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Abstract We survey the use and limitations of some numerical methods for pricing derivative contracts in multidimensional geometric Lévy models.

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1 Introduction

Over the last years financial models with jumps and especially Lévy models have seen a tremendous increase in popularity. By now it is well established that Lévy models are more suitable for capturing market fluctuations than the classical Black–Scholes model [18], see e.g. Cont and Tankov [38] and Schoutens [114] for an overview and empirical justification. The number of financial models with jumps is growing steadily; for the most popular and some recent examples we refer to [12, 32, 33, 49, 74, 86, 87, 114, 116]. However, even in the Black–Scholes setting, analytic solutions to derivative pricing problems are often unavailable or not easily computable, e.g. for American or path-dependent options. Furthermore, in models with jumps one usually cannot even construct analytic solutions for the pricing of plain European vanilla options. Therefore, numerical methods for option pricing have been studied by many authors and several techniques have been developed to obtain efficient pricing algorithms. In particular, models with jumps give rise to previously unconsidered numerical challenges which have led to a number of innovative numerical tools.

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In this survey we shall focus on the use and limitations of numerical methods in multidimensional Lévy models for asset pricing. Naturally, there are many other areas where numerical tools can be applied to Lévy processes, e.g. portfolio optimization, but these will be considered elsewhere. In Sects. 2–4 we describe the basic ideas and mathematical background of the most important numerical pricing techniques and further illustrate some of their recent developments. In Sect. 5 we give a qualitative comparison of the described methods and discuss their advantages and shortcomings. For a comprehensive introduction to Lévy models and numerical methods (mainly in dimension $d = 1$) we refer to Cont and Tankov [38].

Throughout this survey, we focus on the pricing of European and American options on $d \geq 1$ assets with maturity $T < \infty$ and Lipschitz-continuous payoff $g(S)$. Unless stated otherwise, we shall assume that the d-dimensional underlying is modeled by an *exponential* Lévy process S with state space $\mathbb{R}^d_{>0}$. The risk-neutral dynamics of $S = (S^1, \ldots, S^d)$ are given by

$$S_t^i = S_0^i e^{rt + X_t^i}, \quad i = 1, \ldots, d, \tag{1.1}$$

where X is an $\mathbb{R}^d$-valued Lévy process with characteristic triplet (γ, Q, ν) under a (non-unique) risk-neutral measure $\mathbb{P}$ such that $(e^{X^1}, \ldots, e^{X^d})$ is a martingale with respect to the canonical filtration $\mathcal{F}_t^0 := \sigma(X_s, s \leq t)$, $t \geq 0$, of the multivariate process X. By the fundamental theorem of asset pricing (see [46]), an arbitrage-free price $V(t, S)$ of the European option is given by

$$V(t, S) = \mathbb{E}\big[e^{-r(T-t)} g(S_T) \big| S_t = S\big], \tag{1.2}$$

where the expectation is taken with respect to the risk-neutral measure introduced above.

An arbitrage-free price of the corresponding American-type contract is given by an optimal stopping, free boundary problem, namely

$$V(t, S) = \operatorname*{ess\,sup}_{t \leq \tau \leq T} \mathbb{E}\big[e^{-r(T-\tau)} g(S_\tau) \big| S_t = S\big], \tag{1.3}$$

with τ ranging over stopping times. Note that the martingale condition implies

$$\int_{|z|>1} e^{z_i}\, \nu(\mathrm{d}z) < \infty, \quad i = 1, \ldots, d.$$

We therefore assume that the Lévy measure ν of X admits *semiheavy tails* in the following sense. Let ν_i, $i = 1, \ldots, d$ denote the marginal Lévy measures of the Lévy measure ν of X. Then we assume that there are constants $\beta_i^- > 0$, $\beta_i^+ > 0$, $i = 1, \ldots, d$ such that

$$\int_1^\infty e^{\beta_i^+ z} \nu_i(dz) < \infty, \quad \text{and} \quad \int_{-\infty}^{-1} e^{-\beta_i^- z} \nu_i(dz) < \infty. \tag{1.4}$$

Note that this assumption is satisfied by a wide range of Lévy models, cf. e.g. [91]. By [104, Proposition 3.2], it carries over to the multidimensional case:

Lemma 1.1 *Let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν such that the marginal measures ν_i satisfy (1.4). Then the Lévy measure ν also decays exponentially, i.e.,*

$$\int_{|z|>1} e^{\eta(z)}\nu(\mathrm{d}z) < \infty, \quad \text{with } \eta(z) = \sum_{i=1}^{d}\big(\mu_i^+ 1_{\{z_i>0\}} + \mu_i^- 1_{\{z_i<0\}}\big)|z_i|,$$

and $0 < \mu_i^- < \frac{\beta_i^-}{d}$ *and* $0 < \mu_i^+ < \frac{\beta_i^+}{d}$, $i = 1, \ldots, d$.

To fix notation, we recall some essential definitions and properties of Lévy processes. For an extensive description we refer to the monographs [16, 112].

A càdlàg stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ such that $X_0 = 0$ a.s. is called a Lévy process if it has independent and stationary increments and is stochastically continuous. For the characteristic function $\Phi_t(\cdot)$ of X at time $t \geq 0$ we have the Lévy–Khinchin representation (cf. [112]),

$$\Phi_t(\xi) := \mathbb{E}\big[e^{i\langle \xi, X_t\rangle}\big] = e^{t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$
$$\text{with } \psi(\xi) = i\langle \gamma, \xi\rangle - \frac{1}{2}\langle \xi, Q\xi\rangle + \int_{\mathbb{R}^d}\big(e^{i\langle \xi, z\rangle} - 1 - i\langle \xi, z\rangle 1_{\{|z|\leq 1\}}\big)\nu(\mathrm{d}z), \quad (1.5)$$

where $Q \in \mathbb{R}^{d\times d}$ denotes the covariance matrix, $\gamma \in \mathbb{R}^d$ the drift vector and ν is the Lévy measure which satisfies

$$\int_{\mathbb{R}^d}\big(1 \wedge |z|^2\big)\nu(\mathrm{d}z) < \infty. \quad (1.6)$$

The triplet (γ, Q, ν) is called characteristic triplet of the process X and ψ the characteristic exponent.

Also note that the dependence structure of the jump part of a Lévy process X can be described by a Lévy copula F. These were introduced in Tankov [116] and developed in Kallsen and Tankov [71]. Lévy copulas are functions $F : \bar{\mathbb{R}}^d \to \bar{\mathbb{R}}$ which are finite (except at infinity), grounded, d-increasing and preserve the margins. There is a Lévy copula associated to each process X and it satisfies the relationship

$$U(x_1, \ldots, x_d) = F\big(U_1(x_1), \ldots, U_d(x_d)\big), \quad (1.7)$$

where U denotes the tail integral and U_i, $i = 1, \ldots, d$ are the marginal tail integrals of the Lévy process X. Therefore, the jump part of a Lévy process is always uniquely defined by its d univariate marginal processes together with the Lévy copula.

2 Monte Carlo methods

One does not make a mistake by saying that the Monte Carlo method is the most used simulation tool in financial engineering practice. The popularity of Monte Carlo stems mainly from its simplicity, the applicability for parallel computing and the

independence of the convergence rate with respect to the dimension of the underlying problem. In the context of derivative pricing under diffusion driven market models, the Monte Carlo method is especially attractive since it is very simple to simulate Brownian motions. However, when market models are extended to Lévy processes, Monte Carlo simulation becomes more subtle since the laws of the increments of Lévy processes are in general not known explicitly. This fact becomes even more pronounced in $d > 1$ dimensions. As a consequence, the paths of most Lévy processes can only be simulated approximately, see e.g. [9, 38, 114].

A Monte Carlo approximation $\bar{V}$ of the price V of a European option in (1.2) with $t = 0$ consists of the following three steps:

1. Let $N \in \mathbb{N}$. For $j = 1, \ldots, N$ simulate the realizations $\tilde{X}_{T,j}$ of the log-stock price process X at maturity.
2. For each realization $\tilde{X}_{T,j}$, evaluate the payoff

$$g_j := g\big(S_0^1 e^{rT+\tilde{X}_{T,j}^1}, \ldots, S_0^d e^{rT+\tilde{X}_{T,j}^d}\big).$$

3. Take the discounted mean of $g_1, \ldots, g_N$ to obtain $\bar{V} = e^{-rT} N^{-1} \sum_{j=1}^N g_j$.

The key in the above algorithm is the first step. The simulation of $\tilde{X}_{T,j}$ is an easy task as long as the law of the increments of X is known explicitly. However in dimension $d > 1$, except for subordinated Brownian motion, explicit formulas to simulate the increments of multidimensional processes are not available in general and one has to rely on approximation methods. We discuss Gaussian approximation, where the small jumps of the Lévy process are approximated by a Brownian motion, as well as series expansions.

Subordinated Brownian motion A popular class of processes is obtained by subordinating a Brownian motion with drift with an independent positive Lévy process G. Then the resulting process is given by

$$X_t = \Sigma W_{G_t} + \gamma G_t, \quad \Sigma \in \mathbb{R}_{\geq 0}^{d\times d},\ \gamma \in \mathbb{R}^d,\ t \in [0, T],$$

where $W = (W^1, \ldots, W^d)$ is a vector of d independent Brownian motions. If the subordinator is a gamma or a generalized inverse Gaussian process we obtain a multidimensional variance gamma [86] or generalized hyperbolic [49] process.

Such processes can directly be simulated at fixed times $t_0, \ldots, t_M$. To do so, generate the increments of the subordinator $\Delta G_m = G_{t_m} - G_{t_{m-1}}$, $m = 1, \ldots, M$ with $G_{t_0} = 0$. These can be obtained easily as described in e.g. [114]. After that, draw independent random variables $N_m^i \sim \mathcal{N}(0, 1)$, $m = 1, \ldots, M$, $i = 1, \ldots, d$ and set $\Delta X_m = \Sigma N_m \sqrt{\Delta G_m} + \gamma \Delta G_m$. The discretized path of X is then given by $X(t_m) = \sum_{j=1}^m \Delta X_m$. In particular, to obtain the realizations $\tilde{X}_{T,j}$ of X at maturity one may set $M = 1$ and $t_M = T$.

