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Publication Date:
2017-11-01

Permanent Link:
https://doi.org/10.3929/ethz-b-000204152

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Linear/Quadratic Programming-Based Optimal Power Flow using Linear Power Flow and Absolute Loss Approximations

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Abstract—This paper presents a novel method to approximate the nonlinear AC optimal power flow (OPF) into tractable linear/ quadratic programming (LP/QP) based OPF problems that can be used for power system planning and operation. We derive a linear power flow approximation and include a convex reformulation of the power losses in the form of absolute value functions. We show three ways how we can incorporate this approximation into an LP/QP based OPF problem. The usefulness of our OPF methods is analyzed for the IEEE test case grids. As a result, the errors on voltage magnitudes and angles are reasonable, while obtaining near-optimal results. We find that our methods reduce significantly the computational complexity compared to the nonlinear AC-OPF making them a good choice for planning purposes.

Index Terms—Optimal Power Flow, Linear/ Quadratic Programming, Powerflow Approximation

I. INTRODUCTION

A. Motivation

OPTIMAL Power Flow (OPF) is indispensable for current research in power system operation and planning. OPF is widely used to find optimal expansion schemes [1], [2], [3] for transmission networks in planning problems. Furthermore, at operation level when OPF is incorporated as multi-period problems, OPF enables to find optimal generator setpoint schedules in unit commitment (UC) problems that minimize operational costs [4], [5], [6].

Especially for planning problems it is crucial to have tractable formulations of the multi-period OPF problem, since the incorporation of the nonlinear original OPF problem would impose a high computational burden. This is due to the intertemporal coupling of long investment horizons. Moreover, transmission planning methods need to incorporate a power flow approximation, since the nonlinear OPF cannot deal with binary placement constraints. Therefore, current planning methods include either the well-known lossless DC power flow approximation [7], a lossless approximation of the power flow in the full decision variable domain (active and reactive power, voltage magnitudes and angles) [3] or a power flow representation that does not operate in the full decision variable domain [1], [2]. This can result in near-optimal or infeasible solutions, when neither network losses nor the full solution space are considered.

Also UC problems require often binary decisions e.g. consideration of startup costs in the full decision variable space. This can only be achieved by mixed-integer programming (MIP) frameworks that are either able to incorporate linear or semidefinite programming (SDP) constraints. However, relaxing planning or UC problems into a second order cone (SOC) programming problem [8] or into an SDP problem [9] is still a complex optimization problem. Consequently, linear approximations are often the first choice to deal with the complexity issue. This also explains why UC problems are often divided into several stages [6] that reflect the binary decisions in the first stage, using a linear DC approximation, and then use the nonlinear OPF in the following stages at the cost of optimality, and/ or computation time.

In sum, there is still a clear need of linear OPF approximations that work in the full decision variable space and capture power losses. Hence, the objective of this paper is to find a linear and tractable approximation of the OPF problem in the full decision variable space of active/ reactive power and voltage magnitudes/ angles for universal grid topologies.

B. Related Work

Finding reasonable linear power flow approximations for OPF problems is not a new research field. The first approaches included linearizing the power flow equations and passing this information to an LP solver. However, since this approximation does not hold for the entire operating range, the LP problem needs to be solved in an iterative way. Several papers [10], [11], [12] have used this solution approach, where they build the Jacobian of the power flow equations at a given operating point. The work featured in [13], [14], [15] derives a linear approximation of the AC power flow equations, but does not show how these approximations can be incorporated into an OPF problem. Mhanna et al. approximate the second order cone constraints with linear constraints resulting in a high number of constraints [16]. Castillo et al. [17] use also an iterative approach to compute the optimal generator setpoints. The linear OPF method of [18] does not capture losses and
only operates in the decision variable domain of voltage angles and active power.

C. Contribution

The contribution of this paper is the development of a novel tractable Linear /Quadratic Programming (LP/QP) based OPF methods that approximates the power flow over the entire operating range and can be solved by off-the-shelf solvers. This means that we do not require any iterations to solve the OPF problem. Our problem links the full decision variable domain with linear power approximations and captures the power losses by using absolute value loss approximations. Due to the linear constraint formulations we can also account directly for binary decisions that are present in UC commitment problems. Our approach is universal to reflect any grid topology (meshed and radial) and any voltage levels (low voltage, distribution, and transmission grids). Unlike [16], we reduce the number of constraints.

