On the topological 4-genus of torus knots

ON THE TOPOLOGICAL 4-GENUS OF TORUS KNOTS

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Abstract. We prove that the topological locally flat slice genus of large torus knots takes up less than three quarters of the ordinary genus. As an application, we derive the best possible linear estimate of the topological slice genus for torus knots with non-maximal signature invariant.

1. Introduction

The Thom conjecture asserts that algebraic curves in $\mathbb{CP}^2$ are genus-minimising within their homology class [KM94]. More precisely, no smooth embedded surface in $\mathbb{CP}^2$ has smaller genus than an algebraic curve homologous to that surface. Regularity plays an important role here. In fact, Rudolph proved the existence of topological locally flat surfaces with strictly smaller genus than all algebraic curves homologous to it [Rud84]. A precise quantitative measure of the drop in genus for locally flat surfaces was given in [LW97]. The knot theoretic version of the Thom conjecture asserts that the smooth slice genus of a positive braid knot coincides with the ordinary genus [Rud93]. Much less is known about the topological locally flat slice genus g_4 of positive braid knots, or even torus knots. Positive braid knots have non-zero signature invariant σ [Rud82], whence $g_4 > 0$, by the following signature bound: $|\sigma| \leq 2g_4$. This bound was proven smoothly in [Mur65], and for the locally flat slice genus in [KT76]. Using the existence of quasipositive knots with Alexander polynomial 1, Rudolph showed that the torus knot $T(5,6)$ has $g_4 < g$, where g is the classical minimal genus of knots [Rud84]. The main purpose of this paper is to show that the genus defect $\Delta g = g - g_4$ takes up a large portion of the genus for most torus knots.

Theorem 1. *Let $K = T(p,q)$ be a torus knot with non-maximal signature invariant, i.e. $K \neq T(2,n)$, $T(3,4)$, $T(3,5)$. Then*

$$g_4(K) \leq \frac{6}{7}\, g(K).$$

This result is sharp, since the torus knot $T(3,8)$ has $g_4 = 6$ and $g = 7$. However, a larger genus defect is attained for torus knots with

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large parameters $p, q \in \mathbb{N}$. The classical genus formula $g(T(p,q)) = \frac{1}{2}(p-1)(q-1)$ yields

$$\lim_{p,q\to\infty} \frac{2}{pq} g(T(p,q)) = 1.$$

Here the limit is understood as $\lim \min\{p,q\} \to \infty$ (i.e. both parameters must be taken to infinity). As we will see, the corresponding limit for g_4 drops by at least one quarter. The existence of this limit follows from the subadditivity of the function $g_4(T(p,q))$ in both parameters (see the proof of Proposition 9 in the Appendix of [Liv10] for the one-variable case known as Fekete's Lemma; the two-variable case follows from an analogous estimate between the ratios $\frac{g_4(T(p,q))}{pq}$ and $\frac{g_4(T(N,N))}{N^2}$, where $p = aN + b$ and $q = cN + d$).

Theorem 2.

$$\lim_{p,q\to\infty} \frac{2}{pq} g_4(T(p,q)) < \frac{3}{4}.$$

To the best of our knowledge, no attempt at determining the actual limit has been made so far. The signature bound $|\sigma| \leq 2g_4$ potentially allows a drop down to one-half, since

$$\lim_{p,q\to\infty} \frac{1}{pq} \sigma(T(p,q)) = \frac{1}{2}.$$

The latter is an easy consequence of the signature formula for torus knots by Gordon, Litherland and Murasugi [GLM81].

We will prove Theorems 1 and 2 in Sections 4 and 3, respectively. The reason for the reverse order is simple: Theorem 2 implies Theorem 1, up to finitely many values of the braid index $\min\{p,q\}$, since $\frac{3}{4} < \frac{6}{7}$. The main tool for proving Theorem 2 is a homological improvement of Rudolph's method, which we will explain in Section 2.

