is a plane. As the graph of the functions of the bounds of eccentricity is clearly non-linear, the actual deviations from this assumption are assessed below.

The membrane stress resultants of a top cover element $n_{x1}, n_{y1}, n_{yx1}$ define a linear vector function in stress space dependent on the eccentricity value $e$ (fig. 6.8a). In case of shell structures and planar slabs, the explicit notation of this vector function simplifies to eqs. 6.92 and 6.93. The slope of the linear functions is defined by the ratios of the normal stress resultant components of the loaded curved stress fields $n'_y/n'_x$ and $n'_{yx}/n'_x$. Provided that these two ratios were constant throughout a curved stress field, the stress states of the stress field would form a set of parallel lines in stress space, due to changes of the magnitudes of $n_{vx}, n_{vy}$ and $n_{vyx}$. As the magnitudes of $n_{vx}, n_{vy}$ and $n_{vyx}$ are rather small compared to the membrane stress resultants of the cover layer, the intersections of the set of parallel lines and the corresponding yield surface of the top cover element only cover limited areas close to the intersection points $B^+_1$ and $B^-_1$ (see fig. 7.11a). For the further investigation, the influence of $n_{vx}, n_{vy}$ and $n_{vyx}$ is, thus, ignored, such that the lines in stress space are presumed to intersect the yield surface for all stress states of the curved stress field in the points $B^+_1$ and $B^-_1$.

Nonetheless, the eccentricity values of the bounds of eccentricity $e_{min}^l$ and $e_{max}^l$ still vary with the linearly changing magnitudes of the membrane stress resultants of the curved stress field. For given intersection points of a top cover element $B^+_1$ and $B^-_1$, the corresponding eccentricity value $e^l$ is determined by:

Shell: $e^l = \frac{n_{x1}(s_1 + s_2) + n_{yx}s_2}{n'_x \cos \angle(\vec{e}_z, \vec{d})} - \frac{s_2}{\cos \angle(\vec{e}_z, \vec{d})}$ (7.2)

Planar slab: $e^l = \frac{n_{x1}(s_1 + s_2) + n_{yx}s_2}{n'_x \cos \angle(\vec{e}_z, \vec{d})}$ (7.3)

Assuming fixed intersection points of a top cover element $B^+_1$ and $B^-_1$, the determination of the eccentricity value $e^l$ was, according to eqs. 7.2 and 7.3, dependent on the stress resultants $n'_x$ and $n_{yx}$. Regarding the discussed cases of shells and planar slabs, the transverse shear stress resultants...
and, consequently, the membrane stress resultants of the core layer are only dependent on the stress state defined by the loaded curved stress field, such that the stress resultant $n_{\text{tx}}$ can be expressed by a linear function of $n_{\text{tx}}'$. By substituting a linear function of $n_{\text{tx}}'$ for $n_{\text{tx}}$, eqs. 7.2 and 7.3 would, then, only depend on $n_{\text{tx}}'$. The membrane stress resultants of the top cover element would be constant, due to the assumed fixed intersection points $B_{\text{tx}}^+$ and $B_{\text{tx}}^-$. The eccentricity value $e'$ is inversely proportional to the normal stress resultant $n_{\text{tx}}'$, due to the assumption made. The absolute values of the bounds of eccentricity will, thus, fall below their linear interpolation (see fig. 7.10).

In contrast to the assumption made above, now the case with linearly changing ratios $n_{\text{tx}}'/n_{\text{tx}}$ and $n_{\text{tx}x}/n_{\text{tx}}'$ is investigated, while one of the membrane stress resultants of the curved stress field is assumed to be constant. The effect of this assumption is best observed, if $n_{\text{tx}}'$ is set to a constant value throughout the stress field.

The equations for the determination of the eccentricity value $e'$ may also be expressed as functions of the membrane stress resultants $n_{\text{tx}1}$ and $n_{\text{tx}1}'$, such that:

- **Shell:**
  \[
e' = \frac{n_{\text{tx}1}(s_1 + s_2) + n_{\text{tx}x} s_2}{n'_{\text{tx}1} \cos \angle(\vec{e}_z, \vec{d})} - \frac{s_2}{\cos \angle(\vec{e}_z, \vec{d})} \tag{7.4}\]

- **Planar slab:**
  \[
e' = \frac{n_{\text{tx}1}(s_1 + s_2) + n_{\text{tx}x} s_2}{n'_{\text{tx}1} \cos \angle(\vec{e}_z, \vec{d})} \tag{7.5}\]

Due to the assumed constant value for $n_{\text{tx}}'$, the eccentricity is now directly proportional to the membrane stress resultant $n_{\text{tx}1}$ of the top cover element, besides the influence of $n_{\text{tx}x}$. Equations 6.89, 6.90 and 6.91 define the stress resultants of a top cover element as functions of the eccentricity value $e'$. The three equations can be interpreted as the components of a linear vector function of $e'$ and describe a line in stress space. For the corner points $P_A$, $P_B$ and $P_C$ of a triangular field on the middle surface result three lines, defined by the corresponding linear vector functions $\vec{L}_{1A}$, $\vec{L}_{1B}$ and $\vec{L}_{1C}$ (fig. 7.11b). The intersection points of the three lines with the yield surface of the considered top cover element $B_{1A}^+$, $B_{1B}^+$ and $B_{1C}^+$ define the membrane stress resultants for an eccentricity at
the upper bound.

The stress states at an edge of a linear stress field can be linearly interpolated from the stress states of the end points (see sec. 5.2). Consequently, the lines, which represent the functions of the membrane stress resultants of the top cover element at the points of an edge, can also be determined by linear interpolation from the lines of the end points. The lines in stress space corresponding to all points of a linear stress field edge form a double curved surface. The surfaces of the three stress field edges enclose all lines corresponding to the points of the stress field. The intersection curves of these three surfaces and the yield surface surround a yield surface slice, which defines the membrane stress resultants of the top cover element for an eccentricity at the upper bound for the complete field on the middle surface. The upper bound of eccentricity of the corner points $P_A$, $P_B$ and $P_C$, described by the intersection points $B_{1A}^+$, $B_{1B}^+$ and $B_{1C}^+$, define a plane in stress space. The part of this plane, which represents the linear interpolation corresponding to the points of the stress field, is enclosed by the same three surfaces, that also surround the corresponding yield surface slice. Due to the convex shape of the yield surface, the linear interpolation of the bounds of eccentricity cannot exceed the actual values for any constellation of the intersection points under the assumption made.

