Doctoral Thesis

Wavelet Galerkin schemes for option pricing in multidimensional Lévy models

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Wavelet Galerkin schemes for option pricing in multidimensional Lévy models

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Christoph Winter
Abstract

We consider a wavelet Galerkin scheme for solving partial integrodifferential equations where sparse tensor product spaces are applied for the discretization to reduce the complexity in the number of degrees of freedom. The resulting matrices are dense, since the jump operator is non-local. Therefore, wavelet compression methods are used to decrease the number of non-zero matrix entries.

We focus on algorithmic details of the scheme, in particular on the numerical integration of the matrix coefficients. Since the multidimensional Lévy densities have singularities at the origin and on the axes, variable order composite Gauss quadrature formulas are employed. We show that the quadrature rule leads to exponential convergence for Lévy densities which are piecewise analytic. Using an hierarchical data structure, an adaptive numerical scheme is developed which computes each matrix entry with a given accuracy. The accuracy is chosen by an a-priori numerical analysis of the scheme such that the solution of the perturbed problem still converges at the optimal rate.

We give numerical examples. In particular, the regularization of the multidimensional Lévy measure is considered where small jumps are either neglected or approximated by an artificial Brownian motion. We study and compare the impact of these approximations on various financial contracts in multidimensional Lévy market models.
Zusammenfassung


Wir geben numerische Bespiele. Im Besonderen wird die Regularisierung des mehrdimensionalen Lévy-Maßes betrachtet, bei der kleine Sprünge vernachlässigt oder durch eine künstliche Brownsche Bewegung approximiert werden. Wir untersuchen und vergleichen den Einfluss dieser Näherungen auf verschiedene Finanzderivate in mehrdimensionalen Lévy-Marktmodellen.
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Introduction

Over the last years financial models with jumps, and especially Lévy models, have increased tremendously in popularity. By now it is well established that Lévy models are more suitable for capturing market fluctuations than the classical Black-Scholes model [6], see, e.g., Cont and Tankov [16] and Schoutens [61] for an overview and empirical justification. The number of financial models with jumps is growing steadily, for the most popular as well as some recent examples we refer to [2, 9, 10, 26, 38, 41, 42, 61, 65]. However, even in the Black-Scholes setting, closed form solutions to derivative pricing problems are often unavailable or not easily computable. Furthermore, in models with jumps one cannot construct analytic solutions generally, not even for the pricing of plain European vanilla options. Therefore, numerical methods for option pricing have been analyzed by many authors and several techniques have been developed to obtain efficient pricing algorithms. Especially in models with jumps, numerical challenges occur which have given rise to a number of innovative numerical tools. In this work we shall focus on the so-called wavelet method for asset pricing in multidimensional Lévy models.

Consider a basket of $d \geq 1$ risky assets whose log returns $X_t = (X_t^1, \ldots, X_t^d)^\top \in \mathbb{R}^d$ at time $t > 0$ are modeled by a Lévy process $X = \{X_t : t \geq 0\}$ with state space $\mathbb{R}^d$. By the fundamental theorem of asset pricing (see, e.g., [24]), arbitrage free prices $u$ of European contingent claims on such baskets with “reasonable” payoffs $g(\cdot)$ and maturity $T$ are given by the conditional expectation

$$u(t, x) = \mathbb{E}\left( e^{-r(T-t)} g(X_T) \mid X_t = x \right). \quad (0.1)$$

Here, the expectation is taken with respect to an a-priori chosen martingale measure equivalent to the historical measure (see, e.g., [23, 25] for some measure selection criteria).

As it is well-known [34, Chapter 4], the family $\{T_t\}_{t \geq 0}$ of maps $T_t : g(\cdot) \to u(t, \cdot)$ is a one-parameter semigroup. We denote by $\mathcal{A}$ its associated infinitesimal generator, i.e.,

$$\mathcal{A}u := \lim_{t \to 0^+} \frac{1}{t} (T_t u - u), \quad (0.2)$$

for all functions $u \in D(\mathcal{A})$ in the domain

$$D(\mathcal{A}) := \left\{ u \in C_{\infty}(\mathbb{R}^d) : \lim_{t \to 0^+} \frac{1}{t} (T_t u - u) \text{ exists as strong limit} \right\},$$
where $C_\infty(\mathbb{R}^d)$ is the space of continuous functions vanishing at infinity (see, e.g., [34]). Sufficiently smooth value functions $u$ in (0.1) can be obtained as solutions of a partial integrodifferential equation (PIDE), the Kolmogorov equation
\begin{equation}
\partial_t u + \mathcal{A} u - ru = 0,
\end{equation}
where $\mathcal{A}$ is the infinitesimal generator of the process $X$ defined by (0.2). Among several possible notions of solution (classical, variational and viscosity solutions, to name the most frequently employed), we opt for variational solutions which are the basis for variational discretization methods such as finite element discretizations. To convert (0.3) into variational form, we formally integrate against a test function $v$ and obtain (assuming $r = 0$ for convenience)
\begin{equation}
(\partial_t u, v) + \mathcal{E}(u; v) = 0.
\end{equation}
Here, the bilinear expression $\mathcal{E}(u, v)$ denotes the extension of the $L^2(\mathbb{R}^d)$ inner product $(\mathcal{A} u, v)$ corresponding to $X$ from $u, v \in C_0^\infty(\mathbb{R}^d)$ by continuity to the domain $\mathcal{D}(\mathcal{E})$. For the class of Lévy processes considered, we show in this paper that $\mathcal{E}(\cdot, \cdot)$ is in fact a Dirichlet form.

In the univariate case, i.e., for a Lévy process $X$ with state space $\mathbb{R}$, equations (0.3), (0.4) and methods for their numerical solution have been studied by several authors, see, e.g., [8, 17, 43, 45] and the references therein. The numerical methods investigated were either finite difference methods [8, 17] approximating viscosity solutions or variational methods [43, 45] approximating weak (or variational) solutions. Both solution concepts coincide for sufficiently smooth solutions, but the resulting numerical schemes have essentially different properties. In [28], the univariate variational setting was extended to $d > 1$ dimensions for pure jump processes built from 1-homogeneous Lévy copulas and univariate marginal Lévy processes with symmetric tempered stable margins. The domain of the infinitesimal generator $\mathcal{A}$ was characterized and it was shown that the corresponding variational problem is well-posed. The multivariate, nonsymmetric case was studied in [54], i.e., when the univariate marginal Lévy processes are tempered stable, but with possibly nonsymmetric margins. Furthermore, analytical results were provided which are required for an efficient numerical implementation of (0.4). Under these models option pricing using Fourier methods as in [11] is generally not possible since the characteristic functions are not given in closed form.

Following [28, 54] we consider a wavelet Galerkin scheme where sparse tensor product spaces are applied for the discretization to reduce the complexity in the number of degrees of freedom from $\mathcal{O}(h^{-d})$ to $\mathcal{O}(h^{-1} \log h^{d-1})$. Here, $h$ denotes the mesh width of the finite element discretization. The resulting matrices are dense since the jump operator is non-local. Therefore, wavelet compression methods are used to reduce the number of non-zero matrix entries. We focus on algorithmic details of the scheme, in particular on the numerical integration of the matrix coefficients. Since the multidimensional Lévy densities have singularities at the origin and on the axes, variable order composite Gauss quadrature formulas are employed.
The outline is as follows. We first introduce some notation in Chapter 1 and state the function spaces which are used in this work. The basic finite element method is briefly explained. In Chapter 2 we recall essential definitions of Lévy processes and Lévy copulas. Several examples of multivariate Lévy models are given and important properties of the Lévy measure are proved, in particular the so-called sector condition. In Chapter 3 we derive the partial integrodifferential equation corresponding to the option pricing problem. We show that the variational formulation has a unique solution in an anisotropic Sobolev space. Furthermore, the unbounded log-price domain is localized to a bounded domain and an error bound for the localization error is obtained. In Chapter 4 wavelets are explained and the localized problem is discretized in log price domain on a sparse tensor product space. A wavelet compression strategy for the resulting stiffness matrix is given which yields essentially the optimal complexity of non-zero matrix entries. Additionally, a multilevel preconditioner is presented. In Chapter 5 a composite quadrature is derived which combines elementary Gauss quadrature formulas on subdomains decreasing geometrically towards the singular support of the Lévy measure. We show that the quadrature rule leads to exponential convergence for Lévy densities which are piecewise analytic. The computational scheme is explained in Chapter 6. Using a hierarchical data structure, an adaptive numerical scheme is developed which computes each matrix entry with a given accuracy. The accuracy is chosen by an a-priori numerical analysis of the scheme such that the solution of the perturbed problem still converges at the optimal rate. In Chapter 7 we show how to compute model sensitivities, in particular the so-called Greeks. The solution of the sensitivity problem has the same convergence rate as the solution of the initial option pricing problem. Finally, in Chapter 8 numerical examples are given. In particular, the regularization of the multidimensional Lévy measure $\nu$ is considered where small jumps are either neglected or approximated by an artificial Brownian motion. We study and compare the impact of these approximations on various financial contracts in multidimensional Lévy market models.
Introduction
1 Preliminaries

We first set some notational conventions to limit subsequent interruptions. Since the numerical analysis requires tools from functional analysis, in particular Sobolev spaces, we also state several function spaces which are used throughout this work. Furthermore, we briefly explain the basic finite element method for parabolic partial differential equations. Here, the choice of the basis functions is crucial for efficient computation. It is shown in Chapter 4 that choosing a wavelet basis provides several advantages.

1.1 Notation

Let $D$ be a non-empty open subset of $\mathbb{R}^d$. If a function $u : D \to \mathbb{R}$ is sufficiently smooth, we denote the partial derivatives of $u$ by $\partial^n u = \partial_1^{n_1} \cdots \partial_d^{n_d} u$, where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ is a multiindex. The order of the partial derivative is given by $|\mathbf{n}| = \sum_{i=1}^d n_i$. The Euclidean norm of $x \in \mathbb{R}^d$ is denoted by $|x|$ and the Borel $\sigma$-algebra of $\mathbb{R}^d$ is given by $\mathcal{B}(\mathbb{R}^d)$. We set $\bar{\mathbb{R}} = (-\infty, \infty]$ and write $a \leq b$ for $a, b \in \mathbb{R}^d$ if $a_j \leq b_j$, $j = 1, \ldots, d$. In this case $(a, b]$ is a half-open interval $(a, b] = (a_1, b_1] \times \cdots \times (a_d, b_d]$ and correspondingly for the closed interval $[a, b]$. Throughout, we write $x \lesssim y$ to express that the scalar $x$ is bounded by a constant multiple of $y$, i.e., there exists a $c > 0$ such that $x \leq cy$. Correspondingly $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. We denote the indicator function of the set $B \subseteq \mathbb{R}^d$ by $1_B : \mathbb{R}^d \to \{0, 1\}$.

For a non-empty set $I \subseteq \{1, \ldots, d\}$ we define its complement by $I^c = \{1, \ldots, d\} \setminus I$. We set $x^I = (x_i)_{i \in I}$ and use the notation

$$x + y^I = z \in \mathbb{R}^d \quad \text{with} \quad z_i = \begin{cases} x_i & \text{if } i \notin I \\ x_i + y_i & \text{else} \end{cases}$$

for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^{|I|}$. Furthermore, $\partial^I = \partial_{i_1} \cdots \partial_{i_k}$ for $I = \{i_1, \ldots, i_k\}$, $k \in \{1, \ldots, d\}$.

A tensor product for matrices is given by the Kronecker product. For matrices $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{l \times k}$ or vectors $y \in \mathbb{R}^m$, $z \in \mathbb{R}^l$ their Kronecker product $A = B \otimes C \in \mathbb{R}^{ml \times nk}$, $x = y \otimes z \in \mathbb{R}^{ml}$ is defined by (see [29, Section 4.5.5])

$$A = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} y_1 z_1 \\ y_2 z_2 \\ \vdots \\ y_m z_m \end{bmatrix}$$
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To simplify notation we also define the vector multiplication for \( z, y \in \mathbb{R}^n \) by \( x = y \cdot z \in \mathbb{R}^n \) where \( x_j = y_j z_j, \ j = 1, \ldots, n \).

1.2 Function spaces

The numerical analysis and the variational formulation (0.4) require tools from functional analysis, in particular Sobolev spaces. For any integer \( n \in \mathbb{N} \) we define 

\[
C^n(D) = \{ u : \partial^n u \text{ exists and is continuous on } D \text{ for } |n| \leq n \}
\]

and set \( C^\infty(D) = \bigcap_{n \geq 0} C^n(D) \). The support of \( u \) is denoted by \( \text{supp} \ u \) and we define \( C_0^n(D), C_0^\infty(D) \) consisting of all functions \( u \in C^n(D), C^\infty(D) \) with compact support \( \text{supp} \ u \subset D \).

We denote by \( H^n(\mathbb{R}^d), n \in \mathbb{N} \) the usual Sobolev consisting of all functions in \( L^2(\mathbb{R}^d) \) with partial derivatives up to order \( n \) in \( L^2(\mathbb{R}^d) \) and norm

\[
\|u\|_{H^n(\mathbb{R}^d)}^2 := \sum_{|n| \leq n} \|\partial^n u\|_{L^2(\mathbb{R}^d)}^2.
\]

These spaces can naturally be extended to isotropic Sobolev spaces \( H^s(\mathbb{R}^d) \) for non-integer \( s \geq 0 \) as the spaces of all \( \mathcal{S}^s \)-functions with finite norm

\[
\|u\|_{H^s(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2 \, d\xi,
\]

where \( \hat{u} \) is the Fourier transform of \( u \). Similarly, we can define anisotropic Sobolev spaces \( H^s(\mathbb{R}^d) \) with norm

\[
\|u\|_{H^s(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} \prod_{j=1}^d (1 + \xi_j^{s_j})^{s_j} |\hat{u}(\xi)|^2 \, d\xi, \tag{1.1}
\]

for any multiindex \( s \geq 0 \). It is useful to notice that by [50, Section 9.2] the anisotropic Sobolev spaces admit an intersection structure and we have

\[
H^s(\mathbb{R}^d) = \bigcap_{j=1}^d H^{s_j}(\mathbb{R}^d) \quad \text{and} \quad \|u\|_{H^s(\mathbb{R}^d)}^2 \sim \sum_{j=1}^d \|u\|_{H^{s_j}}^2, \tag{1.2}
\]

with

\[
\|u\|_{H^{s_j}(\mathbb{R}^d)} = \left\| (1 + \xi_j^{s_j})^{s_j/2} \hat{u} \right\|_{L^2(\mathbb{R}^d)}.
\]

Furthermore, we have a mixed Sobolev space \( H^n(\mathbb{R}^d) = H^{n_1}(\mathbb{R}) \otimes \cdots \otimes H^{n_d}(\mathbb{R}) \), \( n \in \mathbb{N}^d \) with the norm

\[
\|u\|_{H^n(\mathbb{R}^d)}^2 := \sum_{0 \leq m \leq n} \|\partial^m u\|_{L^2(\mathbb{R}^d)}^2, \tag{1.3}
\]
For any $s \geq 0$ we define $H^s(\mathbb{R}^d)$ by interpolation. Finally, we define the space
\[ \tilde{H}^s(D) := \{ u|_D : u \in H^s(\mathbb{R}^d), u|_{\mathbb{R}^d \setminus D} = 0 \}. \] (1.4)
For $s_j + 1/2 \notin \mathbb{N}, j = 1, \ldots, d$, the space $\tilde{H}^s(D)$ coincides with $H^s_0(D)$, the closure of $C^\infty(\mathbb{R}^d)$ with respect to the norm of $H^s(\mathbb{R}^d)$ [46, Theorem 3.33].

### 1.3 Finite element method

We briefly explain the basic finite element method for parabolic partial differential equations where we opt for variational solutions. Therefore, we first state the abstract variational formulation and then describe the space and time discretization. Convergence rates for the finite element approximation are also given. For more details we refer to the monographs [27, 66].

#### 1.3.1 Variational formulation

Let $V \subset H$ be two Hilbert spaces with continuous, dense embedding. We identify $H$ with its dual $H^*$ and obtain the triplet
\[ V \subset H \equiv H^* \subset V^*. \] (1.5)
In this Gelfand triplet we consider a bilinear mapping $E : V \times V \to \mathbb{R}$. For $f \in L^2((0,T);V^*)$ and $u_0 \in H$, consider the following abstract problem:
\[
\begin{aligned}
\text{Find } u \in L^2((0,T);V) \cap H^1((0,T);V^*) \text{ such that } \\
\langle \partial_t u, v \rangle_{V^*, V} + E(u, v) = \langle f, v \rangle_{V^*, V}, \forall v \in V, \text{ a.e. in } (0,T), \\
u(0) &= u_0.
\end{aligned}
\] (1.6)

The bilinear form $E(\cdot, \cdot)$ can also be a Dirichlet form with domain $D(E)$.

**Definition 1.3.1.** Let $(E, D(E))$ be a closed form on $L^2(\mathbb{R}^d)$, i.e., $D(E)$ is a Hilbert space with inner product $E_1^{\text{sym}}(\cdot, \cdot) := E^{\text{sym}}(\cdot, \cdot) + \langle \cdot, \cdot \rangle$ and $E(\cdot, \cdot)$ is continuous with respect to $E_1^{\text{sym}}(\cdot, \cdot)$, i.e.,
\[ |E(u, v)| \lesssim (E_1^{\text{sym}}(u, u))^{1/2}(E_1^{\text{sym}}(v, v))^{1/2}, \forall u, v \in D(E). \]

The form $(E, D(E))$ is called a Dirichlet form if for all $u \in D(E)$ it follows that $u^+ \wedge 1 \in D(E)$ and
\[
\begin{aligned}
E(u + (u^+ \wedge 1), u - (u^+ \wedge 1)) &\geq 0, \\
E(u - (u^+ \wedge 1), u + (u^+ \wedge 1)) &\geq 0.
\end{aligned}
\]
Here $u^+ := u \vee 0$ denotes the positive part of the function $u$. 

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The following theorem provides criteria for the existence and uniqueness of the solution $u$ of (1.6).

**Theorem 1.3.2.** Assume the bilinear form $E(\cdot, \cdot)$ satisfies the following properties. There exist some constants $C_1, C_2 > 0$ and $C_3 \geq 0$ such that for all $u, v \in V$ there holds

\[
|E(u, v)| \leq C_1 \|u\|_V \|v\|_V, \quad (1.7)
\]
\[
E(u, u) \geq C_2 \|u\|_V^2 - C_3 \|u\|_H^2. \quad (1.8)
\]

Then, the abstract parabolic problem (1.6) admits a unique solution.

**Proof.** See, e.g., [40, Theorem 4.1]. 

**Remark 1.3.3.** It is always possible to modify (1.8) such that $C_3 = 0$ by setting $\tilde{u} = e^{-C_3 t} u$. Therefore, we assume throughout $E$ is coercive, i.e., $C_3 = 0$.

For the discretization we use the method of lines where first (1.6) is only discretized in space to obtain a system of coupled ordinary differential equations (ODEs). Second, a time stepping scheme is applied to solve the ODEs.

1.3.2 Space discretization

Let $V_h$ be a one-parameter family of subspaces $V_h \subset V$ with finite dimension $N_h = \dim V_h < \infty$. For each fixed $t \in (0, T)$ we approximate the solution $u(t, x)$ of (1.6) by a function $u_h(t) \in V_h$. Furthermore, let $u_{h,0} \in V_h$ be an approximation of $u_0$. Then, the semidiscrete form of (1.6) is the initial value problem,

\[
\begin{align*}
\dot{u}_h &\in C^1([0, T]; V_h) \text{ such that}\\
&\langle \partial_t u_h, v_h \rangle_H + E(u_h, v_h) = \langle f, v_h \rangle_{V^*, V}, \quad \forall v_h \in V_h, \; t \in (0, T),
\end{align*}
\]

(1.9)

for the approximate solution function $u_h(t) : [0, T] \to V_h$. Let $V_h$ be generated by a finite element basis $\Phi_h := \{ \phi_{h,k} : k \in \Delta_h \}$ with index set $\Delta_h = \{1, \ldots, N_h\}$. Efficient computation depends on the choice of the basis functions $\phi_{h,k}$. If the operator $A$ is local, we can use, e.g., the so-called hat functions as explained in Examples 1.3.4 and 1.3.5. For non-local operators, wavelets, which are explained in Chapter 4, provide several advantages.

**Example 1.3.4.** Let $D = [0, 1]$ and consider $V = \tilde{H}^1([0, 1])$. The interval is partitioned into a uniform mesh $T_h$ with mesh width $h$. The uniform mesh $T_h$ is then given by the equidistant nodes $x_{h,k} := kh$ with $k = 0, \ldots, N_h + 1$, where $N_h = 1/h - 1$. We define $V_h$ as the space of piecewise polynomials of degree $p - 1 \in \mathbb{N}$ on the mesh $T_h$ which vanish at the endpoints. For piecewise linear continuous functions, $p = 2$, $V_h$ is generated by the so-called hat functions $\hat{\phi}_{h,k} = \max(1 - |x - x_{h,k}| / h, 0)$, $k = 1, \ldots, N_h$. 

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1.3.3 Matrix formulation

We write $u_h \in V_h$ in terms of the basis functions of $\Phi_h$, $u_h(t, x) = \sum_{k \in \Delta_h} u_{h,k}(t) \phi_{h,k}(x)$, and obtain the matrix form of the semidiscretization (1.9)

\[
\text{Find } \underline{u}_h \in C^1([0,T]; \mathbb{R}^{N_h}) \text{ such that } \\
M \frac{\partial}{\partial t} \underline{u}_h(t) + A \underline{u}_h(t) = f(t), \quad t \in (0,T), \quad (1.10)
\]

where $\underline{u}_{h,0}$ denotes the coefficient vector of $u_{h,0}$, $f(t) = (f_k(t))$ the entries $f_k(t) = (f, \phi_{h,k})$ and $M = (M_{k',k})$, $A = (A_{k',k})$ the mass- and stiffness matrix with respect to $\Phi_h$,

\[
M_{k',k} = \langle \phi_{h,k}, \phi_{h,k'} \rangle, \quad A_{k',k} = \mathcal{E}(\phi_{h,k}, \phi_{h,k'}), \quad (1.11)
\]

with $k, k' \in \Delta_h$.

1.3.4 Time discretization

There are various time-stepping methods to approximate the solution of the ODEs (1.10). Here, we only use the $\theta$-scheme. One can also apply finite elements for the time discretization as in [32, 60] where an $hp$-discontinuous Galerkin method is used. It yields exponential convergence rates instead of only algebraic ones as in the $\theta$-scheme.

We consider an uniform grid with time step $\Delta t = T/M$ and time points $t_m = m \Delta t, m = 0, \ldots, M$, $M \in \mathbb{N}$. Applying the $\theta$-scheme we obtain the fully discrete form

\[
\text{Find } u_h^m \in V_h \text{ such that for } m = 0, \ldots, M - 1 \\
\langle \Delta t^{-1}(u_h^{m+1} - u_h^m), v_h \rangle_H + \mathcal{E}(u_h^{m+\theta}, v_h) = \langle f^{m+\theta}, v_h \rangle_{V', V}, \forall v_h \in V_h, \quad (1.12)
\]

\[
u_h^0(0) = u_{h,0},
\]

where $u_h^{m+\theta} = \theta u_h(t_{m+1}) + (1 - \theta) u_h(t_m)$ and $f^{m+\theta} = \theta f(t_{m+1}) + (1 - \theta) f(t_m)$.
We can again write (1.12) in matrix notation
\begin{equation}
\begin{aligned}
&\text{Find } u^{m+1}_h \in \mathbb{R}^{N_h} \text{ such that for } m = 0, \ldots, M - 1 \\
&\Delta t^{-1} M (u^{m+1}_h - u^m_h) + \theta A u^{m+1}_h + (1 - \theta) A u^m_h = f^{m+\theta}, \\
&u^0_h = u_{h,0}.
\end{aligned}
\end{equation}
(1.13)

For \( \theta = 1/2 \), the scheme in (1.12), (1.13) coincides with the popular Crank-Nicholson scheme.

### 1.3.5 Convergence rates

We want to determine the accuracy of the approximation (1.9) and (1.12). Toward this end, let \( D \subset \mathbb{R}^d \) be bounded with Lipschitz boundary \( \partial D \) and assume \( \mathcal{V} = \tilde{H}^r(D) \), \( r \geq 0 \) and \( H = L^2(D) \). Let \( T_h \) be a regular conforming partition of \( D \) into simplices with uniform mesh width \( h \) and let \( V_h \subset \mathcal{V} \) with approximation order \( p \), i.e., containing all polynomials of degree \( p - 1 \). It is well-known in the classical finite element approximation theory (see, e.g., [14]) that for \( u \in \tilde{H}^r(D) \) with \( r \leq s \leq p \)
\begin{equation}
\inf_{u_h \in V_h} \| u - u_h \|_\mathcal{V} \lesssim h^{s-r} \| u \|_{\tilde{H}^s(D)}.
\end{equation}
(1.14)

We have the following error estimate for the semidiscrete problem.

**Theorem 1.3.6.** Let \( u, u_h \) be the solutions of (1.6), (1.9) with \( \mathcal{V}, \mathcal{H} \) and \( V_h \) as defined above. Assume \( u \in C^1([0,T], \tilde{H}^s(D)) \). Then, for \( r \leq s \leq p \)
\begin{equation}
\| u - u_h \|_{L^2([0,T], \mathcal{V})} \leq C(u) \left( \| u_0 - u_{h,0} \|_\mathcal{H} + h^{s-r} \right),
\end{equation}
with a constant \( C(u) > 0 \) depending on \( u \).

**Proof.** See [27, Theorem 6.14]. \[ \square \]

Similarly, for the fully discrete problem, where the Crank-Nicholson scheme is used for the time discretization, we get the following result.

**Theorem 1.3.7.** Let \( u, u^m_h \) be the solutions of (1.6), (1.12) with \( \mathcal{V}, \mathcal{H} \) and \( V_h \) as defined above. Furthermore, let \( \theta = 1/2 \) and assume \( u \in C^1([0,T], \tilde{H}^s(D)) \cap C^3([0,T], \mathcal{V}^s) \) and \( \| u_0 - u_{h,0} \|_\mathcal{H} \lesssim h^s \). Then, for \( r \leq s \leq p \)
\begin{equation}
\| u^M - u^M_h \|_{L^2(D)}^2 + \Delta t \sum_{m=0}^{M-1} \| u^{m+1/2} - u_h^{m+1/2} \|_\mathcal{V}^2 \leq C(u) \left( \Delta t^4 + h^{2(s-r)} \right),
\end{equation}
with a constant \( C(u) > 0 \) depending on higher space and time derivatives of \( u \).

**Proof.** See [33, Theorem 3.3]. \[ \square \]
Remark 1.3.8. For rough initial data $u_0 \notin \tilde{H}^s(D)$ we only have $u(t) \in \tilde{H}^s(D)$, $t > 0$. To compensate for the time singularity at $t = 0$ we need nonuniform time steps as considered e.g., in [44, 59]. For the Euler or Crank-Nicholson scheme algebraically graded meshes yield again optimal convergence rates [59, Remark 3.11].

As seen in the two previous theorems, we obtain for piecewise linear finite elements, $p = 2$, the $L^2$-convergence rate $O(h^2)$ in space, provided the solution $u$ is smooth enough. Here, we expressed the convergence rate in terms of the mesh width $h$. Writing it in terms of the degrees of freedom $N_h = \dim V_h$ with $h = O(N_h^{-\frac{1}{d}})$ the convergence rate decreases with the dimension which is called the “curse of dimension”. In Chapter 4 we show that using sparse tensor product space we can again obtain the optimal convergence rate up to logarithmic terms.
1 Preliminaries
2 Multidimensional Lévy models

In this chapter we give several examples of multidimensional Lévy models, including Lévy copula models, and show important properties of the Lévy measure. Lévy copulas were first introduced by Tankov [65] and were further developed by Kallsen and Tankov [37]. Since the law of a Lévy process $X$ is time-homogeneous, it is completely characterized by its characteristic triplet $(\mathbb{Q}, \nu, \gamma)$. The drift $\gamma$ has no effect on the dependence structure between the components of $X$. The dependence structure of the Brownian motion part of $X$ is given by its covariance matrix $\mathbb{Q}$. For purposes of financial modeling it remains to specify a parametric dependence structure of the purely discontinuous part of $X$ which can be done by using Lévy copulas.

2.1 Lévy processes

We start by recalling essential definitions and properties of Lévy processes. For more details we refer to the monographs [16, 52, 57]. Since Lévy processes can have discontinuous sample paths, we need the class of cadlag functions.

Definition 2.1.1. A function $f := [0, T] \to \mathbb{R}^d$ is said to be cadlag if it is right-continuous with left limits, i.e., for each $t \in [0, T]$ the limits

$$f(t-) = \lim_{s \uparrow t} f(s), \quad f(t+) = \lim_{s \downarrow t} f(s),$$

exist and $f(t) = f(t+)$. 

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq \infty\}$. We assume $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfy the usual hypotheses (see [52]). A stochastic process $X = \{X_t : t \geq 0\}$ is said to be adapted if $X_t \in \mathcal{F}_t$, i.e., is $\mathcal{F}_t$ measurable, for each $t$.

Definition 2.1.2. An adapted, càdlàg stochastic process $X = \{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called a Lévy process if it has the following properties.

1. Independent increments: $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$.
2. Stationary increments: $X_t - X_s$ has the same distribution as $X_{t-s}$, $0 \leq s < t < \infty$.
3. Stochastically continuous: $\lim_{t \downarrow s} X_t = X_s$, where the limit is taken in probability.
In what follows we denote by $X^i$, $i = 1, \ldots, d$, the coordinate projection of $X = (X^1, \ldots, X^d)^\top \in \mathbb{R}^d$. We can associate to $X = \{X_t : t \in [0, T]\}$ a random measure $J_X$ on $[0, T] \times \mathbb{R}^d$, $J_X(\omega, \cdot) = \sum_{t \in [0, T]} 1_{t}(\Delta X_t)$, which is called jump measure. For any measurable subset $B \subset \mathbb{R}^d$, $J_X([0, t] \times B)$ counts then the number of jumps of $X$ occurring between 0 and $t$ whose amplitude belongs to $B$. The intensity of $J_X$ is given by the Lévy measure.

**Definition 2.1.3.** Let $X$ be a Lévy process with state space $\mathbb{R}^d$. The measure $\nu$ on $\mathbb{R}^d$ defined by

$$
\nu(B) = \mathbb{E}(\#\{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in B\}), \quad B \in \mathcal{B}(\mathbb{R}^d),
$$

is called the Lévy measure of $X$. $\nu(B)$ is the expected number, per unit time, of jumps whose size belongs to $B$.

The Lévy measure satisfies $\int_{\mathbb{R}^d} \mathbb{1}_\{z^2 \leq 1\} \cdot \nu(dx) < \infty$. Using the Lévy-Itô decomposition we see that every Lévy process is uniquely defined by a drift vector $\gamma$, a positive definite matrix $Q \in \mathbb{R}^{d \times d}_{sym}$, and the Lévy measure $\nu$. The triplet $(Q, \nu, \gamma)$ is the characteristic triplet of the process $X$.

