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Generation-by-generation dissection of the response function in long memory epidemic processes

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Abstract. In a number of natural and social systems, the response to an exogenous shock relaxes back to the average level according to a long-memory kernel $\sim 1/t^{1+\theta}$ with $0 \leq \theta < 1$. In the presence of an epidemic-like process of triggered shocks developing in a cascade of generations at or close to criticality, this "bare" kernel is renormalized into an even slower decaying response function $\sim 1/t^{1-\theta}$. Surprisingly, this means that the shorter the memory of the bare kernel (the larger $1 + \theta$), the longer the memory of the response function (the smaller $1 - \theta$). Here, we present a detailed investigation of this paradoxical behavior based on a generation-by-generation decomposition of the total response function, the use of Laplace transforms and of "anomalous" scaling arguments. The paradox is explained by the fact that the number of triggered generations grows anomalously with time at $\sim t^{\theta}$ so that the contributions of active generations up to time t more than compensate the shorter memory associated with a larger exponent θ. This anomalous scaling results fundamentally from the property that the expected waiting time is infinite for $0 \leq \theta \leq 1$. The techniques developed here are also applied to the case $\theta > 1$ and we find in this case that the total renormalized response is a **constant** for $t < 1/(1 - n)$ followed by a cross-over to $\sim 1/t^{1+\theta}$ for $t \gg 1/(1 - n)$.

1 Introduction

1.1 General formulation of the studied dynamics

Many systems in the natural and social worlds are characterized by activities whose level $A(t)$ at some time t is a function of its past levels $\{A(\tau),\ \text{for } 0 \leq \tau < t\}$. This can be described by the generic integral equation

$$A(t) = f(t) + n \int_0^t A(\tau)\Phi(t-\tau)d\tau, \qquad (1)$$

where $f(t)$ is some source, news or perturbation term whose impact is instantaneous. The second integral term describes the propagation of past activity levels $A(\tau)$ to the present time t mediated by the kernel $\Phi(t-\tau)$. The summation describes that all past activities have an impact in the present activity level, but with a weaker and weaker weight $\Phi(t-\tau)$ as they recede more in the past ($\tau \to +\infty$). The "bare" kernel function must satisfy the normalization condition

$$\int_0^{\infty} \Phi(t)dt = 1. \qquad (2)$$

Finally, the parameter n, which we impose here to be in $(0, 1)$ to ensure the absence of explosive solutions, describes the relative strength of triggering of future activity by past activity, as will become clear in the sequel. As equation (1) can be obtained as the statistical average of a large class of epidemic branching models [1,2], it is natural to refer to n as the "branching ratio".

We are interested in the class of systems for which the kernel $\Phi(t)$ expresses the existence of a long memory and, for the sake of concreteness, our calculations will use the specific form

$$\Phi(t) = \frac{\theta \varrho^{\theta}}{(t+\varrho)^{\theta+1}}, \qquad (3)$$

corresponding to $\Phi(t) \sim 1/t^{1+\theta}$ at long times.

Expression (1) with (3) corresponds to a mean-field or statistical averaged description of the dynamics of many systems [3], such as the following examples. Present seismicity is in large part triggered by past seismicity over long time scales described by the Omori law [2,4–6], $A(t)$ being the seismic rate in a given region above some magnitude threshold. Commercial and social successes have been shown to promote success over very long time scales [7–9]. The activity $A(t)$ here corresponds to the number of products sold or the number of downloads, views, attendence and so on, per unit time. Past financial volatility has a very long influence on future volatility, leading to bursty

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intermittent behaviors also characterized by long-memory power law decaying kernels [10–12]. Here the activity $A(t)$ is simply a measure of the financial volatility.

1.2 Justification of the power law decay of the memory kernel $\Phi(t)$

As already mentioned, we investigate here the case where the memory kernel $\Phi(t)$ has a long power law tail defined by (3). This assumption is justified by many observations in a variety of different systems. For instance in natural sciences, the average seismicity rate in a given region is characterized by a memory kernel, called the Omori law, which has been documented to take precisely this mathematical form [2,4–6]. In social sciences, the long memory of the kernel $\Phi(t)$ is supported by

(i) the many observations of fat-tailed distributions $P(t)$ of waiting times between cause and action;
(ii) the fact that $P(t)$ is proportional to the "bare" response function $R_{\rm bare}(t)$ of a large group of agents subjected to the same short-lived stimulus; and
(iii) the fact that the "bare" response function $R_{\rm bare}(t)$ is asymptotically equal to the memory kernel $\Phi(t)$ for subcritical dynamics $n < 1$.

Supporting item (i), many studies have documented that the distribution $P(t)$ of waiting times between cause and action performed by humans is a power law $P(t) \sim 1/t^{\alpha}$ with an exponent α less than 2, so that the mathematical expectation of the waiting time (τ) between consecutive events is infinite. This power law behavior applies to the waiting time until an email message is answered [13], to the time intervals between consecutive e-mails sent by a single user and time delays for e-mail replies [14], to the waiting time between receipt and response in the correspondence of Darwin and Einstein [15], and to the waiting times associated with other human activity patterns which extend to web browsing, library visits and stock trading [16].

To prove item (ii), consider the example discussed in reference [17] of the donations following the tsunami that occurred on December 26, 2004. A donation associated with this event can be considered as a task that was triggered (but not necessarily executed) on that day simultaneously for a large population of potential donors. This task competes with many others associated with the jobs, private lives and other activities of each individual in the entire population. The social experiment provided by the tsunami illustrates a general class of experiments in which the same "singular task" is presented at approximately the same time to all potential actors (here the donors), but the priority value of this singular task can be expected to be widely distributed among different individuals. Since the singular task has been initiated at nearly the same time for all individuals, the activity (number of donations) at a time t after this initiation time is then simply equal to $R_{\rm bare}(t) = N \times P(t)$, where N is the number of individuals who will eventually act (donate) in the population and $P(t)$ is the previously defined distribution of waiting times before a task is executed.

Item (iii) derives directly from equation (1): a burst of activity $A(t) = A_0\delta(t - t_0)$ occurring at time t_0 leads to a subsequent triggered activity equal

$$A(t) = n\int_0^t A_0\delta(\tau - t_0)\,\Phi(t-\tau)d\tau = nA_0\Phi(t - t_0)\,. \quad (4)$$

Combining these three elements (i)−(iii), we conclude that the memory kernel of human activity following some perturbation or some shock indeed decays according to a power law of the form (3), justifying our interest in this class of processes.

We note however that the precise mapping between the decay of the activity rate and the distribution of waiting times is valid only in the sub-critical regime characterized by a branching ratio less than 1 [2,9]. In the critical regime where cascades of triggering occur, i.e., when a mother's activity triggers her daughters' activities which themselves trigger their daughters' activities and so on along an avalanche, the decay in time of the activity rate is renormalized by the social epidemic process [2], as we explain below in some details.

The above precise mapping or its renormalized version apply, for instance, in the relaxation of the rate of downloads after the publication of an interview [18], in the relaxation dynamics of book sales on Amazon.com and video views on YouTube™ [7–9] as already mentioned, in the dynamics of visitations of a major news portal [19], and in the decay of popularity of internet blogs posts [20], as well as in the relaxation of financial volatility after a peak [12].

1.3 Relevance of the critical regime $n \simeq 1$

The solution of equation (1) has the form

$$A(t) = f(t) + n\int_0^t R(t-\tau)f(\tau)d\tau, \quad (5)$$

where $R(t)$ is the *resolvent*, also known at the renormalized kernel or response function, satisfying the equation

$$R(t) = \Phi(t) + n\int_0^t \Phi(t-\tau)R(\tau)d\tau. \quad (6)$$

The name "response function" refers to the fact that, if $f(t) = M\delta(t - t_0)$ is a delta function describing a sudden impulse of amplitude M applied to the system at some time t_0 then,

$$A(t) = M\ n\ R\,(t - t_0) \quad (7)$$

for any $t > t_0$. The resolvent $R(t)$ thus describes the response of the system to an impulsive perturbation.

For the class of systems with long memory with $0 \leq \theta < 1$, starting with Montroll and Scher [21], a number of authors have shown that

$$R(t) \sim 1/t^{1-\theta} \quad \text{for } n = 1 \quad \text{or if } n < 1, \text{ for } t < t^*, \quad (8)$$

where

$$t^* = \varrho \left(\frac{n\,\Gamma(1-\theta)}{|1-n|} \right)^{1/\theta} \tag{9}$$

is the time scale delineating the critical region within the subcritical regime $n < 1$. For $t > t^*$, the response function is proportional to the "bare" memory kernel $\Phi(t)$: $R(t) \sim \Phi(t) \sim 1/t^{1+\theta}$ [2,22,23] (see in particular [24] for a synthesis). The value $n = 1$ corresponds to the exact critical regime while values of $n < 1$ correspond to the subcritical regime.

