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Disentangling risk aversion and elasticity of substitution without giving up preference monotonicity

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**Author(s):**

Bommier, Antoine; [Le Grand, François Louis](#) 

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# A Robust Approach to Risk Aversion:

Disentangling risk aversion and elasticity of substitution without giving  
up preference monotonicity\*

Antoine Bommier

François Le Grand<sup>†</sup>

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We formalize the notion of monotonicity with respect to first-order stochastic dominance in the context of preferences defined over the set of temporal lotteries. It is shown that the only Kreps and Porteus (1978) preferences which are both stationary and monotone are Uzawa preferences and risk-sensitive preferences introduced by Hansen and Sargent (1995). We also extend our results to smooth recursive ambiguity models. Focusing on monotone preferences enables a much better understanding of the role of risk aversion. As an application, we derive new general results on the determinants of precautionary savings and asset prices in dynamic settings.

**Keywords:** recursive models, monotonicity, first-order stochastic dominance, temporal lotteries, risk aversion, ambiguity aversion, precautionary savings, asset pricing.

**JEL codes:** D90, D81.

## 1 Introduction

Since Koopmans (1960)'s article, the assumption of preference stationarity plays a central role in the modeling of intertemporal choice under uncertainty. For many problems, it is

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<sup>†</sup>Bommier: ETH Zurich, [abommier@ethz.ch](mailto:abommier@ethz.ch); Le Grand: EMLyon Business School and ETH Zurich, [legrand@em-lyon.com](mailto:legrand@em-lyon.com). Both authors gratefully acknowledge financial support from Swiss-Re.

indeed meaningful to assume that the agent’s objective is independent of past events and of the calendar year. Preference stationarity is then required to generate time consistent planning. The economic literature abounds in works focusing on stationary preferences. In decision theory, Epstein (1983), Epstein and Zin (1989), Klibanoff, Marinacci, and Mukerji (2009) have made significant contributions by extending Koopmans’ initial contribution to more general settings.

This paper investigates how stationarity can be combined with monotonicity with respect to first-order stochastic dominance (mentioned hereafter as “FSD-monotonicity”). FSD-monotonicity is a consistency requirement between risk preferences and preferences over certain prospects. This property intuitively states that an agent cannot prefer a lottery to another one if the latter provides better outcomes than the former in all states of the world. As an illustrative example, consider the case of a driver who likes to drive fast, but dislikes paying speeding fines and has to consider that he may be caught in a radar trap with probability  $p \in (0, 1)$ . FSD-monotonicity involves assuming that the driver will neither go faster than if  $p = 0$  (no speed control) nor slower than if  $p = 1$  (speed is controlled for sure). Similarly, in a standard two-period consumption-saving problem, with uncertain second period income distributed in an interval  $[y_{min}, y_{max}]$ , FSD-monotonicity (combined with the usual assumptions of good normality and preference convexity) stipulates that an agent will not save more than what he would do if anticipating second period income  $y_{min}$  for sure, or less than what he would do if anticipating  $y_{max}$  for sure. FSD-monotonicity is therefore akin to the elimination of dominated strategies in game theory, as it rules out the possibility of an agent taking decisions that would be dominated in all states of the world.<sup>1</sup>

Like the elimination of dominated strategies, FSD-monotonicity can be viewed as a natural requirement in the formalization of rationality. This assumption was in fact included –under different names– in a number of early works suggesting an axiomatic construction of choice under uncertainty. It appears in Wald (1950), Hurwicz (1951), Chernoff (1954), Milnor (1954), Luce and Raiffa (1957), or Fishburn and Vickson (1978) among many others. Arrow (1951, p. 429) considers FSD-monotonicity as being an “extremely reasonable” assumption. However, more recent developments do not include FSD-monotonicity as an assumption anymore. Though present in Chew and Epstein (1990), it does not appear in the influential works of Kreps and Porteus (1978), Selden (1978), or Epstein and Zin (1989). In fact, to the best of our knowledge, the assumption of FSD-monotonicity has never been discussed in works that use a recursive approach to model rational behavior, one of the difficulties being to define stochastic orders in the context of temporal lotteries.

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<sup>1</sup>The parallel between choice under uncertainty and game theory, where the agent plays against nature, was already emphasized in Milnor (1954).

Imposing FSD-monotonicity drastically restricts the set of admissible preference specifications. In particular, our first representation result shows that within the whole set of Kreps-Porteus preferences, the only ones to be stationary and FSD-monotone are the expected utility preferences à la Uzawa (1968), or the risk-sensitive preferences à la Hansen and Sargent (1995). As a corollary, Epstein-Zin preferences are not FSD-monotone, except in the very specific cases when they are also risk-sensitive.<sup>2</sup> Moreover, we show that within the set of stationary and FSD-monotone preferences, only risk-sensitive preferences enable risk preferences and preferences over deterministic consumption paths to be disentangled. In addition, our results can be extended to choice under ambiguity. Focusing on the general framework of Jiu and Miao (2012), axiomatized by Hayashi and Miao (2011), which encompass most popular recursive ambiguity aversion models, we can identify all the specifications which are FSD-monotone.

Assuming FSD-monotonicity leads to focusing on frameworks where the understanding of attitude towards risk and risk aversion becomes much more intuitive. The key feature is that risk-sensitive preferences are well-ordered in risk aversion, not only “in the large” (i.e. in terms of willingness to pay to eliminate all risks) but also “in the small” (i.e. in terms of willingness to pay for marginal risk reductions). This proves to be crucial for addressing problems where complete risk elimination is not possible, or simply not optimal. We provide three concrete examples.

In the first application, we study the demand for insurance in an infinite-horizon economy and show that it is increasing with risk aversion. The second application bears on precautionary savings in a general infinite-horizon setup, in which income uncertainty may follow any kind of (non necessarily stationary) stochastically monotone process. We prove that in such a general framework, more risk averse agents save more. To our knowledge, the role of risk aversion on precautionary savings in an infinite horizon setting has only been studied while assuming a specific –fully parametrized– income process, so as to derive closed-form solutions (Van der Ploeg, 1993 or Weil, 1993). Moreover, studies based on non-monotone preferences found that no general result holds even in two-period problems (Kimball and Weil, 2009). Therefore, assuming FSD-monotonicity makes it possible to achieve a significant contribution to the literature on precautionary savings and to clarify the fundamental link that exists between risk aversion and prudence. The third application relates to asset pricing in a random endowment economy. Again, using risk-sensitive preferences enables us to opt for a non-parametric approach, in which general results can be derived without computing closed-forms solutions. In particular we show that, whenever consumption follows a stochasti-

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<sup>2</sup>This happens to be the case when the intertemporal elasticity of substitution is equal to one, as in Tallarini (2000), or when the risk aversion parameter is set equal to the inverse of the intertemporal elasticity of substitution, providing the standard additive model.

cally monotone process, risk aversion has a negative impact on the risk free rate and a positive one on the market price for risk.

The remainder of the paper is organized as follows. In Section 2, we describe the setup, while Section 3 introduces FSD-monotonicity. In Section 4, we present our main representation result. Section 5 shows that risk-sensitive preferences are well-ordered with respect to risk aversion. We develop the applications in Section 6, with concluding remarks provided in Section 7. The extension to ambiguity, which requires the introduction of a different setting, is given in Section A of the Appendix.

## 2 Setting

We consider preferences defined over the set of temporal lotteries in an infinite horizon setting. Time is discrete and indexed by  $t = 0, 1, \dots$ . For the sake of simplicity, we assume that per-period consumption is bounded and belongs to the compact set  $C = [\underline{c}, \bar{c}]$ , where  $0 < \underline{c} < \bar{c}$ . We denote by  $C^\infty$  the set of possible deterministic consumption paths, which is also compact (by Tychonoff's theorem).

We construct the set of temporal lotteries following Kreps and Porteus (1978) and Epstein and Zin (1989). Wakai (2007) also provides a precise construction of a similar preference domain. First, we define  $D_0$  as the set of all singleton subsets of  $C^\infty$  (with a slight abuse of notation,  $D_0 = C^\infty$ ). Then, for all  $t \geq 1$ , we define  $D_t$  by induction with:

$$D_t = C \times M(D_{t-1}), \quad (1)$$

where  $M(D_{t-1})$  is the space of probability measures on  $D_{t-1}$  endowed with the Prohorov metric (metric of weak convergence).<sup>3</sup> The sequence of sets  $(D_t)_{t \geq 0}$  is increasing (in the sense that  $D_t \subset D_{t+1}$ ). The space of temporal lotteries  $D$ , which is constructed as the closure of  $\bigcup_{t \geq 0} D_t$ , can be shown to be compact and homeomorphic to  $C \times M(D)$ .<sup>4</sup>

The set  $D_t$  contains all temporal lotteries for which no uncertainty is left at date  $t$ . In a symmetric way, we introduce sets of temporal lotteries, for which no uncertainty is resolved during a number of initial periods. These sets play a natural role in dynamic settings, where past consumptions appear as realized deterministic consumptions. Formally, for any  $\tau \geq 0$ ,

$$\Delta_\tau = C^{\tau+1} \times M(D)$$

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<sup>3</sup>More generally, for any metric space  $X$ ,  $M(X)$  denotes the space of Borel probability measures on  $X$  endowed with the weak convergence topology.

<sup>4</sup>The topological notions needed to define the closure as well as formal proofs are not discussed here, but can be found in Epstein and Zin (1989).

denotes the set of lotteries for which the consumptions between dates 0 and  $\tau$  are deterministic. By construction  $\Delta_0 = D$ . Although there is no natural mixture operation defined over the whole set  $D$ , two elements  $(c, m)$  and  $(c, m')$  of  $\Delta_\tau$  with the same vector of initial deterministic consumptions  $c \in C^{\tau+1}$  can be mixed as follows:

$$\lambda(c, m) + (1 - \lambda)(c, m') = (c, \lambda m + (1 - \lambda)m') \quad (2)$$

This mixture plays a central role in the Kreps-Porteus framework, and will also be key for defining mixture-stability in Section 3.2.2. Last, for any  $0 \leq \tau \leq t$ , we define:

$$D_{\tau,t} = \Delta_\tau \cap D_t \quad (3)$$

the set of lotteries where consumptions between dates 0 and  $\tau$  are deterministic and whose uncertainty is resolved at date  $t$ . We thus have  $D_{\tau,t} = C^{\tau+1} \times M(D_{t-\tau-1})$ . A typical element of  $D_{\tau,t}$  will be denoted  $(c, m)$ , where  $c = (c_0, \dots, c_\tau) \in C^{\tau+1}$  is the vector of deterministic consumptions between dates 0 and  $\tau$ , and  $m \in M(D_{t-\tau-1})$ .

Intuitively, an element of  $D_{0,t} = D_t$  can be seen as describing at date 0 the life of an agent born at date 0, and who knows that all uncertainty regarding his consumption will be resolved before date  $t$ . When such an agent reaches date  $\tau$ , his consumption  $(c_0, \dots, c_\tau)$  becomes known, and his (past and future) life is described by an element of  $D_{\tau,t}$ , whose uncertainty resolves between dates  $\tau + 1$  and  $t$ . Once the agent reaches date  $t$ , there is no uncertainty left: formally  $D_{t,t} = D_0 = C^\infty$  is the set of deterministic consumption paths. For a given  $t$ , the sequence of sets  $(D_{\tau,t})_{0 \leq \tau \leq t}$  is decreasing with  $\tau$ , which reflects that as time goes by, the amount of remaining uncertainty shrinks.

As an illustration, Figure 1 shows *simple* temporal lotteries (i.e., with finite support), represented by probability trees. For lotteries  $L_1$  and  $L_3$ , all the uncertainty is revealed at date 2, while nothing is learned at dates 0 and 1:  $L_1, L_3 \in D_{1,2}$ . For lottery  $L_2$ , the uncertainty is completely resolved at date 1:  $L_2 \in D_{0,1} = D_1$ . For lottery  $L_4$ , no information is revealed at dates 0 and 1, and no uncertainty remains at date 4:  $L_4 \in D_{1,4}$ .

As emphasized by Kreps and Porteus (1978), temporal lotteries offer the possibility to account for the timing of resolution of uncertainty. For example, lotteries  $L_1$  and  $L_2$  of Figure 1 provide exactly the same distribution of consumption paths:  $(c_0, c_1, c_2^u, c_3^u, \dots)$  with probability  $p_1$  or  $(c_0, c_1, c_2^d, c_3^d, \dots)$  with probability  $1 - p_1$ . However, they are distinct elements of  $D$ , as they differ with respect to the timing of the resolution of uncertainty.

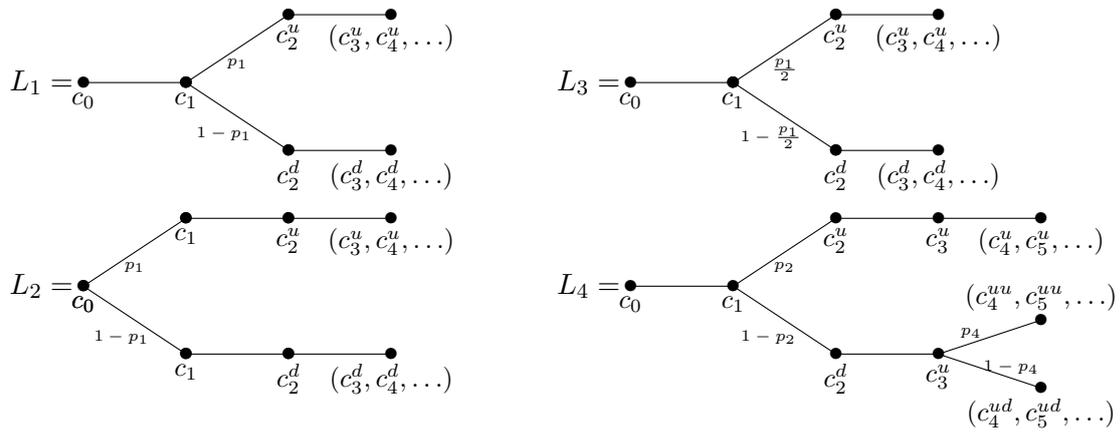


Figure 1: Some temporal lotteries

### 3 Monotonicity with respect to first-order stochastic dominance

The notion of FSD-monotonicity is central in our paper. It is a consistency requirement between risk preferences and preferences over deterministic consumption paths (that we also call “ordinal preferences”). Roughly speaking, it says that if we compare two temporal lotteries with one providing systematically better consumption paths than the other (in the sense of ordinal preferences) then the former should be preferred to the latter.

The assumption of FSD-monotonicity is by nature orthogonal to other assumptions that can be made regarding the ranking of deterministic consumption paths. It just “inherits” these assumptions and “extends” them to the set of temporal lotteries. For example, in the economic tradition, one may want to require that a (deterministic) increase in consumption has a positive impact on welfare. Such an assumption can be stated when discussing the desirable properties of ordinal preferences, as will be done later on in the paper. FSD-monotonicity will then impose –as an extension– that a random positive increase in consumption will also be welfare improving.

Quite insightful is the parallel suggested in Milnor (1954) between choice under uncertainty and playing a game against nature. In a game theory setting, we may distinguish two kinds of assumptions: (i) assumptions about the ranking of payoffs (e.g. “larger payoffs are preferred”); (ii) strategic rationality assumptions (e.g. “elimination of dominated strategies”). In our setting, assumptions regarding the ranking of payoffs are reflected in ordinal preferences, while FSD-monotonicity mirrors the elimination of dominated strategies.

Although the assumption of FSD-monotonicity looks fairly straightforward, its formalization in the context of temporal lotteries has not been, to the best of our knowledge, achieved yet. Section 3.1 presents the intuitions driving our definition. In Section 3.2, we state the formal definition. Finally, in Section 3.3, we illustrate using a two-period saving example how predictions may vary depending on whether FSD-monotonicity holds or not.

#### 3.1 First-order stochastic dominance for simple temporal lotteries

Using graphical representations similar to Figure 1, we explain how a relation of preferences over  $C^\infty$  generates an intuitive notion of “first-order stochastic dominance” (FSD, hereafter) over  $D$ , the set of temporal lotteries. For the sake of simplicity, we focus on cases where the consumption remains constant after date 2. We also assume that ordinal preferences can be represented by a utility function  $U_\infty$  such that  $U_\infty(c_0, c_1, c_2, c_2, \dots) = c_0 + \frac{1}{2}c_1 + \frac{1}{2}c_2$ , which is consistent with an infinite elasticity of substitution and a constant discount factor equal to  $\frac{1}{2}$ .<sup>5</sup> For each path of the probability tree, we compute the associated lifetime utility, reported in

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<sup>5</sup>This specification simply helps to have well-rounded numbers in computations. The argument could of course be developed for any kind of preferences over the set of deterministic consumption paths.

square brackets in the graphs. For example, in the lottery  $L_5$  of Figure 2, there are two possible scenarios: The upper one gives the consumption path  $(3, 2, 5, 5, \dots)$  providing a lifetime utility of 6.5, while the lower one generates a consumption path  $(3, 2, 8, 8, \dots)$  providing a lifetime utility of 8.

In order to provide insights on our notion of FSD, we focus on cases in which the lotteries to be compared are graphically represented with two probability trees that have exactly the same structure, in other words the same ramifications occurring with the same probabilities. One lottery is said to dominate another one if for all paths of the probability tree, it provides a higher lifetime utility than the other lottery. For example, Figure 2 provides two examples, in which the lottery on the left dominates the one on the right. The top example of Figure 2 is straightforward since at every date, lottery  $L_5$  pays off greater consumption levels than  $L_6$ . The bottom example illustrates the role of ordinal preferences.  $L_7$  pays larger outcomes than  $L_8$  at dates 0 and 1 but smaller ones in all subsequent periods. However, due to the assumed relation of ordinal preferences, it turns out that for any path of the probability tree, lifetime utilities are larger with  $L_7$  than with  $L_8$ . Thus, in all circumstances,  $L_7$  generates better consumption paths than  $L_8$ . In other words,  $L_7$  stochastically dominates  $L_8$  at the first-order.

Figure 3 plots two elements of  $D_2$  (or equivalently of  $D_{0,2}$ ), again assuming identical probability trees. Comparison of  $L_9$  and  $L_{10}$  can be made in two equivalent ways. One possibility involves, as before, comparing the lifetime utilities associated to all paths of the probability tree. For each of the four possible paths,  $L_9$  provides a higher lifetime utility than  $L_{10}$ , indicating that  $L_9$  dominates  $L_{10}$ . The alternative possibility is to check at the first node whether the temporal lotteries that are obtained in the different realizations of  $L_9$  (which are elements of  $D_{1,2}$ ) dominate those obtained in  $L_{10}$ . The comparison is straightforward. Indeed, the upper scenarios of  $L_9$  and  $L_{10}$  in Figure 3 replicate the lotteries  $L_5$  and  $L_6$  shown in the top of Figure 2, while the lower scenarios replicate the lotteries  $L_7$  and  $L_8$  that are represented at the bottom of Figure 2. The dominance comparisons for lotteries of Figure 2 can therefore be used to get dominance comparisons regarding the lotteries in Figure 3. We can generalize this method to a recursive approach, in which we determine the dominance between elements of  $D_{\tau-1,t}$  by looking at the dominance of all realizations at date  $\tau$ , which are elements of  $D_{\tau,t}$ . The recursion ends for  $\tau = t$ , when elements of  $D_{t,t}$  are deterministic consumption paths, that can be compared using ordinal preferences. We formally present this recursive approach in Section 3.2.1.

We can also formalize the above reasoning in terms of mixture. Indeed, we have  $L_9 = p_0L_5 + (1 - p_0)L_7$  and  $L_{10} = p_0L_6 + (1 - p_0)L_8$ , with  $L_5$  dominating  $L_6$  and  $L_7$  dominating  $L_8$ . Lotteries  $L_9$  and  $L_{10}$  can thus be viewed as two binary lotteries, such that the outcomes

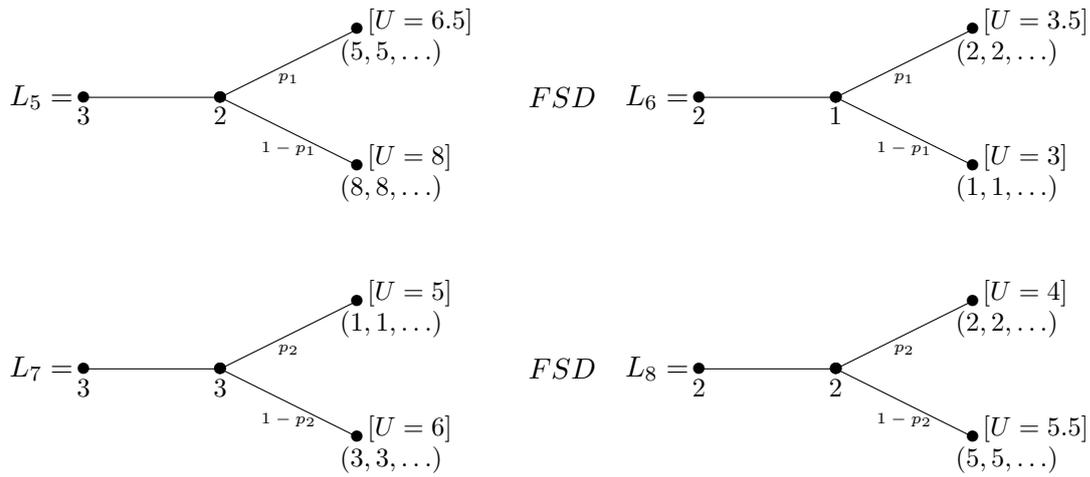


Figure 2: Two illustrations of first-order stochastic dominance for elements of  $D_{1,2}$

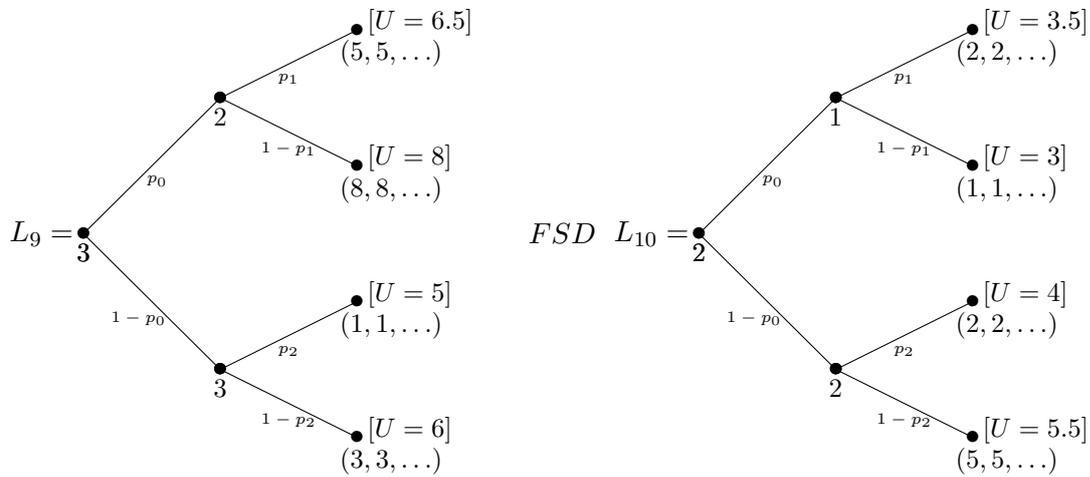


Figure 3: Illustration of first-order stochastic dominance for elements of  $D_{0,2}$

of  $L_9$  (which are lotteries) dominate those of  $L_{10}$ . A well-defined notion of FSD should thus indicate that  $L_9$  dominates  $L_{10}$ . More generally, FSD should fulfill an assumption of “mixture-stability”. In Section 3.2.2, we show how such a property may be used to define FSD in a concise but non-constructive way.