**Theorem 2.1.4** (Lévy-Itô decomposition). Let $X$ be a Lévy process with state space $\mathbb{R}^d$ and $\nu$ its Lévy measure. Then, there exist a vector $\gamma$ and a $d$-dimensional Brownian motion $W$ with covariance matrix $Q$ such that

$$
X_t = \gamma t + W_t + \int_0^t \int_{|x| \leq 1} xJ_X(ds, dx) + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{1 \leq |x| \leq \varepsilon} x(J_X(ds, dx) - \nu(dx)ds)
$$

$$
= \gamma t + W_t + \sum_{0 \leq s \leq t} \Delta X_s 1_{(|\Delta X_s| \geq 1)} + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{1 \leq |x| \leq \varepsilon} xJ_X(ds, dx), \quad (2.1)
$$

where $J_X$ the jump measure of $X$.

Proof. See [57, Theorem 19.2].

The characteristic triplet could also be derived from the Lévy-Khinchin representation.

**Theorem 2.1.5** (Lévy-Khinchin representation). Let $X$ be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(Q, \nu, \gamma)$. Then for $t \geq 0$,

$$
\mathbb{E}(e^{i\langle \xi, X_t \rangle}) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d, \quad \psi(\xi) = -i\langle \gamma, \xi \rangle + \frac{1}{2} \langle \xi, Q\xi \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle 1_{\{|z| \leq 1\}}\right) \nu(dz). \quad (2.2)
$$

Proof. See [57, Theorem 8.1] or [34, Theorem 3.7.7].

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Remark 2.1.6. The characteristic exponent $\psi(\xi)$ of $X$ is a continuous, negative definite function. Based on (2.2), it is well-known (see, e.g., [34]) that the infinitesimal generator $A$ in (0.2) corresponding to the Lévy process $X$ is a pseudodifferential operator acting on $u \in C_0^\infty(\mathbb{R}^d)$ by the oscillatory integral

$$\langle Au \rangle (x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} \hat{u}(\xi) \, dx,$$

where $\hat{u}(\xi) = (2\pi)^{-d} \int e^{-i\langle \xi, z \rangle} u(z) \, dz$ denotes the Fourier transform of $u$.

Note that in (2.2) the integral with respect to the Lévy measure exists since the integrand is bounded outside of any neighborhood of 0 and

$$1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle 1_{|z| \leq 1} = O(|z|^2) \quad \text{as} \quad |z| \to 0.$$  

But there are many other ways to obtain an integrable integrand. We could, for example, replace $1_{|z| \leq 1}$ by any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $f(z) = 1 + O(|z|)$ as $|z| \to 0$ and $f(z) = O(1/|z|)$ as $|z| \to \infty$. Different choices of $f$ do not affect $Q$ and $\nu$. But $\gamma$ depends on the choice of the truncation function. If the Lévy measure satisfies $\int_{|z| \leq 1} |z| \nu(dz) < \infty$, we can use the zero function as $f$ and get

$$\psi(\xi) = -i\langle \gamma_0, \xi \rangle + \frac{1}{2} \langle \xi, Q\xi \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z \rangle} \right) \nu(dz),$$

with $\gamma_0 \in \mathbb{R}^d$. If the Lévy measure satisfies $\int_{|z| > 1} |z| \nu(dz) < \infty$, then, letting $f$ be a constant function 1, we obtain

$$\psi(\xi) = -i\langle \gamma_c, \xi \rangle + \frac{1}{2} \langle \xi, Q\xi \rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle \right) \nu(dz),$$

with triplet $(Q, \nu, \gamma_c)$ where $\gamma_c$ is called the center of $X$ since $E(X_t) = \gamma_c t$. We use the representation (2.4) instead of (2.2) throughout this work but omit the subscript $c$ for simplicity. The requirement $\int_{|z| \leq 1} |z| \nu(dz) < \infty$ defines a special group of Lévy processes.

Proposition 2.1.7. A Lévy process is of finite variation if and only if its characteristic triplet $(Q, \nu, \gamma)$ satisfies,

$$Q = 0 \quad \text{and} \quad \int_{|z| < 1} |z| \nu(dz) < \infty.$$  

Proof. See [16, Proposition 3.9].}

Coordinate projections of Lévy processes are again Lévy processes.
Proposition 2.1.8. Let \( X = (X^1, \ldots, X^d) \) be a Lévy process with state space \( \mathbb{R}^d \) and characteristic triplet \((Q, \nu, \gamma)\). Assume \( \int_{|z|>1} |z| \nu(dz) < \infty \). Then, the marginal processes \( X^j, j = 1, \ldots, d \), are again Lévy processes with the characteristic triplet \((Q_{jj}, \nu_j, \gamma_j)\) where the marginal Lévy measures are defined as
\[
\nu_j(B) := \nu(\{x \in \mathbb{R}^d : x_j \in B \setminus \{0\}\}), \quad \forall B \in \mathcal{B}(\mathbb{R}), \quad j = 1, \ldots, d.
\]

Proof. See [57, Proposition 11.10].

No arbitrage considerations require Lévy processes employed in mathematical finance to be martingales. The following result gives sufficient conditions of the characteristic triplet to ensure this.

Lemma 2.1.9. Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \) and characteristic triplet \((Q, \nu, \gamma)\). Assume, \( \int_{|z|>1} |z| \nu(dz) < \infty \) and \( \int_{|z|>1} e^{\varepsilon z_j} \nu_j(dz) < \infty, j = 1, \ldots, d \). Then, \( e^{X^j}, j = 1, \ldots, d \), are martingales with respect to the filtration \( \mathcal{F} \) of \( X \) if and only if
\[
\frac{Q_{jj}}{2} + \gamma_j + \int_{\mathbb{R}} (\varepsilon z_j - 1 - z_j) \nu_j(dz) = 0, \quad j = 1, \ldots, d.
\]

Proof. We obtain for \( 0 \leq t < s \) using the independent and stationary increments property
\[
\mathbb{E} \left( e^{X^j_s} \mid \mathcal{F}_t \right) = \mathbb{E} \left( e^{X^j_t + X^j_{s-t}} \mid \mathcal{F}_t \right) = e^{X^j_t} \mathbb{E} \left( e^{X^j_{s-t}} \right) = e^{X^j_t} e^{(s-t)\psi(-ie_j)}.
\]

Therefore, setting \( \psi(-ie_j) = 0, j = 1, \ldots, d \), and using the Lévy-Khinchin formula (2.4) yields
\[
\frac{Q_{jj}}{2} + \gamma_j + \int_{\mathbb{R}^d} (\varepsilon z_j - 1 - z_j) \nu(dz) = 0, \quad j = 1, \ldots, d.
\]
The result follows with the definition of the marginal Lévy density \( \nu_j \) given in Proposition 2.1.8.

Remark 2.1.10. Lemma 2.1.9 also holds for general semimartingales as shown in [51].

2.2 Lévy copulas

We first give some definitions. The \( F \)-volume of \( (a, b], a, b \in \mathbb{R}^d \) for a function \( F : S \to \mathbb{R} \), \( S \subseteq \mathbb{R}^d \) is defined by
\[
V_F ((a, b]) := \sum_{u \in \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),
\]
where \( N(u) = |\{k : u_k = a_k\}| \).
2.2 Lévy copulas

**Definition 2.2.1.** A function \( F : S \rightarrow \mathbb{R} \), \( S \subset \mathbb{R}^d \) is called \( d \)-increasing if \( V_F((a,b]) \geq 0 \) for all \( a, b \in S \) with \( a \leq b \) and \( (a,b] \subset S \).

Examples of \( d \)-increasing functions are distribution functions of random vectors \( X \in \mathbb{R}^d \),
\[
F(x_1, \ldots, x_d) = \mathbb{P}[X_1 \leq x_1, \ldots, X_d \leq x_d],
\]
or more general
\[
F(x_1, \ldots, x_d) = \int_{(-\infty, x_1]} \cdots \int_{(-\infty, x_d]} du \cdot \mathbb{P}(\mathbf{X} \in \text{d}u),
\]
where \( \mathbb{P} \) is a finite measure on \( \mathcal{B}(\mathbb{R}^d) \). \( F \) is clearly \( d \)-increasing since the \( F \)-volume is just \( V_F((a,b]) = \mu((a,b]) \) for every \( a \leq b \). For modeling dependence structure, margins play an important role.

**Definition 2.2.2.** Let \( F : \mathbb{R}^d \rightarrow \mathbb{R} \) be a \( d \)-increasing function which satisfies \( F(u) = 0 \) if \( u_i = 0 \) for at least one \( i \in \{1, \ldots, d\} \). For any index set \( I \subset \{1, \ldots, d\} \) the \( I \)-margin of \( F \) is the function \( F^I : \mathbb{R}^{|I|} \rightarrow \mathbb{R} \)
\[
F^I(u^I) := \lim_{a \rightarrow -\infty} \sum_{(u_j)_{j \in I^c} \in (-\infty,a]^{|I^c|}} \left( \prod_{j \in I} \text{sgn} u_j \right) F(u_1, \ldots, u_d).
\]

Since the Lévy measure is a measure on \( \mathcal{B}(\mathbb{R}^d) \), it is possible to define a suitable notion of a copula. However, one has to take into account that the Lévy measure is possibly infinite at the origin.

**Definition 2.2.3.** A function \( F : \mathbb{R}^d \rightarrow \mathbb{R} \) is called Lévy copula if
1. \( F(u_1, \ldots, u_d) \neq \infty \) for \( (u_1, \ldots, u_d) \neq (\infty, \ldots, \infty) \),
2. \( F(u_1, \ldots, u_d) = 0 \) if \( u_i = 0 \) for at least one \( i \in \{1, \ldots, d\} \),
3. \( F \) is \( d \)-increasing,
4. \( F^{(i)}(u) = u \) for any \( i \in \{1, \ldots, d\} \), \( u \in \mathbb{R} \).

Lévy copulas have analogous properties as ordinary copulas (see, e.g., [49] for an introduction to ordinary copulas).

**Lemma 2.2.4.** Let \( F \) be a Lévy copula. Then,
\[
0 \leq \prod_{j=1}^d \text{sgn} u_j F(u_1, \ldots, u_d) \leq \min\{|u_1|, \ldots, |u_d|\} \quad \forall u \in \mathbb{R}^d,
\]
and \( \prod_{j=1}^d \text{sgn} u_j F(u) \) is nondecreasing in the absolute value of each argument \( |u_j| \).

Furthermore, Lévy copulas are Lipschitz continuous, i.e.,
\[
|F(v_1, \ldots, v_d) - F(u_1, \ldots, u_d)| \leq \sum_{i=1}^d |v_i - u_i| \quad \forall u, v \in \mathbb{R}^d.
\]
2 Multidimensional Lévy models

Proof. Let \( u \in \mathbb{R}^d \), \( u_1 \geq 0 \) and \( 0 \leq a_1 \leq u_1 \). Set \( b_1 = u_1 \) and for \( 2 \leq j \leq d \) set \( a_j = 0 \), \( b_j = u_j \) if \( u_j \geq 0 \) otherwise \( a_j = u_j \), \( b_j = 0 \). Since \( F \) is \( d \)-increasing and grounded

\[
V_F ((a, b)) = \sum_{v \in \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}} (-1)^{N(v)} F(v)
\]

\[
= \prod_{j=2}^d \text{sgn } u_j F(u_1, \ldots, u_d) - \prod_{j=2}^d \text{sgn } u_j F(a_1, u_2 \ldots, u_d) \geq 0.
\]

Similarly for \( u_1 < 0 \). This gives the lower bound with \( a_1 = 0 \).

Let \( \mathcal{I} = \{i\} \subset \{1, \ldots, d\} \). Then,

\[
\prod_{j=1}^d \text{sgn } u_j F(u_1, \ldots, u_i, \ldots, u_d)
\]

\[
\leq \text{sgn } u_i \lim_{n \to \infty} \left( \prod_{j \in \mathcal{I}^c} \text{sgn } u_j \right) F(n \text{sgn } u_1, \ldots, u_i, \ldots, n \text{sgn } u_d)
\]

\[
\leq \text{sgn } u_i \lim_{n \to \infty} \sum_{(v_j)_{j \in \mathcal{I}^c} \in \{-n, \infty\}^{|\mathcal{I}^c|}} \left( \prod_{j \in \mathcal{I}^c} \text{sgn } v_j \right) F(v_1, \ldots, u_i, \ldots, v_d)
\]

\[
= \text{sgn } u_i F^i(u_i) = |u_i|.
\]

Since \( i \in \{1, \ldots, d\} \) is arbitrary we obtain the upper bound. Lipschitz continuity is shown in [37, Lemma 3.2].

We also need tail integrals of Lévy processes.

**Definition 2.2.5.** Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \) and Lévy measure \( \nu \). The tail integral of \( X \) is the function \( U : \mathbb{R}^d \setminus \{0\} \to \mathbb{R} \) given by

\[
U(x_1, \ldots, x_d) = \prod_{i=1}^d \text{sgn}(x_j) \nu \left( \prod_{j=1}^d I(x_j) \right),
\]

where

\[
I(x) = \begin{cases} (x, \infty) & \text{for } x \geq 0 \\ (-\infty, x) & \text{for } x < 0 \end{cases}
\]

Furthermore, for \( \mathcal{I} \subset \{1, \ldots, d\} \) nonempty the \( \mathcal{I} \)-marginal tail integral \( U^\mathcal{I} \) of \( X \) is the tail integral of the process \( X^\mathcal{I} := (X^i)_{i \in \mathcal{I}} \).

The next result, [37, Theorem 3.6], shows that essentially any Lévy process \( X = (X^1, \ldots, X^d)^\top \) can be built from univariate marginal processes \( X^j, j = 1, \ldots, d \) and Lévy copulas. It can be viewed as a version of Sklar’s theorem for Lévy copulas.
2.2 Lévy copulas

**Theorem 2.2.6 (Sklar’s theorem for Lévy copulas).** For any Lévy process $X$ with state space $\mathbb{R}^d$ there exists a Lévy copula $F$ such that the tail integrals of $X$ satisfy

$$U^I(x^I) = F^I((U_i(x_i))_{i \in I}),$$

for any nonempty $I \subset \{1, \ldots, d\}$ and any $x^I \in \mathbb{R}^{|I|} \setminus \{0\}$. The Lévy copula $F$ is unique on $\prod_{i=1}^d \text{Ran} U_i$.

Conversely, let $F$ be a $d$-dimensional Lévy copula and $U_i$, $i = 1, \ldots, d$, tail integrals of univariate Lévy processes. Then, there exists a $d$-dimensional Lévy process $X$ such that its components have tail integrals $U_i$ and its marginal tail integrals satisfy (2.5). The Lévy measure $\nu$ of $X$ is uniquely determined by $F$ and $U_i$, $i = 1, \ldots, d$.

Using partial integration we can write the multidimensional Lévy measure in terms of the Lévy copula.

**Lemma 2.2.7.** Let $f(z) \in C^\infty(\mathbb{R}^d)$ be bounded and vanishing on a neighborhood of the origin. Furthermore, let $X$ be a $d$-dimensional Lévy process with Lévy measure $\nu$, Lévy copula $F$ and marginal Lévy measures $\nu_j$, $j = 1, \ldots, d$. Then,

$$\int_{\mathbb{R}^d} f(z) \nu(dz) = \sum_{j=1}^d \int_{\mathbb{R}} f(0 + z_j) \nu_j(dz_j)$$

$$+ \sum_{j=2}^d \sum_{I \in \mathcal{P}_d} \int_{\mathbb{R}^j} \partial^I f(0 + z^I) F^I((U_k(z_k))_{k \in I}) dz^I.$$

(2.6)

**Proof.** We proceed by induction with respect to the dimension $d$:

For $d = 1$, integration by parts yields

$$\int_0^\infty f(z) \nu(dz) = - \lim_{b \to \infty} f(b) \nu(I(b)) + \lim_{a \to 0^+} f(a) \nu(I(a)) + \int_0^\infty \partial_1 f(z) \nu(I(z)) dz,$$

$$\int_{-\infty}^0 f(z) \nu(dz) = \lim_{a \to 0^-} f(a) \nu(I(a)) - \lim_{b \to -\infty} f(b) \nu(I(b)) - \int_{-\infty}^0 \partial_1 f(z) \nu(I(z)) dz,$$

and since $f$ is bounded

$$\int_{\mathbb{R}} f(z) \nu(dz) = f(0) \lim_{a \to 0^+} (\nu(I(a)) + \nu(I(-a))) + \int_{\mathbb{R}} \partial_1 f(z) \text{sgn}(z) \nu(I(z)) dz.$$  

Abusing notation, we write

$$\nu(\mathbb{R}) := \lim_{a \to 0^+} (\nu(I(a)) + \nu(I(-a))).$$

With $f$ vanishing on a neighborhood of 0 we therefore find $f(0) \nu(\mathbb{R}) = 0$. 

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2 Multidimensional Lévy models

For the multidimensional case, i.e., for \( d > 1 \), we use the Lévy measure of \( X^I \) which is given by

\[
\nu^I (B) := \nu \left( \{ x \in \mathbb{R}^d : (x_i)_{i \in I} \in B \setminus \{0\} \} \right), \quad \forall B \in \mathcal{B} \left( \mathbb{R}^{|I|} \right).
\]

We show by induction with respect to the dimension \( d \) that

\[
\int_{\mathbb{R}^d} f(z) \nu(dz) = f(0, \ldots, 0) \nu(\mathbb{R}, \ldots, \mathbb{R})
+ \sum_{i=1}^{d-1} \int_{\mathbb{R}} \partial_i f(0, \ldots, z_i, \ldots, 0) \text{sgn}(z_i) \nu(I(z_i)) \, dz_i
+ \sum_{i=2}^{d} \sum_{x_1 < \cdots < x_i} \int_{\mathbb{R}^{|I|}} \partial^I f(0 + z^I) \prod_{j \in I} \text{sgn}(z_j) \nu^I \left( \prod_{j \in I} I(z_j) \right) \, dz^I.
\]

With \( f(0, \ldots, 0) \nu(\mathbb{R}, \ldots, \mathbb{R}) = 0 \), the definition of the tail integrals and Theorem 2.2.6 we then have the required result.

For the induction step \( d - 1 \to d \), using integration by parts and the induction hypothesis we obtain

\[
\int_{\mathbb{R}^d} f(z) \nu( dz ) = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} f(z', z_d) \nu( dz', dz_d )
+ \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \partial_d f(z', z_d) \text{sgn}(z_d) \nu \left( dz', I(z_d) \right) \, dz_d
+ \sum_{i=1}^{d-1} \int_{\mathbb{R}} \partial_i f(0, \ldots, z_i, \ldots, 0) \text{sgn}(z_i) \nu(I(z_i)) \, dz_i
+ \sum_{i=2}^{d} \sum_{x_1 < \cdots < x_i} \int_{\mathbb{R}^{|I|}} \partial^I f(0 + z^I) \prod_{j \in I} \text{sgn}(z_j) \nu^I \left( \prod_{j \in I} I(z_j) \right) \, dz^I
+ \int_{\mathbb{R}} \partial_d f(0, \ldots, 0, z_d) \text{sgn}(z_d) \nu( \mathbb{R}, \ldots, \mathbb{R}, I(z_d) )
+ \sum_{i=1}^{d-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_i \partial_d f(0, \ldots, z_i, \ldots, 0, z_d) \text{sgn}(z_i) \text{sgn}(z_d) \nu_{i,d}(I(z_i), I(z_d)) \, dz_i \, dz_d
+ \sum_{i=2}^{d} \sum_{x_1 < \cdots < x_i} \int_{\mathbb{R}^{|I|}} \int_{\mathbb{R}} \partial^I \partial_d f( z^I, z_d ) \prod_{j \in (I,d)} \text{sgn}(z_j) \nu^{(I,d)} \left( \prod_{j \in (I,d)} I(z_j) \right) \, dz^I \, dz_d,
\]

which is the claimed result. \( \square \)
2.2 Lévy copulas

**Remark 2.2.8.** The boundedness assumption on \( f \) in Lemma 2.2.7 can be weakened to certain unbounded \( f \in C^d(\mathbb{R}^d) \), if the Lévy measure \( \nu \) decays sufficiently fast.

Using Lemma 2.2.7 we immediately obtain

**Corollary 2.2.9.** Let \( X = (X^1, \ldots, X^d)^T \) be a \( d \)-dimensional Lévy process with characteristic triplet \((0, \nu, \gamma)\). Then,

\[
\text{Cov}(X^i, X^j) = \int_{\mathbb{R}^d} z_iz_j \nu(dz) = \int_{\mathbb{R}^2} F^{(i,j)}(U_i(z_i), U_j(z_j)) dz_i dz_j, \quad \forall i \neq j,
\]

where \( F \) is the Lévy copula from Theorem 2.2.6.

We conclude with examples of Lévy copulas.

**Example 2.2.10.** Examples of Lévy copulas are:

1. Independence Lévy copula

\[
F(u_1, \ldots, u_d) = \sum_{i=1}^d u_i \prod_{j \neq i} 1_{\{u_j \}}(u_j). \tag{2.7}
\]

2. Complete dependence Lévy copula

\[
F(u_1, \ldots, u_d) = \min\{|u_1|, \ldots, |u_d|\} 1_K(u_1, \ldots, u_d) \prod_{j=1}^d \text{sgn}(u_j), \tag{2.8}
\]

where \( K := \{x \in \mathbb{R}^d : \text{sgn}(x_1) = \ldots = \text{sgn}(x_d)\} \).

3. Clayton Lévy copulas

\[
F(u_1, \ldots, u_d) = 2^{2-d} \left( \sum_{i=1}^d |u_i|^{-\vartheta} \right)^{-\frac{1}{\vartheta}} \left( \eta 1_{\{u_1 \geq 0, \ldots, u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \leq 0, \ldots, u_d \leq 0\}} \right), \tag{2.9}
\]

where \( \vartheta > 0 \) and \( \eta \in [0,1] \). For \( \eta = 1 \) and \( \vartheta \to 0 \), \( F \) converges to the independence Lévy copula, for \( \eta = 1 \) and \( \vartheta \to \infty \) to the complete dependence Lévy copula. In Figure 2.1 the Clayton copula in \( d = 2 \) for \( \vartheta = 0.5, 1.5 \) and \( \eta = 1 \) is plotted. We include the upper bound \( \min\{|u_1|, |u_2|\} \) and additionally give the corresponding contour plot.

An important class of Lévy copulas are so-called 1-homogeneous copulas.

**Definition 2.2.11.** A Lévy copula is called 1-homogeneous if for any \( r > 0 \) there holds

\[
F(ru_1, \ldots, ru_d) = rF(u_1, \ldots, u_d),
\]

for all \((u_1, \ldots, u_d)^T \in \mathbb{R}^d \).

For further details and examples of Lévy copulas, we refer to [28, 37].
2 Multidimensional Lévy models

2.3 Lévy models

Financial models with jumps fall into two categories: *Jump-diffusion* models have a nonzero Gaussian component and a jump part which is a compound Poisson process with finitely many jumps in every time interval. An example of such a model is the Merton jump-diffusion model with Gaussian jumps [47]. On the other hand, *infinite activity* models have an infinite number of jumps in every interval of positive measure. A Brownian motion component is not necessary for infinite activity models since the dynamics of the jumps is already rich enough to generate nontrivial small-time behavior [9]. For a more detailed comparison of the two modelling approaches see, e.g., [16]. We give several examples of multidimensional Lévy models with infinite activity.

2.3.1 Stable processes

If \( \{X_t : t \geq 0\} \) is a Brownian motion on \( \mathbb{R}^d \) then, for any \( r > 0 \) the process \( \{X_{rt} : t \geq 0\} \) is identical in law with the process \( \{r^{1/2}X_t : t \geq 0\} \) [57, Theorem 5.4]. This property is called *selfsimilarity* of a stochastic process with index 2. There are many selfsimilar Lévy processes other than the Brownian motion, the so-called stable processes.

**Definition 2.3.1.** Let \( 0 < \alpha < 2 \). A Lévy process \( X = \{X_t : t \geq 0\} \) with state space \( \mathbb{R}^d \) is called \( \alpha \)-stable if the distribution \( \mu \) of \( X \) at \( t = 1 \) is \( \alpha \)-stable, i.e., for any \( r > 0 \) there exists \( c \in \mathbb{R}^d \) such that

\[
\hat{\mu}(z)^r = \hat{\mu}(r^{1/\alpha}z)e^{i(c,z)}.
\]
2.3 Lévy models

It is shown in [57, Theorem 14.3] that any Lévy process with the characteristic triplet 
\((Q, \nu, \gamma)\) has a \(\alpha\)-stable probability measure if and only if 
\(Q = 0\) and if there is a finite measure \(\lambda\) on the unit sphere 
\(S = \{x \in \mathbb{R}^d : |x| = 1\}\) such that

\[
\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{1}{r^{1+\alpha}} dr, \quad B \in \mathcal{B}(\mathbb{R}^d).
\]

A simple example of a \(\alpha\)-stable Lévy process on \(\mathbb{R}^d\) is given by the Lévy measure

\[
\nu(dz) = \sum_{j=1}^{2^d} c_j |z|^{-d-\alpha} 1_{Q_j} dz,
\]

where \(c_j \geq 0, \sum_{j=1}^{2^d} c_j > 0\) and \(Q_j\) denoting the \(j\)-th quadrant. Note that for \(d = 1\) this is
the only possible \(\alpha\)-stable process. The corresponding marginal processes \(X^i, i = 1, \ldots, d\)
of \(X\) are again \(\alpha\)-stable processes in \(\mathbb{R}\) with Lévy measure \(\nu_i(dz) = \tilde{c}_i |z|^{-1-\alpha} dz\) where \(\tilde{c}_i\) depend on \(\alpha, d\) and \(c_j, j = 1, \ldots, 2^d\).

For \(d > 1\) the notation of stable processes can be extended by using nonsingular matrices
for scaling [63].

**Definition 2.3.2.** Let \(Q \in \mathbb{R}^{d \times d}\) be a matrix with positive eigenvalues. A Lévy process \(X = \{X_t : t \geq 0\}\) with state space \(\mathbb{R}^d\) is called \(Q\)-stable if for any \(r > 0\) there exist a \(c \in \mathbb{R}^d\) such that the distribution \(\mu\) of \(X\) at \(t = 1\) satisfies

\[
\hat{\mu}(z)^r = \hat{\mu}(rQ^Tz)e^{i(c,z)},
\]

where \(r^Q = \sum_{n=0}^\infty (n!)^{-1}(\log r)^n Q^n\).

For \(Q = \text{diag}\((1/\alpha, \ldots, 1/\alpha)\), 0 < \alpha < 2,\) we again obtain \(\alpha\)-stable processes. An extension of (isotropic) \(\alpha\)-stable processes, are anisotropic \(\alpha\)-stable processes for an \(\alpha = (\alpha_1, \ldots, \alpha_d)\) with \(0 < \alpha_i < 2, i = 1, \ldots, d\).

**Definition 2.3.3.** Let \(0 < \alpha_i < 2, i = 1, \ldots, d\) and \(Q = \text{diag}\{\alpha_i^{-1} : i = 1, \ldots, d\}\). A Lévy process \(X = \{X_t : t \geq 0\}\) with state space \(\mathbb{R}^d\) is called \(\alpha\)-stable if the distribution \(\mu\) of \(X\) at \(t = 1\) is \(Q\)-stable, i.e., for any \(r > 0\) there exist \(c \in \mathbb{R}^d\) such that

\[
\hat{\mu}(z)^r = \hat{\mu}(r^{1/\alpha_1} z_1, \ldots, r^{1/\alpha_d} z_d)e^{i(c,z)}.
\]

Since we have for the characteristic function \(\hat{\mu}(z) = e^{-\psi(z)}\) if follows from (2.11) that
the characteristic exponent of an \(\alpha\)-stable process satisfies for any \(r > 0\)

\[
\Re\psi(r^{1/\alpha_1} \xi_1, \ldots, r^{1/\alpha_d} \xi_d) = r \Re\psi(\xi), \quad \forall \xi \in \mathbb{R}^d.
\]

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We assume that the Lévy measure $\nu$ has a Lévy density $k$, i.e., $\nu(dz) = k(z)dz$ and we define the symmetric part of the Lévy density by $k^{\text{sym}}(z) = (k(z) + k(-z))/2$. Similar to [28] we obtain for $Q = 0$ that

$$
\mathbb{R}\psi(r^{\frac{1}{\alpha_1}}\xi_1, \ldots, r^{\frac{1}{\alpha_d}}\xi_d) = \int_{\mathbb{R}^d} \left(1 - \cos \left(\sum_{i=1}^{d} r^{\frac{1}{\alpha_i}}\xi_i z_i\right)\right) k^{\text{sym}}(z) dz
$$

$$
= \int_{\mathbb{R}^d} (1 - \cos(\xi, z)) k^{\text{sym}}(r^{-\frac{1}{\alpha_1}}z_1, \ldots, r^{-\frac{1}{\alpha_d}}z_d) r^{-\frac{1}{\alpha_1} \cdots - \frac{1}{\alpha_d}} dz.
$$

Now using (2.12) the Lévy density has to satisfy

$$
k^{\text{sym}}(r^{-\frac{1}{\alpha_1}}z_1, \ldots, r^{-\frac{1}{\alpha_d}}z_d) = r^{1 + \frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_d}} k^{\text{sym}}(z_1, \ldots, z_d). \quad (2.13)
$$

A simple example of a $\alpha$-stable Lévy process on $\mathbb{R}^d$ is given by the Lévy measure

$$
\nu(dz) = \sum_{j=1}^{2d} c_j \left(\sum_{i=1}^{d} |z_i|^{\alpha_i}\right)^{-1 - \frac{1}{\alpha_1} \cdots - \frac{1}{\alpha_d}} 1_{Q_j} dz, \quad (2.14)
$$

where $c_j \geq 0$, $\sum_{j=1}^{2d} c_j > 0$. The corresponding marginal processes $X^i, i = 1, \ldots, d$ of $X$ are again $\alpha$-stable processes in $\mathbb{R}$ with Lévy measure $\nu_i(dz) = \tilde{c}_i |z|^{-1-\alpha_i} dz$ where $\tilde{c}_i$ depend on $d, \alpha$ and $c_j, j = 1, \ldots, 2^d$. We plot the density (2.14) for $d = 2$, $\alpha = (0.5, 1.2)$ and $c_j = 1, j = 1, \ldots, 4$ in Figure 2.2.

![Figure 2.2: Anisotropic $\alpha$-stable density in $d = 2$ for $\alpha = (0.5, 1.2)$ and corresponding contour plot](image)

2.3.2 Subordinated Brownian motion

A popular class of processes is obtained by *subordination* of a Brownian motion with drift. For $d > 1$ there are two possibilities. Using a one-dimensional increasing process or subordinator $G = \{G_t : t \geq 0\}$, the resulting process is given by

$$
X^i_t = W^i_{G_t} + \theta_i G_t, \quad \theta_i \in \mathbb{R}, \quad t \in [0, T],
$$
for \( i = 1, \ldots, d \) where \( W = (W^1, \ldots, W^d)^\top \) is a vector of \( d \) Brownian motions with covariance matrix \( Q = (\sigma_i \sigma_j \rho_{ij})_{1 \leq i, j \leq d} \). Here, \( \sigma_i^2 \), \( i = 1, \ldots, d \), is the variance of the one-dimensional Brownian motions \( W^i \), and \( \rho_{ij} \) the correlation of the Brownian motions \( W^i \) and \( W^j \). But we can also use a \( d \)-dimensional Lévy process \( G = (G^1, \ldots, G^d)^\top \) which is componentwise increasing in each coordinate to obtain

\[
X^i_t = W^i_{G^i_t} + \theta_i G^i_t, \quad \theta_i \in \mathbb{R}, \quad t \in [0, T],
\]

for \( i = 1, \ldots, d \). This is called multivariate subordination and was introduced in [3].