Many systems quoted above are characterized by exponents in the range $0.1 \leq \theta \leq 0.5$, so that t^* remains several order of magnitudes larger then ρ as long as $1-n$ is not too large, so that result (8) ideally valid only for the exact critical regime $n = 1$ actually holds and can be observed over an extended time interval. For instance, for $(n = 0.9, \theta = 0.1)$, we find $t^*/\varrho \sim 7 \times 10^9$; for $(n = 0.7, \theta = 0.1)$, we find $t^*/\varrho \sim 10^4$; for $(n = 0.9, \theta = 0.2)$, we find $t^*/\varrho \sim 10^5$; for $(n = 0.8, \theta = 0.2)$, we find $t^*/\varrho \sim 2 \times 10^3$ and for $(n = 0.9, \theta = 0.5)$, we find $t^*/\varrho \sim 250$. Direct observations [3,7,9] of the existence in several systems of the subcritical decay law $R(t) \sim \Phi(t) \sim 1/t^{1+\theta}$ and of the critical law (8) support the relevance of the critical regime discussed here.

1.4 Formulation of the two problems

The focus of the present paper is to analyze the properties of the response function $R(t)$ by addressing two specific questions, referred to below respectively as Paradox 1 and Problem 2.

- PARADOX 1: *at criticality, the shorter the memory of the bare kernel, the longer the memory of the response function!*
  While the derivation of result (8) is rather straightforward when using the Laplace transform operator as we recall below, this result is paradoxical. Indeed, at face value, it means that the larger θ is, the shorter is the memory encoded by the bare kernel $\Phi(t)$, the longer is the memory described by the response function $R(t)$. A first goal of this paper is to solve this paradox by a detailed analysis of the role played by the cascade of triggering events intrinsically embodied in equation (1) for $n = 1$.
- PROBLEM 2: *power law exponent of the resolvent for* $1 < \theta < 2$ *at criticality* $(n = 1)$ *and in the subcritical regime* $(n < 1)$.
  Recently, an empirical example of the regime where $1 < \theta < 2$ has been discovered in the humanitarian response to the destruction brought by the tsunami generated by the Sumatra earthquake of December 26, 2004, as measured by donations [17]. The data suggests that $n < 1$ so that the observed response is $R(t) \sim \Phi(t) \sim 1/t^{1+\theta}$, i.e., the effect of multiple triggering occurring for $n \simeq 1$ did not occurred in this episode. Here, we ask what would be the response function of a such a system in which the bare kernel is of the form (3) with $1 < \theta < 2$ if an epidemic cascade characterized by $n \simeq 1$ occurred. Clearly, the solution (8) cannot apply for $1 < \theta < 2$, since it would lead to a growth of the response function with time! The solution for this regime that we provide below further illuminates the solution of Paradox 1.

In a nutshell, the solution of the paradox developed in the sequel of this paper is based on the decomposition of the total activity as the sum over a time-varying number $K(t) \sim t^{\theta}$ of generations which have been activited until time t. Since each generation k, with $1 \leq k \leq K(t)$, contributes to the total activity with an amplitude which is proportional to $\sim kn^k/t^{1+\theta}$, the total activity is therefore $\sim \frac{1}{t^{1+\theta}} \times (\sum_{k=1}^{K(t)} kn^k) \sim [K(t)]^2/t^{1+\theta} \sim t^{2\theta}/t^{1+\theta} \sim 1/t^{1-\theta}$. Therefore, the larger θ is, the shorter the memory of $1/t^{1+\theta}$, but the faster growing is the number of generations that are triggered up to time t. Thus, the resulting slower decaying renormalized response function $\sim 1/t^{1-\theta}$ results from the fact that the number of triggered generations grows sufficiently fast so as to more than compensate the shorter memory associated with a larger θ. This anomalous scaling results fundamentally from the property that the expected waiting time is infinite for $0 \leq \theta \leq 1$. Adaptation of this reasoning to the case $1 < \theta$ implies that the renormalized activity is a *constant* for $t < 1/(1-n)$ followed by a cross-over to $\sim 1/t^{1+\theta}$ for $t \gg 1/(1-n)$.

The rest of the paper develops the derivations of these results and is organized in the three following sections. The next Section 2 constructs the mathematical building blocks used in the subsequent sections. In particular, Section 2.3 summarizes the main results obtained using the Laplace transform applied to expression (6). Section 3 presents a detailed derivation of the solution of Paradox 1 in terms of a generation-by-generation decomposition. Specifically, Section 3.2 dissects the two key contributions in the global activity and Section 3.3 presents the intuitive derivation of Paradox 1 based on scaling arguments for long waiting times. Section 4 applies the same approach in terms of the generation-by-generation decomposition for the case $1 < \theta < 2$.

2 Mathematical building blocks

2.1 General relations obtained by using the Laplace transform operator

The standard way to solve the integral equation (6) is to apply the Laplace transform. This transforms the integral equation into an algebraic one

$$\tilde{R}(s) = \tilde{\Phi}(s) + n\tilde{\Phi}(s)\tilde{R}(s), \tag{10}$$

where the tilde denotes that the corresponding function is the Laplace image of the original function. For instance, the Laplace image of the *kernel* $\Phi(t)$ is

$$\tilde{\Phi}(s) = \int_0^{\infty} \Phi(t) e^{-st} dt. \tag{11}$$

It follows from (10) that the Laplace image of the resolvent is given by

$$\tilde{R}(s) = \frac{\tilde{\Phi}(s)}{1 - n\tilde{\Phi}(s)} = \frac{1}{n}\sum_{k=1}^{\infty} n^k \tilde{\Phi}^k(s). \tag{12}$$

Accordingly, one can represent the solution $A(t)$ of (7) in the form

$$A(t) = \sum_{k=1}^{\infty} A_k(t), \tag{13}$$

where

$$A_k(t) = Mn^k \Phi_k(t) \tag{14}$$

and $\Phi_k(t)$ is the inverse Laplace image of $\tilde{\Phi}^k(s)$.

The series (13) has a transparent meaning, when interpreted in the context of epidemic processes. Interpreting $f(t)$ defined in (5) as some ancestor event of amplitude M, then $A(t)$ is the mean birth rate of its offsprings. Correspondingly, the kth term $A_k(t)$ in the series (13) is the mean birth rate of those offsprings of the kth generation. In this context, n is the critical parameter of the corresponding branching process. In what follows, it is convenient to use the terminology of branching processes in order to describe the characteristic properties of the resolvent $R(t)$ and the corresponding solution $A(t)$ of the integral equation (1). For simplicity, but without loss of generality, we put $M = 1$.

2.2 Scaled kernel and resolvent

Specific calculations will be performed with the form (3) of the kernel. But, whenever possible, we will keep the discussion as general as possible. In particular, we will consider the general class of kernels $\Phi(t)$ of the integral equation (1) which has the form

$$\Phi(t) = \frac{1}{\varrho}\varphi\left(\frac{t}{\varrho}\right), \tag{15}$$

where $\varrho > 0$ is some unique characteristic time scale, while the kernel $\varphi(x)$ is some given function of its dimensionless argument, satisfying to normalization condition

$$\int_0^{\infty} \varphi(x)dx = 1. \tag{16}$$

For the choice (3), we have

$$\varphi(x) = \frac{\theta}{(x+1)^{\theta+1}}. \tag{17}$$

We will restrict our study to the case of kernels possessing the power asymptotics $\Phi(t) \sim t^{-\theta-1}$ for $t \to \infty$, with $0 < \theta < 2$. The case $\theta = 1$ requires a special treatment. We will not present it for the sake of conciseness, while it is clear that the method presented below allows one to easily provide the needed detailed description in this case. We note that the main scaling laws obtained below remain valid for $\theta = 1$, while some details and corrections to scaling differ.