From a dynamic problem perspective, choosing  $L_9$  rather than  $L_{10}$  may be viewed as making a choice that generates continuations  $L_5$  or  $L_7$  (depending on which states of the world realizes) rather than a choice generating  $L_6$  or  $L_8$ . If  $L_5$  dominates  $L_7$  and  $L_6$  dominates  $L_8$ , a preference for  $L_9$  over  $L_{10}$  is then consistent with the elimination of dominated strategies.

It is worth emphasizing that the notion of FSD does not involve looking at the distribution of lifetime utilities in a way that would make abstraction of the structure of the probability tree, and in particular of the timing of resolution of uncertainty. In fact, prior to comparing lifetime utilities, one must represent the temporal lotteries to be compared with probability trees having exactly the same structure (that is the same nodes and the same probabilities), which is always possible with simple temporal lotteries. For example, in order to compare lotteries  $L_1$  and  $L_2$ , or  $L_1$  and  $L_3$ , one should not use the representations shown in Figure 1 that involve different probability trees, but the equivalent ones shown in Figures 4 and 5, with which comparison becomes straightforward. Our formal definition will not depend on the choice of a particular graphical representation.

Importantly, assuming that preferences are FSD-monotone is not constraining risk aversion or preferences for the timing of resolution of uncertainty. In fact temporal lotteries that assume the same distribution of consumption paths, but differ because of the timing of uncertainty resolution, are in general not comparable in terms of FSD. An example is provided in Figure 5, in which  $L_1$  and  $L_2$  are not comparable in terms of FSD dominance (unless  $u^u = u^d$ ), in the sense that neither  $L_1$  nor  $L_2$  dominates the other one. An agent with FSD-monotone preferences might very well prefer  $L_1$  to  $L_2$  (preference for late resolution of uncertainty) or  $L_2$  to  $L_1$  (preference for early resolution of uncertainty).

### 3.2 Formal definition of FSD-monotonicity

In order to define FSD-monotonicity, we need to formalize how a relation of ordinal preferences  $\succeq_0$  on  $C^\infty$  generates a natural notion of stochastic dominance on the set  $D$  of temporal lotteries. As previously explained, we provide two ways of doing so. In Section 3.2.1, we recursively build FSD using intuitions provided in Section 3.1. Section 3.2.2 suggests an (equivalent) non-constructive definition based on a closure strategy. As detailed in Section 3.2.3, this latter approach helps emphasizing the parallel between our notion of FSD, which applies to temporal lotteries, and the usual one found in atemporal settings. It will also prove to be useful to define a notion of second-order stochastic dominance for temporal lotteries

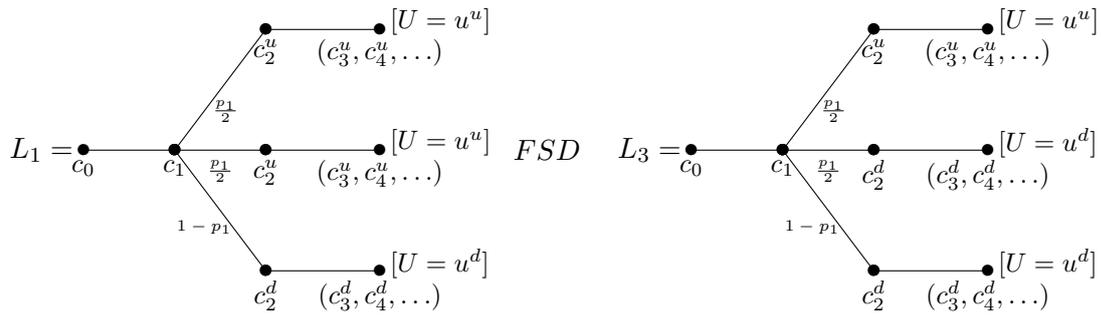


Figure 4: Lottery  $L_1$  dominates  $L_3$  (when  $u^u > u^d$ ).

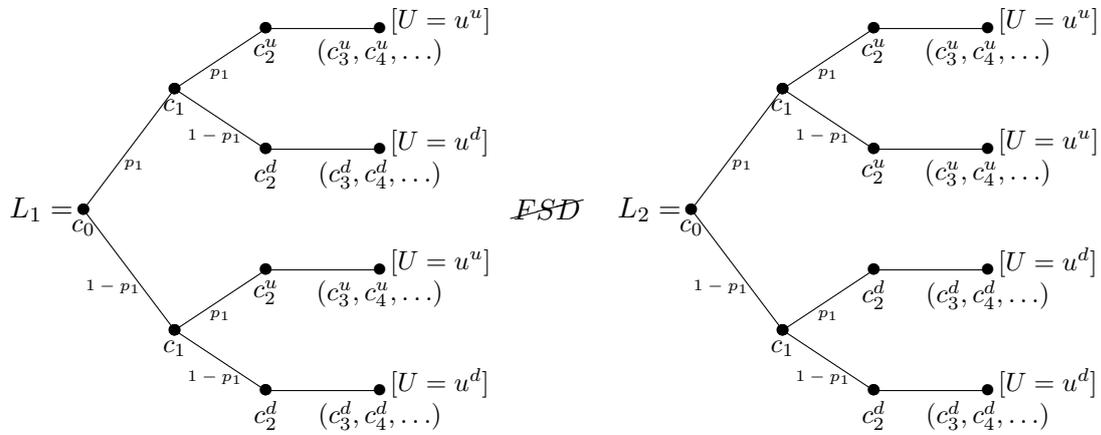


Figure 5: Lotteries  $L_1$  and  $L_2$  from Figure 1 are not FSD comparable when  $u^u \neq u^d$ .

(see Section 5.2.2).

We start by defining some standard or intuitive notions on binary relations that will be helpful for our constructions of the FSD relationship.

**Preamble: binary relations.** Formally speaking, a binary relation  $R$  on a set  $Y$  is a subset of  $Y^2$ , where the notation  $xRy$  means  $(x, y) \in R$ . Since binary relations are subsets of  $Y^2$ , we can use the usual notions of intersection, union or inclusion. For example  $R_1 \subset R_2$  means that  $xR_1y \Rightarrow xR_2y$  (the relation  $R_2$  is then an “extension” of  $R_1$ ), while  $x(R_1 \cap R_2)y$  means  $(xR_1y \text{ and } xR_2y)$ . Moreover, if  $Y$  is a topological space, a binary relation  $R$  is continuous if  $R$  is a closed subset of  $Y^2$  (endowed with the product topology). Given a mixture operation on  $Y$ , the relation  $R$  will be said to be “mixture-stable” if, for all  $\lambda \in [0, 1]$ ,  $x_1Ry_1$  and  $x_2Ry_2$  implies  $((1 - \lambda)x_1 + \lambda x_2)R((1 - \lambda)y_1 + \lambda y_2)$ .<sup>6</sup> We will use the acronym “RTCMS” for “reflexive, transitive, continuous and mixture-stable”.

### 3.2.1 A constructive definition of first-order stochastic dominance

A preference relation  $\succeq_0$  on the set  $C^\infty$  of deterministic consumption paths induces a natural ranking of degenerate lotteries. We construct an associated notion of FSD in two steps: (i) we define by induction relations  $FSD_t$  that apply to elements of  $D_t$ ; (ii) with a continuity argument, we use these relations  $FSD_t$  to define FSD over the whole set  $D$ .

**Definition 1 (First-order stochastic dominance for lotteries in  $D_t$ )** Consider a preference relation  $\succeq_0$  on  $C^\infty$ . For any  $t \geq 0$ , we define the relation of stochastic dominance  $FSD_t$  on  $D_t$  as follows:

1. (starting point of the recursion)  $FSD_{t,t} = \succeq_0$ ;
2. (backward induction on  $\tau$  to define  $FSD_{\tau-1,t}$  from  $FSD_{\tau,t}$ ) For any  $1 \leq \tau \leq t$  and any  $((c, m), (c', m')) \in (D_{\tau-1,t})^2 = (C^\tau \times M(D_{t-\tau}))^2$ , we have  $(c, m)FSD_{\tau-1,t}(c', m')$  if and only if for all continuous functions  $\phi : C^\tau \times D_{t-\tau} (= D_{\tau,t}) \rightarrow \mathbb{R}$  such that  $yFSD_{\tau,t}y' \Rightarrow \phi(y) \geq \phi(y')$ , we have:

$$\int_{D_{t-\tau}} \phi(c, x) dm(x) \geq \int_{D_{t-\tau}} \phi(c', x) dm'(x). \quad (4)$$

3. ( $FSD_t$  defined as the terminal point of the recursion)  $FSD_t = FSD_{0,t}$ .

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<sup>6</sup>When the relation  $R$  is reflexive and transitive, mixture-stability is equivalent to the independence axiom of von Neumann-Morgenstern. However, we will consider binary relations over the set of temporal lotteries, where not all elements can be mixed. Mixture-stability is then an extension of the temporal independence axiom of Kreps and Porteus (1978).

The intuitions lying behind Definition 1 have been explained in Section 3.1. Note that the relation  $FSD_{\tau-1,t}$  is an extension of  $FSD_{\tau,t}$  in the sense that  $FSD_{\tau,t} \subset FSD_{\tau-1,t}$ . By induction, we eventually obtain that for any  $t$  and any  $\tau \leq t$ , we have  $FSD_{\tau,t} \subset FSD_t$ . We can analogously check that  $FSD_{t+1}$  is an extension of  $FSD_t$ . Finally, we can show by induction that the relations  $FSD_{\tau,t}$  (and hence  $FSD_t$ ) are transitive.

We can now provide our definition of FSD over the whole set of temporal lotteries.

**Definition 2 (Stochastic dominance for temporal lotteries)** *We define the first-order stochastic dominance relation  $FSD$  over  $D$  as the topological closure of  $\cup_{t \geq 0} FSD_t$ .*

### 3.2.2 A non-constructive definition of first-order stochastic dominance

In Section 3.1, we explained when discussing lotteries  $L_9$  and  $L_{10}$  of Figure 3 that any well-defined notion of FSD should fulfill an assumption of mixture-stability. We now prove that the relation  $FSD$  provided in Definition 2 is the smallest mixture-stable relation that extends the incomplete order induced by  $\succeq_0$ . We first formally define mixture-stability for temporal lotteries, accounting for the fact that only elements admitting identical deterministic vectors of initial consumptions can be mixed, as in equation (2). Formally:

**Definition 3 (Mixture-stability for temporal lotteries)** *A binary relation  $R$  on  $D$  is mixture-stable if for all  $\tau \geq 0$ ,  $\lambda \in (0, 1)$ ,  $(c_1, c_2) \in (C^{\tau+1})^2$  and  $(m_1, m'_1, m_2, m'_2) \in M(D)^4$ :*

$$\left. \begin{array}{l} (c_1, m_1)R(c_2, m_2) \\ (c_1, m'_1)R(c_2, m'_2) \end{array} \right\} \Rightarrow (c_1, \lambda m_1 + (1 - \lambda)m'_1)R(c_2, \lambda m_2 + (1 - \lambda)m'_2).$$

We can now state the following result:

**Lemma 1 (FSD as a closure of ordinal preferences)** *The relation  $FSD$  introduced in Definition 2 is the smallest RTCMS extension of  $\succeq_0$ .*

**Proof.** The proof, detailed in Appendix B.1 is very much inspired by the contribution of Dubra, Maccheroni and Ok (2004). Actually, it works just like that of Lemma 2, detailed below, which –restricted to atemporal settings– may be considered as a direct application of their paper. However, as the set of temporal lotteries is not a mixture space, proving Lemma 1 involves establishing a similar result to the one of Dubra, Maccheroni and Ok (2004) (also based on the hyperplane separation theorem) that applies to binary relations that are not necessarily transitive or reflexive (Lemma 6 in the Appendix). ■

Definition 2 of FSD takes a “bottom-up approach”: we start from  $\succeq_0$  and construct an increasing sequence of binary relations with the objective to comply with the mixture-stability

property. Lemma 1 suggests an equivalent “top-down” approach, in which FSD could be introduced as the “smallest” binary RTCMS relation extending  $\succeq_0$ .<sup>7</sup>

### 3.2.3 Parallel with the notion of FSD in atemporal settings

In order to provide further insights on the link between our constructive and non-constructive approaches, as well as to emphasize the parallel with the usual notion of FSD found in atemporal settings, we state the following result:

**Lemma 2 (FSD in atemporal settings)** *Consider a compact metric ordered space  $(X, \geq)$ . Denote by  $M(X)$  the set of probability measures over  $X$ . The relation of FSD on  $M(X)$ , denoted  $FSD^X$ , can be defined in three equivalent ways:*

1. (cumulative distribution function approach)  $\forall(m, m') \in M(X)^2$ :

$$mFSD^X m' \iff \forall y \in X, m(\{x \in X | x \geq y\}) \geq m'(\{x \in X | x \geq y\}). \quad (5)$$

2. (dual approach)  $\forall(m, m') \in M(X)^2$ , we have  $mFSD^X m'$ , if for all continuous non-decreasing functions  $\phi : X \rightarrow X$ , the following inequality holds:

$$\int_X \phi(x) dm(x) \geq \int_X \phi(x) dm'(x).$$

3. (closure approach) The relation  $FSD^X$  is the smallest RTCMS extension of  $\geq$ .<sup>8</sup>

**Proof.** Equivalence between 1 and 2 was shown by Fishburn (1974). To show that  $2 \Leftrightarrow 3$ , denote by  $\Omega$  the set of RTCMS extensions of  $\geq$ . It is straightforward to check that  $FSD^X$ , when defined as in point 2, belongs to  $\Omega$ . It remains to be shown that it is the smallest element of  $\Omega$ . From Dubra, Maccheroni and Ok (2004), we know that for any  $R \in \Omega$  there exists a class of continuous utility functions  $\mathcal{U}_R$  such that  $m_1 R m_2 \Leftrightarrow E_{m_1}[u] \geq E_{m_2}[u]$  for all  $u \in \mathcal{U}_R$ . Denote  $R_0$  the smallest element of  $\Omega$ , given by  $R_0 = \bigcap_{R \in \Omega} R$ , to which corresponds the class  $\mathcal{U}_{R_0} = \bigcup_{R \in \Omega} \mathcal{U}_R$ . Since any  $R \in \Omega$  extends  $\geq$ , all classes  $\mathcal{U}_R$  must contain non-decreasing functions. Therefore,  $\mathcal{U}_{R_0}$  is included in the set of all continuous non-decreasing functions. This means  $\mathcal{U}_{R_0} \subset \mathcal{U}_{FSD^X}$  implying  $FSD^X \subset R_0$ . ■

Lemma 2 provides different characterizations of FSD in atemporal settings. When working with temporal lotteries, the first two characterizations look problematic as realizations of

<sup>7</sup>We qualify this approach of non-constructive, as FSD is obtained as the intersection of elements in a set of binary relations on  $D$  (i.e., a set of subsets of  $D^2$ ), whose existence is granted by the axiom of the power set.

<sup>8</sup>As it is standard,  $\geq$  also denotes (with a slight abuse of notation) the order on the set of degenerate lotteries.

temporal lotteries are still random objects, for which there is no natural ranking.<sup>9</sup> In the constructive definition, we circumvented this difficulty by using a recursive approach, making it possible to extend step by step the incomplete order induced by  $\succeq_0$ . The other route, involves using the third characterization, based on the closure approach, which can be readily applied to temporal lotteries, once the notion of mixture-stability is properly defined. This is the idea of our non-constructive definition. Lemma 1 shows that just like in atemporal settings, both routes lead to the same notion of FSD.

### 3.2.4 FSD-monotonicity

Having explained how a preference relation  $\succeq_0$  on  $C^\infty$  generates an FSD relation on temporal lotteries, we now state our definition of FSD-monotonicity.

**Definition 4 (FSD-monotonicity)** *Consider a preference relation  $\succeq$  defined on  $D$ . Denote  $\succeq_0$  the restriction of  $\succeq$  to  $C^\infty$  and FSD the first-order stochastic dominance relation obtained from Definition 2. The preference relation  $\succeq$  is said to be FSD-monotone if and only if we have:*

$$\forall((c, m), (c', m')) \in D^2, (c, m)FSD(c', m') \Rightarrow (c, m) \succeq (c', m').$$

The above definition states that a preference relation is FSD-monotone whenever first-order dominating lotteries are preferred. However FSD is derived from  $\succeq_0$ , the restriction of  $\succeq$  to the set of deterministic consumption paths. FSD-monotonicity is therefore a consistency requirement between ordinal and risk preferences. Moreover, since by construction  $\succeq$  extends  $\succeq_0$ , a consequence of Lemma 1 is that a continuous preference relation which is mixture-stable is necessarily FSD-monotone. The reciprocal however is false.<sup>10</sup>

In order to illustrate the interest of assuming FSD-monotonicity, we consider below a two-period saving problem and emphasize the unintuitive findings that may be obtained when using non FSD-monotone preferences.

### 3.3 FSD-monotonicity and the elimination of dominated-strategies in a two-period saving problem

Consider a two-period economy. At date 0, agents receive the certain income  $y_0$  that they allocate between consumption and savings. At date 1, there are two states of the world,  $h$  and  $l$  that determine both the date 1 income (equal to  $y_1^h$  or  $y_1^l$ ) and the gross return of saving (equal to  $R_h$  or  $R_l$ ).

<sup>9</sup>One cannot use the ranking generated by risk preferences, as the objective is precisely to have a notion of FSD that only depends on preferences over deterministic consumption paths.

<sup>10</sup>In atemporal settings, rank dependent expected utility and cumulative prospect theory provide examples of preferences which are FSD-monotone but not mixture-stable.

Throughout this section, we assume that preferences over deterministic consumption paths are represented by  $U_\infty(c_0, c_1) = (c_0^\rho + \beta c_1^\rho)^{\frac{1}{\rho}}$ , where  $1 > \rho \neq 0$  and  $\beta > 0$ . Risk preferences will either be unspecified, though assumed to be FSD-monotone (for Lemma 3), or preferences à la Epstein-Zin (for Lemma 4 and Figure 6).

In the presence of uncertainty, agents have to choose their savings before observing whether state  $h$  or  $l$  will occur. We denote by  $c_0^*$  the optimal consumption at date 0 and  $s^*$  the optimal amount of savings. The budget constraints can be expressed as follows:

$$\begin{aligned} y_0 - s^* &= c_0^* \geq 0, \\ y_1^\kappa + R_\kappa s^* &= c_{1,\kappa}^* \geq 0 \text{ for } \kappa = h, l, \end{aligned}$$

where  $c_{1,\kappa}^*$  denotes date 1 consumption if state  $\kappa$  occurs.

For  $\kappa = h, l$ , denote by  $s_\kappa$  the amount of savings that the agents would choose if they knew that the state  $\kappa$  would occur for sure. We have  $s_\kappa = \frac{y_0 - y_1^\kappa (\beta R_\kappa)^{\frac{1}{\rho-1}}}{1 + R_\kappa (\beta R_\kappa)^{\frac{1}{\rho-1}}}$ . Using a game theory terminology, the levels  $s_h$  and  $s_l$  are the (state-specific) agents' best responses.

**Lemma 3 (Savings with FSD-monotone preferences)** *Consider the saving problem described above. If agents have FSD-monotone preferences, then  $\min(s_h, s_l) \leq s^* \leq \max(s_h, s_l)$ .*

**Proof.** Since ordinal preferences are strictly convex, choosing  $s^* < \min(s_h, s_l)$  is dominated by the choice of  $\min(s_h, s_l)$  in both states of the world. Analogously, choosing  $s^* > \max(s_h, s_l)$  is dominated by the choice of  $\max(s_h, s_l)$ . ■

This intuitive result does not hold with non FSD-monotone preferences. Indeed, let us now assume that agents have Epstein-Zin preferences, represented by:

$$U^{EZ}(c_0, \tilde{c}_1) = \left( c_0^\rho + \beta (E[\tilde{c}_1^\alpha])^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}}, \quad (6)$$

where  $E[\cdot]$  is an expectation operator and  $\alpha \neq 0$  a parameter driving risk aversion. The optimal amount of savings  $s^{EZ}$  is the unique maximum of the following agents' program:

$$s^{EZ} = \arg \max_{s \in (-\min(\frac{y_1^l}{R_l}, \frac{y_1^h}{R_h}), y_0)} U^{EZ}(y_0 - s, \tilde{y}_1 + \tilde{R}s) \quad (7)$$

where the notation  $\tilde{y}_1$  and  $\tilde{R}$  denote the random income and asset returns.

**Lemma 4 (Savings with Epstein-Zin preferences)** *Consider the saving problem described in equation (7). If  $\rho \neq \alpha$ , there exist values of  $R_\kappa$  and  $y_1^\kappa$  (for  $\kappa = h, l$ ) for which agents end up choosing a dominated saving strategy  $s^{EZ} \notin [\min(s_h, s_l), \max(s_h, s_l)]$ .*

**Proof.** Assume  $y_1^h \neq y_1^l$  and  $R_\kappa = \frac{1}{\beta} \left(\frac{y_1^\kappa}{y_0}\right)^{1-\rho}$  in state  $\kappa = h, l$ . In that case  $s_h = s_l = 0$ , so that the only non-dominated strategy is  $s = 0$ . However, we have

$$\frac{d}{ds} \left( \log U^{EZ}(y_0 - s, \tilde{y}_1 + \tilde{R}s) \right) \Big|_{s=0} = \frac{y_0^{\rho-1}}{U^{EZ}(y_0, \tilde{y}_1)} \left( \frac{E[\tilde{z}^{1-\frac{\rho}{\alpha}}]}{E[\tilde{z}]^{1-\frac{\rho}{\alpha}}} - 1 \right), \quad (8)$$

where  $\tilde{z} = \tilde{y}_1^\alpha$ . Since  $\rho \neq 0$  and  $\rho \neq \alpha$ , the function  $x \mapsto x^{1-\frac{\rho}{\alpha}}$  is either strictly concave or strictly convex. Using Jensen inequality, the derivative (8) cannot be equal to zero. Thus  $s = 0$  is not optimal for the agent. ■

Lemma 4 shows that an agent endowed with Epstein-Zin preferences may choose dominated saving strategies. The proof uses the fact that both income and asset returns are random. However, when the inverse of intertemporal elasticity of substitution is smaller than one and smaller than the coefficient of relative risk aversion, similar examples of dominated choices can be obtained in a simpler consumption-saving problem where only income is random. Actually, whenever the income risk is large enough, or risk aversion strong enough, an agent with such Epstein-Zin preferences can be shown to save more than what he would do in the worst case. In others words, the agent who normally earns  $y_1^l$  but who is told that he might receive a large positive bonus ( $y_1^h - y_1^l$ ) at the end of the year may react to this information by saving more! Figure 6 illustrates such an unintuitive saving behavior.<sup>11</sup> In fact, the possibility of receiving a positive bonus in the second period generates uncertainty regarding the second period utility. An increase in savings is then “rationally” chosen to reduce, in relative terms, this uncertainty. The extent to which this relative uncertainty reduction regarding second period utility is valuable is open to question. One might argue that an anxious agent may be happier to have less uncertainty left for the future. But should that lead the agent to take actions that lower welfare in all circumstances? Such an unappealing feature would be ruled out by the assumption of FSD-monotonicity.

