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Pathwise superhedging on prediction sets

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Abstract

In this paper we provide a pricing-hedging duality for the model-independent superhedging price with respect to a prediction set $\Xi \subseteq C[0,T]$, where the superhedging property needs to hold pathwise, but only for paths lying in Ξ. For any upper-semicontinuous claim ξ which is bounded from below, the superhedging price coincides with the supremum over all pricing functionals $\mathbb{E}_{\mathbb{Q}}[\xi]$ with respect to martingale measures $\mathbb{Q}$ concentrated on the prediction set Ξ. This allows to include beliefs in future paths of the price process expressed by the set Ξ, while eliminating all those which are seen as impossible. Moreover, we provide several examples to justify our setup.

Keywords Model-Independent Superhedging; Pricing-Hedging Duality; Modeling Beliefs
AMS 2010 Subject Classification 91B24; 91G20; 60G44

1 Introduction

In this paper we study the problem of pathwise superhedging on a prediction set $\Xi \subseteq C[0,T]$ of continuous price paths, i.e. finding a predictable trading strategy which super-replicates a given contingent claim $\xi : C[0,T] \to \mathbb{R}$ simultaneously for all possible future price paths in Ξ.

Unlike the famous Black-Scholes model, most financial models cannot exactly replicate every contingent claim. This phenomenon, called incompleteness of the market is equivalent to the failure of uniqueness of equivalent local martingale measures. Since there is not an unique price which financial agents are willing to accept, the concept of superhedging starting with [16] has been well established in the financial literature. Here one wants to find the smallest initial capital for which a trading strategy exists which superhedges the claim ξ.

To be more precise, in classical finance, one assigns probabilities to all events by fixing a probability measure $\mathbb{P}$. Then, the superhedging property is required to hold $\mathbb{P}$-a.s.

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Recently, motivated by the early works of [17, 10], one started to consider a set of probability measures $\mathcal{P}$, rather than an unique one, where each element represents the candidates for the possible right law. In the so-called *quasi-sure setting*, one then requires the superhedging property to hold true $\mathcal{P}$-quasi surely, which means $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$. This problem under volatility uncertainty was motivated by the early works of [2, 19] and later has also been solved in [7, 28, 22, 27].

In the *model-independent* (or *pathwise*) approach, one wants to go away from the classical assumption of assigning probabilities to events related to the financial market by fixing one probability measure, or a set of probability measures allowing for model ambiguity. In this setting, superhedging is required to hold true for every possible future path in $C[0,T]$ of the price process. Such an approach has started with the seminal work [13] and has been recently lead to attention in various other works; we refer to [1, 3, 5, 9], to name but a few.

However, it turns out that the concept of superhedging is too robust leading to too high prices. In fact, for stochastic volatility or rough volatility models, it turns out that the classical superhedging price coincides with the model-independent one and is so high that for Markovian payoffs of the form $\Gamma(S_T)$, like e.g. the European Call and Put option, the optimal superhedging strategy can be chosen to be of buy-and-hold type, see [8, 21]. To reduce the model-independent superhedging price, inspired by the work of [20], [14] introduced the concept of prediction sets, where agents may allow to exclude paths which they consider to be impossible to model future price paths. Hence they require the superhedging property only to hold true on every path in $\Xi \subseteq C[0,T]$ of their prediction set.

Whereas the pricing-hedging duality is well-understood for the pathwise superhedging with respect to all paths in $C[0,T]$, it turns out that the problem becomes considerably more difficult when requiring the superhedging property only to hold true on the prediction set $\Xi \subseteq C[0,T]$. To illustrate the difficulty, consider the examples where the agent may believe in the Black-Scholes model, or is uncertain about the volatility like in the G-expectation (see [25]) and hence models his/her beliefs by requiring

$$\Xi_{BS} := \big\{\omega \in C_0[0,T] : \, d\langle\omega\rangle_t = \sigma^2\omega(t)^2\, dt\big\}, \quad \text{or} \quad \Xi_G := \Big\{\omega \in C_0[0,T] : \, \frac{d\langle\omega\rangle_t}{dt} \in [\underline{\sigma}^2, \overline{\sigma}^2]\Big\}.$$

Observe that these sets are neither closed, nor σ-compact. In [14], they get an asymptotic pricing-hedging duality result. More precisely, the asymptotic price being defined as the limit (when $\varepsilon \to 0$) of superhedging prices on ε-varied prediction sets Ξ^ε turns out to coincide with the limit of the supremum of the pricing functionals with respect to the martingale measures having support on the ε-varied prediction sets Ξ^ε. However, typically, Ξ is not closed and hence Ξ^ε might be far away from the original set Ξ. Indeed, one can show that in the canonical example of the paths Ξ_{BS} of the Black-Scholes model, $\Xi^\varepsilon = C[0,T]$ for any $\varepsilon > 0$.

In [3], they obtain a superhedging duality with respect to a prediction set $\Xi \subseteq C[0,T]$, where the superhedging price coincides with the supremum over all pricing functionals with respect to martingale measures concentrated on the prediction set Ξ. As trading strategies, they use simple strategies and define the gain process to be the limit inferior of the discrete integral with respect to the simple strategies; we refer to [26, 4, 26, 30, 31] which also applied this setup in the context of superhedging. However, they need to impose the crucial assumption that the prediction set Ξ is σ-compact in a topology which is at least as fine as the usual sup-norm; a property which is in general not satisfied in the examples of paths motivated by financial applications, like e.g. Ξ_{BS} or Ξ_G.

In this paper, we extend the work of [14, 3]. We do not require any topological properties on Ξ such that our pricing-hedging duality also covers e.g., Ξ_{BS} or Ξ_G as examples. We stick to the formulation of [3] for the superhedging price. Next to the stock price, we allow to trade also in the iterated integral $\int S\,dS$, which seems to be a bit unnatural in the context of superhedging at first glance. However, when relaxing the notion of trading strategies by allowing a double approximation, we regain the classical formulation of superhedging; see Remark 2.3.

To remove the topological assumption on the prediction set Ξ, we lift the superhedging problem to the product space $\overline{\Omega} := \Omega \times \Omega$, where the first coordinate represents the original price process and the second one represents its quadratic variation; we refer to [12, 29, 18] where similar enlarged spaces $\overline{\Omega}$ were considered. On the enlarged space $\overline{\Omega}$, we then prove the desired superhedging duality and can then conclude the desired result on the original space. This works, roughly speaking, by observing that for any probability measure $\overline{\mathbb{Q}}$ on the enlarged space for which the first coordinate is a local martingale with the second one as its quadratic variation, the $\overline{\mathbb{Q}}$-completed natural filtration coincides with the $\overline{\mathbb{Q}}$-completed one generated only by the first coordinate. This then leads to a one-to-one correspondence to the original space. We point out that our trading strategies are defined with respect to the (right-continuous) natural filtration, without any completion with respect to a probability measure, such that we retain the framework of pathwise superhedging without any probabilistic beliefs.

The remainder of this paper is organized as follows. In Section 2, we introduce the setup and state the main theorem of this paper. Then in Section 3, we provide several examples to motivate our result. In Section 4, we provide the proof of the main result. Finally, in Section 5, we attach some technical results required in the proof of our main result.

2 Setup and Main Results

Fix a finite time horizon $T \in (0,\infty)$, and let $C[0,T]$ be the space of all continuous paths $\omega : [0,T] \to \mathbb{R}$, which as usual is endowed with the sup-norm $\|\omega\|_\infty := \sup_{0\le t\le T} |\omega(t)|$. Denote by $S = (S_t)_{0\le t\le T}$ the canonical process $S_t(\omega) = \omega(t)$, and define for each $m \ge 1$ the sequence $\sigma_0^m := 0$,

$$\sigma_{k+1}^m := \inf\left\{t \ge \sigma_k^m : |S_t - S_{\sigma_k^m}| \ge 2^{-m}\right\}, \quad k \ge 0.$$

Since S has continuous paths, $\lim_k \sigma_k^m(\omega) = \infty$ holds for all $\omega \in C[0,T]$. Moreover, let $M : [0,T] \times C[0,T] \to \mathbb{R}$ be the processes $M_t(\omega) := \liminf_{m\to\infty} M_t^m(\omega)$, where

$$M_t^m := \sum_{k=0}^{\infty} S_{\sigma_k^m}\big(S_{\sigma_{k+1}^m\wedge t} - S_{\sigma_k^m\wedge t}\big).$$

Define a pathwise quadratic variation $\langle\cdot\rangle : C[0,T] \to C[0,T]$ by

$$\langle\omega\rangle := \begin{cases} S^2(\omega) - 2M(\omega) & \text{if } \omega \in \Omega \\ 0 & \text{else,} \end{cases}$$

where Ω is the Borel set of all $\omega \in C[0,T]$ such that $M^m(\omega) \to M(\omega)$ in the sup-norm and $S^2(\omega) - 2M(\omega)$ is nondecreasing.

Remark 2.1. The construction of the pathwise quadratic variation $\langle\cdot\rangle$ is similar to [23] and goes back to [15, 24]. For every Borel probability measure $\mathbb{Q}$ under which the canonical process is a semimartingale in the raw filtration, it is a consequence of the Burkholder-Davis-Gundy inequalities that

$$\sup_{0\le t\le T} \left| M_t^m - {}^{(\mathbb{Q})}\!\!\int_0^t S_s \, dS_s \right| \to 0 \quad \mathbb{Q}\text{-a.s..}$$

Hence, $\langle S\rangle = S^2 - 2\,{}^{(\mathbb{Q})}\!\!\int S\,dS = \langle S\rangle^{(\mathbb{Q})}$ holds $\mathbb{Q}$-a.s. as an application of the integration-by-parts formula for the Itô integral. In particular, $\mathbb{Q}(\Omega) = 1$. Notice that M_t coincides with Föllmer's pathwise stochastic integral [11] of $\int_0^t S\,dS$ on Ω.

The space Ω is endowed with the relative topology and equipped with the corresponding relative Borel σ-field $\mathcal{F}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the raw filtration generated by the canonical process S on Ω, i.e. $\mathcal{F}_t = \sigma(S_s, s \le t)$, and by $\mathbb{F}_+$ its right-continuous version, $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_{s\wedge T}$, for each t. Let $\mathcal{H}$ be the set of all simple processes $H\colon [0,T]\times\Omega \to \mathbb{R}$ of the form

$$H_t(\omega) = \sum_{l=1}^{L} h_l(\omega) 1_{(\tau_l(\omega),\tau_{l+1}(\omega)]}(t)$$

for $(t,\omega) \in [0,T]\times\Omega$, where $L \in \mathbb{N}$, $0 \le \tau_1 \le \cdots \le \tau_{L+1} \le T$ are stopping times w.r.t. the filtration $\mathbb{F}_+$, and $h_l\colon \Omega \to \mathbb{R}$ are bounded $\mathcal{F}_{\tau_l+}$-measurable functions. For a simple predictable $H \in \mathcal{H}$ the pathwise stochastic integral

$$(H\cdot S)_t(\omega) := \sum_{l=1}^{L} h_l(\omega)(S_{\tau_{l+1}(\omega)\wedge t}(\omega) - S_{\tau_l(\omega)\wedge t}(\omega))$$

is well-defined for all $t \in [0,T]$ and all $\omega \in \Omega$. Similarly the integral $(H\cdot M)_t$ is well-defined.