The remainder of this paper is organized as follows. Section II derives the power flow approximation. Section III shows how this approximation can be included in an LP/QP based OPF problem. Section IV analyzes the accuracy and optimality of our suggested OPF method and Section V draws the conclusion.

II. LINEAR APPROXIMATION

We first derive the linear power flow approximations based on a 2-bus example and extend this result to capture tap ratios, shunt elements, and line charging. Then, we introduce nodal admittance matrices to reflect any grid topology and size and incorporate this representation in an optimal power flow problem that is compliant with an LP/QP framework.

Based on Fig. 1, the nodal active $p_1, p_2$ and reactive $q_1, q_2$ powers are given by the nonlinear AC power flow equations that are for this case

$$
\begin{align*}
\Delta p_1 &= v_1^2 g - v_1 v_2 \cos(\theta_1 - \theta_2) g - v_1 v_2 \sin(\theta_1 - \theta_2) b, \\
\Delta p_2 &= v_2^2 g - v_2 v_1 \cos(\theta_2 - \theta_1) g - v_1 v_2 \sin(\theta_2 - \theta_1) b, \\
\Delta q_1 &= -v_1^2 b + v_1 v_2 \cos(\theta_1 - \theta_2) b - v_1 v_2 \sin(\theta_1 - \theta_2) g, \\
\Delta q_2 &= -v_2^2 b + v_2 v_1 \cos(\theta_2 - \theta_1) b - v_2 v_1 \sin(\theta_2 - \theta_1) g,
\end{align*}
$$

where $v_1, v_2$ are the per unit nodal voltage magnitudes, $\theta_1, \theta_2$ are the voltage angles, $g$ is the per unit line conductance and $b$ the per unit line susceptance.

A. Absolute Loss Approximation

The incurred active $p_1$ and reactive $q_1$ power losses can be calculated by

$$
\begin{align*}
\Delta p_1 &= p_1 + p_2 = (v_1^2 + v_2^2) g - 2v_1 v_2 \cos(\theta_1 - \theta_2) g, \\
\Delta q_1 &= q_1 + q_2 = -(v_1^2 + v_2^2) b + 2v_1 v_2 \cos(\theta_1 - \theta_2) b.
\end{align*}
$$

Let $v_1, v_2 = 1$ then we can find an absolute power loss approximation as a function of the voltage angle difference for active and reactive power as follows

$$
\begin{align*}
\Delta p_1 &= 2(1 - \cos(\theta_1 - \theta_2)) g \approx |\theta_1 - \theta_2| 2K_1 g, \\
\Delta q_1 &= -2(1 - \cos(\theta_1 - \theta_2)) b \approx -|\theta_1 - \theta_2| 2K_1 b,
\end{align*}
$$

where $K_1$ is a constant that represents the gradient of the absolute function associated with the voltage angle difference. Here, we approximate $(1 - \cos(\theta_1 - \theta_2))$ with $K_1 |\theta_1 - \theta_2|$. If we let $\theta_1 - \theta_2 = 0$ then we obtain approximations in terms of absolute values that are dependent on the voltage magnitude difference:

$$
\begin{align*}
\Delta p_1 &= (v_1 - v_2)^2 g \approx |v_1 - v_2| 2K_2 g, \\
\Delta q_1 &= -(v_1 - v_2)^2 b \approx -|v_1 - v_2| 2K_2 b,
\end{align*}
$$

where $K_2$ is a constant to approximate the losses associated with the voltage magnitude difference. Here, we approximate $(v_1 - v_2)^2$ with $|v_1 - v_2| 2K_2$.

By superposing (4), (6) and superposing (5), (7), we approximate the active $P_1^\text{approx}$ and reactive power losses $Q_1^\text{approx}$ as follows:

$$
\begin{align*}
P_1^\text{approx} &= |\theta_1 - \theta_2| 2K_1 g + |v_1 - v_2| 2K_2 g, \\
Q_1^\text{approx} &= -|\theta_1 - \theta_2| 2K_1 b - |v_1 - v_2| 2K_2 b,
\end{align*}
$$

that are convex reformulations of the exact power losses (2) and (3).