This example illustrated in Figure 6 also helps to explain why understanding the role of risk aversion may be difficult when using Epstein-Zin preferences. When  $\alpha = \rho$  (i.e., in the additive model), preferences are FSD-monotone and optimal savings  $s^{EZ}$  lie in the interval  $[s_h, s_l]$ . Increasing risk aversion (i.e., decreasing  $\alpha$ ) first raises savings, which get closer to  $s_l$ , the optimal strategy in the bad state. Increasing further the degree of risk aversion eventually leads the agent to take dominated strategies  $s^{EZ} > s_l$ . For extreme degrees of risk aversion (i.e.  $\alpha \rightarrow -\infty$ ), the agent only cares about the bad state and thus chooses  $s^{EZ} \rightarrow s_l$ . The link between risk aversion and saving behavior is therefore non-monotonic. This occurs because the agent opts for dominated strategies for some degrees of risk aversion. Assuming FSD-

<sup>11</sup>The parameters used to draw the figure are  $\rho = \frac{1}{2} = -\alpha$ ,  $R_l = R_h = \beta = 1$ ,  $y_0 = 100$ ,  $y_1^l = 1$ ,  $y_1^h = 289$ . States  $h$  and  $l$  occur with equal probabilities.

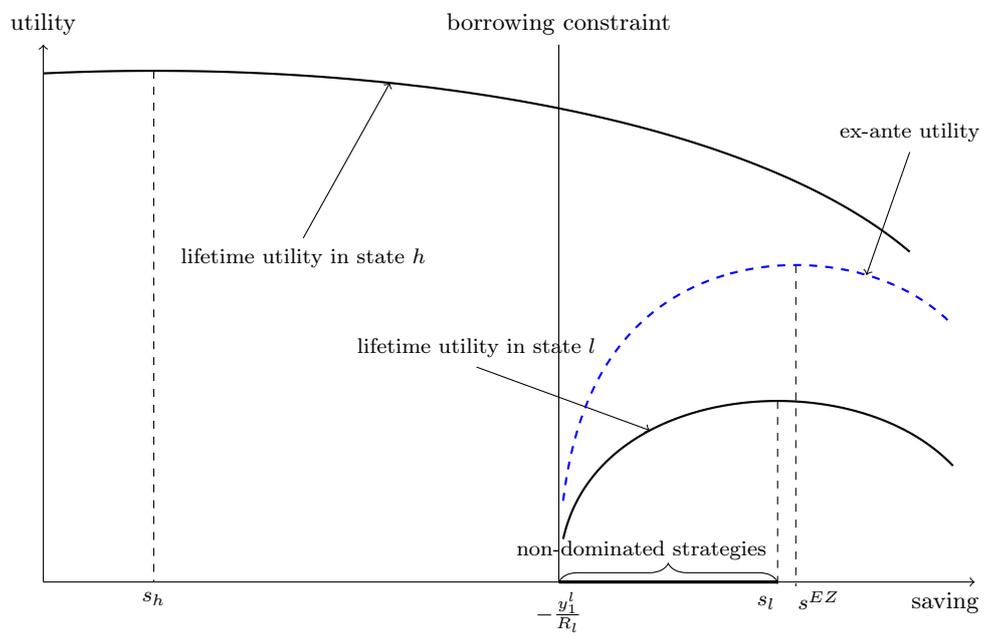


Figure 6: An example of dominated saving strategies with Epstein-Zin preferences

monotone preferences, Proposition 6 provides a much more intuitive monotonic pattern, in which savings increase with risk aversion.

## 4 Monotone Kreps-Porteus recursive preferences

### 4.1 Definition of Kreps-Porteus recursive preferences

Our paper explores the set of stationary preferences defined on  $D$  that fits into the framework introduced by Kreps and Porteus (1978). We restrict our attention to preferences for which, in the absence of uncertainty, greater consumption provides more utility. In other words, ordinal preferences are assumed to be monotonic in the usual sense. Moreover, for technical reasons, we introduce some differentiability requirements, and assume that, when looking at deterministic consumption paths, the marginal rate of substitution between the consumption at two different dates is always well-defined and finite. This leads to the following formal definition:

**Definition 5 (Kreps-Porteus recursive preferences)** *A preference relation is said to be Kreps-Porteus recursive (henceforth KP-recursive) if it can be represented by a utility function  $U:D \rightarrow \mathbb{R}$  fulfilling the following recursion:*

$$U(c_0, m) = W(c_0, E_m[U]), \tag{9}$$

where  $E_m[\cdot]$  denotes the expectation with respect to the probability measure  $m$  and  $W : \mathbb{R}^+ \times \text{Im}(U) \rightarrow \text{Im}(U)$  is a twice continuously differentiable function –called aggregator– with positive derivatives  $W_x$  and  $W_y$ .<sup>12</sup>

Equation (9) reflects both the fact that we consider a Kreps-Porteus framework and the assumption of preference stationarity in the sense of Koopmans (1960). The positivity requirements on the derivatives  $W_x$  and  $W_y$  are necessary conditions for preferences over deterministic consumption paths to be monotonic with well-defined marginal rates of substitution.

By definition, a given KP-recursive preference relation must admit at least one utility representation fulfilling (9). But, not all utility representations need to fulfill (9). The remark below makes the link with other popular representations of KP-recursive preferences formulated in terms of certainty equivalents.

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<sup>12</sup> $\text{Im}(U)$  denotes the image of  $U$ .

**Remark 1** Consider a preference relation  $\succeq$  on  $D$  represented by a utility function  $U : D \rightarrow \mathbb{R}$  fulfilling the following recursion:

$$U(c_0, m) = W(c_0, \phi^{-1}E_m[\phi(U)]), \quad (10)$$

where  $\phi : Im(U) \rightarrow \mathbb{R}$  is a twice differentiable function with a positive derivative and  $W : \mathbb{R}^+ \times Im(U) \rightarrow Im(U)$  is a twice continuously differentiable with positive derivatives.

Then, the preference relation  $\succeq$  is KP-recursive in the sense of Definition 5.

**Proof.** Since  $\phi$  is increasing, the utility function  $V = \phi(U)$  represents the same preference relation as  $U$ . Moreover,  $V$  fulfills the recursion (9) with the aggregator  $\hat{W}(x, y) = \phi(W(x, \phi^{-1}(y)))$ , which is twice continuously differentiable with positive derivatives  $\hat{W}_x$  and  $\hat{W}_y$ . Preferences represented by  $V$  (and by  $U$ ) are therefore KP-recursive. ■

The most common KP-recursive preferences are those represented by additive separable utility functions:

$$U(c_0, m) = (1 - \beta)u(c_0) + \beta E_m[U],$$

where  $0 < \beta < 1$  and  $u : C \rightarrow [0, 1]$  is a twice continuously differentiable function with  $u' > 0$ . Another example is related to Uzawa (1968) utility function with:<sup>13</sup>

$$U(c_0, m) = a(c_0) + b(c_0)E_m[U]. \quad (11)$$

These preferences are KP-recursive, provided that technical conditions on the functions  $a(\cdot)$  and  $b(\cdot)$  are introduced to guarantee that the aggregator is differentiable with positive derivatives.

One of the most popular examples of KP-recursive preferences is the Epstein-Zin isoelastic preferences, usually represented by utility functions fulfilling the following recursion:<sup>14</sup>

$$U(c_0, m) = \begin{cases} \left[ (1 - \beta)c_0^\rho + \beta(E_m[U^\alpha])^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}} & \text{if } 0 \neq \rho < 1, \alpha \neq 0, \\ \exp\left((1 - \beta)\log(c_0) + \frac{\beta}{\alpha}\log(E_m[U^\alpha])\right) & \text{if } \rho = 0, \alpha \neq 0, \\ \left[ (1 - \beta)c_0^\rho + \beta \exp(\rho E_m[\log(U)]) \right]^{\frac{1}{\rho}} & \text{if } 0 \neq \rho < 1, \alpha = 0, \\ \exp((1 - \beta)\log(c_0) + \beta E_m[\log(U)]) & \text{if } \rho = \alpha = 0, \end{cases} \quad (12)$$

with  $0 < \beta < 1$ . The fact that such utility functions represent KP-recursive preferences stems from Remark 1 above.

<sup>13</sup>This utility function represents preferences of the expected utility kind and was introduced in continuous time by Uzawa (1968), and discussed further in discrete time by Epstein (1983).

<sup>14</sup>The cases  $\alpha = 0$  or  $\rho = 0$  are limit cases of the general one. We provide their formulations to simplify the comparison with risk-sensitive preferences introduced in equation (13).

Our purpose in the current paper is to look for classes of KP-recursive preferences that are FSD-monotone. We will prove that this leads to preferences that can be represented by a Uzawa utility function as in (11) or by the following utility function:

$$U(c_0, m) = \begin{cases} (1 - \beta)u(c_0) - \frac{\beta}{k} \log(E_m [e^{-kU}]) & \text{if } k \neq 0, \\ (1 - \beta)u(c_0) + \beta E_m[U] & \text{if } k = 0, \end{cases} \quad (13)$$

for some function  $u$  with positive derivative and a constant  $\beta < 1$ . This specification was introduced by Hansen and Sargent (1995) as a tractable way of having a *risk-adjusted measure of cost* in a problem of optimal control. In Hansen, Sargent and Tallarini (1999), such a specification was used to represent the preferences of robust decision makers.<sup>15</sup> Due to the parallel between robustness and risk-sensitivity analysis, which is discussed in Hansen, Sargent and Tallarini (1999), these preferences might indifferently be called *robust preferences* or *risk-sensitive preferences*. Throughout the paper, we use the latter terminology (risk-sensitive preferences), which is more commonly used in the recent literature, as for example in the survey of Backus, Routledge and Zin (2005).<sup>16</sup>

Note that choosing  $u(c) = \log(c)$  in equation (13) implies that  $V = \exp(U)$ , which represents the same preferences as  $U$ , fulfills the recursion (12) with  $\rho = 0$  and  $\alpha = -k$ . Thus, when the intertemporal elasticity of substitution is equal to one, Epstein-Zin and risk-sensitive preferences coincide with each other, as already noticed by Tallarini (2000). The class of risk-sensitive preferences also intersects with Epstein-Zin's when  $k = 0$  and  $u$  is isoelastic. This corresponds to the standard additively separable model with a constant intertemporal elasticity of substitution. In all other cases, risk-sensitive preferences differ from Epstein-Zin's.

## 4.2 Representation result

Our first representation result shows that imposing recursivity and FSD-monotonicity readily leaves us with a small set of KP-recursive preferences.

**Proposition 1 (Representation result)** *Consider a KP-recursive preference relation  $\succeq$ . The following statements are equivalent:*

1. *the preference relation  $\succeq$  is FSD-monotone;*
2. *the preference relation  $\succeq$  can be represented by a utility function fulfilling one of the following recursive equations:*

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<sup>15</sup>“Robust agents” worry about possible model misspecifications and account for them in their decisions. Hansen and Sargent (2007b) provide a self-contained description of robustness applications in economics.

<sup>16</sup>The terminology *robust preferences* is now used more often to refer to max-min preferences.

(a) (*Uzawa case*)

$$U(c_0, m) = a(c_0) + b(c_0)E_m[U], \quad (14)$$

where  $a, b : C \rightarrow [0, 1]$  are twice continuously differentiable functions such that  $a(\underline{c}) = 0$ ,  $a(\bar{c}) + b(\bar{c}) = 1$  and for all  $x \in C$ ,  $a'(x) > 0$ ,  $a'(x) + b'(x) > 0$  and  $0 < b(x) < 1$ .

(b) (*Risk-sensitive case*)

$$U(c_0, m) = (1 - \beta)u(c_0) - \frac{\beta}{k} \log \left( E_m[e^{-kU}] \right), \quad (15)$$

where  $0 < \beta < 1$ ,  $k \neq 0$  and  $u : C \rightarrow \mathbb{R}$  is a twice continuously differentiable function with a strictly positive derivative.<sup>17</sup>

Moreover, to any function  $a(\cdot)$  and  $b(\cdot)$  or any function  $u(\cdot)$  and scalars  $\beta$  and  $k$  that fulfill the above conditions, corresponds a unique KP-recursive preference relation.

**Proof.** We provide a complete proof in the Appendix. The last statement of the proposition, about existence and uniqueness, relies on a standard fixed-point argument. We outline below the main intuitions underlying the proof  $1 \Leftrightarrow 2$ .

**Intuition for the proof that  $2 \Rightarrow 1$ .** The shortest way to prove  $(2 \Rightarrow 1)$  is to show that Uzawa and risk-sensitive preferences are mixture-stable. The proof given in Appendix B.2.3 follows this path. However, gaining a better intuition of why these preferences are FSD-monotone involves showing that they are monotone with respect to  $FSD_t$  for all  $t \geq 0$ .

Let  $t \geq 0$  and a temporal lottery  $(c_0, m) \in D_t$ . First, consider the Uzawa case. To make intuitions more transparent, we specify the lifetime consumption process denoted  $(c_0, \dots, \tilde{c}_t, \dots)$ . The consumption  $\tilde{c}_i$  of date  $1 \leq i \leq t$  is uncertain from date 0 point of view and the uncertainty is resolved at date  $i$ . The uncertainty for consumptions after date  $t$  is also resolved at date  $t$ , since  $(c_0, m) \in D_t$ . The associated utility is denoted  $U(c_0, \dots, \tilde{c}_t, \dots)$  with a slight abuse of notation. Using the linearity of the expectation operator, iterating forward the recursion yields:

$$U(c_0, \dots, \tilde{c}_t, \dots) = E_0 \left[ \dots E_{t-1} \left[ \sum_{i=0}^{\infty} a(\tilde{c}_i) \prod_{j=0}^{i-1} b(\tilde{c}_j) \right] \dots \right] = E_0 \left[ \sum_{i=0}^{\infty} a(\tilde{c}_i) \prod_{j=0}^{i-1} b(\tilde{c}_j) \right], \quad (16)$$

where  $E_\tau[\cdot]$  ( $0 \leq \tau \leq t - 1$ ) is the expectation conditional on the information available

<sup>17</sup>Remark that in the limit case when  $k$  approaches zero, the recursion (15) converges to  $U(c_0, m) = (1 - \beta)u(c_0) + \beta E_m[U]$ , which is also a particular case of (14), providing the class of additively separable utility functions.

at date  $\tau$  (i.e., the  $\sigma$ -algebra generated by the process  $\tilde{c}_i$  for  $0 \leq i \leq \tau$ ).<sup>18</sup> The second equality in (16) stems from the law of iterated expectations. FSD-monotonicity then becomes straightforward, since the utility  $U(c_0, \dots, \tilde{c}_t, \dots)$  is expressed as the expectation of lifetime utilities given by  $\sum_{i=0}^{\infty} a(c_i) \prod_{j=0}^{i-1} b(c_j)$ . We obtain this expression with a single expectation operator since Uzawa preferences belong to the expected utility framework, in which the reduction of compound lotteries holds.

Second, consider the risk-sensitive case. For any  $\tau \geq 0$ , we denote by  $U_\tau = (1 - \beta)u(c_\tau) - \frac{\beta}{k} \log(E_\tau[e^{-kU_{\tau+1}}])$  the (forward-looking) utility given information available at date  $\tau$ . We also define  $V_0 = U_0$  and for any  $\tau > 0$ ,  $V_\tau = (1 - \beta) \sum_{i=0}^{\tau-1} \beta^i u(c_i) + \beta^\tau U_\tau$ , which can be viewed as the (backward-forward looking) utility given information available at date  $\tau$ . We have:

$$V_\tau = (1 - \beta) \sum_{i=0}^{\tau} \beta^i u(c_i) - \frac{\beta^{\tau+1}}{k} \log(E_\tau[e^{-kU_{\tau+1}}]) = -\frac{\beta^{\tau+1}}{k} \log(E_\tau[e^{-\frac{k}{\beta^{\tau+1}} V_{\tau+1}}]).$$

When  $\tau \geq t$ , there is no uncertainty left since we consider elements of  $D_t$ , and  $V_\tau = (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(c_i)$ . Thus,  $V_0$  (and hence  $U_0$ ) can be obtained as the final point of the backward recursion given by:

$$\begin{cases} V_t &= (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(c_i), \\ V_\tau &= \phi_\tau^{-1} E_\tau[\phi_\tau(V_{\tau+1})] \text{ for any } 0 \leq \tau \leq t - 1. \end{cases} \quad (17)$$

where  $\phi_\tau$  is the function given by  $\phi_\tau : x \mapsto -\frac{\beta^{\tau+1}}{k} e^{-\frac{k}{\beta^{\tau+1}} x}$ . Since functions  $\phi_\tau$  are increasing, FSD-monotonicity holds with lifetime utilities given by  $(1 - \beta) \sum_{i=0}^{\infty} \beta^i u(c_i)$ .

**Intuition for the proof that 1  $\Rightarrow$  2.** For the sake of simplicity, we start with a two-period setting. Consider preferences over the set of certain  $\times$  uncertain consumption pairs that are FSD-monotone, and like KP-recursive preferences, represented by:

$$(c_0, \tilde{c}_1) \in C \times M(C) \mapsto U(c_0, \tilde{c}_1) = W(c_0, E_0[u(\tilde{c}_1)]). \quad (18)$$

Assume for simplicity, that we can find  $c_0^*$  such that for any  $(c_0, c_1)$  there exists  $\eta(c_0, c_1)$ , which makes  $(c_0, c_1)$  and  $(c_0^*, \eta(c_0, c_1))$  equally preferable.<sup>19</sup> We can then extend  $U$ , initially

<sup>18</sup>In the remainder of the paper, we will use either the expectation  $E_m[\cdot]$  with respect to the measure  $m$  or the expectation  $E_\tau[\cdot]$  with respect to the natural filtration generated by the consumption process, depending on whether the consumption process is made explicit or not.

<sup>19</sup>In general, there is no such  $c_0^*$ , but working locally on small neighborhoods, we can assume its existence. The proof contains a technical part mapping local results into global ones.

defined over  $C \times M(C)$  in (18) to  $M(C \times C)$  by setting:

$$(\tilde{c}_0, \tilde{c}_1) \in M(C \times C) \mapsto U(\tilde{c}_0, \tilde{c}_1) = U(c_0^*, \eta(\tilde{c}_0, \tilde{c}_1)). \quad (19)$$

Preferences represented by (19) are FSD-monotone. Moreover, due to (18), they fulfill the von Neuman-Morgenstern independence axiom and thus admit an expected utility representation. We also know that preferences over lotteries on second period consumption are independent of first period consumption and represented by  $E_0[u(\tilde{c}_1)]$ . Using Theorem 5.6 of Keeney and Raiffa (1993), preferences over  $M(C \times C)$  can be represented by:

$$U(\tilde{c}_0, \tilde{c}_1) = E_0[a(\tilde{c}_0) + b(\tilde{c}_0)u(\tilde{c}_1)], \quad (20)$$

for some functions  $a(\cdot)$  and  $b(\cdot)$ . Equations (18) and (20) then provide two utility representations of the same preference relation over  $C \times M(C)$ . It is thus possible to go from one to the other by an increasing transformation. Thus, there must exist an increasing function  $\phi$  such that:

$$W(c_0, E_0[u(\tilde{c}_1)]) = \phi(E_0[a(c_0) + b(c_0)u(\tilde{c}_1)]).$$

FSD-monotonicity therefore implies that the aggregator  $W(x, y)$  must be such that:

$$W(x, y) = \phi(a(x) + b(x)y). \quad (21)$$

To complete the proof, we apply twice the same reasoning as above. Once for lotteries with an uncertain third period consumption, (i.e. of the type  $(c_1, c_2, \tilde{c}_3) \in C^2 \times M(C)$ ) and again for lotteries with an uncertain fourth period consumption (i.e. of the type  $(c_1, c_2, c_3, \tilde{c}_4) \in C^3 \times M(C)$ ). In both cases, FSD-monotonicity implies utility representations similar to (20). However, due to the recursivity assumption, preferences over  $C^2 \times M(C)$  are also represented by  $W(c_1, W(c_2, E_0[u(\tilde{c}_3)]))$  and those over  $C^3 \times M(C)$  by  $W(c_1, W(c_2, W(c_3, E_0[u(\tilde{c}_4)])))$ . This ends up providing a set of restrictions similar to (21) which are all together only compatible with the Uzawa or the risk-sensitive representation.

### 4.3 Discussion

According to Proposition 1, only Uzawa and risk-sensitive preferences are both KP-recursive and FSD-monotone. In this section, we provide more insights on this result and also discuss some properties of these preferences.

### 4.3.1 Stationarity and monotonicity: two different recursivity requirements

Preference stationarity and FSD-monotonicity constrain preferences in two different ways. On the one hand, following the second step of the proof of Proposition 1, one can conjecture that FSD-monotonicity imposes KP-recursive preferences to be represented by the ending point  $V_0$  of the following backward recursion:

$$\begin{cases} V_\infty = V_\infty(c_0, c_1, \dots, c_t, \dots), \\ V_\tau = \phi_\tau^{-1}(E_\tau[\phi_\tau(V_{\tau+1})]), \end{cases} \quad (22)$$

where  $V_\infty$  are lifetime utilities and the monotonic function  $\phi_\tau$  drives the agent's risk aversion with respect to the distribution of lifetime utilities.<sup>20</sup> A temporal lottery can then be viewed as a compound lottery, whose payoffs are the lifetime utilities  $V_\infty$ . This compound lottery is evaluated by a backward induction, in which at each step  $\tau$ , the uncertain future is replaced by its certainty equivalent computed with the function  $\phi_\tau$ . The way these certainty equivalents are computed may depend on the date  $\tau$  because, in the Kreps-Porteus framework, agents are not necessarily indifferent to the timing of uncertainty resolution. Conversely, any preferences defined by the recursion (22) with monotonic functions  $\phi_\tau$  would be FSD-monotone. Without further assumptions about  $\phi_\tau$ , such preferences may be non-stationary for two reasons: (i) the function  $\phi_\tau$  may fail to compensate for time preferences (see the comment at the end of Section 4.3.2), which would generate a "calendar time dependence"; (ii)  $V_\tau$  depends on consumptions prior to date  $\tau$ , which may cause preferences to depend on past history.