As an example we use as the one-dimensional subordinator a gamma process to obtain a multidimensional variance gamma process [41]. As in the one-dimensional case [42] we consider a gamma process \( G \) with Lévy density \( k_G(s) = e^{-\frac{1}{\gamma}(\vartheta s)^{-1}}1_{\{s > 0\}} \). Then, using [57, Theorem 30.1] the Lévy measure of \( X \) is given for \( B \in \mathcal{B}(\mathbb{R}^d) \) by

\[
\nu(B) = \int_{B} \int_{0}^{\infty} \frac{(2\pi)^{-d}}{2} \det Q^{-\frac{1}{2}} s^{-\frac{d}{2}} e^{-\frac{1}{2}Q^{-1}(z-\theta s, Q^{-1}(z-\theta s))/2s} e^{-\frac{1}{2}(\vartheta s)^{-1}} ds dz \\
= \int_{B} (2\pi)^{-d} \vartheta^{-1} \det Q^{-\frac{1}{2}} e^{\vartheta (\vartheta^{-1} + \vartheta'^{-1}) s} \int_{0}^{\infty} s^{-\frac{d}{2}-1} e^{-\frac{1}{2}Q^{-1}(z-\theta s, Q^{-1}(z-\theta s))/2s} e^{-\frac{1}{2}(\vartheta s)^{-1}} ds dz \\
= \int_{B} (2\pi)^{-d} \vartheta^{-1} \det Q^{-\frac{1}{2}} e^{\vartheta (\vartheta^{-1} + \vartheta'^{-1}) s} \int_{0}^{\infty} s^{-\frac{d}{2}-1} e^{-\beta s - \gamma s^2} ds dz,
\]

where \( \beta = (z, Q^{-1}z)/2 \) and \( \gamma = (\theta, Q^{-1}\theta)/2 + 1/\vartheta \). Using [30, Formula 3.471, 9] to integrate the second integral we obtain the Lévy measure

\[
\nu(dz) = 2 (2\pi)^{-d} \vartheta^{-1} \det Q^{-\frac{1}{2}} e^{\vartheta (\vartheta^{-1} + \vartheta'^{-1}) s} \left( \frac{\beta}{\gamma} \right)^{-d/4} K_{-d/2}(2\sqrt{\beta \gamma}) dz, \tag{2.15}
\]

where \( K_{-d/2}(\xi) \) is the modified Bessel function of second kind. For small \( \xi \) we have \( K_{-d/2}(\xi) \sim \xi^{-d/2} \) and therefore \( \nu(dz) \sim (z, Q^{-1}z)^{-d/2} dz \sim |z|^{-d} dz \) since \( Q > 0 \). The marginal processes \( X^i, i = 1, \ldots, d \) of \( X \) are variance gamma processes on \( \mathbb{R} \) with Lévy measure \( \nu_i(dz) = \vartheta^{-1} \rho_i/\sigma_i^2 \vartheta^{-1} |z|^{-1} dz \). We plot the density (2.15) for \( d = 2, \vartheta = (-0.1, -0.2), \sigma = (0.3, 0.4), \rho_{12} = 0.5 \) and \( \vartheta = 1 \) in Figure 2.3.
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2.3.3 Lévy copula models

Lévy copulas $F$ allow parametric constructions of multivariate jump densities from univariate ones. Let $U_1, \ldots, U_d$ be one-dimensional tail integrals with Lévy density $k_1, \ldots, k_d$, and let $F$ be a Lévy copula such that $\partial_1 \ldots \partial_d F$ exists in the sense of distributions. Then,

$$k(x_1, \ldots, x_d) = \partial_1 \ldots \partial_d F|_{\xi_1=U_1(x_1), \ldots, \xi_d=U_d(x_d)} k_1(x_1) \ldots k_d(x_d),$$

(2.16)

is the jump density of a $d$-variate Lévy measure with marginal Lévy densities $k_1, \ldots, k_d$. For example we can use the Clayton Lévy copula (see Definition 2.9)

$$F(u_1, \ldots, u_d) = 2^{-d} \left( \sum_{i=1}^d |u_i|^{-\vartheta} \right)^{-\frac{1}{\vartheta}} \left( \eta \Box_{u_1 \cdots u_d \geq 0} - (1-\eta) \Box_{u_1 \cdots u_d \leq 0} \right),$$

where $\vartheta > 0$, $\eta \in [0, 1]$ and consider $\alpha$-stable marginal Lévy densities, $k_i(z) = |z|^{-1-\alpha_i}$, $0 < \alpha_i < 2$, $i = 1, \ldots, d$. This leads to the $d$-dimensional Lévy density

$$k(z) = 2^{2-d} \prod_{i=1}^d \left( 1 + (i-1)\vartheta \right) \alpha_i^{\vartheta+1} |z_i|^{\alpha_i \vartheta - 1} \left( \sum_{i=1}^d \alpha_i^{\vartheta} |z_i|^{\alpha_i \vartheta} \right)^{-\frac{1}{\vartheta} - d} \cdot \left( \eta \Box_{z_1 \cdots z_d \geq 0} + (1-\eta) \Box_{z_1 \cdots z_d \leq 0} \right).$$

(2.17)

Note that this is again an anisotropic $\alpha$-stable process as shown in [28]. We plot the density (2.17) for $d = 2$, $\vartheta = 0.5$, $\eta = 0.5$ and $\alpha = (0.5, 1.2)$ in Figure 2.4.

Figure 2.4: Anisotropic $\alpha$-stable Lévy copula density (2.17) in $d = 2$ for $\alpha = (0.5, 1.2)$ and corresponding contour plot

2.3.4 Admissible models

We make the following assumptions on our models.

Assumption 2.3.4. Let $X$ be a $d$-dimensional Lévy process with characteristic triplet $(\mathcal{Q}, \nu, \gamma)$, Lévy density $k$ and marginal Lévy densities $k_i$, $i = 1, \ldots, d$. 

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i) There are constants \( \beta_i^- > 0, \beta_i^+ > 0, i = 1, \ldots, d \) such that

\[
k_i(z) \lesssim \begin{cases} 
e^{-\beta_i^- |z|}, & z < -1 \\ 
e^{-\beta_i^+ z}, & z > 1 \end{cases}
\]  

(2.18)

ii) Furthermore, we assume there exist an \( \alpha \)-stable process \( X^0 \) with Lévy density \( k^0 \) such that

\[
k(z) \lesssim k^0(z), \quad 0 < |z| < 1.
\]  

(2.19)

iii) If \( Q \) is not positive definite, we assume additionally that

\[
k^{\text{sym}}(z) \gtrsim k^{0, \text{sym}}(z), \quad 0 < |z| < 1,
\]  

(2.20)

iv) Finally, we require that the density \( k \) is real analytic outside \( z_i = 0, i = 1, \ldots, d \),

\[
|\partial^n k(z)| \lesssim C^n |n!| \|z\|_{\infty}^\alpha \prod_{i=1}^d |z_i|^{-n_i-1}, \quad \forall z_i \neq 0.
\]  

(2.21)

For \( d = 1 \) the assumptions (2.18)–(2.21) coincide with the assumptions (A1)–(A4) in [45, Section 3.2]. It is shown that these are satisfied by a wide range of processes, including the generalized hyperbolic, Meixner and tempered stable processes. In the case that the marginal processes \( X^i, i = 1, \ldots, d \) are independent, (2.18)–(2.21) only needs to hold for the corresponding marginal one-dimensional densities \( k_i, i = 1, \ldots, d \). If one wants to enforce a dependence structure given by a Lévy copula, the assumptions can also be stated in terms of the corresponding Lévy copula and marginal densities.

**Assumption 2.3.5.** Let \( X \) be a \( d \)-dimensional Lévy process with characteristic triplet \((Q, \nu, \gamma)\), Lévy copula \( F \), marginal Lévy densities \( k_i \) with tail integrals \( U_i, i = 1, \ldots, d \).

i) There are constants \( \beta_i^- > 0, \beta_i^+ > 0, i = 1, \ldots, d \) such that

\[
k_i(z) \lesssim \begin{cases} e^{-\beta_i^- |z|}, & z < -1 \\ e^{-\beta_i^+ z}, & z > 1 \end{cases}
\]  

(2.22)

ii) Furthermore, there exist a 1-homogeneous Lévy copula \( F^0 \) and \( \alpha \)-stable densities \( k_i^0 \) with tail integrals \( U_i^0, i = 1, \ldots, d \) such that

\[
k_i(z) \sim k_i^0(z), \quad 0 < |z| < 1, \quad i = 1, \ldots, d,
\]  

(2.23)

\[
\partial_1 \ldots \partial_d F(U(z)) \sim \partial_1 \ldots \partial_d F^0(U^0(z)), \quad 0 < |z| < 1.
\]  

(2.24)
2 Multidimensional Lévy models

iii) Finally, we require that

\[ |\partial^a k_i(z)| \lesssim C^n n! |z|^{-\alpha - n - 1}, \quad \forall z \neq 0, \quad i = 1, \ldots, d, \quad n \in \mathbb{N}_0, \tag{2.25} \]

\[ |\partial^a F(u)| \lesssim C^n |\mathbf{n}|! \min\{|u_1|, \ldots, |u_d|\} \prod_{i=1}^d |u_i|^{-n_i}, \quad \forall u \in \mathbb{R}^d, \quad \mathbf{n} \in \mathbb{N}_0^d. \tag{2.26} \]

for \( C > 0, \alpha = \|\mathbf{\alpha}\|_{\infty}. \)

Any process with a Lévy copula and marginal densities satisfying Assumption 2.3.5 also satisfies Assumption 2.3.4.

**Proposition 2.3.6.** Let \( X \) be a \( d \)-dimensional Lévy process with characteristic triplet \((\mathbb{Q}, \nu, \gamma)\), Lévy copula \( F \) and marginal Lévy densities \( k_i, i = 1, \ldots, d \) satisfying Assumption 2.3.5. Then, the Lévy density \( k \) of \( X \) satisfies Assumption 2.3.4.

**Proof.** The assumption of semiheavy tails (2.18) obviously hold. It is shown in [28, Theorem 3.4] that a Lévy process with a 1-homogeneous Lévy copula and \( \alpha \)-stable margins is \( \alpha \)-stable. Therefore, assumptions (2.19) and (2.20) follow with (2.16), (2.23) and (2.24). To show (2.21) we employ for \( n \in \mathbb{N}_0 \) the formula of Faa di Bruno [55]

\[ \partial^n f(g(z)) = \sum \frac{n!}{m_1! \cdots m_n!} (\partial^m f)(g(z)) \left( \frac{\partial g(z)}{1!} \right)^{m_1} \cdots \left( \frac{\partial^n g(z)}{n!} \right)^{m_n}, \]

where \( m = m_1 + \cdots + m_n \) and the sum is over all \( m_1, \ldots, m_n \) for which \( m_1 + 2m_2 + \cdots + nm_n = n \). Since marginal tail integrals satisfy using (2.25),

\[ |\partial^n U_i(z)| \lesssim C^n n! |z|^{-\alpha - n_i}, \quad \forall z \neq 0, \quad i = 1, \ldots, d, \quad n \in \mathbb{N}_0, \]

we obtain for the composite function \( (\partial_1 \ldots \partial_d F \circ U)(z) \)

\[ |\partial^n (\partial_1 \ldots \partial_d F(U(z)))| \]

\[ = \left| \sum \frac{n!}{m_1! \cdots m_n!} (\partial^m \partial_1 \ldots \partial_d F)(U(z)) \left( \frac{\partial U_i(z)}{1!} \right)^{m_1} \cdots \left( \frac{\partial^n U_i(z)}{n!} \right)^{m_n} \right| \]

\[ \lesssim \sum C^n m! \frac{n!}{m_1! \cdots m_n!} \|z\|^{-\alpha} \prod_{j=1}^d |z_j|^\alpha |z_i|^\alpha |z_i|^{-\alpha m_1 - m_1} \cdots |z_i|^{-\alpha m_n - m_n} \]

\[ \lesssim C^n n! \|z\|^{-\alpha} |z_i|^{-n} \prod_{j=1}^d |z_j|^\alpha. \]

Using the Leibniz rule leads to

\[ |\partial^n k(x)| = |\partial^n (\partial_1 \ldots \partial_d F(U(z))k_1(z_1) \ldots k_d(z_d))| \]

\[ = \left| \sum_{j=1}^n \frac{n!}{j!(n-j)!} \partial^j (\partial_1 \ldots \partial_d F(U(z)) \partial^{n-j} k_i(z_i) \prod_{m=0, m \neq i}^d k_m(z_m) \right|. \]
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\[
\begin{align*}
\lesssim C_3^n n! \sum_{j=1}^{d} |z|_{\infty}^{-\alpha} |z_i|^{-\alpha} \prod_{j=1}^{d} |z_j|^{-\alpha-n+j-1} \prod_{m=0, m \neq i}^{d} |z_m|^{-\alpha-1} \\
&\lesssim C_4^n n! |z|_{\infty}^{-\alpha} |z_i|^{-n} \prod_{m=0}^{d} |z_m|^{-1}.
\end{align*}
\]

\[\Box\]

Proposition 2.3.7. Let \( X \) be a \( d \)-dimensional Lévy process with Clayton Lévy copula

\[
F(u_1, \ldots, u_d) = 2^{2-d} \left( \sum_{i=1}^{d} |u_i|^{-\theta} \right)^{-\frac{d}{\theta}} \left( \eta 1_{\{u_1 \cdots u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \cdots u_d \leq 0\}} \right),
\]

where \( \theta > 0, \eta \in [0,1] \) together with tempered stable marginal densities \([7, 9]\)

\[
k_i(z) = c_i e^{-\beta_i z} 1_{\{z<0\}} + c_i^+ e^{-\beta_i^+ z} \frac{1}{z^{1+\alpha_i}} 1_{\{z>0\}}, \quad i = 1, \ldots, d,
\]

with \( c_i^+, c_i^- \geq 0, c_i^+ + c_i > 0, \beta_i^+, \beta_i^- > 0 \) and \( 0 < \alpha_i < 2, i = 1, \ldots, d \). Then, the Assumption 2.3.5 is satisfied.

Proof. Equations (2.22) and (2.23) obviously hold. Equation (2.24) is also considered in [54]. There it was called equivalence preserving and was shown to be true for the Clayton Lévy copula which is 1-homogeneous, i.e., \( F^0 = F \). Now consider the function \( f(z) = z^\alpha \) with \( \alpha = \max\{\alpha_1, \ldots, \alpha_d\} \). It is straightforward to see that \( f \) satisfies for \( n \in \mathbb{N}_0 \)

\[
|\partial^n f(z)| = |\alpha| \cdots |\alpha - n + 1| |z|^{\alpha-n} \leq (\lceil |\alpha| \rceil + n)! |z|^{\alpha-n} \lesssim C^n n! |z|^{\alpha-n},
\]

with any \( C > 1 \). Using the Leibniz formula yields

\[
\left| \partial^n \left( \frac{e^{-\beta |z|}}{|z|^{1+\alpha}} \right) \right| = \left| \sum_{j=1}^{n} \binom{n}{j} \partial^j e^{-\beta z} \partial^{n-j} |z|^{-1-\alpha} \right| \lesssim C^n n! |z|^{-\alpha-1-n}.
\]

for \( z \neq 0 \) which yields (2.25). To prove (2.26) we again apply the formula of Faà di Bruno as in the proof of Proposition 2.3.6 with \( f(x) = x^{-\frac{1}{\theta}}, \) and \( g(z_i) = \sum_{j=1}^{d} |z_j|^{-\theta} \)

\[
|\partial^n f(g(z))| = \left| \sum_{m_1! \cdots m_n!} \frac{n!}{m_1! \cdots m_n!} (\partial^m f) (g(z)) \left( \frac{\partial g(z_i)}{n!} \right)^{m_1} \cdots \left( \frac{\partial g(z_i)}{n!} \right)^{m_n} \right|
\]

\[
\lesssim C_1^n n! \sum_{m_1! \cdots m_n!} \frac{n!}{m_1! \cdots m_n!} \left( \sum_{j=1}^{d} |z_j|^{-\theta} \right)^{-\frac{d}{\theta} - m} |z_i|^{-\vartheta m-n} \lesssim C_2^n n! \min \{|z_1|, \ldots, |z_d|\} |z_i|^{-\varphi m-n}.
\]

\[\Box\]
2 Multidimensional Lévy models

2.4 Properties of the Lévy measure

In the present section we follow [54] and verify properties of Lévy measures corresponding to Lévy processes $X$ with state space $\mathbb{R}^d$ satisfying Assumption 2.3.4. For the (in general nonsymmetric) bilinear form $\mathcal{E}(\cdot, \cdot)$ corresponding to the generator $A$ of $X$, we prove the so-called sector condition:

$$\exists C > 0 : |3\psi(\xi)| \leq C |\Re \psi(\xi)|, \quad \text{for all } \xi \in \mathbb{R}^d. \quad (2.27)$$

Due to a classical result of Berg and Forst [4] (see also [34, Chapter 4.7]) the sector condition together with the translation invariance of $X$, implies that $\mathcal{E}(\cdot, \cdot)$ is a nonsymmetric Dirichlet form. For Lévy processes, the sector condition also makes an explicit characterization of the domains $D(A)$ and $D(\mathcal{E})$ of $A$ and $\mathcal{E}(\cdot, \cdot)$ in terms of anisotropic Sobolev spaces possible.

First, we show that the tails of the multivariate Lévy processes decay exponentially fast provided the one-dimensional tails decay exponentially.

**Proposition 2.4.1.** Let $X$ be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure $\nu$ such that all marginal measures $\nu_i$ satisfy (2.18). Then, the Lévy measure $\nu$ also decays exponentially

$$\int_{|z| > 1} e^{\eta(z)} \nu(dz) < \infty, \quad \text{with} \quad \eta_i(z) = (\mu_i^+ 1_{\{z_i > 0\}} + \mu_i^- 1_{\{z_i < 0\}}) |z_i|, \quad i = 1, \ldots, d,$$

where $0 < \mu_i^- < \beta_i^-$ and $0 < \mu_i^+ < \beta_i^+$, $i = 1, \ldots, d$.

**Proof.** With Proposition 2.1.8 we obtain

$$\int_{|z| > 1} e^{\mu_i |z_i|} \nu_i(dz_i) < \infty,$$

which verifies the sector condition.

The following proposition provides an upper bound for $|\psi(\xi)|$ and hence for $|3\psi(\xi)|$.

**Proposition 2.4.2.** Let $X$ be a Lévy process with state space $\mathbb{R}^d$, characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ and characteristic exponent $\psi$. Assume $\mathcal{Q} = 0$, $\gamma_i = 0$, $i = 1, \ldots, d$, and the Lévy measure $\nu$ satisfies (2.19) with $\alpha = (\alpha_1, \ldots, \alpha_d)$. Then, there exists $C > 0$ such that for all $\xi \in \mathbb{R}^d$ with $\|\xi\|_\infty > 1$,

$$|\psi(\xi)| \leq C \sum_{j=1}^d |\xi_j|^{\alpha_j}. \quad (2.28)$$
We assume that we can distinguish between \( \alpha_i \) smaller or larger than 1. Let \( 0 \leq j \leq d \) such that

\[
\alpha_1, \ldots, \alpha_j < 1, \quad 1 \leq \alpha_{j+1}, \ldots \alpha_d < 2.
\]

Then characteristic exponent \( \psi \) of \( X \) can be written as

\[
\psi(\xi) = \int_{\mathbb{R}^d_{\leq 0}} \left( 1 - e^{i\langle \xi, z \rangle} + \sum_{k=j+1}^{d} i \xi_k z_k 1_{|z| \leq 1} \right) \nu(dz) + i \sum_{k=1}^{j} \tilde{\gamma}_k \xi_k.
\]

We assume that we can set \( \tilde{\gamma}_k, k = 1, \ldots, j \) to zero. With \( B = [0, \frac{1}{d(\xi_1)}] \times \cdots \times [0, \frac{1}{d(\xi_d)}] \) we obtain for all \( \xi \in \mathbb{R}^d \)

\[
|\psi(\xi)| \lesssim \int_{[0,1]^d} \left| e^{i\langle \xi, z \rangle} - 1 - \sum_{k=j+1}^{d} i \xi_k z_k \right| \nu^0(dz) + 1 \leq \int_{B} \left| e^{i\langle \xi, z \rangle} - 1 - \sum_{k=j+1}^{d} i \xi_k z_k \right| \nu^0(dz) + \int_{[0,1]^d \setminus B} \left( 1 + \sum_{k=j+1}^{d} |\xi_k z_k| \right) \nu^0(dz) + 1.
\]

Since the marginal densities \( k^0_i \) of \( k^0 \) are again \( \alpha \)-stable, we have for the first term

\[
\int_{B} \left| e^{i\langle \xi, z \rangle} - 1 - \sum_{k=j+1}^{d} i \xi_k z_k \right| \nu^0(dz) \lesssim \int_{B} \left( \sum_{k=1}^{j} |\xi_k z_k| + \sum_{k=j+1}^{d} \xi_k^2 z_k^2 \right) \nu^0(dz)
\]

\[
\lesssim \sum_{k=1}^{j} \int_{0}^{T_k} |\xi_k z_k| \nu^0_k(dz_k) + \sum_{k=j+1}^{d} \int_{0}^{T_k} \xi_k^2 z_k^2 \nu^0_k(dz_k)
\]

\[
\lesssim \sum_{k=1}^{j} \int_{0}^{T_k} |\xi_k z_k| \frac{1}{z_k^{\alpha_k+1}} dz_k + \sum_{k=j+1}^{d} \int_{0}^{T_k} \xi_k^2 z_k^2 \frac{1}{z_k^{\alpha_k+1}} dz_k
\]

\[
\lesssim \sum_{k=1}^{d} |\xi_k|^{\alpha_k}.
\]

For the second term note that if \( z \in [0,1]^d \setminus B \) with components \( z_k \) satisfying \( z_k \leq \frac{1}{d(\xi_k)} \), \( k = j+1, \ldots, d \), there exists \( l_k \) such that \( z_{l_k} \geq \frac{1}{d(\xi_{l_k})} \). Then,

\[
\int_{[0,1]^d \setminus B} \left( 1 + \sum_{k=j+1}^{d} |\xi_k z_k| \right) \nu^0(dz) \leq \sum_{k=j+1}^{d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{1} (1 + |\xi_k z_k|) \nu^0(dz)
\]

\[
+ \sum_{k=j+1}^{d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\frac{1}{d(\xi_k)}} \cdots \int_{-\infty}^{\frac{1}{d(\xi_k)}} (1 + |\xi_k z_k|) \nu^0(dz)
\]

\[
2.4 \text{ Properties of the Lévy measure}
\]
2 Multidimensional Lévy models

\[
\begin{align*}
\leq & \sum_{k=j+1}^{d} \int_{\frac{1}{d|\xi_k|}}^{1} (1 + |\xi_k z_k|) \nu_1^0 (dz_k) \\
+ & \sum_{k=j+1}^{d} \int_{-\infty}^{0} \int_{0}^{\frac{1}{d|\xi_k|}} \int_{-\infty}^{1} \int_{-\infty}^{0} \left(1 + \frac{1}{d}\right) \nu_1^0 (dz) \\
\leq & 1 + \sum_{k=j+1}^{d} |\xi_k|^\alpha_k + \sum_{k=j+1}^{d} |\xi_k| + \sum_{k=j+1}^{d} \int_{\frac{1}{d|\xi_k|}}^{1} \nu_1^0 (dz_k) \\
\leq & 1 + \sum_{k=1}^{d} |\xi_k|^\alpha_k + \sum_{k=j+1}^{d} |\xi_k| .
\end{align*}
\]

Therefore, we obtain \(|\psi(\xi)| \leq 1 + \sum_{k=1}^{d} |\xi_k|^\alpha_k\) for all \(\xi \in \mathbb{R}^d\) and thus for \(||\xi||_\infty > 1\) the upper bound \(|\psi(\xi)| \leq \sum_{k=1}^{d} |\xi_k|^\alpha_k\).

In order to prove (2.27), we also require a lower bound on \(\Re \psi(\xi)\).

**Proposition 2.4.3.** Let \(X\) be a Lévy process with state space \(\mathbb{R}^d\), characteristic triplet \((\mathcal{Q}, \nu, \gamma)\) and characteristic exponent \(\psi\). Assume \(\mathcal{Q} = 0, \gamma_i = 0, i = 1, \ldots, d\) and the Lévy measure \(\nu\) satisfies (2.20) with \(\alpha = (\alpha_1, \ldots, \alpha_d)\). Then, there exists \(C > 0\) such that for \(||\xi||_\infty\) sufficiently large

\[
|\Re \psi(\xi)| \geq C \sum_{j=1}^{d} |\xi_j|^\alpha_j .
\]

**Proof.** Since \(k^{0,\text{sym}}\) is \(\alpha\)-stable it satisfies (2.13), i.e.,

\[
k^{0,\text{sym}}(r^{-\frac{1}{\alpha_1}} x_1, \ldots, r^{-\frac{1}{\alpha_d}} x_d) = r^{1 + \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_d}} k^{0,\text{sym}}(x_1, \ldots, x_d),
\]

for all \(r > 0\) and \(x \in \mathbb{R}^d\) such that \(x_i \neq 0\). Herewith, using [28, Theorem 3.3] one obtains that \(\psi^0(\xi) := \int_{\mathbb{R}^d} (1 - \cos (\xi, x)) k^{0,\text{sym}} (x) dx\) is an anisotropic distance function of order \((1/\alpha_1, \ldots, 1/\alpha_d)\). Since all anisotropic distance functions of the same order are equivalent (cf., e.g., [21, Lemma 2.2]), there exists some constant \(C_1 > 0\) such that

\[
\psi^0(\xi) \geq C_1 (|\xi_1|^\alpha_1 + \ldots + |\xi_d|^\alpha_d), \quad \text{for all } \xi \in \mathbb{R}^d.
\]

Hence, with (2.20) there exists \(C_2 > 0\) such that

\[
|\Re \psi(\xi)| = \int_{\mathbb{R}^d} (1 - \cos (\xi, x)) k^{\text{sym}} (x) dx \geq C_2 \int_{B_1(0)} (1 - \cos (\xi, x)) k^{0,\text{sym}} (x) dx
\]

\[
\geq C_2 \psi^0(\xi) - C_2 \int_{\mathbb{R}^d \setminus B_1(0)} (1 - \cos (\xi, x)) k^{0,\text{sym}} (x) dx
\]

\[
\geq C_2 \psi^0(\xi) - C_2 \int_{\mathbb{R}^d} (1 - \cos (\xi, x)) k^{0,\text{sym}} (x) dx
\]

\[
\geq C_2 \psi^0(\xi) - C_2 \psi^0(\xi),
\]

\[
\geq C_2 \psi^0(\xi).
\]

Therefore,\( |\Re \psi(\xi)| \geq C \sum_{j=1}^{d} |\xi_j|^\alpha_j .\)
\[ \geq C_2 \psi^0(\xi) - 2C_2 \int_{\mathbb{R}^d \setminus B_1(0)} k^{0,\text{sym}}(x) \, dx \]

\[ \geq C_2 \psi^0(\xi) - C_3 \geq C_2 C_1 \sum_{i=1}^d |\gamma_i| - C_3. \]

Since \( \psi \) is continuous we immediately obtain the sector condition.

**Theorem 2.4.4.** Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \), characteristic triplet \((Q, \nu, \gamma)\) and characteristic exponent \( \psi \). Assume either \( Q > 0 \) or \( Q = 0 \), \( \gamma_i = 0 \), \( i = 1, \ldots, d \), and the Lévy measure \( \nu \) satisfies (2.20), (2.19). Then

\[ |\Im \psi(\xi)| \lesssim |\Re \psi(\xi)|, \quad \forall \xi \in \mathbb{R}^d. \]

**Proof.** For \( Q = 0 \) the result follows with Propositions 2.4.2 and 2.4.3. For \( Q > 0 \) there exists a \( C_1 > 0 \) such that for all \( \xi \in \mathbb{R}^d \)

\[ |\Re \psi(\xi)| = \frac{1}{2} \langle \xi, Q\xi \rangle + \int_{\mathbb{R}^d} (1 - \cos \langle \xi, z \rangle) \nu(dz) \geq C_1 \sum_{j=1}^d \xi_j^2, \quad (2.30) \]

and for \( \|\xi\|_{\infty} > 1 \) there exists \( C_2 > 0 \) such that

\[ |\psi(\xi)| \leq |\langle \gamma, \xi \rangle| + \langle \xi, Q\xi \rangle + \int_{\mathbb{R}^d} e^{i\langle \xi, z \rangle} - 1 - i\langle \xi, z \rangle 1_{|z| \leq 1} |\nu(dz) \leq C_2 \sum_{j=1}^d |\xi_j|^2. \quad (2.31) \]

The result follows from the continuity of \( \psi \). \( \square \)
2 Multidimensional Lévy models
3 Option pricing

In this chapter we derive the partial integrodifferential equation corresponding to the option pricing problem. It is shown that the variational formulation has a unique solution in an anisotropic Sobolev space. Furthermore, the unbounded log-price domain is localized to a bounded domain and the error incurred by the truncation is estimated. Throughout, we assume the risk-neutral dynamics of $d \geq 1$ assets are given by

$$S_t^i = S_0^i e^{r t + X_t^i}, \quad i = 1, \ldots, d,$$

where $X$ is a $d$-dimensional Lévy process with characteristic triplet $(\mathcal{Q}, \nu \mathcal{Q}, \gamma)$ under a risk-neutral measure $\mathbb{Q}$ such that $e^{X_t^i}$ is a martingale with respect to the canonical filtration $\mathcal{F}_t^0 := \sigma(X_{\tau}, \tau \leq t), \ t \geq 0$, of the multivariate process $X$. As shown in Lemma 2.1.9 this martingale condition implies $\int_{|z| > 1} e^{zi} \nu \mathcal{Q}(dz) < \infty, \ i = 1, \ldots, d$. This equation holds for semiheavy tails satisfying (2.18) with $\beta_i^+ > 1, \ i = 1, \ldots, d$, as shown in Proposition 2.4.1. We assume in the following that $\mathcal{Q}$ was fixed by some procedure (see, e.g., [12, 24, 36]) and drop the subscript $\mathcal{Q}$ in the following.

3.1 Partial integrodifferential equation

We consider a European option with maturity $T < \infty$ and payoff $g(S_T)$ which is assumed to be Lipschitz. An arbitrage free value $V(t, s)$ of this option is given by

$$V(t, s) = \mathbb{E} \left( e^{-r(T-t)} g(S_T) | S_t = s \right). \quad (3.1)$$

It can be characterized as the solution of a PIDE.