For the family of kernels given by (15) with a single characteristic scale ϱ, the resolvent $R(t)$ and the mean activity $A(t)$ can be represented in forms analogous to (15):

$$R(t) = \frac{1}{\varrho}\mathcal{R}\left(\frac{t}{\varrho}, n\right),$$
$$A(t) = \frac{1}{\varrho}\mathcal{A}\left(\frac{t}{\varrho}, n\right),$$
$$\mathcal{A}(x,n) = n\mathcal{R}(x,n). \tag{18}$$

The Laplace images of the resolvent $\mathcal{R}(x,n)$ and of the mean activity $\mathcal{A}(x,n)$ are given by

$$\tilde{\mathcal{R}}(y,n) = \frac{\tilde{\varphi}(y)}{1 - n\tilde{\varphi}(y)} = \frac{1}{n}\sum_{k=1}^{\infty} n^k \tilde{\varphi}^k(y),$$
$$\tilde{\mathcal{A}}(y,n) = \sum_{k=1}^{\infty} n^k \tilde{\varphi}^k(y), \tag{19}$$

where

$$\tilde{\varphi}(y) = \int_0^{\infty} \varphi(x)e^{-xy}dx \tag{20}$$

is the Laplace image of the bare kernel $\varphi(x)$. In particular, the Laplace image of the kernel given by (17) is equal to

$$\tilde{\Phi}(s) = \theta\,(\varrho s)^{\theta}\, e^{\varrho s}\,\Gamma(-\theta, \varrho s),$$
$$\Rightarrow \tilde{\varphi}(y) = \theta y^{\theta} e^{y} \Gamma(-\theta, y). \tag{21}$$

The asymptotic power law of the kernel $\varphi(x) \sim x^{-1-\theta}$ ($x \to \infty$) leads to the following asymptotics for the Laplace image $\tilde{\varphi}(y)$ for small y values:

$$\tilde{\varphi}(y) \simeq 1 + \alpha y - \beta y^{\theta}, \qquad |y| \ll 1. \tag{22}$$

For the particular case (17), we obtain

$$\tilde{\varphi}(y) \simeq 1 + \frac{y}{1-\theta} - y^{\theta}\,\Gamma(1-\theta), \qquad |y| \ll 1. \tag{23}$$

In this case

$$\alpha \equiv \alpha(\theta) = \frac{1}{1-\theta}, \qquad \beta \equiv \beta(\theta) = \Gamma(1-\theta). \tag{24}$$

Notice that $\alpha(\theta)$ and $\beta(\theta)$ change sign as θ crosses the value $\theta = 1$. Specifically, $\alpha(\theta)$ and $\beta(\theta)$ are negative for $\theta \in (1,2)$ and positive for $\theta \in (0,1)$. These signs are the consequence of the fact that $\Phi(t)$ given by (17) is one-sided, i.e., identically equal to zero for any $t < 0$. In the following, we will thus consider the general asymptotic expression (22) with coefficients α and β of the same sign.

2.3 Direct derivation of the resolvent R(t)

Substituting (22) into (12) and using its first equality leads to

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{q - \alpha y + \beta y^{\theta}}, \qquad q = 1 - n. \tag{25}$$

2.3.1 Case $0 < \theta < 1$

In this case, the term αy can be neglected in (25), which becomes

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{q+|\beta|y^\theta}. \quad (26)$$

We have intentionally replaced β by $|\beta|$ to stress the important fact that, for $\theta \in (0,1)$, the parameter β is positive as can be checked from its explicit value given in (24). For very small y, expression (26) can be further expanded into the following asymptotic relation

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{q} - \frac{|\beta|}{q^2}\, y^\theta, \qquad \frac{|\beta|}{q}\, |y|^\theta \ll 1. \quad (27)$$

Using the standard correspondence between functions and their Laplace transforms

$$y^\theta \quad \mapsto \quad \frac{x^{-\theta-1}}{\Gamma(-\theta)}, \qquad 1 \quad \mapsto \quad \delta(x), \quad (28)$$

we obtain the asymptotic time dependence of the resolvent:

$$\mathcal{R}(x,n) \simeq \frac{\theta}{q^2}\, x^{-1-\theta}, \qquad x \gg \frac{|\beta|^{1/\theta}}{q^{1/\theta}}. \quad (29)$$

The intermediate asymptotic regime corresponds to an interval of still small values of y, but not too small so that the following inequality $|\beta|y^\theta \gg q$ holds. Then, one can neglect the term q in the denominator of expression (26) to obtain the following intermediate asymptotics

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{|\beta|y^\theta} \quad \mapsto \quad \mathcal{R}(x,n) \simeq \frac{\sin(\pi\theta)}{\pi}\, x^{-1+\theta},$$

$$x \ll \frac{|\beta|^{1/\theta}}{q^{1/\theta}}. \quad (30)$$

The two regimes (29) and (30) are asymptotics of the inverse Laplace transform of (26) whose explicit expression reads [25]

$$\mathcal{R}(x,n) = \frac{1}{q}\left(\frac{q}{|\beta|}\right)^{1/\theta} \mathcal{Q}\left(\left(\frac{q}{|\beta|}\right)^{1/\theta} x, \theta\right). \quad (31)$$

where

$$\mathcal{Q}(x,\theta) = \frac{\sin(\pi\theta)}{\pi} x^{\theta-1} \int_0^\infty \frac{u^\theta e^{-u} du}{u^{2\theta}+x^{2\theta}+2x^\theta u^\theta \cos(\pi\theta)}. \quad (32)$$

For instance, for $\theta = 1/2$,

$$\mathcal{Q}(x,1/2) = \frac{1}{\sqrt{\pi x}} - e^x \mathrm{erfc}\left(\sqrt{x}\right). \quad (33)$$

2.3.2 Case $1 < \theta < 2$

We rewrite (25) as

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{q+|\alpha|y-|\beta|y^\theta}. \quad (34)$$

For

$$\frac{1}{q}\, |\alpha y| \ll 1, \quad (35)$$

then

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{q} - \frac{|\alpha|}{q^2} y + \frac{|\beta|}{q^2} y^\theta, \quad (36)$$

which is the Laplace image of

$$\mathcal{R}(x,n) \simeq \frac{|\beta|}{q^2\Gamma(-\theta)}\, x^{-\theta-1} = \frac{\theta}{q^2}\, x^{-\theta-1}, \qquad x \gg \frac{1}{q}. \quad (37)$$

The intermediate asymptotic describes the time interval such that $q \ll |\alpha|y$, leading to

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{|\alpha|y-|\beta|y^\theta} = \frac{1}{y}\frac{1}{|\alpha|-|\beta|y^{\theta-1}}. \quad (38)$$

For sufficiently small $|y|$, one gets

$$\tilde{\mathcal{R}}(y,n) \simeq \frac{1}{|\alpha|}\frac{1}{y} + \left|\frac{\beta}{\alpha^2}\right| y^{\theta-2}, \quad (39)$$

which is the Laplace image of

$$\mathcal{R}(x,n) \simeq \frac{1}{|\alpha|} + \left|\frac{\beta}{\alpha^2}\right| \frac{1}{\Gamma(2-\theta)} x^{-(\theta-1)}, \quad 1 \ll x \ll \frac{1}{q}. \quad (40)$$

Expressions (40) and (37) show that the resolvent $\mathcal{R}(x,n)$ is a constant plus a weak power law correction $\sim 1/x^{\theta-1}$ for $1 \ll x \ll 1/(1-n)$ which crosses over to $\sim 1/x^{-\theta-1}$ for $x \gg 1/(1-n)$. A function proportional to the resolvent is plotted in Figure 4 at the end of the paper, which shows these different regimes (40) and (37).

2.4 Asymptotic of the mean activity

While the previous subsection provides the expressions of the different regimes of the resolvent, their derivation using the Laplace transform augmented by different expansions do not provide an understanding of the derived terms, which we would wish to be based on the underlying mechanism of cascades of triggering over different generations. To achieve this goal and remove Paradox 1, we have to explore the asymptotic behavior of the mean activity $\mathcal{A}(x,n)$ for large x values. For this, we study the asymptotics of the corresponding Laplace image $\tilde{\mathcal{A}}(y,n)$ for small y values. Substituting in the series (19) the asymptotic expression (22) and using the asymptotic relation

$$\left(1+\alpha y-\beta y^\theta\right)^k \simeq e^{k\alpha y - k\beta y^\theta}, \qquad |\alpha y - \beta y^\theta| \ll 1, \quad (41)$$

we obtain

$$\tilde{\mathcal{A}}(y,n) \simeq \sum_{k=1}^{\infty} n^k \tilde{\psi}\left(|\beta|^{1/\theta} k^{1/\theta} y; \theta\right) e^{\alpha k y}, \qquad (42)$$

where

$$\tilde{\psi}(y;\theta) = e^{-\mathrm{sign}(\beta) y^{\theta}}. \qquad (43)$$

Taking the inverse Laplace transform of the series (42), we obtain a series representation of the sought mean activity as

$$\mathcal{A}(x,n) \simeq \sum_{k=1}^{\infty} \mathcal{A}_k(x,n), \qquad (44)$$

where

$$\mathcal{A}_k(x,n) = \frac{n^k}{k^{1/\theta}|\beta|^{1/\theta}} \psi\left(\frac{x+\alpha k}{k^{1/\theta}|\beta|^{1/\theta}}; \theta\right), \qquad (45)$$

and $\psi(x;\theta)$ is a stable distribution, whose Laplace image is given by expression (43).