![fig. 7.11 Yield surface of a top cover element with a) the linear vector function $\vec{L}_1$ dependent on $e^l$ and the influence of changing membrane stress resultants of the core element, b) Slice of the yield surface and its linear interpolation of a top cover element under the assumption of a constant value for $n_{lx}^l$](image)

The two main influences on the development of the eccentricity $e^l$ have been investigated: the magnitude of the membrane stress resultant components of the curved stress field and the ratios among them.

For fixed ratios among the membrane stress resultant components, the linear interpolation of the bounds of eccentricity results values exceeding the actual bounds of eccentricity. On the other hand, the interpolated values fall below the actual bounds of eccentricity for cases with a fixed magnitude of one of the membrane stress resultant components. The actual influence of these two effects cannot be assessed in general as it depends on the individual stress state of a curved stress field. As these effects may partially cancel each other, the linear interpolation is considered as an acceptable estimate for the bounds of eccentricity.
7.4 Discussion

The implementation of the application as self-programmed Python™ script in the 3D design software Rhinoceros© has shown to be as an appropriate strategy for all steps, providing an intuitive user input and an illustrative graphical output. Outsourcing the actual computations to Mathematica© with its powerful solvers and methods simplified the creation of the application significantly and allows acceptable calculation periods.

The stress field edge, which connects the middle corner point of a composed nodal zone edge with a corner point of the internal nodal zone, is parallel to the corresponding member of the equilibrated strut and tie network. In fig. 7.12a this refers to the edges connecting the points $P_2$ and $P_{11}$, $P_4$ and $P_{12}$, $P_6$ and $P_{13}$ etc. The same applies for the boundary forces of the composed nodal zone edge. These definitions and the condition of equality of shear lead to the fact that the boundary stress at the outside end of this stress field edge is always zero. This can potentially create a relevant disturbance in the distribution of stresses. Alternative divisions of the nodal zone into triangular stress fields, like shown in fig. 7.12b, would avoid this phenomenon, but have not been investigated in this thesis.

In Sandwich model cover layers with a varying depth, the reinforcement is ignored. In contrast to cover layers with a depth twice the depth of the concrete cover, the centroids of the reinforcement layer and the concrete do not match for this type of cover layer. If reinforcement was considered, the centroid of the cover layer would be dependent on the distribution of stresses to the reinforcement and the concrete part. Based on the decisive yield criterion of the cover layer, the overall stress resultants applied at the cover layer $n_{x1/2}$ and $n_{y1/2}$ could each be split up into a reinforcement and a concrete component. In contrast, only the concrete part of the cover layers is subjected to the stress resultants $n_{yx1/2}$ and $n_{xy1/2}$. Although the depth of the cover layer is equal in all directions, the lever arm for each of the stress resultants would be different. As the lever arms are dependent on the still unknown stress resultants, the equations illustrating the relation of lever arms, stress resultants and stress resultant components applied at the reinforcement and the concrete part, would have to be added to the equation system to determine the bounds of eccentricity.

The investigations made indicated problems with the solvability of such equation systems. An iterative approach, setting starting values for the lever arms, showed convergence problems, especially in case of stress resultant components of reinforcement and concrete with opposite signs. Thus, the reinforcement is ignored in this type of cover layers.
8 Examples

In chapters 4, 5 and 6, the theoretical base for strut and tie networks, discrete curved stress fields and the according bounds of eccentricity has been developed. The implementation of these three steps in a software application is presented in chapter 7. With this software application the use of the developed method is demonstrated.

The aim of the method developed in chapters 4 to 6 was to create shapes for shell structures that consciously deviate from the statically optimal form within the bounds of the inherent bending resistance, which is illustrated in the example in section 8.2. However, the developed method may also be applied to form-finding of e.g. grid shells (see sec. 8.1) or to the analysis of slabs (see sec. 8.3).

For the determination of the structural resistance, it has been assumed that the concrete compression and shear strengths are $f_{cd} = 20\, N/mm^2$ and $\tau_{cd} = 1.1\, N/mm^2$ and that the yield stress of the reinforcement is $f_{sd} = 435\, N/mm^2$.

8.1 Form finding of grid shell structure

At an exemplary pavilion, a method for form-finding based on strut and tie networks is illustrated. It is presumed that the pavilion is a grid shell structure with its members only subjected to compression or tension. The pavilion is a concrete structure with post-tensioning in the tensile members and without an actual roof cover. For this example, the structural behavior with respect to dead loads and horizontal loads from earthquake is considered. The aspired grid shell structure is meant to transfer the applied loads to only four single supports and to span over 20 m in length and 12 m in width.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{images/fig81.png}
\caption{Different arrangements of initial networks and corresponding equilibrated strut and tie networks based on varying values for the initial member forces}
\end{figure}
The number of possible solutions is infinite. Figure 8.1 shows three types of initial network arrangements with four different sets of initial member forces. Each has been used to generate strut and tie networks subjected to an uniform vertical load. The generation steps, which are basically identical for all solutions in fig. 8.1, are illustrated for one of the systems in figs. 8.2, 8.3 and 8.4.

*fig. 8.2* Initial network with uniform loads

*fig. 8.3* Initial network with the defined values for all initial member forces

*fig. 8.4* Generated strut and tie network subjected to uniform loads
Out of the networks subjected to uniform vertical loads, one is picked for further investigation with a first assumption of the actual dead loads. Figure 8.5 shows its initial arrangement, the applied external loads, the values for the initial member forces and the resulting equilibrated network. The strut and tie network, subjected to uniform vertical loads, provides the initial network for this next step. The chosen set of additional unknowns only consists of displacement components of the nodes and a very rough assumption for the expected member forces. Thereby, a strut and tie network has been generated, which possesses a shape very similar to the strut and tie network subjected to uniform vertical loads.

\[ \begin{array}{c}
\begin{array}{c}
\text{a)} \\
\text{b)} \\
\text{c)} \\
\text{d)} \\
\end{array}
\end{array} \]

\textbf{fig. 8.5} Generation of a strut and tie network subjected to the assumed dead loads: a) initial network with loads, b) additional unknowns, c) initial guess, d) comparison of initial network (black) and strut and tie network (colored)
To assure that the members of the aspired grid shell structure are only subjected to compression and tension, the structural body must inherit the strut and tie networks of all load cases. Accordingly, in the next step strut and tie networks subjected to a combination of dead and earthquake loads are generated. The earthquake loads, with an intensity 10% of the dead loads, are assumed to be distributed evenly. The strut and tie network for the assumed dead loads is now used as initial network (fig. 8.5). The chosen set of additional unknowns only consists of components of displacement including the definition of actual values and relations among groups of nodes (fig. 8.7). Thereby, the generated equilibrated network shows only small deviations from the initial network (fig. 8.10). Due to the choice of the set of additional unknowns, a non-linear equation system results, which requires an initial guess for the values of the remaining unknowns to obtain a solution. A rough estimate of the components of displacement and the reaction forces is sufficient as initial guess (fig. 8.9). The initial guess for the initial member forces is provided by the member forces of the strut and tie network subjected to vertical loads only (fig. 8.8).