**Theorem 3.1.1.** Let $X$ be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q}, \nu, \gamma)$. Assume that the function $V(t, s)$ in (3.1) satisfies

$$V(t, s) \in C^{1,2} \left( (0, T) \times \mathbb{R}^d_{>0} \right) \cap C^0 \left( [0, T] \times \mathbb{R}^d_{>0} \right).$$

Then, $V(t, s)$ is a classical solution of the backward Kolmogorov equation

$$\partial_t V(t, s) + \frac{1}{2} \sum_{i,j=1}^d s_i s_j \mathcal{Q}_{ij} \partial_{s_i s_j} V + r \sum_{i=1}^d s_i \partial_{s_i} V(t, s) - r V(t, s) \quad (3.2)$$
3 Option pricing

\[ + \int_{\mathbb{R}^d} \left( V(t, se^z) - V(t, s) - \sum_{i=1}^d s_i (e^{z_i} - 1) \partial_{s_i} V(t, s) \right) \nu(dz) = 0, \]

on \((0, T) \times \mathbb{R}^d_{\geq 0}\) where \(V(t, se^z) := V(t, s_1 e^{z_1}, \ldots, s_d e^{z_d})\), and the terminal condition is given by

\[ V(T, s) = g(s), \quad \forall s \in \mathbb{R}^d_{\geq 0}. \]  

Proof. We first calculate the risk-neutral dynamics of \(S_t^i\). Let \(\Sigma = (\Sigma_{ij})_{1 \leq i, j \leq d}\) be given such that \(\Sigma \Sigma^T = \mathcal{Q}\). With the Itô formula for multidimensional Lévy processes and the Lévy-Itô decomposition we obtain

\[
dS_t^i = rS_t^i dt + S_t^{-\gamma_i} dt + S_t^{-\sum_{k=1}^d \Sigma_{ik} dW_t^k} + \int_{|z| < 1} S_t^{i \cdot \mathbb{1}_{\{\Delta X_t^i > 1\}}} \mathbb{J}(dt, dz) + \frac{1}{2} \mathcal{Q}_{ii} S_t^i dt
\]

\[
= rS_t^i dt + S_t^{-\gamma_i} dt + S_t^{-\sum_{k=1}^d \Sigma_{ik} dW_t^k} + \frac{1}{2} \mathcal{Q}_{ii} S_t^i dt
\]

Since \(e^{X_t^i}\) is a martingale, we have

\[
dS_t^i = rS_t^i dt + S_t^{-\sum_{k=1}^d \Sigma_{ik} dW_t^k} + \int_{\mathbb{R}^d} S_t^{- (e^{z_i} - 1)} \mathbb{J}(dt, dz).
\]

We now apply the Itô formula for semimartingales [35, Theorem 4.57] to the discounted values \(e^{-rt} V_t\). We denote by \([X, Y]_t^i\) the continuous part of the quadratic covariation

\([X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{\tau^-} dY_\tau - \int_0^t Y_{\tau^-} dX_\tau\). Then, we calculate

\[
d(e^{-rt} V_t) = -re^{-rt} V_t dt + e^{-rt} \left( \partial_t V(t, S_t) dt + \sum_{i=1}^d \partial_{s_i} V(t, S_t) dS_t^i 
\right.
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^d \partial_{s_i s_j} V(t, S_t) d[S_t^i, S_t^j]_{\tau^-} + V(t, S_t - e^{\Delta X_t^i})
\]

\[
- V(t, S_t^-) - \sum_{i=1}^d S_t^- \left( e^{\Delta X_t^i} - 1 \right) \partial_{s_i} V(t, S_t^-)
\]

\[
= a(t) dt + dM_t,
\]
3.1 Partial integrodifferential equation

where

\[
\alpha(t) = -re^{-rt}V + e^{-rt} \left( \partial_t V + \sum_{i=1}^d \partial_{s_i} V S_{t-}^i + \frac{1}{2} \sum_{i,j=1}^d Q_{ij} S_{t-}^i S_{t-}^j \partial_{s_i} s_j V \right) \\
+ \int_{\mathbb{R}^d} \left( V(t, S_{t-} e^z) - V(t, S_{t-}) - \sum_{i=1}^d S_{t-}^i (e^{z_i} - 1) \partial_{s_i} V(t, S_{t-}) \right) \nu(dz)
\]

\[
dM_t = e^{-rt} \left( \sum_{i=1}^d \partial_{s_i} V(t, S_{t-}) S_{t-}^i \sum_{k=1}^d \Sigma_{ik} dW_t^k \right) \\
+ \int_{\mathbb{R}^d} (V(t, S_{t-} e^z) - V(t, S_{t-})) \tilde{J}(dt, dz)
\]

Since \( g \) is Lipschitz, \( V \) is also Lipschitz with respect to \( s \) and \( \partial_{s_i} V \) is bounded for \( i = 1, \ldots, d \). With

\[
\mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d} (V(t, S_{t-} e^z) - V(t, S_{t-}))^2 \nu(dz) dt \right) \\
\leq \mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d} (S_{t-}^i)^2 (e^{2z_i} + 1) \nu(dz) dt \right) \\
\leq \sum_{i=1}^d \int_{\mathbb{R}} (e^{2z_i} + 1) \nu_i(dz_i) \mathbb{E} \left( \int_0^T (S_{t-}^i)^2 dt \right) < \infty,
\]

and

\[
\mathbb{E} \left( \int_0^T (S_{t-}^i)^2 |\partial_{s_i} V(t, S_{t-})| dt \right) \leq \mathbb{E} \left( \int_0^T (S_{t-}^i)^2 dt \right) < \infty,
\]

for \( i = 1, \ldots, d \), \( M_t \) is a square-integrable martingale, by [16, Proposition 8.6]. Therefore, \( e^{-rt} V - M_t \) is a martingale and since \( e^{-rt} V - M_t = \int_0^t \alpha(\tau) d\tau \) is also a continuous process with bounded variation, we have \( \alpha(t) = 0 \) almost surely, by [16, Proposition 8.9]. This yields the desired PIDE (3.2).

The PIDE (3.2) can further be transformed into a simpler form:

**Corollary 3.1.2.** Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \) and characteristic triplet \((Q, \nu, \gamma)\) and marginal Lévy measures \( \nu_i, i = 1, \ldots, d \) satisfying (2.18) with \( \beta_i^+ > 1, \beta_i^- > 0, i = 1, \ldots, d \). Furthermore, let

\[
u(\tau, x) = e^{\gamma_i \tau + x_i - \gamma_i \tau} \cdot e^{x_i - \gamma_i \tau}, \ldots, e^{x_d - \gamma_i \tau} \right),
\]

where

\[
\gamma_i = -\frac{Q_{ii}}{2} - \int_{\mathbb{R}} (e^{z_i} - 1 - z_i) \nu_i(dz_i).
\]
3 Option pricing

Then, \( u \) satisfies the PIDE

\[
\partial_t u + A_{BS}[u] + A_J[u] = 0,
\]

in \((0,T) \times \mathbb{R}^d\) with initial condition \( u(0,x) := u_0 \). The differential operator \( A_{BS} \) is defined for \( \varphi \in C^2_0(\mathbb{R}^d) \) by

\[
A_{BS}[\varphi] = -\frac{1}{2} \sum_{i,j=1}^d Q_{ij} \partial_{x_i x_j} \varphi,
\]

and the integrodifferential operator \( A_J \) is given by

\[
A_J[\varphi] = -\int_{\mathbb{R}^d} (\varphi(x+z) - \varphi(x) - z \cdot \nabla_x \varphi(x)) \nu(dz).
\]

The initial condition is given by

\[
u_0 = g(e^x) := g(e^{x_1}, \ldots, e^{x_d}).
\]

Proof. To obtain constant coefficients we set in (3.2) \( x_i = \log s_i \). Furthermore, we change to time to maturity \( \tau = T - t \) and set \( u(\tau, x) = V(T - \tau, e^{x_1}, \ldots, e^{x_d}) \). The resulting differential operator is given by

\[
A_{BS}[\varphi] = -\frac{1}{2} \sum_{i,j=1}^d Q_{ij} \partial_{ij} \varphi + \sum_{i=1}^d \left( \frac{1}{2} Q_{ii} - r \right) \partial_i \varphi + r \varphi, \quad \varphi \in C^2_0(\mathbb{R}^d),
\]

and the integrodifferential operator by

\[
A_J[\varphi] = -\int_{\mathbb{R}^d} (\varphi(x+z) - \varphi(x) - \sum_{i=1}^d (e^z i - 1) \partial_i \varphi(x)) \nu(dz), \quad \varphi \in C^2_0(\mathbb{R}^d).
\]

The interest rate \( r \) can be set to zero by transforming \( u \) to \( \tilde{u} \) using

\[
u(\tau, x) = e^{-r \tau} \tilde{u}(\tau, x + r \tau).
\]

Furthermore, the integrodifferential operator can be rewritten as

\[
A_J[\varphi] = -\int_{\mathbb{R}^d} (\varphi(x+z) - \varphi(x) - z \cdot \nabla_x \varphi(x)) \nu(dz) + \tilde{\gamma} \cdot \nabla_x \varphi(x),
\]

where the coefficients of the drift vector \( \tilde{\gamma} \) are given by

\[
\tilde{\gamma}_i = \int_{\mathbb{R}} (e^z i - 1 - z_i) \nu_i(dz_i), \quad i = 1, \ldots, d.
\]

We remove the drift in the integrodifferential and in the diffusion operator by setting

\[
u(\tau, x) = \tilde{u}(\tau, x + \gamma_1 \tau, \ldots, x_d + \gamma_d \tau).
\]
We next derive the PIDE for knock-out barrier options (see, e.g., [16, Section 12.1.2] for the one-dimensional case). The prices of corresponding knock-in and other barrier contracts with the same barrier can herewith be obtained using superposition and linearity arguments (see, e.g., [7, Section 6]). Let $D \subset \mathbb{R}^d_{\geq 0}$ be an open subset and let $\tau_D = \inf \{ t \geq 0 : X_t \in D^c \}$ be the first hitting time of the complement set $D^c = \mathbb{R}^d \setminus D$ by $X$. Then, the price of a knock-out barrier option with payoff $g$ is given by

$$V_D(t,s) = \mathbb{E} \left( e^{-r(T-t)} g(S_T) 1_{\{ T < \tau_D \}} | S_t = s \right).$$  

(3.9)

If $V_D$ is sufficiently smooth, it can again be computed as the solution of a PIDE.

**Theorem 3.1.3.** Assume $V_D(t,s)$ in (3.9) satisfies

$$V_D(t,s) \in C^{1,2} \left( (0,T) \times \mathbb{R}^d_{\geq 0} \right) \cap C^0 \left( [0,T] \times \mathbb{R}^d_{\geq 0} \right).$$  

(3.10)

Then, $V_D(t,s)$ satisfies the following PIDE:

$$\begin{align*}
\partial_t V_D(t,s) + \frac{1}{2} \sum_{i,j=1}^d s_i s_j Q_{ij} \partial_{s_i s_j} V_D + r \sum_{i=1}^d s_i \partial_{s_i} V_D(t,s) - r V_D(t,s) \\
+ \int_{\mathbb{R}^d} \left( V_D(t,se^z) - V_D(t,s) - \sum_{i=1}^d s_i (e^{z_i} - 1) \partial_{s_i} V_D(t,s) \right) \nu(dz) = 0,
\end{align*}$$

(3.11)

on $(0,T) \times D$ where the terminal condition is given by

$$V_D(T,s) = g(s), \quad \forall s \in D,$$

(3.12)

and the “boundary” condition reads

$$V_D(t,s) = 0, \quad \text{for all } (t,s) \in (0,T) \times D^c.$$

(3.13)

**Proof.** Define the deterministic value function $\bar{g}(s) := g(s) 1_{\{ s \in D \}}$, and consider the European vanilla-type price function

$$\bar{V}(t,s) = \mathbb{E} \left( e^{-r(T-t)} \bar{g}(S_{T \wedge \tau_D}) | S_t = s \right).$$

Since $S$ is a strong Markov process, there holds $V_D(t,S_t) = \bar{V}(t,S_t)$ for all $t \leq T \wedge \tau_D$. Thus, applying the Itô formula as in the proof of Theorem 3.1.1 one obtains that $V_D$ satisfies (3.11) on $(0,T) \times D$. By definition there holds $V_D(t,s) = 0$ for all $(t,s) \in (0,T) \times D^c$.

**Remark 3.1.4.** Note that in contrast to plain European vanilla contracts, the price $V_D$ of a barrier contract does not satisfy the smoothness condition (3.10) for general Lévy models. The validity of (3.10) can however be shown in case the process $X$ admits a non-vanishing diffusion component, i.e., $Q > 0$. Also for market models satisfying the ACP condition of [57, Definition 41.11], Theorem 3.1.3 can be shown to hold, see [7].
3 Option pricing

3.2 Variational formulation

For $u, v \in C_0^\infty(\mathbb{R}^d)$ we associate with $\mathcal{A}_{\text{BS}}$ the bilinear form
\[
\mathcal{E}_{\text{BS}}(u, v) = \frac{1}{2} \sum_{i,j=1}^d Q_{ij} \int_{\mathbb{R}^d} \partial_i u(x) \partial_j v(x) \, dx.
\] (3.14)

To the jump part $\mathcal{A}_J$ we associate the bilinear canonical jump form
\[
\mathcal{E}_J^C(u, v) = - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x + z) - u(x) - z \cdot \nabla u(x)) \, v(x) \, dx \, \nu(dz),
\] (3.15)

and set $\mathcal{E}(u, v) = \mathcal{E}_{\text{BS}}(u, v) + \mathcal{E}_J^C(u, v)$. We can now formulate the abstract problem (1.6) for European contracts with Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, $\mathcal{V} = \mathcal{D}(\mathcal{E})$ and $\mathcal{H} = L^2(\mathbb{R}^d)$:

Find $u \in L^2((0, T); \mathcal{D}(\mathcal{E})) \cap H^1((0, T); \mathcal{D}(\mathcal{E})^*)$ such that
\[
\langle \partial_t u, v \rangle_{\mathcal{D}(\mathcal{E})^*, \mathcal{D}(\mathcal{E})} + \mathcal{E}(u, v) = 0, \quad u(0) = u_0,
\] (3.16)

where $u_0$ is defined as in (3.8)

Remark 3.2.1. We require that $u_0 \in \mathcal{H} = L^2(\mathbb{R}^d)$ which implies a growth condition on the payoff $g$. In Section 3.3 below we reformulate the problem on a bounded domain where this condition can be weakened. It is explicitly given by (3.19).

The well-posedness of (3.16) is ensured by

Theorem 3.2.2. Let $X$ be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(Q, \nu, \gamma)$ and Dirichlet form $\mathcal{E}(\cdot, \cdot)$. Assume either $Q > 0$ or $Q = 0$, $\gamma_i = 0$, $i = 1, \ldots, d$, and the Lévy measure $\nu$ satisfies (2.19), (2.20), with $\alpha = (\alpha_1, \ldots, \alpha_d)$. Then, the variational equation (3.16) with $u_0 \in L^2(\mathbb{R}^d)$ admits a unique solution in $\mathcal{D}(\mathcal{E})$. For $Q > 0$ there holds $\mathcal{D}(\mathcal{E}) = H^1(\mathbb{R}^d)$ and for $Q = 0$ one obtains the anisotropic Sobolev space $\mathcal{D}(\mathcal{E}) = H^{\alpha/2}(\mathbb{R}^d)$ as defined in (1.1).

Proof. Since a Lévy process $X$ is stationary, its infinitesimal generator is translation invariant. As Theorem 2.4.4 shows, the characteristic exponent $\psi$ of $X$ satisfies the sector condition (2.27). Therefore, the bilinear form $\mathcal{E}(u, v)$ is a Dirichlet form and, by [34, Example 4.7.32], it can be written as
\[
|\mathcal{E}(u, v)| = (2\pi)^d \left| \int_{\mathbb{R}^d} \psi(\xi) \hat{g}(\xi) \overline{\hat{v}(\xi)} \, d\xi \right|.
\]

According to Theorem 1.3.2, for existence and uniqueness of a solution of (3.16) we need to show that $\mathcal{E}(\cdot, \cdot)$ satisfies the continuity condition (1.7) and the Gårding inequality (1.8).
3.2 Variational formulation

At first, consider the case $Q = 0$. Through Propositions 2.4.2 and 2.4.3, there exist positive constants $C_1, C_2, C_3 > 0$ such that

$$\Re \psi(\xi) \geq C_1 \sum_{j=1}^{d} |\xi_j|^{\alpha_j} - C_2, \quad |\psi(\xi)| \leq C_3 \left( \sum_{j=1}^{d} |\xi_j|^{\alpha_j} + 1 \right) \quad \text{for all } \xi \in \mathbb{R}^d. \quad (3.17)$$

The continuity of $E(\cdot, \cdot)$ is ensured by

$$|E(u, v)| = \left| \int_{\mathbb{R}^d} \psi(\xi) \overline{\tilde{u}(\xi)} d\xi \right| \leq C_3 \int_{\mathbb{R}^d} \left( 1 + \sum_{i=1}^{d} |\xi_i|^{\alpha_i} \right) \left| \tilde{u}(\xi) \overline{\overline{\tilde{v}(\xi)}} \right| d\xi$$

$$\leq \tilde{C}_3 \int_{\mathbb{R}^d} \sum_{i=1}^{d} \left( 1 + |\xi_i|^2 \right)^{\alpha_i/2} \left| \tilde{u}(\xi) \overline{\tilde{v}(\xi)} \right| d\xi$$

$$\leq \tilde{C}_3 \left( \int_{\mathbb{R}^d} \sum_{i=1}^{d} \left( 1 + |\xi_i|^2 \right)^{\alpha_i/2} |\tilde{u}(\xi)|^2 d\xi \right)^{1/2} \left( \int_{\mathbb{R}^d} \sum_{i=1}^{d} \left( 1 + |\xi_i|^2 \right)^{\alpha_i/2} |\tilde{v}(\xi)|^2 d\xi \right)^{1/2}$$

$$\lesssim \|u\|_{H^{\alpha/2}(\mathbb{R}^d)} \|v\|_{H^{\alpha/2}(\mathbb{R}^d)},$$

where we used the fact that there exists a $c > 0$ such that

$$0 < c \leq \frac{\sum_{i=1}^{d} \left( 1 + |\xi_i|^2 \right)^{\alpha_i/2}}{1 + \sum_{i=1}^{d} |\xi_i|^{\alpha_i}} \leq \frac{1}{c} < \infty, \quad \forall \xi \in \mathbb{R}^d.$$

Furthermore, to prove the Gårding inequality one finds

$$E(u, u) = \int_{\mathbb{R}^d} \Re \psi(\xi) |\tilde{u}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^d} (C_1 + C_2 + \Re \psi(\xi)) |\tilde{u}(\xi)|^2 d\xi - (C_1 + C_2) \int_{\mathbb{R}^d} |\tilde{u}(\xi)|^2 d\xi,$$

and

$$\int_{\mathbb{R}^d} (C_1 + C_2 + \Re \psi(\xi)) |\tilde{u}(\xi)|^2 d\xi \geq C_1 \int_{\mathbb{R}^d} \left( 1 + \sum_{i=1}^{d} |\xi_i|^{\alpha_i} \right) |\tilde{u}(\xi)|^2 d\xi$$

$$\geq \tilde{C}_1 \int_{\mathbb{R}^d} \sum_{i=1}^{d} \left( 1 + |\xi_i|^2 \right)^{\alpha_i/2} |\tilde{u}(\xi)|^2 d\xi.$$

Theorem 1.3.2 thus yields the existence and uniqueness of a solution $u \in H^{\alpha/2}(\mathbb{R}^d)$.

If $Q > 0$ one obtains the required results using the same arguments: With (2.30) and (2.31), instead of (3.17) holds

$$\Re \psi(\xi) \gtrsim \sum_{j=1}^{d} |\xi_j|^2, \quad |\psi(\xi)| \lesssim \sum_{j=1}^{d} |\xi_j|^2, \quad \text{for all } \|\xi\|_{\infty} > 1,$$

and the result follows as above. \(\square\)
Remark 3.2.3. We omitted the partially degenerate case $Q \neq 0$ but $Q \neq 0$ in Theorem 3.2.2. Here, the domain $D(\mathcal{E})$ can be obtained by writing

$$Q = (\sigma_i \sigma_j \rho_{ij})_{1 \leq i, j \leq d},$$

where $\rho_{ij}$ is the correlation of the Brownian motion $W^i$ and $W^j$. Suppose $\sigma_i = 0$ for all $i \in \mathcal{I} \subset \{1, \ldots, d\}$ and $\sigma_j > 0$ for all $j \notin \mathcal{I}$. Using the intersection structure (1.2) one obtains

$$D(\mathcal{E}) = \bigcap_{i \in \mathcal{I}} H^{a_i/2}(\mathbb{R}^d) \cap \bigcap_{j \notin \mathcal{I}} H^1(\mathbb{R}^d).$$

Remark 3.2.4. Theorem 3.2.2 was also obtained in $d = 1$ by [45]. For $d > 1$, Theorem 3.2.2 was proved in [28] for symmetric tempered stable margins.

We convert the canonical form $\mathcal{E}_j^C(\cdot, \cdot)$ of (3.15) into the integrated jump form $\mathcal{E}_j(\cdot, \cdot)$ by using Lemma 2.2.7,

$$\mathcal{E}_j(u, v) = -\sum_{i=1}^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x + z_i) - u(x) - z_i \partial_i u(x)) v(x) k_i(z_i) dx dz_i$$

$$-\sum_{i=2}^d \sum_{|I| = 1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial^I u(x + z^I)v(x)U^I(z^I) dx dz^I. \quad (3.18)$$

The next result states that for the integrals in (3.18) to exist it is sufficient that $u \in H^1(\mathbb{R}^d) = H^1(\mathbb{R}) \otimes \cdots \otimes H^1(\mathbb{R})$ and $u$ has compact support. Note that tensor products of one-dimensional continuous, piecewise linear finite element basis functions satisfy these requirements.

Proposition 3.2.5. Let $u \in H^1(\mathbb{R}^d)$, $v \in H^1(\mathbb{R}^d)$ and suppose $u$, $v$ have compact supports. Then $|\mathcal{E}_j(u, v)| < \infty$.

Proof. For $u, v \in H^1(\mathbb{R}^d)$ with compact supports there holds

$$\left| \int_{\mathbb{R}^d} (u(x + z_i) - u(x) - z_i \partial_i u(x)) v(x) dx \right| \lesssim z_i^2 \|u\|_{H^1(\mathbb{R}^d)} \|v\|_{H^1(\mathbb{R}^d)} \quad i = 1, \ldots, d.$$

With

$$\left| \int_{\mathbb{R}^d} \partial^I u(x + z^I)v(x) dx \right| \leq \|u\|_{H^1(\mathbb{R}^d)} \|v\|_{H^1(\mathbb{R}^d)} \quad \forall z \in \mathbb{R}^d, \mathcal{I} \subset \{1, \ldots, d\},$$

and

$$\left| \int_{\mathbb{R}^d} U^I(z^I) dz^I \right| < \infty \quad \forall \mathcal{I} \subset \{1, \ldots, d\},$$

we obtain the asserted result. \hfill \square
3.3 Localization

In this section we show how one may localize the unbounded log-price space domain \( \mathbb{R}^d \) in (3.5) to a bounded domain \( D \) at the expense of the so-called localization error. To analyze the error introduced by this localization procedure on the option price, we require the following polynomial growth condition on the payoff function: There exists some \( q \geq 1 \) such that

\[
g(s) \lesssim \left( \sum_{i=1}^{d} s_i + 1 \right)^q, \quad \text{for all } s \in \mathbb{R}^d_{\geq 0}.
\]  

(3.19)

This condition is satisfied by all standard multi-asset options like, e.g., basket, rainbow, spread and power options.

The unbounded log-price domain \( \mathbb{R}^d \) of the log price \( x = \log s \) is truncated to a bounded domain \( D \subseteq \mathbb{R}^d \), \( R > 0 \). In terms of financial modeling, this corresponds to approximating the solution \( V \) of the problem (3.2) by a barrier option \( V_R \) which is the solution of the problem (3.11). In log price the European barrier option is given by

\[
u_R(t,x) = \mathbb{E} \left( g(e^{X_T})1_{(T<\tau_D)} | X_t = x \right),
\]

where, for notational convenience, we have set \( r = 0 \). We show that if the probability density \( p_t \) of the Lévy process has semi-heavy tails, the solution of the localized problem converges pointwise exponentially to the solution of the original problem.

**Lemma 3.3.1.** Let \( X = \{X_t : t \geq 0\} \) be a Lévy process with state space \( \mathbb{R}^d \) and Lévy measure \( \nu \) such that the marginal measures \( \nu_i \) satisfy (2.18). Then, the probability density \( p_t(x) \), \( t > 0 \), of the process \( X \) decays exponentially independent of time \( t \)

\[
\int_{\mathbb{R}^d} e^{\eta_i(x)} p_t(x) dx < \infty, \quad \text{with } \eta_i(x) = \left( \mu_i^+ 1_{\{x_i > 0\}} + \mu_i^- 1_{\{x_i < 0\}} \right) |x_i|,
\]

(3.20)

and \( 0 < \mu_i^- < \beta_i^- \) and \( 0 < \mu_i^+ < \beta_i^+ \), \( i = 1, \ldots, d \).

**Proof.** Using [57, Theorem 25.3], we know (3.20) is true if and only if

\[
\int_{|z|>1} e^{\eta_i(z)} \nu(dz) < \infty.
\]

The result (3.20) then follows from Proposition 2.4.1. \( \square \)

**Theorem 3.3.2.** Suppose the payoff function \( g : \mathbb{R}^d \to \mathbb{R} \) satisfies (3.19). Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \) and Lévy measure \( \nu \) such that the marginal measures \( \nu_i \) satisfy (2.18) with \( \beta_i^+ > q \), \( \beta_i^- > q \), \( i = 1, \ldots, d \), with \( q \) as in (3.19). Then,

\[
|u(t,x) - u_R(t,x)| \lesssim e^{-\gamma_1 R + \gamma_2 \|x\|_{\infty}},
\]

with \( 0 < \gamma_1 < \min_i \min(\beta_i^+, \beta_i^-) - q \) and \( \gamma_2 = \gamma_1 + q \).
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Proof. Let \( \eta_i(x) \) be as in (3.20) and \( M_T = \sup_{t \in [t,T]} \| X_t \|_\infty \). Then with (3.19)

\[
|u(t, x) - u_R(t, x)| \leq \mathbb{E} \left( g(e^{X_T}) 1_{\{T \geq \tau_D R \}} |X_t = x| \right) \leq \mathbb{E} (e^{qM_T} 1_{\{M_T > R \}} |X_t = x|).
\]

Using [57, Theorem 25.18] it suffices to show that

\[
\mathbb{E} (e^{q\|X_T\|_\infty} 1_{\{|X_T\|_\infty > R\}} |X_t = x|) = \int_{\mathbb{R}^d} e^{q\|z+x\|_\infty} 1_{\{|z+x\|_\infty > R\}} p_{T-t}(z) dz
\]

\[
\lesssim e^{q\|x\|_\infty} \sum_{i=1}^{d} \int_{\mathbb{R}^d} e^{q|z_i|} e^{-\eta_i(z)} 1_{\{|z+x\|_\infty > R\}} e^{\eta_i(z)} p_{T-t}(z) dz
\]

\[
\lesssim e^{q\|x\|_\infty} \sum_{i=1}^{d} \int_{\mathbb{R}^d} e^{-\min_j (\min_j (\mu^+_j - \nu^-_j) - q) (R-\|x\|_\infty)} e^{\eta_i(z)} p_{T-t}(z) dz
\]

\[
\lesssim e^{-\alpha R + \beta \|x\|_\infty} \sum_{i=1}^{d} \int_{\mathbb{R}^d} e^{\eta_i(z)} p_{T-t}(z) dz
\]

The result follows with (3.20).

\[\blacksquare\]

Remark 3.3.3. In \( d = 1 \) a similar proof is given in [17].

For any function \( u \) with support in \( D_R \) we denote by \( \tilde{u} \) its extension by zero to all of \( \mathbb{R}^d \) and define

\[\mathcal{E}_R(u, v) = \mathcal{E}(\tilde{u}, \tilde{v}).\]

Thus, we obtain continuity and a Gårding inequality of \( \mathcal{E}_R(u, v) \) on the domain \( \mathcal{D}(\mathcal{E}_R) = \dot{H}^{\alpha/2}(D_R) \subset \mathcal{D}(\mathcal{E}) \) as defined in (1.4). Now we can restate the problem (3.16) on the bounded domain:

\[
\text{Find } u_R \in L^2((0, T); \mathcal{D}(\mathcal{E}_R)) \cap H^1((0, T); \mathcal{D}(\mathcal{E}_R)^*) \text{ such that } (\partial_t u_R, v)_{\mathcal{D}(\mathcal{E}_R)^*, \mathcal{D}(\mathcal{E}_R)} + \mathcal{E}_R(u_R, v) = 0, \tau \in (0, T), \forall v \in \mathcal{D}(\mathcal{E}_R), \tag{3.21}
\]

\[u_R(0) = u_0|_{D_R}.\]

By Theorem 3.2.2, the problem (3.21) is well-posed, i.e., there exists a unique solution \( u_R \in L^2(0, T; \mathcal{D}(\mathcal{E}_R)) \cap C^0((0, T]; L^2(D_R)) \) which can now be approximated by a finite element Galerkin scheme.
4 Wavelet basis

Straightforward application of standard finite element schemes to calculate the stiffness matrix $A = (\mathcal{E}(\phi_{h,k}, \phi_{h,k'}))_{k,k' \in \Delta_h}$ (1.11) as explained in Section 1.3 is inefficient due to two reasons. For high-dimensional models we have the “curse of dimension”: The number of degrees of freedom on a tensor product finite element mesh of uniform mesh width $h$ in dimension $d$ grows like $O(h^{-d})$ as $h \to 0$. For jump models the non-locality of the underlying operator implies that the standard finite element stiffness matrix $A$ consists of $O(h^{-2d})$ non-zero entries as $h \to 0$, which is not practical even for a single asset with small mesh widths.

As we show here, spline wavelets can overcome both problems while still being easy to compute. In particular, choosing wavelet bases has three main advantages. Firstly, we can break the curse of dimension using sparse tensor products to obtain essentially dimension independent complexity. Secondly, the use of wavelets allows a multiscale compression of the jump measure of $X$. Then, the complexity of jump models can asymptotically be reduced to Black-Scholes complexity. Finally, we show that wavelets provide norm equivalences in fractional order spaces which lead to efficient preconditioning even for pure jump operators.

4.1 Wavelets

We start by explaining wavelets in one dimension, following the construction described in [19]. The $d$-variate bases are obtained by tensor product construction.