The asymptotic validity of relation (44) for the dependence of $\mathcal{A}(x,n)$ for $x \gg 1$, describing in particular the case where the kernel is given by expression (17), transforms into an exact equality for the mean activity if the kernel of the integral equation (1) coincides with the stable distribution

$$\Phi(t) = \frac{1}{\varrho}\psi\left(\frac{t}{\varrho}; \theta\right), \qquad (46)$$

whose Laplace image is given by (43).

2.5 Properties of the stable distribution $\psi(x;\theta)$ defined by (43) and (46)

Formulas (44) and (45) imply that a better understanding of the asymptotic shape of the mean activity $\mathcal{A}(x,n)$ is dependent on a detailed knowledge of the properties of the stable distribution $\psi(x;\theta)$. This subsection is devoted to this question.

There are many integral representations of the stable distribution $\psi(x;\theta)$. In particular, one can show that

$$\psi(x;\theta) = \frac{1}{\pi}\int_0^{\infty} \exp\left(-\left|\cos\left(\frac{\pi\theta}{2}\right)\right| u^{\theta}\right) \times \cos\left(ux + u^{\theta}\sin\left(\frac{\pi\theta}{2}\right)\mathrm{sign}(\theta-1)\right) du, \quad 0<\theta<2, \quad \theta \neq 1. \qquad (47)$$

Explicit analytic expressions of the stable distribution $\psi(x;\theta)$ exist for some specific values of the parameter θ. For illustrative purposes, we will use below two such stable distributions. The first one is the famous Levy stable law

$$\psi(x;1/2) = \frac{1}{2x\sqrt{\pi x}} \exp\left(-\frac{1}{4x}\right), \qquad (\theta = 1/2), \qquad (48)$$

and the other is

$$\psi(x;3/2) = \frac{1}{\pi\sqrt{3}}\left[\Gamma\left(\frac{2}{3}\right) {}_1F_1\left(\frac{5}{6}, \frac{2}{3}, \frac{4x^3}{27}\right) - x\Gamma\left(\frac{4}{3}\right) {}_1F_1\left(\frac{7}{6}, \frac{4}{3}, \frac{4x^3}{27}\right)\right], \qquad (\theta = 3/2), \qquad (49)$$

where ${}_1F_1(a,b,c)$ denote a confluent hypergeometric function of the first kind.

All stable laws possess in common *long* and *short* tails. By definition, the long tail is their power law behavior at $x \to \infty$:

$$\psi(x;\theta) \simeq \frac{x^{-\theta-1}}{|\Gamma(-\theta)|}, \qquad x \to \infty. \qquad (50)$$

These short tail of stable distributions consists in a very fast decay of $\psi(x;\theta)$ to zero as $x \to 0$ for $0 < \theta < 1$ corresponding to an essential singularity at $x = 0$, and in their super-exponentially fast decay as $x \to -\infty$ for $1 < \theta < 2$. The following asymptotic formula is true [26]

$$\psi(x;\theta) \simeq \frac{1}{\sqrt{2\pi\theta|\theta-1|}}\left(\frac{|x|}{\theta}\right)^{\frac{2-\theta}{2\theta-2}} \times \exp\left(-|\theta-1|\left(\frac{|x|}{\theta}\right)^{\frac{\theta}{\theta-1}}\right), \quad \begin{cases} x \to 0_+, & \text{if} \quad 0<\theta<1, \\ x \to -\infty, & \text{if} \quad 1<\theta<2. \end{cases} \qquad (51)$$

Curiously, for $\theta = 1/2$, this asymptotic formula coincides with the Levy stable law (48). According to the asymptotic relation (51), the stable distribution $\psi(x;3/2)$ decays at $x \to -\infty$ according to

$$\psi(x;3/2) \simeq \sqrt{\frac{4|x|}{9\pi}} \exp\left(-\frac{4}{27}|x|^3\right), \qquad x \to -\infty. \qquad (52)$$

In practice, the asymptotic expression such as (52) can be verified to be extremely accurate already for $x < -2$.

3 Solution of paradox 1 for $0 < \theta < 1$

The asymptotic behavior for $x \gg 1$ of the mean activity $\mathcal{A}(x,n)$ given by (44) is qualitatively different for $\theta \in (0,1)$ and for $\theta \in (1,2)$. In this section, we focus on the former case $\theta \in (0,1)$.

3.1 Integral approximation of the mean activity $\mathcal{A}(x,n)$

As summarized in the statement of Paradox 1, for $\theta \in (0,1)$ and close to criticality ($n \lesssim 1$), the mean activity

$\mathcal{A}(x,n)$ exhibits a double power law behavior, with the coexistence of the power law asymptotic

$$\mathcal{A}(x,n) \sim x^{-1-\theta}, \qquad x \gg 1, \tag{53}$$

for very large x values, and of an intermediate asymptotic regime for smaller x (but still remaining large)

$$\mathcal{A}(x,n) \sim x^{-1+\theta}, \qquad \text{intermediate asympotics.} \tag{54}$$

The goal of this subsection is to show that the intermediate power asymptotic (54) results from the fast decay of the short tail part of the stable distribution $\psi(x;\theta)$, which controls the mean activity $\mathcal{A}_k(x,n)$ defined by (45) of the activity resulting from the kth generation. In contrast, we will show that the power asymptotic (53) is due to the corresponding power asymptotic of the bare kernel $\Phi(t) \sim t^{-\theta-1}$.

The first step consists in noting that, for $0 < \theta < 1$, the shift in expression (45) for $\mathcal{A}_k(x,n)$ can be written as

$$\frac{\alpha k}{|\beta|^{1/\theta} k^{1/\theta}} \sim k^{\frac{\theta-1}{\theta}}. \tag{55}$$

It thus tends to zero for $k \to \infty$, so that one may neglect it as it will not impact the asymptotic law of the mean activity $\mathcal{A}(x,n)$ for large x's. In other words, for $0 < \theta < 1$, one may without essential error replace the kth generation mean activity $\mathcal{A}(x,n)$ by

$$\mathcal{A}_k(x,n) = \frac{n^k}{k^{1/\theta}|\beta|^{1/\theta}} \psi\left(\frac{x}{k^{1/\theta}|\beta|^{1/\theta}};\theta\right). \tag{56}$$

The next step is to notice that, if x large enough, then $\mathcal{A}_k(x,n)$ becomes a sufficiently smooth function of the argument k and one may, without essential error, replace the series (44) by the integral

$$\mathcal{A}(x,n) \simeq \int_1^\infty \mathcal{A}_k(x,n)dk \simeq \int_1^\infty \frac{e^{-\gamma k}}{|\beta|^{1/\theta}k^{1/\theta}} \psi\left(\frac{x}{|\beta|^{1/\theta}k^{1/\theta}};\theta\right) dk, \tag{57}$$

in which we have defined

$$\gamma = \ln\left(\frac{1}{n}\right). \tag{58}$$

We suppose everywhere below that $|\gamma| \ll 1$ (n is close to its critical value 1), so that the exponential function $e^{-\gamma k}$ in the integral (57) is also a smooth function of k.

Using the change of variable

$$u = \frac{x}{|\beta|^{1/\theta}k^{1/\theta}} \iff k = \frac{1}{|\beta|}\left(\frac{x}{u}\right)^\theta, \quad dk = -\frac{\theta}{|\beta|}\left(\frac{x}{u}\right)^\theta \frac{du}{u}, \tag{59}$$

the integral (57) becomes

$$\mathcal{A}(x,n) \simeq \frac{1}{x^{1-\theta}} \frac{\theta}{|\beta|} \int_0^{u(x)} \exp\left(-\frac{\gamma}{|\beta|}\left(\frac{x}{u}\right)^\theta\right) \psi(u;\theta)\frac{du}{u^\theta}, \tag{60}$$

where

$$u(x) = \frac{x}{|\beta|^{1/\theta}}. \tag{61}$$

As $u(x) \gg 1$ for $x \gg 1$, one may replace without essential error the upper limit in the integral (60) by infinity:

$$\mathcal{A}(x,n) \simeq \frac{1}{x^{1-\theta}} \frac{\theta}{|\beta|} \int_0^{\infty} \exp\left(-\frac{\gamma}{|\beta|}\left(\frac{x}{u}\right)^\theta\right) \psi(u;\theta)\frac{du}{u^\theta}. \tag{62}$$

For $\gamma = 0$ ($n = 1$), we obtain the mean activity given by (62) as the power law (54) corresponding to the intermediate asymptotics:

$$\mathcal{A}_{\text{int}}(x,n) \simeq \frac{C(\theta)}{x^{1-\theta}} \frac{\theta}{|\beta|}, \tag{63}$$

where the index 'int' refers to "intermediate asymptotics" and

$$C(\theta) = \int_0^\infty \psi(u;\theta)\frac{du}{u^\theta} = \frac{1}{\Gamma(1+\theta)}. \tag{64}$$

It is reasonable to choose $\beta = \Gamma(1-\theta)$ defined in (24) and using the well-known identity $\Gamma(1-\theta)\Gamma(1+\theta) \equiv \pi\theta \csc(\pi\theta)$, we obtain

$$\mathcal{A}_{\text{int}}(x,n) \simeq \frac{\sin(\pi\theta)}{\pi} x^{-1+\theta}. \tag{65}$$

For $\gamma = 0$ ($n = 1$), the intermediate asymptotic (65) invades the whole large x regime, so that its determination is clearer.