**fig. 8.6** Generation of a strut and tie network subjected to the assumed dead loads and half of the earthquake load: initial network with loads

**fig. 8.7** Generation of a strut and tie network subjected to the assumed dead loads and half of the earthquake load: initial network with additional unknowns
fig. 8.8 Generation of a strut and tie network subjected to the assumed dead loads and half of the earthquake load: initial member forces (rounded) as part of the initial guess

fig. 8.9 Generation of a strut and tie network subjected to the assumed dead loads and half of the earthquake load: components of displacement and reaction forces as part of the initial guess

fig. 8.10 Generation of a strut and tie network subjected to the dead vertical loads and an earthquake load of 5% of the vertical loads: comparison of the initial network (black) to the strut and tie network (colored)
The best results are obtained if a step-by-step procedure is used, in which the horizontal loads are added in parts. In the investigated example, a sequence of two steps, in which 50% of the horizontal loads are added in each step, was suitable, while the initial guess for the initial member forces matches the member forces of the previously generated strut and tie network. Figures 8.11 and 8.12 illustrate the sequence for both directions of earthquake loading.

**fig. 8.11** Sequence of strut and tie networks subjected to vertical loads and (from left to right) no, half and full horizontal earthquake loads in lateral direction

**fig. 8.12** Sequence of strut and tie networks subjected to vertical loads and (from left to right) no, half and full horizontal earthquake loads in longitudinal direction
In fig. 8.13, the generated strut and tie networks subjected to dead loads and four combinations of dead and earthquake loads have been overlayed. The designed shape of the pavilion structure encloses all networks, such that it is predominantly subjected to compression and tension. In order to obtain a more precise analysis of the pavilion structure, the generation of the five strut and tie networks could be repeated with the actual dead loads of the now designed shape of the structure, but will only show small deviations from the strut and tie networks generated so far.

8.2 Form finding of a shell

Shell form finding is mainly ruled by statically optimal forms. The example presented here, will determine such an optimal form using the developed methods of strut and tie networks and curved stress fields, and will go beyond the statical optimum by involving the structure’s bending resistance in the design process. The four single supports of the aspired shell are irregularly arranged. The spans between the supports lie within a range of approximately 15 m to 20 m. The depth of the shell is assumed to be constant with 20 cm throughout the structure. A strong reinforcement of 2093 mm²/m, which matches bars of 20 mm every 15 cm, is presumed for both directions and all layers to obtain a significant contribution of the bending resistance to the design process. For the actual structure, the reinforcement can be reduced to the necessary percentage after the form finding. Stability issues are not taken into account for the design process.

In this example the proposed method for the form finding of shells is divided into five steps:

- Step 1 - Form finding using strut and tie networks with uniform loads
- Step 2 - Adaptation of the strut and tie network of step 1 to approximate loads (optional)
- Step 3 - Optimization of the load transfer (optional)
- Step 4 - Determination of curved stress fields and bounds of eccentricity
- Step 5 - Formal adaptation of the shell within the bounds of eccentricity and its verification
Step 1 - Form finding using strut and tie networks with uniform loads

Based on different arrangements of planar initial networks, the approximate shape of the aspired shell is found. Besides the arrangement of the initial network, the definition of all initial member forces allows the best control at this stage of the form finding process (fig. 8.14). The supported nodes are defined to be fixed to their initial position in this and the following steps of the form finding process. The strut and tie network in fig. 8.14d was selected as basis for the further steps. As the actual loads are dependent on the yet unknown shape of the shell, the initial network is subjected to a uniform load (fig. 8.15). With the definition of all initial member forces as the set of additional unknowns (fig. 8.16), the shape and the load transfer of the aspired strut and tie network are controlled (fig. 8.17).

fig. 8.14  Step 1: strut and tie networks as first approach to the shell form finding, based on the same initial network, but with varying initial member forces as additional unknowns

fig. 8.15  Step 1: initial network with loads and support zones (gray)
Step 2 - Adaptation of the strut and tie network of step 1 to approximate loads

The plan view of the initial network and the strut and tie network of step 1 differ significantly. This results in an enormous difference in the actual loads, applied in the network nodes, in contrast to the assumed uniform loads in step 1. Thus, the strut and tie network generation is repeated with loads, which better approximate the actual loads. For the generation of the curved stress fields and the bounds of eccentricity, the displacement of the network nodes must be limited to the direction of the applied loads. By updating the loads, the geometry of the resulting strut and tie network fits the loads of the final structure better and, thereby, reduces the risk of complications in the further steps. Nonetheless, this step is optional and may also be skipped to shorten the form finding process.

To determine the loads, a middle surface is derived from the shape of the strut and tie network of step 1, using the provided tools of the design software Rhinoceros©. A sensible division of the surface provides load zones, which possess an area centroid close to the corresponding network node (fig. 8.18). As the supported nodes lie on the border of the middle surface, they cannot

fig. 8.16  Step 1: initial network with values for the additional unknowns

fig. 8.17  Step 1: strut and tie network, a) axonometry, b) front view, c) side view, d) top view with member forces (rounded)
be subjected to external loads. The load zones of the neighboring nodes must, accordingly, be extended. The resulting loads are, in turn, applied at the planar initial network of step 1 (fig. 8.19). By successive adaptation of the initial member forces, as the set of additional unknowns, a similar strut and tie network compared to the one of step 1 is obtained (figs. 8.20 and 8.21). With the definition of the initial member forces, the displacement of these nodes is optimized, which, so far, were placed in a big distance to the area centroid of the corresponding load zone.

*fig. 8.18*  Step 2: middle surface, which has been manually derived from the strut and tie network of step 1, split into load zones

*fig. 8.19*  Step 2: initial network with loads and support zones (gray)

*fig. 8.20*  Step 2: initial network with values for the additional unknowns
Step 3 - Optimization of the load transfer

Due to the aspired small horizontal displacements along two edges of the strut and tie network compared to the initial network (fig. 8.21d), the load transfer is yet concentrated in one direction. The load distribution can be improved by strengthening the load transfer in the other direction, while approximately keeping the strut and tie network’s shape. In this optional step, diagonal members along the edges are introduced in the resulting network of step 2 (fig. 8.22). The initial member forces in the, until now, only barely loaded direction are doubled. In contrast to the last steps, only one initial member force per continuous curve is defined as part of the set of additional unknowns (fig. 8.22). To obtain a strut and tie network, which only differs minimally from the initial network, the displacement of all nodes is limited to the direction of the loads and the nodes in the middle of the edge curves are fixed to their initial position. The external loads determined in step 2 are applied at the corresponding nodes of the initial network.