Gaussian approximation Let X be an $\mathbb{R}^d$-valued Lévy process with characteristic triplet $(\gamma, 0, \nu)$. For $\varepsilon \in (0, 1]$ let $\nu_\varepsilon \leq \nu$ be a measure such that $\nu^\varepsilon := \nu - \nu_\varepsilon$ is a finite

measure and $\int_{\mathbb{R}^d} |x|^2 \nu_\varepsilon(\mathrm{d}x) < \infty$. Decompose X correspondingly into its small and large jump parts as $X = X_\varepsilon + X^\varepsilon$. The process X^ε can be written as

$$X_t^\varepsilon = \gamma_\varepsilon t + N_t^\varepsilon, \tag{2.1}$$

where $\gamma_\varepsilon := \gamma + \int_{|z|>1} z\nu_\varepsilon(\mathrm{d}z) - \int_{|z|\le 1} z\nu^\varepsilon(\mathrm{d}z)$ and N^ε is a compound Poisson process. A first approximation of X_t is $X_t \approx \gamma_\varepsilon t + N_t^\varepsilon$ where jumps of magnitude smaller than ε are neglected or replaced by their expected values in the finite activity case. This approximation is reasonable when the intensity of small jumps is low. If this is not the case, the small jump part X_ε can be approximated by an $\mathbb{R}^d$-valued standard Brownian motion W independent of N^ε. It is shown in [10, 109] that, under certain assumptions on the covariance matrix $Q_\varepsilon := \int_{\mathbb{R}^d} zz^\top \nu_\varepsilon(\mathrm{d}z)$, the process $Q_\varepsilon^{-1/2} X_\varepsilon$ converges in distribution to W as $\varepsilon \to 0$. Then, by [37], there holds

Theorem 2.1 *Let X be an $\mathbb{R}^d$-valued Lévy process with characteristic triplet $(\gamma, 0, \nu)$. Assume that Q_ε is non-singular for every $\varepsilon \in (0, 1]$ and that for every $\delta > 0$ there holds*

$$\int_{\langle Q_\varepsilon^{-1} z, z\rangle > \delta} \langle Q_\varepsilon^{-1} z, z\rangle \nu_\varepsilon(\mathrm{d}z) \to 0, \quad \text{as } \varepsilon \to 0.$$

Assume further that for some family of non-singular matrices $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,1]}$ there holds

$$\Sigma_\varepsilon^{-1} Q_\varepsilon \Sigma_\varepsilon^{-\top} \to I_d, \quad \text{as } \varepsilon \to 0,$$

where I_d denotes the identity matrix in $\mathbb{R}^d$. Then for all $\varepsilon \in (0, 1]$ there exists a càdlàg process Y^ε such that (in the sense of finite dimensional distributions)

$$X_t \overset{(d)}{=} \gamma_\varepsilon t + \Sigma_\varepsilon W_t + N_t^\varepsilon + Y_t^\varepsilon, \tag{2.2}$$

and such that for all $T > 0$, $\sup_{t\in[0,T]} |\Sigma_\varepsilon^{-1} Y_t^\varepsilon| \xrightarrow{(\mathbb{P})} 0$, as $\varepsilon \to 0$. Here, γ_ε, N^ε are given in (2.1), and W is an $\mathbb{R}^d$-valued standard Brownian motion independent of N^ε.

Thus, a more refined approximation of X_t is given by $X_t \approx \Sigma_\varepsilon W_t + \gamma_\varepsilon t + N_t^\varepsilon$. The compound Poisson process N_t^ε can be simulated using a series representation.

For X being a tempered α-stable process, $\alpha \in (0, 2)$, it is shown in [37] that the remainder process Y^ε in (2.2) satisfies $\varepsilon^{\alpha/2-1} \sup_{t\in[0,T]} |Y_t^\varepsilon| \to 0$ as $\varepsilon \to 0$. Hence, the convergence rate of the Monte Carlo method may be rather low as α is close to 2.

Series representations In this subsection, we briefly discuss the representation of an $\mathbb{R}^d$-valued pure jump Lévy process with characteristic triplet $(0, 0, \nu)$ by an (infinite) series of random variables. Truncating such a series gives the possibility to simulate the process. In particular, truncated series become useful in connection with Lévy copulas. We closely follow [108] and set $T = 1$ for simplicity.

In order to illustrate the basic idea of series approximation we begin by constructing an infinite series representation for the Poisson process. This representation can then be generalized to more complex processes. Let $\Gamma_1, \Gamma_2, \ldots$ be the jumping times

of a Poisson process with unit rate and $V_1, V_2, \ldots$ i.i.d. uniform random variables independent of the Γ_m. Then, a Poisson process with intensity λ can be written as

$$X_t = \sum_{m=1}^{\infty} 1_{\{\Gamma_m \le \lambda\}} 1_{\{V_m < t\}} = \sum_{m=1}^{\infty} U^{(-1)}(\Gamma_m) 1_{\{V_m < t\}},$$

since the tail integral $U(\cdot)$ of a Poisson process is given by $U(z) = \lambda 1_{\{z \le 1\}}$. Here, V_m can be interpreted as the jump times and $U^{(-1)}(\Gamma_m)$ as the jump sizes of X. To get an implementable algorithm, the series expansion has of course to be truncated. The two most obvious possibilities are either to fix a number M and only consider M jumps or to fix a tolerance ε and only consider $M_\varepsilon = \inf\{m : U^{-1}(\Gamma_m) > \varepsilon\}$ jumps. In the latter case jumps with jump size smaller than ε are omitted. Both approaches yield a compound Poisson approximation of the Lévy process X.

In general, split X into $X = X_\varepsilon + X^\varepsilon$ as above and consider the Lévy–Itô decomposition

$$X_t^\varepsilon = \int_0^t \int_{|z|>1} z J(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\varepsilon \le |z| \le 1} z\big(J(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}z)\,\mathrm{d}s\big).$$

The Poisson random measure J can be represented in the form $J = \sum_{m=1}^{\infty} \delta_{(V_m, G_m)}$ as in [70, Proposition II.1.14], where $\{G_i\}$ is a sequence of random variables independent of the i.i.d. sequence $\{V_i\} \sim U(0,1)$. Using this representation of J, the process X^ε can be rewritten as

$$X_t^\varepsilon = \sum_{m \in \Lambda_\varepsilon(\omega)} G_m 1_{\{V_m \le t\}} - t\gamma_\varepsilon, \quad t \in [0, T],$$

where $\Lambda_\varepsilon(\omega) := \{m \ge 1 : |G_m(\omega)| \ge \varepsilon|\}$ and $\gamma_\varepsilon := \int_{\varepsilon \le |z| \le 1} z\nu(\mathrm{d}z)$. As $\varepsilon \to 0$, $X^\varepsilon \xrightarrow{\text{a.s.}} X$. One obtains a series representation of X as

$$X_t = \sum_{m=1}^{\infty} G_m 1_{\{V_m \le t\}} - t\gamma_m, \tag{2.3}$$

with a suitable sequence of centers $\{\gamma_m\}$. Note that the sequences $\{G_m\}$ and $\{\gamma_m\}$ in (2.3) are not unique. There are several methods to represent them, e.g. LePage's method, Bondesson's method, the rejection method and the shot noise method. For details we refer to [108]. Here, for the sake of brevity, we only illustrate LePage's series representation (see [38, 108]) and focus on the representation of G_m.

LePage's method is based on the radial decomposition of the Lévy measure as

$$\nu(A) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(zs)\mu(\mathrm{d}z, s)\lambda(\mathrm{d}s), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \tag{2.4}$$

where λ is a probability measure on the unit sphere $\mathbb{S}^{d-1}$ and $\mu(\cdot, s)$ is a Lévy measure on $(0, \infty)$ for each $s \in \mathbb{S}^{d-1}$. Define the generalized inverse tail integral

$$U^{(-1)}(x, s) := \inf\big\{z > 0 : U(z, s) < x\big\}, \quad \text{where } U(z, s) := \int_z^\infty \mu(\mathrm{d}x, s).$$

With this notation, the G_m in (2.3) are given by $G_m = U^{(-1)}(\Gamma_m, Y_m)Y_m$, where $\{\Gamma_m\}$ is a sequence of jumping times of some standard Poisson process and $\{Y_m\}$ is an independent sequence of i.i.d. random vectors taking values in $\mathbb{S}^{d-1}$ with distribution λ given by (2.4). There are two difficulties arising in the practical simulation of G_m. Firstly, it may be hard to draw the random vectors Y_m in $\mathbb{S}^{d-1}$ with distribution λ. Secondly, one needs a closed form expression (or a reasonable method to compute it numerically) for the inverse tail integral $U^{(-1)}$ of the measure $\mu(\cdot, s)$, for each s.

