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A FUNDAMENTAL THEOREM OF ASSET PRICING FOR CONTINUOUS TIME LARGE FINANCIAL MARKETS IN A TWO FILTRATION SETTING

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Abstract. We present a version of the fundamental theorem of asset pricing (FTAP) for continuous time large financial markets with two filtrations in an L^p-setting for $1 \leq p < \infty$. This extends the results of Yuri Kabanov and Christophe Stricker [10] to continuous time and to a large financial market setting, however, still preserving the simplicity of the discrete time setting. On the other hand it generalizes Stricker's L^p-version of FTAP [15] towards a setting with two filtrations. We do neither assume that price processes are semi-martigales, (and it does not follow due to trading with respect to the *smaller* filtration) nor that price processes have any path properties, neither any other particular property of the two filtrations in question, nor admissibility of portfolio wealth processes, but we rather go for a completely general (and realistic) result, where trading strategies are just predictable with respect to a smaller filtration than the one generated by the price processes. Applications range from modeling trading with delayed information, trading on different time grids, dealing with inaccurate price information, and randomization approaches to uncertainty.

1. Introduction

One of the often unanimously accepted hypothesis in modeling financial markets is the following:

Standard Hypothesis. *Observations of prices are perfect and can be immediately included into trading decisions.*

It is the goal of this article to consider a setting beyond this hypothesis: imagine a stock exchange with liquid prices given in continuous time, whose informational content is encoded in a large filtration $\mathbb{G}$ and whose price processes are modeled by a stochastic process S adapted to this filtration. However, like in Platon's famous allegory of the cave, the prices S are not fully revealed to us observers of the market, but only a shadow of them is visible for us traders, i.e. a perturbed observation of S. Nevertheless we (have to) trade in the market given our observational basis. We call this a *platonic stock exchange*.

We believe that the platonic stock exchange helps to encode the idealistic assumption of continuous time models and the actual observational reality in terms of a combined filtering and trading model. There are several instances, where the platonic stock exchange can be directly applied:

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: Prices come on a discrete grid possibly with a certain degree of reliability, hence the observational filtration is smaller than the idealistic continuous time model filtration of the price process.

: Prices can additionally come with frictions (transaction costs or liquidity), where the actually traded prices and the observed prices, upon which trading decisions are based, do not necessarily agree.

: The setting of stochastic portfolio theory, where relative capitalizations (prices relative to the market capitalization taken as numéraire) are quoted on different orders of magnitude with different precision and with different degrees of friction. Here again, actually traded prices and observed prices do not necessarily agree.

: Market models jointly written for underlyings and derivatives, where prices often come on different time grids, e.g. derivatives might be traded on a daily basis whereas underlyings are usually traded on a much finer grid. It is useful to introduce a difference between actually observed prices and traded prices.

: A time delay in receiving market information (or in applying it), which causes the trader's filtration simply being a delayed one in comparison to a price filtration. For instance, different time scales of trading for underlyings, which are traded on a high frequency basis, occur in all major markets.

: Markets with an execution delay of orders.

: To quantify the effect of calibration errors, which appear as additive error variables on market data. For instance in term structure models market prices are only met approximately, which is a consequence of (simple) inter- or extrapolation procedures.

: ... and, of course, the setting of model uncertainty combined with realistic market information as outlined in the sequel.

Formally speaking a *platonic financial market* is given by a stochastic basis $(\Omega, \mathcal{G}, P)$ together with two filtrations $\mathbb{F} \subset \mathbb{G}$ and a family of stochastic processes S (without any assumption on path properties) adapted to $\mathbb{G}$. Trading in the given assets is possible but only with $\mathbb{F}$-predictable, simple strategies (and limits of such strategies, which is made precise later). The smaller filtration $\mathbb{F}$ corresponds to the agent's information which actually enters into trading decisions, whereas the larger filtration $\mathbb{G}$ encodes all information from price processes which is not necessarily instantaneously available for trading decisions. The precise setting that we introduce in Section 2 is even more general in terms of the involved filtrations, but for the sake of simplicity we do not enter into details here. We emphasize (and outline this more precisely in the sequel) that this framework is also a way to incorporate model *uncertainty*, since a lack of observations actually leads to uncertainty in the model choice itself. In any case the measure P does not necessarily have the meaning of a historical measure: one more natural interpretation is a randomization by a subjective prior of a class of models among which one cannot distinguish by actual observations.

As a novelty here we do neither assume that the price processes are semimartingales (and there is also no reason to do so since trading is only using strategies predictable with respect to the smaller filtration $\mathbb{F}$), nor do we pose a standard filtering problem by filtering a "true price". We consider the non-semimartingality (and the absence of path properties) of price processes as a particular challenge as well as an advantage of our approach, for instance in view of explaining well known econometric evidence, as we encounter it for instance in high frequency data.

The main goal of the present article is to investigate all foundational questions of Mathematical Finance, i.e. fundamental theorems, superhedging and duality in this new setting of two filtrations. Our main result states that a certain "No-arbitrage" condition (see Definition 3.3) for *large* financial markets in a two filtration setup, which we call (NAFLp), is equivalent to the existence of an equivalent measure under which the optional projections of price processes S on the smaller filtration $\mathbb{F}$ are martingales (see Theorem 3.9 and Corollary 3.10). Our "No-arbitrage" condition involves L^p-integrability and L^p-convergence of terminal portfolio values with respect to some measures P', for $1 \leq p < \infty$, and turns out to be equivalent to the classical (NFLVR) condition in the case of one filtration and bounded price processes. The concept is appealing since topologies are actually quite strong, economically reasonable and no further admissibility assumptions are needed.

To the best of our knowledge this is the first *fundamental theorem of asset pricing* (FTAP) in continuous time when trading only with respect to a smaller filtration is considered. It thus extends the results of Y. Kabanov and C. Stricker [10] to continuous time and to a large financial market setting. On the other hand it generalizes Stricker's L^p-version of FTAP [15] towards a setting with two filtrations. This L^p-setting allows to tranfer the simplicity of discrete time Mathematical Finance to continuous time, i.e. no assumptions on paths and no assumptions on admissibility. Our setup also allows to overcome the disadvantage of Stricker's setting that the no-arbitrage condition depends on the measure and not only on its equivalence class, however, there are different ways to do so: we have therefore provided several competing (and equivalent) versions of our "No-arbitrage" condition. The obtained two filtration FTAP then constitutes the basis for superhedging results, where trading is again only allowed with respect to the smaller filtration. As shown in Example 1.1, our setting naturally embeds semi-static hedging. Combined with *Bayesian uncertainty modeling* as explained below these superhedging results give a new flavor to the corresponding results in the area of robust finance (see e.g. the work of Bruno Bouchard and Marcel Nutz [3]).

The remainder of the article is organized as follows. The following Subsection 1.1 introduces the concept of *Bayesian uncertainty modeling*, while Subsection 1.2 underpins the relevance of our approach by means of examples. In Section 2 we introduce the formal setting of platonic large financial markets, while Section 3 and 4 are dedicated to "No-arbitrage" definitions, FTAP and superheding results.

1.1. **Bayesian uncertainty modeling.** The above described two filtration setting is the first step towards a *dynamic* framework for modeling uncertainty where a real-time inclusion of new information and thus a decrease (or increase) in model uncertainty can be analyzed. We understand uncertainty here from a Bayesian viewpoint, i.e. we randomize over different measures [1]. We refer to this as *Bayesian uncertainty modeling* and call the whole approach *Bayesian Finance*, since optional projections will play a key role such as in Bayesian Filtering.

Let us outline what we mean by *Bayesian uncertainty modeling* and how this is naturally linked to a two-filtration framework: assume here some path space D together with its canonical filtration $\widetilde{\mathbb{F}}$ and a family of (canonical) price processes S, adapted to $\widetilde{\mathbb{F}}$. Furthermore we are given a family of probability measures P^θ for a parameter $\theta \in \Theta$. We assume an a priori given probability measure ν on Θ (of course we could at this point also introduce some time dependence on Θ but we leave this away for the sake of simplicity). Formally, our two filtration setup can be introduced in the following way: consider first on $D \times \Theta$ the (full information)

[1] Josef Teichmann is grateful to Dima Kramkov for many discussions during morning runs about this viewpoint.

filtration $\mathbb{G} := \widetilde{\mathbb{F}} \otimes \mathcal{B}(\Theta)$, and the probability measure

$$\mathbb{P}(A \times B) = \int_B P^\theta(A)\nu(d\theta)\,.$$

Of course price processes can also be defined on $D\times\Theta$, namely simply via $S_t(\omega,\theta) = \omega(t)$. In the case of perfect information we find ourselves again in a classical setting as far as trading is concerned, since the trader could use the information generated by S.