B. Selection of $K_1, K_2$

We have two degrees of freedom to approximate the power losses with the constants $K_1, K_2$. To parametrize those parameters we define the design parameters $\Delta \theta_d = \theta_1 - \theta_2, \Delta v_d = v_1 - v_2$. They specify a usual voltage magnitude and angle difference between two nodes that are connected by a line. If we solve the equations (4), (5) for $K_1$ and (6), (7) for $K_2$, we obtain the following parametrizations for $K_1, K_2$

$$
\begin{align*}
K_1 &= \frac{1 - \cos \Delta \theta_d}{|\Delta \theta_d|} \approx \frac{|\Delta \theta_d|}{2}, \\
K_2 &= \frac{|\Delta v_d|}{2}.
\end{align*}
$$

We choose reasonable values for $\Delta \theta_d = 0.08 \text{ rad}, \Delta v_d = 0.02 \text{ p.u.}$ With this approach we normalize the losses with respect to the susceptances and conductances. In this way it is possible to consider universal grid topologies ranging from low voltage grids usually having a high R/X ratio to transmission grids possessing a high X/R ratio. Figures 2a and 2b show

\[\text{Fig. 1. Two-bus system to illustrate and derive the linear power flow and absolute loss approximations.}\]
the exact and the approximated power losses for the 2 bus transmission system (Fig. 1) with the per unit susceptance \( b = -10 \) and the per unit conductance \( g = 1 \). Note that this formulation is an approximation and not a relaxation, since there are loss regions that are underestimated above the values \( \Delta \theta_i \) and \( \Delta \phi_i \). This means that these errors translate to an underestimation of voltage angles and magnitudes. Hence, there is no guarantee that the approximated OPF solutions will lie inside the feasible original solution space.

C. Linear Power Flow Approximation

We linearize the nonlinear power flow equations (1) by using the following approximations \( \cos(\theta_i - \theta_j) \approx 1 \), \( v_i^2 - v_i v_j \approx v_i - v_j \), \( v_i v_j \sin(\theta_i - \theta_j) \approx (\theta_i - \theta_j) \) to obtain

\[
\begin{align*}
    p_1 & \approx (v_1 - v_2)g - (\theta_1 - \theta_2)b + p_1^\Delta + p_1^\Theta, \\
    p_2 & \approx (v_2 - v_1)g - (\theta_2 - \theta_1)b + p_2^\Delta + p_2^\Theta, \\
    q_1 & \approx -(v_1 - v_2)b - (\theta_1 - \theta_2)g + q_1^\Delta + q_1^\Theta, \\
    q_2 & \approx -(v_2 - v_1)b - (\theta_2 - \theta_1)g + q_2^\Delta + q_2^\Theta,
\end{align*}
\]

in which we also add the convex reformulations of the power losses \( p_1^\Delta \), \( q_1^\Delta \), \( p_1^\Theta \), \( q_1^\Theta \) derived from the previous Section II-A. As a result, the power flow approximations (12) are convex, which can also be graphically verified in Figures 3a and 3b.

D. Extension for Line Charging, Transformer Tap Ratios and Shunts

The aforementioned two-bus example considers only a series admittance \( y \). In this section, we aim to extend our approach to incorporate line charging, transformer tap ratios and shunt elements. To capture these features, we use the standard \( \pi \) branch model. The nodal admittance matrix \( Y_b \) for the two-bus system is then e.g.