The strength of this method is demonstrated in Proposition 7, which provides a sharp estimate of the topological slice genus for positive fibred arborescent links. In particular, we find prime positive braid links of arbitrarily large genus with $\frac{g_4}{g} = \frac{1}{2}$.

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2. Construction of locally flat surfaces

Let us first briefly fix notation and conventions. We assume all Seifert surfaces to be connected. The genus $g(L)$ and Betti number $b_1(L)$ of a link L are the minimal genus and Betti number of a Seifert surface of L, respectively. Homology groups are considered over the integers. The topological slice genus $g_4(L)$ is the minimal genus of a *slice surface* of L, i.e. of a connected oriented compact surface, properly and locally flatly embedded into the 4-ball, whose boundary is L. For any surface

Σ, a subsurface $\Sigma' \subset \Sigma$ is simply a surface contained in Σ, assuming neither that Σ' is connected, nor that it is embedded properly into Σ. We write $a_1, \dots, a_{n-1}$ for the standard generators of the braid group on n strands. If a braid is given by a braid word β, we write $\widehat{\beta}$ for its closure. A non-split braid word β yields a canonical Seifert surface for $\widehat{\beta}$, which we denote by $\Sigma(\beta)$. If, in addition, β is positive, $\Sigma(\beta)$ is in fact the fibre surface of $\widehat{\beta}$ [Sta78]. The Alexander polynomial of a bilinear integral form represented by a matrix M is $\det(t \cdot M - M^\top) \in \mathbb{Z}[t^\pm]$; this does not depend on the chosen matrix, and is considered up to multiplication with a unit.

Our main tool uses Freedman's celebrated result [Fre82, FQ90] to construct slice surfaces of lower genus from Seifert surfaces by ambient surgery. See [Fel16, BL15, FM16] for other applications of this method. Here we prove a version for multi-component links.

Proposition 3. *Let L be a link with a Seifert surface Σ. Let $V \subset H_1(\Sigma)$ be a subgroup. If the Seifert form of Σ restricted to V has Alexander polynomial* 1*, then L has a slice surface of genus $g(\Sigma) - \operatorname{rk} V/2$.*

We will call such a subgroup V *Alexander-trivial.* Before the proof, let us show a sample application.

Example 4. The link L given as the closure of the positive 4-braid $a_1a_3a_2^2a_1a_3a_2^3$ has topological slice genus one. Calculating the signature yields $1 = \frac{|\sigma(L)|-2}{2} \leq g_4(L)$ by the bound provided in [KT76]. We now show that L has a genus one slice surface. For this, let $\widetilde{X}$ be the canonical fibre surface $\Sigma(a_1a_3a_2^2a_1a_3a_2^3)$. Observe that $\widetilde{X}$ is a plumbing of six positive Hopf bands along an X-shaped tree, as shown in Figure 1. Here we use the fact that fibre surfaces with the same boundary link are isotopic. The two additional simple closed curves in the figure (red and dashed blue) represent homology classes $[\gamma_1]$ and $[\gamma_2]$ in $H_1(\widetilde{X})$. We claim that the subspace V generated by $[\gamma_1]$ and $[\gamma_2]$ is Alexander-trivial. A matrix for the Seifert form of the boundary link is given by the 6×6 matrix A, where

$$A_{ii} = A_{12} = A_{23} = A_{43} = A_{53} = A_{63} = 1$$

and $A_{ij} = 0$ otherwise. In the chosen basis, $[\gamma_1]$ and $[\gamma_2]$ are represented by the vectors $(0, 1, -2, 1, 1, 1)^\top$ and $(1, 0, 0, 0, 0, 0)^\top$, respectively. A direct computation yields

$$[\gamma_1]^\top A[\gamma_1] = [\gamma_1]^\top A[\gamma_2] = 0,$$
$$[\gamma_2]^\top A[\gamma_2] = [\gamma_2]^\top A[\gamma_1] = 1,$$