We now encode the very nature of market data, i.e. the basis for trading and for calibration, which usually come

- on a discrete, not necessarily equidistant and possibly unpredictable time grid;
- with an additional degree of non-reliability;
- with an idea on acceptable calibration accuracy (i.e. liquid prices should be perfectly calibrated, less liquid ones less perfectly, etc);

by specifying a smaller filtration $\mathcal{F}_t \subset \sigma\big((S_s)_{0\leq s\leq t}\big) \subset \widetilde{\mathcal{F}}_t \otimes \mathcal{B}(\Theta)$, for $t \geq 0$, on the product probability space, which corresponds to the information of the actual observations being often strictly smaller than the filtration generated by S. Even if data are fully reliable, already the discrete character of observations causes a filtration shrinkage. If data are additionally not fully reliable (due to market frictions or simple observational issues or delays), the smaller filtration corresponds to the actual observation whereas the larger filtrations account for model uncertainty and additionally for some noise perturbing, e.g. additively, the market data. Now we find ourselves in a full-fledged two-filtration setting (actually a three filtration setting: full information on price process S and θ, only full information on S and actual observation information on S), where trading strategies are predictable with respect to a smaller filtration $\mathbb{F}$ than the one generated by the actual (idealistic) price processes S, and where full knowledge on price processes and perturbing noise is encoded in the largest filtration $\mathbb{G}$. Hence we can analyze in this setting how market observations lead to less uncertainty on the one hand, i.e. updates of ν, and how market observations are used for trading on the other hand.

This Bayesian uncertainty setting can be simplified to a *mixture model* setting where we work in contrast to the above one not on the product space $D \times \Theta$ but on D itself and consider a probability measure of the form

$$\mathbb{P}(A) = \int_\Theta P^\theta(A)\nu(d\theta).$$

While this setting already allows to give answers to many questions coming up in robust finance (see Example 1.1), it can for instance not include derivatives whose payoff depends on the parameter θ (see Example 1.4). Note that not only the name of this setup but the whole modeling is in spirit of mixture models as considered by Damiano Brigo and Fabio Mercurio [4].

This Bayesian approach to uncertainty encodes subjective believes on Θ as probabilities, which are by no means probabilities in the sense of risk with respect to the physical measure. However, in finance we are anyway accustomed to probabilities, which do not necessarily have the meaning of statistical risk, namely, risk-neutral probabilities. Here we add a third dimension what a probability actually means, namely a subjective belief in the validity of a model.

Of course, we are not the first ones to introduce Bayesian viewpoints in Finance, see, e.g. the pioneering paper by Per Mykland [12] or the large literature on dynamic risk measures (e.g., the work by Beatrice Acciaio and Irina Penner [1] and the references therein) where updating and time consistency plays a major role. In this respect we refer in particular to [5, 13, 2]. However, up to our knowledge we are the first ones to combine Bayesian methods consequently with FTAP and duality concepts. In other words, a long term goal is a combination of the two seminal works [10] and [9].

1.2. **Examples.** The following examples illustrate the universal applicability of our two-filtrations setting and its strength in combination with Bayesian uncertainty modeling.

Example 1.1 (**Finitely many assets with a large family of semi-statically traded options**). Suppose we are in the setting of finitely many liquidly traded assets, i.e. we consider a finite number of adapted processes $S^1, \ldots S^n$ adapted to a filtration $\mathbb{F}$, together with a (large) family of option price processes $\pi(f^j)$ paying off at time $t = 1$ the payoff f^j, for $j \in J$ with J some index set. Let us be more precise here: the process $(\pi_t(f^j))_{t \in I^j}$ comes with a time grid $I^j \subset [0, 1]$ being a finite union of disjoint closed intervals, where the option is actually traded. A particular example is $I^j = \{t^j_1, \ldots, t^j_n\}$ meaning that the option j can be only traded at times t^j_k, $k \in \{1, \ldots, n\}$.

Additionally we apply the mixture setting of Bayesian uncertainty as in the last part of Subsection 1.1: consider a family of probability measures P^θ for a parameter $\theta \in \Theta$, where Θ carries an *a priori measure* ν. Assume that

$$\mathbb{P}(A) = \int_\Theta P^\theta(A)\nu(d\theta)$$

is well-defined. Notice that nullsets with respect to this mixture measure are nullsets with respect to ν-almost every P^θ.

The filtration $\mathbb{G}$ is just chosen to be the constant filtration $\mathcal{G}_t := \mathcal{F}_1$, for $t \in [0, 1]$. Assume that prices S^i_t on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, P)$ lie in some $L^p(\Omega)$ with trading based on $\mathbb{F}$. With respect to this filtration we can of course extend the price to a $\mathbb{G}$-adapted process on $[0, 1]$ via

$$\pi_t(f^j) := \pi_{\min\{s \geq t \mid s \in I^j\}}(f^j)\,.$$

This process is *not* adapted anymore to the filtration $\mathbb{F}$ but rather to $\mathbb{G}$. Since we do not require path properties it does not matter that actually this process is càglàd.

The advantage of this construction is that semi-static hedging on I^j is now expressible just via the standard stochastic integral[2]. Here $\mathbb{G}$ needs to be a filtration as large to contain already information on the values of the options at the next future trading time. Since we do not have any assumptions on $\mathbb{G}$ this can of course be assumed and our chosen filtration clearly does the job.

The super-replication result from Section 4 then reads as follows: for every $f \in L^p(\mathbb{P})$

$$\sup_{Q \in \mathbb{M}^q} E_Q[f] = \inf\{x : \exists\, g \in \overline{C_p} \text{ with } x + g \geq f\}\,.$$

The set $\mathbb{M}^q$ consists of measures $Q \sim \mathbb{P}$, $\frac{dQ}{d\mathbb{P}} \in L^q(\mathbb{P})$ such that

$$E_Q[\pi_t(f^j) \mid \mathcal{F}_s] = \pi_s(f^j)$$

for $s \leq t$ in I^j (sic!) and for all $j \in J$, and, of course, such that the optional projection (which conincides with S itself) of every price process S^i on $\mathbb{F}$ is a Q-martingale. On the other hand the super-hedge is understood $\mathbb{P}$-almost surely, i.e. for each $f \in L^p(\mathbb{P})$ there are sequences of simple trades in S and semi-static trades in finitely many options minus some consumptions converging in $L^p(\mathbb{P})$ to a limit g which dominates f minus the super-hedging price x $\mathbb{P}$-a.s., i.e. on a measurable set $A \subset \Omega$ satisfying $P^\theta(A) = 1$ for ν-almost all θ (for details see

[2]We are grateful to Matteo Burzoni and Martin Larsson for pointing this out.

Section 4). This is a first super-hedging result in continuous time with a large family of options traded on different time grids in a robust setting interpreted in a Bayesian way.

Remark 1.2. We could of course have considered a full-fledged Bayesian setting of robustness where also the price processes are not adapted to the filtration $\mathbb{F}$.

Remark 1.3. The trick to include semi-static hedges by writing piece-wise constant processes with anticipating information (depending on the amount of static properties the hedging should have: if one wants to have static hedging on the interval $]s,t] \subset [0,1]$, then time t information must already be present at $s+$ which due to our generality of $\mathbb{G}$ is feasible) works in general and one can therefore model *trading on different grids for each single asset* within a two filtration setting. This makes our setting extremely general in scope. This is in line with discrete time small financial market setting in [3].

Example 1.4 (**finitely many assets with a large family of semi-statically traded options and uncertainty swaps**). Assume the setting of Bayesian uncertainty, as of Section 1.1, which allows to include artificial derivatives $\pi(f^j)$, for $j \in J$ being traded at time $t = 0$ at an $\mathcal{F}_0$-measurable price. We imagine here payoffs f^j which depend on the uncertainty parameter θ and call them *uncertainty swaps* (even though they do not necessarily have a price 0 at time $t = 0$). These swaps represent risks related to uncertainty, i.e. how likely certain areas of Θ are. One can think of uncertain volatility, i.e. the parameter θ represents all possible volatility configurations in the market, or of risks not fully reflected in price behavior like, e.g., temperature in energy markets.

Example 1.5 (**An asset with uncertain price**). Probably the simplest and most extreme example of a two filtration setting is the following: assume a standard price process Y on a filtered probability space $(\Omega, \mathbb{G}, P)$, for instance a Black-Scholes or Heston price model, and assume additionally the existence of a centered (i.e. expectations with respect to P are vanishing), uniformly bounded $\mathbb{G}$-adapted process Z fully independent of Y whose values at each time are *independent* of all other values at any time, as it is often assumed when modeling micro structure noise. Of course Z cannot have any reasonable path properties. Define $S := Y + Z$, then the price looks like Y when observed over some time interval, but trading might lead to surprises. Our setting allows to derive super-replication results in such extreme but realistic cases.