It is also possible to use the series representation (2.3) to simulate Lévy processes given by Lévy copulas [38, 117]. Here, the jump sizes G_m^j of each component $j = 1, \ldots, d$ are calculated conditionally on the components $i = 1, \ldots, j-1$. For the sake of simplicity, we discuss the method for $\mathbb{R}^2$-valued Lévy processes having positive jumps, following [38]. For a more general treatment, we refer to [117]. Let F be the Lévy copula associated to X as given in (1.7) with marginal tail integrals U_1, U_2. Assume F is continuous on $[0, \infty]^2$. We again let $\Gamma_1^1, \Gamma_2^1, \ldots$ be the jumping times of a Poisson process with unit rate and $V_1, V_2, \ldots$ i.i.d. uniform random variables independent of Γ_m^1, $m \in \mathbb{N}$. Now, additionally consider $\Gamma_1^2, \Gamma_2^2, \ldots$, which are independent of all other variables and distributed according to $\partial_u F(u, v)\,|_{u=\Gamma_m^1}$. Then, in law there holds

$$X_t^j = \sum_{m=1}^{\infty} U_j^{(-1)}\big(\Gamma_m^j\big) 1_{\{V_m \le t\}}.$$

To simulate X^1, X^2 we start as explained above. We fix a tolerance ε and only consider $M_\varepsilon = \inf\{m : U_1^{-1}(\Gamma_m) > \varepsilon\}$ jumps. Then, for the m-th jump, $m \in \{1, \ldots, M_\varepsilon\}$, compute the jump time of the Poisson process, $\Gamma_m^1 = \sum_{j=1}^m T_j$, where T_j are standard exponentially distributed random variables. Then, compute $\Gamma_m^2 \sim \partial_u F(u, v)\,|_{u=\Gamma_m^1}$ and the jump time $V_m \sim U(0, 1)$. The discretized path is given by

$$X_t^j = \sum_{m=1}^{M_\varepsilon} U_j^{(-1)}\big(\Gamma_m^j\big) 1_{\{V_m \le t\}}.$$

Remark 2.2 Suppose the underlying stochastic process S is not modeled explicitly as in (1.1), but as the solution of a stochastic differential equation (SDE) of the form

$$S_t = S_0 + \int_0^t f(S_{s-})\,\mathrm{d}X_s, \quad t \in [0, T], \tag{2.5}$$

where f is some suitable given function and X is an $\mathbb{R}^d$-valued Lévy process. The process S might for instance arise from a local or stochastic volatility model. In general, it is no longer a Lévy process. To solve the pricing equation (1.2) in such models, one may use the well-known Euler–Maruyama scheme [89] to approximate the solution of (2.5). This allows to use Monte Carlo methods also in this setting. To implement the Euler–Maruyama scheme, one has to simulate increments of the driving Lévy process. As discussed above, in general this can be done only approximately, so that the approximation of the solution of the SDE bears two errors, one stemming

from the discretization through the Euler–Maruyama scheme, and one coming from the approximation of the underlying Lévy process, see e.g. [68, 69, 110] and the references therein.

In the Black–Scholes setting this approach has been widely studied; for an introductory overview we refer to [73].

Remark 2.3 Quasi-Monte Carlo as well as randomized Quasi-Monte Carlo methods have gained a lot of interest over the last decade; see e.g. [58, 79] and the references therein. A recent survey by L'Ecuyer can be found in [76]. These methods rely on a careful choice of random numbers that drive the simulation and can improve the $O(N^{-1/2})$ convergence rate of ordinary Monte Carlo methods. Recently, they have been applied in the context of Lévy processes in [11, 80].

Variance reduction In order to obtain more precise estimates, an alternative to simply increasing the number of simulations N is to use variance reduction techniques. For a description of the most relevant variance reduction techniques like control variates technique, antithetic sampling, importance sampling, stratified and bridge sampling, moment matching and low-discrepancy sequences we refer to the monographs [9, 58] as well as [120].

Here, we only illustrate the basic idea of control variates. Let $\bar{V} = N^{-1} \sum_{m=1}^{N} X_m$ be a Monte Carlo estimator of the expectation $V = \mathbb{E}[X]$ of a random variable X. The basic idea of the control variates technique is to look for a random variable Y which is highly correlated with X and has *known mean* $\mathbb{E}[Y]$. Then one may use the empirical means $\bar{V}, \bar{Y}$ to obtain an estimator with lower variance. To do so, for $\alpha \in \mathbb{R}$ consider the random variable

$$X_\alpha := X + \alpha\big(Y - \mathbb{E}[Y]\big).$$

Its variance is $\mathrm{Var}[X_\alpha] = \mathrm{Var}[X] + \alpha^2 \mathrm{Var}[Y] + 2\alpha \mathrm{Cov}(X, Y)$, which is minimized by the value $\alpha^* := -\frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}[Y]}$. In this case there holds $\mathrm{Var}[X_{\alpha^*}] = (1 - \rho^2)\mathrm{Var}[X]$, where ρ denotes the correlation between X and Y. Thus, the variance of X is reduced by a factor of $1 - \rho^2$ and the variance reduction performs well if ρ is close to ± 1. The *unbiased* control variate estimator $\bar{V}_\alpha$ for $V = \mathbb{E}[X]$ is hence defined by

$$\bar{V}_\alpha := \bar{V} + \alpha\big(\bar{Y} - \mathbb{E}[Y]\big).$$

In practice, α^* is usually not known and is therefore replaced by its sample counterpart

$$\bar{\alpha}^* := -\frac{\sum_{m=1}^{N}(X_m - \bar{X})(Y_m - \bar{Y})}{\sum_{m=1}^{N}(Y_m - \bar{Y})^2}.$$

In the context of option pricing, the control variate Y may, for example, be chosen to be lower and/or upper bounds for the option price itself (see e.g. [38]) or the price of the underlying asset at maturity S_T (see e.g. [58]). A combination of control variates and importance sampling for Lévy processes has been applied in [72].

American options Monte Carlo evaluation of American options has a "Monte Carlo on Monte Carlo" feature, since the determination of the optimal exercise time depends on an average over future events. To see this, consider a point (X_t, t) on a single simulated path X. In order to decide whether to exercise at this point, one has to evaluate the expectation (1.3). This requires continuation from (X_t, t) on many branching paths, and makes this direct approach infeasible.

In the Black–Scholes setting several methods have been suggested to overcome this difficulty. For a comparison of the main approaches we refer to [55]. The following have been introduced. The path-bundling technique provides lower and upper bounds for the true price of an American contract; see e.g. [26, 27]. The martingale optimization approach replaces the maximization over stopping times by minimizing over martingales and provides also an upper bound, see [35, 62, 107]. In [84], the future expectation is replaced by a least square interpolation. This method has been applied to value an American option with S in (1.1) following the Merton model ($d = 1$). Furthermore, in [107] some Monte Carlo results for the valuation of American options under spectrally one-sided Lévy processes are presented.

3 Fast Fourier methods

Contrary to the classical Black–Scholes case in Lévy models there are usually no closed form option prices since the probability density of a Lévy process is typically not known in closed form. However, the characteristic function of this density can be expressed in terms of elementary functions for the majority of one-dimensional Lévy processes discussed in the literature. This has led to the development of Fourier-based option pricing methods where the Fourier transform and its inverse,

$$\mathcal{F}\big[g(x)\big](z) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} g(x)\,\mathrm{d}x, \quad z \in \mathbb{R}^d,$$

$$\mathcal{F}^{-1}\big[g(z)\big](x) = \int_{\mathbb{R}^d} e^{-i\langle x,z\rangle} g(z)\,\mathrm{d}z, \quad x \in \mathbb{R}^d,$$

are efficiently evaluated numerically by using the FFT algorithm, see [19, 20, 34, 47, 67, 78, 83, 85, 95, 100]. We present two approaches here. First, following the line of [20, 34, 47, 78] we transform the option value with respect to the log strike price. Secondly, it is also possible to transform the option value with respect to the log spot price as in [67, 83, 100]. For simplicity we set the interest rate $r = 0$.

Transformation with respect to the log strike price Consider an option with strike $K = e^k$, payoff $g(k) = (e^{\langle a, X_T\rangle} - e^k)^+$ with $a \in [0, 1]^d$ and maturity T. For the sake of notational simplicity suppose $t = 0$. The option price is given by

$$V(0, k) = \int_{\mathbb{R}^d} \big(e^{\langle a,s\rangle} - e^k\big)^+ p_T(s)\,\mathrm{d}s, \quad k \in \mathbb{R},$$

where $p_T(s)$ denotes the corresponding joint density of $X_1, \ldots, X_d$ at T.

Since the payoff $g(k)$ tends to a positive constant as $k \to -\infty$, the Fourier transformation of $g(k)$ does not exist in general. Therefore, instead of $g(k)$ one has to consider the damped payoff $e^{\alpha k} g(k)$ with a damping constant $\alpha > 0$. The Fourier transformation of the damped option price can then be written as

$$\begin{aligned}
\mathcal{F}\big[e^{\alpha k} V(0,k)\big](z) &= (2\pi)^{-1} \int_{\mathbb{R}} e^{(\alpha+iz)k} V(0,k)\,\mathrm{d}k \\
&= (2\pi)^{-1} \int_{\mathbb{R}^d} p_T(s) \int_{-\infty}^{\langle a,s\rangle} e^{(\alpha+iz)k} \big(e^{\langle a,s\rangle} - e^k\big)\,\mathrm{d}k\,\mathrm{d}s \\
&= (2\pi)^{-1} \int_{\mathbb{R}^d} p_T(s) \frac{e^{(1+\alpha+iz)\langle a,s\rangle}}{(\alpha+iz)(1+\alpha+iz)}\,\mathrm{d}s \\
&= (2\pi)^{-1} \frac{\Phi_T((1+\alpha+iz)a)}{(\alpha+iz)(1+\alpha+iz)}
\end{aligned}$$

and therefore

$$V(0,k) = \frac{e^{-\alpha k}}{2\pi} \mathcal{F}^{-1}\left[\frac{\Phi_{T-t}((1+\alpha+iz)a)}{(\alpha+iz)(1+\alpha+iz)}\right], \quad k \in \mathbb{R}.$$

Note that, by Lemma 1.1, for $\beta_i^- > (1+\alpha)d$, $i = 1, \ldots, d$ the characteristic function $\Phi_T((1+\alpha+iz)a)$ exists.