so a matrix B of the Seifert form restricted to V is given by

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

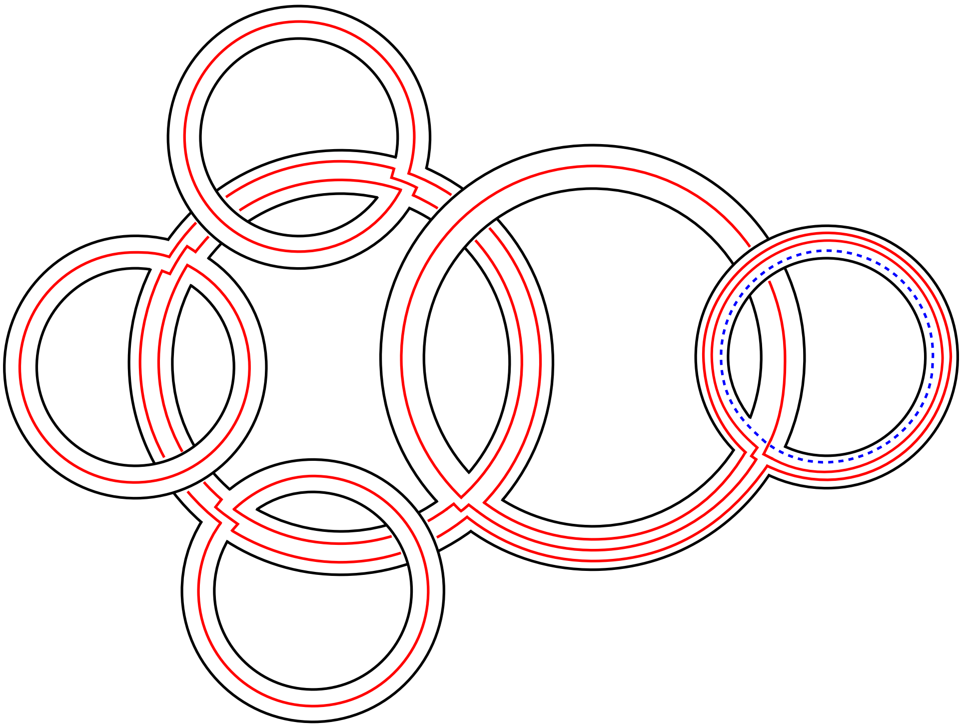

Figure 1. The surface $\widetilde{X}$, which is a plumbing of six positive Hopf bands. For simplicity, the twists of the individual Hopf bands are not drawn. Two curves representing homology classes of interest are drawn in red and dashed blue – see Example 4 for details.

Indeed, we now have $\det(t \cdot B - B^\top) = t$, which is a unit in $\mathbb{Z}[t^{\pm 1}]$. Thus, by Proposition 3, the boundary link $\partial \widetilde{X}$ possesses a slice surface of genus $g(\widetilde{X}) - 1 = 1$. Geometrically, what happens if we apply Proposition 3 is the following: starting from $\widetilde{X}$ we cut out the punctured torus T defined by the union of the thickened red and dashed blue curves. Then, using Freedman's disc theorem, we reglue a disc whose interior lies in the 4-dimensional unit ball along ∂T, obtaining a slice surface with smaller genus than $\widetilde{X}$. For this we use that ∂T is a knot with Alexander polynomial 1.

Let us now turn to the proof of Proposition 3. A crucial ingredient is the following fact about the mapping class group of surfaces, which is well-known for surfaces with at most one boundary component (see e.g. [FM12]).

Lemma 5. *Let Σ be a connected oriented compact surface of genus g with n boundary components. An automorphism φ of $H_1(\Sigma)$ is induced by an orientation-preserving diffeomorphism $\widetilde{\varphi}$ of Σ if and only if φ preserves the intersection form of Σ and permutes the homology classes of the boundary curves.*

Proof. Clearly every orientation-preserving diffeomorphism preserves the intersection form and maps boundary curves to boundary curves (preserving the orientations), which induces a permutation of the corresponding homology classes.