2. Large Platonic financial markets

We consider a large platonic financial market model in continuous time in the following way. Let I be an arbitrary parameter space which can be any set, countable or uncountable. Let $T = 1$ denote the time horizon and let $(\Omega, \mathcal{G}, P)$ be a probability space with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,1]}$. On this probability space we are given a family of $\mathbb{G}$-adapted stochastic processes $(S^i_t)_{t\in[0,1]}$, $i \in I$. In particular no path properties are needed. P-almost surely is understood with respect to $\mathcal{G}$ being the largest σ-algebra in this setting.

We define, for each $n \geq 1$, a family $\mathcal{A}^n$ of subsets of I, which contain exactly n elements:

$$\mathcal{A}^n = \{\text{all/some subsets } A \subseteq I, \text{ such that } |A| = n\}, \tag{2.1}$$

where $|A|$ denotes the cardinality of the set A. Moreover, we assume that if $A^1, A^2 \in \bigcup_{n\geq 1} \mathcal{A}^n$, then $A^1 \cup A^2 \in \bigcup_{n\geq 1} \mathcal{A}^n$ (*refining property*).

We consider a family of filtrations $\mathbb{F}^A = (\mathcal{F}^A_t)_{t\in[0,1]}$, indexed by $A \in \bigcup_{n\geq 1} \mathcal{A}^n$, which are all contained in (and possibly smaller than) the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,1]}$

introduced above. Additionally we suppose for two sets $A^1, A^2 \in \bigcup_{n\geq 1} \mathcal{A}^n$, such that $A^1 \subseteq A^2$, that $\mathbb{F}^{A^1} \subseteq \mathbb{F}^{A^2}$, i.e., for each t, $\mathcal{F}_t^{A^1} \subset \mathcal{F}_t^{A^2} \subseteq \mathcal{G}_t$ (*monotonicity property*).

For each $A \in \bigcup_{n\geq 1} \mathcal{A}^n$ we define the following set of portfolio wealth processes based on simple strategies for deterministic time points in the *small financial market* A that are predictable with respect to the smaller filtration $\mathbb{F}^A = (\mathcal{F}_t^A)_{t\in[0,1]}$. To be precise:

Definition 2.1. *Let $t_0, \dots, t_l \in [0,1]$ denote a set of deterministic time points and consider a small market indexed by $A = \{\alpha_1, \dots, \alpha_n\} \in \mathcal{A}^n$. Denote $\mathbb{F}^A$-simple and bounded processes by*

$$\mathbf{H}^A = \sum_{i=1}^{l} H_{t_{i-1}}^A 1_{(t_{i-1}, t_i]}$$

with $\mathbf{H}_{t_{i-1}}^A = (H_{t_{i-1}}^{\alpha_1}, \dots, H_{t_{i-1}}^{\alpha_n})^\top \in \mathcal{F}_{t_{i-1}}^A$ for all $i \in \{1, \dots, l\}$. Then the set of simple portfolio wealth processes obtained from bounded, $\mathbb{F}^A$-simple trading strategies is defined as

$$\mathcal{X}^A = \{(\mathbf{H}^A \cdot \mathbf{S}^A)_{t\in[0,1]} : \mathbf{H}^A \ \mathbb{R}^n\text{-valued, bounded, } \mathbb{F}^A\text{-simple}\},$$

where $\mathbf{S}^A = (S^{\alpha_1}, \dots, S^{\alpha_n})^\top$ and

$$(\mathbf{H}^A \cdot \mathbf{S}^A)_t = \sum_{j=1}^{n} \sum_{i=1}^{l} H_{t_{i-1}\wedge t}^{\alpha_j} (S_{t_i\wedge t}^{\alpha_j} - S_{t_{i-1}\wedge t}^{\alpha_j}),$$

meaning that trading is done in an $\mathbb{F}^A$-predictable way.

Next we define the set $\mathcal{X}^n$ of all portfolio wealth processes with respect to simple strategies that include at most n assets (but all possible different choices of n assets). Indeed, for each $n \geq 1$, we consider the following set $\mathcal{X}^n$

$$\mathcal{X}^n = \bigcup_{A\in\mathcal{A}^n} \mathcal{X}^A. \tag{2.2}$$

Note that the sets $\mathcal{X}^n$ are neither convex nor do they satisfy a concatenation property in the sense of [11], because in both cases $2n$ assets could be involved in the combinations. Therefore the result would rather be in the larger set $\mathcal{X}^{2n}$ than in $\mathcal{X}^n$.

Definition 2.2. *We introduce the convex sets of ($\mathbb{F}$-simple) portfolio wealth processes, its terminal evaluation and the convex cone of all super-replicable claims:*

(i) *Define the set of all wealth processes defined on $\mathbb{F}$-simple strategies involving a finite number of assets in the large financial market as $\mathcal{X} = \bigcup_{n\geq 1} \mathcal{X}^n$.*
(ii) *We denote by $K_0 = \{X_1 : X \in \mathcal{X}\}$ the evaluations of elements of $\mathcal{X}$ at terminal time $T = 1$.*
(iii) *We denote by C the convex cone of all super-replicable claims (by $\mathbb{F}$-simple strategies) in the large financial market, that is,*

$$C = K_0 - L_+^0(\Omega, \mathcal{G}, P).$$

Remark 2.3. So far our setting is not only completely general but also extraordinarily realistic in the sense that it can fully capture all desired features mentioned in the introduction, in particular trading with delay and market frictions. Note that we do not need to assume any path properties for price processes.

Note that the above setting includes as examples the large financial market based on a sequence of assets as in the work of Marzia DeDonno, Paolo Guasoni and Maurizio Pratelli [7] as well as bond markets (with a continuum of assets), with trading as specified in Definition 2.1. For a more detailed discussion see [6].

3. No asymptotic L^p-free lunch and FTAP

For $1 \leq p < \infty$ we denote in the sequel $L^p(\Omega, \mathcal{G}, P) = L^p(\mathcal{G}, P)$. Moreover, for some set $E \subset L^p(\mathcal{G}, P)$ we denote by $\overline{E} = \overline{E}^{\|\cdot\|_{L^p(\mathcal{G},P)}}$ the $L^p(\mathcal{G}, P)$-closure of E.

The crucial assumption which allows us to work in an L^p-setting, as Stricker did in his work [15] in the setting of one filtration and small markets, is:

Assumption 3.1. *For some fixed p, $1 \leq p < \infty$, we denote by $\mathcal{P}_p$ the following set of measures*

$$\mathcal{P}_p = \{P' \sim P \,|\, S_t^i \in L^p(\mathcal{G}, P') \text{ for all } i \in I, t \in [0,1]\}.$$

We assume that $\mathcal{P}_p \neq \emptyset$, i.e. that there is an equivalent probability measure $P' \sim P$ such that $S_t^i \in L^p(\mathcal{G}, P')$, for all $i \in I$, $t \in [0,1]$.

Note that in the case of countably many assets, i.e., when I is countable, Assumption 3.1 is always satisfied (for each p).

Remark 3.2. Assumption 3.1 can be slightly weakened. It is enough to assume that there exists some $P' \sim P$ such that $(S_u^i - S_t^i)^- \in L^p(\mathcal{G}, P')$, for all $i \in I$, $t \leq u \in [0,1]$ for some fixed $1 \leq p < \infty$ when we only consider long-only investments in the assets. The corresponding result will then be slightly weaker in the sense that we will only get a measure such that the optional projections are supermartingales (and not martingales as in the case of Assumption 3.1). We refer to Section 3.2 for the corresponding result.

We can now define a notion of absence of arbitrage, notably without applying stochastic integration, which – at this point – would not be available in full generality.

Definition 3.3. *We say that the large financial market satisfies the condition no asymptotic L^p-free lunch (NAFLp) if there is a probability measure $P' \sim P$ as in Assumption 3.1 such that the following holds:*

$$\overline{C_p(P')} \cap L^p_+(\mathcal{G}, P') = \{0\}, \tag{3.1}$$

where $C_p(P') = C \cap L^p(\mathcal{G}, P')$ with C introduced in Definition 2.2 (iii).

From [10] we know that C is closed in L^0 when dealing with finitely many assets and finitely many trading times. Hence, elements in $\overline{C_p(P')}$ which do not lie in $C_p(P')$ necessarily involve infinitely many assets and /or infinitely many trading times.