On the other hand, preference stationarity in the sense of Koopmans (1960) requires the utility function  $U_t$  to follow a recursion of the form:

$$U_t = W(c_t, E_t[U_{t+1}]), \quad (23)$$

where the utility  $U_t$  at date  $t$  is independent from consumptions prior to date  $t$ , and where the aggregator  $W$  is independent of the date  $t$ .

Recursions (22) and (23) respectively associated to FSD-monotonicity and stationarity are only compatible for Uzawa and risk-sensitive preferences. A more general framework would need to depart from FSD-monotonicity (as in Epstein and Zin, 1989, with the unpleasant consequences highlighted in Section 3.3) or from stationarity. In the latter case, we could for example still impose FSD-monotonicity, but accept a weaker notion of recursivity, where the aggregator in (23) could depend on  $t$ . In a dynamic framework, this would involve assuming history independence, time consistency and FSD-monotonicity but would

<sup>20</sup>Formally, the recursion (22) only makes it possible to define  $V_0$  on  $\bigcup_{t \geq 0} D_t$ . Extension to  $D$  would require continuity arguments to be considered in an approach similar to that of Streufert (1990).

make time-dependent evaluations possible. The frameworks of Pye (1973) and Van der Ploeg (1993) fit in that category. They correspond to the case where  $V_\infty = (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(c_i)$  and  $\phi_\tau(x) = -\frac{1}{k} e^{-kx}$  (independent of  $\tau$ ) in the recursion (22). More generally, all cases where  $V_\infty = (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(c_i)$  and the  $\phi_\tau$  are exponential (but possibly depending on  $\tau$ ) would correspond to FSD-monotone preferences fulfilling a weaker notion of recursivity, implying history independence (but allowing for calendar time dependence).

We could further weaken the recursivity requirement and allow for history dependence, while keeping sufficient structure to maintain reasonable tractability. A solution could be to consider an additive function  $V_\infty$  for lifetime utilities, together with any kind of increasing functions  $\phi_t$ . Though generating some history dependence, the history could be summarized in a single variable, the stock of accumulated welfare. Economic problems with such preferences could still be addressed using standard dynamic programming techniques with the introduction of only one additional state variable.<sup>21</sup>

### 4.3.2 Preference for the timing

As explained in Kreps and Porteus (1978), concavity (or convexity) with respect to the second argument of the aggregator introduced in Definition 5 dictates: (i) the preference for the timing of uncertainty resolution and (ii) whether risk aversion is increasing or decreasing with time distance. A convex aggregator ( $W_{yy} > 0$ ) is associated with preferences for an early resolution of uncertainty and a greater risk aversion for lotteries resolving in the distant future. Conversely, a concave aggregator ( $W_{yy} < 0$ ) generates preferences for a late resolution of uncertainty and a lower risk aversion for lotteries resolving late.

Uzawa preferences assume a linear aggregator and therefore exhibit no preferences for the timing. Risk-sensitive preferences allow for preferences for the timing. Indeed, define  $V(c, m) = -\frac{e^{-kU(c, m)}}{k}$ . The utility function  $V$  represents the same preferences as  $U$  and fulfills the recursion (9) with the aggregator  $W(x, y) = -\frac{1}{k} e^{-k(1-\beta)u(x)}(-ky)^\beta$ . We obtain  $\frac{W_{yy}(x, y)}{W_y(x, y)} = k(1 - \beta)(-ky)^{-1}$ , which has the same sign as  $k$ . Agents, who are more risk averse than in the standard additive model (whenever  $k > 0$ ) have preferences for an early resolution of uncertainty, while the reverse holds when  $k < 0$ . When  $k = 0$ , agents are indifferent to the timing of uncertainty resolution. To gain a better insight, one may also notice that if preferences were defined on a smaller domain, so that preferences with zero or negative time preferences (i.e.,  $\beta \geq 1$ ) could be considered, we would obtain the opposite relation between risk aversion and preference for the timing.<sup>22</sup> Time preference, risk aversion and preference

<sup>21</sup>Bommier (2008) considers such a possibility in the expected utility framework (i.e., with functions  $\phi_t$  independent of  $t$ ) to study life-cycle behavior.

<sup>22</sup>The case where  $\beta \geq 1$  can for example be considered when assuming that all consumption paths converge to an exogenous  $c^*$  within a finite amount of time. Risk-sensitive preferences can still be defined by the

for the timing appear therefore to be intertwined.

The above interrelation can be interpreted as an intuitive consequence of the assumptions of stationarity and FSD-monotonicity. Let us consider an agent comparing temporal lotteries that provide the same consumption  $c$  during  $N$  periods, but may differ afterwards. On the one hand, with stationary preferences, the  $N$  periods of constant consumption  $c$  do not matter and the ranking is independent of  $c$  and  $N$ . On the other hand, FSD-monotonicity implies that risk aversion is considered with respect to lifetime utility, including the utility derived from the first  $N$  periods of life. For the ranking to be independent of  $c$ , preferences must exhibit a constant absolute risk aversion with respect to lifetime utility, so that the utility of the first  $N$  periods does not impact the evaluation of what may be consumed afterwards. This explains the exponential functional form of risk-sensitive preferences, which incidentally makes them extremely tractable in dynamic problems.

Moreover, the larger the  $N$ , the smaller the utility risk the agent is facing, because of the discount factor  $\beta < 1$  (the reverse would hold if  $\beta > 1$ ). This generates a kind of non-stationarity of preferences, unless an “amplification” mechanism of risk attitudes regarding “utility risk” in the future is introduced. In consequence, an agent, who is risk averse with respect to lifetime utility ( $k > 0$ ) and who has positive time preferences ( $\beta < 1$ ) must exhibit greater risk aversion for lotteries resolving in the distant future in order to preserve preference stationarity. This greater risk aversion should precisely compensate the discount of future risks due to the time preference parameter  $\beta$ . Similarly an agent, who is risk prone with respect to lifetime utility ( $k < 0$ ) and has positive time preferences ( $\beta < 1$ ) has to exhibit more risk loving (and thus smaller risk aversion) for lotteries resolving in the future. Symmetric arguments would hold in the case of negative time preferences ( $\beta > 1$ ). In the case of zero time preferences ( $\beta = 1$ ), agents would be indifferent to the timing of uncertainty resolution.<sup>23</sup> Preferences for the timing (or, equivalently, a degree or risk aversion that depends on time distance) is a necessary ingredient to compensate for the existence of time preferences, as soon as temporal risk aversion is introduced and preferences are stationary. This compensation mechanism is explicit in recursion (17), in which certainty equivalents at date  $\tau$  are computed with  $\phi_\tau(x) = -\frac{\beta^{\tau+1}}{k}e^{-\frac{kx}{\beta^{\tau+1}}}$  exhibiting a coefficient of absolute risk aversion equal to  $\frac{k}{\beta^{\tau+1}}$ . As time goes by, this coefficient increases at a rate  $\frac{1}{\beta}$ , which exactly offsets the decrease of utility due to time preferences.

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recursion  $V(c, m) = u(c_t) - \frac{\beta}{k} \log(E_m [e^{-kV}])$ , where  $u$  is such that  $u(c^*) = 0$ , avoiding any problem of convergence.

<sup>23</sup>The case  $\beta = 1$  (zero time preferences) precisely corresponds to the multiplicative model of Bommier (2013), which fits into the expected utility framework and exhibits no preference for the timing.

### 4.3.3 Weak separability and utility independence

Risk-sensitive preferences fulfill an assumption of (mutual) utility independence similar to the one discussed by Keeney and Raiffa (1993), that may significantly help for the resolution of a number of economic problems. Preferences regarding what may happen at dates  $t_0$  and  $t_1$ , conditional on having consumption at another date  $t$  being equal to a given level  $c_t$ , are independent of  $c_t$ . The property also holds when  $c_t$  is random and independently distributed. In a dynamic setting, preference stationarity implies independence of past history. The utility independence property in addition requires that preferences have to be independent of the (exogenous) future.

Utility independence is not a direct consequence of stationarity and FSD-monotonicity. Indeed, when the function  $b$  is non-constant, Uzawa preferences associated with the aggregator (14) are both stationary and FSD-monotone but not utility independent. In fact, when considering monotone KP-recursive preferences, utility independence appears as a requirement for preferences to disentangle ordinal and risk preferences, as will be shown in Proposition 2. This comes from the –so far unnoticed– fact that, in the expected utility framework, history independence together with non-trivial risk aversion comparability imposes independence with respect to the future, and therefore mutual utility independence in the sense of Keeney and Raiffa (1993).

Mutual utility independence has, in turn, significant implications. Indeed, as has been known since Koopmans (1960), the combination of stationarity and independence with respect to the future (called “period independence” in Koopmans’ article) implies weak separability of preferences and constant time discounting. This explains why discussing risk aversion eventually requires us to consider preferences over deterministic consumption paths that can be represented by an additive utility function with a constant time discounting.<sup>24</sup> The property of weak separability, which is often introduced as a technical assumption or justified by the long tradition of research that makes use of it –as in Epstein and Zin (1989), Chew and Epstein (1990) or Klibanoff, Marinacci, and Mukerji (2009)– is in fact a necessary condition to study risk aversion while assuming preference stationarity and FSD-monotonicity.

Utility independence is a very powerful property that simplifies the analysis from both a theoretical and a numerical point of view in many intertemporal problems. Consider for example independently distributed (but not necessarily identically distributed) periodic consumptions  $(\tilde{c}_t)_{t \geq 1}$ , where uncertainty regarding date  $t$  consumption is revealed at date  $t$ .

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<sup>24</sup>The restriction of risk-sensitive preferences to  $C^\infty$  is represented by the utility function  $U_\infty(c_0, c_1, \dots) = (1 - \beta) \sum_{t=0}^{\infty} \beta^t u(c_t)$ .

Setting  $V_t = e^{-kU_t}$ , we have:

$$\begin{aligned} V_t &= e^{-k(1-\beta)u(c_t)} (E_t[V_{t+1}])^\beta = e^{-k(1-\beta)u(c_t)} \left( E_t \left[ e^{-k(1-\beta)u(\tilde{c}_{t+1})} E_{t+1}[V_{t+2}]^\beta \right] \right)^\beta, \\ &= e^{-k(1-\beta)u(c_t)} E_t[e^{-k(1-\beta)u(\tilde{c}_{t+1})}]^\beta E_t[V_{t+2}]^{\beta^2}, \end{aligned}$$

where the independence property has been used for getting the last equality. By induction, we have  $V_0 = e^{-k(1-\beta)u(c_0)} \prod_{i=1}^{\infty} (E_0[e^{-k(1-\beta)u(\tilde{c}_i)}])^{\beta^i}$  and thus:

$$U_0 = (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(\hat{c}_i), \quad (24)$$

where  $\hat{c}_0 = c_0$  and for any  $i \geq 1$ ,  $\hat{c}_i = \psi^{-1}(E_0[\psi(\tilde{c}_i)])$  is the certainty equivalent of random consumption  $\tilde{c}_i$  computed with the utility index:

$$x \mapsto \psi(x) = -\frac{1}{k(1-\beta)} e^{-k(1-\beta)u(x)}. \quad (25)$$

Thus, when per-period consumptions are independently distributed, we fall back to the original proposition of Selden (1978) who suggested replacing random per-period consumptions with their certainty equivalents. However, the certainty equivalent is computed with a particular utility index (25), whose degree of risk aversion increases with  $k$ . The additive structure of equation (24) guarantees that most results holding in a two-period framework will extend to an infinite horizon setting when per-period consumptions are independently (but not necessarily identically) distributed. We provide an illustration in Proposition 5.

## 5 Studying risk aversion with monotone recursive preferences

Departure from the standard additively separable expected utility model was mainly justified to obtain a framework flexible enough to study risk aversion. Kihlstrom and Mirman (1974) have shown that it requires a setup in which risk aversion can be varied without impacting ordinal preferences. In this section, we investigate whether preferences obtained in Proposition 1 achieve such a separation between risk and ordinal preferences, and to what extent they are well-ordered in terms of risk aversion.

### 5.1 Disentangling ordinal and risk preferences

The following proposition states our result about the separation between ordinal and risk preferences.

**Proposition 2 (Comparability of preferences)** *Consider two KP-recursive preference relations  $\succeq^A$  and  $\succeq^B$  on  $D$ , which are FSD-monotone and whose restrictions to  $C^\infty$  are identical. Then:*

- *either both preference relations are identical:  $\succeq^A = \succeq^B$ ,*
- *or the preference relations  $\succeq^A$  and  $\succeq^B$  can be represented by utility functions  $U^A$  and  $U^B$  fulfilling the following recursion (for  $i = A, B$ ):*

$$\forall (c, m) \in D, U^i(c, m) = \begin{cases} (1 - \beta)u(c) - \frac{\beta}{k_i} \log \left( E_m[e^{-k_i U^i}] \right), & k_i \neq 0, \\ (1 - \beta)u(c) + \beta E_m[U^i], & k_i = 0, \end{cases}$$

*where  $0 < \beta < 1$ ,  $k_A, k_B$  in  $\mathbb{R}$  and  $u : C \rightarrow \mathbb{R}$  is a continuously differentiable function with a strictly positive derivative.*

**Proof:** See Appendix.

This proposition shows, that among KP-recursive preferences, risk-sensitive preferences are the only ones that make it possible to disentangle risk and ordinal preferences. By contrast, preferences à la Uzawa, though stationary and FSD-monotone, are inappropriate to study the role of risk aversion. Indeed, with Uzawa preferences, risk aversion cannot be varied without changing ordinal preferences.

## 5.2 Risk-sensitive preferences and risk aversion

We now explore whether risk-sensitive preferences exhibit risk aversion, and can be ordered in terms of risk aversion. Following the literature initiated by Pratt (1964), Rothschild and Stiglitz (1970) and Diamond and Stiglitz (1974), we explore in turns the notions of weak and strong risk aversion. Weak risk aversion refers to the willingness to fully eliminate all risks, while strong risk aversion refers to the willingness to marginally decrease risk.

### 5.2.1 Weak risk aversion

A first way to look at risk aversion involves looking at how agents compare risky lotteries with deterministic prospects.

**Proposition 3 (Weak risk aversion)** *Consider two KP-recursive preference relations represented by utility functions  $U^A$  and  $U^B$  fulfilling the recursion (for  $i = A, B$ ):*

$$\forall (c, m) \in D : U^i(c, m) = \begin{cases} (1 - \beta)u(c) - \frac{\beta}{k_i} \log \left( E_m[e^{-k_i U^i}] \right), & k_i \neq 0, \\ (1 - \beta)u(c) + \beta E_m[U^i], & k_i = 0, \end{cases} \quad (26)$$

with  $k_A, k_B$  in  $\mathbb{R}$ ,  $0 < \beta < 1$ , and  $u : C \rightarrow [0, 1]$  a strictly increasing continuously differentiable concave function. Then:

1. (weak risk aversion) If  $k_A \geq 0$ , then for all  $(c, m) \in D$ , we have  $U^A(E[(c, m)]) \geq U^A(c, m)$ .<sup>25</sup>
2. (weak comparative risk aversion) If  $k_A \geq k_B$ , then for all  $(c, m) \in D$  and  $(c', m') \in C^\infty$ , we have:

$$U^A(c, m) \geq U^A(c', m') \Rightarrow U^B(c, m) \geq U^B(c', m').$$

This proposition first states that risk preferences with a non-negative constant  $k$  and a concave function  $u$  exhibit weak risk aversion, in the sense that the expectation of a temporal lottery is always viewed as at least as good as the temporal lottery itself. Therefore, we will say that an agent endowed with a positive  $k$  and a concave  $u$  is weakly risk averse. The second point indicates that if two agents with risk-sensitive preferences differ only with respect to the constant  $k$  that enters in the recursion (26), the one with a greater  $k$  will always associate a lower certainty equivalent (lower in terms of lifetime utility) than the one with a smaller  $k$ . In other words, the willingness to pay for eliminating all risks increases with  $k$ . The parameter  $k$  therefore allows us to compare agents in terms of weak risk aversion. Since Pratt (1964) and the notion of risk premium, this comparison of certainty equivalents is often used to compare agents' risk aversion. For example, Epstein and Zin (1989, pp. 949-950) rely on such an approach to conclude that decreasing the parameter  $\alpha$  in (12) involves increasing risk aversion.

### 5.2.2 Second-order stochastic dominance and strong risk aversion

Weak risk aversion and weak comparative risk aversion, as presented above, relate to comparisons of lotteries with deterministic outcomes. These notions are however not fully informative about the willingness of agents to marginally reduce risk. For example, in atemporal settings, it is well known that weakly risk averse agents may fail to prefer risk reductions in the sense of "second-order stochastic dominance" (SSD, hereafter).<sup>26</sup> This calls therefore for notions of strong risk aversion and strong comparative risk aversion, that reflect agent's preferences for marginal risk reduction.

**Second-order stochastic dominance.** We use insights from the closure approach introduced in Section 3.2 to explain how SSD can be defined when considering temporal lotteries.

<sup>25</sup>As already explained, a temporal lottery  $(c_0, m) \in D$  can also be represented as the sequence of uncertain future consumptions  $(c_0, \tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_t, \dots)$ . The expectation of the temporal lottery is then  $(c_0, E[\tilde{c}_1], E[\tilde{c}_2], \dots, E[\tilde{c}_t], \dots)$ , which is an element of  $C^\infty$ .

<sup>26</sup>See Cohen (1995) for an excellent survey on this topic and related issues.

We first state a result, similar to that of Lemma 2, related to the notion of SSD in atemporal settings.

**Lemma 5 (SSD in atemporal settings)** *Consider a compact metric ordered space  $(X, \geq)$ . Denote by  $M(X)$  the set of probability measures over  $X$ , on which the binary relation of SSD (denoted  $SSD^X$ ) is –as usual– defined as follows. For  $(m, m') \in M(X)^2$ :*

$$mSSD^X m' \Leftrightarrow \forall z \in X, \int_{y \leq z} m(\{x \in X | x \geq y\}) dy \leq \int_{y \leq z} m'(\{x \in X | x \geq y\}) dy. \quad (27)$$

Let  $R_a^X$  be the binary relation on  $X$  defined by  $mR_a^X m'$  if and only if  $m$  is a (degenerate) lottery paying off the average outcome of  $m'$  with probability one.

Then, the relation  $SSD^X$  is the smallest RTCMS extension of  $(\geq \cup R_a^X)$ .

**Proof.** The proof is similar to that of Lemma 2, except that extending  $(\geq \cup R_a^X)$  rather than  $\geq$ , constrains the set  $\mathcal{U}_R$  to only include increasing and concave functions. We can then use the result from Fishburn (1974) showing that SSD is obtained for  $\mathcal{U}_R$  being the class of continuous, increasing and concave utility functions. ■

In the case of temporal lotteries, characterization (27) is not relevant because of the lack of natural order on  $D$ . A similar difficulty was encountered for defining FSD in Section 3.2. However, here again, we can use the closure approach to bypass it:

**Definition 6 (SSD for temporal lotteries)** *Consider  $R_a$  the binary relation on  $D$  defined by  $(c, m)R_a(c', m') \Leftrightarrow (c, m) = E[(c', m')]$  and  $\succeq_0$  a preference relation on  $C^\infty$ .*

*We define SSD, the relation of second-order stochastic dominance on  $D$ , as the smallest RTCMS extension of  $(\succeq_0 \cup R_a)$ .*

This definition of SSD exactly replicates the closure approach suggested by Lemma 5. Moreover, since  $\succeq_0 \subset (\succeq_0 \cup R_a)$ , we deduce by taking the RTCMS extension that  $FSD \subset SSD$  just like in atemporal settings.

**Strong risk aversion.** Monotonicity with respect to SSD is, as usual, called “strong risk aversion”: a strongly risk averse agent should prefer  $(c, m)$  to  $(c', m')$  if  $(c, m)SSD(c', m')$ .

It is insightful to write the difference between weak and strong notions of risk aversion in terms of inclusions of binary relations. In the first point of Proposition 3, weak risk aversion was formalized as  $R_a \subset \succeq$ , or equivalently  $(R_a \cup \succeq_0) \subset \succeq$  since by construction  $\succeq_0 \subset \succeq$ . Strong risk aversion stipulates that not only  $(R_a \cup \succeq_0)$ , but also its smallest RTCMS extension, should be included in  $\succeq$ . Thus, loosely speaking, strong risk aversion involves formulating a “mixture-stable” statement of weak risk aversion.

Similarly, we can reformulate the second point of Proposition 3 on weak comparative risk aversion in terms of inclusions of binary relations, so as to derive a natural strong extension. Precisely, we define the binary relation  $R_d$  on  $D$  by

$$(c, m)R_d(c', m') \Leftrightarrow (c', m') \in C^\infty, \quad (28)$$

which can be interpreted as  $(c, m)$  is at least as risky as  $(c', m')$ , since the latter is risk-free. The second point of Proposition 3 writes then as: if  $k_A \geq k_B$ , then  $(R_d \cap \succeq^A) \subset (R_d \cap \succeq^B)$ . This approach to weak comparative risk aversion parallels the one of Yaari (1969) based on inclusions of acceptance sets. The idea is that if a given agent prefers a risky lottery to a deterministic one, so should it be for any less risk averse agent. However, since  $R_d$  is not mixture-stable, the binary relations  $(R_d \cap \succeq^A)$  and  $(R_d \cap \succeq^B)$  are not mixture-stable either. In order to define strong comparative risk aversion, we introduce a mixture-stable notion of acceptable increases in risk. Formally:

**Definition 7 (Acceptable increase in risk)** Consider a preference relation  $\succeq$  on  $D$ . We define the relation of acceptable increase in risk  $R_d^{\succeq}$  as the smallest RTCMS extension of  $(R_d \cap \succeq)$ .

We will say that agent  $A$  is more strongly risk averse than agent  $B$  if  $R_d^{\succeq^A} \subset R_d^{\succeq^B}$ . This way of defining strong comparative risk aversion parallels the definition proposed by Diamond and Stiglitz (1974) in atemporal settings.<sup>27</sup> We can now summarize the behavior of risk-sensitive preferences with respect to strong risk aversion:

**Proposition 4 (Strong risk aversion)** Consider two KP-recursive preference relations  $\succeq^A$  and  $\succeq^B$  represented by utility functions  $U^A$  and  $U^B$  as in Proposition 3. Assume that  $\succeq^A$  and  $\succeq^B$  have the same restriction to  $C^\infty$ , denoted  $\succeq_0$ . Then:

1. (strong risk aversion) If  $k_A \geq 0$ , then  $(c, m)SSD(c', m') \Rightarrow (c, m) \succeq^A(c', m')$ .
2. (strong comparative risk aversion) If  $k_A \geq k_B$ , then  $R_d^{\succeq^A} \subset R_d^{\succeq^B}$ .