In the following we fix a function

$$\overline{Z} : C[0,T]\times C[0,T] \to [1,\infty]$$

with compact sublevel sets, such that $\overline{Z}(\omega,\nu) \ge \|\omega\|_\infty + \|\nu\|_\infty$ for all (ω,ν), and ν is nondecreasing with $\nu(0) = 0$ whenever $\overline{Z}(\omega,\nu) < \infty$. Moreover, define $Z := \overline{Z}\circ\psi$ on Ω, for the Borel mapping

$$\psi\colon \Omega \to C[0,T]\times C[0,T], \quad \omega \mapsto (\omega, \langle\omega\rangle).$$

Our goal is to study superhedging for a given (measurable) prediction set $\Xi \subseteq \Omega$ of price paths. Let $\mathfrak{P}(\Omega)$ be the set of all Borel probability measures on Ω, and denote by

$$\mathcal{M}_Z(\Xi) := \{\mathbb{Q} \in \mathfrak{P}(\Omega) \colon S \text{ is } \mathbb{Q}\text{-}\mathbb{F}\text{-martingale}, \mathbb{E}_{\mathbb{Q}}[Z] < \infty \text{ and } \mathbb{Q}(\Xi) = 1\}$$

the set of all martingale measures on Ξ which integrate Z.

The following is our main result.

Theorem 2.2. *Suppose $\Xi = \{\omega \in \Omega : Z(\omega) < \infty\}$ and $\mathcal{M}_Z(\Xi)$ is nonempty. Then, for every upper semicontinuous $\xi \colon C[0,T] \to (-\infty, \infty]$ which is bounded from below*

$$\begin{aligned}\phi(\xi) &:= \inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \textit{there is } c \geq 0 \textit{ and sequences } (H^n), (G^n) \textit{ in } \mathcal{H} \textit{ so that} \\ \textit{for all } n, (H^n \cdot S)_T + (G^n \cdot M)_T \geq -cZ \textit{ on } \Xi \\ \textit{and } \lambda + \liminf_{n\to\infty} \big((H^n \cdot S)_T + (G^n \cdot M)_T\big) \geq \xi \textit{ on } \Xi \end{array} \right\} \\ &= \sup_{\mathbb{Q} \in \mathcal{M}_Z(\Xi)} \mathbb{E}_{\mathbb{Q}}[\xi].\end{aligned}$$

Moreover, the infimum is attained whenever $\phi(\xi) < \infty$.

Remark 2.3. [Double approximation] Given a sequence (G^n) of simple predictable integrands of the form $G^n = \sum_{l=1}^{L^n} g_l^n 1_{(\tau_l^n, \tau_{l+1}^n]}$ it follows that

$$G^n \cdot M^m = \widetilde{G}^{m,n} \cdot S$$

for the integrands $\widetilde{G}^{m,n} = \sum_{k=0}^{\infty} \sum_{l=1}^{L^n} S_{\sigma_k^m} g_l^n 1_{(\sigma_k^m \vee \tau_l^n, \sigma_{k+1}^m \wedge \tau_{l+1}^n]}$. Since $M_T^m \to M_T$ we have $\lim_{m\to\infty} (\widetilde{G}^{m,n} \cdot S)_T = ((G^n S) \cdot S)_T$ where the latter can be defined pathwise, see Remark 2.1. This shows

$$(G^n \cdot M)_T = \lim_{m\to\infty} (\widetilde{G}^{m,n} \cdot S)_T = ((G^n S) \cdot S)_T$$

and therefore for integrands $\widetilde{H}^n := H^n + G^n S$,

$$\begin{aligned}\liminf_{n\to\infty} \big((H^n \cdot S)_T + (G^n \cdot M)_T\big) &= \liminf_{n\to\infty} \lim_{m\to\infty} \big((H^n + \widetilde{G}^{m,n}) \cdot S\big)_T \\ &= \liminf_{n\to\infty} \big((H^n + G^n S) \cdot S\big)_T \\ &= \liminf_{n\to\infty} \big(\widetilde{H}^n \cdot S\big)_T.\end{aligned}$$

Remark 2.4. Our result and its proof is stated on $\mathbb{R}$ for the sake of readability, however we point out that by the same arguments, it remains valid for a d-dimensional price process.

Remark 2.5. At first glance, it seems to be unintuitive how to obtain the functions $\overline{Z}$ or Z. By providing several examples motivated by applications in finance, we will see in Section 3 that in fact one can define $\overline{Z}$ and then also Z in a natural way in regard to the prediction set Ξ which is of interest.

3 Examples

In this section we provide several examples of prediction sets and explain the role of the growth function Z. For notational convenience, probability measures $\mathbb{P} \in \mathfrak{P}(\Omega)$ are viewed as probability measures $\mathbb{P} \in \mathfrak{P}(C[0,T])$ without changing notation; this is possible since $\Omega \subseteq C[0,T]$ is Borel. In the following consider the space of all Hölder continuous functions,

$$C^{\text{Hölder}}[0,T] = \bigcup_{n\in\mathbb{N}} \left\{\omega \in C[0,T] : \|\omega\|_{\frac{1}{n}} \leq n\right\},$$

where, for $\alpha \in (0,1]$, the α-Hölder norm is given by $\|\omega\|_\alpha := |\omega(0)| + \sup_{s\neq t} \frac{|\omega(t)-\omega(s)|}{|t-s|^\alpha}$. Notice that most of the usually used financial models $\mathbb{P} \in \mathfrak{P}(\Omega)$ are concentrated on $C^{\text{Hölder}}[0,T]$, i.e. $\mathbb{P}(C^{\text{Hölder}}[0,T]) = 1$; see [3, Section 3] for a discussion.

Remark 3.1. Let $\mathcal{P} \subseteq \mathfrak{P}(C[0,T])$ be a set of probability measures such that

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}\left[|S_t - S_s|^4\right] \leq M|t-s|^2 \quad \text{for all } s,t \in [0,T] \tag{3.1}$$

and $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[S_0^4] \leq M$ for some constant $M > 0$. Then there exists a function ζ from $C[0,T]$ to $[1,\infty]$ with compact level sets such that $\{\zeta < \infty\} = C^{\text{Hölder}}[0,T]$ and $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[\zeta] < \infty$. Indeed, by a version of Kolmogorov's continuity criterion (see e.g. [3, Theorem A.1]), for any $\alpha \in (0,1/4)$ there is a constant C such that $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[|S|_\alpha^4] \leq C$, where $|\omega|_\alpha := \sup_{s\neq t} |\omega(t)-\omega(s)|/|t-s|^\alpha$. Using the elementary inequality $(a+b)^4 \leq 8a^4 + 8b^4$ as well as $|\cdot|_{1/n} \leq (1 \vee T^\alpha)|\cdot|_\alpha$ for $n \geq 1/\alpha$, it follows that $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[\|S\|_{1/n}^4] \leq C'$ for all $n \geq 1/\alpha$ and a new constant C'. Therefore, by Markov's inequality, there is a sequence (a_n) which increases to ∞ such that $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(\|S\|_{1/n} > a_n) \leq 1/n^3$. Then the function

$$\zeta(\omega) := \inf\{n \geq 1/\alpha : \|\omega\|_{1/(n+1)} \leq a_{n+1}\} \quad \text{for } \omega \in C[0,T]$$

has compact level sets by the Arzelà-Ascoli theorem (see also [3, Lemma 3.1]) and satisfies $\{\zeta < \infty\} = C^{\text{Hölder}}[0,T]$. Moreover, as $\{\zeta = n\} \subseteq \{\|S\|_{1/n} > a_n\}$ and the latter set has probability less than n^{-3} under all $\mathbb{P} \in \mathcal{P}$, one gets $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[\zeta] < \infty$.

Notice that as $\|\omega\|_\infty \leq |\omega(0)| + T^\alpha\|\omega\|_\alpha$ the assumptions yield $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[\|\cdot\|_\infty] < \infty$. Therefore, possibly replacing ζ by $\zeta + \|\cdot\|_\infty$, one may assume that $\zeta \geq \|\cdot\|_\infty$.

Example 3.2 (Brownian motion)**.** Let $\mathbb{P} \in \mathfrak{P}(\Omega)$ be the Wiener measure, i.e. S is a standard Brownian motion under $\mathbb{P}$. For $\mathcal{P} := \{\mathbb{P}\}$ it follows from Lévy's theorem that

$$\mathcal{P} = \{\mathbb{Q} \in \mathfrak{P}(\Omega) : S \text{ is } \mathbb{Q}\text{-}\mathbb{F}\text{-martingale and } \mathbb{Q}((S,\langle S\rangle) \in \overline{\mathfrak{G}}) = 1\},$$

where

$$\overline{\mathfrak{G}} := \{(\omega,\nu) \in C[0,T] \times C[0,T] : \omega(0) = 0 \text{ and } \nu(t) = t \text{ for all } t \in [0,T]\}.$$

We claim that $\mathcal{P} = \mathcal{M}_Z(\Xi)$ for a suitable function Z and $\Xi := \{Z < \infty\}$. By the Burkholder-Davis-Gundy inequality

$$\mathbb{E}_{\mathbb{P}}[|S_t - S_s|^4] \leq M\mathbb{E}_{\mathbb{P}}\Big[\Big(\int_s^t 1\, d\langle S\rangle\Big)^2\Big] = M|t-s|^2, \tag{3.2}$$

for a constant M, so that Remark 3.1 guarantees the existence of a function $\zeta\colon C[0,T] \to [1,\infty]$ with compact level sets such that $\{\zeta < \infty\} = C^{\text{Hölder}}[0,T]$ and $\mathbb{E}_{\mathbb{P}}[\zeta] < \infty$ as well as $\zeta \geq \|\cdot\|_\infty$. Define

$$\overline{Z}(\omega,\nu) := \zeta(\omega) + \|\nu\|_\infty + \infty\, 1_{\overline{\mathfrak{G}}^c}(\omega,\nu) \quad \text{for } (\omega,\nu) \in C[0,T] \times C[0,T].$$

Since $\psi(S) = (S, \langle S\rangle^{\mathbb{P}})$ $\mathbb{P}$-almost surely by Remark 2.1 and Z is defined as $Z := \overline{Z} \circ \psi$, it follows that $\mathbb{E}_{\mathbb{P}}[Z] < \infty$. In particular $\mathbb{P}(\Xi) = \mathbb{P}(Z < \infty) = 1$, where

$$\begin{aligned} \Xi := \{Z < \infty\} &= \big\{\omega \in C^{\text{Hölder}}[0,T] :\ \omega(0) = 0 \text{ and } (\omega, \langle\omega\rangle) \in \overline{\mathfrak{G}}\big\} \\ &= \big\{\omega \in C^{\text{Hölder}}[0,T] :\ \omega(0) = 0 \text{ and } \langle\omega\rangle_t = t \text{ for all } t \in [0,T]\big\} \end{aligned}$$

so that $\mathbb{P} \in \mathcal{M}_Z(\Xi)$ by definition. On the other hand, if $\mathbb{Q} \in \mathcal{M}_Z(\Xi)$, then $\mathbb{E}_{\mathbb{Q}}[Z] < \infty$ by definition which implies $\mathbb{Q}((S,\langle S\rangle) \in \overline{\mathfrak{G}}) = 1$ and therefore $\mathbb{Q} \in \mathcal{P}$. Note that $\overline{Z}$ satisfies all assumptions of Theorem 2.2.