Remark 3.4.
(i) It is obvious that $C_p(P') \neq \emptyset$, since strategies are bounded. Indeed, $K_0 \subseteq L^p(\mathcal{G}, P')$ and therefore

$$C_p(P') = K_0 - L^p_+(\mathcal{G}, P').$$

(ii) It would also be possible to consider the set $\mathcal{C}_p := C \cap \bigcap_{P' \in \mathcal{P}_p} L^p(\mathcal{G}, P')$. Since $K_0 \subseteq \bigcap_{P' \in \mathcal{P}_p} L^p(\mathcal{G}, P')$, we have similarly as above

$$\mathcal{C}_p := K_0 - \bigcap_{P' \in \mathcal{P}_p} L^p_+(\mathcal{G}, P').$$

For some fixed P', it then holds that the $L^p(\mathcal{G}, P')$-closures of $C_p(P')$ and $\mathcal{C}_p$ are the same. Indeed if g in the closure is the $L^p(P')$-limit of $g_n = f_n - h_n$ where $f_n \in K_0$ and $h_n \geq 0$ we can always choose $h_n \in L^\infty_+(\mathcal{G}) \subseteq \bigcap_{\widetilde{P} \in \mathcal{P}_p} L^p_+(\mathcal{G}, \widetilde{P})$ as $L^\infty_+(\mathcal{G})$ is dense in $L^p_+(\mathcal{G}, P')$ for the $L^p(P')$-norm.

Furthermore, (3.1) is equivalent to

$$\overline{\mathcal{C}_p}^{L^p(\mathcal{G},P')} \cap \bigcap_{\widetilde{P}\in\mathcal{P}_p} L^p_+(\mathcal{G},\widetilde{P}) = \{0\}\,. \tag{3.2}$$

Indeed, suppose (3.2) holds but (3.1) does not hold. By the above there is $g \in \overline{\mathcal{C}_p}^{L^p(\mathcal{G},P')}$ with $g \geq 0$, $g \neq 0$, which is the $L^p(P')$-limit of $g_n = f_n - h_n \in \bigcap_{\widetilde{P}\in\mathcal{P}_p} L^p(\mathcal{G},\widetilde{P})$. Then, clearly $\widetilde{g_n} = g_n - (g_n - 1)\mathbb{1}_{\{g_n\geq 1\}} \in \bigcap_{\widetilde{P}\in\mathcal{P}_p} L^p(\mathcal{G},\widetilde{P})$ as well and converges in $L^p(\mathcal{G},P')$ to $g \wedge 1$ which lies in $\cap_{\widetilde{P}\in\mathcal{P}_p} L^p_+(\mathcal{G},\widetilde{P}) \setminus \{0\}$, yielding thus a contradiction. The other direction is clear.

The following example illustrates that the choice of $\overline{C_p(P')}$ instead of $\overline{K_0}$ in the definition of (NAFLp) is crucial beyond the setting of small financial markets in discrete time.

Example 3.5. A careful reading of Example 3.3 of [14] shows that it is not possible to replace $\overline{C_p(P')}$ by $\overline{K_0}$ in the definition of (NAFLp). Indeed, let $p = 1$ and consider a one period market with countably many derivatives given at time $t = 1$ by the random variables f_n of the Example 3.3 in [14] and at price 0 at time $t = 0$. Again, we can create, as in the introduction, a two filtration setting, where hedging is actually buy & hold in this large financial market, and the filtration $\mathbb{F}$ is trivial. In this setting K_0 contains all (finite) linear combinations of f_n. As in [14] we can show that $g_n = \sum_{k=1}^n f_k \in K_0$ is bounded below by -1, for each n, and $P'(g_n \geq 1) \to 1$. Hence $\tilde{g_n} = g_n - (g_n - 1)\mathbb{1}_{\{g_n\geq 1\}} \to 1$ in L^1 by dominated convergence. Therefore $1 \in \overline{C_1(P')}$ and (3.1) is not satisfied. However, analogously as in [14] we can show that $\overline{K_0} \cap L^1_+(\mathcal{G},P') = \{0\}$.

Remark 3.6. We emphasize that we do not assume any admissibility for our portfolio wealth processes, instead we assume L^p-integrability with respect to a measure P' equivalent to the physical measure P. This follows the setting of [15]. However it does *not* share the disadvantage that the respective no arbitrage condition depends on P itself, but only on the equivalence class of P which is a desirable feature and in particular the case for the classical NFLVR condition introduced by Freddy Delbaen and Walter Schachermayer in [8]. Furthermore we do not need a stochastic integration theory at hand, which, in our general setting and in contrast to the setting of [15] is not (yet) available.

Fix now $1 < q \leq \infty$ dual to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, for p given in Assumption 3.1. We define the set of L^q-probability measures for which the optional projection of the process $(\mathbf{S}^A_t)$ with respect to the filtration $\mathbb{F}^A$ is a martingale, for all finite subsets A of I, as follows:

$$\mathcal{M}^q = \{Q \sim P \,|\, \exists\, P' \in \mathcal{P}_p \text{ s.t. } \frac{dQ}{dP'} \in L^q(\mathcal{G},P') \text{ and } E_Q[S^{\alpha_i}_u|\mathcal{F}^A_t] = E_Q[S^{\alpha_i}_t|\mathcal{F}^A_t] \text{ a.s.,}$$
$$\text{for all } A = \{\alpha_1,\ldots,\alpha_l\} \in \bigcup_{n\geq 1}\mathcal{A}^n, 1 \leq i \leq l \text{ and all } t \leq u \in [0,1]\}.$$

Moreover, for $q = 1$, we define the analogous set of equivalent probability measures without additional property on the qth moments of the Radon Nikodym density, i.e.,

$$\mathcal{M}^1 = \{Q \sim P \,|\, E_Q[S^{\alpha_i}_u|\mathcal{F}^A_t] = E_Q[S^{\alpha_i}_t|\mathcal{F}^A_t] \text{ a.s.,}$$
$$\text{for all } A = \{\alpha_1,\ldots,\alpha_l\} \in \bigcup_{n\geq 1}\mathcal{A}^n, 1 \leq i \leq l \text{ and all } t \leq u \in [0,1]\}$$

Let us remark that the only instance where the filtrations $\mathbb{F}^A$ introduced in Section 2 actually occur explicitly is in the above definition of the dual objects $\mathcal{M}^q$ for $1 \leq q \leq \infty$.

Remark 3.7. Clearly, $\mathcal{M}^1$ is convex. For $1 < q \leq \infty$, the sets $\mathcal{M}^q$ are convex as well. Indeed, for this purpose we consider the following slightly more general statement: For all $Q_i \ll P$ with $\frac{dQ_i}{dP_i} \in L^q(\mathcal{G}, P_i)$ for measures $P_i \sim P$, $i \in \{1, 2\}$, the convex combinations $Q_s := sQ_2 + (1-s)Q_1$ satisfy $\frac{dQ_s}{d\widetilde{P}} \in L^q(\mathcal{G}, \widetilde{P})$ for $0 \leq s \leq 1$, where $\widetilde{P} = \frac{1}{2}(P_1 + P_2)$. Suppose $1 < q < \infty$, since the assertion is clear for $q = \infty$. Indeed we have

$$E_{\widetilde{P}}\left[\left|\frac{dQ_i}{d\widetilde{P}}\right|^q\right] \leq 2^{q-1} E_{\widetilde{P}}\left[\left|\frac{dQ_i}{d\widetilde{P}}\right|^q \left|\frac{d\widetilde{P}}{dP_i}\right|^{q-1}\right] = 2^{q-1} E_{P_i}\left[\left|\frac{dQ_i}{dP_i}\right|^q\right] < \infty$$

for $i = 0, 1$, since $2\frac{d\widetilde{P}}{dP_i} \geq 1$. The rest follows by the triangle inequality.

Remark 3.8. The above convexity assertion is related to the fact that locally convex vector spaces formed as intersection of spaces $L^p(\mathcal{G}, P')$, where $P' \sim P$ runs over a set of probability measures subject to additional constraints (e.g., as in our case probability measures P' such that all price processes are p-integrable), has as a (strong) dual space the union of $L^q(\mathcal{G}, P')$ with respect to the same family of measures P'. The corresponding topologies are the projective and injective locally convex topologies, i.e. the initial and final topologies making all canonical maps continuous. Let us formulate this more directly in case of $X := \cap_{P' \in \mathcal{P}_p} L^p(P')$: note that $\mathcal{P}_p$ is a directed set inheriting its (reflexive, transitive and anti-symmetric) relation "$\leq$" from reversing the inclusion of the spaces $L^p(\mathcal{G}, P')$. Indeed, $P' \leq \hat{P}$, if $L^p(\mathcal{G}, \hat{P}) \subseteq L^p(\mathcal{G}, P')$. For $P' \leq \hat{P}$, consider the inclusions from $L^p(\mathcal{G}, \hat{P}) \to L^p(\mathcal{G}, P')$. Then, X is the projective limit with respect to these mappings. The topology of X is now the coarsest topology on X which makes the inclusion maps from X to $L^p(\mathcal{G}, P')$ continuous. Therefore, intersections of $L^p(\mathcal{G}, P')$ balls of some radius around 0 with X constitute a neighborhood base for this topology. Hence any linear functional ℓ with respect to this topology can be extended some $L^p(\mathcal{G}, \hat{P})$ with $\hat{P}$ in $\mathcal{P}_p$, just by the fact that the open neighborhood $\ell^{-1}(]-1, 1[)$ of 0 has to contain some intersection of $L(\mathcal{G}, P')$ balls with X. Therefore, ℓ can be represented as

$$\ell : X \to \mathbb{R},\ f \mapsto \int fg d\hat{P}$$

for some $g \in L^p(\mathcal{G}, \hat{P})$. Combining this with the fact that all linear functionals of this form for some $\hat{P} \in \mathcal{P}_p$ and some $g \in L^q(\mathcal{G}, \hat{P})$ are elements of the dual, yields the assertion that the strong dual of X is actually

$$\bigcup_{P' \in \mathcal{P}} L^q(\mathcal{G}, P').$$

The strong topology on the strong dual just means that convergence always takes place in some $L^q(\mathcal{G}, P')$.