**Proof.** If  $k_A \geq 0$  then  $R_a \subset \succeq^A$  (from Proposition 3) and  $\succeq_0 \subset \succeq^A$  (by construction of  $\succeq_0$ ). Since  $\succeq^A$  is RTCMS, it implies that  $SSD \subset \succeq^A$ , which proves the first point.

If  $k_A \geq k_B$ , we have  $(R_d \cap \succeq^A) \subset (R_d \cap \succeq^B)$  (from Proposition 3). Considering the RTCMS extension of this inclusion readily implies that  $R_d^{\succeq^A} \subset R_d^{\succeq^B}$ , which proves the second point. ■

<sup>27</sup>In Diamond and Stiglitz (1974), expected utility is assumed and acceptable increases in risk are introduced as “mean utility preserving increases in risk”. Then,  $A$  is said to be more risk averse than  $B$  if a mean utility preserving increase in risk for  $A$  is utility increasing for  $B$ .

Proposition 4 makes it clear that with risk-sensitive preferences “weak” notions systematically extend to “strong” notions, just like in the (atemporal) expected utility framework.<sup>28</sup> This is generally false in non mixture-stable frameworks. For example, Epstein-Zin preferences exhibit weak aversion, but not strong risk aversion (unless they are risk-sensitive preferences).<sup>29</sup>

In many concrete and practical problems, full risk elimination is either impossible, or simply not optimal, while marginal risk reduction is more likely to be available. To address such problems, it is then important to use a framework that is well-behaved in terms of strong risk aversion, like risk-sensitive preferences. The applications provided below will show some results that can be obtained when using risk-sensitive preferences.

## 6 Applications

We study the role of risk aversion in three applications: demand for insurance, precautionary savings in infinite horizon and asset pricing. We will see that in these three cases, using risk-sensitive preferences makes it possible to derive intuitive results. All these results could be shown to fail with non FSD-monotone preferences.

### 6.1 The demand for insurance

Our first result bears on the demand for insurance, in a setting where agents face independent future income risks that can be insured at date 0.

**Proposition 5 (Risk aversion and the demand for insurance)** *Consider a class of agents with risk-sensitive preferences, with a concave function  $u$  and differing only by the risk aversion parameter  $k \geq 0$ . Agents are endowed with an independently distributed income process  $(\tilde{y}_t)_{t \geq 0} \in Y^\infty$  where  $Y$  is a compact interval. Agents can purchase at date 0 quantities  $(q_t)_{t \geq 1}$  of insurance assets  $(\tilde{\alpha}_t)_{t \geq 1}$ . For each  $t \geq 1$ , the insurance asset  $\tilde{\alpha}_t$  has a price  $p_t > 0$  at date 0 and pays off  $\tilde{\alpha}_t = E[\tilde{y}_t] - \tilde{y}_t$  at date  $t$ . We denote  $(q_t^k)_{t \geq 1}$  the optimal insurance demand of agent  $k$ , which is the solution of:*

$$(q_t^k)_{t \geq 1} = \arg \max_{(q_t)_{t \geq 1}} U^k(y_0 - \sum_{i=1}^{\infty} p_i q_i, \tilde{y}_1 + q_1 \tilde{\alpha}_1, \dots, \tilde{y}_t + q_t \tilde{\alpha}_t, \dots),$$

where  $U^k$  denotes the utility of the agent with risk aversion  $k$ .

<sup>28</sup>In the atemporal expected utility framework, Rothschild and Stiglitz (1971) demonstrated the equivalence between weak and strong risk aversion, and Diamond and Stiglitz (1974) the equivalence between weak comparative risk aversion and strong comparative risk aversion.

<sup>29</sup>Actually, we know from Proposition 1 (or from Lemma 4) that Epstein-Zin preferences are not monotonic with respect to FSD. Thus, they cannot be monotonic with respect to SSD.

Then, for all  $t \geq 1$ , we have  $\frac{\partial q_t^k}{\partial k} \geq 0$ .

**Proof.** The formal proof is in the Appendix. It relies on equation (25), which implies that the marginal rate of substitution between consumption in period 0 and consumption in period  $t$  is independent of consumption in other periods. Due to this strong separability assumption, the problem is akin to a standard problem of insurance demand in a two-period setting. Moreover, since preferences are well-ordered in terms of strong risk aversion, this two-period problem leads to intuitive predictions, stating that the demand for insurance increases with risk aversion. ■

Proposition 5 shows that with risk-sensitive preferences, the greater the risk aversion the stronger the insurance demand, similar to what was found in the expected utility framework by Briys and Schlesinger (1985) and Dionne and Eeckhoudt (1985).

## 6.2 Precautionary savings in infinite horizon

Since the early works of Leland (1968) and Sandmo (1970), the problem of savings when facing uncertain future income has received a lot of attention. In particular, Carroll (1997) and Carroll and Samwick (1998) emphasize that, quantitatively speaking, precaution could be one of the main motives for savings. On the theoretical side, the question of precautionary savings has been mostly investigated in two-period frameworks, as in Drèze and Modigliani (1972), Kimball (1990), Kimball and Weil (2009), or Bommier, Chassagnon and LeGrand (2012), among many others. The extension to many periods or to an infinite horizon has only been addressed in very few cases. Most analytical studies are based on standard additively separable life-cycle models where the role of risk aversion cannot be explored. The only two exceptions we are aware of are Van der Ploeg (1993) and Weil (1993), who consider very specific parametrized forms of uncertainty so as to derive closed-form solutions.<sup>30</sup> Apart from these studies, a couple of papers numerically address the problem, as for example in the recent work of Wang, Wang and Yang (2013).

In the particular cases considered by Van der Ploeg (1993) and Weil (1993), risk aversion was found to have a positive impact on precautionary savings. However, it is impossible to tell from these studies, whether this positive correlation reflects a fundamental link between risk aversion and prudence, or whether this is a consequence of the specific risks that are considered. The two-period analysis of Kimball and Weil (2009) shows that the relation between risk aversion and prudence is ambiguous in the Epstein-Zin framework. As was

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<sup>30</sup>Both studies assume autoregressive income processes. Weil uses particular recursive Kreps-Porteus preferences, which are not FSD-monotone. Van der Ploeg investigates the case of a (non-stationary) multiplicative expected utility model, with instantaneous quadratic utility functions.

explained in Section 3.3, this is directly related to the fact that Epstein-Zin preferences are not FSD-monotone.

In this section, we show that a general result holds when studying risk aversion with KP-recursive FSD-monotone preferences. Rather than introducing specific parametrized forms of uncertainty, we assume that agents are endowed with a *stochastically monotone* income process  $(\tilde{y}_t)_{t \geq 0}$ , meaning that for all  $t \geq 0$  and  $x \in \mathbb{R}$  the function  $(y_0, y_1, \dots, y_t) \rightarrow \text{Prob}(\tilde{y}_{t+1} \geq x | y_0, y_1, \dots, y_t)$  is non-decreasing (i.e., non-decreasing in each of the  $y_\tau$ , for  $\tau \leq t$ ). The assumption of stochastic monotonicity is a general way to formulate that good news at dates 0 to  $t$  cannot convey bad information for the subsequent periods. Most income processes used in the literature comply with such an assumption.

As in Proposition 5, we consider agents with risk-sensitive preferences who only differ by the risk aversion parameter  $k$ . The function  $u$  is assumed to be concave and  $k$  always positive, as we restrict our attention to risk averse agents. Technically speaking, the assumption of a stochastically monotone income process does not rule out extremely rapid income growth or income decline, which could result in existence and convergence problems. Rather than introducing a set of technical assumptions, we simply assume that the income process and preference parameters (in particular  $\beta$  and  $u$ ) are such that convergence problems do not occur.

We consider the saving decision at time  $t$  of agents with wealth  $w_t$  and realized income trajectory that we denote  $y^t = (y_0, \dots, y_t)$ .<sup>31</sup> Let  $V_t^k(w_t, y^t)$  be the indirect utility at time  $t$  of the agent with risk aversion  $k$ . We have:

$$V_t^k(w_t, y^t) = \max_{s_t} u(c_t) - \frac{\beta}{k} \log E_t e^{-kV_{t+1}^k(w_{t+1}, y^{t+1})} \quad (29)$$

s.t.  $w_{t+1} = Rs_t$  and  $y_t + w_t - s_t = c_t \geq 0$

The optimum is reached at  $s_t^k(w_t, y^t)$ , which is the amount that the agent with risk aversion  $k$  chooses to save. We can now state the following result:

**Proposition 6 (Precautionary savings)** *For all  $t \geq 0$ , we have  $\frac{\partial s_t^k(w_t, y^t)}{\partial k} \geq 0$ .*

**Proof.** The formal proof is in the Appendix. Even in the presence of binding consumption positivity constraints, consumption, income and continuation utility can be shown to be comonotone at all dates: states of the world can be ranked from good to bad, with good states corresponding to high consumption levels. The impact on continuation utility of a marginal increase in  $s_t$  is thus larger if information at date  $t + 1$  reveals a bad state than if

<sup>31</sup>We indicate the whole income history, as the income process is not necessarily Markov. Expectations regarding future income may then depend on the whole income history.

it reveals a good state. Increasing savings therefore transfers welfare from good states to bad ones, achieving a risk reduction. A given agent saves then more than a less risk averse one. ■

Proposition 6 makes a clear statement about the relationship between risk aversion and prudence in dynamic frameworks. Greater risk aversion implies greater prudence. The key feature of our result is that it holds for any stochastically monotone income process. The result is thus much more general than those of Van der Ploeg (1993) and Weil (1993), who specify income process to be able to derive closed form solutions. To our knowledge, Proposition 6 provides the first general result on the determinants of precautionary savings in an infinite horizon model that was derived without assuming either a specific random income process or additive separability of preferences.

### 6.3 Asset pricing

The third application investigates the impact of risk aversion on the risk free rate and market price of risk. This is to some extent a dual problem to the one in Section 6.2. Instead of studying consumption-saving decisions with exogenous prices, we now analyze endogenous asset prices, while consumption is exogenous.

We consider an economy, with a single agent endowed with an exogenous random consumption. Here again, we do not assume a specific form of uncertainty but take a non-parametric approach. In line with Section 6.2, we assume that the consumption process  $(\tilde{c}_t)_{t \geq 0}$  is stochastically monotone. This embeds most standard consumption processes used in the asset pricing literature, like for example the trend stationary specification and the random walk dynamics considered in Tallarini (2000).

At date  $t$ , the agent with risk aversion  $k$  has a utility function  $U_t^k$  fulfilling:

$$U_t^k = (1 - \beta)u(c_t) - \frac{\beta}{k} \log \left( E_t[e^{-kU_{t+1}^k}] \right). \quad (30)$$

As noted in the previous section, stochastic monotonicity does not rule out extremely rapid consumption growth or consumption decline, which could result in existence and convergence problems. Rather than introducing a set of technical assumptions on the preference parameters or on the consumption process, we simply assume that the recursion (30) always has a unique solution fulfilling all required properties of differentiability and integrability for the computations shown in the proof of Proposition 7.

The intertemporal marginal rate of substitution  $M_{t,t+1}$  between dates  $t$  and  $t + 1$ , also

known as the pricing kernel, can be expressed as follows:

$$M_{t,t+1} = \frac{\beta}{1-\beta} \frac{u'(\tilde{c}_{t+1})}{u'(c_t)} \frac{e^{-kU_{t+1}^k}}{E_t \left[ e^{-kU_{t+1}^k} \right]}.$$

The one period risk-free rate  $R_t$  (between dates  $t$  and  $t+1$ ) can be deduced from  $M_{t,t+1}$ :

$$\frac{1}{R_t} = E_t [M_{t,t+1}], \quad (31)$$

as well as the market price of risk  $\sigma_t$ :

$$\sigma_t = \frac{V_t [M_{t,t+1}]^{1/2}}{E_t [M_{t,t+1}]}, \quad (32)$$

where  $V_t [M_{t,t+1}]$  denotes the variance of the marginal rate of substitution  $M_{t,t+1}$  conditional on the information available at date  $t$ .

The next proposition summarizes our findings regarding the impact of a change in the risk aversion parameter  $k$  on the risk free rate and the market price of risk.

**Proposition 7 (Asset prices and risk aversion)** *In the set-up described above, an increase in the risk aversion parameter  $k$  of the agent implies:*

- a smaller risk free rate;
- a greater market price of risk.

**Proof.** The proof is given in Appendix. ■

Risk aversion in the risk-sensitive framework has an intuitive effect on the risk free rate and the market price of risk. A more risk averse agent is willing to pay more to transfer resources from a certain state of the world (today) to an uncertain one (tomorrow), which raises the price of riskless savings and thus reduces the riskless interest rate. By the same token, a more risk averse agent requires a larger discount to hold a risky asset, which increases the market price of risk.

The results of Proposition 7 also have interesting consequences when discussing policy issues. For example, the on-going debate as to the cost of climate change is strongly influenced by the choice of the appropriate discount rate, and on how the risk and the planner's risk aversion affect this rate. Our result clearly states that the more risk averse the planner, the lower the discount rate that should be used for policy evaluation.

## 7 Concluding remarks

The primary objective of this paper was twofold: (i) to show which recursive specifications would rule out the choice of dominated strategies; (ii) to emphasize the possible gains for economists to opt for such specifications. The answer to the first point is clear-cut. Within the Kreps-Porteus framework, Uzawa preferences and risk-sensitive preferences introduced by Hansen and Sargent are the only ones to rule out preferences for dominated strategies. The results extend to smooth recursive ambiguity models (see Appendix A) where –in addition to the Uzawa model– the only well-behaved specifications are provided by the “robust model with hidden states” of Hansen (2007) and Hansen and Sargent (2007a). As for the second point, we have shown that imposing FSD-monotonicity leads to a much better understanding of the role of risk aversion. In particular, we have established the existence of a fundamental link between risk aversion and prudence, with clear consequences on precautionary savings, asset pricing, or on the choice of an appropriate discount rate in cost-benefit evaluation.

Fulfilling our primary objective has led us to derive side results which are interesting in themselves. We conclude our paper by mentioning two of them. First, we have shown that FSD and SSD can be characterized as the smallest binary relations fulfilling some intuitive properties, including mixture-stability. This proves to be a very efficient approach for defining FSD and SSD on complex sets, such as the one of temporal lotteries, and to establish that some preferences are monotone with respect to FSD and SSD. Such an approach could be used in other settings or to define higher order stochastic orders.

Second, we have obtained that weak separability, on the one hand, and allowing for preferences for the timing, on the other hand, are necessary requirements to study the role of risk aversion in recursive settings. Weak separability is needed to combine past independence with the separation of ordinal and risk preferences. Preferences for the timing have to be introduced to compensate for time preferences. These findings complete the initial results of Koopmans (1960, 1965) by showing some key features that appear when applying the assumption of preference stationarity in an infinite horizon setting. In line with the comments of Fisher (in Koopmans, 1965),<sup>32</sup> these results can be seen as providing arguments to abandon the infinite horizon assumption, to include terminal constraints as in Ramsey (1928) or Bommier (2013), or to weaken the notion of preference recursivity as was discussed in Section 4.3.1. The alternative is to accept risk aversion, time preferences and preferences for the timing to be intertwined.

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<sup>32</sup>According to Fisher, “The obvious conclusion from Koopmans’ paper, therefore, seems to me to be that one ought to abandon the use of infinite horizons – not that one ought to abandon certain ethical notions.”

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# Appendix

## A On monotone recursive models of ambiguity

In this section, we extend our main representation result (Proposition 1) to choice under ambiguity. To do so, we investigate the class of smooth recursive ambiguity models axiomatized by Hayashi and Miao (2011) and studied further by Ju and Miao (2012). We consider compound lottery acts, that can be viewed as a function from the set of states of the world into the set of lotteries whose outcomes are the combination of a per-period consumption and a compound lottery act. Denoting by  $\mathcal{G}$  the set of compound lottery acts and by  $\Omega$  the set of the states of the world we have:

$$\mathcal{G} = (M(C \times \mathcal{G}))^\Omega,$$

where  $M(C \times \mathcal{G})$  denotes the set of probability measures on  $C \times \mathcal{G}$ . Hayashi and Miao (2011) provide an axiomatic construction of preferences, where the agent evaluates elements of  $\mathcal{C} \times \mathcal{G}$  as follows: (i) the agent relies on a subjective probability measure over the set of states of the world, which generates a correspondence between compound lottery acts and compound lotteries; (ii) using this probability measure, the agent evaluates the compound lottery through a sequence of expected utility evaluations. Formally, the representation result of Hayashi and Miao stipulates that there exist increasing functions  $\psi_1$  and  $\psi_2$  and an aggregator  $\hat{W}$  increasing in its second argument such that the utility of an element  $(c, g) \in C \times \mathcal{G}$  can be expressed as follows:

$$V(c, g) = \hat{W} \left( c, \psi_1^{-1} \left( \int_{M(\Omega)} \psi_1 \left( \psi_2^{-1} \left( \sum_{\omega \in \Omega} [\pi](\omega) \int_{C \times \mathcal{G}} \psi_2(V) dg(\omega) \right) \right) d\pi([\pi]) \right) \right),$$

where  $\pi \in M(M(\Omega))$  is a probability measure over the set of objective distributions, and  $[\pi] \in M(\Omega)$  is one of the possible “realizations” of  $\pi$ . Following Ju and Miao (2012), we rewrite the above expression in a more synthetic way:

$$V(c, g) = \hat{W} \left( c, \psi_1^{-1} \left( E_\pi \left[ \psi_1 \left( \psi_2^{-1} \left( E_{[\pi]}[\psi_2(V)] \right) \right] \right) \right) \right). \quad (33)$$

In the recursion (33), the notation  $E_\pi[\cdot]$  refers to the expectation operator when considering the distribution of objective probability measures and the notation  $E_{[\pi]}[\cdot]$  refers to the expectation operator relating to the probability measure  $[\pi]$ .

Since  $\psi_2$  is an increasing function, the utility function  $U = \psi_2(V)$  represents the same

preferences as  $V$ . Moreover,  $U$  fulfills the following recursion:

$$U(c, g) = W(c, \phi^{-1} E_{\pi}[\phi(E_{[\pi]}[U])]), \quad (34)$$

where  $\phi(\cdot) = \psi_1(\psi_2^{-1}(\cdot))$  is an increasing function and  $W(x, y) = \psi_2(\hat{W}(x, \psi_2^{-1}(y)))$  is an aggregator, which is increasing in its second argument. Our discussion uses the representation (34) rather than the equivalent one (33) as it makes it easier to visualize the connections with our previous results.

In that setting, FSD can be defined as in Section 3.2, with minor technical differences to account for compound lotteries. We do not provide a formal definition of FSD as the extension of Definition 2 to compound lotteries is straightforward. Note that the notion of stochastic dominance is subjective because the beliefs are. The corresponding FSD-monotonicity (which could be defined as in Definition 4) becomes a consistency requirement between ordinal preferences, subjective beliefs and preferences over compound lottery acts.

The following proposition explains which specifications of the recursion (34) represent FSD-monotone preferences.

**Proposition 8 (Monotone ambiguity-averse preferences)** *Consider a recursive model of choice under ambiguity, where the preference relation  $\succeq$  defined over  $C \times \mathcal{G}$  can be represented by the recursion (34), where  $\phi$  and  $W$  are twice continuously differentiable with positive derivatives. We assume that subjective beliefs can take any value in  $M(M(\Omega))$ . Then, the following statements are equivalent:*

1. *The preference relation  $\succeq$  is FSD-monotone.*
2. *The preference relation  $\succeq$  can be represented by a utility function fulfilling one of the following recursions:*

(a) *(Subjective expected utility à la Uzawa)*

$$U(c, g) = a(c) + b(c)E_{\pi}E_{[\pi]}[U], \quad (35)$$

where  $a(\cdot)$  and  $b(\cdot)$  fulfill the same conditions as in Proposition 1.

(b) *(Robust model with hidden states)*

$$U(c, g) = (1 - \beta)u(c) - \frac{\beta}{k_2} \log \left( E_{\pi} \left[ e^{-k_2 \left( -\frac{1}{k_1} \log E_{[\pi]}[e^{-k_1 U}] \right)} \right] \right), \quad (36)$$

where  $0 < \beta < 1$ ,  $k_1$  and  $k_2$  in  $\mathbb{R}$  (the cases  $k_1 = 0$  or  $k_2 = 0$  being obtained by considering the limit in (36)) and  $u : C \rightarrow \mathbb{R}$  is a twice continuously differentiable

function with a strictly positive derivative.

**Proof.**

**Necessary part** ( $1 \Rightarrow 2$ ). We consider preferences whose utility representation  $U$  follows the recursion (34). First, we consider cases without ambiguity, where the subjective probability measure  $\pi$  is degenerate and equal for sure to some  $[\pi] \in M(\Omega)$ . For preferences represented by (34) to be FSD-monotone, KP-recursive preferences represented by  $U^A(c, g) = W(c, E_{[\pi]}[U^A])$  must also be FSD-monotone. Second, we consider the case where the support of  $\pi$  is included in the set of degenerated lotteries over  $\Omega$ . As in the first case, preferences represented by  $U^B(c, g) = W(c, \phi^{-1}E_{\pi}[\phi(U^B)])$  must also be FSD-monotone. The functions  $U^A$  and  $U^B$  represent the same preferences over the set of deterministic consumption paths and both represent FSD-monotone recursive preferences. Thus, from Proposition 2, the associated preferences are either (i) identical and Uzawa preferences, or (ii) risk-sensitive preferences with the same  $u$  and  $\beta$ .