Example 3.3 (G-Brownian motion)**.** Fix two real numbers $0 < \underline{\sigma} < \overline{\sigma}$ and consider the set

$$\mathcal{P} := \{\mathbb{Q} \in \mathfrak{P}(\Omega) : S \text{ is } \mathbb{Q}\text{-}\mathbb{F}\text{-martingale and } \mathbb{Q}((S,\langle S\rangle) \in \overline{\mathfrak{G}}) = 1\}$$

where

$$\overline{\mathfrak{G}} := \left\{(\omega,\nu) \in C[0,T] \times C[0,T] :\ \begin{array}{l} \omega(0) = \nu(0) = 0 \text{ and } \nu \text{ is nondecreasing} \\ \text{with } d\nu \ll dt \text{ and } d\nu/dt \in [\underline{\sigma}^2, \overline{\sigma}^2] \end{array}\right\}.$$

Using the Burkholder-Davis-Gundy inequality as in (3.2), one shows that $\mathcal{P}$ satisfies (3.1). Therefore, by the same arguments as in Example 3.2, one has $\mathcal{P} = \mathcal{M}_Z(\Xi)$ for $\overline{Z}(\omega,\nu) := \zeta(\omega) + \|\nu\|_\infty + \infty\, 1_{\overline{\mathfrak{G}}^c}(\omega,\nu)$, $Z := \overline{Z} \circ \psi$ on Ω, and

$$\begin{aligned} \Xi := \{Z < \infty\} &= \big\{\omega \in C^{\text{Hölder}}[0,T] :\ \omega(0) = 0 \text{ and } (\omega, \langle\omega\rangle) \in \overline{\mathfrak{G}}\big\} \\ &= \big\{\omega \in C^{\text{Hölder}}[0,T] :\ \omega(0) = 0 \text{ and } d\langle\omega\rangle_t/dt \in [\underline{\sigma}^2, \overline{\sigma}^2]\big\}. \end{aligned}$$

Further, as every Lipschitz continuous function is absolutely continuous, one has

$$\overline{\mathfrak{G}} = \left\{(\omega,\nu) \in C[0,T] \times C[0,T] :\ \begin{array}{l} \omega(0) = \nu(0) = 0 \text{ and for } 0 \leq s < t \leq T \\ \underline{\sigma}^2 \leq (\nu(t) - \nu(s))/(t-s) \leq \overline{\sigma}^2 \end{array}\right\}$$

showing that $\overline{\mathfrak{G}}$ is closed. In fact, the projection of $\overline{\mathfrak{G}}$ on the second component is compact (e.g. by the Arzelà-Ascoli theorem), so that $\overline{Z}$ has compact level set.

Example 3.4 (Stopped G-Brownian motion)**.** Let $\mathcal{P}$, $\overline{\mathfrak{G}}$, and $\overline{Z}$ be the sets and function of Example 3.3, where we may assume without loss of generality that ζ is of the form $\zeta(\omega) = \inf\{n : \|\omega\|_{1/n+1} \leq a_{n+1}\} + \|\omega\|_\infty$; see Remark 3.1. For an $\mathbb{F}_+$-stopping time τ and $\mathbb{P} \in \mathcal{P}$, denote by $\mathbb{P}^\tau$ the law of the stopped process $S^\tau_t := S_{\tau\wedge t}$. Then

$$\mathcal{P}_{ST} := \{\mathbb{P}^\tau : \mathbb{P} \in \mathcal{P} \text{ and } \tau \text{ is } \mathbb{F}_+\text{-stopping time}\} = \mathcal{M}_{Z_{ST}}(\Xi)$$

for the function $\overline{Z}_{ST}(\omega, \nu) := \zeta(\omega) + \|\nu\|_\infty + \infty\, 1_{\overline{\mathfrak{G}}^c_{ST}}(\nu)$ with

$$\overline{\mathfrak{G}}_{ST} := \{(\omega(s \wedge \cdot), \nu(s \wedge \cdot)) : (\omega, \nu) \in \overline{\mathfrak{G}} \text{ and } s \in [0, T]\}.$$

Indeed, for $\mathbb{Q} := \mathbb{P}^\tau \in \mathcal{P}_{ST}$, as $\zeta(\omega(s \wedge \cdot)) \leq \zeta(\omega)$ and thus $\overline{Z}_{ST}(\omega(s\wedge \cdot), \nu(s \wedge \cdot)) \leq \overline{Z}(\omega, \nu)$ for all (ω, ν) and s, it follows that $\mathbb{E}_\mathbb{Q}[Z_{ST}] \leq \mathbb{E}_\mathbb{P}[Z]$ and so $\mathbb{Q} \in \mathcal{M}_{Z_{ST}}(\Xi)$. On the other hand, let $\mathbb{Q} \in \mathcal{M}_{Z_{ST}}(\Xi)$ and define the $\mathbb{F}_+$-stopping time τ as the minimal time such that $t \mapsto \langle S\rangle^\mathbb{Q}_t$ is constant afterwards. This is indeed an $\mathbb{F}_+$-stopping time as $\langle S\rangle^\mathbb{Q}$ is strictly increasing till it remains constant as $\underline{\sigma} > 0$, see the definition of $\overline{\mathfrak{G}}_{ST}$. Taking $\mathbb{P}$ to be the measure which equals $\mathbb{Q}$ up to time τ and the law of the Brownian motion with volatility $\underline{\sigma}$ after τ, one has $\mathbb{P} \in \mathcal{P}$ and $\mathbb{Q} = \mathbb{P}^\tau$ by definition. Thus $\mathbb{Q} \in \mathcal{P}_{ST}$.

Example 3.5 (Black-Scholes)**.** Fix $\sigma > 0$ and let $\mathbb{P} \in \mathfrak{P}(\Omega)$ be the measure under which S has the dynamics $dS = \sigma S dB$ with $S_0 = s_0 \in \mathbb{R}$ fixed, where B is a Brownian motion. Then

$$\mathcal{P} := \{\mathbb{P}\} = \big\{\mathbb{Q} \in \mathfrak{P}(\Omega) : S \text{ is } \mathbb{Q}\text{-}\mathbb{F}\text{-martingale and } \mathbb{Q}((S, \langle S\rangle) \in \overline{\mathfrak{G}}) = 1\big\}$$

for the set

$$\overline{\mathfrak{G}} := \left\{(\omega, \nu) \in C[0, T] \times C[0, T] : \omega(0) = s_0 \text{ and } \nu(\cdot) = \int_0^\cdot \sigma^2 \omega(t)^2\, dt\right\}.$$

Similar arguments as before (see e.g. [3, Example 3.1.3]) yield the existence of a function $\zeta\colon C[0, T] \to [1, \infty]$ growing faster than $\|\cdot\|_\infty$ with compact level sets such that $\mathbb{E}_\mathbb{P}[\zeta] < \infty$ and $\{\zeta < \infty\} = C^{\text{Hölder}}[0, T]$. Now, using the same arguments as in Example 3.2, it follows that $\mathcal{P} = \mathcal{M}_Z(\Xi)$ for $\overline{Z}(\omega, \nu) := \zeta(\omega) + \|\nu\|_\infty + \infty\, 1_{\overline{\mathfrak{G}}^c}(\omega, \nu)$ and $\Xi := \{Z < \infty\}$. Moreover, as $\overline{\mathfrak{G}}$ is closed, one can show that $\overline{Z}$ has compact level sets and in turn satisfies all assumptions of Theorem 2.2.

4 Proof of Theorem 2.2

The idea of the proof is to lift and prove the main result in a suitable enlarged space and then conclude in the original space. Let us start with introducing the enlarged space

$$\overline{\Omega} := C[0, T] \times C[0, T],$$

which is endowed with the sup-norm $\|\overline{\omega}\|_\infty := \|\omega\|_\infty + \|\nu\|_\infty$ for $\overline{\omega} = (\omega, \nu) \in \overline{\Omega}$, and equipped with its Borel σ-field $\overline{\mathcal{F}}$. Denote by $\mathfrak{P}(\overline{\Omega})$ the set of all probability measures on $(\overline{\Omega}, \overline{\mathcal{F}})$. Recall the Borel mapping $\psi : \Omega \to \overline{\Omega}$, $\omega \mapsto (\omega, \langle\omega\rangle)$, and define

$$\overline{\Delta} := \psi(\Omega) = \{(\omega, \nu) \in \overline{\Omega} : \omega \in \Omega \text{ and } \langle\omega\rangle = \nu\}.$$

Moreover, we write $\overline{\Delta}^c$ for the complement of the set $\overline{\Delta}$. On $\overline{\Omega}$ we consider the canonical process $(\overline{S}, \overline{V})$ given by $\overline{S}_t(\overline{\omega}) = \omega(t)$ and $\overline{V}_t(\overline{\omega}) = \nu(t)$ for $\overline{\omega} = (\omega, \nu) \in \overline{\Omega}$. Denote by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0 \le t \le T}$ the raw filtration generated by $(\overline{S}, \overline{V})$, by $\overline{\mathbb{F}}_+$ its right-continuous version, and define the corresponding $\overline{\Delta}^c$-augmentations

$$\overline{\mathbb{F}}^{\overline{\Delta}} := \overline{\mathbb{F}} \vee \{\overline{N} \subseteq \overline{\Omega} : \overline{N} \subseteq \overline{\Delta}^c\} \quad \text{and} \quad \overline{\mathbb{F}}_+^{\overline{\Delta}} := \overline{\mathbb{F}}_+ \vee \{\overline{N} \subseteq \overline{\Omega} : \overline{N} \subseteq \overline{\Delta}^c\}.$$