In this direction one could also work with the locally convex vector space

$$\bigcap_{p \geq 1} \bigcap_{P' \in \mathcal{P}_p} L^p(\mathcal{G}, P')$$

and its natural projective limit topology. The corresponding "No arbitrage" condition would be to replace the fixed p in Definition 3.3 by " there exists some p" and corresponding set of martingale measures would then be $\cup_{q>1}\mathcal{M}^q$.

We have now collected all ingredients to formulate a *fundamental theorem of asset pricing* in the present context of two filtrations.

Theorem 3.9. *Suppose that Assumption 3.1 holds for some fixed* $1 \le p < \infty$. *Then the condition (NAFLp) holds if and only if* $\mathcal{M}^q \neq \emptyset$ *where* q *satisfies* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Assume first that (NAFLp) holds. This means that there exists some $P' \sim P$ such that (3.1) holds. Note that this then implies Condition (ii) of Theorem A.1 of the Appendix for $M = C_p(P')$. This in turn is equivalent to Condition (iii) in Theorem A.1 and thus yields some $Z \in L^q(\mathcal{G}, P')$ such that $Z > 0$ a.s. and $\sup_{f \in \overline{C_p(P')}} E_{P'}[Zf] < \infty$. As $C_p(P')$ is a convex cone this implies that $\sup_{f \in \overline{C_p(P')}} E_{P'}[Zf] \le 0$. Define now Q with $\frac{dQ}{dP'} = \frac{Z}{E_{P'}[Z]}$. We have that $E_Q[f] \le 0$ for all $f \in \overline{C_p(P')}$. In particular, $\pm 1\!\!1_B(S^{\alpha_i}_u - S^{\alpha_i}_t) \in C_p(P')$ for $t \le u \in [0,1]$ and $A = \{\alpha_1, \dots, \alpha_l\} \in \bigcup_{n \ge 1} \mathcal{A}^n$, $1 \le i \le l$ and $B \in \mathcal{F}^A_t$. Hence we get $E_Q[1\!\!1_B(S^{\alpha_i}_u - S^{\alpha_i}_t)] = 0$, $i = 1, \dots, l$ and so

$$E_Q[\mathbf{S}^A_u | \mathcal{F}^A_t] = E_Q[\mathbf{S}^A_t | \mathcal{F}^A_t] \tag{3.3}$$

almost surely. This shows the first direction of the theorem.

Concerning the other direction, let $Q \in \mathcal{M}^q$. By the definition of $\mathcal{M}^q$ there thus exists some $P' \in \mathcal{P}_p$ such that $\frac{dQ}{dP'} \in L^q(\mathcal{G}, P')$. Assume now that (3.1) does not hold for this P' and the dual p. Then there exists $f \neq 0$, $f \in \overline{C_p(P')} \cap L^p_+(\mathcal{G}, P')$. By definition $f = \lim_{n \to \infty} f^n$ where the limit is in $L^p(\mathcal{G}, P')$ and $f^n = X^n_1 - h^n$ with $h^n \in L^p_+(\mathcal{G}, P')$ and $X^n_1 \in K_0$. Clearly, $E_Q[X^n_1] = 0$, hence $E_Q[f^n] \le 0$ for all n. The convergence of f^n to f in $L^p(\mathcal{G}, P')$ implies that f^n converges to f in $L^1(\mathcal{G}, Q)$ and hence $E_Q[f] \le 0$. This is a contradiction to $f \ge 0$ and $f \neq 0$. $\square$

The above fundamental theorem can also be reformulated in the following way, showing that in a one filtration setting the current no arbitrage condition is equivalent to the NFLVR condition of [8] in the case of small financial markets and to the NAFLVR condition of [6] for large financial markets, whenever there exists an equivalent martingale measure for all S^i (this is the case, for example, if all S^i are bounded). Note in particular that the result therefore only depends on the equivalence class of P but not on P itself.

Corollary 3.10. *The condition (NAFLp) holds for some* $1 \le p < \infty$ *if and only if* $\mathcal{M}^1 \neq \emptyset$.

Remark 3.11. If $\mathcal{M}^1 \neq \emptyset$ then there always exists some $q > 1$ such that $\mathcal{M}^q \neq \emptyset$. Indeed, take any $Q \in \mathcal{M}^1$ and let $P' = Q$. Then $Q \in \mathcal{M}^\infty$ as $\frac{dQ}{dP'} = 1$ and all $S^i \in L^1(\mathcal{G}, Q)$.

Proof. If (NAFLp) holds, then by Theorem 3.9 $\mathcal{M}^q \neq \emptyset$. And, clearly $\mathcal{M}^q \subseteq \mathcal{M}^1$. The reverse direction follows by Remark 3.11 as (NAFL1) holds for $Q \in \mathcal{M}^1$. $\square$

3.1. **Equivalent formulations for (NAFLp).** In the spirit of Remark 3.8 one can introduce a slightly weaker versions of (NAFLp) by considering the projective locally convex topology on $\cap_{P' \in \mathcal{P}_p} L^p(\mathcal{G}, P')$. For a set $E \in \cap_{P' \in \mathcal{P}_p} L^p(\mathcal{G}, P')$ we denote the closure with respect to this topology by $\overline{E}^{\cap}$.

Corollary 3.12. *The following conditions are equivalent.*

$$\begin{aligned} \text{(NAFLp)} \Leftrightarrow & \bigcap_{P' \in \mathcal{P}_p} \overline{C_p(P')} \cap \bigcap_{P' \in \mathcal{P}_p} L^p_+(\mathcal{G}, P') = \{0\} \\ \Leftrightarrow & \overline{\bigcap_{P' \in \mathcal{P}_p} C_p(P')}^{\cap} \cap \bigcap_{P' \in \mathcal{P}_p} L^p_+(\mathcal{G}, P') = \{0\}\,. \end{aligned} \tag{3.4}$$

Remark 3.13. Note that $\overline{\bigcap_{P'\in\mathcal{P}_p} C_p(P')}^{\cap} = \overline{\mathcal{C}_p}^{\cap}$ where $\mathcal{C}_p$ was introduced in Remark 3.4 (ii).

Proof. Indeed, the implications

$$\begin{aligned}\text{(NAFLp)} &\Rightarrow \bigcap_{P'\in\mathcal{P}_p} \overline{C_p(P')} \cap \bigcap_{P'\in\mathcal{P}_p} L^p_+(\mathcal{G},P') = \{0\}\\ &\Rightarrow \overline{\bigcap_{P'\in\mathcal{P}_p} C_p(P')}^{\cap} \cap \bigcap_{P'\in\mathcal{P}_p} L^p_+(\mathcal{G},P') = \{0\}\end{aligned}$$

hold since

$$\overline{C_p(P')} \supseteq \bigcap_{P'\in\mathcal{P}_p} \overline{C_p(P')} \supseteq \overline{\bigcap_{P'\in\mathcal{P}_p} C_p(P')}^{\cap}.$$

and since we can replace $L^p_+(\mathcal{G},P)$ by $\cap_{P'\in\mathcal{P}_p} L^p_+(\mathcal{G},P')$ in the Definition of (NAFLp) (see Remark 3.4 (ii)). In order to prove that the last condition implies (NAFLp) we apply the Hahn-Banach theorem, in this locally convex case to construct an element $Q \in \mathcal{M}^q$, i.e. a normalized, separating continuous linear functional, which – by an exhaustion argument – maps characteristic functions 1_A for measurable sets A with $P(A) > 0$ to positive numbers (compare with the proof of Theorem A.1 in [16, 15]). Note here that the relevant fact used here is that $\cup_{P'\in\mathcal{P}_p} L^q(\mathcal{G},P')$ is the dual of $\cap_{P'\in\mathcal{P}_p} L^p(\mathcal{G},P')$ as shown in Remark 3.8. The existence of a measure $Q \in \mathcal{M}^q$ means by Theorem 3.9 that (NAFLp) holds true. □