**Sufficient part** ( $2 \Rightarrow 1$ ). Since aggregators are very similar to those in Proposition 1, a similar proof can be provided to show that preferences corresponding to these aggregators are FSD-monotone. The Uzawa case is straightforward. For the other one, defining  $V_t = (1 - \beta) \sum_{i=0}^{t-1} \beta^i u(c_i) + \beta^t U_t$ , we have:

$$V_{\infty} = (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(c_i),$$

$$V_t = \phi_t^{-1} (E_{\pi_t} [\phi_t (\psi_t^{-1} (E_{[\pi_t]} [\psi_t (V_{t+1})]))]),$$

where  $\psi_t(x) = -\frac{\beta^{t+1}}{k_1} e^{-\frac{k_1}{\beta^{t+1}}x}$  and  $\phi_t(x) = -\frac{\beta^{t+1}}{k_2} e^{-\frac{k_2}{\beta^{t+1}}x}$ . FSD-monotonicity then becomes straightforward, as a temporal lottery act is evaluated as a compound lottery on lifetime utilities. ■

As a preliminary comment, we emphasize that Proposition 8 should be carefully interpreted, since the statement involves considering all possible beliefs in  $M(M(\Omega))$ , while models of choice under ambiguity may restrict the set of beliefs to some subsets of  $M(M(\Omega))$ . As a consequence, there may exist preferences fulfilling the representation (34) and constraining the set of possible beliefs, which are not presented by any utility function of Proposition 8 but still FSD-monotone. Proposition 8 provides a set of sufficient (but not necessary) restrictions on the aggregator guaranteeing that no dominated strategy would be chosen. It provides moreover the weakest restriction that can be derived without any additional information on how subjective beliefs are formed.

The representations of Proposition 8 connect with well-known models of the literature. Specification (35) corresponds to the subjective (ambiguity neutral) version of Uzawa. If  $k_1 = k_2$  in equation (36), we obtain the subjective (ambiguity neutral) version of risk-sensitive preferences. The case  $k_1 = 0$  corresponds to the model of Klibanoff, Marinacci and Mukerji (2005), with a constant absolute ambiguity aversion. This specification, for which we provide a formal argument, is the one actually chosen in most applications, as for instance in Collard, Mukerji, Sheppard and Tallon (2011). The general case with different and non-zero  $k_1$  and  $k_2$  has been introduced by Hansen (2007) and Hansen and Sargent (2007a) in their work on robustness with hidden states. Our assumption of differentiability prevents us from considering the max-min recursive model initialized by Gilboa and Schmeidler (1989) and studied in Epstein and Wang (1994) and Epstein and Schneider (2003). However, this model can be obtained for  $k_1 \rightarrow \infty$  in (36) and is clearly FSD-monotone.

## B Technical Appendix

When explicitly specified, KP-recursive preferences will be represented by *normalized* utility functions. These are functions which, in addition to the requirements of Definition 5, fulfill  $U(\underline{c}, \underline{c}, \underline{c}, \dots) = 0$  and  $U(\bar{c}, \bar{c}, \bar{c}, \dots) = 1$ . There is no loss of generality in this normalization, as an affine transformation makes it possible to recover the general case.

Moreover we introduce a new notation. For any measure  $m$ , any strictly positive and bounded random variable  $w$  such that  $E_m[w] = 1$ , we define the weighted expectation of a random variable  $X$  with respect to weight  $w$  as follows (provided that the expectation exists):

$$\tilde{E}_m^w[X] = E_m[Xw]. \quad (37)$$

We also define (when they exist) the variance and the covariance with respect to  $w$ :

$$\tilde{V}_m^w[X] = E_m[(X)^2w] - E_m[Xw]^2, \quad (38)$$

$$\tilde{cov}_m^w(X, Y) = E_m[XYw] - E_m[Xw]E_m[Yw]. \quad (39)$$

### B.1 Proof of Lemma 1

We first state a general result, similar to that of Dubra, Maccheroni and Ok (2004), but which applies to non-reflexive and non-transitive binary relations.

**Lemma 6 (Representation of a continuous mixture-stable binary relation)** *We consider a compact metric space  $X$  and a continuous mixture-stable binary relation  $R$  on the set  $M(X)$  of Borel probability measures over  $X$ . Then, there exists a set  $\mathcal{U}$  of pairs of continuous*

functions over  $X$ , such that for any  $(m, m') \in M(X)^2$ ,

$$mRm' \Leftrightarrow E_m[u] \geq E_{m'}[v] \text{ for all } (u, v) \in \mathcal{U}. \quad (40)$$

**Proof.** We define:

$$\mathcal{U} = \{(u, v) \in \mathcal{C}(X)^2 : \int_X u dm \geq \int_X v dm' \text{ for all } (m, m') \in M(X)^2 \text{ such that } mRm'\},$$

where  $\mathcal{C}(X)$  is the set of continuous functions on  $X$ . The set  $M(X)$  is endowed with the weak (Prohorov) topology.

We wish to show that for any  $mRm' \Leftrightarrow E_mu \geq E_{m'}v$  for all  $(u, v) \in \mathcal{U}$ . One direction ( $\Rightarrow$ ) is straightforward. Let us now consider  $(m_1, m_2) \in M(X)^2$ , such that  $(m_1, m_2) \notin R$  (i.e.,  $m_1Rm_2$  does not hold) but such that  $\int_X u dm_1 \geq \int_X v dm_2$  for all  $(u, v) \in \mathcal{U}$ . It implies that  $\{(m_1, m_2)\} \cap R = \emptyset$ . Note that  $\mathcal{U}$  is a convex subset of  $\mathcal{C}(X)^2$  and  $R$  is a closed convex subset of  $M(X)^2$ , because  $R$  is continuous and mixture-stable. The Strong Separating Hyperplane Theorem (see proof of Theorem 3 in Vind, 2000) implies that there exists a continuous affine map  $T : M(X)^2 \rightarrow \mathbb{R}$  such that for all  $(m, m') \in R$ :

$$T(m_1, m_2) < 0 \leq T(m, m'). \quad (41)$$

To characterize  $T$ , we consider the product space of all finite signed measures on  $X$  that we denote  $ca(X)^2$ . The set  $ca(X)$  is endowed with the weak\*-topology (which induces the standard weak topology on  $M(X)$ ) and is the dual of  $\mathcal{C}(X)$ . Dubra, Maccheroni and Ok (2004) provides a detailed exposure of these topologies. The application  $T(\cdot, \cdot) - T(0, 0)$  is linear on  $ca(X)^2$ . We deduce that there exists  $(\hat{u}, \hat{v}) \in \mathcal{C}(X)^2$ , such that for any  $(m, m') \in ca(X)^2$ ,  $T(m, m') = \alpha \int_X \hat{u} dm + \beta \int_X \hat{v} dm' + \gamma$ . Restricting our attention to  $M(X)^2$  and defining  $\tilde{u} = \alpha \hat{u} + \gamma$  and  $\tilde{v} = -\beta \hat{v}$ , we have for all  $(m, m') \in M(X)^2$ ,  $T(m, m') = \int_X \tilde{u} dm - \int_X \tilde{v} dm'$ . Inequality (41) implies then that  $(\tilde{u}, \tilde{v}) \in \mathcal{U}$  and  $\int_X \tilde{u} dm_1 < \int_X \tilde{v} dm_2$ , which contradicts the definition of  $(m_1, m_2)$ . We deduce that for any  $(m, m') \in M(X)^2$  the equivalence (40) holds. ■

We now prove Lemma 1. Denote by  $\Omega$  the set of RTCMS extensions of  $\succeq_0$ . This set is non-empty as  $FSD \subset \Omega$  and has thus a minimal element  $R_0 = \bigcap_{R \in \Omega} R$ . We need to show that  $FSD \subset R_0$ . Since  $R_0$  is a closed subset of  $D^2$  and  $FSD$  is the closure of  $\bigcup_t FSD_t = \bigcup_{t, \tau \leq t} FSD_{\tau, t}$ , it is sufficient to prove by backward induction on  $\tau$  that for all  $\tau \leq t$  we have  $FSD_{\tau, t} \subset R_0$ . For  $\tau = t$ , we have  $FSD_{t, t} = \succeq_0 \subset R_0$ .

We now assume that for  $0 < \tau \leq t$ , we have  $FSD_{\tau, t} \subset R_0$ . Consider  $c, c' \in C^\tau$  as given and define the binary relation  $R$  on  $M(D_{t-\tau})$  by  $mRm' \Leftrightarrow (c, m)R_0(c', m')$ . Lemma 6 shows that there exists a set  $\mathcal{U}$  of pairs of continuous functions over  $D_{t-\tau}$ , such that equivalence (40) holds for any  $(m, m') \in M(D_{t-\tau})^2$ .

For all pairs of degenerate elements of  $M(D_{t-\tau})^2$  (that is for all  $(m_1, m_2) \in (D_{t-\tau})^2$ ), the induction hypothesis together with equivalence (40) imply that the class  $\mathcal{U}$  may only contain pairs of functions  $(u, v)$ , such that  $(c, m_1)FSD_{\tau,t}(c', m_2) \Rightarrow u(m_1) \geq v(m_2)$ .

Let us now consider  $(c, m)$  and  $(c', m')$  in  $D_{\tau-1,t} = C^\tau \times M(D_{t-\tau})$  that are such that  $(c, m)FSD_{\tau-1,t}(c', m')$ . Definition 1 implies that for all continuous functions  $\phi : C^\tau \times D_{t-\tau} \rightarrow \mathbb{R}$  such that  $(c, m_1)FSD_{\tau,t}(c', m_2) \Rightarrow \phi(c, m_1) \geq \phi(c', m_2)$ , we have  $E_m[\phi(c, \cdot)] \geq E_{m'}[\phi(c', \cdot)]$ . By setting  $u(\cdot) = \phi(c, \cdot)$  and  $v(\cdot) = \phi(c', \cdot)$ , we can obtain any pair  $(u, v) \in \mathcal{U}$  verifying both (i)  $(c, m_1)FSD_{\tau,t}(c', m_2) \Rightarrow u(m_1) \geq v(m_2)$  for  $(m_1, m_2) \in (D_{t-\tau})^2$  and (ii)  $E_m[u] \geq E_{m'}[v]$  for  $(m, m') \in M(D_{t-\tau})^2$ . Equivalence (40) implies then that  $mRm'$  or  $(c, m)R_0(c', m')$ . Thus  $FSD_{\tau-1,t} \subset R_0$ .

## B.2 Proof of Proposition 1

### B.2.1 Necessary conditions

We first prove that FSD-monotone KP-recursive preferences admit a linear or a risk-sensitive aggregator (i.e., that (14) or (15) hold).

**A preliminary lemma.** The lemma provides a first set of restrictions on aggregators.

**Lemma 7** *Consider KP-recursive preferences represented by a normalized utility function  $U$  fulfilling (9) with an aggregator  $W$ . If preferences are FSD-monotone, then the aggregator can be expressed as follows:*

$$\forall (x, y) \in C \times [0, 1], \quad W(x, y) = \phi(a(x) + yb(x)),$$

where  $a, b : C \rightarrow [0, 1]$ , and  $\phi : [0, 1] \rightarrow [0, 1]$  are continuously differentiable, with  $a(\underline{c}) = \phi(0) = 0$ ,  $a(\bar{c}) + b(\bar{c}) = \phi(1) = 1$ ,  $a' > 0$ ,  $(a' + b') > 0$  and  $\phi' > 0$ .

#### Proof.

We define preferences over  $C \times M([0, 1])$  by considering the utility function  $\tilde{U}$  given by:  $\forall (c, m) \in C \times M([0, 1])$ ,  $\tilde{U}(c, m) = W(c, E_m[x])$ , where  $E_m[x] \in [0, 1]$  denotes the expected payment associated with probability measure  $m$ . Due to the normalization conditions, when  $c$  varies in  $C$ ,  $U(c, c, \dots)$  covers  $[0, 1]$  and the utility function  $U$  –applied to constant consumption paths– generates an isomorphism from  $C$  into  $[0, 1]$ . As a consequence, if preferences over  $D$  fulfill FSD-monotonicity, preferences over  $C \times M([0, 1])$  represented by the utility function  $\tilde{U}$  also fulfill FSD-monotonicity.

**First step.** We consider  $x_0 \in (c, \bar{c})$  and  $y_0 \in (0, 1)$  (i.e.,  $(x_0, y_0)$  lies in the interior of the definition domain of  $W$ ). Since  $W$  is a continuously differentiable function with  $W_y > 0$ , the implicit function theorem states that there exist  $\tilde{B}_{x_0}$  and  $\tilde{B}_{y_0}$ , respective neighborhoods of  $x_0$  and  $y_0$ , and a continuously differentiable function  $\eta_{x_0, y_0}$  from  $\tilde{B}_{x_0}$  into  $\tilde{B}_{y_0}$  such that:

$\forall x \in \widetilde{B}_{x_0}$ ,  $W(x_0, y_0) = W(x, \eta_{x_0, y_0}(x))$ . Let  $y_1 \in \widetilde{B}_{y_0}$ . By the same token, there exists a continuously differentiable function  $\eta_{x_0, y_1}$  from  $\widehat{B}_{x_0}$  into  $\widehat{B}_{y_1}$  such that,  $\forall x \in \widehat{B}_{x_0}$ ,  $W(x_0, y_1) = W(x, \eta_{x_0, y_1}(x))$ . We define  $B_{x_0} = \widetilde{B}_{x_0} \cap \widehat{B}_{x_0}$  and  $B_{y_0, y_1} = \widetilde{B}_{y_0} \cap \widehat{B}_{y_1}$ , which are non-empty and open sets. For all  $x \in B_{x_0}$ , we have:

$$W(x_0, y_0) = W(x, \eta_{x_0, y_0}(x)) \text{ and } W(x_0, y_1) = W(x, \eta_{x_0, y_1}(x)). \quad (42)$$

With the assumption of FSD-monotonicity, (42) implies that for all  $p \in [0, 1]$ :

$$\forall x \in B_{x_0}, W(x, p\eta_{x_0, y_0}(x) + (1-p)\eta_{x_0, y_1}(x)) = W(x_0, py_0 + (1-p)y_1). \quad (43)$$

Derivation with respect to  $x$  of equations (42)–(43) yields:

$$\begin{aligned} W_x(x, \eta_{x_0, y_0}(x)) + W_y(x, \eta_{x_0, y_0}(x)) \frac{\partial \eta_{x_0, y_0}(x)}{\partial x} &= 0, \\ W_x(x, \eta_{x_0, y_1}(x)) + W_y(x, \eta_{x_0, y_1}(x)) \frac{\partial \eta_{x_0, y_1}(x)}{\partial x} &= 0, \\ W_x(x, p\eta_{x_0, y_0}(x) + (1-p)\eta_{x_0, y_1}(x)) \\ + W_y(x, p\eta_{x_0, y_0}(x) + (1-p)\eta_{x_0, y_1}(x)) \left( p \frac{\partial \eta_{x_0, y_0}(x)}{\partial x} + (1-p) \frac{\partial \eta_{x_0, y_1}(x)}{\partial x} \right) &= 0. \end{aligned}$$

By substituting the first two equalities in the last one we deduce that:

$$\forall x \in B_{x_0}, \frac{W_x(x, p\eta_{x_0, y_0}(x) + (1-p)\eta_{x_0, y_1}(x))}{W_y(x, p\eta_{x_0, y_0}(x) + (1-p)\eta_{x_0, y_1}(x))} = p \frac{W_x(x, \eta_{x_0, y_0}(x))}{W_y(x, \eta_{x_0, y_0}(x))} + (1-p) \frac{W_x(x, \eta_{x_0, y_1}(x))}{W_y(x, \eta_{x_0, y_1}(x))},$$

which implies that the restriction of  $\frac{W_x(x, y)}{W_y(x, y)}$  on  $B_{x_0} \times B_{y_0, y_1}$  is linear in  $y$ .

Thus, for any  $(x_0, y_0) \in (\underline{c}, \bar{c}) \times (0, 1)$ , there exists a neighborhood  $B_{x_0, y_0}$  and two functions  $\hat{a}_{x_0, y_0}$  and  $\hat{b}_{x_0, y_0}$  such that for all  $(x, y) \in B_{x_0, y_0}$  we have:

$$\frac{W_x(x, y)}{W_y(x, y)} = \hat{a}_{x_0, y_0}(x) + \hat{b}_{x_0, y_0}(x)y. \quad (44)$$

**Second step.** Let  $y_1 \in (0, 1)$ . For all  $x \in (\underline{c}, \bar{c})$ , we define  $\hat{a}(x)$  and  $\hat{b}(x)$  by:

$$((x, y_1) \in B_{x_0, y_0} \text{ for some } (x_0, y_0)) \Rightarrow (\hat{a}(x) = \hat{a}_{x_0, y_0}(x) \text{ and } \hat{b}(x) = \hat{b}_{x_0, y_0}(x)).$$

The functions  $\hat{a}$  and  $\hat{b}$  are well defined. Indeed, firstly, from (44), we know that for any  $x \in (\underline{c}, \bar{c})$ , there exists a pair  $(x_0, y_0)$  such that  $(x, y_1) \in B_{(x_0, y_0)}$ . Secondly, if for some  $x_1 \in (\underline{c}, \bar{c})$  there are two pairs  $(x_0, y_0)$  and  $(x'_0, y'_0)$ , such that  $(x_1, y_1) \in B_{(x_0, y_0)}$  and  $(x_1, y_1) \in B_{(x'_0, y'_0)}$  then for all  $(x, y) \in B_{(x_0, y_0)} \cap B_{(x'_0, y'_0)}$ , we have:

$$\frac{W_x(x, y)}{W_y(x, y)} = \hat{a}_{x_0, y_0}(x) + \hat{b}_{x_0, y_0}(x)y = \hat{a}_{x'_0, y'_0}(x) + \hat{b}_{x'_0, y'_0}(x)y,$$

which implies  $\hat{a}_{x_0, y_0}(x_1) = \hat{a}_{x'_0, y'_0}(x_1)$  and  $\hat{b}_{x_0, y_0}(x_1) = \hat{b}_{x'_0, y'_0}(x_1)$ .

The continuity of  $\frac{W_x}{W_y}$  implies that  $\hat{a}$  and  $\hat{b}$  are continuous functions. Let us consider the set  $\Gamma = \{(x, y) \in C \times [0, 1] \mid \frac{W_x(x, y)}{W_y(x, y)} = \hat{a}(x) + \hat{b}(x)y\}$ . This set is non-empty by construction and closed by continuity of  $(x, y) \mapsto \frac{W_x(x, y)}{W_y(x, y)} - \hat{a}(x) - \hat{b}(x)y$ .

Assume that there exists  $(x_0, y_0) \in (\underline{c}, \bar{c}) \times (0, 1)$  which does not belong to  $\Gamma$ . The set  $\Lambda = \{\lambda \in [0, 1] \mid (x_0, \lambda y_0 + (1 - \lambda)y_1) \in \Gamma\}$  is closed, and thus compact, contains 0 and does not contain 1. Let  $\lambda_1$  be the supremum of  $\Lambda$ . By compactness,  $\lambda_1 \in \Lambda$  and  $\lambda_1 < 1$ . We know from (44) that there exists  $\varepsilon > 0$ , such that for all  $\lambda$  with  $|\lambda - \lambda_1| < \varepsilon$ , we have:

$$\frac{W_x(x_0, \lambda y_0 + (1 - \lambda)y_1)}{W_y(x_0, \lambda y_0 + (1 - \lambda)y_1)} = \hat{a}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) + \hat{b}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) (\lambda y_0 + (1 - \lambda)y_1). \quad (45)$$

Moreover since  $\lambda_1 \in \Omega$ , we have for all  $\lambda \leq \lambda_1$ :

$$\frac{W_x(x_0, \lambda y_0 + (1 - \lambda)y_1)}{W_y(x_0, \lambda y_0 + (1 - \lambda)y_1)} = \hat{a}(x_0) + \hat{b}(x_0) (\lambda y_0 + (1 - \lambda)y_1). \quad (46)$$

We deduce that  $\hat{a}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) = \hat{a}(x_0)$  and  $\hat{b}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) = \hat{b}(x_0)$ . Using (45), equation (46) extends to some  $\lambda > \lambda_1$ , contradicting  $\lambda_1$  being the supremum of  $\Lambda$ . We conclude that  $\Gamma = C \times [0, 1]$ .

**Third step.** Given  $x_0$ , we define  $w_0 : y \mapsto w_0(y) = W(x_0, y)$ , which is increasing continuously differentiable on  $[0, 1]$ . From equation (44), we know that  $W$  solves:

$$\begin{aligned} \forall (x, y) \in C \times [0, 1], \quad W_x(x, y) &= (\hat{a}(x) + \hat{b}(x)y)W_y(x, y), \\ \forall y \in [0, 1], \quad W(x_0, y) &= w_0(y). \end{aligned}$$

The method of characteristics shows the existence and uniqueness of the solution given by:

$$\begin{aligned} W(x, y) &= w_0 \left( \tilde{a}(x) + \tilde{b}(x)y \right), \quad (47) \\ \text{with: } \tilde{a}(x) &= \int_{x_0}^x \left( \exp \left( \int_{x_0}^s \hat{b}(\tau) d\tau \right) \right) \hat{a}(s) ds \text{ and } \tilde{b}(x) = \exp \left( \int_{x_0}^x \hat{b}(s) ds \right). \end{aligned}$$

The functions  $\tilde{a}$  and  $\tilde{b}$  are continuously differentiable on  $C$ . The normalization of  $U$  implies that  $W(\bar{c}, 1) = 1$  and  $W(\underline{c}, 0) = 0$ . The representation of Lemma 7 is therefore obtained for  $a(x) = \frac{\tilde{a}(x) - \tilde{a}(\underline{c})}{\tilde{a}(\bar{c}) - \tilde{a}(\underline{c}) + \tilde{b}(\bar{c})}$ ,  $b(x) = \frac{\tilde{b}(x)}{\tilde{a}(\bar{c}) - \tilde{a}(\underline{c}) + \tilde{b}(\bar{c})}$  and  $\phi(z) = w_0((\tilde{a}(\bar{c}) - \tilde{a}(\underline{c}) + \tilde{b}(\bar{c}))z + \tilde{a}(\underline{c}))$ . Moreover, since  $w_0$  and  $W$  are differentiable and strictly increasing, we have that  $a$ ,  $a + b$  and  $\phi$  are also differentiable, increasing and with values in  $[0, 1]$  (in  $(0, 1)$  for  $b$ ). ■

**Proof of the representation result in Proposition 1.** From Lemma 7, we know that KP-recursive preferences admit an aggregator  $W(x, y) = \phi(a(x) + b(x)y)$  with  $a(\underline{c}) = \phi(0) = 0$

and  $a(\bar{c}) + b(\bar{c}) = \phi(1) = 1$ . Let  $x_1 \in C$  and let the function  $\widetilde{W}$  be defined by:

$$\widetilde{W}(x, y) = W(x, W(x_1, y)) \text{ for all } (x, y) \in C \times [0, 1].$$

With the same proof strategy as in Lemma 7, one may show that for all  $(x, y) \in C \times [0, 1]$ ,  $\widetilde{W}(x, y) = \phi(a(x) + b(x)\phi(a(x_1) + b(x_1)y))$ . By derivation:

$$\frac{\widetilde{W}_x(x, y)}{\widetilde{W}_y(x_0, y)} = \frac{a'(x) + b'(x)\phi(a(x_1) + b(x_1)y)}{b(x)b(x_1)\phi'(a(x_1) + b(x_1)y)},$$

which has to be linear in  $y$ . Since  $a' > 0$ , we deduce that  $\frac{1 + \frac{b'(x)}{a'(x)}\phi(z)}{\phi'(z)}$  has to be linear in  $z$ .