4.1.1 Spline wavelets on the interval

The one-dimensional interval $D = [0,1]$ is partitioned into an equidistant mesh $T_\ell$ with mesh width $h_\ell = 2^{-\ell}$, $\ell \in \mathbb{N}$. We define $\mathcal{V}_\ell$ as the space of piecewise polynomials of degree $p-1 \in \mathbb{N}$ on the mesh $T_\ell$ which vanish at the endpoints and denote with $N_\ell = \dim \mathcal{V}_\ell = O(2^\ell)$. The spaces $\mathcal{V}_\ell$ are nested, $\mathcal{V}_\ell \subset \mathcal{V}_{\ell+1}$, and generated by single scale bases $\Phi_\ell := \{\phi_{\ell,k} : k \in \Delta_\ell\}$ with suitable index set $\Delta_\ell$. Here, we change notation and write $\phi_{\ell,k}$ instead of $\phi_{h_\ell,k}$ for simplicity. We assume that the basis functions $\phi_{\ell,k} \in \Phi_\ell$, $\ell \in \mathbb{N}$, have compact support of size $|\text{supp} \phi_{\ell,k}| \lesssim 2^{-\ell}$ and are normalized in $L^2$, $\|\phi_{\ell,k}\|_{L^2([0,1])} = 1$. The approximation order of $\Phi_\ell$ is given by $p$. 
4 Wavelet basis

In addition, we associated with \( \Phi_{\ell,k} \) a dual basis, \( \tilde{\Phi}_{\ell} := \{ \tilde{\phi}_{\ell,k} : k \in \Delta \ell \} \), i.e., one has \( \langle \phi_{\ell,k}, \tilde{\phi}_{\ell,k'} \rangle = \delta_{k,k'}, k,k' \in \Delta \ell \). The approximation order of \( \tilde{\Phi}_{\ell} \) is denoted by \( \tilde{p} \), and we assume \( p \leq \tilde{p} \).

Given the single-scale basis \( \Phi_{\ell} \), we can construct a biorthogonal complement or wavelet basis \( \Psi_{\ell} = \{ \psi_{\ell,k} : k \in \nabla \ell \} \), \( \Psi_{\ell} = \{ \hat{\psi}_{\ell,k} : k \in \nabla \ell \} \) with \( \nabla \ell = \Delta \ell+1 \backslash \Delta \ell \) such that

\[
V_{\ell+1} = V_{\ell} \oplus W_{\ell}, \quad \tilde{V}_{\ell+1} = \tilde{V}_{\ell} \oplus \tilde{W}_{\ell}, \quad \ell \in \mathbb{N},
\]

and

\[
V_{\ell} = W_0 \oplus \cdots \oplus W_{\ell-1}, \quad \ell \in \mathbb{N},
\]

where the increment spaces \( W_{\ell}, \tilde{W}_{\ell} \) are the span of \( \Psi_{\ell}, \tilde{\Psi}_{\ell} \) for \( \ell > 0 \), and \( W_0 := V_1 \), \( \tilde{W}_0 := \tilde{V}_1 \). We suppose the wavelets \( \psi_{\ell,k} \) have compact support \( |\text{supp} \psi_{\ell,k}| \lesssim 2^{-\ell} \) and are normalized in \( L^2([0,1]) \).

Any function \( u \in V_{L+1} \) has the representation

\[
u = \sum_{\ell=0}^{L} \sum_{k \in \nabla \ell} u_{\ell,k} \psi_{\ell,k} = \sum_{\ell=0}^{L} \sum_{k \in \nabla \ell} \langle u, \hat{\psi}_{\ell,k} \rangle \psi_{\ell,k}.
\]

For \( u \in \tilde{H}^s([0,1]), 0 \leq s \leq p \) one obtains an infinite series

\[
u = \sum_{\ell=0}^{\infty} \sum_{k \in \nabla \ell} u_{\ell,k} \psi_{\ell,k}, \quad (4.2)
\]

which converges in \( \tilde{H}^s([0,1]) \). There holds the norm equivalence

\[
\|u\|_{\tilde{H}^s([0,1])}^2 \lesssim \sum_{\ell=0}^{\infty} \sum_{k \in \nabla \ell} 2^{2\ell s} |u_{\ell,k}|^2 \lesssim \|u\|_{\tilde{H}^s([0,1])}^2, \quad 0 \leq s < p - 1/2. \quad (4.3)
\]

**Example 4.1.1.** We give an example of wavelet basis for \( \tilde{H}^s([0,1]), 0 \leq s < 3/2 \) using piecewise linear continuous functions, \( p = 2 \), on \([0,1] \) vanishing at the endpoints. The mesh \( T_{\ell} \) is defined by the nodes \( x_{\ell,k} := 2^{-\ell} \cdot k \) with \( k = 0, \ldots, 2^\ell - 1 \). Let \( N_{\ell} = 2^{\ell+1} - 1 \) and \( c_{\ell} := \sqrt{3} \cdot 2^{\ell/2 - 1}, \ell \in \mathbb{N}_0 \). We define the wavelets \( \psi_{\ell,k} \) for level \( \ell \in \mathbb{N}_0, k = 1, \ldots, 2^\ell \).

For \( \ell = 0 \) we have \( N_0 = 1 \) and \( \psi_{0,1} \) is the function with value \( 2c_0 \) at \( x_{1,0} = 1/2 \). For \( \ell \geq 1 \) the wavelet \( \psi_{\ell,1} \) has the values \( \psi_{\ell,1}(x_{\ell,1}) = 2c_{\ell}, \psi_{\ell,1}(x_{\ell,2}) = -c_{\ell} \) and zero at all other nodes. The wavelet \( \psi_{\ell,2} \) has the values \( \psi_{\ell,2}(x_{\ell,N_{\ell}}) = 2c_{\ell}, \psi_{\ell,2}(x_{\ell,N_{\ell}-1}) = -c_{\ell} \) and zero at all other nodes. The wavelet \( \psi_{\ell,k} \) with \( 1 < k < 2^\ell \) has the values \( \psi_{\ell,k}(x_{\ell,2k-2}) = -c_{\ell}, \psi_{\ell,k}(x_{\ell,2k-1}) = 2c_{\ell}, \psi_{\ell,k}(x_{\ell,2k}) = -c_{\ell} \) and zero at all other nodes. For \( \ell = 0, \ldots, 3 \) these wavelets are plotted in Figure 4.1 for the space \( V_4 \) and its decomposition.
4.1 Wavelets

4.1.2 Sparse tensor product space

In $D = [0,1]^d$, $d > 1$ we define as in Example 1.3.5 the subspace $V_L$ as the full tensor product of the one-dimensional spaces $V_{L+1} = \bigotimes_{1 \leq i \leq d} V_{L+1}$ which can be written as

$$V_{L+1} = \text{span} \{ \psi_{\ell,k} : 0 \leq \ell_i \leq L, k_i \in \nabla_{\ell_i}, i = 1, \ldots, d \},$$

with basis functions $\psi_{\ell,k} = \psi_{\ell_1,k_1} \ldots \psi_{\ell_d,k_d}$, $0 \leq \ell_i \leq L, k_i \in \nabla_{\ell_i}, i = 1, \ldots, d$. Using (4.1) we can write $V_{L+1}$ again in terms of increment spaces

$$V_{L+1} = \bigoplus_{0 \leq \ell_i \leq L} W^{d_1} \otimes \ldots \otimes W^{d_d}.$$

Therefore, together with (4.2) for any function $u \in L^2([0,1]^d)$ we have the series representation

$$u = \sum_{\ell_i=0}^{\infty} \sum_{k_i \in \nabla_{\ell_i}} u_{\ell,k} \psi_{\ell,k}.$$

Using the norm equivalences (4.3) and the intersection structure (1.2) we obtain

$$\|u\|_{\tilde{H}^{s_1}(\{0,1\}^d)}^2 \lesssim \sum_{\ell_i=0}^{\infty} \sum_{k_i \in \nabla_{\ell_i}} (2^{2s_1\ell_1} + \ldots + 2^{2s_1d_1}) \|u_{\ell,k}\|^2 \lesssim \|u\|_{\tilde{H}^{s_1}(\{0,1\}^d)}^2,$$

for $0 \leq s_i \leq p - 1/2, i = 1, \ldots, d$.

**Remark 4.1.2.** To obtain a multilevel preconditioner we only need these norm equivalences for $H^{\alpha/2}([0,1]^d)$, i.e., $0 \leq s_i = \alpha_i/2 \leq 1, i = 1, \ldots, d$. Therefore, $p = 2$ is sufficient.
4 Wavelet basis

The space $V_L$ has $O(2^{dL})$ degrees of freedom which grow exponentially with increasing dimension $d$. To avoid this “curse of dimension” we introduce the sparse tensor product space

$$\hat{V}_{L+1} := \text{span}\{\psi_{\ell,k} : 0 \leq \ell_1 + \cdots + \ell_d \leq L, k_i \in \nabla_{\ell_i}, i = 1, \ldots, d\}$$

$$= \bigoplus_{0 \leq \ell_1 + \cdots + \ell_d \leq L} W_{\ell_1} \otimes \cdots \otimes W_{\ell_d}.$$ 

The difference between the tensor product space $V_L$ and the sparse tensor product space $\hat{V}_L$ is shown in Figure 4.2 for level $L = 3$ using wavelets as described in Example 4.1.1.

![Tensor product (left) and sparse tensor product (right) for $d = 2$](image)

As $L \to \infty$ we have $N = \text{dim}(V_{L+1}) = O(2^{dL})$ and $\hat{N} = \text{dim}(\hat{V}_{L+1}) = O(L^{d-1}2^L)$, i.e., the spaces $\hat{V}_L$ have considerably smaller dimension than $V_L$. On the other hand, they do have similar approximation properties as $V_L$, provided the function to be approximated is sufficiently smooth. As shown in [68] there holds for $u \in H^s(D)$ with $0 \leq r < p - 1/2$, $r \leq s \leq p$,

$$\inf_{u_L \in \hat{V}_L} \|u - u_L\|_{H^r} \lesssim \begin{cases} h^{s-r} \log h \frac{d-1}{2} & \text{if } r = 0, s = p \\ h^{s-r} & \text{else} \end{cases}$$  (4.5)

Please note that we can also state the approximation rate in terms of the level index $L$, i.e., $h^{s-r} = 2^{-L(s-r)}$ and $\|\log h\| \lesssim L$.

4.2 Wavelet discretization

We cast the variational form on bounded domain (3.21) into the matrix form (1.13) for the finite-dimensional subspace $\hat{V}_{L+1}$.

Here, we use the integrated jump form (3.18) and integrate the first sum by parts

$$- \int_{R} \int_{R^d} (u(x + z_i) - u(x) - z_i \partial_i u(x)) v(x) k_i(z_i) \, dx \, dz_i$$
We define the one-dimensional mass matrix

\[ M_i(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \delta^2 u(x + z_i) v(x) k_i^{-2}(z_i) \, dx \, dz_i \]

\[ = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \delta_i u(x + z_i) \delta_i v(x) k_i^{-2}(z_i) \, dx \, dz_i, \]

for \( i = 1, \ldots, d \), where \( k_i^{-2}(x) = \operatorname{sgn}(x) \int_{I_i} U(z) \, dz \) is the second antiderivative of \( k \) vanishing at \( \pm \infty \). Therefore, the jump part of the bilinear form can be written as

\[ \mathcal{E}_j(u, v) = \sum_{i=1}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \delta_i u(x + z_i) \delta_i v(x) k_i^{-2}(z_i) \, dx \, dz_i \]

\[ - \sum_{i=2}^{d} \sum_{|z| = i} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \delta^2 u(x + z^I) v(x) U^I(z^I) \, dx \, dz^I. \]

Using the basis \( \psi_{\ell, k} = \psi_{\ell_1, k_1} \cdots \psi_{\ell_d, k_d}, 0 \leq \ell_1 + \cdots + \ell_d \leq L, k_i \in \nabla \ell_i \) of \( \tilde{V}_{L+1} \) we need to compute the stiffness matrix for the diffusion part

\[ A^{\text{BS}}_{(\ell', k'), (\ell, k)} = \mathcal{E}_{\text{BS}}(\psi_{\ell, k}, \psi_{\ell', k'}) = \sum_{i,j=1}^{d} \int_{\Omega} \partial_i \psi_{\ell, k} \partial_j \psi_{\ell', k'} \, dx, \]

and also for the jump part

\[ A^j_{(\ell', k'), (\ell, k)} = \sum_{i=1}^{d} \int_{\Omega} \partial_i \psi_{\ell, k} \partial_i \psi_{\ell', k'} \, dx \]

\[ - \sum_{i=2}^{d} \sum_{|z| = i} \int_{\Omega} \int_{\Omega} \delta^2 \psi_{\ell, k} \partial_i \psi_{\ell', k'} \, dx \, dz^I. \]

We define the one-dimensional mass matrix \( M^i \), stiffness matrix \( S^i \), cross matrix \( C^i \) as

\[ M^i_{(\ell', k'), (\ell, k)} := \int_{-R}^{R} \psi_{\ell, k} \psi_{\ell', k'} \, dx, \quad S^i_{(\ell', k'), (\ell, k)} := \int_{-R}^{R} \psi'_{\ell, k} \psi'_{\ell', k'} \, dx, \]

\[ C^i_{(\ell', k'), (\ell, k)} := \int_{-R}^{R} \psi_{\ell, k} \psi_{\ell', k'} \, dx, \quad \text{(4.6)} \]

for \( 0 \leq \ell \leq L, k \in \nabla \ell \). Then, we can write an entry in the diffusion stiffness matrix as

\[ A^{\text{BS}}_{(\ell', k'), (\ell, k)} = \sum_{i=1}^{d} \frac{1}{2} Q_{ij} \sum_{j \neq i} S^i_{(\ell', k'), (\ell, k)} \prod_{j \neq i} M^j_{(\ell', k'), (\ell, k)} \]

\[ - \sum_{i < j} Q_{ij} C^i_{(\ell', k'), (\ell, k)} C^j_{(\ell', k'), (\ell, k)} \prod_{r \neq i,j} M^r_{(\ell', k'), (\ell, k)}. \]
4 Wavelet basis

Let $S_{i',i}^{l}$ denote the block matrix with entries $(S_{i',k',i,k}^{l})_{k',k} \in \nabla_{l'} \cap \nabla_{l}$. We use the same notation when we refer to the matrix of the same size as $S^{l}$ but with zero entries except the block matrix $S_{i',i}^{l}$. With this convention $S^{l}$ can be written as

$$S^{l} = \sum_{0 \leq l',l \leq L} S_{i',i}^{l}.$$

The full tensor product and the sparse tensor product of two matrices with multilevel structure are defined as

$$S^{i} \otimes M^{j} = \sum_{0 \leq l',l \leq L} S_{i',i}^{l} \otimes M_{j,j}^{l}, \quad S^{i} \rhd M^{j} = \sum_{0 \leq l',l \leq L} S_{i',i}^{l} \otimes M_{j,j}^{l},$$

respectively. Therefore, the stiffness matrix $A^{BS}$ can be computed as a $d$ iterated sparse tensor product using one-dimensional matrices

$$A^{BS} = \sum_{i=1}^{d} \frac{1}{2} Q_{ii} \bigotimes_{1 \leq j \leq i-1} M_{j}^{i} \bigotimes_{i+1 \leq j \leq d} M_{j}^{i} - \sum_{i<j} Q_{ij} \bigotimes_{1 \leq r \leq i-1} M_{r}^{i} \bigotimes_{i+1 \leq r \leq j-1} M_{r}^{j} \bigotimes_{j+1 \leq r \leq d} M_{r}^{j}.$$

Additionally, for the jump part we define

$$A^{i}_{(i',k'),(i,k)} := - \int_{-R}^{R} \int_{-R}^{R} \psi'_{x,k}(x+z) \psi'_{i',k'}(x) k_{i}^{-2}(z) \, dx \, dz,$$

$$A^{j}_{(i',k'),(i,k)} := \int_{|I|} \int_{|I|} \partial^{2} \psi_{x,k}(x+z) \psi_{x,k}(x) U_{x}^{j}(z) \, dx \, dz. \quad (4.7)$$

where $\ell_{I} = (\ell_{i})_{i \in I}$, $0 \leq \ell_{i} \leq L$, $k_{I} = (k_{i})_{i \in I}$, $k_{i} \in \nabla_{l_{i}}$, $I \subset \{1, \ldots, d\}$, $|I| > 1$, and write the jump stiffness matrix as

$$A^{j}_{(i',k'),(i,k)} = - \sum_{i=1}^{d} \sum_{|I|=i} A^{j}_{(i',k'),(i,k)} \prod_{j \in I} M_{j}^{i}_{(\ell_{j},k_{j}), (\ell_{j},k_{j})}.$$

As in the diffusion case we can then compute the jump stiffness matrix $A^{j}$ as a sparse tensor product using the matrices $A^{j}$ and $M^{j}$. Applying the $\theta$-scheme in time, we can write the problem (3.21) in fully discrete matrix form, similar to (1.13)

$$\text{Find } \{u_{L}^{m+1}\} \in \mathbb{R}^{N} \text{ such that for } m = 0, \ldots, M - 1$$

$$\Delta t^{-1} M(u_{L}^{m+1} - u_{L}^{m}) + \theta A u_{L}^{m+1} + (1 - \theta) A u_{L}^{m} = 0,$$

$$u_{L}^{0}(0) = u_{L,0}.$$
4.3 Wavelet compression of the Lévy measure

Wavelet compression for isotropic domains has been studied extensively by various authors, e.g., \[20, 19, 31, 69\]. It is shown that compression yields asymptotically optimal complexity (on not necessarily tensor product domains) in the sense that the number of non-zero entries in the resulting matrices grows linearly with the number of degrees of freedom. These results are extended to anisotropic spaces on sparse tensor product spaces in \[53\].

To define the compression scheme we need to introduce some notation. Consider tensor product wavelets \( \psi_{\ell,k} = \psi_{\ell_1,k_1} \otimes \cdots \otimes \psi_{\ell_d,k_d} \) and \( \psi_{\ell',k'} = \psi_{\ell'_1,k'_1} \otimes \cdots \otimes \psi_{\ell'_d,k'_d} \). The distance of support in each coordinate direction is denoted by

\[
\delta_{x_i} := \text{dist}\{\text{supp} \psi_{\ell_i,k_i}, \text{supp} \psi_{\ell'_i,k'_i}\},
\]

for \( i = 1, \ldots, d \) and the distance of singular support

\[
\delta^\text{sing}_{x_i} := \begin{cases} 
\text{dist}\{\text{singsupp} \psi_{\ell_i,k_i}, \text{supp} \psi_{\ell'_i,k'_i}\}, & \text{if } \ell_i \leq \ell'_i \\
\text{dist}\{\text{supp} \psi_{\ell_i,k_i}, \text{singsupp} \psi_{\ell'_i,k'_i}\}, & \text{else}
\end{cases}
\]

Let \( 0 < \alpha < p - \frac{1}{2} \), define

\[
\tilde{L}_{\ell,\ell'} := \begin{cases} 
L(p - \alpha/2) - p |\ell| & \text{if } p(L - |\ell|) \geq \alpha/2(L - |\ell|_\infty) \\
-\alpha/2 |\ell|_\infty & \text{else}
\end{cases}
+ \begin{cases} 
L(p - \alpha/2) - p |\ell'| & \text{if } p(L - |\ell'|) \geq \alpha/2(L - |\ell'|_\infty) \\
-\alpha/2 |\ell'|_\infty & \text{else}
\end{cases}
\]

and \( m_i := \ell_i + \ell'_i - 2 \min\{\ell_i, \ell'_i\} \). Furthermore, we denote the index sets \( \mathcal{I}^c_{\ell,\ell'} \subset \mathcal{I}_{\ell,\ell'} \subset \{1, \ldots, d\} \) by

\[
\mathcal{I}^c_{\ell,\ell'} = \left\{ i \in \{1, \ldots, d\} : \delta_{x_i} > 2^{-\min(\ell_i, \ell'_i)} \right\}, \quad \mathcal{I}_{\ell,\ell'} = \{1, \ldots, d\} \setminus \mathcal{I}^c_{\ell,\ell'},
\]

and set

\[
P_{\ell,\ell'} = \tilde{L}_{\ell,\ell'} - \tilde{p}(\ell_i + \ell'_i) + \alpha \sum_{j \neq i} \min\{\ell_j, \ell'_j\} + \frac{1}{2} \sum_{j \in \mathcal{I}_{\ell,\ell'} \setminus \{i\}} m_j - \tilde{p} \sum_{j \in \mathcal{I}_{\ell,\ell'} \setminus \{i\}} m_j,
\]

\[
\tilde{P}_{\ell,\ell'} = \tilde{L}_{\ell,\ell'} - \tilde{p} \max\{\ell_i, \ell'_i\} + \alpha \sum_{j \neq i} \min\{\ell_j, \ell'_j\} + \frac{1}{2} \sum_{j \in \mathcal{I}_{\ell,\ell'} \setminus \{i\}} m_j - \tilde{p} \sum_{j \in \mathcal{I}_{\ell,\ell'}} m_j.
\]
4 Wavelet basis

The cut-off parameter are now defined by
\[
B^i_{\ell,\ell'} = a \max \left\{ 2^{-\min(\ell,\ell')}, 2^{g_{\ell,\ell}/(2\bar{p}+\alpha)} \right\}, \quad a > 0,
\]
\[
\tilde{B}^i_{\ell,\ell'} = a' \max \left\{ 2^{-\max(\ell,\ell')}, 2^{\bar{g}_{\ell,\ell}/(\bar{p}+\alpha)} \right\}, \quad a' > 0.
\]

The compression scheme is based on the fact that the matrix entries \( A_{(\ell',k'),(\ell,k)} = \mathcal{E}(\psi_{\ell,k}, \psi_{\ell,k'}) \) can be estimated a-priori and therefore neglected if these are smaller than some cut-off parameter. There are two reasons for an entry to be omitted. Either the distance of the supports \( \text{supp} \psi_{\ell,k} \) and \( \text{supp} \psi_{\ell',k'} \) or the distance of the singular supports is large enough for some \( i \in \{1, \ldots, d\} \).

**Theorem 4.3.1.** Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \) and characteristic triplet \( (\mathcal{Q}, \nu, \gamma) \) and Dirichlet form \( \mathcal{E}(\cdot, \cdot) \). Assume \( \mathcal{Q} > 0 \) and that the Lévy density \( k \) satisfies (2.21) with \( 0 < \alpha < p - 1/2 \). Define the compression scheme by
\[
\tilde{\mathbf{A}}_{(\ell',k'),(\ell,k)} = \begin{cases} 
0, & \text{if } \exists i \in \mathbb{I}_{\ell,\ell'}^d : \delta_{x_i} > B^i_{\ell,\ell'} \\
0, & \text{if } \exists i \in \mathbb{I}_{\ell,\ell'}^d : \delta^{\text{sing}}_{x_i} > \tilde{B}^i_{\ell,\ell'} \\
A_{(\ell',k'),(\ell,k)}, & \text{else}
\end{cases}
\]

If \( \bar{p} > 2dp - (d+1)\alpha \) and \( \alpha \leq 2/d \), the number of non-zero entries for the compressed matrix \( \tilde{\mathbf{A}} \) is \( O(2^L L^{2(d-1)}) \).

**Proof.** See [53, Theorem 4.6.3].

**Remark 4.3.2.** Here, we only stated the isotropic case \( \alpha_1 = \ldots = \alpha_d = \alpha \), i.e., in each direction the same compression is used. Although we still get asymptotically optimal complexity, the number of matrix entries can further be reduced using anisotropic compression. The corresponding compression scheme is defined in [53].

We give an example for the matrix compression

**Example 4.3.3.** Let \( a = 1 \), \( a' = 1 \), \( p = 2 \), \( \bar{p} = 2 \), \( \alpha = 0.5 \) and \( L = 7 \). The corresponding compression scheme is plotted in Figure 4.3 for \( d = 1, 2, 3 \). Zero entries due to the first compression are left white, zero entries due to the second compression are colored red and non-zero entries blue regardless of their size. For \( d = 1 \) there are 18% non-zero entries, for \( d = 2 \) it is 35% and for \( d = 3 \), we have 52%.

We now consider the fully discrete, i.e., space and time, problem in matrix form (4.8) where we replace the matrix \( \mathbf{A} \) with the compressed matrix \( \tilde{\mathbf{A}} \).

Find \( \tilde{u}_{L}^{m+1} \in \mathbb{R}^N \) such that for \( m = 0, \ldots, M - 1 \)
\[
\Delta t^{-1} \mathbf{M}(\tilde{u}_{L}^{m+1} - \tilde{u}_{L}^{m}) + \theta \tilde{\mathbf{A}} \tilde{u}_{L}^{m+1} + (1 - \theta) \tilde{\mathbf{A}} \tilde{u}_{L}^{m} = 0,
\]
(4.12)
\[
\tilde{u}_{L}^{0}(0) = \tilde{u}_{L,0}.
\]

There exists a unique solution \( \tilde{u}_{L}^{m} \) of the perturbed scheme (4.12) and the solution converges at the optimal rate.
4.3 Wavelet compression of the Lévy measure

4.3.1 Wavelet compression of the Lévy measure

Figure 4.3: Wavelet compression of the Lévy measure for level \( L = 7 \) in \( d = 1, 2, 3 \)

**Theorem 4.3.4.** Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \) satisfying Assumption 2.3.4. Consider \( \tilde{A} \) as given in Theorem 4.3.1 and let all assumptions of Theorem 4.3.1 hold. Then, there exists a unique solution \( \tilde{u}_M \) of the perturbed \(-\)scheme (4.12). Furthermore, if \( u \in C^1([0, T], H^p(D_R)) \cap C^3([0, T], V^r) \) and the approximation \( \hat{u}_{L,0} = \tilde{V}_{L+1} \) of the initial data \( u_0 \) is quasi-optimal in \( L^2(D_R) \), then for \( \theta = 1/2 \)

\[
\|u^M - \hat{u}_L^M\|^2_{L^2(D_R)} + \Delta t \sum_{m=0}^{M-1} \left\| u^{m+1/2} - \hat{u}_L^{m+1/2} \right\|_{V^r}^2 \leq C(u) \left( \Delta t^4 + 2^{-2L(p-\alpha/2)} \right),
\]

where \( u \) is the solution of (3.21) and the constant \( C(u) > 0 \) depends on higher space and time derivatives of \( u \).

**Proof.** See [53, Theorem 2.2.3, Theorem 3.3.8].

**Remark 4.3.5.** As already noted in Remark 1.3.8 for rough initial data we need to use nonuniform time steps to obtain optimal convergence rates. Furthermore, for Barrier contracts the solution may not be smooth at the barrier \( \partial D \) as indicated in Remark 3.1.4. Nonuniform mesh widths in space can be applied for again optimal convergence rates.

These convergence rates are shown in the next example. We only look at independent margins because here one can obtain an exact solution with which to compare the finite element solution.

**Example 4.3.6.** Let \( d = 2 \) and consider two independent tempered stable marginal densities

\[
k_i(z) = c_i \frac{e^{-\beta_i z^+}}{|z|^{1+\alpha_i}} 1_{\{z < 0\}} + c_i \frac{e^{-\beta_i z^-}}{|z|^{1+\alpha_i}} 1_{\{z > 0\}}, \quad i = 1, 2.
\]

We solve the elliptic problem

\[
A[u] = f \quad \text{on} \quad \Omega = [0, 1]^2,
\]

(4.13)
where \( f \) is chosen such that the exact solution is

\[
    u(x) = \begin{cases} 
        (x_1^2 - 2x_1^3 + x_1^4)(x_2^2 - 2x_2^3 + x_2^4) & \text{if } x \in \Omega \\
        0 & \text{else} 
    \end{cases}
\]

We set the model parameter \( c_1 = c_2 = 1, \beta_1^- = 10, \beta_1^+ = 15, \beta_2^- = 9, \beta_2^+ = 16, \alpha_1 = 0.5, \alpha_2 = 0.7 \) and the compression parameter \( a = 1, a' = 1, p = 2, \overline{p} = 2 \). For \( L = 8 \) the absolute value of the entries in the stiffness matrix \( A \) and the compressed matrix \( \hat{A} \) are shown in Figure 4.4. Here, large entries are colored red. For the stiffness matrix blue entries are small but non-zero whereas for the compressed matrix blue entries are zero either due to the first or second compression. One clearly sees that the compression scheme neglects small entries.

We solve problem (4.13) for various mesh widths \( h_L = 2^{-L} \) and plot the convergence rate in Figure 4.5. To compare the rates we also solved the problem on full grid. In the left picture it can be seen that sparse grid has (up to log terms) the same rate as full grid and that the compression scheme preserves the convergence rate. To better show the advantage of sparse grid we additionally plot the convergence rate in terms of degrees of freedom. For full grid we have \( N = O(2^L) \) and for sparse grid \( \hat{N} = O(L 2^L) \). The convergence rate in full grid shows the “curse of dimension”, whereas for the sparse grid we still obtain the optimal rate (up to log terms).

Since in general the matrix entries \( A_{(e',k'),(e,k)} \) cannot be computed exactly, we need to approximate these with a numerical quadrature rule. To still retain the optimal order of convergence, we require a certain accuracy.

**Theorem 4.3.7.** Consider \( \hat{A} \) as given in Theorem 4.3.1 and let \( \tilde{A} \) be a perturbed matrix such that

\[
    \left| (\hat{A} - \tilde{A})_{(e',k'),(e,k)} \right| \lesssim \epsilon_{e,e'}, \quad \text{with} \quad \epsilon_{e,e'} \lesssim 2^{-(|e'|+|e|)/p} 2^{-L}. 
\]

Then, Theorem 4.3.4 still holds with \( \tilde{A} \) instead of \( \hat{A} \).
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Figure 4.5: Convergence rate of the wavelet discretization in terms of the mesh width $h$ (left) and in terms of degrees of freedom (right).

Proof. We need to show that the error satisfies $\|S\|_2 \lesssim 2^{-L_{\ell,\ell'}}$, as shown in [53, Theorem 2.5.2]. Let $S = [\tilde{A}_{\ell',\ell} - \hat{A}_{\ell',\ell}]$ where $\hat{A}_{\ell',\ell}$ is the block matrix with entries $(\hat{A}_{\ell',\ell})_{(\ell',\ell') \in \mathcal{C}, (\ell,\ell) \in \mathcal{C}}$. Estimating for each row (or column), the sum over all entries yields

\[
\sum_{k_1 \in \mathcal{C}_{\ell_1}} \cdots \sum_{k_d \in \mathcal{C}_{\ell_d}} S_{k',k} \lesssim 2^{\ell'} \epsilon_{\ell,\ell'} \quad \text{and} \quad \sum_{k_1 \in \mathcal{C}_{\ell_1}} \cdots \sum_{k_d \in \mathcal{C}_{\ell_d}} S_{k',k} \lesssim 2^{\ell} \epsilon_{\ell,\ell'},
\]

We can rewrite this as

\[
\sum_{k_1 \in \mathcal{C}_{\ell_1}} \cdots \sum_{k_d \in \mathcal{C}_{\ell_d}} w_k S_{k',k} \lesssim w_k 2^{(\ell_1+\ell')/2} \epsilon_{\ell,\ell'} ,
\]

with weights $w_k = 2^{(\ell_1-\ell')/4}$ and $w_k' = 2^{(\ell - |\ell'|)/4}$. Using the Schur lemma [48, Lemma 4] we obtain the required result.