The bipolar theorem now allows to show equality of the following sets

$$\overline{\bigcap_{P'\in\mathcal{P}_p} C_p(P')}^{\cap} = \bigcap_{P'\in\mathcal{P}_p} \overline{C_p(P')}$$

under (NAFLp), yielding a nice characterization of the closure of $\cap_{P'\in\mathcal{P}_p} C_p(P')$ in the projective locally convex topology. To this end, let us introduce the polar cone of a convex cone $E \in \cap_{P'\in\mathcal{P}_p} L^p(\mathcal{G},P')$ denoted by E°:

$$E^\circ = \left\{ g \in \bigcup_{P'\in\mathcal{P}_p} L^q(\mathcal{G},P') : E[fg] \le 0, \text{ for all } f \in E \right\}.$$

Theorem 3.14. *Under (NAFLp) (or one of the equivalent conditions in* (3.4)*) it holds that*

$$\overline{\bigcap_{P'\in\mathcal{P}_p} C_p(P')}^{\cap} = \bigcap_{P'\in\mathcal{P}_p} \overline{C_p(P')}.$$

Proof. Let us show first that

$$(\underbrace{\overline{\bigcap_{P'\in\mathcal{P}_p} C_p(P')}^{\cap}}_{:=V})^\circ = (\underbrace{\bigcap_{P'\in\mathcal{P}_p} \overline{C_p(P')}}_{:=W})^\circ = \bigcup_{\lambda\ge 0} \lambda \overline{\mathcal{M}^q}^{\cup},$$

where $\overline{\mathcal{M}^q}^{\cup}$ denotes closure with respect to the injective locally convex topology on $\cup_{P'\in\mathcal{P}_p} L^q(\mathcal{G},P')$. First assume that $Z = \frac{dQ}{dP'}$ for some $Q \in \overline{\mathcal{M}^q}^{\cup}$ and $P' \in \mathcal{P}_p$. Let $f \in V, W$. Then $f \in \overline{C_p(P')}$ and thus $E_Q[f] \le 0$. This shows $\bigcup_{\lambda\ge 0} \lambda \overline{\mathcal{M}^q}^{\cup} \subseteq V^\circ, W^\circ$.

Assume now $Z \in V^\circ, W^\circ$. As $-\cap_{P'\in\mathcal{P}_p} L^p_+(\mathcal{G},P') \subseteq V, W$ this immediately implies that $Z \ge 0$ a.s. Assume the non-trivial case that $P(Z > 0) > 0$ and define a probability measure $Q \ll P'$ for some $P' \in \mathcal{P}_p$ via $\frac{dQ}{dP'} = \frac{Z}{E_{P'}[Z]}$. Hence we get $E_Q[f] \le 0$ for all $f \in V, W$. As all S^i_t are in $\cap_{P'\in\mathcal{P}_p} L^p(\mathcal{G},P')$ we have that, for $t \le u$, $\pm 1\!\!1_B(S^i_u - S^i_t) \in V, W$ for B in an appropriate $\mathcal{F}^i_t$ and all $i \in I$. This shows that $Q \in \overline{\mathcal{M}^q}$ and proves the above claim. By the bipolar theorem applied in this locally convex case, we then have

$$V^{\circ\circ} = V, \quad W^{\circ\circ} = \overline{W}^{\cap}.$$

As the polars $V^\circ = W^\circ$ it follows that $V = \overline{W}^{\cap}$. But since $W \supseteq V = \overline{W}^{\cap}$ it follows that $W = \overline{W}^{\cap} = V$. □

Remark 3.15. Working with the topologically more involved setting of intersections and unions of L^p- and L^q-spaces, we see that (NAFLp) could have been defined via

$$\bigcap_{P' \in \mathcal{P}_p} \overline{C_p(P')} \cap \bigcap_{P' \in \mathcal{P}_p} L^p_+(\mathcal{G}, P') = \{0\}.$$

In case of bounded price processes this is already on the level of the "no-arbitrage" condition similar to (NFLVR) (in a one filtration setup with finitely many assets), however without an (explicit) admissibility assumption which is however implicit due to our simple bounded trading strategies.

3.2. **No asymptotic L^p free lunch for long only portfolios and FTAP.** We will now make the setting of Remark 3.2 precise and proceed in an analogous way. Indeed, in this subsection we will assume the following.

Assumption 3.16. *We assume that there is an equivalent probability measure $P' \sim P$ such that*

$$(S^i_u - S^i_t)^- \in L^p(\mathcal{G}, P'), \text{ for all } i \in I,\ t \le u \in [0,1]$$

for some fixed $1 \le p < \infty$. We denote the set of all measures $P' \sim P$ satisfying this property by $\mathcal{P}_p^{long}$.

Let us define the set of all equivalent L^q-measures such that the optional projections of each S^i are supermartingales:

$$\mathcal{S}^q = \{Q \sim P \,|\, \exists P' \in \mathcal{P}_p^{\text{long}} \text{ s.t. } \frac{dQ}{dP'} \in L^q(\mathcal{G}, P') \text{ and } E_Q[S^{\alpha_i}_u|\mathcal{F}^A_t] \le E_Q[S^{\alpha_i}_t|\mathcal{F}^A_t] \text{ a.s.},$$
$$\text{for all } A = \{\alpha_1, \dots, \alpha_l\} \in \bigcup_{n \ge 1} \mathcal{A}^n, 1 \le i \le l \text{ and all } t \le u \in [0,1]\}$$

In the definition of the wealth processes $\mathcal{X}^A$ as in Definition 2.1, for all sets A, we assume now that, additionally, $H^{\alpha_j}_{t_{i-1}} \ge 0$, for all i, j. This means that we are only allowed to have long positions in all assets. The corresponding definitions of K_0, $C_p(P')$, $\overline{C_p(P')}$ and (NAFLp) are then analogous as in Definition 2.2 and Definition 3.3.

Note that by the Assumption 3.16 it is clear that $C_p(P') \neq \emptyset$. Indeed, for $f \in K_0$ defined with a *bounded* nonnegative $\mathbb{F}$-simple integrand we have that, for example,

$$f \wedge 1 = f - (f-1)\mathbb{1}_{\{f>1\}} \in C_p(P'),$$

as, by the boundedness of the integrand and Assumption 3.16 we have that $(f \wedge 1)^- \in L^p(\mathcal{G}, P')$ and by definition $f \wedge 1 \le 1$.

Theorem 3.17. *Suppose that Assumption 3.16 holds for some fixed $1 \le p < \infty$. Then (NAFLp) holds with non-negative strategies if and only if $\mathcal{S}^q \neq \emptyset$ where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Assume that (NAFLp) holds. We proceed exactly as in the proof of Theorem 3.9. But in the last step of this direction we only get that $\mathbb{1}_B(S^{\alpha_i}_u - S^{\alpha_i}_t)^- \in$

$L^p(\mathcal{G}, P')$ which immediately implies that

$$\begin{aligned}&(\mathbb{1}_B(S_u^{\alpha_i} - S_t^{\alpha_i})) \wedge n \\ &\quad = \mathbb{1}_B(S_u^{\alpha_i} - S_t^{\alpha_i}) - (\mathbb{1}_B(S_u^{\alpha_i} - S_t^{\alpha_i}) - n)\, \mathbb{1}_{\{\mathbb{1}_B(S_u^{\alpha_i} - S_t^{\alpha_i}) > n\}} \in C_p(P'),\end{aligned}$$

for all $n \geq 1$. Hence $E_Q[(\mathbb{1}_B(S_u^{\alpha_i} - S_t^{\alpha_i})) \wedge n] \leq 0$, for all $n \geq 1$. By Fatou's Lemma we get that $E_Q[\mathbb{1}_B(S_u^{\alpha_i} - S_t^{\alpha_i})] \leq 0$ which shows the first direction of the theorem.