\[
Y_b = \begin{bmatrix}
\frac{y + j \frac{b}{2}}{\tau} & \frac{-y}{\tau} \\
\frac{-y}{\tau} & \frac{y + j \frac{b}{2}}{\tau}
\end{bmatrix}
\begin{bmatrix}
y_{sh,1} & 0 \\
0 & y_{sh,2}
\end{bmatrix},
\]

where \( \tau \) is the per unit tap ratio, \( \theta_s \) is the transformer shift angle, \( b_c \) is the per unit capacitive reactance of the line, and \( y_{sh,1} \) are the per unit shunt admittances. We also define the matrix \( Y'_b \), in which we only consider the series admittance with the complex tap ratios as follows

\[
Y'_b = \begin{bmatrix}
y \frac{1}{\tau} & -y \frac{1}{\tau} \\
-y \frac{1}{\tau} & y \frac{1}{\tau}
\end{bmatrix}
\]
generators, and $n_l$ lines. Any topology can be specified by constructing the node-branch incidence matrix $C_{br} \in \mathbb{Z}^{n_l \times n_b}$. We introduce the generator active and reactive power injections $p_g, q_g \in \mathbb{R}^{n_g \times 1}$. The generator to bus mapping is specified with the matrix $C_g \in \mathbb{Z}^{n_g \times n_b}$. We now can extend $Y_b \in \mathbb{C}^{n_b \times n_b}, Y'_b \in \mathbb{C}^{n_b \times n_b}, \theta, v \in \mathbb{R}^{n_b \times 1}, p_1^\Delta, p_0^\Delta, q_0^\Delta, q_1^\Delta \in \mathbb{R}^{n_b \times 1}$ to reflect any grid topology. The nodal active and reactive power injections $p, q \in \mathbb{R}^{n_b \times 1}$ are split into

$$p = C_g p_g - p_d,$$

$$q = C_g q_g - q_d,$$

where $p_d, q_d \in \mathbb{R}^{n_b \times 1}$ are the active and reactive load vectors.

Under these definitions, we can find a more general matrix representation for the active power flow approximation as

$$\left[-\Im{\{Y'_b\}}\right] \mathbb{R}\{Y_b\} - C_g [C_{br}]^T [C_{br}]^T \begin{bmatrix} \theta \\ v \\ p_{\Delta q} \\ p_{\Delta v} \end{bmatrix} = -p_d,$$

and for the reactive power flow as

$$\left[-\Re{\{Y'_b\}} - \Im{\{Y_b\}}\right] - C_g [C_{br}]^T [C_{br}]^T \begin{bmatrix} \theta \\ v \\ q_{\Delta q} \\ q_{\Delta v} \end{bmatrix} = -q_d.$$

### V. Approximated Tractable OPF Problems

#### A. Lossy LP/QP based OPF Problem

With the introduced power flow approximations we can now formulate the OPF problem within a standard LP/QP framework. We specify the decision vector $x = [\theta \ v \ p_g \ q_g \ p_\Delta^\theta \ p_\Delta^v \ q_\Delta^\theta \ q_\Delta^v]^T$. The objective of the OPF problem is to find the optimal active and reactive generator powers that minimize either a linear or quadratic cost objective. The approximated lossy LP/QP based Optimal Power Flow (LOLIN-OPF) problem is

$$\text{LOLIN-OPF:} \min_{x} f_p(p_g) + f_q(q_g) \quad \text{s.t. (17), (18)}$$

(a) $k_1 \text{diag} \{g\} C_{br} \theta - p_{\Delta q}^\theta \leq 0$
(b) $-k_1 \text{diag} \{g\} C_{br} \theta - p_{\Delta v}^\theta \leq 0$
(c) $k_2 \text{diag} \{g\} C_{br} v - p_{\Delta q}^v \leq 0$
(d) $-k_2 \text{diag} \{g\} C_{br} v - p_{\Delta v}^v \leq 0$
(e) $k_3 \text{diag} \{b\} C_{br} \theta - q_{\Delta q}^\theta \leq 0$
(f) $-k_3 \text{diag} \{b\} C_{br} \theta - q_{\Delta v}^\theta \leq 0$