For the computation we use sparse tensor products to obtain $\hat{A}$. Theorem 4.3.7 still holds.

Corollary 4.3.8. If the perturbed matrix $\hat{A}$ satisfies

\[
\|\hat{A}_{\ell',\ell} - \hat{A}_{\ell',\ell}\|_2 \lesssim 2^{-L_{\ell,\ell'}},
\]

with $\ell, \ell' \in \mathbb{N}^{d-1}$ then

\[
\|\hat{A}_{\ell',\ell} \otimes M^d_{\ell',\ell'} - \hat{A}_{\ell',\ell} \otimes M^d_{\ell',\ell'}\|_2 \lesssim 2^{-L_{(\ell,\ell'),(\ell',\ell')}} .
\]

Proof. Follows immediately since $2^{-L_{(\ell,\ell'),(\ell',\ell')}} \geq 2^{-L_{\ell,\ell'}}$. 

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4.4 Multilevel preconditioning

We have to solve a linear system

\[
(M + \theta \Delta t \mathbf{A}) \mathbf{u}^{m+1} = (M - \Delta t (1 - \theta) \Delta t \mathbf{A}) \mathbf{u}^m,
\]

at each time step \( m = 0, \ldots, M - 1 \). For an iterative solution of these systems, \( \mathbf{B} \mathbf{u} = \mathbf{b} \),
we use multilevel preconditioning. The preconditioner is obtained by using the wavelet norm equivalences. With (4.4) for \( s = 0 \) we have for every \( u \in \mathcal{V}_{L+1} \) with coefficient vector \( \mathbf{u} \in \mathbb{R}^N \) that

\[
\langle \mathbf{u}, \mathbf{u} \rangle \lesssim \langle \mathbf{u}, \mathbf{M} \mathbf{u} \rangle \lesssim \langle \mathbf{u}, \mathbf{u} \rangle.
\]

Denote by \( \mathbf{D}_A \) the diagonal matrix with entries \( 2^{\alpha_1 \ell_1} + \ldots + 2^{\alpha_d \ell_d} \) for an index corresponding to level \( \ell = (\ell_1, \ldots, \ell_d) \). Then, Theorem 4.3.4 and (4.4) for \( s_i = \alpha_i / 2, i = 1, \ldots, d \) imply that

\[
\langle \mathbf{u}, \mathbf{D}_A \mathbf{u} \rangle \lesssim \langle \mathbf{u}, \mathbf{A} \mathbf{u} \rangle \lesssim \langle \mathbf{u}, \mathbf{D}_A \mathbf{u} \rangle,
\]

Thus, we have \( \langle \mathbf{u}, \mathbf{D} \mathbf{u} \rangle \lesssim \langle \mathbf{u}, \mathbf{B} \mathbf{u} \rangle \lesssim \langle \mathbf{u}, \mathbf{D} \mathbf{u} \rangle \), with the diagonal matrix \( \mathbf{D} = \mathbf{I} + \theta \Delta t \mathbf{D}_A \). Written in terms of \( \hat{\mathbf{u}} = \mathbf{D}^{-1/2} \mathbf{u} \) we finally obtain

\[
\| \hat{\mathbf{u}} \|^2 \lesssim \langle \hat{\mathbf{u}}, \mathbf{D}^{-1/2} \mathbf{B} \mathbf{D}^{-1/2} \hat{\mathbf{u}} \rangle \lesssim \| \hat{\mathbf{u}} \|^2.
\]

The linear system \( \hat{\mathbf{B}} \hat{\mathbf{u}} = \hat{\mathbf{b}} \) with preconditioned matrix \( \hat{\mathbf{B}} = \mathbf{D}^{-1/2} \mathbf{B} \mathbf{D}^{-1/2} \) and right hand side \( \hat{\mathbf{b}} = \mathbf{D}^{-1/2} \mathbf{b} \) can be solved with GMRES [56] in a number of steps which is independent of level index \( L \).

**Lemma 4.4.1.** For the linear system \( \hat{\mathbf{B}} \hat{\mathbf{u}} = \hat{\mathbf{b}} \) let \( \hat{\mathbf{u}}_j \) denote the iterate obtained by the GMRES method with initial guess \( \hat{\mathbf{u}}_0 \). There is a constant \( 0 < r < 1 \) independent of \( L \) and \( \Delta t \) such that

\[
| \hat{\mathbf{u}} - \hat{\mathbf{u}}_j | \lesssim r^j | \hat{\mathbf{u}} - \hat{\mathbf{u}}_0 |.
\]

**Proof.** See [28].

**Example 4.4.2.** Let \( d = 2 \) and consider two independent tempered stable marginal densities as in Example 4.3.6. We compute the price of a basket option, \( g(s_1, s_2) = (\frac{1}{2}s_1 + \frac{1}{2}s_2 - K)_+ \), with maturity \( T = 0.5 \), strike \( K = 100 \) and interest rate \( r = 0.01 \). We set \( c_1 = c_2 = 1, \beta_1^- = 10, \beta_1^+ = 15, \beta_2^- = 9, \beta_2^+ = 16, \alpha_1 = 0.5, \alpha_2 = 0.7 \) and compute the maximum number of GMRES iterations for \( m = 0, \ldots, M - 1 \) where \( \Delta t = 0.005 \). The values are shown in Table 4.1.

<table>
<thead>
<tr>
<th>Level L</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. Iterations</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 4.1: Number of GMRES iterations
5 Composite Gauss quadrature rules

We have to evaluate integrals \( \int_{[1,1]} f(z^I)U^I(z)dz \) for \( I \subset \{1, \ldots, d\} \) as seen in the last section. The tail integrals \( U^I(z) \) have a singularity at the origin and possibly on each axis as shown in Example 2.3.3. Therefore, we can not use standard quadrature rules for integration since these depend on the smoothness of the function. Instead, we use a composite Gauss quadrature rule proposed in [62]. Elementary Gauss quadrature formulas of varying orders on subdomains are combined. The size of these subdomains decreases geometrically towards the singular support of the integrand. Multidimensional quadrature rules are obtained by using tensor products of one-dimensional quadrature formulas. We start recalling error estimates for the basic Gauss-Legendre quadrature rules.

5.1 Gauss-Legendre quadrature

For a given function \( f \in C([0,1]) \) we set \( I^{[0,1]} = \int_0^1 f(s)ds \) and denote the \( g \)-point Gauss-Legendre integration rule on \([0,1]\) by \( Q_g^{[0,1]} = \sum_{j=1}^g \omega_{g,j} f(\xi_{g,j}) \). If \( f \in C^{2g}([0,1]) \) we obtain the following error estimate (see, e.g., [22])

\[
E_g^{[0,1]} f := \left| I^{[0,1]} - Q_g^{[0,1]} f \right| = \frac{(g!)^4}{(2g + 1)!(2g)!^2} \max_{\xi \in [0,1]} \left| f^{(2g)}(\xi) \right|, \quad \xi \in [0,1].
\]

We use the Stirling formula \( g! \sim \sqrt{2\pi g} g^g e^{-g} \) to obtain the estimate

\[
E_g^{[0,1]} f \lesssim \frac{2^{-4g}}{(2g)!^2} \max_{\xi \in [0,1]} \left| f^{(2g)}(\xi) \right|.
\] (5.1)

On \([0,1]^d\) we approximate the integral, \( I^{[0,1]^d} f := \bigotimes_{1 \leq i \leq d} I^{[0,1]} f = \int_{[0,1]^d} f(s)ds \), for \( f \in C([0,1]^d) \) by a tensor product Gauss-Legendre quadrature rule

\[
Q_g^{[0,1]^d} f := \bigotimes_{1 \leq i \leq d} Q_g^{[0,1]} f = \sum_{j_1, \ldots, j_d=1}^g \prod_{i=1}^d \omega_{g,j_i} f(\xi_{g,j_1}, \ldots, \xi_{g,j_d}),
\]

and obtain the following error bound.
5 Composite Gauss quadrature rules

Lemma 5.1.1. If \( f \in C^{2g}([0, 1]^d) \), the quadrature error \( E_g^{[0, 1]^d} f := I^{[0, 1]^d} f - Q_g^{[0, 1]^d} f \) is bounded by

\[
\left| E_g^{[0, 1]^d} f \right| \leq \frac{2^{-4g}}{(2g)!^d} \max_{\xi \in [0, 1]^d} \left| \partial^2 \xi \cdot f(\xi) \right|.
\]  

(5.2)

Proof. We prove this lemma by induction over the dimension \( d \). With (5.1) it is true for \( d = 1 \). For \( d > 1 \) we have

\[
\left| E_g^{[0, 1]^d} f \right| = \left( \bigotimes_{1 \leq i \leq d} I^{[0, 1]} - \bigotimes_{1 \leq i \leq d} Q_g^{[0, 1]} \right) f
\]

\[
= \left( \bigotimes_{1 \leq i \leq d} I^{[0, 1]} - \bigotimes_{1 \leq i \leq d-1} (I^{[0, 1]} \otimes Q_g^{[0, 1]}) + \bigotimes_{1 \leq i \leq d-1} (I^{[0, 1]} \otimes Q_g^{[0, 1]}) - \bigotimes_{1 \leq i \leq d} Q_g^{[0, 1]} \right) f
\]

\[
\leq \frac{2^{-4g}}{(2g)!^d} \max_{\xi \in [0, 1]^d} \left| \partial^2 \xi \cdot f(\xi) \right| + \frac{2^{-4g}}{(2g)!^d} \sum_{i=1}^{d-1} \max_{\xi \in [0, 1]^d} \left| \partial^2 \xi \cdot f(\xi) \right|.
\]

\[\square\]

For our analysis we consider a class of functions which have singularities on the origin and on the axes.

Assumption 5.1.2. Let \( f \in L^1([0, 1]^d) \). There exist \( 0 < \alpha < d, \alpha \notin \mathbb{N} \), \( C_f > 0 \), such that for \( k \in \mathbb{N}_0, i = 1, \ldots, d \)

\[
\left| \partial_i^k f(\xi) \right| \lesssim k! C_f^k \| \xi \|^{-\alpha} \xi_i^{-k}, \quad \forall \xi \in (0, 1)^d.
\]  

(5.3)

Equation (5.3) is satisfied by all tail integrals corresponding to a Lévy process which satisfy Assumption 2.3.4, in particular (2.18) and (2.19). We introduce the notation \( I^{[0, 1]} f_{\xi_i} := \int_1^1 f(\xi_1, \ldots, \xi_{i-1}, s_i, \ldots, \xi_d) ds_i \) where we just integrate over the \( i \)-th dimension, \( i \in \{1, \ldots, d\} \). Similarly \( Q_g^{[0, 1]} f_{\xi_i} \) and \( E_g^{[0, 1]} f_{\xi_i} := I^{[0, 1]} f_{\xi_i} - Q_g^{[0, 1]} f_{\xi_i} \). We can now state the basic error estimates on rectangular domains.

Proposition 5.1.3. Let \( i \in \{1, \ldots, d\} \), interval \([a, b]\) with \( a, b \in \mathbb{R}, 0 \leq a \leq b \leq 1 \) and \( h = b - a \). Assume \( f \) satisfies (5.3) and set \( I = \{1, \ldots, d\} \setminus i \). Then for

\[
\left| E_g^{[a, b]} f_{\xi_i} \right| \lesssim \| \xi \|^{-\alpha} h^{2g} \frac{C_f h}{4a} a^{-\frac{\alpha}{2}}, \quad \text{for } a > 0, \quad \xi \in (0, 1)^{d-1},
\]  

(5.4)

\[
\left| E_0^{[a, b]} f \right| \lesssim \| \xi \|^{-\alpha} h^{1-\frac{\alpha}{2}} a^{-\frac{\alpha}{2}}, \quad \text{for } a = 0, \quad \xi \in (0, 1)^{d-1}.
\]  

(5.5)
5.2 Composite Gauss quadrature

\textbf{Proof.} Consider the transformation \( \varphi : [0, 1] \rightarrow [a, b] \), \( \varphi(\xi) = a + h\xi \). Then, with \( I^{[a,b]} f_\xi = I^{[0,1]} (f_\xi \circ \varphi) h \) and \( \partial^k f(\xi_1, \ldots, \varphi(\xi_i), \ldots, \xi_d) = \partial^k f h^k \) we get (5.4) by

\[
\left| E_{[a,b]}^{[a,b]} f_\xi \right| = h \left| E_{[0,1]}^{[0,1]} (f_\xi \circ \varphi) \right| \lesssim h \frac{2^{-4q}}{(2g)!} \max_{\xi_i \in [a,b]} \left| \partial^g f(\xi) \right| h^{2g} \lesssim \| \xi^q \|_\infty^{-\alpha + \frac{\nu}{2}} h \left( \frac{C_f h}{4a} \right)^{2q} a^{-\frac{\nu}{2}}.
\]

With \( |f| \lesssim \| \xi \|_\infty^{-\alpha} \) one obtains (5.5) since

\[
\left| E_{[0,h]}^{[0,h]} f_\xi \right| = \left| \int_{[0,h]} f(\xi_1, \ldots, s_i, \ldots, \xi_d) ds_i \right| \lesssim \| \xi^q \|_\infty^{-\alpha + \frac{\nu}{2}} \int_{[0,h]} s_i^{-\frac{\alpha}{q}} ds_i \lesssim \| \xi^q \|_\infty^{-\alpha + \frac{\nu}{2}} h^{1-\frac{\nu}{q}}.
\]

\( \square \)

5.2 Composite Gauss quadrature

On \([0, 1]\) a geometric partition is given by \( 0 < n < \sigma^n < \sigma^{n-1} < \ldots < \sigma < 1 \) for \( n \in \mathbb{N} \), \( \sigma \in (0, 1) \). We denote the subdomains by \( \Lambda_j := [\sigma^{n+1-j} \sigma^{n-j}] \), with \( j = 1, \ldots, n \) and \( \Lambda_0 := [0, \sigma^n] \). Given a linear degree vector \( \mathbf{q} \in \mathbb{N}^n \), \( q_j = [\mu j] \) with slope \( \mu > 0 \), we use on each subdomain \( \Lambda_j \), \( j = 1, \ldots, n \), a Gauss quadrature with degree \( q_j \) and no quadrature points in \( \Lambda_0 \). The subdomains and the quadrature points are plotted in Figure 5.1 for the grading factor \( \sigma = 0.3 \), refinements \( n = 4 \) and a linear degree vector with slope \( \mu = 1 \).

![Figure 5.1: Composite Gauss quadrature in d = 1](image)

The composite Gauss quadrature rule in the \( i \)-th direction is defined by

\[
Q_{\sigma^q}^{\mathbf{q}} f_\xi = \sum_{j=1}^n Q_{q_j}^{\Lambda_j} f_\xi, \quad i \in \{1, \ldots, d\},
\]

and converges exponentially.

\textbf{Theorem 5.2.1.} Let \( i \in \{1, \ldots, d\} \) and \( f \) satisfy (5.3). Consider

\[
\sigma \in (0, 1), \quad \text{such that} \quad w = \frac{C_f(1-\sigma)}{4\sigma} < 1,
\]

and converges exponentially.
5 Composite Gauss quadrature rules

and a linear degree vector $q = (q_1, \ldots, q_n)$,

$$q_j = \lceil \mu j \rceil, \quad \text{with slope } \mu > \frac{(1 - \frac{d}{2}) \ln \sigma}{2 \ln w}. \quad (5.8)$$

Then,

$$|I_{[0,1]} f_{\xi_j} - Q_n^w f_{\xi_j}| \lesssim \|\xi\|_\infty^{-\alpha + \frac{d}{2}} \sigma^{n(1 - \frac{d}{2})}. \quad (5.9)$$

Proof. On each $\Lambda_j$, $j = 1, \ldots, n$ we have the following estimate using (5.4) with $a = \sigma^{n+1-j}$ and $h = \sigma^{n-j}(1 - \sigma)$

$$|E_{\Lambda_j} f_{\xi_j}| \lesssim \|\xi\|_\infty^{-\alpha + \frac{d}{2}} h \left( C_f (1 - \sigma) \right)^{2g} \sigma^{-(n+1-j)\frac{d}{2}} \lesssim \|\xi\|_\infty^{-\alpha + \frac{d}{2}} w^{2g} \sigma^{(n-j)(1 - \frac{d}{2})}.$$

Summing over all subdomains $j = 1, \ldots, n$, yields

$$\sum_{j=1}^n |E_{\Lambda_j} f_{\xi_j}| \lesssim \|\xi\|_\infty^{-\alpha + \frac{d}{2}} \sum_{j=1}^n w^{2g} \sigma^{(n-j)(1 - \frac{d}{2})} \lesssim \|\xi\|_\infty^{-\alpha + \frac{d}{2}} \sigma^{n(1 - \frac{d}{2})} \sum_{j=1}^\infty \left( w^{2g} \sigma^{-1} \right)^j.$$

The last sum converges since $\mu > \frac{(1 - \frac{d}{2}) \ln \sigma}{2 \ln w}$. We neglected the subdomains $\Lambda_0$ in the composite Gauss quadrature. Using (5.5) we have

$$|E_{[0,1]} f_{\xi_j}| \lesssim \|\xi\|_\infty^{-\alpha + \frac{d}{2}} \sigma^{n(1 - \frac{d}{2})}.$$

\[ \square \]

Remark 5.2.2. Condition (5.7) is suboptimal. Using [62, Theorem 4.1] or [13, Proposition 2.8] we can obtain exponential convergence for any $\sigma \in (0, 1)$.

We define the composite Gauss quadrature on $[0,1]^d$ by the tensor product of one-dimensional composite Gauss quadrature rules $Q_n^w \sigma^{q_1, \ldots, q_d} f = \bigotimes_{1 \leq i \leq d} Q_n^w \sigma^{q_i} f_{\xi_i}$. The subdomains and the quadrature points on $[0,1]$ are plotted in Figure 5.2 for the grading factor $\sigma = 0.3$, refinements $n = 4$ and linear degree vectors with slope $\mu_1 = \cdots = \mu_d = 1$ in $d = 2, 3$.

The composite Gauss quadrature rule converges exponentially with respect to the number $N$ of Gauss points.

Theorem 5.2.3. Let $f$ satisfy (5.3). Consider a grading factor $\sigma \in (0,1)$ satisfying (5.7) and linear degree vectors $(q_1, \ldots, q_d)$ satisfying (5.8). Then, there exist a $\gamma > 0$ such that the quadrature error decays exponentially

$$|f^{[0,1]^d} f - Q_n^w \sigma^{q_1, \ldots, q_d} f| \lesssim e^{-\gamma 2^N N}.$$
5.2 Composite Gauss quadrature

Figure 5.2: Composite Gauss quadrature in $d = 2, 3$

Proof. We prove this theorem in two steps.

1. As in proof of Lemma 5.1.1 we prove

$$\left| I_{[0,1]^d} f - Q^n_{\sigma(q_1,\ldots,q_d)} f \right| \lesssim e^{-\gamma n},$$

by induction over the dimension $d$. With (5.9) it is true for $d = 1$. For $d > 1$ we have with (5.9)

$$\left| I_{[0,1]^d} f - Q^n_{\sigma(q_1,\ldots,q_d)} f \right| = \left( \bigotimes_{1 \leq i \leq d} I_{[0,1]} - \bigotimes_{1 \leq i \leq d} Q^n_{\sigma(q_i)} \right) f$$

$$= \bigotimes_{1 \leq i \leq d-1} I_{[0,1]} \otimes \left( I_{[0,1]} - Q^n_{\sigma(q_i)} \right) f$$

$$+ \left( \bigotimes_{1 \leq i \leq d-1} I_{[0,1]} - \bigotimes_{1 \leq i \leq d-1} Q^n_{\sigma(q_1,\ldots,q_{d-1})} \right) \otimes Q^n_{\sigma(q_d)} f$$

$$\lesssim \int_{[0,1]^{d-1}} \left\| \xi^T \right\|_{\infty}^{-\alpha n / \pi} d\xi^T \sigma^{n(1-\alpha n / \pi)} + e^{-\gamma n} \sum_{j=1}^n \sum_{m=1}^n q_{j,m} \omega_{j,m} \xi_j \xi_m$$

$$\lesssim e^{-\gamma n}.$$

2. Let $\mu_1 = \max\{\mu_1, \ldots, \mu_d\}$. We estimate the number of quadrature points by

$$N \leq \left( \sum_{j=1}^n q_{j,1} \right)^d \lesssim \left( \sum_{j=1}^n j \right)^d \lesssim n^{2d}.$$
We give a numerical example which shows the exponential convergence of the composite Gauss quadrature formula.

**Example 5.2.4.** Consider the function \( f(x) = \left( \sum_{i=1}^{d} x^{\beta_i} \right)^{-\frac{1}{\theta}} \) on the domain \([0, 1]^d\) for \( \alpha = \beta_1 = \ldots = \beta_d = 0.5 \). We apply a composite Gauss quadrature formula with grading factor \( \sigma = 0.2 \) and linear degree vectors with slope \( \mu_1 = \ldots = \mu_d = 0.5 \). For \( \theta = 0.5 \) the relative quadrature error \( \left| I_{[0,1]^d} f - Q_{\sigma}^{(\mu_1, \ldots, \mu_d)} f \right| / \left| I_{[0,1]^d} f \right| \) versus \( \frac{1}{2} \sqrt{N} \) is plotted in logarithmic scale in Figure 5.3. Additionally, we also plot the relative error for \( d = 2 \) and various \( \sigma \). As already seen in the proof of Theorem 5.2.3 the convergence rate depends on \( 1 - \alpha/d \) which increases in \( d \).

![Figure 5.3: Exponential convergence of the composite Gauss quadrature for \( \theta = 0.5 \), \( \sigma = 0.2 \) and \( d = 2, 3 \) (left) and \( d = 2 \) and various \( \sigma \) (right)](image)
6 Computational scheme

As seen in Chapter 4 we need to compute matrix entries of the type

\[ B_{(\ell', k'), (\ell, k)} = \int_{\mathbb{R}^d} \int_{D_R} \partial_1 \cdots \partial_d \psi_{\ell, k}(x + z) \psi_{\ell', k'}(x) \kappa(z) \, dx \, dz, \tag{6.1} \]

where the kernel \( \kappa \) satisfies (5.3), i.e.,

\[ \left| \partial_i^k \kappa(z) \right| \lesssim k! C_f^k \|z\|_{\infty}^{-\alpha} z_i^{-k}, \quad \forall z \in \mathbb{R}^d, \quad k \in \mathbb{N}_0, \quad i = 1, \ldots, d, \]

for \( 0 < \alpha < d, \alpha \notin \mathbb{N} \) and \( C_f > 0 \). Introducing a new variable \( y = x + z \) we can write the integral (6.1) as

\[ B_{(\ell', k'), (\ell, k)} = \int_{\Sigma_{\ell, k}} \int_{\Sigma_{\ell', k'}} \partial_1 \cdots \partial_d \psi_{\ell, k}(y) \psi_{\ell', k'}(x) \kappa(y - x) \, dy \, dx, \tag{6.2} \]

where \( \Sigma_{\ell, k} = \text{supp} \psi_{\ell, k} \). Similar equations have been studied for the boundary element methods, although only in the isotropic setting. Several schemes have been developed to solve these problems in dimension \( d \leq 3 \), see [31, 58, 67] and the references therein. We adapt these methods to the anisotropic case for \( d \geq 1 \). Throughout this section we consider wavelets as described in Example 4.1.1 which are piecewise linear.

6.1 Hierarchical data structure

For an efficient implementation of the compression scheme it is necessary to have an hierarchical data structure. Therefore, we introduce an hierarchical element tree up to a given level \( L \in \mathbb{N} \).

6.1.1 Element tree

We start with \( D_{(0, \ldots, 0), (1, \ldots, 1)} = D_R \) as the first generation. On the \( \ell \)-th generation we consider the elements \( D_{\ell, k} \) where the multiindices are given by \( \ell = (\ell_1, \ldots, \ell_d), \ell_i = 0, \ldots, \ell, \quad i = 1, \ldots, d \) with \( \|\ell\|_{\infty} = \ell \), and \( k = (k_1, \ldots, k_d), k_i = 1, \ldots, 2^{\ell_i}, \quad i = 1, \ldots, d \). Each element \( D_{\ell, k} \) has sons \( D_{\ell + \tilde{\ell}, k + k} \) where \( \|\ell + \tilde{\ell}\| \leq L \),

\[ \tilde{\ell}_i = \begin{cases} 0 & \text{if } \ell_i \neq \ell \\ \in \{0,1\} & \text{if } \ell_i = \ell \end{cases}, \quad i = 1, \ldots, d, \quad \text{with } \|\tilde{\ell}\|_{\infty} = 1, \]
6 Computational scheme

and $\tilde{k}_i = 2^{\tilde{\ell}_i}(k_i - 1) + 1, \ldots, 2^{\tilde{\ell}_i}k_i$, $i = 1, \ldots, d$. Since there exists a bijective mapping which indicates each element $D_{\ell,k}$ uniquely by an integer $\lambda$, we write shortly $D_{\lambda} = D_{\ell,k}$ and set $|\lambda| = |\ell|$, $\|\ell\|_{\infty} = \|\ell\|_{\infty}$.

**Example 6.1.1.** To illustrate this data structure we let $d = 2$, $L = 2$ and plot the corresponding elements in Figure 6.1. The initial element $D_1$ has 8 sons, $D_{2,1}^{\ell_1}$, $D_{2,2}^{\ell_1}$, $D_{2,3}^{\ell_1}$, $D_{2,4}^{\ell_1}$, $D_{2,5}^{\ell_1}$, $D_{2,6}^{\ell_1}$, $D_{2,7}^{\ell_1}$, $D_{2,8}^{\ell_1}$. The element $D_2$ has only 2 sons $D_{10}^{\ell_2}$, $D_{11}^{\ell_2}$ and so on. Note that the elements $D_{\lambda}, \lambda = 6, \ldots, 9$ do not have any sons due to the sparse tensor product spaces.

![Figure 6.1: Hierarchical element tree for $d = 2$ and $L = 2$](image)

Similar to the standard single-scale finite element method, we do not compute the matrix entry (6.2) directly over $\text{supp} \psi_{\ell,k}$, since $\psi_{\ell,k}$ is not smooth. Instead, we decompose $\Sigma_{\ell,k}$ into a set of elements $\bigcup D_{\lambda}$ such that $\psi_{\ell,k}|_{D_{\lambda}}$ is smooth. More precisely, consider the set

$$\Sigma_{\ell,k} = \text{supp} \psi_{\ell,k} = \bigcup_{D_{\lambda} \in \Sigma_{\ell,k}} D_{\lambda}, \quad \Sigma_{\ell,k}^{\text{sing}} = \text{singsupp} \psi_{\ell,k} = \bigcup_{D_{\lambda} \in \Sigma_{\ell,k}} \partial D_{\lambda},$$

and

$$\psi_{\ell,k}(x)|_{D_{\lambda}} = \sum_{n_1, \ldots, n_d = 1}^{2^{\tilde{\ell} - \ell}/2} \omega_{\ell,k,n,\lambda} \hat{\phi}_{n}(\varphi_{\lambda}^{-1}(x)), \quad D_{\lambda} \in \Sigma_{\ell,k},$$

with weights $\omega_{\ell,k,n,\lambda} = \prod_{i=1}^{d} \omega_{\ell_i,n_i,\lambda}$, shape functions $\hat{\phi}_{n}(z) = \prod_{i=1}^{d} \hat{\phi}_{n_i}(z_i)$ and diffeomorphism $\varphi_{\lambda} : D_{\lambda} \to [0, 1]^d$. The one-dimensional weights follow immediately from Example 4.1.1 and the one-dimensional shape functions are $\hat{\phi}_1(z) = 1 - z$, $\hat{\phi}_2(z) = z$. 

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6.1 Hierarchical data structure

6.1.2 Compression pattern

To set up the compression scheme we need to check the distance criteria for each matrix coefficient. Checking these conditions would require \(O(N^2)\) operations. For an efficient computation we exploit the tree structure described above. We denote by \(\sigma_{i,k_i} = \text{supp} \psi_{i,k_i}\), \(\sigma_{i,k_i} = \text{singsupp} \psi_{i,k_i}\), \(i = 1,\ldots,d\) and define a wavelet tree by

**Definition 6.1.2.** The wavelet \(\psi_{i,\text{son}}\) is the son of \(\psi_{i,\text{father}}\) if \(\sigma_{i,\text{son}} \subseteq \sigma_{i,\text{father}}\), \(i = 1,\ldots,d\) and there exists \(i \in \{1,\ldots,d\}\) such that \(\tilde{\ell}_i = \ell_i + 1\).

Then, the following lemmas hold.

**Lemma 6.1.3.** Let \(\sigma_{i,\text{father}}, \sigma_{i',\text{father}} > B^i_{\ell,\ell'}\) for \(i \in \mathcal{I}_{\ell,\ell'}\) and \(\sigma_{i+1,\text{son}} \subseteq \sigma_{i,\text{father}}\), \(i, \sigma_{i+1,\text{son}} \subseteq \sigma_{i',\text{father}}\). Then \(\sigma_{i+1,\text{son}}, \sigma_{i',\text{father}} \rangle B^i_{\ell,\ell'}\) and \(\sigma_{i+1,\text{son}}, \sigma_{i',+1,\text{son}} \rangle B^i_{\ell,\ell'}\) where \(\tilde{\ell}' = (\ell_1, \ldots, \ell_i + 1, \ldots, \ell_d)\) and \(\tilde{\ell} = (\ell_1', \ldots, \ell_i', 1, \ldots, \ell_d')\).

**Proof.** The result follows from \(B^i_{\ell,\ell'} \geq B^i_{\ell,\ell'} \geq B^i_{\ell,\ell'}\).

**Lemma 6.1.4.** Let \(\sigma_{i,\text{father}}, \sigma_{i',k_i'} > \tilde{B}^i_{\ell,\ell'}\) for \(i, \ell_i > \ell_i'\) and \(\sigma_{i+1,\text{son}} \subseteq \sigma_{i,\text{father}}\). Then \(\sigma_{i+1,\text{son}}, \sigma_{i',k_i'} \rangle \tilde{B}^i_{\ell,\ell'}\) where \(\tilde{\ell} = (\ell_1, \ldots, \ell_i + 1, \ldots, \ell_d)\) and \(\tilde{\ell} = (\ell_1', \ldots, \ell_i' + 1, \ldots, \ell_d')\).