Concerning the other direction, let $Q \in \mathcal{S}^q$. By the definition of $\mathcal{S}^q$ there thus exists some $P' \in \mathcal{P}_p^{\text{long}}$ such that $\frac{dQ}{dP'} \in L^q(\mathcal{G}, P')$. Assume now that (3.1) (for non-negative strategies) does not hold for this P' and the dual p. Proceeding exactly as in the proof of Theorem 3.9, due to the non-negativity of the strategies, the boundedness of the integrands and as $Q \in \mathcal{S}^q$ we get $E_Q[X_1^k] \leq 0$. The rest follows analogously as in the proof of Theorem 3.9. □

4. A Super-replication result

This section is dedicated to present a super-replication results in the present L^p-setting. Throughout this section we assume that, for some fixed $1 \leq p < \infty$, (3.1) holds for the original measure P, and we say (NAFLp) holds for P and write C_p for $C_p(P)$. This means in particular that $P \in \mathcal{P}_p$. Let us also introduce the following sets of measures

$$\begin{aligned}\mathbb{M}^q = \mathcal{M}^q(P) = \{Q \sim P \,|\, \frac{dQ}{dP} \in L^q(\mathcal{G}, P) \text{ and } E_Q[S_u^{\alpha_i}|\mathcal{F}_t^A] = E_Q[S_t^{\alpha_i}|\mathcal{F}_t^A] \text{ a.s.,}\\ \text{for all } A = \{\alpha_1, \ldots, \alpha_l\} \in \bigcup_{n \geq 1} \mathcal{A}^n, 1 \leq i \leq l \text{ and all } t \leq u \in [0,1]\},\end{aligned}$$

which play a key role in our super-replication result. Note that $\mathcal{M}^q = \bigcup_{P' \in \mathcal{P}_p} \mathcal{M}^q(P')$ and that the proof of Theorem 3.9 implies the following assertion.

Corollary 4.1. *The condition (NAFLp) holds for P if and only if $\mathbb{M}^q \neq \emptyset$.*

We henceforth identify measures $Q \in \mathbb{M}^q$ with their density $\frac{dQ}{dP}$ so that we can consider $\mathbb{M}^q$ as a subset of $L^q(\mathcal{G}, P)$. Recall that $\overline{\mathbb{M}^q}$ is the closure of $\mathbb{M}^q$ in $L^q(\mathcal{G}, P)$.

Remark 4.2. The closure of $\mathbb{M}^q$ in L^q just consists of the corresponding absolutely continuous measures, that is,

$$\begin{aligned}\overline{\mathbb{M}^q} = \{Q \ll P, \frac{dQ}{dP} \in L^q(\mathcal{G}, P) : E_Q[S_u^{\alpha_i}|\mathcal{F}_t^A] = E_Q[S_t^{\alpha_i}|\mathcal{F}_t^A] \text{ a.s.,}\\ \text{for all } A = \{\alpha_1, \ldots, \alpha_l\} \in \bigcup_{n \geq 1} \mathcal{A}^n, 1 \leq i \leq l \text{ and all } t \leq u \in [0,1]\}\end{aligned} \tag{4.1}$$

Indeed, take any $Q \in \overline{\mathbb{M}^q}$ and $Q' \in \mathbb{M}^q$. Then $Q^n = (1 - 2^{-\frac{n}{q}})Q + 2^{-\frac{n}{q}} Q' \in \mathbb{M}^q$ converges to Q with respect to $L^q(\mathcal{G}, P)$-norm.

Analogously to Section 3.1, denote by E° the polar cone of a convex cone $E \subseteq L^p(\mathcal{G}, P)$, i.e.,

$$E^\circ = \{g \in L^q(\mathcal{G}, P) : E[fg] \leq 0, \text{ for all } f \in E\}.$$

Similarly as in the proof of Theorem 3.14, we can show the following duality result.

Lemma 4.3. *For the polar cone the following identity holds true*

$$(\overline{C_p})^\circ = (C_p)^\circ = \bigcup_{\lambda \geq 0} \lambda \overline{\mathbb{M}^q}\,.$$

Proof of Lemma 4.3. First assume that $Z = \frac{dQ}{dP}$ for some $Q \in \overline{\mathbb{M}^q}$. Let $f \in \overline{C_p}$. So $f = \lim_{n\to\infty} f^n$ in $L^p(\mathcal{G}, P)$ with $f^n \in C_p$. Hence $E_Q[f^n] \leq 0$ for all n and the same holds for the $L^1(\mathcal{G}, P)$-limit of Zf^n, and so $E_Q[f] \leq 0$. This shows $\bigcup_{\lambda\geq 0} \lambda\overline{\mathbb{M}^q} \subseteq (\overline{C_p})^\circ$. And clearly $(\overline{C_p})^\circ \subseteq (C_p)^\circ$.

Assume now $Z \in (C_p)^\circ$. As $-L^p_+(\mathcal{G}, P) \subseteq C_p$ this immediately implies that $Z \geq 0$ a.s. Assume the non-trivial case that $P(Z > 0) > 0$ and define a probability measure $Q \ll P$ via $\frac{dQ}{dP} = \frac{Z}{E_P[Z]}$. Hence we get $E_Q[f] \leq 0$ for all $f \in C_p$ and in the $L^p(\mathcal{G}, P)$-closure $\overline{C_p}$. As all S^i_t are in $L^p(\mathcal{G}, P)$ as $P \in \mathcal{P}_p$ we have that, for $t \leq u$, $\pm\mathbb{1}_B(S^i_u - S^i_t) \in C_p$ for B in an appropriate $\mathcal{F}^i_t$ and all $i \in I$. This shows that $Q \in \overline{\mathbb{M}^q}$. This finishes the proof. □

We can now prove the following super-replication result.

Theorem 4.4. *Let $f \in L^p(\mathcal{G}, P)$. Then*

$$\sup_{Q\in\mathbb{M}^q} E_Q[f] = \inf\left\{x \in \mathbb{R} \mid \exists\, g \in \overline{C_p} \textit{ with } x + g \geq f\right\}$$

Proof of Theorem 4.4. By Lemma 4.3 it is clear that sup $\leq$ inf. Let now $x_0 = \sup_{Q\in\mathbb{M}^q} E_Q[f]$ and suppose that $x_0 < \inf\{x \in \mathbb{R} \mid \exists\, g \in \overline{C_p} \text{ with } x + g \geq f\}$. Then $f - x_0 \notin \overline{C_p}$. Hence there exists $Z \in L^q(\mathcal{G}, P)$ such that $\sup_{g\in\overline{C_p}} E[Zg] \leq 0$ and $E[Z(f - x_0)] > 0$. This implies that $Z \in (\overline{C_p})^\circ$ and $Z \neq 0$ therefore we can define a measure $Q \in \mathbb{M}^q$ by $\frac{dQ}{dP} = \frac{Z}{E_P[Z]}$. We get

$$x_0 < E_Q[f] \leq \sup_{R\in\mathbb{M}^q} E_R[f].$$

This implies that inf $\leq$ sup. □

Remark 4.5. As shown in the course of the proof in Theorem 3.14

$$\left(\overline{\bigcap_{P'\sim\mathcal{P}_p} C_p(P')}^{\cap}\right)^\circ = \left(\bigcap_{P'\sim\mathcal{P}_p} \overline{C_p(P')}\right)^\circ = \bigcup_{\lambda\geq 0} \lambda\overline{\mathcal{M}^q}^{\cup}$$

holds true. Recall here that $\overline{E}^{\cap}$ and $\overline{E}^{\cup}$ denote the closure of a set E with respect to the projective respectively injective locally convex topology on $\cap_{P'\in\mathcal{P}_p} L^p(\mathcal{G}, P')$ respectively $\cup_{P'\in\mathcal{P}_p} L^q(\mathcal{G}, P')$. This gives rise to another slightly different super-replication result, namely for $f \in \cap_{P'\in\mathcal{P}_p} L^p(\mathcal{G}, P')$, we have

$$\sup_{Q\in\mathcal{M}^q} E_Q[f] = \inf\left\{x \in \mathbb{R} \mid \exists\, g \in \bigcap_{P'\in\mathcal{P}_p} \overline{C_p(P')} \text{ with } x + g \geq f\right\}$$

In the setting of Subsection 3.2 under the weaker Assumption 3.16 we get an analogous super-replication result for long only strategies and measures in $\mathbb{S}^Q$ defined as follows:

$$\begin{aligned}\mathbb{S}^q = \{Q \sim P \mid \frac{dQ}{dP} \in L^q(\mathcal{G}, P) \text{ and } E_Q[S^{\alpha_i}_u|\mathcal{F}^A_t] \leq E_Q[S^{\alpha_i}_t|\mathcal{F}^A_t] \text{ a.s.,}\\ \text{for all } A = \{\alpha_1, \ldots, \alpha_l\} \in \bigcup_{n\geq 1} \mathcal{A}^n, 1 \leq i \leq l \text{ and all } t \leq u \in [0, 1]\}.\end{aligned}$$

Theorem 4.6. *Let $f \in L^p(\mathcal{G}, P)$. Then*

$$\sup_{Q\in\mathbb{S}^q} E_Q[f] = \inf\left\{x \in \mathbb{R} \mid \exists\, g \in \overline{C_p} \textit{ for long only strategies such that } x + g \geq f\right\}.$$

Proof. For the proof of Theorem 4.6 we have to adapt Lemma 4.3 by replacing $\overline{\mathbb{M}^q}$ by $\overline{\mathbb{S}^q}$. In the proof we get by the boundedness of the integrands and as $Q \in \overline{\mathbb{S}^q}$ that $E_Q[f^n] \leq 0$ for $f^n = X_1^n - h^n$ with $f^n \in L^p(\mathcal{G}, P)$, $X_1^n \in K_0$ and $h^n \in L^0_+(\mathcal{G}, P)$. The rest is identical. □

The next theorem represents elements of $\overline{C_p}$ as L^0-limits of replicable claims minus consumptions, and clarifies additionally that

$$\overline{C_p} \cap -\overline{C_p} \subset \overline{K_0}^{L^0} \cap L^p(\mathcal{G}, P)$$

as well as that proper intervals of arbitrage-free prices are open. Here we denote by $\overline{K_0}^{L^0}$ the L^0-closure of K_0. These considerations are of course almost classical, their proofs do not differ much from classical counterparts.