### III. Approximated Tractable Optimal Power Flow Problems

In this section we derive the formulations of the approximated OPF problem. First, we present a lossy LP/QP based OPF problem that incorporates the power flow approximations as linear constraints. Hence, we call this method LOLIN-OPF. Secondly, we also define a lossless version of the OPF problem that we call LIN-OPF. Thirdly, we also provide a mixed integer formulation of the problem, where we include the power flow approximations as linear constraints with binary decision variables. This method we call MIP-OPF.
Then, the Mixed Integer LP/QP based Optimal Power Flow (MIP-OPF) problem is

\[
\begin{align*}
\text{MIP-OPF:} \quad & \min_{x'} f_p(p_g) + f_q(q_g) \\
\text{s.t.} \quad & (23), (21i-l) \\
& (a) -M(1 - b^v) \leq -C_{br} \theta \leq M b^v \\
& (b) \quad 0 \leq -C_{br} \theta + \Delta \theta \leq 2 M b^v \\
& (c) \quad 0 \leq C_{br} \theta + \Delta \theta \leq 2 M(1 - b^v) \\
& (d) \quad -M(1 - b^v) \leq -C_{br} v \leq M b^v \\
& (e) \quad 0 \leq -C_{br} v + \Delta v \leq 2 M b^v \\
& (f) \quad 0 \leq C_{br} v + \Delta v \leq 2 M(1 - b^v), \\
\end{align*}
\]

where the constraints (24a-f) specify a big M formulation of the absolute value functions in (8) and (9). The variable \( M \) has a considerable influence on the feasibility of the problem. It needs to be chosen sufficiently large to approximate the real (practical) range of the absolute values. Too large values might result in weak relaxations leading to branching the problem and hence to an increased computation time.

IV. RESULTS

In this section we aim to show the performance of our suggested OPF methods compared to existing OPF methods.

A. Implementation

We implemented the suggested OPF methods within the MATPOWER framework [19] and use the GUROBI [20] solver for solving the LP/QP, MILP/MIQP problems. We test our methods based on the IEEE reference grid cases that are bundled within MATPOWER. Since all test cases comprise quadratic generator cost functions, we solve for the LIN-OPF/LOLIN-OPF methods a QP problem and for the MIP-OPF method an MIQP problem.

B. Voltage Angle and Magnitude Errors

First, we aim to analyze the accuracy of our methods. Here, we compare the approximated OPF solution of voltage magnitudes and angles of our methods with the exact power flow solution that calculates the true voltage magnitudes and angles. For the comparison, we set the PV buses in the power flow according to the OPF solution and compute the root mean square (RMS) errors on the magnitude and angle deviations. Figures 5a and 5b show the angles and magnitudes for the LOLIN-OPF method and for the power flow (PF) program for the IEEE 118 test grid. Although there is a small deviation in voltage angles, it can be observed that the curves match well. For the comparison, we set the PV buses in the power flow solution that calculates the true voltage magnitudes and angles. For the comparison, we set the PV buses in the power flow according to the OPF solution and compute the root mean square (RMS) errors on the magnitude and angle deviations. Figures 5a and 5b show the angles and magnitudes for the LOLIN-OPF method and for the power flow (PF) program for the IEEE 118 test grid. Although there is a small deviation in voltage angles, it can be observed that the curves match well.
find a MIP-OPF solution for the IEEE 300 test case, since the generators reactive power capability margin was not sufficient to cover the overestimated reactive power losses.

We observe a higher angle error ($\approx 20^\circ$) for the LIN-OPF for the IEEE 300 test case. This can be explained in that the LIN-OPF version translates the underestimated active power generation setpoints to higher errors in voltage angles. Note that the DC-OPF solution would also generate such error. However, for the LOLIN-OPF and MIP-OPF methods the errors are reasonable and can be accepted.

The error patterns follow the same ordering from low errors (MIP-OPF) to higher errors (LIN-OPF) except for the cases IEEE 30 and IEEE 39. The different orderings of these special cases can be explained by higher voltage magnitude differences that lead to higher approximation errors for the LOLIN-OPF and MIP-OPF.