Then

$$\overline{\mathcal{F}}_t^{\overline{\Delta}} = \{(\overline{A} \cap \overline{\Delta}) \cup \overline{N} : \overline{A} \in \overline{\mathcal{F}}_t,\, \overline{N} \subseteq \overline{\Delta}^c\}, \quad \text{and} \quad \overline{\mathcal{F}}_{t+}^{\overline{\Delta}} = \{(\overline{A} \cap \overline{\Delta}) \cup \overline{N} : \overline{A} \in \overline{\mathcal{F}}_{t+},\, \overline{N} \subseteq \overline{\Delta}^c\}.$$

Lemma 4.1. *For every $t \in [0, T]$ one has*

(a) $\overline{\mathcal{F}}_{t+}^{\overline{\Delta}} = \bigcap_{s>t} \overline{\mathcal{F}}_{s\wedge T}^{\overline{\Delta}}$,

(b) $\{\overline{\omega} \in \overline{\Omega} : \omega \in A\} \in \overline{\mathcal{F}}_t^{\overline{\Delta}}$ *for all* $A \in \mathcal{F}_t$,

(c) $\{\overline{\omega} \in \overline{\Omega} : \omega \in A\} \in \overline{\mathcal{F}}_{t+}^{\overline{\Delta}}$ *for all* $A \in \mathcal{F}_{t+}$,

(d) $\psi^{-1}(\overline{A}) \in \mathcal{F}_t$ *for all* $\overline{A} \in \overline{\mathcal{F}}_t^{\overline{\Delta}}$,

(e) $\psi^{-1}(\overline{A}) \in \mathcal{F}_{t+}$ *for all* $\overline{A} \in \overline{\mathcal{F}}_{t+}^{\overline{\Delta}}$.

Proof. (a) By definition, we have $\overline{\mathcal{F}}_{t+}^{\overline{\Delta}} \subseteq \bigcap_{s>t} \overline{\mathcal{F}}_{s\wedge T}^{\overline{\Delta}}$. For the reverse inclusion, fix $t \in [0, T]$ and $\overline{B} \in \bigcap_{s>t} \overline{\mathcal{F}}_{s\wedge T}^{\overline{\Delta}}$. Then, for every $s > t$, there exist $\overline{A}_s \in \overline{\mathcal{F}}_s$ and $\overline{N}_s \subseteq \overline{\Delta}^c$ such that

$$\overline{B} = (\overline{B} \cap \overline{\Delta}) \uplus (\overline{B} \cap \overline{\Delta}^c) = (\overline{A}_s \cap \overline{\Delta}) \uplus \overline{N}_s.$$

This implies for each $s > t$ that $\overline{A}_s \cap \overline{\Delta} = \overline{B} \cap \overline{\Delta}$, as well as $\overline{N}_s = \overline{B} \cap \overline{\Delta}^c$. Define

$$\overline{A} := \bigcap_{s>t,\, s\in\mathbb{Q}} \; \bigcup_{t<r<s,\, r\in\mathbb{Q}} \overline{A}_r \quad \in \overline{\mathcal{F}}_{t+}$$

Then by construction $\overline{B} = (\overline{A} \cap \overline{\Delta}) \uplus (\overline{B} \cap \overline{\Delta}^c)$, proving the reverse inclusion.

(b) Fix $t \in [0, T]$ and $A \in \mathcal{F}_t$, so that

$$A = \big\{\omega \in \Omega : (\omega(s))_{s\in S} \in B\big\}$$

for a countable set $S \subseteq [0,t]$ and a Borel set $B \subseteq \mathbb{R}^S$. Let $\overline{N} := \{\overline{\omega} \in \overline{\Delta}^c : \omega \in A\} \in \overline{\mathcal{F}}_t^{\overline{\Delta}}$. Since $\overline{\omega} \mapsto \omega(s)$ is $\overline{\mathcal{F}}_t$-measurable for all $s \le t$, one has

$$\left\{\overline{\omega} \in \overline{\Omega} : \omega \in A\right\} = \left\{\overline{\omega} \in \overline{\Delta} : \omega \in A\right\} \cup \overline{N} = \left\{\overline{\omega} \in \overline{\Delta} : (\omega(s))_{s \in S} \in B\right\} \cup \overline{N} \in \overline{\mathcal{F}}_t^{\overline{\Delta}}.$$

(c) If $A \in \mathcal{F}_{t+}$ then $A \in \mathcal{F}_{s \wedge T}$ for every $s > t$, so that by (a)

$$\{\overline{\omega} \in \overline{\Omega} : \omega \in A\} \in \bigcap_{s>t} \overline{\mathcal{F}}_{s \wedge T}^{\overline{\Delta}} = \overline{\mathcal{F}}_{t+}^{\overline{\Delta}}.$$

(d) Fix $t \in [0,T]$ and $\overline{A} \in \overline{\mathcal{F}}_t^{\overline{\Delta}}$. Then

$$\overline{A} = \left\{\overline{\omega} \in \overline{\Delta} : (\omega(r), \nu(s))_{(r,s) \in S} \in B\right\} \cup \overline{N}$$

for a countable set $S \subseteq [0,t] \times [0,t]$, a Borel set $B \subseteq \mathbb{R}^S$, and $\overline{N} \subseteq \overline{\Delta}^c$. Since $\psi(\omega) \in \overline{\Delta}$ for every $\omega \in \Omega$, it follows that

$$\psi^{-1}(\overline{A}) = \left\{\omega \in \Omega : (\omega(r), \langle\omega\rangle(s))_{(r,s) \in S} \in B\right\} \in \mathcal{F}_t,$$

since $\omega \mapsto (\omega(r), \langle\omega\rangle(s))$ is $\mathcal{F}_t$-measurable for every $r, s \le t$.

(e) The argumentation is similar to (c). $\square$

Remark 4.2. Let $h : \Omega \to \mathbb{R}$ be $\mathcal{F}_{t+}$-measurable and τ a $\mathbb{F}_+$-stopping time. It follows from Lemma 4.1 that $\overline{h}(\overline{\omega}) := h(\omega) 1_{\overline{\Delta}}(\overline{\omega})$ is $\overline{\mathcal{F}}_{t+}^{\overline{\Delta}}$-measurable and $\overline{\tau}(\overline{\omega}) := \tau(\omega)\, 1_{\overline{\Delta}}(\overline{\omega}) + T\, 1_{\overline{\Delta}^c}(\overline{\omega})$ is a $\overline{\mathbb{F}}_+^{\overline{\Delta}}$-stopping time.

In the following, we assume the conditions on the function $\overline{Z} : \overline{\Omega} \to [1, \infty]$ and the set Ξ imposed in Section 2 and Theorem 2.2 hold true, and introduce the σ-compact set

$$\overline{\Xi} := \left\{\overline{\omega} \in \overline{\Omega} : \overline{Z}(\overline{\omega}) < \infty\right\}.$$

Note that $\psi(\Xi) = \overline{\Xi} \cap \overline{\Delta}$, and ν is nondecreasing with $\nu(0) = 0$ for all $(\omega, \nu) \in \overline{\Xi}$.

Define the process $\overline{M} = (\overline{M}_t)_{0 \le t \le T}$ by

$$2\overline{M}_t(\overline{\omega}) = \overline{S}_t(\overline{\omega})^2 - \overline{V}_t(\overline{\omega}) = \omega(t)^2 - \nu(t), \quad \overline{\omega} \in \overline{\Omega}.$$

For every $\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega})$ such that $\overline{S}$ and $\overline{M}$ are $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-local martingales and $\overline{\mathbb{Q}}(\overline{\Xi}) = 1$, it follows from standard results on the quadratic variation that $\langle \overline{S} \rangle^{\overline{\mathbb{Q}}} = \overline{V}$ $\overline{\mathbb{Q}}$-a.s., and $\overline{\mathbb{Q}}(\overline{S} \in \Omega) = 1$ by Remark 2.1, which shows $\overline{\mathbb{Q}}(\overline{\Delta}) = 1$. Hence, $\overline{\mathbb{Q}}(\overline{N}) = 0$ for all $\overline{N} \subseteq \overline{\Delta}^c$, and therefore $\overline{\mathbb{F}}^{\overline{\Delta}} \subseteq \overline{\mathbb{F}}^{\overline{\mathbb{Q}}}$ and $\overline{\mathbb{F}}_+^{\overline{\Delta}} \subseteq \overline{\mathbb{F}}_+^{\overline{\mathbb{Q}}}$.

Lemma 4.3. *One has*

$$\begin{aligned}\overline{\mathcal{M}}_{\overline{Z}}(\overline{\Xi}) := &\left\{\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega}) : \overline{S} \text{ and } \overline{M} \text{ are } \overline{\mathbb{Q}}\text{-}\overline{\mathbb{F}}\text{-martingales}, \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}] < \infty \text{ and } \overline{\mathbb{Q}}(\overline{\Xi}) = 1\right\} \\ = &\left\{\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega}) : \overline{S} \text{ and } \overline{M} \text{ are } \overline{\mathbb{Q}}\text{-}\overline{\mathbb{F}}\text{-local martingales}, \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}] < \infty \text{ and } \overline{\mathbb{Q}}(\overline{\Xi}) = 1\right\}.\end{aligned}$$

Moreover, for each $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Xi})$, the process $\overline{S}$ is a square-integrable $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale.

Proof. The inclusion "$\subseteq$" is obvious. For the other one, fix a $\overline{\mathbb{Q}}$ of the right-hand side. Since $\overline{Z}(\overline{\omega}) \geq \|\omega\|_\infty + \|\nu\|_\infty$ for all $\overline{\omega} \in \overline{\Omega}$, we obtain that $\mathbb{E}_{\overline{\mathbb{Q}}}[\sup_t |\overline{S}_t|] \leq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}] < \infty$, hence $\overline{S}$ is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale. Next, we can use the estimate

$$\mathbb{E}_{\overline{\mathbb{Q}}}[\langle \overline{S} \rangle_T] = \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{V}_T] \leq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}] < \infty,$$

showing that $\overline{S}$ is a square-integrable $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale. But then the identity $2\overline{M} = \overline{S}^2 - \langle \overline{S} \rangle$ $\overline{\mathbb{Q}}$-a.s. implies that $\overline{M}$ is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale. □

Lemma 4.4. *The set $\overline{\mathcal{M}}_{\overline{Z}}(\Xi)$ is nonempty. Moreover,*

$$\sup_{\mathbb{Q} \in \mathcal{M}_Z(\Xi)} \mathbb{E}_{\mathbb{Q}}[\xi(S)] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\Xi)} \mathbb{E}_{\overline{\mathbb{Q}}}[\xi(\overline{S})]$$

for every Borel function $\xi \colon C[0,T] \to (-\infty, \infty]$ which is bounded from below.