Transformation with respect to the log spot price We now consider a general European option with maturity T and payoff $g(x)$ in log spot price x. The option price is then given by

$$V(t,x) = \mathbb{E}\big[g(X_T)\big|X_t = x\big] = \int_{\mathbb{R}^d} g(x+s)\, p_{T-t}(s)\,\mathrm{d}s, \quad x \in \mathbb{R}^d,\ t \geq 0.$$

Now, if $\mathcal{F}[g]$ exists, the Fourier transform of the option price can be written as

$$\begin{aligned}
\mathcal{F}\big[V(t,x)\big](z) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle z,x\rangle} V(t,x)\,\mathrm{d}x \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle z,y\rangle} g(y)\,\mathrm{d}y \int_{\mathbb{R}^d} e^{-i\langle z,s\rangle} p_{T-t}(s)\,\mathrm{d}s \\
&= \mathcal{F}\big[g(y)\big](z) \cdot \Phi_{T-t}(-z),
\end{aligned}$$

where $y = x + s$. Therefore

$$V(t,x) = \mathcal{F}^{-1}\big[\mathcal{F}\big[g(y)\big](z) \cdot \Phi_{T-t}(-z)\big], \quad x \in \mathbb{R}^d,\ t \geq 0. \tag{3.1}$$

The restriction for the Fourier transformation of the payoff to exist is quite strong, since it is not even satisfied for a simple basket option. To weaken this assumption one may again try to dampen the payoff. But this approach is only practicable in dimension $d = 1$, since for most multivariate payoff functions there exist some coordinate directions $j \in \{1, \ldots, d\}$ such that $\lim_{y_j \to \pm\infty} g(y) > 0$, e.g. for basket options. Therefore, in the d-dimensional case one usually has to localize the payoff to

a bounded domain. One strategy of doing so with an explicit error analysis is described in Sect. 4 below. Thereafter, in most cases the Fourier transformation of $g(y)$ must be evaluated numerically and one hence has to calculate both $\mathcal{F}$ and $\mathcal{F}^{-1}$ in (3.1) numerically. The computational cost is doubled. In dimension $d = 1$ however the Fourier transformation of most payoffs can be obtained analytically and only one Fourier transformation, i.e., $\mathcal{F}^{-1}$ in (3.1), has to be evaluated numerically.

Discretization The multidimensional discrete Fourier transform of a given series of data points $f_{\underline{j}}$ is given by the collection

$$\hat{f}_{\underline{k}} = \sum_{j_1=0}^{N-1} \cdots \sum_{j_d=0}^{N-1} e^{2\pi i \langle \underline{k}, \underline{j} \rangle / N} f_{\underline{j}}, \quad k_n = 0, \ldots, N-1, \ n = 1, \ldots, d.$$

To compute $\hat{f}_{\underline{k}}$, $k_n = 0, \ldots, N-1, n = 1, \ldots, d$ one a priori needs $\mathcal{O}(N^{2d})$ operations. Utilizing the so-called *fast Fourier transform* [41, 98] this computational cost can be reduced to $\mathcal{O}(N^d \log N)$. For instance, suppose we want to approximate the inverse Fourier transform of a function $f(z)$ with a discrete Fourier transform (to solve (3.1) one may choose $f(z) = \mathcal{F}[g(y)](z) \cdot \Phi_{T-t}(-z)$). Then, the integral can be truncated and discretized using the trapezoidal rule, to give

$$\begin{aligned} \mathcal{F}^{-1}\big[f(z)\big](x) &= \int_{\mathbb{R}^d} e^{-i\langle x,z\rangle} f(z)\,\mathrm{d}z \approx \int_{[-R,R]^d} e^{-i\langle x,z\rangle} f(z)\,\mathrm{d}z \\ &\approx \sum_{j_1=0}^{N-1} \cdots \sum_{j_d=0}^{N-1} \omega_{\underline{j}} f(z_{\underline{j}}) e^{-i\langle x, z_{\underline{j}}\rangle}, \end{aligned}$$

with discretization step $\Delta z = \frac{2R}{N-1}$, $z^n_{j_n} = -R + j_n \Delta z$ in Fourier space and suitable weights $w_{\underline{j}}$; see e.g. [67].

Herewith, in order to obtain an approximate value of $V(t,x)$ in (3.1) for any $x \in \mathbb{R}^d$, we also have to discretize the spot price or x-domain $\mathbb{R}^d$. For this, we define an additional grid by setting $x^n_{j_n} = -R_2 + k_n \Delta x$ with step size $\Delta x = \frac{2R_2}{N-1}$ and given $R_2 > 0$. With the relation

$$\Delta z \cdot \Delta x = \frac{2\pi}{N} \tag{3.2}$$

we then find

$$\begin{aligned} \mathcal{F}^{-1}\big[f(z)\big](x_{\underline{k}}) &\approx e^{iR\langle x_{\underline{k}}, \mathbf{1}\rangle} \sum_{j_1=0}^{N-1} \cdots \sum_{j_d=0}^{N-1} e^{-i2\pi \langle \underline{k}, \underline{j}\rangle / N} \omega_{\underline{j}} f(z_{\underline{j}}) e^{iR_2 \Delta z \langle \underline{1}, \underline{j}\rangle} \\ &= e^{iR\langle x_{\underline{k}}, \underline{1}\rangle} \hat{f}_{\underline{k}}. \end{aligned}$$

This expression can now be evaluated very efficiently using the fast Fourier transform as mentioned above. Also note that by (3.2) the discretization of the Fourier space and the spot price (or strike price) space are related and cannot be chosen independently. No time stepping is required and for $d = 1$ dimension only $\mathcal{O}(N \log N)$

work is needed to obtain the price at N spot (or strike) prices. For convergence rates and error analysis see [78].

American options In order to compute American option prices, one can approximate them by a sequence of Bermudan options with increasing number of exercise dates (as introduced in [56]).

For this, let $t = t^0 < \cdots < t^M = T$ be a time discretization which can be thought of as exercise times of a Bermudan option. Its price can be computed by backward induction as

$$\begin{aligned} V\big(t^M, x\big) &= g(x), \\ V\big(t^m, x\big) &= \max\big(\mathbb{E}\big[V\big(t^{m+1}, X_{t^{m+1}}\big)\big|X_{t^m} = x\big], g(x)\big), \quad m = M-1, \ldots, 0, \end{aligned} \tag{3.3}$$

where at each time point equation (3.1) is solved with time step $t^{m+1} - t^m$ as in [67, 85]. The overall computational cost for this approximation then is $\mathcal{O}(MN^d \log N)$. In [19] the Wiener–Hopf factorization is used to compute the values of perpetual American options.

4 PIDE-based methods

In the Black–Scholes setting, when the underlying process S is a geometric Brownian motion, it is well known that the solution $V(t, S)$ of the pricing equation (1.2) can be described as the solution of a parabolic partial differential equation (PDE) also known as the Black–Scholes equation, namely

$$\frac{\partial V}{\partial t}(t, S) + \frac{1}{2}\sum_{i,j=1}^{d} S_i S_j Q_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{d} S_i \frac{\partial V}{\partial S_i}(t, S) - rV(t, S) = 0, \tag{4.1}$$

with suitable boundary conditions depending on the payoff function $g(\cdot)$. Since PDEs of the form (4.1) are well studied objects in engineering and numerical mathematics, there exists a whole zoo of sophisticated and very general so-called finite difference and finite element methods for their numerical solution (at least in dimension $d \leq 3$). For an introduction see e.g. [23] or for a more finance-related perspective [2, 48, 115]. Over the last years many authors have developed several variants of these methods, especially for the efficient treatment of linear and non-linear problems arising in diffusion-driven markets, see e.g. [1, 2, 51, 52, 97, 111, 118] and the references therein.

In this survey article we focus on markets driven by general Lévy processes. Therefore, we illustrate in this section how numerical methods to solve (4.1) can be modified and extended in order to yield a general and efficient pricing technique which can be applied to general Lévy models also in moderate dimension $d > 3$. The main difference is that corresponding to the existence of jumps in a Lévy model an additional integral term has to be introduced to (4.1) resulting in a partial *integro*-differential equation (PIDE). More precisely, if S is driven by an exponential Lévy process X as in (1.1) then, by [104, Theorem 4.2], the solution $V(t, S)$ of (1.2) can be characterized by

Theorem 4.1 *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet (γ, Q, ν). Assume that the function $V(t, S)$ in (1.2) satisfies*

$$V(t, S) \in C^{1,2}\big((0, T) \times \mathbb{R}^d_{>0}\big) \cap C^0\big([0, T] \times \mathbb{R}^d_{\geq 0}\big).$$

Then $V(t, S)$ satisfies the PIDE

$$\frac{\partial V}{\partial t}(t, S) + \frac{1}{2}\sum_{i,j=1}^{d} S_i S_j Q_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{d} S_i \frac{\partial V}{\partial S_i}(t, S) - rV(t, S)$$
$$+ \int_{\mathbb{R}^d} \left(V\big(t, Se^z\big) - V(t, S) - \sum_{i=1}^{d} S_i \big(e^{z_i} - 1\big) \frac{\partial V}{\partial S_i}(t, S) \right) \nu(\mathrm{d}z) = 0 \quad (4.2)$$

in $(0, T) \times \mathbb{R}^d_{\geq 0}$ where $V(t, Se^z) := V(t, S_1 e^{z_1}, \ldots, S_d e^{z_d})$, and the terminal condition is given by

$$V(T, S) = g(S) \quad \forall S \in \mathbb{R}^d_{\geq 0}.$$

For its numerical solution, by [104, Corollary 4.3], the PIDE (4.2) can be transformed into a simpler form.