C. Optimality

Another fact that needs to be discussed is the optimality of our suggested OPF methods. We define optimality as the objective value deviation between the introduced OPF methods and the nonlinear AC-OPF. From Fig. 7 we find that the objective values are higher for our OPF methods (LOLIN-OPF, MIP-OPF), since those methods overestimate the power losses in the grid to some extent. However, the solutions do not deviate more than 4% from the AC-OPF solution. Furthermore, the DC-OPF and LIN-OPF solutions result in the same values and have lower objective values compared to the nonlinear OPF solution. This is due to the fact that these methods do not incorporate power losses. In contrast to the DC-OPF, the LIN-OPF also includes a voltage projection. The errors introduced by MIP-OPF and LOLIN-OPF are almost identical. We can see that on average, the absolute deviation for the LOLIN-OPF (1.3%) and the MIP-OPF (1.6%) is lower than for the DC-OPF and LIN-OPF (2.5%).

D. Complexity

To assess the computational complexity of our method, we solve the same multi-period storage siting problem as proposed in [21]. Here, we replace the linear Forward-Backward Sweep (FBS) OPF with our proposed LOLIN-OPF and the nonlinear AC-OPF. Note that the LOLIN-OPF is a more general OPF method than the FBS-OPF, since it can be used for transmission and meshed networks. We extend the decision vector $X = [x(1), \ldots, x(N-1)]^T$ to reflect multiple timesteps within the investment horizon $N$. The main objective of the problem is to find the battery locations and sizes
we achieve an improvement in computation time with the introduction of the nonlinear AC-OPF is almost shifted in parallel problem (25) with IPOPT and pardiso [22]. The computation for comparison, we solve the nonlinear AC-OPF version of computing cluster using two 12-core Intel Xeon E5 processors. 

Penetration. More information on the full scenario definition and parameters can be found in [21].

PV curtailment is represented as financial transactions associated with the import grid and the generation units. In this example, the cost function that minimizes PV curtailment. The general problem can be represented as

$$J^* = \min_{X, z} \left\{ T \sum_{k=0}^{N-1} c_k^T p_k(k) + c_s^T z \right\}$$

s.t. 
(a) \( \forall x : \ (21a) - (21i) \) 
(b) \( A_s \begin{bmatrix} X \\ z \end{bmatrix} \leq b_s \) 
(c) \( X_{\text{min}} \leq X \leq X_{\text{max}} \) 

where \( z \) are the battery capacities, \( c_s \) are the storage prices and \( c_k(k) \) are the linear generator costs. The constraint (25a) incorporates \( N \) single-shot OPF problems, which are coupled across different time instants via (25b). The matrix \( A_s \) and vector \( b_s \) reflect the storage coupling and are specified in [21]. The constraint (25c) determines the operating bounds of the grid and the generation units. In this example, the cost function represents the financial transactions associated with the import and export of energy at the feeder node. PV curtailment is mitigated and storage is installed when the storage prices are low and the price spreads of importing electricity are rather high. The grid has 18 buses and is configured with a high PV penetration. More information on the full scenario definition and parameters can be found in [21].

Problem (25) is a large-scale LP problem in which the effective grid size grows as \( n_1N \) with \( n_1 = 18 \) and the decision variable size as \( N(7n_1 + 2n_1) \) with \( n_1 = 17 \). The LP problem is solved with gurobi [20] on a high-performance computing cluster using two 12-core Intel Xeon E5 processors. For comparison, we solve the nonlinear AC-OPF version of problem (25) with IPOPT and pardiso [22]. The computation time results for both versions are shown in Fig. 8, where the investment horizon corresponds to hours. The lin-log regression of the nonlinear AC-OPF is almost shifted in parallel towards higher computation times. This means that on average we achieve an improvement in computation time with the LOLIN-OPF method of one order of magnitude with respect to the nonlinear AC-OPF.

Fig. 8. Computation time comparison between LOLIN-OPF and nonlinear AC-OPF and their corresponding lin-log regressions.

V. CONCLUSION

In this paper we presented novel tractable OPF methods that work in the full decision domain of active/ reactive power and voltage magnitudes/ angles. We linearly approximate the power flow over the entire operating range avoiding the need to iterate the OPF problem. Our OPF methods can be used by efficient off-the-shelf LP/QP solvers. The obtained accuracy in terms of voltage magnitudes and angles is reasonable and we achieve near-optimal solutions compared to the nonlinear AC-OPF. In a planning problem, we have shown that the LP/QP based methods outperform the nonlinear AC-OPF in terms of computational complexity.

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