Since the system of equations remains non-linear with the chosen set of additional unknowns, an initial guess for the other unknowns is necessary. The member forces of the strut and tie network of step 2 serve as start values for the still undefined initial member forces (fig. 8.23). For the remaining displacement components and the reaction forces, a rough initial guess is sufficient to obtain a desirable solution (fig. 8.24). The resulting strut and tie network only differs slightly from the initial network (fig. 8.25).
**fig. 8.22**  Step 3: Initial network with values for the additional unknowns

**fig. 8.23**  Step 3: Initial network with initial guess: initial member forces (rounded)
**fig. 8.24** Step 3: Initial network with initial guess: displacement components and reaction forces

**fig. 8.25** Step 3: strut and tie network, a) axonometry with initial network, b) front view, c) side view, d) top view with member forces (rounded)
Step 4 - Determination of curved stress fields and bounds of eccentricity

Based on the strut and tie network of step 3, a new middle surface is developed. By dividing this surface into load zones, the distributed loads are assigned as single loads to the network nodes (fig. 8.26). In this step, the displacement of the nodes is again limited to the direction of the loads. Therefore, the area centroids of the load zones are used as nodes of the initial network. The arrangement of members remains unchanged compared to step 3 (fig. 8.27). The member forces of the strut and tie network of step 3 are used as initial member forces as part of the set of additional unknowns (fig. 8.28) and the initial guess (fig. 8.29). The resulting strut and tie network deviates only slightly from the initial network (fig. 8.31).

*fig. 8.26*  Step 4: middle surface, which has been manually derived from the strut and tie network of step 3, split into load zones and area centroids of the surface slices

*fig. 8.27*  Step 4: initial network with loads and support zones (gray)
**fig. 8.28** Step 4: Initial network with values for the additional unknowns

**fig. 8.29** Step 4: Initial network with initial guess: initial member forces (rounded)
**fig. 8.30** Step 4: Initial network with initial guess: displacement components and reaction forces

**fig. 8.31** Step 4: strut and tie network, a) axonometry with initial network, b) front view, c) side view, d) top view with member forces (rounded)
With the information provided by the strut and tie network and the middle surface (fig. 8.26), the curved stress fields are generated. Figure 8.32 shows the curved stress fields with the boundary forces illustrated by colored lines, indicating the type of force. The optimization of the stress field geometry within a nodal zone results in a maximization of the size of the internal stress fields of nodal zones with four corresponding network members, which meet in an angle of approximately $90^\circ$. In contrast, the stress fields of the nodal zones along the edges possess a more even size. The cause and the effects of this phenomenon are discussed in section 8.3.3.

Based on the stress states defined by the curved stress fields and the structural properties, the bounds of eccentricity are computed, which illustrate the maximal and minimal eccentricity of the curved stress fields towards the stress fields of the middle surface. The application only evaluates the values at the stress field corner points. Based on these values, the bounds of eccentricity are linearly interpolated. A stress field corner point possesses as many bounds of eccentricity pairs as adjoining stress fields, which causes sudden changes of eccentricity limits at the stress field edges. The illustration of such discontinuous surfaces, as shown in fig. 8.37, is difficult to interpret at this stage of the form finding process. As in the next step, the shape of the shell is adapted within the bounds of eccentricity, a continuous illustration is more advantageous (fig. 8.33). Therefore, at each stress field corner point only the bounds of eccentricity with the smallest absolute values are used, which yields conservative results (figs. 8.34 and 8.35).
Step 4: axonometric view of the curved stress fields (gray) and the continuous illustration of the bounds of eccentricity (brown)

Step 4: longitudinal section in axonometric view of the curved stress fields (gray) and the continuous illustration of the bounds of eccentricity (brown)
Step 5 - Formal adaptation of the shell within the bounds of eccentricity and its verification

Within the envelop determined in step 4, the middle surface of the shell is adapted to face further conditions besides statics. In fig. 8.36, a possible shell is illustrated that deviates from the statically optimal form defined by the curved stress fields. The vertical distance between the new middle surface and the curved stress fields is up to 65 cm and is approximately 14 cm on average. The new middle surface lies within the bounds of eccentricity of step 4. Nonetheless, the new shape must be verified, as the bounds of eccentricity illustrate the maximal and minimal eccentricity of the curved stress field towards the underlying middle surface.

The strut and tie network and the curved stress fields of step 5 are computed with the same assumptions as in step 4. The determination of the bounds of eccentricity is, in contrast, based on the new middle surface. For the verification of the new shape, the discontinuous illustration of the
bounds of eccentricity is used (fig. 8.37). The top and the bottom view (fig. 8.38) prove that the middle surface lies within the bounds of eccentricity, except for small areas close to the supports, which indicates that the depth should be increased there.

**fig. 8.37** Step 5: axonometric view of the curved stress fields (gray) and the discontinuous illustration of the bounds of eccentricity

**fig. 8.38** Step 5: curved stress fields (gray) and the discontinuous illustration of the bounds of eccentricity, a) top view, b) bottom view
Simultaneously with the determination of the bounds of eccentricity, the transverse shear stresses within the core element of the Sandwich model are compared with the yield criterion (see sec, 6.4). In fig. 8.39, the results of this investigation are illustrated. Due to the small depth of the core element of only 40\(\text{mm}\) or less, depending on the Sandwich model type, the shear stresses exceed the shear strength in rather large areas. At this stage of the implementation of transverse shear in the method, the results only indicate areas of the structure, which might require a more detailed analysis in this regard.

\[\text{fig. 8.39} \quad \text{Step 5: colored middle surface stress fields, illustrating whether the transverse shear yield criterion is met (green) or violated (red)}\]

### 8.3 Analysis of a slab

The strut and tie networks and the curved stress fields have been developed as form finding tools as shown in the last two sections. However, the developed methods can also be used for analysis of surface structures. As the load bearing behavior of slabs is well understood, the analysis of a slab is used to assess the quality of the methods’ results.

The considered slab is assumed to be a linearly supported, rectangular slab with a clear span of 8\(m\) by 10\(m\). The span differs slightly with the used strut and tie networks and is approximately 8.30\(m\) by 10.30\(m\). The depth of the slab is assumed with 25\(cm\).