Corollary 4.2 *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet (γ, Q, ν) and marginal Lévy measures ν_i, $i = 1, \ldots, d$ satisfying (1.4) with $\beta_i^+ > 1$, $\beta_i^- > 0$, $i = 1, \ldots, d$. Furthermore, let*

$$u(\tau, x) = e^{r\tau} V\big(T - \tau, e^{x_1 + (\gamma_1 - r)\tau}, \ldots, e^{x_d + (\gamma_d - r)\tau}\big), \quad (4.3)$$

where

$$\gamma_i = \frac{Q_{ii}}{2} + \int_{\mathbb{R}} \big(e^{z_i} - 1 - z_i\big) \nu_i(\mathrm{d}z_i).$$

Then u satisfies the PIDE

$$\frac{\partial u}{\partial \tau} + \mathcal{A}_{\mathrm{BS}}[u] + \mathcal{A}_{\mathrm{J}}[u] = 0 \quad (4.4)$$

in $(0, T) \times \mathbb{R}^d$ with initial condition $u(0, x) := u_0$. The differential operator is defined for $\varphi \in C_0^2(\mathbb{R}^d)$ by

$$\mathcal{A}_{\mathrm{BS}}[\varphi] = -\frac{1}{2}\sum_{i,j=1}^{d} Q_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \quad (4.5)$$

and the integro-differential operator by

$$\mathcal{A}_{\mathrm{J}}[\varphi] = -\int_{\mathbb{R}^d} \left(\varphi(x + z) - \varphi(x) - \sum_{i=1}^{d} z_i \frac{\partial \varphi}{\partial x_i}(x) \right) \nu(\mathrm{d}z). \quad (4.6)$$

The initial condition is given by

$$u_0 = g\big(e^x\big) := g\big(e^{x_1}, \ldots, e^{x_d}\big). \tag{4.7}$$

Furthermore, for American options instead of a PIDE-based characterization one obtains an *inequality* representation provided the price $u(\cdot,\cdot)$ solves a partial integro-differential inequality, i.e.,

$$\begin{aligned} \frac{\partial u}{\partial \tau} + (\mathcal{A}_{\mathrm{BS}} + \mathcal{A}_{\mathrm{J}})[u] &\leq 0, \\ u(\tau,\cdot) &\geq \tilde{g}_\tau, \\ (u - \tilde{g}_\tau)\left(\frac{\partial u}{\partial \tau} + (\mathcal{A}_{\mathrm{BS}} + \mathcal{A}_{\mathrm{J}})[u]\right) &= 0, \end{aligned} \tag{4.8}$$

with $\mathcal{A}_{\mathrm{BS}}$ and $\mathcal{A}_{\mathrm{J}}$ as in (4.4), where $\tilde{g}_\tau$ denotes the payoff function g transformed according to (4.3), i.e.,

$$\tilde{g}_\tau(x) = e^{r\tau} g\big(e^{x_1+(\gamma_1-r)\tau}, \ldots, e^{x_d+(\gamma_d-r)\tau}\big), \quad x \in \mathbb{R}^d. \tag{4.9}$$

For the derivation of (4.8) see e.g. [90] and [38, Sect. 12.1.3] based on the methodology of [14, 15].

The implementation of any finite difference or finite element scheme for (4.4), (4.8) requires the *localization* of the log price domain $\mathbb{R}^d$ to a bounded domain $B_R := [-R, R]^d$, $R > 0$. For this, we find that in finance truncation of $\mathbb{R}^d$ to B_R corresponds to approximating the solution u of (4.4) by the price u_R of a corresponding barrier option on B_R. In log-prices u_R is given by

$$u_R(t,x) = \mathbb{E}\big[g\big(e^{X_T}\big)1_{\{T<\tau_{B_R,t}\}}\big|X_t = x\big],$$

where $\tau_{B_R,t} = \inf\{s \geq t | X_s \notin B_R\}$ denotes the first exit time of X from B_R after time t. In case the underlying stochastic process X admits semiheavy tails (1.4), the solution u_R of the localized problem converges pointwise exponentially to the solution u of (4.4), i.e., there exist constants $c_1, c_2 > 0$ such that

$$\big|u(t,x) - u_R(t,x)\big| \lesssim e^{-c_1 R + c_2 \|x\|_\infty}.$$

It therefore indeed suffices to replace the original price space domain $\mathbb{R}^d$ by B_R with sufficiently large $R > 0$. For details we refer to [104]. Furthermore, note that for a barrier option on B_R there is no localization error.

4.1 Finite difference methods

After localization of the original space domain $\mathbb{R}^d$ to B_R as described above, the numerical solution of (4.4) by finite differences is obtained in three main steps:

1. The integration domain $\mathbb{R}^d$ in (4.6) must also be localized to a bounded domain.
2. The small jumps must be approximated by a Brownian motion.
3. The solution is computed at discrete grid points and the derivatives in (4.4)–(4.6) are replaced by finite differences.

Localization of the integration domain The integration domain $\mathbb{R}^d$ in (4.6) is truncated to a bounded domain $\Lambda_Z = [-Z, Z]^d$, $Z > 0$. Similarly to the localization of the spatial domain it can be shown that the error decays exponentially with respect to Z; see [39, 104].

Approximation of small jumps In order to numerically integrate the jump measure in the finite difference discretization described below, the small jumps of the Lévy process need to be truncated. By doing this, the Lévy measure becomes finite. More precisely, we introduce a truncation parameter $\varepsilon \in (0, 1]$ and split the process X into $X = X_\varepsilon + X^\varepsilon$ as defined in Sect. 2. The process X_ε is approximated by a Brownian motion and for the remainder process X^ε we then obtain the triplet $(\gamma + \gamma_\varepsilon, Q + Q_\varepsilon, \nu^\varepsilon)$, where the Lévy measure ν^ε is now a finite measure, i.e., $\nu^\varepsilon(\mathbb{R}^d) < \infty$.

Note that the truncation of the small jumps based on the non-financial parameter $\varepsilon > 0$ introduces an additional error to the discretization which can have a significant impact on the accuracy and stability of the numerical scheme. For examples, such as barrier contracts under pure jump Lévy models, we refer to [82, Sects. 6.2, 8.3].

Discretization Consider a uniform grid on $[0, T] \times [-R, R]^d$ with time step $\Delta t = \frac{T}{M}$ and mesh width $\Delta x = \frac{2R}{N}$ for $N, M \in \mathbb{N}$. Then, the time and space points are given by $t_m = m\Delta t$ and $x^j_{n_j} = -R + n_j \Delta x$ where $m = 0, \ldots, M$, $n_j = 0, \ldots, N$ and $j = 1, \ldots, d$. Let $\underline{u}^m_{\underline{n}} = u(t_m, x^1_{n_1}, \ldots, x^d_{n_d})$ be the solution on the grid which is zero outside of $[-R, R]^d$. The spatial derivatives in (4.5) can be approximated using finite differences by

$$\frac{\partial^2 u}{\partial x^j \partial x^j}(x^j) \approx \frac{\underline{u}_{n_j+1} - 2\underline{u}_{n_j} + \underline{u}_{n_j-1}}{(\Delta x)^2},$$

$$\frac{\partial^2 u}{\partial x^j \partial x^i}(x^j, x^i) \approx \frac{\underline{u}_{(n_j+1,n_i+1)} - \underline{u}_{(n_j+1,n_i-1)} - \underline{u}_{(n_j-1,n_i+1)} + \underline{u}_{(n_j-1,n_i-1)}}{4(\Delta x)^2},$$

and the integral in (4.6) is numerically integrated using a trapezoidal quadrature rule with the same grid resolution Δx. Using this, the jump operator (4.6) reads

$$-\int_{\Lambda_Z} \big(u(x_{\underline{n}} + z) - u(x_{\underline{n}})\big)\nu^\varepsilon(\mathrm{d}z) \approx -\sum_{\ell_1} \cdots \sum_{\ell_d} (\underline{u}_{\underline{n}+\underline{\ell}} - \underline{u}_{\underline{n}})\underline{\nu}_{\underline{\ell}},$$

where $\underline{\nu}_{\underline{\ell}} = \int_{(\ell_1-1/2)\Delta x}^{(\ell_1+1/2)\Delta x} \cdots \int_{(\ell_d-1/2)\Delta x}^{(\ell_d+1/2)\Delta x} \nu^\varepsilon(\mathrm{d}z)$. Note that $\underline{u}_{\underline{n}+\underline{\ell}} = 0$ for $n_j + \ell_j \notin \{0, \ldots, N\}$. Non-zero boundary conditions are treated in [39]. Denoting by $\mathbf{A}_{\mathrm{BS}}$ and $\mathbf{A}_{\mathrm{J}}$ the discretization matrices representing the differential and jump part, we can write a θ-time stepping scheme as

$$\frac{\underline{u}^{m+1} - \underline{u}^m}{\Delta t} + \theta_1 \mathbf{A}_{\mathrm{BS}}\underline{u}^{m+1} + (1-\theta_1)\mathbf{A}_{\mathrm{BS}}\underline{u}^m + \theta_2 \mathbf{A}_{\mathrm{J}}\underline{u}^{m+1} + (1-\theta_2)\mathbf{A}_{\mathrm{J}}\underline{u}^m = 0. \tag{4.10}$$

Note that the matrix $\mathbf{A}_{\mathrm{BS}}$ is sparse whereas the matrix $\mathbf{A}_{\mathrm{J}}$ is densely populated. For $\theta_1 = \theta_2 = 0$ the scheme is explicit but not unconditionally stable. Therefore, to obtain stability small time steps are required. For $\theta_1 = \theta_2 = 1$ the scheme is implicit and

there are no stability constrains. However, at each time step a linear system with a full matrix has to be solved. Therefore, Cont and Voltchkova [39] propose an explicit-implicit scheme with $\theta_1 = 1$, $\theta_2 = 0$ in dimension $d = 1$ and prove convergence of the fully discrete scheme. In particular, they show that under certain smoothness assumptions on the payoff function $g(\cdot)$ one obtains first order convergence in Δx.

Similar techniques for $d = 1$ and $d = 2$ are also shown in [24]. In dimension $d = 1$, it is also possible to use the fast Fourier transform and exploit the Toeplitz structure of the matrix $\mathbf{A}_J$, cf. [3, 7].

However, the described discretization suffers from the so-called "curse of dimension," i.e., the number of grid points grows like $\mathcal{O}(N^d)$. Therefore, high-dimensional problems with $d > 3$ cannot be solved. To break this curse, for the Black–Scholes equation Reisinger and Wittum [105] employed so-called sparse grids (see [30, 125]) based on the combination technique (see e.g. [31]). Here, the number of grid points only grows like $\mathcal{O}(N(\log N)^{d-1})$. To our knowledge, no finite difference schemes for Lévy models of dimension $d > 2$ have been considered in the literature up to now.