First, assume that  $\frac{b'(x)}{a'(x)}$  is not constant and takes at least two values  $\lambda_1 \neq \lambda_2$ . Since  $\frac{1 + \lambda_1\phi(z)}{\phi'(z)}$  and  $\frac{1 + \lambda_2\phi(z)}{\phi'(z)}$  are linear in  $z$ ,  $\frac{(\lambda_2 - \lambda_1)\phi(z)}{\phi'(z)}$  and  $\frac{\phi(z)}{\phi'(z)}$  are also linear. Together with  $\phi(0) = 0$  and  $\phi(1) = 1$ , we obtain that there exists  $\nu \in \mathbb{R}$ , such that  $\forall y \in [0, 1]$ ,  $\phi(y) = y^\nu$ . The case where  $\nu < 1$  contradicts the continuous differentiability of  $\phi$ , while  $\nu > 1$  contradicts  $W_y(x, 0) > 0$ . Therefore  $\phi$  has to be linear providing  $W(x, y) = a(x) + b(x)y$ . The regularity and monotonicity conditions imposed in Definition 5, and the assumed normalization of  $U$ , lead to  $a(\underline{c}) = 0$ ,  $a(\bar{c}) + b(\bar{c}) = 1$ ,  $a'(x) > 0$  and  $a'(x) + b'(x) > 0$ . The condition  $b(x) < 1$  comes from  $a + b$  and  $b$  being strictly increasing together with  $a(\underline{c}) = 0$  and  $a(\bar{c}) + b(\bar{c}) = 1$ . We are thus left with the linear aggregators (14).

Second, we assume that  $\frac{b'(x)}{a'(x)}$  is constant, necessarily larger than  $-1$  (since  $W_x > 0$ ) and different from zero (since  $W_y > 0$ ). We define  $k \neq 0$  with  $\frac{b'(x)}{a'(x)} = e^{-k} - 1$ . By integration:

$$b(x) = \left(e^{-k} - 1\right) a(x) + b_0, \quad \text{with } b_0 \in \mathbb{R}. \quad (48)$$

Let  $h(z) = 1 + (e^{-k} - 1)\phi(z)$ . Since  $\frac{1 + \frac{b'(x)}{a'(x)}\phi(z)}{\phi'(z)}$  is linear in  $z$ , so is  $\frac{h(z)}{h'(z)}$ .  $\phi(0) = 0$  and  $\phi(1) = 1$  imply  $h(0) = 1$  and  $h(1) = e^{-k}$ . By integration, there exists  $\beta > 0$  such that:

$$\forall z \in [0, 1], \quad h(z) = \left(1 + \left(e^{-\frac{k}{\beta}} - 1\right)z\right)^\beta \text{ and } \phi(z) = \frac{1 - \left(1 - (1 - e^{-\frac{k}{\beta}})z\right)^\beta}{1 - e^{-k}}. \quad (49)$$

We can remark that for all  $z \in [0, 1]$ :

$$\frac{1 + (e^{-k} - 1)\phi(z)}{\phi'(z)} = \frac{1}{\beta} \frac{1 - e^{-k}}{1 - e^{-\frac{k}{\beta}}} \left(1 - (1 - e^{-\frac{k}{\beta}})z\right). \quad (50)$$

Let  $x_2 \in C$ . Similarly to the definition of  $\widetilde{W}$ , we introduce a function  $\widehat{W}$  as follows:

$$\forall (x, y) \in C \times [0, 1], \quad \widehat{W}(x, y) = W(x, W(x_1, W(x_2, y))).$$

As in Lemma 7 and as with  $\widetilde{W}$ , one can show that FSD-monotonicity imposes that  $\frac{\widehat{W}_x}{\widehat{W}_y}$  has to be linear in  $y$ . Writing  $\widehat{W}(x, y) = \phi(a(x) + b(x)\phi(a(x_1) + b(x_1)\phi(a(x_2) + b(x_2)y)))$  to compute  $\frac{\widehat{W}_x}{\widehat{W}_y}$  and using (50), we obtain that:  $\frac{1-(1-e^{-\frac{k}{\beta}})(a(x_1)+b(x_1)\phi(a(x_2)+b(x_2)y))}{\phi'(a(x_2)+b(x_2)y)}$  is linear in  $y$ . However we know that  $\frac{1-(1-e^{-k})\phi(a(x_2)+b(x_2)y)}{\phi'(a(x_2)+b(x_2)y)}$  is linear in  $y$ . So, after substitution in the above expression, we have  $\left(1 - (1 - e^{-\frac{k}{\beta}})a(x_1) - \frac{(1-e^{-\frac{k}{\beta}})b(x_1)}{(1-e^{-k})}\right) \frac{\phi(a(x_2)+b(x_2)y)}{\phi'(a(x_2)+b(x_2)y)}$  is linear in  $y$ . Thus, either  $\phi$  is linear (case already considered) or  $1 - (1 - e^{-\frac{k}{\beta}})a(x_1) = \frac{(1-e^{-\frac{k}{\beta}})b(x_1)}{(1-e^{-k})}$  for any  $x_1 \in C$ . Equation (48) implies that  $b_0 = \frac{1-e^{-k}}{1-e^{-\frac{k}{\beta}}}$  and

$$\forall x \in C, b(x) = (1 - e^{-k}) \left( \frac{1}{1 - e^{-\frac{k}{\beta}}} - a(x) \right).$$

Then (49) implies that for all  $x$  and  $y$ :

$$\phi(a(x) + b(x)y) = \frac{1 - \left[ \left(1 - (1 - e^{-\frac{k}{\beta}})a(x)\right) (1 - (1 - e^{-k})y) \right]^\beta}{1 - e^{-k}}.$$

We define  $u : [0, 1] \rightarrow \mathbb{R}$  as follows:

$$u(x) = -\frac{\beta}{k(1-\beta)} \log \left( 1 - (1 - e^{-\frac{k}{\beta}})a(x) \right), \text{ or } 1 - (1 - e^{-\frac{k}{\beta}})a(x) = e^{-\frac{k(1-\beta)u(x)}{\beta}}, \quad (51)$$

which, after renormalization, leads to the specification (15). Equation (51) implies  $u(\underline{c}) = 0$ . Moreover  $a(\bar{c}) + b(\bar{c}) = 1$  imposes that  $a(\bar{c}) = \frac{1-e^{-k\frac{1-\beta}}{1-e^{-\frac{k}{\beta}}}}$  and  $u(\bar{c}) = 1$ . The condition  $\beta < 1$  results from the fact that the restriction of  $U$  to  $C^\infty$  has to be monotonic. Indeed from (15) and  $u(\underline{c}) = 0$ , the utility associated with the consumption of the same  $c \in C$  for  $N$  periods and  $\underline{c}$  afterwards is  $(1 - \beta) \sum_{i=0}^{N-1} \beta^i u(c)$ , which is monotone in  $c$  if and only if  $\beta < 1$ .

## B.2.2 Existence and uniqueness

We wish to use the Banach fixed point theorem to show that the aggregator (15) defines a unique utility function.<sup>33</sup> We define  $\mathcal{C}(D, [0, 1])$  the set of continuous functions from  $D$  into  $[0, 1]$ . We know that  $C^\infty$  endowed with the product metric is a compact Polish space and that  $D$  endowed with the Prohorov metric is also a compact Polish space. The metric space  $(\mathcal{C}(D, [0, 1]), \|\cdot\|_\infty)$  is thus a Banach space (e.g., Theorem 9.3 in Aliprantis and Burkinshaw (1998)).

Denote by  $\mathcal{F}(D, \mathbb{R})$  the set of all functions from  $D$  to  $\mathbb{R}$ . We consider the mapping  $T$  from

<sup>33</sup>The proof in the case of the aggregator (14) is not provided as it would follow the same arguments.

$\mathcal{C}(D, [0, 1])$  into  $\mathcal{F}(D, \mathbb{R})$  such that for  $F \in \mathcal{C}(D, [0, 1])$ :

$$\forall (c, m) \in D, TF(c, m) = (1 - \beta)u(c) - \frac{\beta}{k} \log E_m[e^{-kF}]. \quad (52)$$

To use the fixed point theorem, we prove that  $T$  verifies Blackwell's (1965) sufficient conditions and is a contraction.

1. Let  $F \in \mathcal{C}(D, [0, 1])$  and  $(c, m) \in D$ . Since  $0 \leq u(c) \leq 1$  and  $0 \leq -\frac{1}{k} \log(E_m[e^{-kF}]) \leq 1$ , we also have  $0 \leq TF(c, m) \leq 1$ . Consider  $(c_n, m_n)_{n \geq 0} \in D^\infty$  that converges towards  $(c, m) \in D$ . Since  $F$  is bounded and continuous,  $e^{-kF}$  is also bounded (below by  $e^{-k}$ ) and continuous, we have:  $\log E_{m_n}[e^{-kF}] \rightarrow \log E_m[e^{-kF}]$ . Since  $u$  is continuous,  $TF(c_n, m_n) \rightarrow TF(c, m)$  and  $TF \in \mathcal{C}(D, [0, 1])$ .
2. Let  $F_1, F_2 \in \mathcal{C}(D, [0, 1])$ , such that  $\forall x \in D, F_2(x) \geq F_1(x) \geq 0$ . We have for all  $k$  and for all  $m \in M(D)$ ,  $E_m[e^{-kF_2}]^{-\frac{1}{k}} \geq E_m[e^{-kF_1}]^{-\frac{1}{k}} > 0$  and  $-\frac{1}{k} \log E_m[e^{-kF_2}] \geq -\frac{1}{k} \log E_m[e^{-kF_1}]$ . Since  $\beta > 0$ , we deduce that  $T$  is increasing.
3. Let  $0 \leq A \leq 1$ ,  $F \in \mathcal{C}(D, [0, 1])$  and  $(c, m) \in D$ . Noticing that  $-\frac{\beta}{k} \log(E_m[e^{-k(F+A)}]) = -\frac{\beta}{k} \log(E_m[e^{-kF}]) + \beta A$ , it is straightforward that  $T(F+A)(c, m) - TF(c, m) = \beta A$ .

The map  $T$  is a contraction of modulus  $\beta \in (0, 1)$  on  $(\mathcal{C}(D, [0, 1]), \|\cdot\|_\infty)$ . The Banach fixed point theorem implies that (15) admits a unique solution.

### B.2.3 Sufficient conditions

We conduct the proof for the risk-sensitive case, showing that preference relation  $\succeq$  is mixture-stable. The Uzawa case is analogous. Let us consider  $\tau \geq 0$ ,  $\lambda \in (0, 1)$ ,  $c_1, c_2$  in  $C^{\tau+1}$  and  $m_1, m'_1, m_2, m'_2$  in  $M(D)$ , such that  $(c_1, m_1) \succeq (c_2, m_2)$  and  $(c_1, m'_1) \succeq (c_2, m'_2)$ . Denoting  $c_i = (c_i^0, \dots, c_i^\tau)$  with  $i = 1, 2$ , we have

$$\begin{aligned} U^A(c_i, m_i) &= (1 - \beta) \sum_{j=0}^{\tau} \beta^j u(c_i^j) - \frac{\beta}{k_A} \log \left( E_{m_i} \left[ e^{-k_A U^A} \right] \right), \\ &= -\frac{\beta}{k_A} \log \left( e^{-\frac{1-\beta}{\beta} k_A \sum_{j=0}^{\tau} \beta^j u(c_i^j)} E_{m_i} \left[ e^{-k_A U^A} \right] \right). \end{aligned} \quad (53)$$

Since  $E_{\lambda m_i + (1-\lambda)m'_i}[\cdot] = \lambda E_{m_i}[\cdot] + (1 - \lambda)E_{m'_i}[\cdot]$ , we deduce that

$$\begin{aligned} U^A(c_i, \lambda m_i + (1 - \lambda)m'_i) &= -\frac{\beta}{k_A} \log \left( \lambda e^{-\frac{1-\beta}{\beta} k_A \sum_{j=0}^{\tau} \beta^j u(c_i^j)} E_{m_i} \left[ e^{-k_A U^A} \right] \right. \\ &\quad \left. + (1 - \lambda) e^{-\frac{1-\beta}{\beta} k_A \sum_{j=0}^{\tau} \beta^j u(c_i^j)} E_{m'_i} \left[ e^{-k_A U^A} \right] \right). \end{aligned} \quad (54)$$

Therefore, if  $U^A(c_1, m_1) \geq U^A(c_2, m_2)$  and  $U^A(c_1, m'_1) \geq U^A(c_2, m'_2)$ , (53) and (54) readily imply that  $U^A(c_1, \lambda m_1 + (1 - \lambda)m'_1) \geq U^A(c_2, \lambda m_2 + (1 - \lambda)m'_2)$ .

### B.3 Proof of Proposition 2

Consider normalized utility functions  $U^A$  and  $U^B$  (and their corresponding aggregators  $W^A$  and  $W^B$ ) that represent two FSD-monotone KP-recursive preference relations, whose restrictions to  $C^\infty$  are identical. Let us assume that one of them (e.g.,  $U^A$ ) is not a risk-sensitive utility function. Proposition 1 implies that  $W^A(x, y) = a_1(x) + b_1(x)y$  for some non-constant function  $b_1$ . Such preferences do not fulfill the assumption of weak separability and therefore cannot have the same restriction over  $C^\infty$  as risk-sensitive preferences, which are weakly separable. Thus,  $U^B$  must also correspond to an aggregator  $W^B$  such that  $W^B(x, y) = a_2(x) + b_2(x)y$  for some non-constant function  $b_2$ . In order to prove that  $W^A = W^B$ , we first state the following simple result:

**Lemma 8 (Aggregators with identical ordinal preferences)** *Consider  $W^A$  and  $W^B$  two aggregators associated to normalized KP-recursive utility functions representing the same preferences over  $C^\infty$ .*

*Then there exists an increasing function  $\psi$  such that  $\psi(0) = 0$ ,  $\psi(1) = 1$  and:*

$$\forall (x, y) \in C \times [0, 1], \psi(W^A(x, y)) = W^B(x, \psi(y)).$$

**Proof.** Consider the utility functions  $U^A$  and  $U^B$  corresponding to the aggregators  $W^A$  and  $W^B$ . Since they represent the same ordinal preferences, there exists an increasing function  $\psi$ , such that for all  $x \in C^\infty$  we have  $U^B(x) = \psi(U^A(x))$ . From  $U^i(\underline{c}, \underline{c}, \dots) = 0$  and  $U^i(\bar{c}, \bar{c}, \dots) = 1$  ( $i = A, B$ ), we have  $\psi(0) = 0$ ,  $\psi(1) = 1$ . For any  $(c_0, c_1, \dots) \in C^\infty$ , we have:  $U^B(c_0, c_1, \dots) = W^B(c_0, U^B(c_1, \dots)) = W^B(c_0, \psi(U^A(c_1, \dots)))$ . We also have:  $U^B(c_0, c_1, \dots) = \psi(U^A(c_0, c_1, \dots)) = \psi(W^A(c_0, U^A(c_1, \dots)))$ . Noting  $y = U^A(c_1, c_2, \dots)$  which covers  $[0, 1]$ , we deduce that  $\psi(W^A(c_0, y)) = W^B(c_0, \psi(y))$  for all  $(c_0, y) \in C \times [0, 1]$ . ■

We now terminate the proof. Lemma 8 implies that there exists an increasing function  $\psi$  with  $\psi(0) = 0$  and  $\psi(1) = 1$ , such that for all  $(x, y) \in C \times [0, 1]$ :

$$\psi(a_1(x) + b_1(x)y) = a_2(x) + b_2(x)\psi(y). \quad (55)$$

Since  $W^A$  and  $W^B$  are continuously differentiable on  $C \times [0, 1]$ ,  $a_i$  and  $b_i$  also are (on  $C$ ) for  $i = 1, 2$ .

As a preliminary remark, it would be straightforward to show that for any  $y \in [0, 1]$ , the application  $x \mapsto a_i(x) + b_i(x)y$  ( $i = 1, 2$ ) is a strictly increasing bijection from  $C$  into  $[b_i(\underline{c})y, a_i(\bar{c}) + b_i(\bar{c})y]$ . Moreover  $a_i(\bar{c}) + b_i(\bar{c})y < 1$  if and only if  $y < 1$ , and  $b_i(\underline{c})y > 0$  if and only if  $y > 0$ . Remark also that the properties of  $a_i$  and  $b_i$  imply that  $y \in [b_i(\underline{c})y, a_i(\bar{c}) + b_i(\bar{c})y]$  for any  $y \in [0, 1]$ .

Let  $Y = \{z \in [0, 1] \mid \psi \text{ is continuously derivable on } [0, z]\}$ . Using (55) and our preliminary remark with  $y = 0$ , we deduce that  $\psi$  is continuously differentiable on  $[0, a_1(\bar{c})]$ . So,  $Y \neq \emptyset$  and  $Y$  is also bounded by 1. Let  $\bar{y} = \sup Y$  with  $0 < a_1(\bar{c}) \leq \bar{y} \leq 1$ . If  $\bar{y} < 1$ , our preliminary

remark with  $y = \bar{y}$  and (55) contradict the definition of  $\bar{y}$ . So  $\bar{y} = 1$  and  $\psi$  is continuously differentiable on  $[0, 1]$ . After taking the derivative of (55) with respect to  $y$ , the same strategy can be used to show that  $\psi'$  is continuously differentiable and strictly positive. Since  $b_1$  and  $b_2$  share the same properties, we can compute the second-order derivative of (55), providing  $b_1(x) \frac{\psi''}{\psi'} (a_1(x) + b_1(x)y) = \frac{\psi''}{\psi'}(y)$ . The preliminary remark implies that for any  $y \in [0, 1]$ , there exists  $x_y \in C$  such that  $a_1(x_y) + b_1(x_y)y = y$ . We thus obtain  $(1 - b_1(x_y)) \frac{\psi''}{\psi'}(y) = 0$ . Since  $b_1(x_y) < 1$ , we deduce that  $\psi'' = 0$  which, with  $\psi(0) = 0$  and  $\psi(1) = 1$ , imply that  $\psi(x) = x$ . From (55), it follows that  $a_1(x) = a_2(x)$  and  $b_1(x) = b_2(x)$  and finally  $W^A = W^B$ .

## B.4 Proof of Proposition 3

### B.4.1 Weak risk aversion

We prove that if  $k_A \geq 0$ , we have  $U^A(E[(c, m)]) \geq U^A(c, m)$  for any  $(c, m) \in \cup_{t \geq 0} D_t$ , the result extending to any  $(c, m) \in D$  by continuity. Let us consider  $t \geq 0$ ,  $(c, m) \in D_t$  and use representation (17). We have  $V_t^A = (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(c_i)$ , and for any  $0 \leq \tau \leq t - 1$ :  $V_\tau^A = \phi_{A,\tau}^{-1} E_\tau [\phi_{A,\tau}(V_{\tau+1}^A)]$  with  $\phi_{A,\tau} : x \mapsto -\frac{\beta^{\tau+1}}{k_A} e^{-\frac{k_A}{\beta^{\tau+1}} x}$ . Since  $\phi_{A,\tau}$  is concave for any  $\tau$  whenever  $k_A \geq 0$ , Jensen's inequality implies that  $V_\tau^A \leq E_\tau [V_{\tau+1}^A]$ , or after iteration  $V_0^A \leq E_0[V_t^A] = E_0[(1 - \beta) \sum_{i=0}^{\infty} \beta^i u(\tilde{c}_i)]$ . The function  $u$  being concave, we eventually obtain  $V_0^A \leq (1 - \beta) \sum_{i=0}^{\infty} \beta^i u(E_0[\tilde{c}_i])$ , which concludes the proof.

### B.4.2 Weak comparative risk aversion

We assume that  $k_A > k_B$  and  $k_A, k_B \neq 0$  (the case where either  $k_A$  or  $k_B$  equals zero can be treated similarly). For  $i = A, B$ , set  $V^i = -\frac{e^{-k_i U^i}}{k_i}$ , which represents the same preferences as  $U^i$  and fulfills the recursion (9) with the aggregator  $W^i(x, y) = -\frac{1}{k_i} e^{-k_i(1-\beta)u(x)} (-k_i y)^\beta$ . Consider the function  $\psi_{A,B}$  defined on  $Im(V^B)$  by  $\psi_{A,B}(y) = \frac{-(-k_B y)^{k_A/k_B}}{k_A}$ . This function is increasing and concave. For any deterministic consumption path  $(c_0, c_1, \dots) \in C^\infty$ , we have  $V^i(c_0, c_1, \dots) = \frac{-\exp(-k_i(1-\beta) \sum_{t=0}^{\infty} \beta^t u(c_t))}{k_i}$  and therefore  $V^A(x) = \psi_{A,B}(V^B(x))$ . Moreover for all  $c \in C$  and  $y \in Im(V^B)$ , we have  $\psi_{A,B}(W^B(c, y)) = W^A(c, \psi_{A,B}(y))$ .

We now show by induction that for all  $t \geq 0$  and all  $(c, m) \in D_t$ ,  $\psi_{A,B}(V^B(c, m)) \geq V^A(c, m)$ . It holds for  $t = 0$  since  $V^A(x) = \psi_{A,B}(V^B(x))$  for all  $x \in D_0 = C^\infty$ . Assume that it holds up to  $t - 1$  and let  $(c, m) \in D_t$ :

$$\begin{aligned} \psi_{A,B}(V^B(c, m)) &= \psi_{A,B}(W^B(c, E_m[V^B])) = W^A(c, \psi_{A,B}(E_m[V^B])) \\ &\geq W^A(c, E_m[\psi_{A,B}(V^B)]) \geq W^A(c, E_m[V^A]) = V^A(c, m), \end{aligned}$$

where the first inequality is a consequence of Jensen inequality and of  $W^B$  being increasing with respect to its second argument, while the second comes from the induction hypothesis.

Since  $\cup_{t \geq 0} D_t$  is dense in  $D$ , we deduce by continuity that  $\psi_{A,B}(V^B(c, m)) \geq V^A(c, m)$  for all  $(c, m) \in D$ . We deduce that for all  $x \in C^\infty$  and  $(c, m) \in D$ ,  $V^A(c, m) \geq V^A(x) \Rightarrow$

$V^B(c, m) \geq V^B(x)$ , which terminates the proof.