Now let us prove that the conditions are sufficient. Let

$$\gamma_{1,1}, \gamma_{2,1}, \dots, \gamma_{g,1}, \gamma_{1,2}, \dots, \gamma_{g,2}, \delta_1, \dots, \delta_{n-1}$$

be a *geometric basis* on Σ (a term taken from [GT04]); that is, the δ_i are boundary curves, $\gamma_{i,j}$ intersects $\gamma_{i,3-j}$ once (geometrically), and there are no other geometric intersections between any of these curves. The homology classes of these curves then form a basis of $H_1(\Sigma)$.

One easily finds a simple closed curve $\zeta \subset \Sigma$ with the following properties: it intersects $\gamma_{1,1}$ once, does not intersect any other curve in the geometric basis, and $[\zeta] = [\gamma_{1,2}] + [\delta_1]$. So the only basis curve affected by a Dehn twist along ζ is $\gamma_{1,1}$, whose homology class is sent to $[\gamma_{1,1}] + [\gamma_{1,2}] + [\delta_1]$. Composing with another Dehn twist along $\gamma_{1,2}$, one finds a diffeomorphism that sends $[\gamma_{1,1}] \mapsto [\gamma_{1,1}] + [\delta_1]$. Similarly, for all $i \in \{1, \dots, g\}, j \in \{1, 2\}, k \in \{1, \dots, n-1\}$, there is a diffeomorphism sending $[\gamma_{i,j}] \mapsto [\gamma_{i,j}] + [\delta_k]$. Composing these diffeomorphisms, one may realise automorphisms of $H_1(\Sigma)$ with a matrix of the following kind:

$$\left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline M & \mathbb{1} \end{array}\right),$$

where M is an arbitrary $(n-1) \times 2g$ matrix. Next we make use of the fact that for a surface Σ' of genus g with one boundary component, the mapping class group surjects onto the symplectic group; see e.g. [FM12], where this is established for closed surfaces, which essentially implies the result for surfaces with one boundary component. Since Σ contains Σ' as a subsurface, the following matrices may be realised as orientation-preserving diffeomorphisms:

$$\left(\begin{array}{c|c} X & 0 \\ \hline 0 & \mathbb{1} \end{array}\right),$$

where X is symplectic. Finally, it is easy to see that boundary curves may be permuted (though by diffeomorphisms not coming from Dehn twists). So, composing, one may realise any matrix of the form

$$\left(\begin{array}{c|c} X & 0 \\ \hline M & P \end{array}\right),$$

where X is symplectic, M is arbitrary, and P is a permutation matrix. This completes the proof since such matrices are precisely those which represent an automorphism of $H_1(\Sigma)$ that preserves the intersection form and permutes the homology classes of the boundary. □

A Seifert surface Σ may inherit genus defect from an *incompressible* subsurface, i.e. a subsurface $\Sigma' \subset \Sigma$ such that the induced map on the first homology group is injective.

Lemma 6. *Let Σ be a Seifert surface of a link L, and let $\Sigma' \subset \Sigma$ be an incompressible subsurface with boundary link L'. If L' bounds a slice surface S', then L bounds a slice surface S of genus $g(\Sigma) - g(\Sigma') + g(S')$. In particular, if $g(\Sigma) = g(L)$, then $\Delta g(L) \geq \Delta g(L')$.*

Proof. To construct S, simply cut out Σ' and glue in S'. □

Proof of Proposition 3. Let B be a matrix of the Seifert form of Σ restricted to V, with respect to an arbitrary basis of V. Setting $t = 1$ gives $\det(B - B^\top) = \pm 1$ and, in fact, $+1$: indeed, $B - B^\top$ is antisymmetric, so $\det(B - B^\top)$ is the square of its Pfaffian. It also follows that V is of even rank. Because $B - B^\top$ is antisymmetric and unimodular, we may assume the basis $x_{1,1}, \ldots, x_{k,1}, x_{1,2}, \ldots, x_{k,2}$ of V has been chosen such that $B - B^\top$ is the $2k \times 2k$ matrix

$$J_k = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right).$$

Let $\delta_1, \ldots, \delta_n$ be the boundary curves of Σ. Note that the intersection form of Σ is unimodular on V (in fact it is represented by the matrix $B - B^\top = J_k$), and identically zero on $\langle [\delta_1], \ldots, [\delta_{n-1}] \rangle$. This implies that one can extend the basis of V to a basis of $H_1(\Sigma)$ of the form

$$x_{1,1}, \ldots, x_{k,1}, x_{1,2}, \ldots, x_{k,2},$$
$$y_{1,1}, \ldots, y_{g-k,1}, y_{1,2}, \ldots, y_{g-k,2}, [\delta_1], \ldots, [\delta_{n-1}].$$