Remark 4.3 (Multinomial tree methods) Among the most popular and intuitive numerical pricing techniques in the Black–Scholes setting are the so-called multinomial tree or Markov chain methods. The tree method originally dates back to Cox et al. [42] and was subsequently extended to finite activity jump diffusion models [5, 93]. The pricing of European and American options with multinomial approximation has thereafter been considered by several authors, see e.g. [6, 63, 88]. For multidimensional models however the method fails due to a rapid, exponential growth of complexity with the dimension. An overview can be found in e.g. [75]. The basic idea reads as follows. For the exponential Lévy process S one can construct a discrete-time Markov chain s by setting

$$s^{n+1} = S_{t+\Delta t}^{\Delta t, \Delta x} = S_t^{\Delta t, \Delta x} \exp\bigl(\varepsilon^n\bigr) = s^n \exp\bigl(\varepsilon^n\bigr),$$

where ε^n denotes an i.i.d. family of random variables taking $k \in \mathbb{N}$ values. Usually, the values of ε^n are chosen to be multiples of the given step size Δx,

$$\varepsilon^n \in \{-k_1 \Delta x, \ldots, -\Delta x, 0, \Delta x, \ldots, k_2 \Delta x\}.$$

Then there holds $k = k_1 + k_2 + 1$. The numbers $k_1, k_2 \in \mathbb{N}$ are allowed to differ to account for asymmetry of jumps. The paths of the Markov chain $\ln s$ fall on a lattice with step size $(\Delta t, \Delta x)$. They can therefore be seen as an explicit finite difference scheme in $(t, \ln S)$-space. The values of k_1, k_2 and the transition probabilities have to be chosen in such a way that absence of arbitrage is guaranteed, cf. [5]. For example, if we denote quantities on the lattice by A_j^n, where n denotes the time index of a node and j the price space index, the approximate value V of a European option can be computed by backward induction; starting at maturity from the final node $N \Delta t = T$, at each node in the tree the option value is given by the discounted expectation of the values on the branches, i.e.,

$$V_j^n = e^{-r \Delta t} \sum_{i=-k_1}^{k_2} q_i V_{j+i}^{n+1},$$

where q_i, $i = -k_1, \dots, k_2$, denote the transition probabilities $q_i = \mathbb{P}[\varepsilon^n = i\Delta x]$. For American options this backward step has to be replaced by taking the maximum of V_j^n and the corresponding payoff from exercising at the current time. See also [93]. If the transition probabilities are chosen such that $S^{\Delta t, \Delta x}$ converges weakly to S as $(\Delta t, \Delta x) \to 0$, convergence of the discrete time European and American option prices to their continuous time counterparts can be shown to hold, cf. [75, 99].

4.2 Finite element methods

Instead of solving the PIDE (4.4) directly, the finite element method is based on the reformulation of (4.4) into the corresponding variational Galerkin equation. To this end, for $u, v \in C_0^\infty(\mathbb{R}^d)$ we associate with $\mathcal{A}_{\mathrm{BS}}$ in (4.5) the bilinear form

$$\mathcal{E}_{\mathrm{BS}}(u, v) = \frac{1}{2} \sum_{i,j=1}^{d} Q_{ij} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x.$$

To the jump part $\mathcal{A}_{\mathrm{J}}$ in (4.6) we associate the bilinear jump form

$$\mathcal{E}_{\mathrm{J}}(u, v) = -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(u(x+z) - u(x) - \sum_{i=1}^{d} z_i \frac{\partial u}{\partial x_i}(x) \right) v(x) \, \mathrm{d}x \, \nu(\mathrm{d}z), \tag{4.11}$$

and set

$$\mathcal{E}(u, v) = \mathcal{E}_{\mathrm{BS}}(u, v) + \mathcal{E}_{\mathrm{J}}(u, v).$$

Denoting by $\mathcal{D}(\mathcal{E})$ the domain of $\mathcal{E}(\cdot, \cdot)$, the variational problem associated to (4.4) reads

$$\begin{aligned} &\text{Find } u \in L^2\big((0,T); \mathcal{D}(\mathcal{E})\big) \cap H^1\big((0,T); \mathcal{D}(\mathcal{E})^*\big) \text{ such that} \\ &\left\langle \frac{\partial u}{\partial \tau}, v \right\rangle_{\mathcal{D}(\mathcal{E})^*, \mathcal{D}(\mathcal{E})} + \mathcal{E}(u, v) = 0, \quad \text{a.e. } \tau \in (0,T), \ \forall v \in \mathcal{D}(\mathcal{E}), \\ &u(0) = u_0, \end{aligned} \tag{4.12}$$

where u_0 is defined as in (4.7). For the well-posedness of (4.12) we refer to [91] for one-dimensional and to [50, 104] for certain multidimensional Lévy models. For instance, if $Q > 0$ in (1.5) the domain $\mathcal{D}(\mathcal{E})$ coincides with the Sobolev space $H^1(\mathbb{R}^d)$, and for $Q = 0$ and tempered stable margins $\mathcal{D}(\mathcal{E})$ can be seen to be some anisotropic Sobolev space.

Remark 4.4 One key advantage of the variational formulation (4.12) is that the bilinear form $\mathcal{E}(\cdot, \cdot)$ allows for a *naturally singularity free* discretization of general Lévy measures, i.e., the small jumps of the process do not need to be approximated by a Brownian motion. In contrast to Sect. 4.1, no additional truncation error is introduced. Consider, for example, a symmetric Lévy measure ν. Then, by [50, Proposition 4.1],

the bilinear form $\mathcal{E}_J(\cdot,\cdot)$ in (4.11) can be rewritten as

$$\mathcal{E}_J(u,v) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(u(x+z)-u(x)\big)\big(v(x+z)-v(x)\big)\nu(\mathrm{d}z)\,\mathrm{d}x. \tag{4.13}$$

Using (1.6), one readily infers that merely Lipschitz-continuous functions u, v are sufficient for the integrals in (4.13) to exist in the Lebesgue sense. In addition, in [104, Proposition 4.11] it is shown that the original bilinear form (4.11) exists for all standard, continuous finite element basis functions and general non-symmetric Lévy measures.

The finite element method for solving the pricing equations (4.4) and (4.12) has been studied by Achdou and Pironneau using adaptive mesh refinement techniques, see [2, Chaps. 4 and 5]. In one dimension, Matache et al. [91, 92] have introduced a very general wavelet-based finite element scheme to solve (4.12). This was subsequently applied to American-type contracts (cf. [90, 121]) as well as stochastic volatility models (cf. [64]). In [50, 101, 104] the wavelet-based approach was extended to multidimensional models based on sparse tensor products and wavelet compression techniques as described in [96, 101] and the references therein.

In the following we explain the basic finite element approach (cf. [2, 115]) and further illustrate the use of wavelet basis functions in this context. After localization of the original space domain $\mathbb{R}^d$ to B_R as described above, the numerical solution of (4.12) by the finite element method is obtained in two main steps:

1. The infinite dimensional space $\mathcal{D}(\mathcal{E})$ needs to be *discretized* by finite dimensional subspaces $V_N \subset \mathcal{D}(\mathcal{E})$ corresponding to a finite element mesh with N degrees of freedom.
2. A *time stepping* scheme has to be applied to discretize in time.

Space discretization Let $V_N \subset \mathcal{D}(\mathcal{E})$ be a subspace of dimension $N^d := \dim V_N$ generated by a finite element basis $\Phi := \{\phi_j : j = 1,\ldots,N^d\}$ on a tensor product mesh of width $\Delta x = \frac{2R}{N}$ on B_R. For classical examples of basis functions see e.g. [115, Chapter 5] or [23]. We use the Galerkin approach and approximate the solution u by a function $u_N(t,x) = \sum_{j=1}^{N^d} \underline{u}_j(t)\phi_j(x) \in V_N$. Then, for each time $t \in [0,T]$ the semidiscrete problem of finding the coefficient vector $\underline{u}(t)$ is an initial value problem for N^d ordinary differential equations

$$\mathbf{M}\frac{\partial}{\partial t}\underline{u}(t) + \mathbf{A}\underline{u}(t) = 0, \quad \underline{u}(0) = \underline{u}^0, \tag{4.14}$$

where $\underline{u}^0$ denotes the coefficient vector of u_0, and $\mathbf{M}$, $\mathbf{A}$ denote the mass and stiffness matrices with respect to the basis of V_N, i.e.,

$$\mathbf{M} = \big((\phi_i,\phi_j)\big)_{1\le i,j\le N^d}, \qquad \mathbf{A} = \big(\mathcal{E}(\phi_i,\phi_j)\big)_{1\le i,j\le N^d}. \tag{4.15}$$

Time stepping using the θ-scheme As in the finite difference case, for the time discretization of (4.15) we consider a uniform grid with time step $\Delta t = \frac{T}{M}$ and time

points $t_m = m\Delta t$, $m = 0, \ldots, M$ for some $M \in \mathbb{N}$. Furthermore, we again use the θ-scheme with $\theta = \theta_1 = \theta_2$ to obtain

$$\frac{\underline{u}^{m+1} - \underline{u}^m}{\Delta t}\mathbf{M} + \theta \mathbf{A}\underline{u}^{m+1} + (1-\theta)\mathbf{A}\underline{u}^m = 0, \quad m = 0, \ldots, M-1. \tag{4.16}$$

For $\theta = 1/2$, the scheme in (4.16) coincides with the popular Crank–Nicholson scheme. Note that, in contrast to the finite difference methods, also implicit time stepping schemes are admissible here provided one chooses a suitable (e.g. wavelet) basis for V_N; see e.g. [91]. Furthermore, one can also use finite elements for the time discretization as in [64, 113] where an hp-discontinuous Galerkin method is used. This yields exponential convergence rates instead of only algebraic ones as in the θ-scheme, and therefore only $M = \log(N^d)$ time steps are required.