The fact that expression (65) holds only for an intermediate range of x's values for $\gamma > 0$ ($n < 1$) is now determined from the following derivation. Let $\xi(\theta)$ be such that, for if $x > \xi(\theta)$, the stable distribution $\psi(x;\theta)$ is not different from its long tail (50) within a specified error margin. For example, for the Levy stable distribution $\psi(x;1/2)$ given by (48), $\xi(\theta)$ can be taken equal to 3 when considering an error margin of less than 1%.

From expression (62), one can see that, if the following condition holds,

$$\frac{\gamma}{|\beta|}\left(\frac{x}{\xi(\theta)}\right)^\theta \gtrsim 1 \quad \Rightarrow \quad x \gtrsim \xi(\theta)\left(\frac{|\beta|}{\gamma}\right)^{1/\theta} \tag{66}$$

then, without essential error, the stable distribution $\psi(x;\theta)$ can be replaced by its long tail in the integral (62). This leads to the following approximate relation

$$\mathcal{A}(x,n) \simeq \frac{1}{x^{1-\theta}} \frac{\theta}{|\beta||\Gamma(-\theta)|} \int_0^\infty \exp\left(-\frac{\gamma}{|\beta|}\left(\frac{x}{u}\right)^\theta\right) \frac{du}{u^{2\theta+1}}. \tag{67}$$

Changing the integration variable to $z = u/x$, we obtain

$$\mathcal{A}(x,n) \simeq \frac{1}{x^{1+\theta}} \frac{\theta D(\theta)}{|\beta||\Gamma(-\theta)|}, \tag{68}$$

where

$$D(\theta) = \int_0^\infty \exp\left(-\frac{\gamma}{|\beta|z^\theta}\right) \frac{dz}{z^{2\theta+1}} = \frac{\beta^2}{\theta\gamma^2}. \tag{69}$$

3.2 Dissecting the two key contributions to solve Paradox 1

The asymptotic analysis of the mean activity $\mathcal{A}(x,n)$ presented in the previous subsection, while sufficiently rigorous, does not provide an intuitive understanding of the two intermediate and asymptotic regimes and of the solution of Paradox 1. In the present subsection, we provide a cruder but more transparent analysis, which reveals the hidden springs of the crossover from the intermediate asymptotic power law (54) to the full asymptotic power law (53). The next section will provide a different approach which illuminates even further the mechanism of the transition from the bare kernel time decay to the resolvent time dependence.

For this, we replace in expression (45) the stable distribution $\psi(x;\theta)$ by its "geometrical power law skeleton"

$$\psi_0(x;\theta) = \begin{cases} \dfrac{x^{-1-\theta}}{|\Gamma(-\theta)|}, & x > \xi(\theta), \\ 0, & x < \xi(\theta). \end{cases} \tag{70}$$

Replacing in (45) $\psi(x;\theta)$ by $\psi_0(x;\theta)$ (and neglecting the shift parameter (55)), we obtain

$$\mathcal{A}_k^0(x,n) \simeq x^{-1-\theta} n^k \frac{|\beta| k}{|\Gamma(-\theta)|} \mathbf{1}\left(x - |\beta|^{1/\theta} k^{1/\theta} \xi(\theta)\right), \tag{71}$$

where $\mathbf{1}(z)$ is the unit step function. This term $\mathbf{1}(x - |\beta|^{1/\theta} k^{1/\theta} \xi(\theta))$ is important, as it accounts semi-quantitatively for the fast decaying short tail of the stable distribution $\psi(x;\theta)$, which has the role of effectively truncating the series (44) at large k's.

For concreteness, let us consider the particular kernel $\Phi(t)$ defined by (17), for which $\beta = \Gamma(1-\theta)$. We can then rewrite relation (71) in the more transparent form

$$\mathcal{A}_k^0(x,n) = \theta\ x^{-1-\theta}\ k\ n^k\ \mathbf{1}\left(k(x,\theta) - k\right), \tag{72}$$

where

$$k(x,\theta) = \frac{x^\theta}{\xi^\theta(\theta)\beta} = \frac{x^\theta}{\xi^\theta(\theta)\Gamma(1-\theta)}. \tag{73}$$

Choosing for simplicity $\xi^\theta(\theta) = 1/|\beta|$, we obtain

$$k(x,\theta) = x^\theta. \tag{74}$$

Substituting expression (72) into the series (45), we obtain the mean activity estimated in this geometrical skeleton approximation, denoted as $\mathcal{A}^0(x,n)$, as

$$\mathcal{A}^0(x,n) \simeq \theta\ x^{-1-\theta}\ \mathcal{S}\left(k(x,\theta),n\right) = \theta\ x^{-1-\theta} \mathcal{S}\left(x^\theta, n\right), \tag{75}$$

where

$$\mathcal{S}(\kappa,n) = \sum_{k=1}^{\kappa} k n^k. \tag{76}$$

Expression (75) with (76) allows us to pinpoint the origin of the slow decay $\sim 1/t^{1-\theta}$ of the resolvent in the intermediate asymptotic regime or for $n=1$ as due to the fight between the fast decay $\sim 1/t^{1+\theta}$ of the bare kernel and the growth $\sim t^{2\theta}$ of the contributions to the activity at time t of all generations set in motion up to time t. This later growth is controlled by the short tail of the stable distribution $\psi(x;\theta)$ corresponding to the truncation on the above geometrical skeleton approximation (70). The main contributions to $\mathcal{A}(x,n)$ are provided by the first $k(x,\theta)$ summands, because the mean activities $\mathcal{A}_k(x,n)$ of the highest order generations, for which $k \gtrsim k(x,\theta)$, are, for given $x = t/\varrho$, not yet large enough to influence significantly the total activity level $A(t)$. Roughly speaking, the larger the order k of a generation, the later its contribution $\mathcal{A}_k(x,n)$ is felt.

For $n=1$, the sum (76) reduces to

$$\mathcal{S}\left(\kappa,1\right) = \sum_{k=1}^{\kappa} k = \frac{1}{2}\kappa\left(\kappa+1\right) \simeq \frac{1}{2}\kappa^2. \tag{77}$$

Using (74), we obtain

$$\mathcal{S}\left(k\left(x,\theta\right),1\right) \simeq \frac{1}{2} x^{2\theta}. \tag{78}$$

Substituting this expression (78) into (75), we finally obtain promised power law (54)

$$\mathcal{A}^0(x,n) \simeq \frac{\theta}{2} x^{-1-\theta} x^{2\theta} \simeq \frac{\theta}{2} x^{-1+\theta}. \tag{79}$$

This last equation (79) illuminates the origin of Paradox 1: the larger θ is, the faster the decay of the bare kernel $\sim 1/t^{1+\theta}$, but the larger the number $k(t,\theta) \sim t^\theta$ of generations which are activated up to time t and the greater their combined contribution $\sim [k(t,\theta)]^2 \sim t^{2\theta}$ to the overall activity at time t, so that, all being taken into account, the response function actually develops a longer memory $\sim t^{2\theta} \times 1/t^{1+\theta} = 1/t^{1-\theta}$. Paradox 1 can thus be seen as a result of an "anomalously" slow triggering of successive generations associated with the infinite average waiting time between triggered events. Indeed, the average waiting time between two events, as described by the bare kernel, is defined by

$$\langle t \rangle \sim \lim_{T\to+\infty} \int^T \frac{t}{t^{1+\theta}} dt, \tag{80}$$

which is diverging as the upper bound T of the integral goes to infinity, for $\theta \le 1$. This divergence is a standard diagnostic of the existence of an anomalous trapping time regime [27,28], leading to anomalous scaling laws. In the present case, the "anomalous" scaling law is the "renormalization" of the bare kernel time decay $\sim 1/t^{1+\theta}$ into the resolvent time decay $\sim 1/t^{1-\theta}$. The next Section 3.3 re-derives this result from scratch by using a completely intuitive and straightforward reasoning, exemplifying that the root of Paradox 1 indeed lies on the diverging mean waiting time between triggered events and its associated anomalous diffusion.