Let A be the matrix of the Seifert form of Σ with respect to this basis. Then $A - A^\top$ has the form

$$\left(\begin{array}{c|c|c} J_k & * & 0 \\ \hline * & * & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

Since $A - A^\top$ restricted to the span of the $x_{i,j}$ and $y_{i,j}$ is antisymmetric and unimodular, one may assume w.l.o.g. that the y_i were chosen such that $A - A^\top$ is in fact

$$\left(\begin{array}{c|c|c} J_k & 0 & 0 \\ \hline 0 & J_{g-k} & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

Now let

$$\gamma_{1,1}, \gamma_{2,1}, \ldots, \gamma_{g,1}, \gamma_{1,2}, \ldots, \gamma_{g,2}, \delta_1, \ldots, \delta_{n-1}$$

be a geometric basis on Σ as in Lemma 5. Let φ be the automorphism of $H_1(\Sigma)$ given by

$$\begin{aligned} [\gamma_{i,j}] &\mapsto x_{i,j} && \text{for } 1 \leq i \leq k, \\ [\gamma_{i,j}] &\mapsto y_{i-k,j} && \text{for } k < i \leq g, \\ [\delta_i] &\mapsto [\delta_i]. \end{aligned}$$

As the computation of $A - A^\top$ shows, φ preserves the intersection form. It also acts by permutation (in fact, as the identity) on the homology classes of boundary curves, and is therefore realised by a diffeomorphism $\widetilde{\varphi}$ (see Lemma 5). Take a simple closed curve ζ that separates the curves $\gamma_{1,*}, \ldots, \gamma_{k,*}$ from the curves $\gamma_{k+1,*}, \ldots, \gamma_{g,*}, \delta_1, \ldots, \delta_n$. Then $\widetilde{\varphi}(\zeta)$ is a separating simple closed curve, which bounds an incompressible subsurface Σ' of Σ of genus k. By construction, $H_1(\Sigma') = V \subset H_1(\Sigma)$, and hence the boundary knot $\widetilde{\varphi}(\zeta)$ of Σ' has Alexander polynomial 1. Thus, Freedman's theorem implies that it bounds a slice disc. Using Lemma 6, this concludes the proof. □

Let us come back to Example 4. So far we have proved that $\partial\widetilde{X}$ has topological slice genus equal to one, while its classical genus equals two. Lemma 8 will show how this example can be used to build larger examples with $\Delta g = g_4 = g/2$. As a sample application, we calculate the topological slice genus of the infinite family provided in the proof of Proposition 7. These examples are of particular interest since they maximise the ratio

$$\frac{2\Delta g(L)}{b_1(L)}$$

of genus defect and first Betti number among tree-like plumbings of positive Hopf bands. Indeed, for any plumbing of positive Hopf bands along a tree, this ratio is at most 1/3 by a theorem of the fourth author [Lie16]. Therefore, an infinite family of examples that attain this ratio is sufficient to prove the following proposition.