Wavelet-based finite element methods Wavelet-based finite element methods (or *wavelet methods*) provide a very general PIDE-based numerical pricing technique. The methods owe their name to the choice of a wavelet basis for the spaces V_N in the finite element method. A basic survey of wavelet-based finite element methods in finance can be found in [65], and we shall follow it here.

The motivation for applying wavelet methods rather than classical finite elements can be summarized as follows. As for the finite difference method, in high-dimensional models finite elements suffer from the "curse of dimension," i.e., the number of degrees of freedom on a tensor product finite element mesh grows like $\mathcal{O}(N^d)$. For jump models the non-locality of the underlying operator $\mathcal{A}_J$ implies that the standard finite element stiffness matrix $\mathbf{A}$ consists of $\mathcal{O}(N^{2d})$ non-zero entries, which is not practicable even in one dimension with small mesh widths.

For this reason, wavelet basis functions come into play. They can overcome these issues while still being easy to compute. In addition to great analytical tractability, choosing a wavelet basis for the discrete space V_N has three main advantages in practice:

- Break the curse of dimension using sparse tensor products (see e.g. [29, 96]) $\Rightarrow$ Dimension-independent complexity (up to log-factors).
- Multiscale compression of jump measure of $X \Rightarrow$ Complexity of jump models can asymptotically be reduced to Black–Scholes complexity.
- Efficient preconditioning.

To illustrate what a wavelet basis is, we introduce the notion of a *multiscale basis* for the finite dimensional space $V_N \subset \mathcal{D}(\mathcal{E})$. In one dimension, suppose the mesh width Δx of V_N can be represented by a negative power of two, $\Delta x = 2^{-L}$, corresponding to a level index $L \in \mathbb{N}_0$. To simplify notation, write $V_L := V_N$ with basis $\Phi_L := \Phi$. Then by decreasing the mesh width $\Delta x = 2^{-L}$ one obtains a sequence of spaces

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2(B_R), \qquad \bar{\bigcup_{L\in\mathbb{N}_0}} V_L = L^2(B_R),$$

generated by bases Φ_L, $L \in \mathbb{N}_0$. Hence, bi-orthogonal *complement* or *wavelet bases* $\Psi_L = \{\psi_{j,L} : j \in \nabla_L\}$, with suitable index sets ∇_L, can be constructed from the

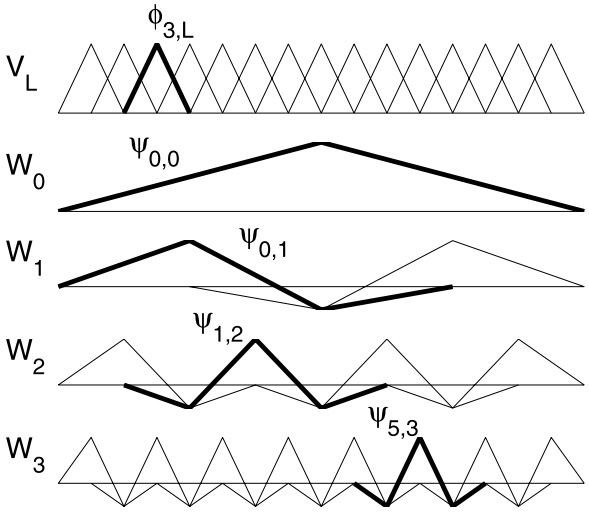

Fig. 1 Schematic of single-scale space V_L and its decomposition into multiscale wavelet spaces W_ℓ

single-scale bases Φ_L; for details see [36, 44, 94]. Denoting by W_L the span of Ψ_L, the spaces V_{L+1} admit a splitting

$$V_{L+1} = W_L \oplus V_L, \quad L > 0. \tag{4.17}$$

Each wavelet space W_L can be thought of as describing the increment of information when refining the finite element approximation from V_L to V_{L+1}. Furthermore, (4.17) implies that for any $L > 0$ the finite element space V_L can be written as a direct *multilevel* sum of the wavelet spaces W_ℓ, $\ell < L$. Thus, any $u_L \in V_L = V_N$ has the representation

$$u_L = \sum_{\ell=0}^{L-1} \sum_{j \in \nabla_\ell} d_{j,\ell} \psi_{j,\ell},$$

with suitable coefficients $d_{j,\ell} \in \mathbb{R}$. Figure 1 illustrates the decomposition of the finite element space V_L, $L = 4$, spanned by continuous, piecewise linear (nodal) basis functions $\phi_{i,L}$ into its increment spaces W_ℓ, $\ell = 0, \ldots, 3$, spanned by wavelets $\psi_{j,\ell}$.

In the multidimensional setting we obtain multivariate wavelet basis functions by using tensor products. The finite element spaces V_L can then be characterized by

$$V_L = \operatorname{span}\big\{\psi_{j_1,\ell_1}(x_1) \cdots \psi_{j_d,\ell_d}(x_d) : \ell_1, \ldots, \ell_d \le L,\ j_i \in \nabla_{\ell_i}\big\}.$$

Since these multivariate wavelet bases consist of products of one-dimensional wavelets, they form *hierarchical bases* as in [61]. Thus, the spaces V_L can be replaced by *sparse tensor product* spaces

$$\hat{V}_L = \operatorname{span}\big\{\psi_{j_1,\ell_1}(x_1) \cdots \psi_{j_d,\ell_d}(x_d) : \ell_1 + \cdots + \ell_d \le L,\ j_i \in \nabla_{\ell_i}\big\}.$$

In [29, 96] it is shown that, under certain smoothness assumptions on the solution u of (4.12), the sparse tensor product spaces preserve the approximation properties of the full tensor product spaces while there holds $\hat{N} := \dim \hat{V}_L = \mathcal{O}(N|\log N|^{d-1}) \ll N^d$. Therefore, the complexity of the finite element stiffness matrix can be reduced to $\mathcal{O}(\hat{N}^2)$ instead of originally $\mathcal{O}(N^{2d})$ non-zero entries.

Furthermore, wavelet basis functions give rise to certain *cancelation properties* and *norm equivalences* as illustrated in e.g. [21, 36]. One therefore obtains sharp estimates for the entries of the corresponding stiffness matrix, cf. [91, 101]. Herewith a priori compression schemes can be defined that further reduce the complexity of the stiffness matrix. The compression exploits the fact that the position of large entries in the stiffness matrix arising from a model with jumps resembles the structure of a Black–Scholes stiffness matrix. The remaining entries can a priori be proved to be negligible. Therefore, the compression scheme (asymptotically) reduces the complexity of a model with jumps to that of the Black–Scholes model.

Combining the compression scheme with the sparse tensor product spaces results in a computational complexity of $\mathcal{O}(\hat{N})$ instead of the original $\mathcal{O}(N^{2d})$. It is proved that these wavelet schemes preserve stability and convergence of the classical finite element schemes, cf. [101–103].

Since the convergence of finite element methods has been studied intensively for many decades, sophisticated numerical analysis is available here. For example, in [50, 101] it is shown that when using piecewise polynomial basis functions of degree $p \geq 1$, the finite element scheme described above converges at rate $p+1$ in Δx (provided sufficient smoothness of the solution u of (4.4)). In particular, employing a piecewise linear basis as illustrated in Fig. 1 one obtains a second order scheme.

American options Similarly to the variational formulation for European contracts, for finite element implementation the partial integro-differential inequality (4.8) is reformulated into the corresponding variational inequality

$$\begin{aligned}&\text{Find } u(\tau,\cdot) \in K_{\tilde{g}_\tau} := \left\{v \in \mathcal{D}(\mathcal{E}) : v \geq \tilde{g}_\tau \text{ a.e.}\right\} \quad \text{such that}\\ &\left\langle \frac{\partial u}{\partial \tau} v - u \right\rangle_{\mathcal{D}(\mathcal{E})^*,\mathcal{D}(\mathcal{E})} + \mathcal{E}(u, v-u) \leq 0 \quad \text{a.e. in } [0,T], \text{ for all } v \in K_{\tilde{g}_\tau},\end{aligned}$$

with $\tilde{g}_\tau$ given by (4.9).

Choosing finite dimensional subspaces $V_N \subset V$, as above, to discretize in space and applying the θ-scheme defined in (4.16) with $\theta = 1$ to discretize in time leads to a sequence of matrix linear complementary problems (LCPs)

$$\begin{aligned}&\text{Given } \underline{u}^0, \text{ find } \underline{u}^m \in \underline{K} := \left\{\underline{v} \in \mathbb{R}^{\dim V_N} : \underline{v} \geq \underline{\tilde{g}}_t\right\} \quad \text{such that for all } \underline{v} \in \underline{K},\\ &\left(\underline{v} - \underline{u}^{m+1}\right)^\top (\mathbf{M} + \Delta t \mathbf{A}) \underline{u}^{m+1} \geq \left(\underline{v} - \underline{u}^{m+1}\right)^\top \mathbf{M} \underline{u}^m, \quad m = 0, \ldots, M-1,\end{aligned} \tag{4.18}$$

with $\mathbf{M}$, $\mathbf{A}$ as in (4.15) and where $\underline{u}^0$ denotes the coefficient vector of u_0. As already described in (3.3) this discretization can be interpreted as approximating the value of the American option by a sequence of Bermudan option values.

For a large number N of degrees of freedom, standard solution methods like projected SOR [43] for the matrix LCP (4.18) are not suitable, since their rate of convergence depends on N. The wavelet-based solution algorithm suggested in [90] relies on a fixed point iteration where in each iteration step a V_L-projection $P_{\underline{K}}$ onto the convex cone $\underline{K}$ has to be realized. Due to norm equivalences of the wavelet basis, the outer fixed point iteration convergences at a rate independent of the number of degrees of freedom. The projection $P_{\underline{K}}$ is based on a wavelet generalization of the classical Cryer algorithm [43].

5 Comparison

So far we have presented the setup and general methodology of the major numerical schemes for multidimensional Lévy models. In this final section we explain the main differences and problem-dependent advantages of these methods.

Admissible exotic contracts The applicability of the different numerical methods to exotic contracts in Lévy models is essentially subject to the same challenges as in the classical Black–Scholes model.