Proof. In the first step, we show that for each $\mathbb{Q} \in \mathcal{M}_Z(\Xi)$, there exists $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\Xi)$ such that

$$\mathbb{E}_{\mathbb{Q}}[\xi(S)] = \mathbb{E}_{\overline{\mathbb{Q}}}[\xi(\overline{S})]. \tag{4.1}$$

Fix $\mathbb{Q} \in \mathcal{M}_Z(\Xi)$ and define

$$\overline{\mathbb{Q}} := \mathbb{Q} \circ \psi^{-1}.$$

Then as $Z = \overline{Z} \circ \psi$, we have $\mathbb{E}_{\overline{\mathbb{Q}}}[\sup_t |\overline{S}_t|] \leq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}] = \mathbb{E}_{\mathbb{Q}}[Z] < \infty$. In particular, $\overline{\mathbb{Q}}(\overline{\Xi}) = 1$. Moreover $\overline{S}$ is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale, since for every $0 \leq s \leq t \leq T$ and $\overline{A} \in \overline{\mathcal{F}}_s$,

$$\mathbb{E}_{\overline{\mathbb{Q}}}[(\overline{S}_t - \overline{S}_s)1_{\overline{A}}] = \mathbb{E}_{\mathbb{Q}}[(S_t - S_s)1_{\psi^{-1}(\overline{A})}] = 0$$

by Lemma 4.1 and the martingale property of S under $\mathbb{Q}$. Similarly, since $\psi(S) = (S, \langle S \rangle^{\mathbb{Q}})$ $\mathbb{Q}$-almost surely, it follows that $\overline{M}$ is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale. In particular, $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\Xi)$ and $\mathbb{E}_{\overline{\mathbb{Q}}}[\xi(\overline{S})] = \mathbb{E}_{\mathbb{Q}}[\xi(S)]$ by definition.

In the second step, we show the converse statement, i.e. for every $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\Xi)$ there exists $\mathbb{Q} \in \mathcal{M}_Z(\Xi)$ satisfying (4.1). Fix $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\Xi)$ and notice that since both $\overline{S}$ and $2\overline{M} = \overline{S}^2 - \overline{V}$ are $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingales, and $\overline{\mathbb{Q}}$-a.s., $\overline{V}$ is nondecreasing with $\overline{V}_0 = 0$, one concludes that $\overline{V} = \langle \overline{S} \rangle^{\overline{\mathbb{Q}}}$ $\overline{\mathbb{Q}}$-a.s., and thus $\psi(\overline{S}) = (\overline{S}, \langle \overline{S} \rangle^{\overline{\mathbb{Q}}})$ $\overline{\mathbb{Q}}$-a.s.. In particular $\overline{\mathbb{Q}}(\overline{S} \in \Omega) = 1$, so that

$$\mathbb{Q} := \overline{\mathbb{Q}} \circ \overline{S}^{-1}$$

defines a Borel probability measure on Ω which satisfies (4.1) by definition. Moreover, as $\overline{\mathbb{Q}}(\overline{\Delta}) = 1$, one has

$$\mathbb{E}_{\mathbb{Q}}[Z] = \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}(\psi(\overline{S}))] = \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}] < \infty, \quad \text{in partiuclar} \quad \mathbb{Q}(\Xi) = \mathbb{Q}(Z < \infty) = 1$$

and for all $0 \leq s \leq t \leq T$ and $A \in \mathcal{F}_s$, it follows from Lemma 4.1 and $\overline{\mathcal{F}}_s^{\overline{\Delta}} \subseteq \overline{\mathcal{F}}_s^{\overline{\mathbb{Q}}}$ that

$$\mathbb{E}_{\mathbb{Q}}[(S_t - S_s)1_A] = \mathbb{E}_{\overline{\mathbb{Q}}}[(\overline{S}_t - \overline{S}_s)1_{\{\overline{\omega} \in \overline{\Omega} : \omega \in A\}}] = 0,$$

which shows that S is a $\mathbb{Q}$-$\mathbb{F}$-martingale. Thus $\mathbb{Q} \in \mathcal{M}_Z(\Xi)$, which completes the proof. □

Remark 4.5. In fact, we we see from the proof of Lemma 4.4 that

$$\mathcal{M}_Z(\Xi) = \big\{\overline{\mathbb{Q}} \circ \overline{S}^{-1} :\ \overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\Xi)\big\},$$
$$\overline{\mathcal{M}}_{\overline{Z}}(\overline{\Xi}) = \big\{\mathbb{Q} \circ \psi^{-1} :\ \mathbb{Q} \in \mathcal{M}_Z(\Xi)\big\}.$$

Denote by $\overline{\mathcal{H}}$ the set of all $\overline{\mathbb{F}}^{\overline{\Delta}}_{+}$-simple processes $\overline{H}\colon [0,T] \times \overline{\Omega} \to \mathbb{R}$ of the form

$$\overline{H}_t(\omega) = \sum_{l=1}^{L} \overline{h}_l(\overline{\omega}) 1_{(\overline{\tau}_l(\overline{\omega}), \overline{\tau}_{l+1}(\overline{\omega})]}(t)$$

for $(t, \overline{\omega}) \in [0,T] \times \overline{\Omega}$, where $L \in \mathbb{N}$, $0 \le \overline{\tau}_1 \le \cdots \le \overline{\tau}_{L+1} \le T$ are stopping times w.r.t. the filtration $\overline{\mathbb{F}}^{\overline{\Delta}}_{+}$, and $\overline{h}_l\colon \overline{\Omega} \to \mathbb{R}$ are bounded $\overline{\mathcal{F}}^{\overline{\Delta}}_{\overline{\tau}_l+}$-measurable functions. For a simple process $\overline{H} \in \overline{\mathcal{H}}$ define the pathwise stochastic integral

$$(\overline{H} \cdot \overline{S})_t(\overline{\omega}) := \sum_{l=1}^{L} \overline{h}_l(\overline{\omega}) \big(\overline{S}_{\overline{\tau}_{l+1}(\overline{\omega}) \wedge t}(\overline{\omega}) - \overline{S}_{\overline{\tau}_l(\overline{\omega}) \wedge t}(\overline{\omega})\big).$$

The integral $(\overline{H} \cdot \overline{M})_t$ is defined analogously.

Given a function $\overline{\xi}\colon \overline{\Omega} \to (-\infty, \infty]$, define the functional

$$\overline{\phi}(\overline{\xi}) := \inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there is } c \ge 0 \text{ and sequences } (\overline{H}^n), (\overline{G}^n) \text{ in } \overline{\mathcal{H}} \text{ so that} \\ \text{for all } n,\ (\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T \ge -c\overline{Z} \quad \text{on } \overline{\Delta} \cap \overline{\Xi} \\ \text{and } \lambda + \liminf_{n\to\infty} \big((\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T\big) \ge \overline{\xi} \quad \text{on } \overline{\Delta} \cap \overline{\Xi} \end{array} \right\}.$$

Recall that $\overline{S} : \overline{\Omega} \to C[0,T]$ is the projection on the first coordinate, i.e. $\overline{S}(\omega, \nu) = \omega$. Note that $\overline{S}(\overline{\Xi} \cap \overline{\Delta}) = \Xi$, however $\overline{S}(\overline{\Xi})$ is not a subset of Ξ in general.

Lemma 4.6. *For every function $\xi\colon C[0,T] \to (-\infty, \infty]$ one has $\phi(\xi) = \overline{\phi}(\xi \circ \overline{S})$.*

Proof. To see the first inequality $\phi(\xi) \ge \overline{\phi}(\xi \circ \overline{S})$, let $\lambda > \phi(\xi)$. By definition, there exist $c > 0$ and sequences $(H^n), (G^n)$ in $\mathcal{H}$ such that

$$\text{for all } n,\ (H^n \cdot S)_T + (G^n \cdot M)_T \ge -cZ \quad \text{on } \Xi$$
$$\text{and } \lambda + \liminf_{n\to\infty} \big((H^n \cdot S)_T + (G^n \cdot M)_T\big) \ge \xi \quad \text{on } \Xi.$$

For each n, one has $H^n = \sum_{l=1}^{L_n} h^n_l\, 1_{(\tau^n_l, \tau^n_{l+1}]}$. Define $\overline{H}^n := \sum_{l=1}^{L_n} \overline{h}^n_l\, 1_{(\overline{\tau}^n_l, \overline{\tau}^n_{l+1}]}$, where

$$\overline{h}^n_l(\overline{\omega}) := h^n_l(\omega)\, 1_{\overline{\Delta}}(\overline{\omega}) \quad \text{and} \quad \overline{\tau}^n_l(\overline{\omega}) := \tau^n_l(\omega)\, 1_{\overline{\Delta}}(\overline{\omega}) + T\, 1_{\overline{\Delta}^c}(\overline{\omega})$$

for all n, l, and $\overline{\omega} \in \overline{\Omega}$, and define $\overline{G}^n$ analogously. By Remark 4.2, $\overline{H}^n, \overline{G}^n \in \overline{\mathcal{H}}$. Then, since $\overline{M} = M \circ \overline{S}$ on $\overline{\Delta}$,

$$(\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T = (H^n \cdot S)_T \circ \overline{S} + (G^n \cdot M)_T \circ \overline{S} \quad \text{on } \overline{\Delta}.$$

Since also $\overline{Z} = Z \circ \overline{S}$ on $\overline{\Delta}$, as $\overline{S}(\overline{\Delta} \cap \overline{\Xi}) = \Xi$, one has

$$\begin{aligned}&\text{for all } n,\ (\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T \geq -c(Z \circ \overline{S}) = -c\overline{Z} \quad \text{on } \overline{\Delta} \cap \overline{\Xi}\\ &\text{and } \lambda + \liminf_{n\to\infty} \big((\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T\big) \geq \xi \circ \overline{S} \quad \text{on } \overline{\Delta} \cap \overline{\Xi}.\end{aligned}$$

As $\lambda > \phi(\xi)$ was arbitrary, the inequality $\phi(\xi) \geq \overline{\phi}(\xi \circ \overline{S})$ follows.