As the methods are based on the lower bound theorem of the theory of plasticity, the number of possible equilibrium solutions is unlimited. To allow a basic assessment of the capability of the curved stress fields, two load transfer strategies have been chosen. The first example is comparable to a solution with the Hillerborg strip method [16], the second example is similar to an elastic solution with radial load transfer and a clamping effect in the corners. The loads of the slab are summarized in table 8.1.
### Table 8.1 Assumed loads for the slab

<table>
<thead>
<tr>
<th>Type</th>
<th>Characteristic value ([kN/m^2])</th>
<th>Safety factor</th>
<th>Design value ([kN/m^2])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dead load</td>
<td>6.25</td>
<td>1.35</td>
<td>8.44</td>
</tr>
<tr>
<td>Permanent load</td>
<td>1.50</td>
<td>1.35</td>
<td>2.03</td>
</tr>
<tr>
<td>Life load</td>
<td>2.00</td>
<td>1.50</td>
<td>3.00</td>
</tr>
<tr>
<td>Sum of loads</td>
<td>-</td>
<td>-</td>
<td>13.47</td>
</tr>
</tbody>
</table>

The reinforcement in all layers and directions is assumed with 1026 \(mm^2/m\), which matches reinforcement bars with a diameter of 14 mm every 15 cm.

**8.3.1 Torsionless load transfer**

To achieve a torsionless load transfer the initial network on the middle surface of the slab is formed by a rectangular grid of straight lines between opposing supports (see fig. 8.40). The supported nodes are positioned along the middle axis of a surrounding support zone, which has a width of 30 cm. Furthermore, the initial network is arranged in a way that the single loads applied at the nodes are statically equivalent to the distributed load in the surrounding area.

![Initial system with external loads and support zone (gray)](image)

The aspired strut and tie network is meant to represent a solution, which provides a load transfer with an even utilization of the structure. Best results are obtained when the vertical reaction force components of the supported nodes along one long and one short edge are set to the same value.
respectively (see fig. 8.41 groups R1 and R2). This definition is combined with a limitation of the node displacement to a vertical and a pre-defined initial member force in both load transfer directions for two members in the middle of the network (see fig. 8.41). A side effect of the latter is that the members in the middle of the network remain horizontal and, thus, do not change their length when the strut and tie network is generated. As the ratio of lengths and member forces in the initial and strut and tie network have been defined to be proportional to each other (see eq. 4.1), the defined values for the initial member forces match the corresponding member forces in the strut and tie network in case of symmetric loads and geometry. Furthermore, the eccentricity of the four nodes in the middle of the network is equal in a symmetric strut and tie network, such that the multiplication of the directly defined member forces and their lever arm towards the middle plane of the slab are identical for both load transfer directions. This assures an even utilization of the slab in combination with the defined groups of supported nodes. The definition of values for the additional unknowns, illustrated in fig. 8.41, refers to the loaded strut and tie network only. The second strut and tie network, which is not subjected to external loads, is generated by the application accordingly.

**Fig. 8.41** Definition of values and relations for the additional unknowns of the equation system

Figures 8.42, 8.43 and 8.44 illustrate the initial guess, which has been used to solve the non-linear equation system. Even with the rough values of this initial guess, the aspired solution for the combined strut and tie networks could be generated (fig. 8.45).
**fig. 8.42** Initial system with the initial guess for the solution of the non-linear equation system: initial member forces

**fig. 8.43** Initial system with the initial guess for the solution of the non-linear equation system: reaction forces
**fig. 8.44** Initial system with the initial guess for the solution of the non-linear equation system: displacement of the initial nodes

**fig. 8.45** Combined strut and tie networks; blue: compression members; red: tension members

Based on the combined strut and tie networks, the application generates the curved stress fields mainly automatically (fig. 8.46) and computes the boundary forces of the linear stress field. The colored lines in fig. 8.46 indicate the orientation and the type of these boundary forces.
In case of the considered network, the optimization of the size of the internal nodal zone yields a reduction to the minimal size of the stress fields along the nodal zone edges (figs. 8.46 and 8.47). Besides the size of the internal nodal zone, also its division into stress fields is varied (see section 7.2).

Due to the chosen torsionless load transfer, the members adjoining the nodes in the corners of the network connect supported nodes only. Thus, their member force is irrelevant and has been set to zero. Therefore, the corner points have been skipped in the generation process of the curved stress fields.

Figures 8.46 and 8.47 show that boundary forces at the same stress field edge are mainly parallel and the angle between the force vector and the stress field edge is close to a right angle. This indicates an even stress distribution in both load transfer directions. The stress distribution is analyzed involving the boundary force intensities in section 8.3.3.

The detail view of the nodal zone illustrates not only the orientation and the type of the stress field boundary forces, but also the orientation of the edge load vectors. Depending on the nodal zone’s geometry and the type of boundary force, the edge load vectors possess partly an opposite orientation compared to the node’s single load. The phenomenon results from the discretized geometry and the equilibrium conditions. Nonetheless, the edge loads of the nodal zone are in force and torque equilibrium with the node’s single load.
Figures 8.48, 8.49, 8.50 and 8.51 show the combined sets of curved stress fields and the surfaces interpolating the bounds of eccentricity. As discussed in section 7.3, the bounds of eccentricity are only evaluated for the stress field corner points. Based on these three points, the surfaces representing the bounds of eccentricity of a stress field are linearly interpolated. The bounds of eccentricity of stress field corner points with zero or very small stresses show extreme values. To display the bounds of eccentricity, these extreme values have been manually reduced to a reasonable size.

For the determination of the bounds of eccentricity, the results of the three Sandwich model types are evaluated (see fig. 7.9). In case of the considered slab, the Sandwich model type AA, with two cover elements of identical depth, provided the best results, as expected. As the bounds of eccentricity are dependent on the underlying stress state, they vary from stress field to stress field. Consequently, there are as many bound of eccentricity pairs at the corner points as they possess adjoining stress fields. As long as the curved stress fields are enclosed by the bounds of eccentricity, the stress state does not violate the yield criteria of the cover elements. Whether this condition is met, is best illustrated by the top view of the curved stress fields and bounds of eccentricity in fig. 8.49. Except for strips close to the supports and a few other points, the eccentricity of the curved stress fields lies within the admissible range.
**fig. 8.48** Axonometric view of the curved stress fields (gray) and the bounds of eccentricity (brown)

**fig. 8.49** Top view of the curved stress fields (gray) and the bounds of eccentricity (brown)
**fig. 8.50** Transverse section in axonometric view in the middle of the slab with curved stress fields (gray) and bounds of eccentricity (brown)

**fig. 8.51** Longitudinal section in axonometric view in the middle of the slab with curved stress fields (gray) and bounds of eccentricity (brown)
The stress intensities in the curved stress fields increase from the middle of the slab to the supports, just as the member forces of the strut and tie network. According to the increase of the stress intensities, the values of the bounds of eccentricity decrease, while the surfaces interpolating the bounds of eccentricity remain in a similar distance to the curved stress fields throughout the slab (see figs. 8.50 and 8.51).