Remark 4.7. (i) Note that $\overline{C_p} \cap -\overline{C_p} \supseteq \overline{K_0}$, where $\overline{K_0}$ is the L^p-closure of K_0.
(ii) The set $\overline{C_p} \cap -\overline{C_p}$ is dually characterized as the set of elements g such that $E_Q[g] = 0$ for all $Q \in \overline{\mathbb{M}^q}$ by the bipolar theorem.

For (ii) note that if g is in $\overline{C_p} \cap -\overline{C_p}$. Then clearly $E_Q[g] = 0$ for all $Q \in \overline{\mathbb{M}^q}$. On the other hand, suppose $E_Q[g] = 0$ for all $Q \in \overline{\mathbb{M}^q}$. Then g and $-g$ are in $\left(\bigcup_{\lambda \geq 0} \lambda \overline{\mathbb{M}^q}\right)^\circ = \overline{C_p}$, where the last equality holds by the bipolar theorem.

Theorem 4.8. *Assume that (NAFLp) holds for P.*

(i) *Every $g \in \overline{C_p}$ can be represented as L^p-limit $g = \lim_{n\to\infty} \left(f^n - h^n\right) = f - h$, where f and h are finitely valued random variables, with $f = \lim_{n\to\infty} f^n$ appearing only as limit in probability of a sequence $f_n \in K_0$, and $\lim_{n\to\infty} h^n = h \geq 0$ being again a limit in probability of finitely valued, non-negative random variables h^n.*
(ii) *Let $\tilde{g} \in L^p(\mathcal{G}, P)$. Then either $\tilde{g}$ is replicable (attainable), i.e.*

$$\tilde{g} - x \in \overline{C_p} \cap -\overline{C_p} = \bigcap_{Q \in \overline{\mathbb{M}^q}} \overline{K_0}^{L^1(Q)} \cap L^p(\mathcal{G}, P) \subset \overline{K_0}^{L^0} \cap L^p(\mathcal{G}, P)$$

for some $x \in \mathbb{R}$, or there are at least two measures $Q, Q' \in \mathbb{M}^q$ such that $E_Q[\tilde{g}] \neq E_{Q'}[\tilde{g}]$. In the second case the super-replication price $x = \sup_{Q \in \mathbb{M}^q} E_Q[\tilde{g}]$ is not attained by any equivalent measure $Q \in \mathbb{M}^q$.

Proof. For the first assertion take $g = \lim_{n\to\infty}(g^n - k^n)$, an L^p-limit, where $g^n \in K_0$ and $k^n \in L^p_+(\mathcal{G}, P)$. By Komlos' theorem we can choose forward convex combinations h^n of elements $k^n, k^{n+1}, \ldots$ such that $h^n \to h$ in probability, where $h \geq 0$ is a not necessarily finitely valued random variable. Take forward convex combinations with the same weights of $g^n, g^{n+1}, \ldots$ and denote them by f^n. Then again $f^n - h^n \to g$ in L^p. Take now any $Q \in \mathbb{M}^q$. Then $E_Q[f^n] = 0$ and by $L^p(\mathcal{G}, P)$-convergence we have that $\lim_{n\to\infty} E_Q[-h^n] = E_Q[g] > -\infty$. By Fatou's Lemma

$$0 \leq E_Q[h] \leq \lim_{n\to\infty} E_Q[h^n] < \infty,$$

whence h is finitely valued as well as the limit $\lim_{n\to\infty} f^n = f$, which is only understood in L^0.

For the second assertion take $\tilde{g} \in L^p(\mathcal{G}, P)$ and $x \in \mathbb{R}$ such that $g = \tilde{g} - x \in \overline{C_p} \cap -\overline{C_p}$. By Remark 4.7, $E_Q[g] = 0$ for all $Q \in \overline{\mathbb{M}^q}$. As in the previous step by passing to forward convex combinations we find two sequences $f^n \in K_0$ and $h^n \geq 0$, each converging in L^0 to finitely valued random variables, such that $f^n - h^n \to g$ in L^p as $n \to \infty$. Take any $Q \in \mathbb{M}^q$, then $0 \leq \lim_{n\to\infty} E_Q[h^n] = -E_Q[g] = 0$ by convergence in L^p. Hence actually $h = 0$. Therefore $g \in \overline{K_0}^{L^0} \cap L^p(\mathcal{G}, P)$. Moreover we have that $h^n \to 0$ in $L^1(\mathcal{G}, Q)$ and hence

$$E_Q[|f^n - g|] \le E_Q[|f^n - h^n - g|] + E_Q[h^n] \to 0,$$

for $n \to \infty$ as $f^n - h^n \to g$ in $L^p(\mathcal{G}, P)$ and $h^n \to 0$ in $L^1(\mathcal{G}, Q)$. So we have that, in fact, $g \in \bigcap_{Q \in \overline{\mathbb{M}^q}} \overline{K_0}^{L^1(Q)}$.

This argument holds also true when there is only one element $Q \in \mathbb{M}^q$ such that $E_Q[\tilde{g}] = x$, whence the last assertion that proper pricing intervals have to be open which in turn only appears in case of non-attainability.

Finally observe that for $g \in \bigcap_{Q \in \overline{\mathbb{M}^q}} \overline{K_0}^{L^1(Q)} \cap L^p$ we have that $E_Q[g] = 0$ for any $Q \in \overline{\mathbb{M}^q}$ and hence g is in $\overline{C_p} \cap -\overline{C_p}$ which shows the equality of the two sets. □

Remark 4.9. Notice that a replicable claim in our setting is replicated by L^0-limits of elements of K_0, not necessarily by its L^p-limits. This subtlety cannot be removed.

The following example shows that $\overline{C_1} \cap -\overline{C_1} \subsetneq \overline{K_0}^{L^0} \cap L^1(\mathcal{G}, P)$:

Example 4.10. Consider a one period market with countably many derivatives $f^j \ge -1$ at time $T = 1$, $j \ge 0$, at price 0 at time $t = 0$. Assume for simplicity that the historical measure P satisfies already the following two conditions:

- $E[f^j] = 0$ for $j \ge 0$.
- The sequence f^j converges to -1 almost surely with respect to P, hence – of course – the convergence is not in L^1.

In this case we can create, as in the introduction, a two filtration setting, where hedging is actually buy & hold in this large financial market, and the filtration $\mathbb{F}$ is trivial. In this setting K_0 contains all (finite) linear combinations of f^j, its closure in L^1 only contains elements with vanishing expectations, however, its L^0 closure even if intersected with L^1 contains the constant function -1 and 1 (the latter by taking $-f_j \in K_0$). However $\{-1, 1\} \notin \overline{C_1} \cap -\overline{C_1}$, whence

$$\overline{C_1} \cap -\overline{C_1} \subsetneq \overline{K_0}^{L^0} \cap L^1.$$

Appendix A. A technical result

The following theorem, which goes back to Jia-An Yan [16] for $p = 1$ and to Jean-Pascal Ansel for the case $1 \le p < \infty$ is taken from [15]:

Theorem A.1. *Let E be a convex subset of $L^p(\mathcal{G}, P)$ with $0 \in E$. Then the following three conditions are equivalent:*

(i) *For every $\eta \in L^p_+(\mathcal{G}, P)$, $\eta \neq 0$, there exits some $c > 0$ such that $c\eta \notin \overline{E - L^p_+(\mathcal{G}, P)}$.*
(ii) *For every $A \in \mathcal{G}$ such that $P[A] > 0$, there exists some $c > 0$ such that $c1_A \notin \overline{E - L^p_+(\mathcal{G}, P)}$.*
(iii) *There exists a random variable $Z \in L^q(\mathcal{G}, P)$ such that $Z > 0$ a.s. and $\sup_{Y \in E} E[ZY] < \infty$.*

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