**Proof.** The result follows from \(\tilde{B}^i_{\ell,\ell'} \geq \tilde{B}^i_{\ell,\ell'}\).

**Remark 6.1.5.** Similar results for different wavelets in \(d = 2\) are given in [31].

Using Lemma 6.1.3 and 6.1.4 we have only to check the distance criteria for coefficients which have a non-zero father. The number of operations for setting up the compression scheme is then obviously of log linear complexity \(O(2^L L^{2(d-1)})\).
6 Computational scheme

6.2 Matrix computation

Replacing the wavelets in (6.2) by the element representation (6.3) leads to

\[ B(\ell', k', \ell, k) = \sum_{D_\lambda \in \mathcal{D}_\ell} \sum_{D_{\lambda'} \in \mathcal{D}_{\ell'}} \sum_{n_1, \ldots, n_d=1}^{2} \sum_{n'_1, \ldots, n'_d=1}^{2} \omega_{\ell, k, n, \lambda} \omega_{\ell', k', n', \lambda'} Q(\lambda, n, \lambda', n'), \]

with

\[ Q(\lambda, n, \lambda', n') = 2^{\lambda/2 + |\lambda'|/2} \int_{D_\lambda} \int_{D_{\lambda'}} \partial_1 \cdots \partial_d \widehat{\phi}_n(\varphi_\lambda^{-1}(y)) \widehat{\phi}_{n'}(\varphi_{\lambda'}^{-1}(x)) \kappa(y-x) dy dx, \]

or in terms of the reference interval

\[ Q(\lambda, n, \lambda', n') = (-1)^{|n|} 2^{\lambda/2 + |\lambda'|/2} \prod_{i=1}^d h'_i \int_{[0,1]^d} \int_{[0,1]^d} \widehat{\phi}_{n'}(\widehat{x}) \widehat{\kappa}_{\lambda, \lambda'}(\widehat{x}, \widehat{y}) d\widehat{x} d\widehat{y}, \quad (6.4) \]

where \( h'_i = R 2^{-\ell_i}, i = 1, \ldots, d, \) and \( \widehat{\kappa}_{\lambda, \lambda'}(\widehat{x}, \widehat{y}) = \kappa(\varphi_\lambda(\widehat{y}) - \varphi_{\lambda'}(\widehat{x})) \). Therefore, computing the matrix entries reduces to computing the element-element interactions \( Q(\lambda, n, \lambda', n') \).

We can again use the hierarchical data structure to obtain an entry of a father element from the son elements. For example, for a father element \( D_{\text{father}} = D(\ell, k) \) with the two sons \( D_{\text{son}_1} = D(\ell_1, \ldots, \ell_i + 1, \ldots, \ell_d), (k_1, \ldots, 2k_i - 1, \ldots, k_d) \) and \( D_{\text{son}_2} = D(\ell_1, \ldots, \ell_i + 1, \ldots, \ell_d), (k_1, \ldots, 2k_i, \ldots, k_d) \), we get

\[ Q(\text{father, n}, \lambda, n', \lambda') = (Q(\text{son}_1, n, \lambda, n') + Q(\text{son}_2, n, \lambda, n')) 2^{-3/2}. \quad (6.5) \]

Similarly,

\[ Q(\lambda, n, \text{father, } n'_1, \ldots, n'_d) = \left( Q(\lambda, n, \text{son}_1, n'_1, \ldots, n'_d) + Q(\lambda, n, \text{son}_1, n'_1, \ldots, n'_d) / 2 \right) 2^{-1/2}. \quad (6.6) \]

6.3 Numerical integration

Consider \( \ell, k, \ell', k' \in \mathbb{N}^d \), the corresponding \( \lambda, \lambda' \), fix \( n, n' \) and introduce the notation \( \delta_i = \text{dist}(D^{\ell_i}, D^{\ell_i'}) \) where \( D_\lambda = D^{\ell_1} \times \cdots \times D^{\ell_d} \). Let \( \epsilon > 0 \), \( i \in \{1, \ldots, d\} \) and set \( I = \{1, \ldots, d\} \setminus i \) and \( z = y-x \). We distinguish several cases: The integrand \( \widehat{\kappa}_{\lambda, \lambda'}(x, y) \) is non-singular in \( y_i - x_i \), i.e., \( \delta_i > 0 \), the elements are identical, \( D_\lambda = D_{\lambda'} \), or the elements share a common vertex.

1. Let \( \delta_i > C_f \max\{h_i, h_i'\}/4 \). Consider

\[ g \gtrsim \frac{\ln \epsilon + \frac{\alpha}{2} \ln \delta_i + (|\lambda'|/2 - |\lambda|/2) \ln 2}{2 \ln w}, \]

\[ g' \gtrsim \frac{\ln \epsilon + (\frac{\alpha}{2} - 1) \ln \delta_i + (|\lambda'|/2 - |\lambda|/2) - \ell'_i \ln 2}{2 \ln w'}, \quad (6.7) \]
number of Gauss points where \( w = \frac{h_i C_L}{4\pi} \) and \( w' = \frac{h_i C_L}{4\pi} \). Furthermore, let the standard Gauss quadrature points and weights on \([0,1]\) be given by \( \xi_g, \omega_g \in \mathbb{R}^g \). Then, we define quadrature points \( \xi_i \in \mathbb{R}^{gg'} \) and weights \( \omega_i \in \mathbb{R}^{gg'} \) by

\[
\xi^i = \xi^\lambda \otimes 1_{g'} - 1_g \otimes \xi^{\lambda'}, \quad \omega^i = 2^{i/2 + \ell_i'/2} h_i^{-1} \phi_n(1_g \otimes \xi^{\lambda'}_g) \cdot \omega^\lambda \otimes \omega^{\lambda'},
\]

(6.8)

where \( 1_g = (1, \ldots, 1)^\top \in \mathbb{R}^g \), \( \xi^\lambda_j = (k_i - 1) h_i + h_i \xi_j \) and \( \omega^\lambda_j = h_i \omega_j \).

2. Let \( \delta_i = 0 \), \( \ell_i = \ell'_i \) and \( k_i = k'_i \). Consider

\[
n \geq \frac{\ln \epsilon + (|\lambda'|/2 - |\lambda|/2 - \ell_i) \ln 2}{(1 - \frac{\delta}{\ell_i}) \ln (h_i \sigma)}.
\]

(6.9)

refinements for the composite Gauss quadrature and \( \sigma, q \) satisfying (5.7), (5.8).

Furthermore, let the composite Gauss quadrature points and weights on \([0,1]\) be given by \( \xi_n, \omega_n \in \mathbb{R}^N \). Then, we define quadrature points \( \xi^i \in \mathbb{R}^{2N} \) and weights \( \omega^i \in \mathbb{R}^{2N} \) by

\[
\begin{align*}
(\xi^i_j)_{1 \leq j \leq N} &= h_i \xi_j, \quad (\xi^i_j)_{N+1 \leq j \leq 2N} = -h_i \xi_j, \\
(\omega^i_j)_{1 \leq j \leq N} &= h_i \int_0^1 \hat{\phi}_n'(x(1 - \xi_j)) dx \cdot (1 - \xi_j) \cdot \omega_j, \\
(\omega^i_j)_{N+1 \leq j \leq 2N} &= h_i \int_0^1 \hat{\phi}_n'(x(1 - \xi_j)) dx \cdot (1 - \xi_j) \cdot \omega_j.
\end{align*}
\]

(6.10)

3. Let \( \delta_i = 0 \), \( \ell_i = \ell'_i \) and \( k_i = k'_i - 1 \). Consider

\[
\begin{align*}
g &\geq \frac{\ln \epsilon + \frac{\delta}{\ell_i} \ln h_i + (|\lambda'|/2 - |\lambda|/2) \ln 2}{2 \ln \omega}, \\
n &\geq \frac{\ln \epsilon + (|\lambda'|/2 - |\lambda|/2 - \ell_i) \ln 2}{(2 - \frac{\delta}{\ell_i}) \ln (h_i \sigma)},
\end{align*}
\]

(6.11)

number of Gauss points or refinements, respectively. We define quadrature points \( \xi^i \in \mathbb{R}^{g+N} \) and weights \( \omega^i \in \mathbb{R}^{g+N} \) by

\[
\begin{align*}
(\xi^i_j)_{1 \leq j \leq g} &= h_i + h_i \xi_j, \quad (\xi^i_j)_{g+1 \leq j \leq g+N} = h_i \xi_j, \\
(\omega^i_j)_{1 \leq j \leq g} &= h_i \int_0^1 \hat{\phi}_n'(x(1 - \xi_j)) dx \cdot (1 - \xi_j) \cdot \omega_j, \\
(\omega^i_j)_{g+1 \leq j \leq g+N} &= h_i \int_0^1 \hat{\phi}_n'(x(1 - \xi_j)) dx \cdot \xi_j \cdot \omega_j.
\end{align*}
\]

(6.12)

Using a tensor product quadrature formula we have the following error estimate.
6 Computational scheme

**Theorem 6.3.1.** Assume that the kernel \( \kappa \) satisfies (5.3). Consider \( \epsilon > 0 \) and assume either \( \delta_i > C_f \max\{h_i, h_i'\}/4 \) or \( \delta_i = 0 \), \( \ell_i = \ell_i' \), \( k_i = k_i' \) or \( \delta_i = 0 \), \( \ell_i = \ell_i' \), \( k_i = k_i' - 1 \) for \( i = 1, \ldots, d \). Define the \( d \)-dimensional quadrature points and weights by

\[
\xi_i = \bigoplus_{1 \leq j \leq 1-i} \xi^j \bigoplus_{i+1 \leq j \leq d} \xi^j, \quad \omega = \bigoplus_{1 \leq j \leq d} \omega^j,
\]

where the one-dimensional quadrature points and weights \( \xi^j, \omega^j \), \( i = 1, \ldots, d \) are given by (6.8), (6.10) or (6.12). Then, we obtain

\[
\left| 2^{\lambda/2+|\lambda'|/2} \prod_{i=1}^d h_i' \int_{[0,1]^d} \phi_{\nu'}(\hat{x}) \hat{\kappa}_{\lambda,\lambda'}(\hat{x}, \hat{y}) d\hat{y} d\hat{x} - \langle \omega, \kappa(\xi_1, \ldots, \xi_d) \rangle \right| \lesssim \epsilon.
\]

**Proof.** We again distinguish three cases.

1. Let \( \delta_i > C_f \max\{h_i, h_i'\}/4 \) and define \( f(\hat{x}, \hat{y}) = 2^{\lambda/2-|\lambda'|/2} \phi_{\nu'}(\hat{x}) \hat{\kappa}_{\lambda,\lambda'}(\hat{x}, \hat{y}) \). Using the standard product rule

\[
\partial_{\hat{x}}^n f(\hat{x}, \hat{y}) = \partial_{\hat{x}}^n \hat{\kappa}_{\lambda,\lambda'}(\hat{x}, \hat{y}) \phi_{\nu'}(\hat{x}) + \partial_{\hat{x}}^{n-1} \hat{\kappa}_{\lambda,\lambda'}(\hat{x}, \hat{y}) \partial_{\hat{y}} \phi_{\nu'}(\hat{x}),
\]

there holds for \( h_i = R 2^{-\ell_i} \),

\[
\left| \partial_{\hat{x}}^n f(\hat{x}, \hat{y}) \right| \lesssim 2^{\lambda/2-|\lambda'|/2} n! (h_i C_f)^n \delta_i^{-\frac{a}{2} - n} \| \hat{z} \|_{\infty}^{-\alpha - \frac{a}{2}}, \quad n \in \mathbb{N}_0,
\]

and for \( \delta_i \gtrsim h_i' \) and \( h_i' = R 2^{-\ell_i} \),

\[
\left| \partial_{\hat{x}}^n f(\hat{x}, \hat{y}) \right| \lesssim 2^{\lambda/2-|\lambda'|/2} n! (h_i' C_f)^{n-1} \delta_i^{-\frac{a}{2} - n+1} \| \hat{z} \|_{\infty}^{-\alpha - \frac{a}{2}}, \quad n \in \mathbb{N}_0.
\]

Therefore, we obtain similar to (5.4)

\[
\left| E_{g,g'}^{[0,1]^2} f_{\hat{x}, \hat{y}} \right| \lesssim \left( h_i C_f \right)^{2g} \left( \frac{h_i C_f}{4 \delta_i} \right)^{2g'} \delta_i^{-\frac{a}{2} - 1} 2^{g'}.
\]

Choosing the number of Gauss points according to (6.7) we have

\[
\left| E_{g,g'}^{[0,1]^2} f_{\hat{x}, \hat{y}} \right| \lesssim \| \hat{z} \|_{\infty}^{-\alpha - \frac{a}{2}} \epsilon.
\]

2. Let \( \delta_i = 0 \), \( \ell_i = \ell_i' \) and \( k_i = k_i' \). The integrand \( \hat{\kappa}_{\lambda,\lambda'}(\hat{x}, \hat{y}) \) is singular on the diagonal \( x_i = y_i \). We first transform this singularity to the axis. Let \( \kappa_i(s - t) = \kappa(z_1, \ldots, h_i(s - t), \ldots, z_d) \) and consider the integral

\[
I = \int_{[0,1]} \int_{[0,1]} \phi(s) \psi(t) \kappa_i(s - t) ds dt.
\]
6.3 Numerical integration

Introducing the variable \( z = s - t \) and splitting the integral yields

\[
I = - \int_{[0,1]} \int_s^{s-1} \phi(s)\psi(s-z)\kappa_i(z)dzds
\]

\[
= \int_{[0,1]} \int_s^{0} \phi(s)\psi(s-z)\kappa_i(z)dzds + \int_{[0,1]} \int_0^s \phi(s)\psi(s-z)\kappa_i(z)dzds.
\]

With \( x = s - z \), \( y = -z \) we have

\[
\int_{[0,1]} \int_{s-1}^{0} \phi(s)\psi(s-z)\kappa_i(z)dzds = \int_{[0,1]} \int_0^x \phi(x-y)\psi(x)\kappa_i(-y)dydx,
\]

and therefore,

\[
I = \int_{[0,1]} \int_0^x \phi(x)\psi(x-y)\kappa_i(y)dydx + \int_{[0,1]} \int_0^x \phi(x-y)\psi(x)\kappa_i(-y)dydx.
\]

Finally setting \( x = \xi + \eta(1-\xi), y = \xi \), we obtain

\[
I = \int_{[0,1]} \int_{[0,1]} \phi(\xi + \eta(1-\xi))\psi(\eta(1-\xi))\kappa_i(\xi)(1-\xi)d\xi d\eta
\]

\[
+ \int_{[0,1]} \int_{[0,1]} \phi(\eta(1-\xi))\psi(\xi + \eta(1-\xi))\kappa_i(-\xi)(1-\xi)d\xi d\eta.
\]

The function

\[
f(\tilde{x}, \tilde{y}) = 2^{\lambda/2-|\lambda'|/2}(1-\tilde{y}_i) \left( \phi_{\eta_i}(\tilde{y}_i + \tilde{x}_i(1-\tilde{y}_i))\kappa_i(\tilde{y}_i) + \phi_{\eta_i}(\tilde{x}_i(1-\tilde{y}_i))\kappa_i(-\tilde{y}_i) \right),
\]

has a singularity at \( \tilde{y}_i = 0 \) and satisfies (5.3) with respect to \( \tilde{y}_i \), i.e.,

\[
|\partial_{\tilde{y}_i} f(\tilde{x}, \tilde{y})| \lesssim 2^{\lambda/2-|\lambda'|/2} k! (h_i C_f)^k (h_i \tilde{y}_i)^{-\alpha - k} \| z^T \|^{-\alpha + \frac{2}{\alpha'}}, \quad k \in \mathbb{N}_0.
\]

The integrand \( f \) is polynomial in the \( \tilde{x}_i \) and can be integrated exactly. Thus, similar to Theorem 5.2.1 we obtain

\[
\left| I^{[0,1]} f_{\tilde{y}_i} - Q_{h_i \sigma}^n f_{\tilde{y}_i} \right| \lesssim 2^{\lambda/2-|\lambda'|/2} 2^{\ell_i} \| z^T \|^{-\alpha + \frac{2}{\alpha'}} (h_i \sigma)^n (1-\frac{2}{\alpha'})
\]

where \( \sigma, q \) satisfy (5.7), (5.8). Choosing the number of refinements according to (6.9) we have

\[
\left| I^{[0,1]} f_{\tilde{y}_i} - Q_{h_i \sigma}^n f_{\tilde{y}_i} \right| \lesssim \| z^T \|^{-\alpha + \frac{2}{\alpha'}} \varepsilon.
\]

3. Let \( \delta_i = 0, \ell_i = \ell_i' \) and \( k_i = k_i' - 1 \). Similar to the case of identical elements we have \( \kappa_i(s + t) = \kappa(z_1, \ldots, h_i(s + t), \ldots, z_d) \) and transform the integral

\[
I = \int_{[0,1]} \int_{[0,1]} \phi(s)\psi(t)\kappa_i(s + t)dsdt.
\]
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\begin{equation}
I = \int_{[0,1]} \int_{[0,1]} \phi(\xi + \eta(1 - \xi))\psi(1 + \eta(\xi - 1))\kappa_i(\xi + 1)(1 - \xi)\,d\xi\,d\eta \\
+ \int_{[0,1]} \int_{[0,1]} \phi(\eta\xi)\psi(\xi(1 - \eta))\kappa_i(\eta)\xi\,d\xi\,d\eta.
\end{equation}

(6.14)

The function

\[ f(\tilde{x}, \tilde{y}) = 2^{|\lambda|/2 - |\lambda'|/2} \phi(\tilde{y} + \tilde{x}(1 - \tilde{y}))\kappa_i(\tilde{y} + 1)(1 - \tilde{y}), \]

can be integrated exactly in the \( \tilde{x}_i \) direction and has no singularity in \( \tilde{y}_i \), i.e.,

\[ \left| \partial_{\tilde{y}_i} f(\tilde{x}, \tilde{y}) \right| \lesssim 2^{|\lambda|/2 - |\lambda'|/2} (h_iC_f)^k h_i^{-\frac{\alpha}{2} - k} \| z^{T} \|_{\infty}^{-\frac{\alpha}{2}}, \quad k \in \mathbb{N}_0. \]

The function

\[ f(\tilde{x}, \tilde{y}) = 2^{|\lambda|/2 - |\lambda'|/2} \phi(\tilde{x} \tilde{y}_i)\kappa_i(\tilde{y}_i)\tilde{y}_i, \]

can again be integrated exactly in the \( \tilde{x}_i \) direction and has singularity in \( \tilde{y}_i \), i.e.,

\[ \left| \partial_{\tilde{y}_i} f(\tilde{x}, \tilde{y}) \right| \lesssim 2^{|\lambda|/2 - |\lambda'|/2} k! (h_iC_f)^k (h_i\tilde{y}_i)^{-\frac{\alpha}{2} - k + 1} \| z^{T} \|_{\infty}^{-\alpha/2}, \quad k \in \mathbb{N}_0. \]

Choosing the number of Gauss points and refinements according to (6.11) we again obtain an error in the \( i \)-th direction of order \( \| z^{T} \|_{\infty}^{-\alpha/2} \epsilon \).

Finally, tensorization arguments as in Theorem 5.2.3 yield the required result. \qed

6.4 Adaptive strategy

As proposed in [31] we define an adaptive strategy to compute the element-element interactions \( Q_{(\lambda,m),(\lambda',m')} \) with the precision \( c_{\epsilon,E} \) given by (4.14).

We loop over the dimension \( i = 1, \ldots, d \). For each \( i \) we do:

1. Starting point. If \( \delta_i > C_f \max\{h_i, h_i'\}/4 \) we define quadrature points in the \( i \)-th direction according to (6.7). Else if \( \delta_i = 0, \ell_i = \ell_i' \) and \( k_i = k_i' \) or \( k_i = k_i' - 1, k_i' = k_i - 1 \) define quadrature points according to (6.9) or (6.11). Otherwise go to item 2 if \( \ell_i > \ell_i' \), item 3 if \( \ell_i' > \ell_i \) and item 4 if \( \ell_i = \ell_i' \).

2. Case \( \ell_i > \ell_i' \). Replace the larger element \( D_{\lambda} \) by its two sons and compute the associated element-element interaction with precision \( 2^{-3/2} \epsilon_{\ell,E} \) according to item 1. The desired element-element interaction is calculated via formula (6.6).

3. Case \( \ell_i' > \ell_i \). Replace the larger element \( D_{\lambda} \) by its two sons and compute the associated element-element interaction with precision \( 2^{-1/2} \epsilon_{\ell,E} \) according to item 1. The desired element-element interaction is calculated via formula (6.5).
Case $\ell_i = \ell'_i$. Replace both elements $D_i$ by their two sons and compute the associated element-element interaction with precision $\epsilon_{\ell,\ell'}$ according to item 1. The desired element-element interaction is calculated via formulas (6.5) and (6.6).

Note that using this strategy we only have to compute an element-element interaction where Theorem 6.3.1 holds. The next lemma shows that the algorithm stops after, at the most, $O(\|\ell_i - \ell'_i\|_\infty)$ steps.

**Lemma 6.4.1.** Let $i \in \{1, \ldots, d\}$. The following statements concerning the computation of the element-element interaction by the above algorithm are valid:

1. The given element-element interaction is subdivided into at most $O(|\ell_i - \ell'_i|)$ interactions $Q_{(\lambda, \mu), (\lambda', \mu')}$ where $\hat{\ell}_i \geq \ell_i$, $\hat{\ell}'_i \geq \ell'_i$.
2. If $\ell_i \leq \ell'_i$, there holds $\ell_i \leq \hat{\ell}_i \leq \hat{\ell}'_i \sim \ell'_i$. The analogous result holds if $\ell'_i \leq \ell_i$.
3. On a fixed level $\hat{\ell}_i$ and $\hat{\ell}'_i$ the number of directly computed as well as subdivided element-element interactions is $O(1)$.

**Proof.** See [31, Lemma 9.7].

Now with formulas (6.5), (6.6), Lemma 6.4.1 and Theorem 6.3.1 it follows that the proposed quadrature algorithm computes the desired element-element interactions with a precision that stays proportional to $\epsilon_{\ell,\ell'}$.

**Corollary 6.4.2.** Let $X$ be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(Q, \nu, \gamma)$. Assume the Lévy density $k(z)$ satisfies (2.21), i.e. is real analytic outside of $z_i = 0$, $i = 1, \ldots, d$. Let $\epsilon_{\ell,\ell'}$ be given by (4.14). Then, the number of quadrature points to compute an entry $A_{(\ell',k'),(\ell,k)}$ is at most $O(L^{2d})$ and the overall operations to compute the stiffness matrix $A$ at most of log linear complexity $O(2^L L^{4d-2})$.

**Proof.** We have for the one-dimesional Gauss points in (6.8), $g, g' \lesssim L$, for the refinements in (6.10), $n \lesssim L$ and for the quadrature points and refinements in (6.12) again, $g, n \lesssim L$. Therefore, we need at most $O(L^2)$ quadrature points in each direction $i = 1, \ldots, d$. 

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7 Model sensitivities and Greeks

Calculating price sensitivities is a central modeling and computational task for risk management and hedging. We distinguish between two classes: Sensitivities of the price $V$ in (3.1) to variations of a model parameter, like the Greek Vega $\partial_\sigma V$, and sensitivities of $V$ to variations of the state space such as the Greek Delta $\partial_S V$.

7.1 Sensitivity with respect to model parameters

Suppose the market model and hence the operator $A = A_{BS} + A_J$ in (3.5) depends on some model parameter $\eta$. We want to calculate the sensitivity of the solution $u$ of (3.5) with respect to $\eta$. To this end, we write $u(\eta_0)$ for a fixed realization $\eta_0$ of $\eta$ in order to emphasize the dependence of $u$ on $\eta_0$ in (3.5). Typical examples are the Greeks Vega $(\partial_\sigma u)$, Rho $(\partial_r u)$ and Vomma $(\partial_{\sigma\sigma} u)$. Other sensitivities which are not so commonly used in the financial community are the sensitivity of the price with respect to the jump intensity or the order of the process that models the underlying. We show that the finite element approximation to such sensitivities satisfies again the scheme (1.12) with a right hand side $f^{m+\theta}$ which depends on the approximation $u_h^{m+\theta}$ of the pricing function $u$. We also show that the approximation of these sensitivities converge with the same rate as $u_h$.

**Definition 7.1.1.** Let $X$ be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(Q, \nu, \gamma)$. We call $X$ a parametric Lévy model with admissible parameter set $S_\eta$, if

(i) for all $\eta \in S_\eta$, $X$ is a Lévy process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

(ii) the mapping $S_\eta \ni \eta \rightarrow \{Q, \nu, \gamma\}$ is infinitely differentiable.

Let $C$ be a Banach space over a domain $D \subset \mathbb{R}^d$. $C$ is the space of parameters or coefficients in the operator $A$ and $S_\eta \subseteq C$ is the set of admissible coefficients. We denote by $u(\eta_0)$ the unique solution to (1.6) and introduce the derivative of $u(\eta_0)$ with respect to $\eta_0 \in S_\eta$ as the mapping $D_{\eta_0} u(\eta_0) : C \rightarrow V$,

$$\tilde{u}(\delta \eta) := D_{\eta_0} u(\eta_0)(\delta \eta) := \lim_{s \rightarrow 0^+} \frac{1}{s} (u(\eta_0 + s\delta \eta) - u(\eta_0)), \quad \delta \eta \in C.$$ 

We also introduce the derivative of $A(\eta_0)$ with respect to $\eta_0 \in S_\eta$

$$\tilde{A}(\delta \eta) \varphi := D_{\eta_0} A(\eta_0)(\delta \eta) \varphi := \lim_{s \rightarrow 0^+} \frac{1}{s} (A(\eta_0 + s\delta \eta) \varphi - A(\eta_0) \varphi), \quad \varphi \in V, \quad \delta \eta \in C.$$
7 Model sensitivities and Greeks

We assume that $$\tilde{A}(\delta \eta) \in \mathcal{L}(\tilde{V},\tilde{V}^*)$$ with $$\tilde{V}$$ being a real and separable Hilbert space satisfying

$$\tilde{V} \subseteq V \xrightarrow{d} H \cong H^* \xrightarrow{d} V^* \subseteq \tilde{V}^*.$$ 

We further assume that there exists a real and separable Hilbert space $$\overline{V} \subseteq \tilde{V}$$ such that $$\tilde{A}v \in V^*, \forall v \in \overline{V}$$. We have the following relation between $$D_{\eta_0} u(\eta_0)(\delta \eta)$$ and $$u$$.

**Lemma 7.1.2.** Let $$\tilde{A}(\delta \eta) \in \mathcal{L}(\tilde{V},\tilde{V}^*), \forall \delta \eta \in \mathcal{C}$$ and $$u(\eta_0) : (0, T] \to \overline{V}, \eta_0 \in \mathcal{S}$$ be the unique solution to

$$\begin{align*}
\partial_t u(\eta_0) + A(\eta_0)u(\eta_0) &= 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \\
u(\eta_0)(0, \cdot) &= g(x) \quad \text{in } \mathbb{R}^d.
\end{align*}$$

(7.1) (7.2)

Then $$\tilde{u}(\delta \eta)$$ solves

$$\begin{align*}
\partial_t \tilde{u}(\delta \eta) + A(\eta_0)\tilde{u}(\delta \eta) &= -\tilde{A}(\delta \eta)u(\eta_0) \quad \text{in } (0, T) \times \mathbb{R}^d, \\
\tilde{u}(\delta \eta)(0, \cdot) &= 0 \quad \text{in } \mathbb{R}^d.
\end{align*}$$

(7.3) (7.4)

**Proof.** Since $$u(\eta_0)(0) = g$$ does not depend on $$\eta_0$$ its derivative with respect to $$\eta$$ is 0. Now let $$\eta_s := \eta_0 + s \delta \eta, s > 0, \delta \eta \in \mathcal{C}$$. Subtract from the equation $$\partial_t u(\eta_s)(t) + A(\eta_s)u(\eta_s)(t) = 0$$ equation (7.1) and divide by $$s$$ to obtain

$$\partial_t \frac{1}{s}(u(\eta_s)(t) - u(\eta_0)(t)) + \frac{1}{s}(A(\eta_s) - A(\eta_0))u(\eta_s)(t) + \frac{1}{s}A(\eta_0)(u(\eta_s)(t) - u(\eta_0)(t)) = 0.$$ 

Taking $$\lim_{s \to 0^+}$$ gives equation (7.3). \(\square\)

We associate to the operator $$\tilde{A}(\delta \eta)$$ the Dirichlet form $$\tilde{E}(\delta \eta; \cdot, \cdot) : \tilde{V} \times \tilde{V} \to \mathbb{R}$$ which is given by $$\tilde{E}(\delta \eta; u, v) = \langle \tilde{A}(\delta \eta)u, v \rangle$$. The variational formulation of (7.3)–(7.4) reads:

Find $$\tilde{u}(\delta \eta) \in L^2((0, T); \mathcal{V}) \cap H^1((0, T); \mathcal{V}^*)$$ such that

$$\langle \partial_t \tilde{u}(\delta \eta), v \rangle + \mathcal{E}(\eta_0; \tilde{u}(\delta \eta), v) = -\tilde{E}(\delta \eta; u(\eta_0), v), \forall v \in \mathcal{V}, \ a.e. \ in \ (0, T),$$

(7.5)

Note that (7.5) has an unique solution $$\tilde{u}(\delta \eta) \in \mathcal{V}$$ due to the assumptions on $$\mathcal{E}(\eta_0, \cdot, \cdot)$$, $$\tilde{A}$$ and $$u(\eta_0) \in \overline{V}$$. As in (1.12) the fully discrete form is given by

Find $$\tilde{u}_h^m \in V_h$$ such that for $$m = 0, \ldots, M - 1$$

$$\langle \Delta t^{-1}(\tilde{u}_h^{m+1} - \tilde{u}_h^m), v_h \rangle_h + \mathcal{E}(\eta_0; \tilde{u}_h^{m+\theta}, v_h) = -\tilde{E}(\delta \eta; u_h^{m+\theta}, v), \forall v_h \in V_h,$$

(7.6)

or in matrix notation $$\Delta t^{-1}M(\tilde{u}_h^{m+1} - \tilde{u}_h^m) + \theta A \tilde{u}_h^{m+1} + (1 - \theta)A \tilde{u}_h^m = -\tilde{A} u_h^{m+\theta},$$ where $$A$$ is matrix of the Dirichlet form $$\tilde{E}(\delta \eta; \cdot, \cdot)$$ with respect to $$\Phi_h$. 

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7.1 Sensitivity with respect to model parameters

Example 7.1.3. Let \( d = 2 \) and consider a pure jump Lévy process as in Proposition 2.3.7 with Clayton Lévy copula
\[
F(u) = 2^{2-d} \left( \sum_{i=1}^{d} |u_i|^{-\vartheta} \right)^{-\frac{1}{\vartheta}} \left( \eta 1_{\{u_1 \cdots u_d \geq 0\}} - (1 - \eta) 1_{\{u_1 \cdots u_d \leq 0\}} \right),
\]
and tempered stable marginal densities
\[
k_i(z) = c_i e^{-\beta_i^0|z|} + c_i e^{-\beta_i^+ z} 1_{\{z < 0\}} 1_{\{z > 0\}}, \quad i = 1, \ldots, d.
\]
The sensitivity of the Lévy copula with respect to \( \vartheta \) is given by
\[
\frac{\partial}{\partial \vartheta} F(u) = \frac{1}{\vartheta^2} F(u) \left( \ln \left( \sum_{i=1}^{d} |u_i|^{-\vartheta} \right) + \frac{\vartheta \sum_{i=1}^{d} |u_i|^{-\vartheta} \ln |u_i|}{\sum_{i=1}^{d} |u_i|^{-\vartheta}} \right).
\]
We compute the sensitivity with respect to \( \vartheta \) in \( d = 2 \) of a basket put option price with payoff \( g(s_1, s_2) = (K - s_1 - \frac{1}{2}s_2)_+ \), where the maturity \( T = 0.5 \), strike \( K = 100 \) and interest rate \( r = 0.01 \). We set \( c_1 = c_2 = 1 \), \( \beta_1^- = 10 \), \( \beta_1^+ = 15 \), \( \beta_2^- = 9 \), \( \beta_2^+ = 16 \), \( \alpha_1 = 0.5 \), \( \alpha_2 = 0.7 \), \( \vartheta = 0.5 \) and \( \eta = 0.5 \). The sensitivity is shown in Figure 7.1.