![Colored middle surface stress fields illustrating whether the transverse shear yield criterion is met (green) or violated (red)](image)

**fig. 8.52** Colored middle surface stress fields illustrating whether the transverse shear yield criterion is met (green) or violated (red)

To verify the determined bounds of eccentricity, the transverse shear stresses of the core element of the Sandwich model are compared with the yield criterion (see sec. 6.4). The results of this investigation are illustrated in fig. 8.52. In the corners of the slab, small areas were identified by the application to exceed the shear strength. These areas indicate disturbances in the general stress distribution, as a torsionless load transfer should create an even distribution of shear stresses without an increase in the corners of the slab.
8.3.2 Radial load transfer

An elastic solution for a rectangular slab is basically characterized by a radial load transfer and a clamping in the slab’s corners. The initial network shown in fig. 8.53 reproduces these basic properties of an elastic solution. The position of the network nodes considers that the applied single loads are statically equivalent to the surrounding distributed load. Thus, the supported nodes could not be placed along the middle axis of the support zone.

Using the initial network, shown in fig. 8.53, desirable solutions were only found with sets of initial values containing a high number of initial member forces. Thus, the used set consists of values for all initial member forces except for the members in the corners of the slab (fig. 8.54). The initial member forces were defined by means of a simple extension of the application computing member forces based on the nodal equilibrium conditions. As the horizontal nodal equilibrium conditions are met, the nodes are only displaced vertically to obtain the strut and tie network. To directly control the clamping effect, all displacement components of one node per corner of the slab are defined. The definition of the additional unknowns, illustrated in fig. 8.54, refers to the loaded strut and tie network only. The second network, which is not subjected to external loads, is generated by the application accordingly.
**fig. 8.54** Definition of values and relations for the additional unknowns of the equation system (initial member forces rounded)

Figures 8.55 and 8.56 illustrate the initial guess, which has been used to solve the non-linear equation system. The roughly approximated values are sufficient to obtain a desirable solution for the strut and tie network (fig. 8.57).

**fig. 8.55** Initial system with the initial guess for the solution of the non-linear equation system: values for the initial member and reaction forces
fig. 8.56  Initial system with the initial guess for the solution of the non-linear equation system: values for the displacement of the initial nodes

Based on the combined strut and tie networks, the application generates the curved stress fields mainly automatically (fig. 8.58) and computes the boundary forces for each linear stress field. The colored lines in fig. 8.58 indicate the orientation and the type of the boundary forces. In case of the considered network, the optimization of the size of the internal nodal zones yields a strong reduction in size of the stress fields along the nodal zone edges (fig. 8.58). This reduction is more distinct in nodes with four adjoining members which form angles close to 90°. Beside the size of the internal nodal zone, its division into stress fields is also varied (see sec. 7.2).

At the composed edges of the nodal zone, the two boundary forces are parallel by definition. The orientation of the two boundary forces at the other stress field edges differs significantly for the major part of the curved stress field set, as illustrated in figs. 8.58 and 8.59. Only at nodes with four adjoining members forming angles close to 90°, curved stress fields with a low variation in the orientation of the two boundary forces of an edge are found. This indicates, without considering the actual intensities of the boundary forces, that the stresses within the linear stress fields vary significantly, which in turn is a sign for an uneven load distribution. The stress distribution is analyzed involving the boundary force intensities in section 8.3.3.

fig. 8.57  Combined strut and tie networks; blue: compression members; red: tension members
**fig. 8.58** Curved stress fields with boundary forces (blue: compression, red: tension)

**fig. 8.59** Detail view of the nodal zone marked in fig. 8.58 with boundary forces (blue: compression, red: tension), edge loads (green) and nodal zone boundary forces (black)
In fig. 8.59, also the edge load vectors are shown. Depending on the nodal zone geometry and the type of boundary force, the edge load vectors possess partly an opposite orientation compared to the node’s single load. The phenomenon results from the discretized geometry and the equilibrium conditions. Nonetheless, the edge loads of the nodal zone are in force and torque equilibrium with the single load of the network node.

Figures 8.60, 8.61, 8.62, 8.63 and 8.64 show the combined sets of curved stress fields and the surfaces interpolating the bounds of eccentricity. As discussed in section 7.3, the bounds of eccentricity are only evaluated for the stress field corner points. Based on these three points, the surfaces representing the bounds of eccentricity of a stress field are linearly interpolated. The bounds of eccentricity of stress field corner points with zero or at least very small stresses show extreme values. To display the bounds of eccentricity, these extreme values have been manually reduced to a reasonable size.

![Axonometric view of the curved stress fields (gray) and the bounds of eccentricity (brown)](image)

**fig. 8.60** Axonometric view of the curved stress fields (gray) and the bounds of eccentricity (brown)

For the determination of the bounds of eccentricity, the results of the three Sandwich model types are evaluated (see fig. 7.9). In case of the considered slab the Sandwich model type AA, with two cover elements of identical depth, provided the best results, as expected. As the bounds of eccentricity are dependent on the underlying stress state, they vary from stress field to stress field, such that at the corner points exist as many bound of eccentricity values as they possess adjoining
stress fields. As long as the curved stress fields are enclosed by the bounds of eccentricity, the stress state does not violate the yield criteria. Whether this condition is met, is best illustrated by the top and the bottom view of the curved stress fields and bounds of eccentricity in figs. 8.61 and 8.62. Regarding the curved stress fields above the slab’s middle surface, the structural resistance is exceeded by the determined stresses in a ring around the center. In the other parts of the slab, the curved stress fields are mainly enclosed by the bounds of eccentricity. Except for small strips, the curved stress fields below the slab’s middle surface lie within the bounds of eccentricity (fig. 8.62).
fig. 8.63 Transverse section in the middle of the slab through the curved stress fields (gray) and the bounds of eccentricity (brown) in axonometric view.

fig. 8.64 Longitudinal section in the middle of the slab through the curved stress fields (gray) and the bounds of eccentricity (brown) in axonometric view.
The bounds of eccentricity are only valid, if the transverse shear stresses of the core element of the Sandwich model are below the shear strength. Thus, the shear stresses are compared with the yield criterion (see sec. 6.4). Figure 8.65 shows the results of this investigation. Due to the conservative yield criterion and the small depth of the core element of only 90\text{mm}, problems concerning transverse shear transfer have been determined for rather large areas.