To prove the opposite inequality, let $\lambda > \overline{\phi}(\xi \circ \overline{S})$. Then, by definition, there exists $c > 0$ and sequences $(\overline{H}^n)$, $(\overline{G}^n)$ in $\overline{\mathcal{H}}$ such that

$$\begin{aligned}&\text{for all } n,\ (\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T \geq -c\overline{Z} \quad \text{on } \overline{\Delta} \cap \overline{\Xi}\\ &\text{and } \lambda + \liminf_{n\to\infty} \big((\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T\big) \geq \xi \circ \overline{S} \quad \text{on } \overline{\Delta} \cap \overline{\Xi}.\end{aligned}$$

For each n, one has $\overline{H}^n = \sum_{l=1}^{L_n} \overline{h}_l^n 1_{(\overline{\tau}_l^n, \overline{\tau}_{l+1}^n]}$. Define $H^n := \sum_{l=1}^{L_n} \overline{h}_l^n \circ \psi\, 1_{(\overline{\tau}_l^n \circ \psi, \overline{\tau}_{l+1}^n \circ \psi]}$, and G^n analogously. Note that

$$M = \overline{M} \circ \psi, \quad \psi(\Omega) = \overline{\Delta}, \quad \text{and} \quad \overline{S} \circ \psi = \mathrm{id}.$$

Therefore

$$\begin{aligned}(H^n \cdot S)_T + (G^n \cdot M)_T &= ((\overline{H}^n \circ \psi) \cdot S)_T + ((\overline{G}^n \circ \psi) \cdot M)_T\\ &= ((\overline{H}^n \circ \psi) \cdot (\overline{S} \circ \psi))_T + ((\overline{G}^n \circ \psi) \cdot (\overline{M} \circ \psi))_T\\ &= (\overline{H}^n \cdot \overline{S})_T \circ \psi + (\overline{G}^n \cdot \overline{M})_T \circ \psi.\end{aligned}$$

In particular, since $Z = \overline{Z} \circ \psi$ and $\psi(\Xi) = \overline{\Delta} \cap \overline{\Xi}$, it follows that

$$\text{for all } n,\ (H^n \cdot S)_T + (G^n \cdot M)_T \geq -c(\overline{Z} \circ \psi) = -cZ \quad \text{on } \Xi$$

and similarly

$$\lambda + \liminf_{n\to\infty} \big((H^n \cdot S)_T + (G^n \cdot M)_T\big) \geq \xi \circ \overline{S} \circ \psi = \xi \quad \text{on } \Xi.$$

As $\lambda > \overline{\phi}(\xi \circ \overline{\pi})$ was arbitrary, the inequality $\overline{\phi}(\xi \circ \overline{S}) \geq \phi(\xi)$ follows. $\square$

Denote by $C_b(\overline{\Omega})$ and $U_b(\overline{\Omega})$ the spaces of bounded continuous and bounded upper semicontinuous functions $\overline{\xi} \colon \overline{\Omega} \to \mathbb{R}$, respectively.

Now we are ready to prove our main result.

Proof of Theorem 2.2. The proof is divided in the following steps.

Step (a): Fix $n \in \mathbb{N}$. For any Borel function $\overline{\xi} \colon \overline{\Omega} \to (-\infty, \infty]$ which is bounded from below, we define

$$\overline{\phi}_n(\overline{\xi}) := \inf \left\{ \lambda \in \mathbb{R} : \begin{array}{l} \text{there is } c > 0 \text{ and } \overline{H}, \overline{G} \in \overline{\mathcal{H}} \text{ such that} \\ (\overline{H} \cdot \overline{S})_T, (\overline{G} \cdot \overline{M})_T \geq -c \text{ on } \overline{\Delta} \cap \overline{\Xi} \\ \lambda + (\overline{H} \cdot \overline{S})_T + (\overline{G} \cdot \overline{M})_T \geq \overline{\xi} - \overline{Z}/n \text{ on } \overline{\Delta} \cap \overline{\Xi} \end{array} \right\}.$$

Then, for all $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})$ the following hold

$$\overline{\phi}_n(\overline{\xi}) \geq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi}] - \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}/n], \tag{4.2}$$

$$\overline{\phi}(\overline{\xi}) \geq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi}]. \tag{4.3}$$

In particular, as $\overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})$ is nonempty by Lemma 4.4, the functional $\overline{\phi}_n$ is real-valued on U_b.

Indeed, let $\lambda > \overline{\phi}_n(\overline{\xi})$ so that there exist $\overline{H}, \overline{G} \in \overline{\mathcal{H}}$ and $c > 0$ such that the inequalities $(\overline{H} \cdot \overline{S})_T, (\overline{G} \cdot \overline{M})_T \geq -c$ and $\lambda + (\overline{H} \cdot \overline{S})_T + (\overline{G} \cdot \overline{M})_T \geq \overline{\xi} - \overline{Z}/n$ hold on $\overline{\Delta} \cap \overline{\Xi}$. Since both $\overline{S}$ and $\overline{M}$ are $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}^{\overline{\Delta}}_+$-martingales and $\overline{H}$, $\overline{G}$ are simple integrands, we see that by denoting $\overline{\mathbb{G}} := \overline{\mathbb{F}}^{\overline{\Delta}}_+$

$$\mathbb{E}_{\overline{\mathbb{Q}}}\big[(\overline{H} \cdot \overline{S})_T\big] = \sum_{l=1}^{L} \mathbb{E}_{\overline{\mathbb{Q}}}\big[\overline{h}_l(\overline{S}_{\overline{\tau}_{l+1}} - \overline{S}_{\overline{\tau}_l})\big] = \sum_{l=1}^{L} \mathbb{E}_{\overline{\mathbb{Q}}}\Big[\overline{h}_l\, \mathbb{E}_{\overline{\mathbb{Q}}}[(\overline{S}_{\overline{\tau}_{l+1}} - \overline{S}_{\overline{\tau}_l}) \,|\, \overline{\mathcal{G}}_{\overline{\tau}_l}]\Big] = 0,$$

and analogously $\mathbb{E}_{\overline{\mathbb{Q}}}\big[(\overline{G} \cdot \overline{M})_T\big] = 0$. Hence $\lambda \geq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi}] - \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}/n]$, which shows (4.2). To prove (4.3), we can use the same arguments together with Fatou's lemma.

Step (b): Each $\overline{\phi}_n$ is continuous from above on $C_b(\overline{\Omega})$, that is, for every sequence $(\overline{\xi}_k)$ in $C_b(\overline{\Omega})$ which decreases pointwise to 0, one has $\overline{\phi}_n(\overline{\xi}_k) \downarrow \overline{\phi}_n(0)$.

Indeed, to that end, fix such a sequence $(\overline{\xi}_k)$ and an arbitrary $\varepsilon > 0$. Then there exist $\overline{H}, \overline{G} \in \overline{\mathcal{H}}$ with $(\overline{H} \cdot \overline{S})_T, (\overline{G} \cdot \overline{M})_T \geq -c$ on $\overline{\Delta} \cap \overline{\Xi}$ for some $c \geq 0$ such that

$$\varepsilon + \overline{\phi}_n(0) + (\overline{H} \cdot \overline{S})_T + (\overline{G} \cdot \overline{M})_T + \overline{Z}/n \geq 0 \quad \text{ on } \overline{\Delta} \cap \overline{\Xi}.$$

Now define $b := \sup_{\overline{\omega} \in \overline{\Omega}} \overline{\xi}_1(\overline{\omega}) - \varepsilon - \overline{\phi}_n(0) + c$, so that

$$b + \varepsilon + \overline{\phi}_n(0) + (\overline{H} \cdot \overline{S})_T + (\overline{G} \cdot \overline{M})_T \geq \overline{\xi}_1 \quad \text{ on } \overline{\Delta} \cap \overline{\Xi}.$$

As $\{\overline{Z} \leq bn\} \subseteq \overline{\Omega}$ is compact by assumption, Dini's lemma yields $\overline{\xi}_k 1_{\{\overline{Z} \leq bn\}} \leq \varepsilon$ for k large enough. Hence for k big enough we have on $\overline{\Delta} \cap \overline{\Xi}$

$$\begin{aligned}
\overline{\xi}_k &\leq \overline{\xi}_k 1_{\{\overline{Z} \leq bn\}} + \overline{\xi}_1 1_{\{\overline{Z} > bn\}} \\
&\leq \varepsilon + \big(\varepsilon + \overline{\phi}_n(0) + (\overline{H} \cdot \overline{S})_T + (\overline{G} \cdot \overline{M})_T + \overline{Z}/n\big) 1_{\{\overline{Z} > bn\}} \\
&\leq 2\varepsilon + \overline{\phi}_n(0) + (\overline{H} \cdot \overline{S})_T + (\overline{G} \cdot \overline{M})_T + \overline{Z}/n.
\end{aligned}$$

Therefore, $\overline{\phi}_n(\overline{\xi}_k) \leq \overline{\phi}_n(0) + 2\varepsilon$ for k large enough which shows that $\overline{\phi}_n(\overline{\xi}_k) \downarrow \overline{\phi}_n(0)$.

Step (c): We proceed to show that for each n and every Borel measure $\overline{\mathbb{Q}}$ on $\overline{\Omega}$ one has

$$\begin{aligned}
\overline{\phi}_n^*(\overline{\mathbb{Q}}) := \sup_{\overline{\gamma} \in C_b(\overline{\Omega})} (\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}] - \overline{\phi}_n(\overline{\gamma})) &= \sup_{\overline{\gamma} \in U_b(\overline{\Omega})} (\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}] - \overline{\phi}_n(\overline{\gamma})) \\
&= \begin{cases} \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}]/n, & \text{if } \overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Xi}), \\ \infty, & \text{else.} \end{cases}
\end{aligned} \tag{4.4}$$

Indeed, if $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})$, then by (4.2) we have for every $\overline{\gamma} \in U_b(\overline{\Omega})$ that $\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}] - \overline{\phi}_n(\overline{\gamma}) \leq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}/n]$. In particular, for every $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})$

$$\overline{\phi}_n^*(\overline{\mathbb{Q}}) \leq \sup_{\overline{\gamma} \in U_b(\overline{\Omega})} (\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}] - \overline{\phi}_n(\overline{\gamma})) \leq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}/n]. \tag{4.5}$$

On the other hand, for every $m \in \mathbb{R}$ it holds that $\overline{\phi}_n(m) \leq m$ by definition, so that $\overline{\phi}_n^*(\overline{\mathbb{Q}}) \geq \sup_m (m\,\overline{\mathbb{Q}}(\overline{\Omega}) - m) = \infty$ whenever $\overline{\mathbb{Q}} \notin \mathfrak{P}(\overline{\Omega})$. Further, since $\overline{Z} \geq 1$ is lower semicontinuous (as it has compact sublevel sets) there exists a sequence of non-negative functions $\overline{Z}_k \in C_b(\overline{\Omega})$ which increase pointwise to $\overline{Z}$. By definition $\overline{\phi}_n(\overline{Z}_k/n) \leq 0$ for all k, hence for every $\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega})$

$$\overline{\phi}_n^*(\overline{\mathbb{Q}}) \geq \sup_k (\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}_k/n] - \overline{\phi}_n(\overline{Z}_k/n)) \geq \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}/n]. \tag{4.6}$$

In particular, as $\{\overline{Z} = \infty\} = \overline{\Xi}^c$, it follows from (4.6) that $\overline{\phi}_n^*(\overline{\mathbb{Q}}) = \infty$ whenever $\overline{\mathbb{Q}}(\overline{\Xi}) < 1$.