![Figure 7.1: Sensitivity of a basket put option with respect to \( \vartheta \) in \( d = 2 \)](image)

We establish convergence rates for the sequence \( \{u^M_m\}_{m=0}^{M-1} \) of sensitivities with respect to model parameters as the discretization parameter \( h \) tends to zero. We show that the computed sensitivities converge essentially at the same rate as the computed prices. For notational simplicity the subscript \( \eta_0 \) is omitted.

Theorem 7.1.4. Let \( \tilde{u}, \tilde{u}_h^M \) be the solutions of (7.5),(7.6) and the assumptions of Theorem 1.3.7 be fulfilled. Additionally assume \( \tilde{u} \in C^1([0, T], \bar{H}^s(D)) \cap C^3([0, T], V^r) \). Then, for \( r \leq s \leq q + 1 \)
\[
\|u^M - \tilde{u}_h^M\|_{L^2(D)}^2 + \Delta t \sum_{m=0}^{M-1} \left\| \tilde{u}_h^{m+1/2} - \tilde{u}_h^{m+1/2} \right\|_{V^r}^2 \leq C(u, \tilde{u}) \left( \Delta t^4 + h^{2q(s-r)} \right),
\]

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where \( C(u, \bar{u}) > 0 \) depends on higher space and time derivatives of \( \bar{u} \) and \( u \).

Proof. See [33].

Theorem 7.1.4 shows that if the error of the approximate price converges with \( O(h^{s-r}) + O(\Delta t^2) \), the error of the approximate sensitivity preserves the same convergence rates both in space and time.

### 7.2 Sensitivity with respect to solution arguments

We also want to calculate the sensitivity of the solution \( u \) to a variation of arguments \( t, x \). Typical examples are the Greeks Theta (\( \partial_t u \)), Delta (\( \partial_x u \)) and Gamma (\( \partial_{xx} u \)).

We show that these sensitivities can directly be obtained by postprocessing the finite element solution \( u_h \) without additional runs. Again our numerical approximations to these sensitivities converge with the same rate as \( u_h \).

Let \( u \) be the solution of the variational problem (1.6). We discuss the computation of \( D^n u = \frac{\partial^{|n|} u}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} \) for arbitrary multiindex \( n \in \mathbb{N}_0^d \). For \( \mu \in \mathbb{Z}^d \) and \( h \in \mathbb{R}_+ \) we define the translation operator \( T^h \varphi(x) = \varphi(x + \mu h) \) and the forward difference quotient \( \partial^h \varphi(x) = h^{-1}(T^h \varphi(x) - \varphi(x)) \), where \( e_j, j = 1, \ldots, d \), denotes the \( j \)-th standard basis vector in \( \mathbb{R}^d \). For \( n \in \mathbb{N}_0^d \) we denote by \( \partial^n \varphi = \partial_{h,1}^{n_1} \cdots \partial_{h,d}^{n_d} \varphi \) and by \( D^n h \) the difference operator of order \( n \geq 0 \)

\[
D^n h \varphi := \sum_{\gamma_1 + \cdots + \gamma_d = n} C_{\gamma} T^n_h \partial^h \varphi.
\]

**Definition 7.2.1.** The difference operator \( D^n h \) of order \( |n| = n \) and mesh width \( h \) is called an approximation to the derivative \( D^n \) of order \( s \in \mathbb{N}_0 \) if for any \( \varphi \in \mathcal{H}^{s+r+n}(D) \) holds

\[
\| D^n \varphi - D^n h \varphi \|_{\mathcal{H}^{s+r+n}(D)} \leq C h^s \| \varphi \|_{\mathcal{H}^{s+r+n}(D)}, \forall \varphi \in \mathcal{H}^{s+r+n}(D).
\]

(7.7)

Given a basis \( \Phi_h \) of \( V_h \), the action of \( D^n h \) to \( v_h \in V_h \) can be realized as matrix-vector multiplication \( v_h \mapsto D^n h v_h \), where

\[
D^n_h = (D^n_{h,1} \phi_{h,1}, \ldots, D^n_{h,N} \phi_{h,N}) \in \mathbb{R}^{N \times N},
\]

and \( v_h \) is the coefficient vector of \( v_h \) with respect to basis \( \Phi_h \), respectively.

**Example 7.2.2.** Let \( V_h \) be as in Example 1.3.4 the space of piecewise linear continuous functions on \( [0, 1] \) vanishing at the end points 0, 1. For \( \alpha, \beta, \gamma \in \mathbb{R} \) and \( \mu \in \mathbb{N}_0 \) we denote by \( \text{diag}_\mu(\alpha, \beta, \gamma) \) the matrices

\[
\text{diag}_\mu(\alpha, \beta, \gamma) = \begin{pmatrix}
\cdots & 0 & \alpha & \beta & \gamma & 0 & \cdots \\
0 & \cdots & 0 & \alpha & \beta & \gamma & 0 & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
7.2 Sensitivity with respect to solution arguments

where the entries $\beta$ are on the $\mu$-th lower diagonal. Then, the matrices $Q_h$ of the forward difference quotient $\partial_h$ and $T_\mu$ of the translation operator $T_\mu^h$ respectively are given by

$$Q_h = h^{-1}\text{diag}(0, -1, 1), \quad T_\mu = \text{diag}(0, 1, 0).$$

Hence, for example, we have for the centered finite difference quotient

$$D_0^2 \varphi(x) = h^{-2}(\varphi(x + h) - 2\varphi(x) + \varphi(x - h)),$$

of order 2 in one dimension $D_0^2 = T_1^Q = h^{-2}\text{diag}(1, -2, 1)$.

**Example 7.2.3.** Let $V_h$ as in Example 1.3.5 be the tensor product of the one-dimensional spaces. Then, the matrix $D^n_h$ is given by

$$D^n_h = \sum_{\gamma, |n| = n} C_{\gamma, n} T_{\gamma_1} \otimes \cdots \otimes T_{\gamma_d} Q_h^{n_1} \otimes \cdots \otimes Q_h^{n_d}.$$

We have the following convergence result for the approximation of sensitivities with respect to solution arguments.

**Theorem 7.2.4.** Let $u, u^m_h$ be the solutions of (1.6), (1.12) and the assumptions of Theorem 1.3.7 be fulfilled. Additionally, assume that $u(x, t)$ is sufficiently smooth in $[0, T] \times \overline{\mathcal{D}}$ and that the approximation $\partial_h^\beta u^0_h$ is quasi-optimal in $L^2(\mathcal{D})$ for all $\beta \leq n$. Assume further that $D^n_h$ approximates $D^n$ in the sense of Definition 7.2.1. Then there holds

$$\|D^n u^M - D^n u^M_h\|_{L^2(\mathcal{D})}^2 + \Delta t \sum_{m=0}^{M-1} \|D^n u^{m+1/2} - D^n u^{m+1/2}_h\|_V^2 \leq C(u) (\Delta t^s + h^{2(s-r)}),$$

where $C(u) > 0$ depends on higher space and time derivatives of $u$.

**Proof.** See [33].

**Remark 7.2.5.** (i) Note that we cannot get higher convergence rates than $s - r$, even if $u$ has a higher regularity.

(ii) Theorem 7.2.4 shows that arbitrary derivatives of $u$ can be approximated with the same rate as $u$ itself, provided $u$ is sufficiently smooth.

**Example 7.2.6.** We consider the same problem as in Example 4.3.6 where for $d = 2$ the elliptic problem

$$\mathcal{A}[u] = f \quad \text{on} \quad \Omega = [0, 1]^2,$$

is solved for the exact solution

$$u(x) = \begin{cases} (x_1^2 - 2x_1 + x_1^4)(x_2^2 - 2x_2^3 + x_2^4) & \text{if } x \in \Omega \\ 0 & \text{else} \end{cases}$$

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with two independent tempered stable marginal densities

\[ k_i(z) = c_i \frac{e^{-\beta_i^- |z|}}{|z|^{1+\alpha_i}} 1_{\{z<0\}} + c_i \frac{e^{-\beta_i^+ z}}{z^{1+\alpha_i}} 1_{\{z>0\}}, \quad i = 1, 2. \]

We set the model parameter \( c_1 = c_2 = 1, \beta_1^- = 10, \beta_1^+ = 15, \beta_2^- = 9, \beta_2^+ = 16, \alpha_1 = 0.5, \alpha_2 = 0.7 \) and compute the sensitivities \( D^{(1,0)} u \) and \( D^{(1,1)} u \) in Figure 7.2. It can be seen that approximation of the sensitivities converges at the same rate as the approximation of the solution \( u \).

Figure 7.2: Convergence rates for the solution \( u \) and the sensitivities \( D^{(1,0)} u, D^{(1,1)} u \)

For more details and numerical examples we refer to [33].
8 Impact of approximations of small jumps

In this chapter we consider a regularization of the multivariate Lévy measure where small jumps are either neglected or approximated by artificial Brownian motion. This Gaussian approximation is often proposed to simulate Lévy processes \cite{1, 15} or to price options using finite differences \cite{18}. Applying the methods developed in Chapter 4 & 6 gives accurate numerical schemes for either model. We use our scheme to study and compare the error of diffusion approximations of small jumps in multivariate Lévy models via accurate numerical solutions of the corresponding PIDEs for various types of contracts.

8.1 Gaussian approximation

Let $X$ be a $d$-dimensional Lévy process with characteristic exponent

$$\psi(\xi) = -i\langle \gamma, \xi \rangle + \int_{\mathbb{R}^d} \left( 1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle \right) \nu(dz),$$

where we assume $\int_{|z|>1} |z| \nu(dz) < \infty$. For $\varepsilon > 0$ let $\nu_\varepsilon$ be a measure such that $\nu_\varepsilon = \nu - \nu_\varepsilon$ is a finite measure and $\int_{\mathbb{R}^d} |z|^2 \nu_\varepsilon(dz) < \infty$. Then, the characteristic exponent can be decomposed into two parts

$$\psi(\xi) = \underbrace{-i\langle \gamma^\varepsilon, \xi \rangle}_{\psi^\varepsilon(\xi)} + \int_{\mathbb{R}^d} \left( 1 - e^{i\langle \xi, z \rangle} \right) \nu^\varepsilon(dz) + \int_{\mathbb{R}^d} \left( 1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle \right) \nu_\varepsilon(dz), \quad (8.1)$$

where $\gamma^\varepsilon_i = \gamma_i - \int_{\mathbb{R}} z_i \nu^\varepsilon_i(dz_i), i = 1, \ldots, d$. Correspondingly we can decompose $X$ into its small and large jump parts

$$X_t = \gamma^\varepsilon_t + N^\varepsilon_t + X_{\varepsilon,t} = X_t^\varepsilon + X_{\varepsilon,t}, \quad (8.2)$$

where $N^\varepsilon$ is a compound Poisson process with jump measure $\nu^\varepsilon$. The small jump part $X_\varepsilon$ is independent of $N^\varepsilon$ and has the covariance matrix $Q_\varepsilon = \int_{\mathbb{R}^d} zz^\top \nu_\varepsilon(dz)$. We assume $Q_\varepsilon$ is non-singular. Let $\Sigma_\varepsilon$ be a non-singular matrix such that $\Sigma_\varepsilon \Sigma_\varepsilon^\top = Q_\varepsilon$. $X_\varepsilon$ can be approximated by a $d$-dimensional standard Brownian motion $W$ independent of $N^\varepsilon$. The next theorem shows that the process $\Sigma_\varepsilon^{-1}X_\varepsilon$ converges in distribution to $W$ as $\varepsilon \to 0$. 

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Theorem 8.1.1. Let \( X \) be a Lévy process with state space \( \mathbb{R}^d \) and characteristic triplet \((0, \nu, \gamma)\). Assume that \( Q_\varepsilon \) is non-singular for every \( \varepsilon \in (0, 1) \) and that for every \( \delta > 0 \) there holds
\[
\int_{(Q_\varepsilon^{-1} z, z) > \delta} (Q_\varepsilon^{-1} z, z) \nu_\varepsilon(dz) \to 0, \quad \text{as} \ \varepsilon \to 0.
\]
Assume further that for some family of non-singular matrices \( \{ \Sigma_\varepsilon \}_{\varepsilon \in (0, 1)} \) there holds
\[
(S_\varepsilon^{-1} Q_\varepsilon S_\varepsilon^\top \to I_d, \quad \text{as} \ \varepsilon \to 0,
\]
where \( I_d \) denotes the identity matrix in \( \mathbb{R}^d \). Then, for all \( \varepsilon \in (0, 1) \) there exists a càdlàg process \( R^\varepsilon \) such that
\[
X_t \overset{(d)}{=} \gamma^\varepsilon t + \Sigma_\varepsilon W_t + N_\varepsilon + R^\varepsilon,
\]
in the sense of equality of finite dimensional distributions. Furthermore, we have for all \( T > 0 \),
\[
\sup_{t \in [0, T]} |S_\varepsilon^{-1} R^\varepsilon| \overset{(P)}{\to} 0, \quad \text{as} \ \varepsilon \to 0
\]
where \( N^\varepsilon \) are given in (8.2) and \( W \) is a \( d \)-dimensional standard Brownian motion independent of \( N^\varepsilon \).

Proof. See [15, Theorem 3.1]. \( \square \)

We give an example of the decomposition (8.1) into small and large jumps.

Example 8.1.2. Let \( X = (X^1, \ldots, X^d)^\top \) be a \( d \)-dimensional Lévy process with Lévy measure \( \nu \) and marginal Lévy measures \( \nu_i, i = 1, \ldots, d \). To obtain \( \nu^\varepsilon \) in \( d = 1 \) we simply cut off the small jumps, i.e.,
\[
\nu^\varepsilon = \nu 1_{\{|z| > \varepsilon\}}.
\]
For \( d > 1 \) the Lévy measure \( \nu^\varepsilon \) could be obtained by \( \nu^\varepsilon = \nu 1_{\{\|z\|_\infty > \varepsilon\}} \) where jumps are neglected if the jump size in all directions is small. But the corresponding one-dimensional Lévy measures \( \nu^\varepsilon_i, i = 1, \ldots, d \) are not then of the form (8.4). If we choose
\[
\nu^\varepsilon = \nu 1_{\{\min\{|z_1|, \ldots, |z_d|\} > \varepsilon\}},
\]
the corresponding one-dimensional Lévy measures \( \nu^\varepsilon_i, i = 1, \ldots, d \) again satisfy (8.4).

We consider the Clayton Lévy copula model as explained in Section 2.3.3 with the density \( k \) given by (2.17) for \( d = 2, \theta = 0.5, \eta = 0.5 \) and \( \alpha = (0.5, 1.2) \). The corresponding regularized density \( k^\varepsilon, \nu^\varepsilon(dz) = k^\varepsilon(z)dz \) as in (8.5) for \( \varepsilon = 0.01 \) is plotted in Figure 8.1.

We now consider a \( d \)-dimensional pure jump process \( X \) with characteristic triplet \((0, \nu, \gamma)\) where the Lévy measure \( \nu \) satisfies (2.18). Let \( \gamma \) be chosen according to Lemma 2.1.9 such that \( e^{\gamma X^j} \), \( j = 1, \ldots, d \) are martingales. The covariance matrix is given by \( Q = \int_{\mathbb{R}^d} zz^\top \nu(dz) \). For any \( \varepsilon > 0 \) the process \( X \) can be approximated by a compound Poisson process \( Y^\varepsilon_1 \) as in (8.2) where the small jumps are neglected as in (8.5),
\[
Y^\varepsilon_{1,t} = \gamma^\varepsilon t + N^\varepsilon_1.
\]
8.2 Basket options

The characteristic triplet of $Y_1^\varepsilon$ is $(0, \nu^\varepsilon, \gamma_1^\varepsilon)$ and $\gamma_1^\varepsilon$ is again such that $\varepsilon^Y_{t,j}$, $j = 1, \ldots, d$ are martingales. A better approximation can be obtained by replacing the small jumps with a Brownian motion which yields a jump-diffusion process $Y_2^\varepsilon$,

$$Y_{2,t}^\varepsilon = \Sigma \varepsilon W_t + \gamma_2^\varepsilon t + N_t^\varepsilon,$$

with characteristic triplet $(Q_\varepsilon, \nu^\varepsilon, \gamma_2^\varepsilon)$. The processes $W$ and $N$ are independent. $Y_2^\varepsilon$ has the same covariance matrix as $X$ and drift $\gamma_2^\varepsilon_j = \gamma_1^\varepsilon_j - Q_{\varepsilon,j}/2$, $j = 1, \ldots, d$. For $\varepsilon \to \infty$ we obtain a diffusion process $Y_\infty^\varepsilon = \Sigma W_t + \gamma_\infty t$ with covariance matrix $Q = \Sigma \Sigma^\top$ and drift $\gamma_\infty_j = -Q_{jj}/2$, $j = 1, \ldots, d$.

There are two sources of error. We have a discretization error using a mesh width $h > 0$ and a modeling error using $\varepsilon > 0$. To assess the impact of $\varepsilon > 0$, we use the discretization (4.12) for $\varepsilon = 0$ and $\varepsilon > 0$. Here, $h$ is chosen so small that the discretization error is negligible in comparison to the truncation error due to cut-off of jumps of size smaller than $\varepsilon$.

**Remark 8.1.3.** To obtain a converging scheme for finite difference methods, $\varepsilon > 0$ was chosen in [18] depending on the mesh width $h$. For a fixed mesh width $h > 0$ the discretization error increases as $\varepsilon \to 0$, i.e., $\varepsilon = 0$ cannot be used.

### 8.2 Basket options

Consider a basket option $u(t,x)$ with payoff $g(x)$ where the log price processes of the underlyings are given by the pure jump process $X = (X^1, \ldots, X^d)^\top$ and correspondingly $u_1^\varepsilon(t, x)$, $u_2^\varepsilon(t, x)$ for the processes $Y_1^\varepsilon$, $Y_2^\varepsilon$. We want to study the error $|u(T, x) - u_i^\varepsilon(T, x)|$ for $\varepsilon \to 0$, $i = 1, 2$. Since we adjusted the drift to preserve the martingale property, we additionally introduce the processes

$$Z_{i,t}^\varepsilon = X_t + (\gamma_i^\varepsilon - \gamma)t, \quad i = 1, 2.$$
which have the same drift as \( Y_i \) and the same Lévy measure as \( X \).

**Proposition 8.2.1.** Assume \( g \) is Lipschitz continuous. Then,

\[
\left| \mathbb{E}(g(x + X_T)) - \mathbb{E}(g(x + Z_{1,T}^\varepsilon)) \right| \lesssim \sum_{j=1}^d \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d, 
\]

(8.8)

\[
\left| \mathbb{E}(g(x + X_T)) - \mathbb{E}(g(x + Z_{2,T}^\varepsilon)) \right| \lesssim \sum_{j=1}^d \int_{-\varepsilon}^{\varepsilon} |z_j|^3 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d. 
\]

(8.9)

**Proof.** We have for \( i = 1, 2 \),

\[
\left| \mathbb{E}(g(x + X_T)) - \mathbb{E}(g(x + Z_{i,T}^\varepsilon)) \right| \leq \mathbb{E} |g(x + X_T) - g(x + X_T + (\gamma_i^\varepsilon - \gamma)T)| 
\]

\[
\leq T \sum_{j=1}^d |\gamma_{i,j}^\varepsilon - \gamma_j|.
\]

Furthermore,

\[
|\gamma_{1,j}^\varepsilon - \gamma_j| = \left| \int_{\mathbb{R}^d} (e^{z_j} - 1 - z_j) \nu_\varepsilon(\mathrm{d}z) \right| \leq \int_{-\varepsilon}^{\varepsilon} \int_{0}^{|z_j|} e^s |z_j - s| \mathrm{d}s \nu_j(\mathrm{d}z) 
\]

\[
\leq \frac{\varepsilon}{2} \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \nu_j(\mathrm{d}z), \quad j = 1, \ldots, d,
\]

\[
|\gamma_{2,j}^\varepsilon - \gamma_j| = \left| \frac{Q_{\varepsilon,j}}{2} - \int_{\mathbb{R}^d} (e^{z_j} - 1 - z_j) \nu_\varepsilon(\mathrm{d}z) \right| \leq \frac{1}{2} \left| \int_{-\varepsilon}^{\varepsilon} \int_{0}^{|z_j|} e^s (z_j - s)^2 \mathrm{d}s \nu_j(\mathrm{d}z) \right| 
\]

\[
\leq \frac{\varepsilon}{6} \int_{-\varepsilon}^{\varepsilon} |z_j|^3 \nu_j(\mathrm{d}z), \quad j = 1, \ldots, d.
\]

\[ \square \]

**Remark 8.2.2.** For \( d = 1 \) a similar proof is given in [17].

The same error estimates are also obtained for the compound Poisson and Gaussian approximation.

**Proposition 8.2.3.** Assume \( g \in C^2(\mathbb{R}^d) \). Then,

\[
\left| \mathbb{E}(g(x + Z_{1,T}^\varepsilon)) - \mathbb{E}(g(x + Y_{1,T}^\varepsilon)) \right| \lesssim \sum_{j=1}^d \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d. 
\]

(8.10)

Furthermore, assume \( g \in C^4(\mathbb{R}^d) \) and \( \int_{\mathbb{R}^d} |z| \nu(\mathrm{d}z) < \infty \). Then,

\[
\left| \mathbb{E}(g(x + Z_{2,T}^\varepsilon)) - \mathbb{E}(g(x + Y_{2,T}^\varepsilon)) \right| \lesssim \sum_{j=1}^d \int_{-\varepsilon}^{\varepsilon} |z_j|^3 \nu_j(\mathrm{d}z_j), \quad \forall x \in \mathbb{R}^d. 
\]

(8.11)
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Proof. Consider the Taylor series expansion of \( g(x) \) at \( x_0 \)

\[
g(x) = g(x_0) + \nabla g(x_0) \cdot (x - x_0) + \frac{1}{2} (x - x_0) \cdot D^2 g(x_0) (x - x_0) + \mathcal{O}(|x - x_0|^3),
\]

where \( D^2 g \) is the Hessian matrix of \( g \). Define \( R^\varepsilon = Z^\varepsilon_1 - Y^\varepsilon_1 \). The Lévy process \( R^\varepsilon \) has Lévy measure \( \nu^\varepsilon \) and is independent of \( Y^\varepsilon \). Since \( Z^\varepsilon_{1,T} \) and \( Y^\varepsilon_{1,T}, j = 1, \ldots, d, \) have the same expected value, we have \( \mathbb{E}(R^\varepsilon_{1,j}) = 0, j = 1, \ldots, d \). Thus, we obtain

\[
\left| \mathbb{E}(g(x + Z^\varepsilon_{1,T})) - \mathbb{E}(g(x + Y^\varepsilon_{1,T})) \right| = \sum_{j=1}^d \mathbb{E}\left( \partial_j g(x + Y^\varepsilon_{1,T}) \right) \mathbb{E}(R^\varepsilon_{1,j}) + \sum_{j=1}^d \sum_{k=1}^d \mathcal{O}\left( \mathbb{E}(R^\varepsilon_{1,j} R^\varepsilon_{1,k}) \right)
\]

\[
\lesssim \sum_{j=1}^d \mathbb{E}\left( (R^\varepsilon_{1,j})^2 \right).
\]

Equation (8.10) follows from

\[
\mathbb{E}\left( (R^\varepsilon_{1,j})^2 \right) = \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \nu_j(dz_j), \quad j = 1, \ldots, d.
\]

Furthermore, \( Y^\varepsilon_{2,t} = Y^\varepsilon_{1,t} + \Sigma_x W_t + (\gamma \varepsilon - \gamma) t, \) where the standard Brownian motion \( W \) is independent of \( Y^\varepsilon \). We set \( \bar{x} = x + (\gamma \varepsilon - \gamma) T \) and obtain

\[
\left| \mathbb{E}(g(x + Z^\varepsilon_{1,T})) - \mathbb{E}(g(x + Y^\varepsilon_{2,T})) \right| = \left| \mathbb{E}(g(\bar{x} + Z^\varepsilon_{1,T})) - \mathbb{E}(g(\bar{x} + Y^\varepsilon_{1,T})) + \mathbb{E}(g(\bar{x} + Y^\varepsilon_{2,T})) - \mathbb{E}(g(x + Y^\varepsilon_{2,T})) \right|
\]

\[
= \sum_{j=1}^d \sum_{k=1}^d \frac{1}{2} \mathbb{E}\left( \partial_j \partial_k g(\bar{x} + Y^\varepsilon_{1,T}) \right) \mathbb{E}\left( (R^\varepsilon_{1,j} R^\varepsilon_{1,k}) \right) + \sum_{j=1}^d \mathcal{O}\left( \mathbb{E}(|R^\varepsilon_{1,j}|^2) \right)
\]

\[
- \sum_{j=1}^d \sum_{k=1}^d \frac{1}{2} \mathbb{E}\left( \partial_j \partial_k g(\bar{x} + Y^\varepsilon_{1,T}) \right) \mathbb{E}(|\Sigma_x W|^2) + \mathcal{O}\left( \mathbb{E}(|\Sigma_x W|^4) \right)
\]

\[
\lesssim \sum_{j=1}^d \int_{-\varepsilon}^{\varepsilon} |z_j|^3 \nu_j(dz_j) + \left( \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \nu_j(dz_j) \right)^2.
\]

Now with \( c_j = \int_{-\varepsilon}^{\varepsilon} |z_j| \nu_j(dz_j) < \infty, j = 1, \ldots, d \) and Jensen’s inequality we have

\[
\left( \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \nu_j(dz_j) \right)^2 = c_j^2 \left( \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \nu_j(dz_j) \right)^2 \leq c_j^2 \int_{-\varepsilon}^{\varepsilon} |z_j|^2 \frac{|z_j| \nu_j(dz_j)}{c_j}
\]

\[
= c_j \int_{-\varepsilon}^{\varepsilon} |z_j|^3 \nu_j(dz_j), \quad j = 1, \ldots, d.
\]

\[\square\]
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Remark 8.2.4. For $d = 1$ similar results are given in [64]. Under less restrictive assumptions on $g(x)$ error estimates for $d = 1$ are also proved in [17]. The results could be extended to $d > 1$ using, e.g., [5]. These error estimates, however, do not appear to be optimal as we show in the numerical examples.

Using Proposition 8.2.1 and 8.2.3 we immediately obtain

Corollary 8.2.5. Assume the Lévy measure $\nu$ satisfies (2.19) with $\alpha = (\alpha_1, \ldots, \alpha_d)$. Then, for $g \in C^4(\mathbb{R}^d)$

$$
|\mathbb{E}(g(x + X^T_{\varepsilon})) - \mathbb{E}(g(x + Y^\varepsilon_{1,T})))| \lesssim \varepsilon^{2 - \max\{\alpha_1, \ldots, \alpha_d\}}, \quad \forall x \in \mathbb{R}^d, \quad 0 < \alpha_j < 2,
$$

$$
|\mathbb{E}(g(x + X^T_{\varepsilon})) - \mathbb{E}(g(x + Y^\varepsilon_{2,T})))| \lesssim \varepsilon^{3 - \max\{\alpha_1, \ldots, \alpha_d\}}, \quad \forall x \in \mathbb{R}^d, \quad 0 < \alpha_j < 1.
$$

These convergence rates can also be shown numerically even for $g \notin C^4(\mathbb{R}^d)$ and $\alpha > 1$.

Example 8.2.6. Let $d = 1$ and consider the tempered stable density as in Example 4.3.6. We compute the price of a put option with maturity $T = 0.5$, strike $K = 100$ and interest rate $r = 0.01$. We set $c_1 = 1$, $\beta^+ = 10$, $\beta^- = 15$ and compute for $Y^\varepsilon_1$, $Y^\varepsilon_2$ the convergence rate with respect to $\varepsilon$ at $s = 100$ using various $\alpha$’s. As shown in Figure 8.2, the rates $2 - \alpha$ and $3 - \alpha$ are always obtained.

![Figure 8.2: Convergence rates with respect to $\varepsilon$ for $Y^\varepsilon_1$ (left) and $Y^\varepsilon_2$ (right) in $d = 1$.](image)

Example 8.2.7. Let $d = 2$ and consider two independent tempered stable marginal densities as in Example 4.3.6. We compute the price of a basket option, $g(s_1, s_2) = (K - s_1 - s_2)^+$, with maturity $T = 0.5$, strike $K = 100$ and interest rate $r = 0.01$. We set $c_1 = c_2 = 1$, $\beta^-_1 = 10$, $\beta^+_1 = 15$, $\beta^-_2 = 9$, $\beta^+_2 = 16$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$. We compute for $Y^\varepsilon_1$, $Y^\varepsilon_2$ the convergence rate with respect to $\varepsilon$ at $s = (100, 100)$. As shown in Figure 8.3 the rates $2 - \alpha_2$ and $3 - \alpha_2$ are obtained.
8.3 Barrier options

Propositions 8.2.1 and 8.2.1 do not hold for barrier options since the option price is not smooth at the boundary $\partial D$. In particular it is shown in $d = 1$ for tempered stable densities with $1 < \alpha < 2$ and $c^+ = c^-$ that the derivative of the option price behaves in log price like $|x - \log B|^{\alpha/2-1}$ as $x \to \log B$ (see, e.g., [39]). Therefore, one obtains a large error at the boundary by approximating $X$ with $Y^+_2$.

Example 8.3.1. Let $d = 2$ and consider again a pure jump process $(Q \equiv 0)$ with two independent tempered stable marginal densities as in Example 8.2.7. We compute the price of a down-and-out basket option, $g(s_1, s_2) = (K - \frac{1}{2}s_1 - \frac{1}{2}s_2)_+$, on the domain $D = [B, \infty]^2$ with barrier $B = 80$, maturity $T = 0.5$, strike $K = 100$ and interest rate $r = 0.01$. We set $c_1 = c_2 = 1$, $\beta^-_1 = 10$, $\beta^+_1 = 15$, $\beta^-_2 = 9$, $\beta^+_2 = 16$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.7$. The option price is shown in Figure 8.4.

![Figure 8.4: Barrier option price in $d = 2$ with barrier $B = 80$ and strike $K = 100'](image)

The relative error for approximating $X$ by $Y^+_2$ is plotted in Figure 8.5. Additionally, we also show the corresponding error for the non-barrier basket option. As expected, the
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relative error is significantly higher for a barrier option close to the barrier.

Figure 8.5: Relative error for various values of $\epsilon$ using $Y_j^x$ (8.7) in place of $X$ in (8.2) for a barrier option (left) and a non-barrier basket option (right) in $d = 2$ with barrier $B = 80$ and strike $K = 100$
References


References


References


References


Curriculum Vitae

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