## B.5 Proof of Proposition 5

We consider an agent with risk-sensitive preferences, whose risk aversion parameter is  $k$ . The utility of an independently distributed consumption stream  $(\tilde{c}_t)_{t \geq 1}$  is (see equation (24)):  $U^k(c_0, \tilde{c}_1, \dots) = (1 - \beta)u(c_0) - \frac{1}{k} \sum_{j=1}^{\infty} \beta^j \log E_{j-1} [e^{-k(1-\beta)u(\tilde{c}_j)}]$ . With this simple additive structure, the optimal insurance demand  $q_t^k$  against the risk of date  $t$  solves:

$$p_t \beta^{-t} = \frac{E_{t-1} [\tilde{\alpha}_t u'(\tilde{y}_t + q_t^k \tilde{\alpha}_t) e^{-k(1-\beta)u(\tilde{y}_t + q_t^k \tilde{\alpha}_t)}]}{u'(y_0 - \sum_{i=1}^{\infty} p_i q_i^k) E_{t-1} [e^{-k(1-\beta)u(\tilde{y}_t + q_t^k \tilde{\alpha}_t)}]}. \quad (56)$$

We want to compute the total derivative of (56) with respect to  $k$ . First, we take the derivative with respect to  $k$  while keeping other parameters constant. We denote  $c_0 = y_0 - \sum_{i=1}^{\infty} p_i q_i^k$  and  $\tilde{c}_t = \tilde{y}_t + q_t^k \tilde{\alpha}_t$ . Since shocks are bounded, there exists a constant  $A_t \geq 0$  such that  $\tilde{a}_t = A_t + \tilde{\alpha}_t \geq 0$  ( $t \geq 1$ ). We define the following functions:

$$V_0(k) = u'(c_0), \quad V_{a,t}(k) = \frac{E_{t-1} [\tilde{a}_t u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]}{E_{t-1} [e^{-k(1-\beta)u(\tilde{c}_t)}]}, \quad V_{A,t}(k) = \frac{A_t E_{t-1} [u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]}{E_{t-1} [e^{-k(1-\beta)u(\tilde{c}_t)}]},$$

such that (56) becomes:  $p_t \beta_t^{-t} = \frac{V_{a,t}(k) - V_{A,t}(k)}{V_0(k)}$ . We have:

$$\frac{V_{a,t}(k)}{V_{A,t}(k)} = \frac{A_t E_{t-1} [u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}] + \text{cov}_{t-1}(\tilde{a}_t, u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)})}{A_t E_{t-1} [u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]} > 1,$$

because  $\tilde{\alpha}_t$  and  $\tilde{c}_t$  (and therefore  $\tilde{a}_t$  and  $\tilde{c}_t$ ) are anticomotone.<sup>34</sup> Moreover:

$$\begin{aligned} \frac{1}{1-\beta} \frac{\partial}{\partial k} \left( \frac{V_{a,t}(k)}{V_{A,t}(k)} \right) &= \frac{E_{t-1} [u(\tilde{c}_t) u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]}{E_{t-1} [u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]} \frac{E_{t-1} [\tilde{a}_t u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]}{E_{t-1} [u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]} \\ &\quad - \frac{E_{t-1} [\tilde{a}_t u(\tilde{c}_t) u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]}{E_{t-1} [u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}]} \\ &= -\widetilde{\text{cov}}_{t-1}^{u'(\tilde{c}_t) e^{-k(1-\beta)u(\tilde{c}_t)}}(u(\tilde{c}_t), \tilde{a}_t) > 0, \end{aligned}$$

again because  $\tilde{a}_t$  and  $\tilde{c}_t$  are anticomotone. Similarly, we have:

$$\frac{1}{1-\beta} \frac{\partial}{\partial k} \left( \frac{V_{A,t}(k)}{V_0(k)} \right) = -\frac{A_t}{u'(c_0)} \widetilde{\text{cov}}_{t-1}^{e^{-k(1-\beta)u(\tilde{c}_t)}}(u(\tilde{c}_t), u'(\tilde{c}_t)) > 0,$$

<sup>34</sup>Purchasing a quantity of asset  $q_t^k$  larger than one cannot be optimal, as this would be dominated (in the sense of SSD) by choosing a quantity equal to one. This imposes that  $q_t^k \leq 1$  and the mentioned anticomotonicity.

where the covariance operators are defined in (39) and  $A_t \geq 0$ . We therefore deduce:

$$\frac{\partial}{\partial k} \left( \frac{V_a(k) - V_A(k)}{V_0(k)} \right) = \frac{\partial}{\partial k} \left( \frac{V_A(k)}{V_0(k)} \left( \frac{V_a(k)}{V_A(k)} - 1 \right) \right) \geq 0. \quad (57)$$

Second, we want to take the derivative of (56) with respect to  $q_t^k$ , while keeping  $k$  constant. Denoting by  $V_k(q_t^k)$  the right-hand side term of (56) and defining the variance  $\tilde{V}_{t-1}^{e^{-k(1-\beta)u(\tilde{c}_t)}}$  as in (38), we obtain after some algebra:

$$\begin{aligned} \left. \frac{\partial V_k(q_t^k)}{\partial q_t^k} \right|_k &= \frac{E_{t-1} \left[ \tilde{\alpha}_t^2 u''(\tilde{y}_t + q_t^k \tilde{\alpha}_t) e^{-k(1-\beta)u(\tilde{y}_t + q_t^k \tilde{\alpha}_t)} \right]}{u'(y_0 - \sum_{i=1}^{\infty} p_i q_i^k) E_{t-1} \left[ e^{-k(1-\beta)u(\tilde{y}_t + q_t^k \tilde{\alpha}_t)} \right]} \\ &\quad + \beta^{-j} u'(y_0 - \sum_{i=1}^{\infty} p_i q_i^k) \frac{u''(y_0 - \sum_{i=1}^{\infty} p_i q_i^k)}{u'(y_0 - \sum_{i=1}^{\infty} p_i q_i^k)} \\ &\quad - k(1-\beta) \frac{\tilde{V}_{t-1}^{e^{-k(1-\beta)u(\tilde{c}_t)}} \left[ \tilde{\alpha}_t u'(\tilde{y}_t + q_t^k \tilde{\alpha}_t) \right]}{u'(y_0 - \sum_{i=1}^{\infty} p_i q_i^k)} < 0. \end{aligned} \quad (58)$$

Using previous notations, the total derivative of (56) with respect to  $k$  yields:

$$\frac{\partial}{\partial k} \left( \frac{V_a(k) - V_A(k)}{V_0(k)} \right) + \left. \frac{\partial V_k(q_t^k)}{\partial q_t^k} \right|_k \frac{\partial q_t^k}{\partial k} = 0.$$

From inequalities (57) and (58), we deduce  $\frac{\partial q_t^k}{\partial k} \geq 0$ .

## B.6 Proof of Proposition 6

One of the technical difficulties is that the consumption positivity constraints generate potentially binding borrowing constraints. Indeed, as consumption has to be nonnegative at all dates, agents cannot borrow more than what they may be able to repay, given their future income. Formally, at any date  $t$ , the saving has to be greater than a (negative) lower bound  $\underline{s}_t$  that depends on the income process. This bound is deterministic and can be time-varying as the income process is not assumed to be stationary. The program (29) can be expressed as follows:

$$\begin{aligned} V_t^k(w_t, y^t) &= \max_{s_t \geq \underline{s}_t} u(y_t - s_t + w_t) - \frac{\beta}{k} \log E_t e^{-kV_{t+1}^k(w_{t+1}, y^{t+1})}, \\ \text{s.t. } w_{t+1} &= R s_t. \end{aligned} \quad (59)$$

To shorten the notation, we introduce  $W_t^k$  defined by:

$$W_t^k(w_t, y^t) = -kV_t^k(w_t, y^t) = \min_{s_t \geq \underline{s}_t} -k u(y_t - s_t + w_t) + \beta \log E_t e^{W_{t+1}^k(R s_t, y^{t+1})}. \quad (60)$$

We denote  $W_{t,w}^k$  and  $W_{t,k}^k$  the derivatives of  $W_t^k$  with respect to wealth  $w_t$  and risk aversion  $k$  respectively. The first-order condition provides:

$$ku'(c_t) \geq -\beta R \frac{E_t \left[ W_{t+1,w}^k(w_{t+1}, y^{t+1}) e^{W_{t+1}^k(w_{t+1}, y^{t+1})} \right]}{E_t \left[ e^{W_{t+1}^k(w_{t+1}, y^{t+1})} \right]}, \quad (61)$$

where the equality holds if  $s_t > \underline{s}_t$ . The envelop theorem yields the following equalities (which are valid whether the constraint  $s_t \geq \underline{s}_t$  is binding or not):

$$W_{t,w}^k(w_t, y^t) = -ku'(y_t - s_t + w_t) = -ku'(c_t) \quad (62)$$

$$W_{t,k}^k(w_t, y^t) = -u(c_t) + \beta E_t \left[ W_{t+1,k}^k \frac{e^{W_{t+1}^k(w_{t+1}, y^{t+1})}}{E_t \left[ e^{W_{t+1}^k(w_{t+1}, y^{t+1})} \right]} \right] \quad (63)$$

We now state following lemma.

**Lemma 9 (Comonotonicity of income and consumption)** *In the setup of Proposition 6, at any date  $t \geq 0$ , the optimal consumption  $\tilde{c}_t$  and the income  $\tilde{y}_t$  are comonotone.*

**Proof.** We prove the result for  $(\tilde{y}_i)_{i \geq 0} \in \cup_{T \geq 0} D_T$ , as it then extends to  $D$  by continuity. Let  $T \geq 0$  and  $(\tilde{y}_i)_{i \geq 0} \in D_T$ . We prove by reverse induction on  $i$  that (i)  $\tilde{c}_i$  and  $\tilde{y}_i$  are comonotone and that (ii)  $\tilde{y}_i$  and  $W_i^k$  are anticomonotone.

At date  $i = T$ , there is no uncertainty left. First, assume that consumption positivity constraints do not bind. Equation (61) implies for any  $\tau \geq 0$ ,  $u'(\tilde{c}_T) = (\beta R)^\tau u'(\tilde{c}_{T+\tau})$ . So  $\tilde{c}_T$  and  $\tilde{c}_{T+\tau}$  (for any  $\tau \geq 0$ ) are comonotone. The intertemporal budget constraint at date  $T$  being  $\tilde{y}_T + \sum_{\tau=1}^{\infty} \frac{\tilde{y}_{T+\tau}}{R^\tau} + R s_{T-1} = \sum_{\tau=0}^{\infty} \frac{\tilde{c}_{T+\tau}}{R^\tau}$ , we deduce that  $\tilde{y}_T$  and  $\tilde{c}_T$  are comonotone. Since  $W_T^k = -k \sum_{\tau=0}^{\infty} \beta^\tau u(\tilde{c}_{T+\tau})$ , it is also clear that  $\tilde{y}_T$  and  $W_T^k$  are anticomonotone.

Second, assume that the consumption positivity constraint binds at date  $T + \tau$  for some  $\tau \geq 0$ . Note that due to the absence of uncertainty after date  $T$ , the consumption positivity constraint also binds at all later dates  $T + \tau' > T + \tau$ . Let  $\underline{y}_{T,\tau}$  be the smallest realization of  $\tilde{y}_T$  for which the consumption positivity constraint binds at date  $t + \tau$ . Because of the stochastic monotonicity assumption, for all realizations of  $\tilde{y}_T$  smaller than  $\underline{y}_{T,\tau}$ , the consumption positivity constraint will also bind at date  $t + \tau$  (and at all later dates). Therefore,  $\tilde{c}_T$  and  $\tilde{c}_{T+\tau}$  (for any  $\tau \geq 0$ ) are also comonotone when consumption positivity constraint binds at some date. We deduce that  $\tilde{c}_T$  and  $\tilde{y}_T$  are comonotone, and that  $\tilde{y}_T$  and  $W_T^k$  are anticomonotone.

We have shown that points (i) and (ii) hold for  $i = T$ . We now proceed by induction showing that if they hold for  $0 < i \leq T$ , they also hold for  $i - 1$ . When consumption positivity constraints do not bind, the Euler equation (61) together with (62) implies:

$$u'(\tilde{c}_{i-1}) = (\beta R) E_{i-1} \left[ u'(\tilde{c}_i) \frac{e^{W_i^k(w_i, y^i)}}{E_{i-1} [e^{W_i^k(w_i, y^i)}]} \right]$$

Using the induction hypothesis, we know that  $u'(\tilde{c}_i) \frac{e^{W_i^k(w_i, y^i)}}{E_{i-1}[e^{W_i^k(w_i, y^i)}]}$  and  $\tilde{y}_i$  are anticomonotone. Since the income process is stochastically monotone, we deduce that  $u'(\tilde{c}_{i-1})$  is non-increasing with  $\tilde{y}_{i-1}$ , meaning that  $\tilde{c}_{i-1}$  and  $\tilde{y}_{i-1}$  are comonotone. Using definition (60) of  $W_{i-1}^k$  together with the induction hypothesis and the comonotonicity of  $\tilde{c}_{i-1}$  and  $\tilde{y}_{i-1}$ , we obtain that  $W_{i-1}^k$  and  $\tilde{y}_{i-1}$  are anticomonotone.

When consumption positivity constraints bind, denote by  $\underline{y}_{i-1}$  the cut-off value of  $\tilde{y}_{i-1}$  below which the Euler equation does not hold. For any realization of  $\tilde{y}_{i-1}$  below  $\underline{y}_{i-1}$ , the saving constraint at date  $i$  is binding: the consumption at date  $i$  varies exactly as  $\tilde{y}_{i-1}$ . Therefore, when the Euler equation does not hold,  $\tilde{c}_{i-1}$  and  $\tilde{y}_{i-1}$  are also comonotone and  $W_{i-1}^k$  and  $\tilde{y}_{i-1}$  are anticomonotone. ■

We now complete the proof of Proposition 6. The case where the inequality (61) is strict, indicating a binding borrowing constraint  $s_t = \underline{s}_t$  is trivial, as we then have by continuity  $\frac{\partial s_t}{\partial k} = 0$ . We now assume that (61) is an equality. We take the derivative of this equality. Dropping the argument  $(w_{t+1}, y^{t+1})$  of  $W_{t+1}^k$  and using the notation defined in (37), we get:

$$-u'(c_t) + ku''(c_t) \frac{\partial s_t}{\partial k} = \beta R \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k]}{\partial s_t} \right|_k \frac{\partial s_t}{\partial k} + \beta R \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k]}{\partial k} \right|_{s_t}. \quad (64)$$

We investigate the signs of both derivatives in (64). We start with  $\left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k]}{\partial s_t} \right|_k$ . Using  $w_{t+1} = R s_t$ , we obtain:

$$\begin{aligned} \frac{1}{R} \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k]}{\partial s_t} \right|_k &= \frac{E_t [W_{t+1, w w}^k e^{W_{t+1}^k}]}{E_t [e^{W_{t+1}^k}]} + \frac{E_t [(W_{t+1, w}^k)^2 e^{W_{t+1}^k}]}{E_t [e^{W_{t+1}^k}]} - \frac{E_t [W_{t+1, w}^k e^{W_{t+1}^k}]^2}{E_t [e^{W_{t+1}^k}]^2}, \\ &= \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w w}^k] + \tilde{V}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k] > 0, \end{aligned} \quad (65)$$

since  $W_{t+1}^k$  is convex in wealth (the indirect utility is concave), and the variance is positive (the variance operator is defined in equation (38)). We then deduce from (64) that the sign of  $\frac{\partial s_t}{\partial k}$  is opposite to the one of  $u'(c_t) + \beta R \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k]}{\partial k} \right|_{s_t}$ . Note that equation (62) implies

that  $\left. \frac{\partial W_{t+1, w}^k}{\partial k} \right|_{s_t} = -u'(c_{t+1})$ . We thus have:

$$\begin{aligned} \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k]}{\partial k} \right|_{s_t} &= -\tilde{E}_t^{e^{W_{t+1}^k}} [u'(c_{t+1})] + \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k W_{t+1, k}^k] \\ &\quad - \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, w}^k] \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1, k}^k]. \end{aligned}$$

Using the Euler equation (61) and equality (62) we get:

$$u'(c_t) + \beta R \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1,w}^k]}{\partial k} \right|_{s_t} = -k\beta R \widehat{\text{cov}}_t^{e^{W_{t+1}^k}} (u'(c_{t+1}), W_{t+1,k}^k). \quad (66)$$

Observe now that, from equation (63), we can derive by iteration:

$$\begin{aligned} W_{t+1,k}^k &= - \sum_{\tau=1}^{\infty} \beta^{\tau-1} E_{t+1} \left[ u(\tilde{c}_{t+\tau}) \frac{e^{\sum_{j=2}^{\tau} W_{t+j}^k}}{\prod_{j=2}^{\tau} E_{t+j-1} [e^{W_{t+j}^k}]} \right], \\ &= - \sum_{\tau=1}^{\infty} \beta^{\tau-1} \widehat{E}_{t+1}^{t+\tau} [u(\tilde{c}_{t+\tau})], \end{aligned} \quad (67)$$

where  $\widehat{E}_{t+1}^{t+\tau} [X] = E_{t+1} \left[ X \frac{e^{\sum_{j=2}^{\tau} W_{t+j}^k}}{\prod_{j=2}^{\tau} E_{t+j-1} [e^{W_{t+j}^k}]} \right]$  is an expectation operator under a new measure ( $\widehat{E}_{t+1}^{t+\tau} [1] = 1$  and  $\frac{e^{\sum_{j=2}^{\tau} W_{t+j}^k}}{\prod_{j=2}^{\tau} E_{t+j-1} [e^{W_{t+j}^k}]} > 0$  for any  $t, \tau$ ). Using this notation, we have:

$$u'(c_t) + \beta R \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1,w}^k]}{\partial k} \right|_{s_t} = k\beta R \sum_{\tau=1}^{\infty} \beta^{\tau-1} \widehat{\text{cov}}_t^{t+\tau} (u'(\tilde{c}_{t+1}), u(\tilde{c}_{t+\tau})) \quad (68)$$

where  $\widehat{\text{cov}}_t^{t+\tau} (\cdot, \cdot)$  is the covariance operator associated to the expectation operator  $\widehat{E}_t^{t+\tau} [\cdot]$ . From Lemma 9, we know that  $\tilde{c}_t$  and  $\tilde{y}_t$  are comonotone for any  $t \geq 0$ . The stochastic monotonicity property of the income process then implies that  $\widehat{\text{cov}}_t^{t+\tau} (u'(\tilde{c}_{t+1}), u(\tilde{c}_{t+\tau})) < 0$  (see Lehman, 1966, for instance). From Equation (68), we get  $u'(c_t) + \beta R \left. \frac{\partial \tilde{E}_t^{e^{W_{t+1}^k}} [W_{t+1,w}^k]}{\partial k} \right|_{s_t} < 0$  meaning that  $\frac{\partial s_t}{\partial k} > 0$ .

## B.7 Proof of Proposition 7

**The risk free rate.** Define:

$$W_t^k = -kU_t^k = -ku(c_t) + \frac{\beta}{k} \log E_t [e^{W_{t+1}^k}]. \quad (69)$$

Taking the derivative of the equation (31) that defines  $R_t$ , we obtain:

$$-\frac{(1-\beta)u'(c_t)\partial R_t}{\beta R_t^2 \partial k} = E_t \left[ \frac{u'(\tilde{c}_{t+1})W_{t+1,k}^k e^{W_{t+1}^k}}{E_t [e^{W_{t+1}^k}]} \right] - E_t \left[ \frac{u'(\tilde{c}_{t+1})e^{W_{t+1}^k}}{E_t [e^{W_{t+1}^k}]} \right] E_t \left[ \frac{W_{t+1,k}^k e^{W_{t+1}^k}}{E_t [e^{W_{t+1}^k}]} \right], \quad (70)$$

where  $W_{t+1,k}$  is the derivative of  $W_{t+1}^k$  with respect to  $k$ . By taking the derivative of (69) we obtain that the  $W_t^k$  fulfill a recursive equation that writes exactly as (63). We can then follow the proof of Proposition 6 to obtain an expression identical to (67) and eventually:

$$\frac{(1-\beta)u'(c_t)}{\beta R_t^2} \frac{\partial R_t}{\partial k} = \sum_{\tau=1}^{\infty} \beta^{\tau-1} \widehat{cov}_t^{t+\tau} (u'(\tilde{c}_{t+1}), u(\tilde{c}_{t+\tau})), \quad (71)$$

where  $\widehat{cov}_t^{t+\tau}(\cdot, \cdot)$  is the covariance operator associated to  $\widehat{E}_t^{t+\tau}[\cdot]$ , as in the proof of Proposition 6. Since the consumption process is stochastically monotone, we deduce that the covariance in (71) is negative and  $\frac{\partial R_t}{\partial k} < 0$ .

**The market price of risk.** The derivation of the expression (32) of the market price of risk yields:

$$\begin{aligned} E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right] \sigma_t \frac{\partial \sigma_t}{\partial k} &= E_t \left[ W_{t+1,k}^k \left( u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right) \frac{\left( u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right)}{E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right]} \right] \\ &\quad - E_t \left[ W_{t+1,k}^k \frac{u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}}{E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right]} \right] E_t \left[ \left( u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right) \frac{u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}}{E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right]} \right]. \end{aligned}$$

Using again that an expression identical to (67) holds, we have:

$$\begin{aligned} E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right] \sigma_t \frac{\partial \sigma_t}{\partial k} &= - \sum_{\tau=1}^{\infty} \beta^{\tau-1} E_t \left[ \widehat{E}_{t+1}^{t+\tau} \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} u(\tilde{c}_{t+\tau}) \right] \frac{u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}}{E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right]} \right] \\ &\quad + \sum_{\tau=1}^{\infty} \beta^{\tau-1} E_t \left[ \widehat{E}_{t+1}^{t+\tau} \left[ u(\tilde{c}_{t+\tau}) \right] \frac{u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}}{E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right]} \right] E_t \left[ \widehat{E}_{t+1}^{t+\tau} \left[ u'(\tilde{c}_{t+1}) \right] \frac{u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}}{E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right]} \right]. \end{aligned} \quad (72)$$

We now define the expectation operator  $\overline{E}_t^{t+\tau}[X] = E_t \left[ \widehat{E}_{t+1}^{t+\tau} \left[ X \frac{u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}}{E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right]} \right] \right]$ . Defining the associated covariance  $\overline{cov}_t^{t+\tau}(\cdot, \cdot)$ , equation (72) becomes:

$$E_t \left[ u'(\tilde{c}_{t+1}) e^{W_{t+1}^k} \right] \sigma_t \frac{\partial \sigma_t}{\partial k} = - \sum_{\tau=1}^{\infty} \beta^{\tau-1} \overline{cov}_t^{t+\tau} \left( u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}, u(\tilde{c}_{t+\tau}) \right). \quad (73)$$

As consumption follows a stochastically monotone process, we can prove, like in Lemma 9, that  $W_{t+1}^k$  and  $\tilde{c}_{t+1}$  are anticomotone. Thus  $u'(\tilde{c}_{t+1}) e^{W_{t+1}^k}$  and  $\tilde{c}_{t+1}$  are anticomotone. Using again that consumption follows a stochastically monotone process, we eventually deduce that the covariance in (73) is negative and  $\sigma_t \frac{\partial \sigma_t}{\partial k} > 0$ .