As we have already seen in the above sections, nowadays *American contracts* can be handled more or less efficiently by all described methods. For Monte Carlo however this was long thought to be an impossible task, since it is based on simulation forward in time. In mesh-based methods one employs backward time stepping. With this, the efficient pricing of American contracts is straightforward. Therefore, Monte Carlo schemes are still considered inferior to mesh-based methods when applied to American contracts.

Contracts with *discontinuous payoffs* such as digital options do not pose any additional challenges for the finite element method, since it solves the variational problem (4.12), which is L^2-based. Under certain conditions, however, discontinuous payoffs can result in a decreased rate of convergence for Monte Carlo methods; for details see e.g. [25].

Pricing of *barrier options* can be handled very easily with a PIDE-based approach, since one simply has to restrict the price-space domain of the discretization to that of the barrier and impose suitable boundary conditions. Here, very complex multidimensional barriers can be handled instantly. If the barrier is monitored continuously, pricing barrier options is somewhat more complicated for Monte Carlo methods, since we simulate forward in time and one cannot see what happens in between sampling dates. Therefore, straightforward Monte Carlo methods overestimate the prices of knock-out options and underestimate those of knock-in contracts. Correction techniques have been considered for several years now; see e.g. [8, 28, 106]. In the case where the barrier is monitored at discrete days (which is common in practice) one may choose the sampling dates to coincide with the observation dates and hence the bias in the Monte Carlo estimates vanishes. For Fourier methods the Wiener–Hopf factorization can be used; see e.g. [19].

Monte Carlo methods are easily applied to *path-dependent derivatives* such as Asian or lookback options, since sample paths are simulated forward in time and

the history at each time step is known. One only has to take into account that, as for barrier options, Monte Carlo methods might result in biased estimates if the contract is monitored continuously. Since mesh-based methods solve backward in time, they are intrinsically not well suited for path-dependent options. However, several methods have been introduced to provide PIDE-based tools for such contracts. For example, to price Asian options one may introduce an additional variable, i.e., increase the problem dimension, to handle the averaging of the underlying's prices; see e.g. [119, 123].

For all numerical methods, a continuously paid *dividend* q can simply be handled by changing the interest rate $r \mapsto r - q$. If the dividend is paid at discrete points in time it can still be handled easily by Monte Carlo methods, since one is proceeding forward in time and payment of a dividend results in an immediate decrease of the underlying's value. For Fourier and PIDE-based techniques the situation of discretely paid dividends is usually more complicated, since one needs to enforce additional no-arbitrage conditions; see e.g. [122].

Admissible market models Monte Carlo methods are applicable to any Lévy process X as long as there is an efficient way of simulating its trajectories. Furthermore, the simulation techniques illustrated in Sect. 2 provide a very general framework for Monte Carlo simulations applicable to more exotic models. Such simulations can however be rather involved for general multidimensional models. They have to be considered separately for each model.

In contrast to constructing a new simulation scheme for each process, PIDE-based methods provide a standard approach whenever the PIDE (4.4) admits a unique solution and the Lévy measure ν of X is available in a suitable form, i.e., if the density or tail integrals of ν are known explicitly. These requirements have been proved for all major one-dimensional models (see [40, 91]) and multidimensional Lévy copula models (see [50, 104]). One major advantage of the PIDE-based methods is that changing the market model only amounts to changing the stiffness matrix $\mathbf{A}$ in (4.15) and (4.16). The matrix $\mathbf{A}$ needs to be assembled only once for each model and can be re-used for different payoffs. PIDE-based methods are hence well suited for the analysis of model risk.

As long as the characteristic function is known (see [114] for an overview) Fourier methods using the fast Fourier transformation provide a very efficient standardized approach in moderate dimensions. Except for driving processes with independent marginals or subordinated Brownian motion, however, the characteristic function is generally not known in Lévy models of dimension $d > 1$.

Implementation It is well known that in the Black–Scholes setting one advantage of the Monte Carlo method is its intuitive and rather simple implementation. In fact for the one-dimensional Black–Scholes model also Fourier, finite difference and classical finite element methods can be implemented by straightforward standard techniques. A great amount of fundamental literature is available in this case; see e.g. [2, 22, 115]. However, for (multidimensional) Lévy models more work is required for all methods. Even though the convergence rate of the Monte Carlo method is dimension-independent, its implementation in general requires special considerations for different techniques; see e.g. [117] for multidimensional simulation based on copulas.

As illustrated in the previous sections, all naive grid-based techniques such as standard Fourier and PIDE methods suffer from an exponential growth of complexity with the dimension. Applying sparse grids (cf. [29, 50]) or the so-called multigrid technique (cf. [17, 105]) to overcome this issue requires the implementation of a non-trivial data structure to handle the degrees of freedom efficiently. Furthermore, being the most general grid-based method, the implementation of wavelet-based finite elements requires additional handling of the compression techniques. For implementation details see [124].

Model sensitivities and Greeks Calculating price sensitivities (e.g. the Greeks) is a central modeling and computational task for risk management and hedging. We distinguish two classes: Sensitivities of the price V in (1.2) to variations of a model parameter, like the Greek Vega $\partial_\sigma V$; and sensitivities of V to variations of the state space such as the Greek Delta $\partial_S V$. Mesh-based methods are known to be well suited for the fast and accurate calculation of sensitivities whereas Monte Carlo methods are facing a certain challenge in this respect.

For PIDE-based methods, for instance, suppose the market model and hence the operator $\mathcal{A} = \mathcal{A}_{\mathrm{BS}} + \mathcal{A}_{\mathrm{J}}$ in (4.4) depends on some model parameter η. We want to calculate the sensitivity of the solution u of (4.4) with respect to η. To this end, write $u(\eta_0)$ for a fixed realization η_0 of η in order to emphasize the dependence of u on η_0 in (4.4). Then, as shown in [66], the derivative $\tilde{u}(\delta\eta)$ of u with respect to η, i.e., $\tilde{u}(\delta\eta) := \lim_{s\to 0^+} \frac{1}{s}(u(\eta_0 + s\delta\eta) - u(\eta_0))$, is the solution of the PIDE

$$\frac{\partial \tilde{u}(\delta\eta)}{\partial \tau} + \mathcal{A}(\eta_0)\tilde{u}(\delta\eta) = -D_\eta \mathcal{A} u(\eta_0), \quad \tilde{u}(\delta\eta)(0,\cdot) = 0 \quad \text{in } \mathbb{R}^d,$$

where $D_\eta \mathcal{A}$ is the derivative of $\mathcal{A}$ with respect to η. Therefore, the derivative of u with respect to η can be obtained as a solution of the same PIDE as the price u itself, where now the right hand side depends on u. Thus, sensitivities with respect to model parameters can be calculated with the same computational effort as the price itself. Furthermore, it is shown in [66] that all computed sensitivities converge with the same rate as the original price u. For sensitivities with respect to a variation of the state space, a finite difference-like differentiation procedure is presented in [66] which allows to obtain the sensitivities from the finite element forward price with the same convergence rate but without additional work.

Using a similar approach, also Fourier methods are capable of calculating sensitivities with respect to state space variation efficiently; see e.g. [4] for some numerical examples.

For Monte Carlo methods the computing time required for the calculation of sensitivities can be significantly greater than the time needed to calculate the prices themselves (to the same accuracy). For example, suppose one wants to estimate the Delta $\partial_S V(S)$. Then, one can compute a Monte Carlo estimator for $V(0, S+\delta)$ for some small perturbation δ. With this, the Delta $\partial_S V(S)$ may then be approximated by a forward finite difference estimator $(V(0, S+\delta) - V(0, S))/\delta$. In the Black–Scholes setting, it is proved in [60] that the best possible convergence rate for such an approximation is $N^{-1/4}$ if the simulations of the two estimators are drawn independently

(here N denotes the number of sample paths). By replacing the forward finite difference estimator with a central difference estimator this rate can be improved to $N^{-1/3}$. Furthermore, using common random numbers for both Monte Carlo estimators, one can obtain $N^{-1/2}$; see [59, 60, 77]. However, an important drawback of using common random numbers is that it may perform very poorly when the payoff is not sufficiently smooth, e.g. for digital options. For a more sophisticated method of calculating the Delta we refer to e.g. [57].

Also in the Black–Scholes setting, Malliavin calculus has become a popular tool for the Monte Carlo calculation of certain Greeks. This approach is based on the observation that most of the sensitivities of interest can be expressed as $\mathbb{E}[g(S_T)\pi]$, where π is a random variable depending on the sensitivity to be calculated. For details we refer to [45, 53, 54, 81] and the references therein. In case the probability density of S is known and differentiable, the likelihood ratio method (see e.g. [58]) may be used to replace the Malliavin calculus. See also [58, Sect. 7.2] for the pathwise differentiation technique.

Extension beyond Lévy models Recently, certain market models have been introduced where the driving process is still Markovian but not Lévy anymore. In [33] for instance, the underlying S is modeled by a time-inhomogeneous so-called Sato process. Furthermore, in certain stochastic volatility models such as the BNS model [13] the driving process is not stationary anymore.

Both time-inhomogeneity and non-stationarity can be handled easily by the PIDE-based methods. Non-stationarity can be treated by efficient evaluation of the entries of the stiffness matrices $\mathbf{A}$ in (4.10) and (4.14). Furthermore, if the stiffness matrix $\mathbf{A}$ in (4.10), (4.14) is time-dependent, then the time stepping scheme (4.16) can still be applied with $\mathbf{A} = \mathbf{A}(t)$ evaluated at each time step.

If the characteristic function of a non-stationary process is known explicitly (as e.g. for certain instances of the BNS model, see [114]), Fourier methods are applicable. Also if the process is time-inhomogeneous, Fourier methods can be applied as in [33].

Monte Carlo discretization can easily be extended to non-stationary or time-inhomogeneous processes, e.g. using suitable Markov chain approximations. With such a rather naive approach, however, the computational complexity grows significantly.

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