But before doing so, we exploit the present analysis to describe the subcritical case $n \lesssim 1$. For arbitrary n, the

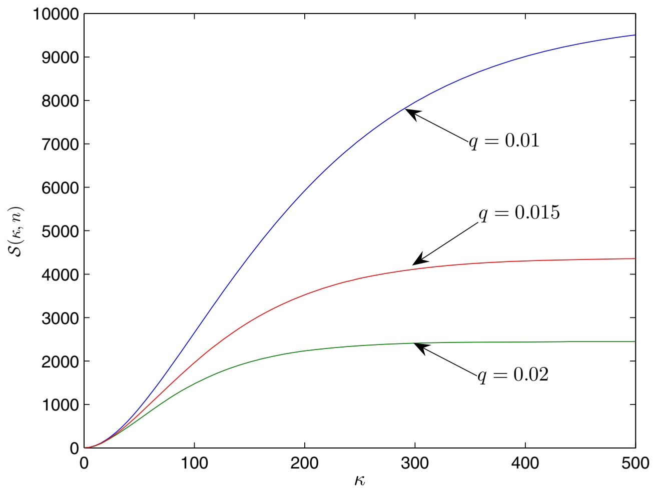

Fig. 1. (Color online) Plot of function $\mathcal{S}(\kappa, n)$ defined by (81) for $q \equiv 1 - n = 0.01; 0.015; 0.02$, illustrating the cross-over between the power quadratic behavior (77) for $\kappa q \ll 1$ to a constant for $\kappa q \gg 1$.

sum (76) is equal to

$$\mathcal{S}\left(\kappa, n\right) = \frac{1-q}{q^2}\left[1 - (1-q)^{\kappa}(1+\kappa q)\right], \qquad q = 1 - n. \tag{81}$$

For small enough q, such that $\kappa q \ll 1$, expression (81) has the same square asymptotic (77) as for $n = 1$. In contrast, for $\kappa q \gg 1$, $\mathcal{S}(\kappa, n)$ tends to a constant limit $\mathcal{S}(\kappa, n) \to \frac{1}{q^2}$. This implies that, when $k(x, \theta)q \gg 1$, the mean activity $\mathcal{A}^0(x, n)$ obeys the same power law as the bare kernel,

$$\mathcal{A}^0(x, n) \simeq \frac{\theta}{q^2} x^{-1-\theta}, \tag{82}$$

as seen from (75). The function $\mathcal{S}(\kappa, n)$ defined by (81) is plotted in Figure 1 for different $q = 1 - n$ values. It demonstrates the crossover from $\mathcal{S}(\kappa, n) \sim \kappa^2$ for $\kappa q \ll 1$ to $\mathcal{S}(\kappa, n) \simeq$ const. for $\kappa q \gg 1$. This crossover governs the crossover of the resolvent from $1/t^{1-\theta}$ for $t \lesssim 1/(1-n)^{1/\theta}$ to $1/t^{1+\theta}$ for $t \gtrsim 1/(1-n)^{1/\theta}$.

3.3 Intuitive derivation explaining Paradox 1 based on scaling arguments for long waiting times

Let us now re-derive all the key results of the previous subsection by a completely different and intuitive route. Our approach is based on the conceptual view of the total activity $A(t)$ at a given time t as the superposition of the activities $\mathcal{A}_k(t)$ coming from all possible generations $k = 1, 2, ...$ that are significant at this time t.

Consider a first burst of activity starting at time 0, constituting the event of zero-th generation. This initial event may lead to an event of first generation at a latter time t_1, which itself may trigger an event of second generation at time t_2, and so on. We assume that these events constitute the starting time for each successive generation to contribute significantly to the overall activity. For a given time t of observation of the activity, our problem is to determine the typical time $t_k(t)$ of occurrence of the kth generation and its corresponding contribution $\mathcal{A}_k(t)$.

For the first question, we use the interpretation that the bare kernel $\Phi(t)$ is nothing but the probability density function (pdf) of the waiting time t from a burst of activity and its first triggered event. Let us suppose that $K(t)$ generations have been triggered over the total time interval t. Time consistency imposes that

$$t_1 + t_2 + ... + t_{K(t)} = t. \tag{83}$$

Let us call $t_{\max}(t)$ the largest waiting time among the $K(t)$ values $t_1, t_2, ..., t_{K(t)}$. Since the probability that a waiting time between two successive generations is equal to or larger than $t_{\max}(t)$ is of the order of $\varrho^{\theta} \int_{t_{\max}(t)}^{+\infty} dt/t^{1+\theta}$, by consistency, one must have

$$K(t) \times \varrho^{\theta} \int_{t_{\max}(t)}^{+\infty} dt/t^{1+\theta} \sim 1. \tag{84}$$

Expression (84) just states that there is typically just one waiting time of the order of the maximum waiting time $t_{\max}(t)$ among the $K(t)$ waiting times between the successive generations. The solution of (84) is

$$t_{\max}(t) \sim \varrho \left[K(t)\right]^{1/\theta}. \tag{85}$$

The dependence $K(t)$ as a function of t is then obtained by estimating the l.h.s of (83) as

$$t_1 + t_2 + ... + t_{K(t)} = K(t) \times \langle t \rangle_t \sim K(t) \times \varrho^{\theta} \int_0^{t_{\max}(t)} dt\ t/t^{1+\theta} \sim \varrho[K(t)]^{1/\theta}. \tag{86}$$

We have used the fact that the average waiting time $\langle t \rangle_t$ between successive generations has to be estimated by the standard sum of all possible t's weighted by their corresponding probability, but with an upper bound since no waiting times larger than $t_{\max}(t)$ are sampled in the finite set of $K(t)$ realizations. This trick is standard to tame the infinities of the unconditional average waiting time $\langle t \rangle$ defined by (80) in the limit $T \to +\infty$, found for $\theta < 1$, leading to anomalous diffusion and other abnormal scaling effects [27,28]. Then, by (83), we obtain $[K(t)]^{1/\theta} \sim t/\varrho$ and thus

$$K(t) \sim \left(t/\varrho\right)^{\theta}, \tag{87}$$

which retrieves (74) obtained in the previous subsection.

The contribution $\mathcal{A}_k(t)$ of the kth active generation at time t has two important terms. The first one is the probability n^k that k generations have occurred. The second one is based on the concept that each activated generation contributes proportionally to the bare kernel $\sim 1/t^{1+\theta}$ but with a characteristic time scale ρ_k equal to the only existing time scale associated with the generation k, namely its waiting time t_k: $\rho_k \sim t_k$ until its happenance. In complete analogy with the shape of $\Phi(t)$ given by (3), this leads finally to

$$\mathcal{A}_k \sim n^k \, \frac{\theta[t_k(t)]^{\theta}}{(t + t_k(t))^{1+\theta}}. \tag{88}$$

Ordering the indices by increasing values of the waiting times $t_k(t)$, expression (87) implies that $t_k(t) \sim \varrho k^{1/\theta}$, and thus

$$\mathcal{A}_k \sim n^k \frac{\theta k}{t^{1+\theta}}, \quad \text{for } t \gg t_k(t). \tag{89}$$

The total activity is thus

$$\mathcal{A}(t) = \sum_{k=1}^{K(t)} \mathcal{A}_k(t) \sim \frac{\theta}{t^{1+\theta}} \times \sum_{k=1}^{K(t)} k n^k, \tag{90}$$

which recovers (75) with (76).