![Fig. 8.65 Colored middle surface stress fields illustrating whether the transverse shear yield criterion is met (green) or violated (red)](image)

### 8.3.3 Assessment of the presented load transfer strategies

Comparing the zones of the two load transfer strategies, in which the curved stress fields are lying outside of the bounds of eccentricity (figs. 8.48 and 8.60), the torsionless solution provides significantly better results. This conclusion is surprising, as the radial load transfer strategy with the clamping effect in the slab’s corners provides, in general, a more effective load distribution. The reason for this phenomenon must, thus, be found in the curved stress fields.

Instead of constant stress fields, linear ones are used to decouple the stress fields’ geometry from the actual load transfer. Thereby, a problematic relation of geometry and load transfer is avoided. The linearly changing stress states in linear stress fields can have a negative impact on the quality of the results. The bigger the difference between the stress states of the corner points, the more disadvantageous is this effect.

In the example, the depth and the reinforcement remain unchanged throughout the slab. The variability in the bounds of eccentricity is, thus, a strong indicator for the variability in stress distribution. Besides the variability of the bounds of eccentricity, the difference between the two boundary...
forces of a stress field edge is an objective measure for the variability in stress distribution. This difference is assessed with respect to the:

**difference vector ratio:** the ratio of the norm of the difference vector between a stress field edge’s boundary forces to the average norm of the two boundary force vectors

**boundary force angle:** the angle enclosed by the boundary force vectors of a stress field edge

In the comparison of the bounds of eccentricity of the two load transfer strategies in figs. 8.48 and 8.60, the variability in stress distribution is, in case of the radial load transfer, obviously bigger. The same result is obtained by the evaluation of the measures that describe the actual differences between the two boundary forces of a stress field edge. Table 8.2 contains the average, the standard deviation and the median of the difference vector ratios of all stress field edges for both load transfer strategies. The average values differ less than expected from the comparison of the bounds of eccentricity in figs. 8.48 and 8.60. The reason is the high standard deviation, which states that the average is distorted by extreme values. Thus, the median has a higher significance for the interpretation of the difference vector ratios. The same applies for the boundary force angles (tab. 8.4).

<table>
<thead>
<tr>
<th>Load transfer strategy</th>
<th>Average [%]</th>
<th>Standard deviation [%]</th>
<th>Median [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsionless</td>
<td>30.39</td>
<td>41.25</td>
<td>5.47</td>
</tr>
<tr>
<td>Radial</td>
<td>40.63</td>
<td>40.87</td>
<td>23.72</td>
</tr>
</tbody>
</table>

*tab. 8.2* Difference vector ratios of all stress field edges

As discussed in section 7.4, the outside boundary force of the stress field edge, which connects the middle corner point of a composed nodal zone edge and the nearest internal corner point, is zero, due to the chosen division principle of the nodal zones into stress fields. To get a clearer impression of the situation without this general disturbance, the corresponding stress field edges have been excluded from the analysis in table 8.3. The difference between both load transfer strategies remains, but the average, the standard deviation and the median of the difference vector ratio reduce largely.

<table>
<thead>
<tr>
<th>Load transfer strategy</th>
<th>Average [%]</th>
<th>Standard deviation [%]</th>
<th>Median [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsionless</td>
<td>14.70</td>
<td>27.31</td>
<td>3.47</td>
</tr>
<tr>
<td>Radial</td>
<td>26.61</td>
<td>32.18</td>
<td>11.53</td>
</tr>
</tbody>
</table>

*tab. 8.3* Difference vector ratios of all stress field edges except those, which connect the middle corner point of a composed nodal zone edge and the nearest internal corner point

Regarding the solution with torsionless load transfer, the medians of the difference vector ratios and the boundary force angles in tables 8.3 and 8.4 state that the two boundary forces are almost identical at 50% of the stress field edges and, thereby, close to a solution with constant stress fields,
when neglecting the special edges with a zero boundary force. For the radial load transfer solution, the corresponding values are significantly bigger. This explains why the curved stress fields lie in a remarkably greater part outside of the bounds of eccentricity than in the first strategy.

<table>
<thead>
<tr>
<th>Load transfer strategy</th>
<th>Average [°]</th>
<th>Standard deviation [°]</th>
<th>Median [°]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsionless</td>
<td>16.17</td>
<td>41.99</td>
<td>1.85</td>
</tr>
<tr>
<td>Radial</td>
<td>31.35</td>
<td>49.52</td>
<td>8.11</td>
</tr>
</tbody>
</table>

*tab. 8.4*  Boundary force angles of all stress field edges except edges with one or two boundary forces that are zero

The reason, why the results for the radial load transfer are worse than those of the torsionless load transfer have been investigated above. To assess the results of the torsionless load transfer strategy in a broader sense, it is compared to Hillerborg’s strip method [16]. The assumptions for the torsionless load transfer are, besides the same geometry, an identical utilization of the slab in both directions and uniform reaction forces along each side. The identical situation is obtained with a general break down of the distributed external load into two parts $q_{1d}$ and $q_{2d}$ (fig. 8.66). With $q_{1d} = 8.16 kN/m^2$ transferred over the short span and $q_{2d} = 5.31 kN/m^2$ over the long span, the maximal bending moments for both directions are identical, $m_{xd} = m_{yd} = 70.3 kN/m$.

Using the Sandwich model type AA with an internal lever arm of $z = 170 mm$, the resistance for torsionless bending results in $m_{Rd} = 75.9 kN/m$. The utilization of the solution of the Hillerborg strip method is $m_{xd}/m_{Rd} = m_{yd}/m_{Rd} = 0.93$. The eccentricity ratios of the curved stress fields $e/e_{max}$ have a similar dimension (fig. 8.67). Only in the strips parallel to the supports ratios greater than 1 are reached.
In addition to the bounds of eccentricity, also the transverse shear stresses are checked. The results are illustrated in figs. 8.52 and 8.65. In the torsionless load transfer strategy, only small areas in the slab’s corners were identified as problematic, while in the radial load transfer strategy, the shear strength is exceeded in larger areas in the corners. As the slab is linearly supported and is subjected to a distributed load of normal intensity, the obtained results are obviously not realistic and will not withstand a comparison with established analysis methods. Nonetheless, it must be considered that the remaining depth for the core layer is strongly limited to only 90 mm, due to the use of the Sandwich model. Additionally, the comparison of transverse shear stress and the yield criterion is only evaluated in the corner points of the stress fields. If the yield criterion is violated at only one corner point, the whole stress field is classified as problematic by the application. This and the, at times, strong variability of the stresses are the main causes for these results. Thus, at this stage of the implementation of transverse shear in the method, the results only indicate areas of the structure, which might require a more detailed analysis in this regard.
9 Discussion / Outlook

For the determination of curved stress fields a continuous and a discrete approach have been taken. The continuous approach is based on the membrane theory [13]. Based on an initial surface with a tangential initial stress state, the developed differential equation describes a curved membrane stress state that is in equilibrium with the external loads. The practical applicability of the continuous curved stress fields is very limited, due to the general mathematical issue with the exact solution of differential equations. Instead of using a numerical approximation for the solution of the differential equation, the continuous approach was omitted in favor of the independent development of discrete curved stress fields.