Now fix $\overline{\mathbb{Q}} \notin \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})$. We need to show that $\phi_n^*(\overline{\mathbb{Q}}) = \infty$. First, note from the arguments provided above, we may assume w.l.o.g. that $\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega})$, $\overline{\mathbb{Q}}(\overline{\Xi}) = 1$, and that $\overline{Z}$ is integrable w.r.t. $\overline{\mathbb{Q}}$. Therefore, Lemma 4.3 implies that $\overline{S}$ or $\overline{M}$ is not a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-local martingale. In either case, by Lemma 5.4 there is a function $\overline{\gamma} \in C_b(\overline{\Omega})$ and $\overline{H} \in \overline{\mathcal{H}}$ such that $\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}] > 0$ and $\overline{\gamma} \leq (\overline{H} \cdot \overline{S})_T$ or $\overline{\gamma} \leq (\overline{H} \cdot \overline{M})_T$. This implies that $\overline{\phi}_n(m\overline{\gamma}) \leq 0$ for all $m > 0$. Therefore, as

$$\overline{\phi}_n^*(\overline{\mathbb{Q}}) \geq \sup_{m>0} \big(\mathbb{E}_{\overline{\mathbb{Q}}}[m\overline{\gamma}] - \overline{\phi}_n(m\overline{\gamma})\big) \geq \sup_{m>0} \mathbb{E}_{\overline{\mathbb{Q}}}[m\overline{\gamma}],$$

we conclude that $\overline{\phi}_n^*(\overline{\mathbb{Q}}) = \infty$, if $\overline{\mathbb{Q}} \notin \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})$. All together, combining with (4.6) and (4.5), we have proven (4.4).

As an application of Proposition 5.1

$$\overline{\phi}_n(\overline{\xi}) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})} \big(\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi}] - \mathbb{E}_{\mathbb{Q}}[\overline{Z}]/n\big) \quad \text{for all } \xi \in U_b(\overline{\Omega}). \tag{4.7}$$

Step (d): Let $\overline{\xi} \colon \overline{\Omega} \to (-\infty, \infty]$ be an upper semicontinuous function which is bounded from below. We show that

$$\overline{\phi}(\overline{\xi}) = \sup_n \overline{\phi}_n(\overline{\xi} \wedge n) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})} \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi}]. \tag{4.8}$$

Indeed, by (4.3) and (4.7) it holds

$$\begin{aligned} \overline{\phi}(\overline{\xi}) &\geq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})} \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi}] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})} \big(\sup_n \big(\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi} \wedge n] - \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{Z}]/n\big)\big) \\ &= \sup_n \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})} \big(\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\xi} \wedge n] - \mathbb{E}_{\mathbb{Q}}[\overline{Z}]/n\big) = \sup_n \overline{\phi}_n(\overline{\xi} \wedge n). \end{aligned} \tag{4.9}$$

On the other hand, if $m := \sup_n \overline{\phi}_n(\overline{\xi} \wedge n)$, then for every n there exists $\overline{H}^n$ and $\overline{G}^n$ in $\overline{\mathcal{H}}$ with

$$m + \frac{1}{n} + (\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T \geq \overline{\xi} \wedge n - \overline{Z}/n \quad \text{on } \overline{\Delta} \cap \overline{\Xi}.$$

Hence, $(\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T \geq -c\overline{Z}$ on $\overline{\Delta} \cap \overline{\Xi}$ for $c := \|\overline{\xi} \wedge 0\|_\infty + m + 2$ and

$$m + \liminf_{n\to\infty} \big((\overline{H}^n \cdot \overline{S})_T + (\overline{G}^n \cdot \overline{M})_T\big) \geq \liminf_{n\to\infty} \Big(\overline{\xi} \wedge n - \overline{Z}/n - \frac{1}{n}\Big) = \overline{\xi} \quad \text{on } \overline{\Delta} \cap \overline{\Xi},$$

where we use that $\overline{\Delta} \cap \overline{\Xi} \subseteq \{\overline{Z} < \infty\}$. Therefore $\overline{\phi}(\overline{\xi}) \leq m$, hence together with (4.9) we obtain that $m = \overline{\phi}(\overline{\xi})$, and the infimum in the definition of $\overline{\phi}(\overline{\xi})$ is attained whenever $\overline{\phi}(\overline{\xi}) < \infty$.

Step (e): If $\xi\colon C[0,T] \to (-\infty, \infty]$ is bounded from below and upper semicontinuous, then $\xi \circ \overline{S}\colon \overline{\Omega} \to (-\infty, \infty]$ is again bounded from below and upper semicontinuous. Therefore, using the identity $\phi(\xi) = \overline{\phi}(\xi \circ \overline{S})$ shown in Lemma 4.6, the representation (4.8), and Lemma 4.4, we conclude

$$\phi(\xi) = \overline{\phi}(\xi \circ \overline{S}) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\overline{Z}}(\overline{\Omega})} \mathbb{E}_{\overline{\mathbb{Q}}}[\xi \circ \overline{S}] = \sup_{\mathbb{Q} \in \mathcal{M}_Z(\Omega)} \mathbb{E}_{\mathbb{Q}}[\xi].$$

Moreover, we know from Step (d) that the infimum in the definition of $\overline{\phi}(\xi \circ \overline{S})$ is attained whenever $\overline{\phi}(\xi \circ \overline{S}) < \infty$. Therefore, we see from the arguments in the proof of Lemma 4.6 that also the infimum in the definition of $\phi(\xi)$ is attained whenever $\phi(\xi) < \infty$. The proof is complete. □

5 Technical Results

In this section, if not explicitly stated otherwise, we use the setting of Section 2.

Proposition 5.1. *Let here Ω be any metric space and $\Psi\colon U_b(\Omega) \to \mathbb{R}$ a convex monotone functional. If $\Psi(f_n) \downarrow \Psi(0)$ for every sequence (f_n) in $C_b(\Omega)$ which decreases pointwise to 0 and $\Psi^*(\mathbb{Q}) := \sup_{f \in C_b(\Omega)}(\mathbb{E}_{\mathbb{Q}}[f] - \Psi(f)) = \sup_{f \in U_b(\Omega)}(\mathbb{E}_{\mathbb{Q}}[f] - \Psi(f))$ holds, then*

$$\Psi(f) = \max_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}[f] - \Psi^*(\mathbb{Q}))$$

for all $f \in U_b(\Omega)$, where the maximum is attained in the set of all Borel measures on Ω.

Proof. That $\Psi(f) = \max_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}[f] - \Psi^*(\mathbb{Q}))$ for all $f \in C_b$ follows from the Hahn-Banach extension theorem and the Daniell-Stone theorem along the same lines as in the proof of [6, Theorem A.1]. The extension to U_b is similar to the proof of [6, Theorem A.5] □

The following results are similar to ones in [3], whose proofs we provide for the sake of completeness.

Lemma 5.2. *Let $[a,b] \subseteq \mathbb{R}$. For every $\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega})$ and every $\overline{\mathcal{F}}_t$-measurable function $\overline{h}\colon \overline{\Omega} \to [a,b]$ there are continuous $\overline{\mathcal{F}}_t$-measurable functions $\overline{h}_k\colon \overline{\Omega} \to [a,b]$ which converge $\overline{\mathbb{Q}}$-almost surely to $\overline{h}$.*

Proof. First notice that both $\overline{\Omega}$ and $\overline{\Omega}^t := \{\overline{\omega}|_{[0,t]} : \overline{\omega} \in \overline{\Omega}\}$, each endowed with the sup-norm, are Polish spaces. Now as $\overline{\mathcal{F}}_t = \{\overline{\pi}^{-1}(\overline{B}) : \overline{B} \subseteq \overline{\Omega}^t \text{ is Borel}\}$ where $\overline{\pi}\colon \overline{\Omega} \to \overline{\Omega}^t$ is defined by $\overline{\pi}(\overline{\omega}) = \overline{\omega}|_{[0,t]}$, one has $\overline{h} = \overline{h}^t \circ \pi$ for some Borel function $\overline{h}^t\colon \overline{\Omega}^t \to \mathbb{R}$. Since $\overline{\Omega}^t$ is Polish and $\overline{\mathbb{Q}}^t := \overline{\mathbb{Q}} \circ \overline{\pi}^{-1}$ a probability measure thereon, there are continuous functions $h_k^t\colon \overline{\Omega}^t \to \mathbb{R}$ such that $\overline{h}_k^t \to \overline{h}^t$ $\overline{\mathbb{Q}}^t$-almost surely. Therefore, we can define $\overline{h}_k\colon \overline{\Omega} \to \mathbb{R}$ by $\overline{h}_k := \overline{h}_k^t \circ \overline{\pi}$. □

Lemma 5.3. *Let here Ω be any metric space and $X : \Omega \to C[0,T]$ continuous. Fix $0 \le s < t \le T$, $m > 0$, and define*

$$\tau := \inf\{r \ge s : X_r > m \text{ or } X_r \le -m\} \wedge T.$$

Then the function $\omega \mapsto X_{\tau(\omega)\wedge t}(\omega)$ is lower semicontinuous.

Proof. Define $\tau_+ := \inf\{r \ge s : X_r > m\} \wedge T$ and $\tau_- := \inf\{r \ge s : X_r \le -m\} \wedge T$, and note that $\tau = \tau_+ \wedge \tau_-$. Moreover, fix ω and a sequence (ω_n) such that $\omega_n \to \omega$.

First, we claim that

$$\limsup_n \tau_+(\omega_n) \le \tau_+(\omega) \quad \text{and} \quad \liminf_n \tau_-(\omega_n) \ge \tau_-(\omega). \tag{5.1}$$

Indeed, for the first inequality, assume without loss of generality that $r := \tau_+(\omega) < T$. Fix any $\varepsilon > 0$. Then, by definition, there is $\delta \in (0,\varepsilon)$ such that $X_{r+\delta}(\omega) > m$. Since $\omega \mapsto X(\omega)$ is continuous, $X_{r+\delta}(\omega_n) > m$ for eventually all n, showing that $\tau_+(\omega_n) \le r + \varepsilon$ for eventually all n. As $\varepsilon > 0$ was arbitrary, the first inequality of the claim follows.