3.4 Validation with the exactly solvable case $\theta = 1/2$

It is always useful to check the validity of asymptotic relations when exact results are available. Here, we compare the above asymptotic relations of Section 3.2 for the mean activity $\mathcal{A}(x, n)$ with the series (44), (45) for $\theta = 1/2$. In this case, the series (44) takes the form

$$\mathcal{A}(x, n) = \sum_{k=1}^{\infty} \frac{1}{\pi k^2} \psi \left(\frac{x + 2k}{\pi k^2}; \frac{1}{2} \right), \tag{91}$$

while its geometrical skeleton is equal to

$$\mathcal{A}^0(x, n) = \frac{1}{2} x^{-3/2} \mathcal{S} \left(\sqrt{x}, n \right). \tag{92}$$

A more accurate approximation of the series (91) than that given by integral (62) in the case $\theta = 1/2$ is equal to

$$\mathcal{A}_{\text{int}}(x, n) = \frac{1}{\pi \sqrt{x}} \left[1 - \gamma \sqrt{x} \exp \left(\frac{x \gamma^2}{\pi} \right) \text{erfc} \left(\gamma \sqrt{\frac{x}{\pi}} \right) \right], \tag{93}$$

where γ is defined by (58). For $\theta = 1/2$, expressions (63) and (68) become

$$\mathcal{A}_{\text{int}}(x, n) \simeq \begin{cases} \dfrac{1}{\pi \sqrt{x}}, & \gamma = 0, \\[2ex] \dfrac{1}{2x\sqrt{x}\gamma^2}, & x \gtrsim \dfrac{3\pi}{\gamma^2}. \end{cases} \tag{94}$$

Figure 2 compares the mean activity $\mathcal{A}(x, n)$ given by (91), its geometric skeleton $\mathcal{A}^0(x, n)$ obtained from (92) and its integral approximation $\mathcal{A}_{\text{int}}(x, n)$ given by (93). One can check that they are practically undistinguishable for any $x \gtrsim 10$. One can also observe the two power law regimes $\mathcal{A} \sim x^{-1 \pm \theta}$ and their cross-over.

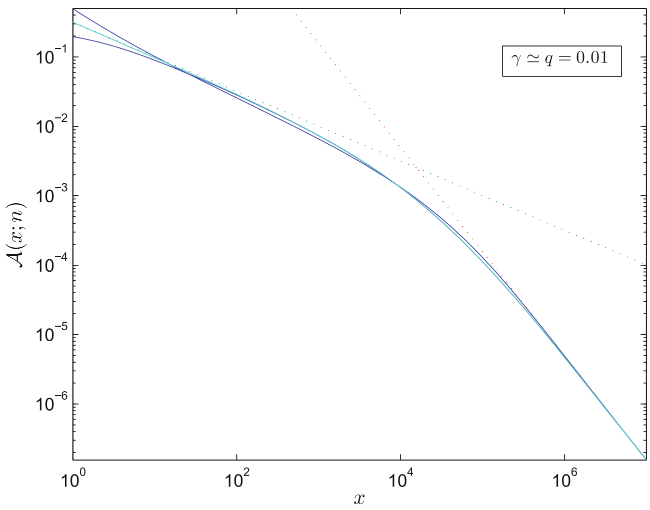

Fig. 2. (Color online) Plots of the mean activity $\mathcal{A}(x, n)$ given by (91), its geometric skeleton $\mathcal{A}^0(x, n)$ obtained from (92) and its integral approximation $\mathcal{A}_{\text{int}}(x, n)$ given by (93), for $\theta = 1/2$ and $\gamma \simeq q = 1 - n = 0.01$. The dotted straight lines show the limiting power asymptotics (94) corresponding to $\mathcal{A} \sim x^{-1 \pm \theta}$.

4 Problem 2: power law exponent for the resolvent for $1 < \theta$

4.1 Integral representation

It is convenient to re-express equations (44) and (45) in a form more adapted to the case $\theta > 1$:

$$\mathcal{A}(x, n) = \sum_{k=1}^{\infty} \frac{n^k}{|\beta|^{1/\theta} k^{1/\theta}} \psi \left(\frac{x - |\alpha| k}{|\beta|^{1/\theta} k^{1/\theta}}; \theta \right). \tag{95}$$

For large $x \gg |\alpha|$, one may, without significant error, replace the series in (95) by the integral

$$\mathcal{A}_{\text{int}}(x, n) \simeq \int_1^{\infty} e^{-\gamma k} \mathcal{A}_k(x; \theta) \, dk, \tag{96}$$

where

$$\mathcal{A}_k(x; \theta) = \frac{1}{|\beta|^{1/\theta} k^{1/\theta}} \psi \left(\frac{x - |\alpha| k}{|\beta|^{1/\theta} k^{1/\theta}}; \theta \right). \tag{97}$$

The "body" of the stable distribution $\psi(u; \theta)$ is concentrated near $u = 0$, which means that the "body" of $\mathcal{A}_k(x; \theta)$ taken as a function of k is concentrated in the vicinity of

$$k_* = \frac{x}{|\alpha|}. \tag{98}$$

As a consequence, the value of the integral (96) is qualitatively different depending on the value of the parameter

$$\varepsilon = \gamma k_* = \frac{\gamma x}{|\alpha|}. \tag{99}$$

4.2 Early time asymptotic

Let us first consider the regime $\varepsilon \ll 1$. In this case, one may, without significant error, put $\gamma = 0$ in (96) to obtain the following approximate relation

$$\mathcal{A}_{\text{int}}(x, n) \simeq \int_1^{\infty} \mathcal{A}_k(x; \theta) \, dk. \tag{100}$$

Given the definition (58), putting $\gamma = 0$ is equivalent to neglecting the difference between the branching ratio n and its critical value 1. In other words, for times sufficiently short such that ε defined by (99) is small, the mean activity is faithfully described as if the system was in the critical regime $n = 1$. Taking into account that the effective width of the function $\mathcal{A}_k(x;\theta)$ of the argument k, defined as the domain in which $\mathcal{A}_k(x;\theta)$ is significantly different from zero, is much smaller than k_*, one can replace k by the constant k_* given by (98) in the denominators of the r.h.s. of expression (97). Then, using the change of integration variable $k \mapsto u = \frac{x-|\alpha|k}{|\beta|^{1/\theta}k_*^{1/\theta}}$, we rewrite the integral (96) in the approximate form

$$\mathcal{A}_{\rm int}(x,n) \simeq \frac{1}{|\alpha|}\int_{-\infty}^{\infty}\psi\left(u;\theta\right)du = \frac{1}{|\alpha|} \qquad (\varepsilon \ll 1)\,, \tag{101}$$

as result of the normalization condition of the stable distribution $\psi(x;\theta)$. In sum, we have

$$\mathcal{A}(x,n) \simeq \frac{1}{|\alpha|} = \theta - 1, \qquad \frac{\gamma x}{|\alpha|} \ll 1 \quad \Longrightarrow \quad x \ll \frac{1}{(\theta-1)\,(1-n)}, \tag{102}$$

where we have used definition (24) for α and (58) for γ, assuming that n is close to 1 so that $\gamma \approx 1-n$. The result (102) expresses that, for $n \simeq 1$, there is plateau in the mean activity $\mathcal{A}(x,n)$ as a function time, for early time $x \ll \frac{1}{(\theta-1)(1-n)}$. As n moves closer and closer to 1, the regime where $\mathcal{A}(x,n) \simeq \theta - 1$ extends to longer and longer times. We note that this constancy of the resolvent at criticality $n = 1$ is well-known for an exponentially decaying kernel function $\Phi(t) \sim \exp[-rt]$, corresponding to the resolvent $R(t) \sim \exp[-r(1-n)t]$. The novel behavior found here is the existence of a non-trivial cross-over to the power law $\sim 1/t^{1+\theta}$, as explained in the next Section 4.3.

This result (102) can be recovered simply by using the arguments of Section 3.3. They extends straightforwardly to the case $\theta > 1$, for which the unconditional mean waiting time $\langle t\rangle$ defined by (80) in the limit $T \to +\infty$ is now finite and well-behaved. This implies that $t_1+t_2+...+t_{K(t)}$ in (83) is well approximated by $K(t)\langle t\rangle$, which, being equal to t, yields $K(t) \sim t$, for all values of $\theta > 1$. This extends the result (87) previously derived only for $0 < \theta < 1$. Then, expression (89) still holds and we obtain finally that expression (90) holds with $K(t) \sim t$. For $n = 1$, we recover that the leading term describing the time dependence of $\mathcal{A}(t)$ is a constant as described by (102) for the case $\theta > 1$. The simple scaling argument of Section 3.3 shows that this result can be actually extended to all positive values of $\theta > 1$. For $n < 1$ but $1 - n$ small, $\mathcal{A}(t) \sim$ const. still holds for times $\ll 1/(1-n)$.