The discrete approach is based on strut and tie networks. These are used to define the load transfer. For the generation of strut and tie networks, a general method has been developed, including an extensive discussion of the fundamental relations and strategies for the solution of the non-linear equation systems. With the means developed to control the generation of strut and tie networks, the method allows an application to form finding and to the analysis of a wide range of member structures.

The presented idea of discrete curved stress fields represents the extension of thrust lines for surface structures. From thrust lines, the internal forces of arch structures can be concluded. The same applies for curved stress fields and shell structures. The combination of a thrust line with a tie (see fig. 4.14) represents the internal forces of a beam. Analogously, the combination of two sets of curved stress fields expresses the internal forces of a slab. The relations of internal forces of surface structures and curved stress fields applies generally to continuous as well as discrete curved stress fields.

The translation of the structural resistance into maximal and minimal values of eccentricity and their graphical illustration extends the conventional form-finding methods. By the presented bounds of eccentricity, the possibilities and limits, which evolve from the structural resistance, are included.

Reinforced concrete slabs and shells provide, like concrete walls and beams, a quasi-infinite statistical indeterminacy. According to the lower bound theorem of the theory of plasticity, the load transfer within these structures can be controlled by the engineer. For structures subjected to in-plane loads, like walls and beams, planar pin-jointed networks and planar stress fields [39] are intuitive methods to practically apply this given possibility. The strip method [16] allows a comparably intuitive control over the load transfer of slabs, but does not consider torsion. With the presented spatial strut and tie networks and discrete curved stress fields, methods for laterally loaded structures have been developed, that are analogous to planar pin-jointed networks and stress fields. With the generation of discrete curved stress fields based on strut and tie networks, an intuitive control is obtained. The method developed for laterally loaded surface structures, can generally be applied to all types of shells and linearly supported slabs without restrictions regarding load transfer.
The generation of the strut and tie networks is completely controlled by the user and is meant to significantly control the load transfer in the discrete curved stress fields, which are generated automatically afterwards. Besides some disturbances in the stress distribution, which arise from the used division of the nodal zones into stress fields (see sec. 5.7) and the, at times, strong variability in the stress states of the linear stress fields (see sec. 8.3.3), the orientation of the boundary forces is mainly analogous to the member forces of the strut and tie network (figs. 9.1, 9.2 and 9.3), such that the intended load transfer is kept.

Disadvantages may arise from the procedure proposed to develop curved stress fields based on strut and tie networks. One, which was already discussed in section 5.7, is that edge loads are no direct representation of the actual distributed loads as they may possess an opposite orientation. In a more general view, also the fact that the geometry of the stress fields has to respect the nodal zone edges can be disadvantageous. An independent stress field geometry may provide an optimization potential for the stress distribution. Nonetheless, in contrast to an approach of direct control over the stress field boundary forces, the strut and tie networks reduce the complexity of the user input significantly. Especially regarding form finding, the proposed procedure provides a significant advantage, as the strut and tie network already allows conclusions on a corresponding surface, such that the elaborate generation of curved stress fields can be omitted for the first form finding steps.
fig. 9.2  Slab analysis with torsionless load transfer from section 8.3.1: strut and tie network (thick blue lines), boundary of the nodal zones (thick black lines), curved stress field (thin black lines) and stress fields boundary forces (short blue and red lines) in top view

fig. 9.3  Slab analysis with radial load transfer from section 8.3.2: strut and tie network (thick blue lines), boundary of the nodal zones (thick black lines), curved stress fields (thin black lines) and stress field boundary forces (short blue and red lines) in top view
With the slab example in section 8.3, the applicability of the method to slabs has generally been proven. The obtained results are, however, not yet competitive to established analysis methods. From the assessment of the slab examples, it can be concluded that in shell form finding the potential for deviations of the shell shape within the bounds of eccentricity is not yet fully utilized. With a further optimization of the stress field geometry, these possibilities could be extended.

In this thesis, as well as in the developed software application, the internal forces of slabs are described by two sets of curved stress fields. With these two sets, an equilibrium solution for any kind of slab, including those with free edges, can be determined. Figure 9.4a shows two combined strut and tie networks that describe a possible load transfer of a slab on four single supports. The possible load transfer strategies at free edges of a slab are, however, limited. With the introduction of a third strut and tie network, this limitation would be eliminated. Figure 9.4b shows three combined strut and tie networks, describing a torsionless load transfer for the considered slab. For each of the strut and tie networks, curved stress fields can be determined. The combination of the three sets of curved stress fields yields the stress state of the slab.

For simple arrangements of initial networks, as used in the slab analysis example in section 8.3.1, a densification only means a minimal additional effort for the specification of the additional unknowns and the initial guess. For more complex initial networks, such as in the shell form finding example in section 8.2 or the radial load transfer strategy for the analysis of a slab in section 8.3.2, a densification of the network requires a significant additional effort from the user. In addition, the determination of a denser strut and tie network, that possesses a similar shape compared to a beforehand generated rougher network, can be time-consuming. To improve this situation, an algorithm would be necessary to automatically densify user-defined strut and tie networks. A densification of the network, in turn, means a reduction of the stress field size and, thereby, increases the accuracy of the results. In the investigated examples in chapter 8, the network arrangements and the size of the according stress fields are sufficient to obtain acceptable results in the major parts, except for the partly strong variability in the stress distribution. However, the part of the strut and tie network of the slab example with radial load transfer (see sec. 8.3.2), which is below the initial network, consists of only three nodes per corner of the slab. Two of these are only slightly and one significantly below the zero level. The arrangement of these three nodes is necessary to obtain a meaningful clamping effect. But the relatively sudden change in the vertical position of the nodes caused by this arrangement results in very uneven curved stress fields in this area. Here, a densification of the network would be useful to obtain better results.
As discussed in section 8.3.3, the transverse shear is only evaluated in the stress field corner points. The highest utilization in these three points is then used for the illustration of the stress field by the application. A denser network, and the according reduction of the size of the stress fields, would in this regard also increase the accuracy of the results.
References