To see the second inequality of the claim, we may assume without loss of generality that $r := \tau_-(\omega) > s$. Then necessarily $X_u(\omega) > -m$ for $u \in [s,r)$. Fix $\varepsilon > 0$. By continuity of $t \mapsto X_t(\omega)$, there exists $\delta > 0$ such that $X_u(\omega) \ge -m + \delta$ for $u \in [s, r-\varepsilon]$. Further, due to continuity of $\omega \mapsto X(\omega)$, it follows that $X_u(\omega_n) \ge -m + \delta/2$ for $u \in [s, r-\varepsilon]$ for eventually all n. Therefore $\tau_-(\omega_n) \ge r - \varepsilon$ for eventually all n and as ε was arbitrary, the second part of the claim follows.

Next, to prove the lower semicontinuity of X_t^τ, we distinguish between several cases:

(a) If $X_t^\tau(\omega) > m$, then by the continuity of the paths of $X.(\omega)$, $\tau(\omega) = \tau_+(\omega) = s$ and $X_s(\omega) > m$. By continuity of $X_s(\cdot)$ on Ω, $X_s(\omega_n) > m$ and $\tau_+(\omega_n) = s$ for eventually all n, hence $\lim_n X_t^\tau(\omega_n) = \lim_n X_s(\omega_n) = X_s(\omega) = X_t^\tau(\omega)$.

(b) If $X_t^\tau(\omega) = m$, then either $\tau_+(\omega) < t$ or $\tau_+(\omega) \ge t$. In the first case it follows that $\tau_+(\omega) < \tau_-(\omega)$ so that by (5.1) $\tau_+(\omega_n) < \tau_-(\omega_n)$ and $\tau_+(\omega_n) < t$ for eventually all n and therefore

$$\liminf_{n\to\infty} X_t^\tau(\omega_n) = \liminf_{n\to\infty} X_{\tau_+(\omega_n)}(\omega) = m = X_t^\tau(\omega).$$

On the other hand, if $\tau_+(\omega) \ge t$, then $X_t(\omega) = m$ and $X_r(\omega) > -m$ for $r \in [s,t]$. This implies that $\tau_-(\omega_n) \ge t$ for eventually all n and therefore

$$\liminf_n X_t^\tau(\omega_n) = \liminf_n X_{t\wedge\tau_+(\omega_n)}(\omega_n) = m = X_t^\tau(\omega).$$

(c) If $X_t^\tau(\omega) \in (-m, m)$, then either $\tau(\omega) > t$ or $\tau(\omega) = T$ (in which case necessarily $t = T$). In the latter case it follows that $X_r(\omega) > -m$ for $r \in [s, T]$, hence $\tau_-(\omega_n) = T$ for eventually all n and thus

$$\liminf_n X_t^\tau(\omega_n) = \liminf_n X_{t\wedge\tau_+(\omega_n)}(\omega_n) \geq X_t(\omega) = X_t^\tau(\omega).$$

If $\tau(\omega) > t$, then again $\tau_-(\omega_n) > t$ for eventually all n so that the same argument shows that $\liminf_n X_t^\tau(\omega_n) \geq X_t^\tau(\omega)$.

(d) If $X_t^\tau(\omega) = -m$, then $X_s(\omega) \geq -m$. Assume that $\liminf_n X_t^\tau(\omega_n) < -m$. Then there is a subsequence still denoted by (ω_n) such that $\tau(\omega_n) = \tau_-(\omega_n) = s$ for eventually all n. However, this contradicts $\liminf_n X_t^\tau(\omega_n) = \lim_n X_s(\omega_n) = X_s(\omega) \geq -m$.

(e) If $X_t^\tau(\omega) < -m$, then $\tau_-(\omega) = s$ and $X_s(\omega) < -m$. This implies $X_s(\omega_n) < -m$ and therefore $\tau_-(\omega_n) = s$ for eventually all n, so that $\lim_n X_t^\tau(\omega_n) = \lim_n X_s(\omega_n) = X_s(\omega) = X_t^\tau(\omega)$. $\square$

Proposition 5.4. *Let $\overline{X}$ be either $\overline{S}$ or $\overline{M}$ and fix a $\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega})$. If $\overline{X}$ is not a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-local martingale, then there exists $\overline{\gamma} \in C_b(\overline{\Omega})$ and $\overline{H} \in \overline{\mathcal{H}}$ such that $\overline{\gamma} \leq (\overline{H} \cdot \overline{X})_T$ and $\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}] > 0$.*

Proof. Consider the set

$$\overline{\Gamma} := \{\overline{\gamma} \in C_b(\overline{\Omega}) : \overline{\gamma} \leq (\overline{H} \cdot \overline{X})_T \text{ for some } \overline{H} \in \overline{\mathcal{H}}\}.$$

We prove that if $\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}] \leq 0$ for all $\overline{\gamma} \in \overline{\Gamma}$, then $\overline{X}$ is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-local martingale with localizing sequence

$$\overline{\tau}_m := \inf\{t \geq 0 : |\overline{X}_t| \geq m\} \wedge T,$$

i.e. for every $m \in \mathbb{N}$, the stopped process

$$\overline{X}_t^{\overline{\tau}_m} := \overline{X}_{t\wedge\overline{\tau}_m}$$

is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale. Fix $m \in \mathbb{N}$ and write $\overline{\tau} := \overline{\tau}_m$. First let us show that $\overline{X}^{\overline{\tau}}$ is an $\overline{\mathbb{F}}$-submartingale. To that end, let $0 \leq s < t \leq T$, and define

$$\overline{\sigma} := \inf\{r \geq s : |\overline{X}_r| \geq m\} \wedge T,$$
$$\overline{\sigma}_\varepsilon := \inf\{r \geq s : \overline{X}_r > m - \varepsilon \text{ or } \overline{X}_r \leq -m + \varepsilon\} \wedge T$$

for $0 < \varepsilon \leq 1$. Since both $\overline{\tau}$ and $\overline{\sigma}$ are hitting times of a closed set and $\overline{X}$ is continuous, they are $\overline{\mathbb{F}}$-stopping times, whereas $\overline{\sigma}_\varepsilon$ is a $\overline{\mathbb{F}}_+$-stopping time.

Now, fix an arbitrary $\overline{\mathcal{F}}_s$-measurable function $\overline{h} : \overline{\Omega} \to [0, 1]$. Notice that $\overline{\sigma} = \overline{\tau}$ on $\{\overline{\tau} \geq s\}$, so that $1_{\{\overline{\tau}\geq s\}}(\overline{X}_t^{\overline{\sigma}} - \overline{X}_s) = \overline{X}_t^{\overline{\tau}} - \overline{X}_s^{\overline{\tau}}$. Moreover, $\overline{\sigma}_\varepsilon$ increases to $\overline{\sigma}$ as ε tends to 0, and therefore $\overline{X}_t^{\overline{\sigma}_\varepsilon} \to \overline{X}_t^{\overline{\sigma}}$ by continuity of $\overline{X}$. Since additionally $|\overline{X}_t^{\overline{\sigma}_\varepsilon} - \overline{X}_s| \leq 2m$, this shows that

$$\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{h}(\overline{X}_t^{\overline{\tau}} - \overline{X}_s^{\overline{\tau}})] = \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{h}\,1_{\{\overline{\tau}\geq s\}}\,(\overline{X}_t^{\overline{\sigma}} - \overline{X}_s)] = \lim_{\varepsilon\to 0} \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{h}\,1_{\{\overline{\tau}\geq s\}}\,(\overline{X}_t^{\overline{\sigma}_\varepsilon} - \overline{X}_s)].$$

Recall that $\overline{g} := \overline{h}1_{\{\overline{\tau}\geq s\}}\colon \overline{\Omega} \to [0,1]$ is $\overline{\mathcal{F}}_s$-measurable. By Lemma 5.2, there exists a sequence of continuous $\overline{\mathcal{F}}_s$-measurable functions $\overline{g}_k\colon \overline{\Omega} \to [0,1]$ which converge $\overline{\mathbb{Q}}$-almost surely to $\overline{g}$. By Lemma 5.3 the function $\overline{\omega} \mapsto \overline{X}_{t\wedge\overline{\sigma}_\varepsilon(\overline{\omega})}(\overline{\omega})$ is lower semicontinuous for every ε. In particular, for every fixed k it holds that

$$\overline{\omega} \mapsto (\overline{H} \cdot \overline{X})_T(\overline{\omega}) \text{ is lower semicontinuous, where } \overline{H} := \overline{g}_k 1_{(s,\overline{\sigma}_\varepsilon\wedge t]} \in \overline{\mathcal{H}}.$$

Since additionally $|\overline{X}_t^{\overline{\sigma}_\varepsilon} - \overline{X}_s| \leq 2m$ and $\overline{\Omega}$ is a Polish space, there exists a sequence of continuous functions $\overline{\gamma}_n\colon \overline{\Omega} \to [-2m, 2m]$ such that $\overline{\gamma}_n \leq (\overline{H}\cdot\overline{X})_T$ and $\overline{\gamma}_n$ increases pointwise to $(\overline{H} \cdot \overline{X})_T$. Therefore $\overline{\gamma}_n \in \overline{\Gamma}$, hence by assumption

$$\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{g}_k(\overline{X}_t^{\overline{\sigma}_\varepsilon} - \overline{X}_s)] = \mathbb{E}_{\overline{\mathbb{Q}}}[(\overline{H} \cdot \overline{X})_T] = \sup_n \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{\gamma}_n] \leq 0.$$

We conclude that

$$\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{h}(\overline{X}_t^{\overline{\tau}} - \overline{X}_s^{\overline{\tau}})] = \lim_{\varepsilon\to 0} \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{h}\, 1_{\{\overline{\tau}\geq s\}}\,(\overline{X}_t^{\overline{\sigma}_\varepsilon} - \overline{X}_s)] = \lim_{\varepsilon\to 0}\lim_{k\to\infty} \mathbb{E}_{\overline{\mathbb{Q}}}[\overline{g}_k\,(\overline{X}_t^{\overline{\sigma}_\varepsilon} - \overline{X}_s)] \leq 0,$$

which implies $\overline{\mathbb{Q}}$-almost surely $\mathbb{E}_{\overline{\mathbb{Q}}}[\overline{X}_t^{\overline{\tau}}|\overline{\mathcal{F}}_s] \leq \overline{X}_s^{\overline{\tau}}$, hence $\overline{X}^{\overline{\tau}}$ is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-supermartingale.

By similar arguments one can also show that $\overline{X}^{\overline{\tau}}$ is a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-submartingale and thus a $\overline{\mathbb{Q}}$-$\overline{\mathbb{F}}$-martingale. □

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