We can combine the results of the intermediate asymptotics valid for $x \ll \frac{1}{(\theta-1)(1-n)}$ for $\theta \in (0,1)$ and for $\theta > 1$ by the following power law dependence of the mean activity

$$\mathcal{A}(x,n) \sim x^{-\delta(\theta)} \tag{103}$$

where the exponent $\delta(\theta)$ is given by

$$\delta(\theta) = (1-\theta)\mathbf{1}(1-\theta) = \begin{cases} 1-\theta, & 0<\theta<1, \\ 0, & \theta > 1. \end{cases} \tag{104}$$

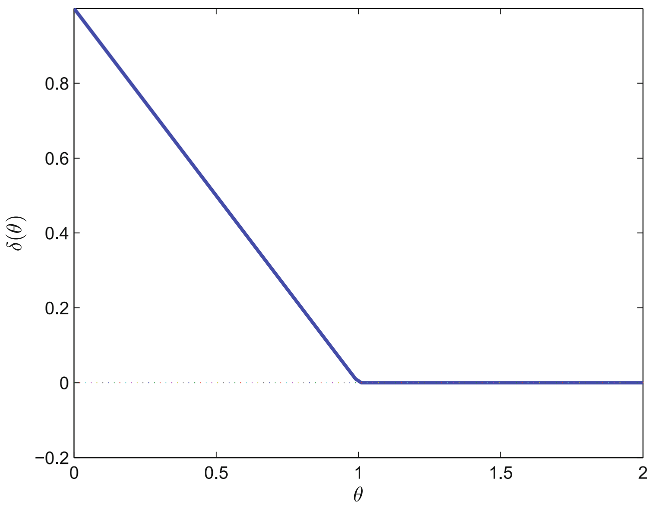

Fig. 3. (Color online) Dependence of the exponent $\delta(\theta)$ given by (104) of the mean activity $\mathcal{A}(x,n)$ as a function of θ, in the intermediate power asymptotic defined $1-n \ll 1$.

Figure 3 shows the exponent $\delta(\theta)$ given by (104) as a function of θ.

4.3 Long time asymptotic

Let us now consider the asymptotic behavior of integral (96) for $\varepsilon \gg 1$, where ε is defined in (99). In this case, the main contribution of the integral is the power law (50) of the stable distribution, so that one may replace the function $\mathcal{A}_k(x;\theta)$ by

$$\mathcal{A}_k(x;\theta) \simeq \frac{|\beta|}{|\Gamma(-\theta)|}\,\frac{k}{(x-|\alpha|k)^{\theta+1}}. \tag{105}$$

Accordingly, integral (96) takes the form

$$\mathcal{A}_{\rm int}(x,n) \simeq \frac{|\beta|}{|\Gamma(-\theta)|}\int_1^{k_*-1}\frac{k\;e^{-\gamma k}\;dk}{(x-|\alpha|k)^{\theta+1}} \qquad \varepsilon = \frac{\gamma x}{|\alpha|} \gg 1. \tag{106}$$

The upper limit of this integral removes the influence of the singularity at k_* defined by (98) which is irrelevant for $\varepsilon \gg 1$. One may interpret the upper limit as resulting from the short tail of the stable distribution.

Due to the fast decaying exponential $e^{-\gamma k}$, (106) can be approximated by

$$\mathcal{A}_{\rm int}(x,n) \simeq \left|\frac{\beta}{\Gamma(-\theta)}\right|\;x^{-\theta-1}\int_0^{\infty} k\;e^{-\gamma k}\;dk \qquad \varepsilon = \frac{\gamma x}{|\alpha|} \gg 1, \tag{107}$$

which finally leads to

$$\mathcal{A}_{\rm int}(x,n) \simeq \left|\frac{\beta}{\Gamma(-\theta)}\right|\;\frac{1}{\gamma^2}\;x^{-\theta-1} \qquad \varepsilon \gg 1, \tag{108}$$

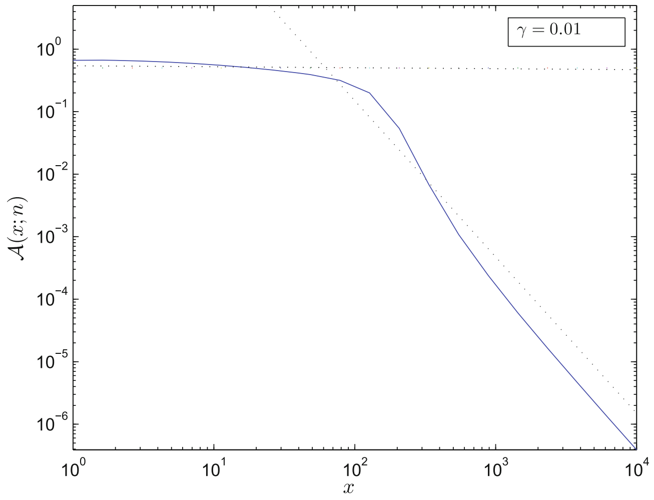

Fig. 4. (Color online) Dependence of expression (111) for the mean activity (solid line) and its two asymptotic regimes (112) (dotted lines), for $\theta = 3/2$. The branching ratio is equal to $n = 0.99$, i.e., $\gamma = 0.01$.

If the kernel $\Phi(t)$ is of the form (17), then $\beta = \Gamma(1-\theta)$ and one has

$$\mathcal{A}_{\rm int}(x,n) \simeq \frac{\theta}{\gamma^2}\, x^{-\theta-1} \qquad \varepsilon \gg 1. \tag{109}$$

4.4 Exact results for $\theta = 3/2$

It is useful to check these results for $\theta = 3/2$, for which we can make use of the explicit expression (49). To be specific, we also assume that the kernel $\Phi(t)$ is given by formula (17), so that the parameters $|\alpha|$ and $|\beta|$ are equal to

$$|\beta| = \left|\Gamma\left(-\frac{1}{2}\right)\right| = 2\sqrt{\pi}, \qquad |\alpha| = 2. \tag{110}$$

Accordingly, the series in (95) reads

$$\mathcal{A}(x,n) = \sum_{k=1}^{\infty} \frac{n^k}{\sqrt[3]{4\pi}\; k^{2/3}} \psi\left(\frac{x-2k}{\sqrt[3]{4\pi}\; k^{2/3}}; 3/2\right), \tag{111}$$

while the power asymptotics (102) and (106) take the form

$$\mathcal{A}(x,n) \simeq \begin{cases} \dfrac{1}{2}, & \dfrac{\gamma x}{2} \ll 1, \\[2ex] \dfrac{3}{2\gamma^2}\; \dfrac{1}{x^{5/2}}, & \dfrac{\gamma x}{2} \gg 1. \end{cases} \tag{112}$$

Figure 4 plots $\mathcal{A}(x,n)$ given by (111) and its asymptotics given by(112) as a function the reduced time x for $n = 0.99$. One can observe the predicted constant plateau for times up to $\simeq 1/(1-n) \approx 10^2$, followed by the power law $\sim 1/t^{1+\theta}$ at larger times.

5 Conclusion

In 1973, Montroll and Scher [21] discovered the phenomenon of the renormalization of the exponent of the power relaxation law in the context of the physics of transport in semiconductors, from the value $1+\theta$ to $1-\theta$ (for $0 \le \theta \le 1$) as a result of a cascade of triggered generations. Since then, this effect has been rediscovered and used several times in several fields with various applications, as documented in Section 1.2. However, while the mathematical mechanics of this renormalization is straightforward when using the apparatus of the Laplace transform, no physical insight is gained on its mechanism from the mathematical derivation. In fact, the opposite holds, in the sense that the mathematical derivation and the result are counter to the intuitive expectation that, the shorter the memory of the bare response function, the shorter should be the memory of the response function renormalized by the existence of the cascade of triggered generations. This problem is stated explicitly in PARADOX 1 in Section 1.4. In this paper, we have resolved this paradox by a detailed generation-by-generation analysis, providing both a rigorous analysis and intuitive scaling arguments. The final physical understanding emerging from our analysis is that the renormalization results from the competition of two opposite effects: the shorter the memory of the bare response function, (i) the shorter the memory of the response function of each generation of triggered events but (ii) the faster is the rate of the number of generations triggered as a function of time. Having more generations triggered implies an impact that extends over a longer time horizon. Combining the two effects (i) and (ii), it turns out that they only partially cancel out and it is the later that ends up dominating, leading to the result that, the shorter the memory of the bare response function, the longer is the memory of the response function renormalized by the existence of the cascade of triggered generations.

In Section 4, this results is further extended to the case where the exponent θ is larger than 1, i.e., the bare response function is even shorter. In agreement with the now intuitive (as well as detailed and rigorous) understanding of the PARADOX 1, we should expect and indeed have found that the memory of the renormalized response function which includes the effect of triggered generations should be even longer: indeed, the renormalized response function does not decay up to a cross-over time $\sim 1/(1-n)$ controlled by the distance to the critical triggering point $n = 1$.

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