Proposition 7. *For the class of links arising as plumbings of positive Hopf bands along a finite tree, we have*

$$\limsup_{b_1(L)\to\infty} \frac{2\Delta g(L)}{b_1(L)} = \frac{1}{3}.$$

Lemma 8. *Let Σ be a Seifert surface. Let Σ' be a plumbing of Σ and $\widetilde{X}$ along a square on the right-most Hopf band of $\widetilde{X}$ (see Figure 1). If there is an Alexander-trivial subgroup $V \subset H_1(\Sigma)$, then there is also an Alexander-trivial subgroup $V' \subset H_1(\Sigma')$ of rank $\operatorname{rk} V' = 2 + \operatorname{rk} V$.*

Proof. Let γ_1, γ_2 be the red and dashed blue curves on $\widetilde{X}$ as in Example 4. Let $V' = V + \langle[\gamma_1], [\gamma_2]\rangle$, where we understand $H_1(\Sigma) \oplus H_1(\widetilde{X})$ as a subgroup of $H_1(\Sigma')$, because Σ and $\widetilde{X}$ are incompressible subsurfaces of Σ', and $H_1(\Sigma) \cap H_1(\widetilde{X}) = \{0\}$. The crucial observation is that, algebraically, γ_1 does not pass through the plumbing location on the right-most Hopf band of $\widetilde{X}$. Therefore, γ_1 algebraically does not intersect curves on Σ; and so, using that Σ' is a plumbing, any small push-off of γ_1 along a normal direction of Σ' has linking number 0 with curves on Σ. Thus the Seifert form of Σ' restricted to V' is represented by the following matrix:

$$M' = \left(\begin{array}{ccc|cc} & & & 0 & * \\ & M & & \vdots & \vdots \\ & & & 0 & * \\ \hline 0 & \cdots & 0 & 0 & 0 \\ * & \cdots & * & 1 & 1 \end{array}\right).$$

Here, M is a matrix of the Seifert form restricted to V. It has Alexander polynomial 1, hence so does M'. $\square$

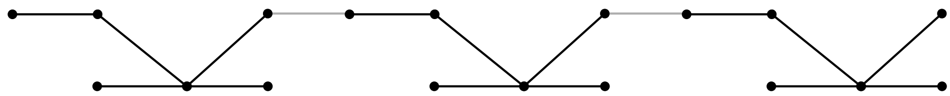

FIGURE 2. How to stick together three copies of the tree corresponding to $\widetilde{X}$ in order to obtain a link with $b_1 = 18$.

Proof of Proposition 7. In Example 4, we considered such a link L with $b_1 = 6$. In order to obtain an example L_n with $b_1 = 6n$, we simply stick together n distinct copies of the tree corresponding to $\widetilde{X}$ and take the corresponding positive tree-like Hopf plumbing; compare Remark 9 for an explicit braid description. This is shown in Figure 2 for the case $n = 3$. By Lemma 8, the corresponding fibre surface has defect $\Delta g \geq n$, which establishes the proposition. $\square$

Remark 9. For all $n \geq 1$, the link L_n used in the proof of Proposition 7 can also be obtained as the closure of the $(3n+1)$-braid

$$a_1(a_1a_3a_2^2a_4a_1a_3a_2^2)(a_4a_6a_5^2a_7a_4a_6a_5^2)\cdots$$
$$(a_{3k-2}a_{3k}a_{3k-1}^2a_{3k+1}a_{3k-2}a_{3k}a_{3k-1}^2)\cdots(a_{3n-2}a_{3n}a_{3n-1}^2a_{3n-2}a_{3n}a_{3n-1}^2).$$

Furthermore, if we compare the topological slice genus with the classical genus (instead of the first Betti number), the quotient becomes even larger: since the links L_n have topological slice genus n and genus $2n$, they form an infinite family of examples of positive braid links with $\Delta g = g_4 = g/2$.

Next, let us focus on braids. Incompressible subsurfaces of canonical Seifert surfaces of positive braids will typically be constructed as in the following lemma, whose proof we leave to the reader. We call a braid word β' a *subword* of a braid word β if the former arises from the latter by deleting some occurrences of generators.

Lemma 10. *If β' is a subword of β, then $\Sigma(\beta')$ is an incompressible subsurface of $\Sigma(\beta)$.* $\square$

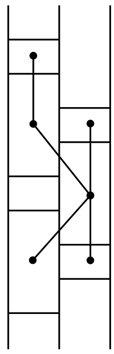

FIGURE 3. A fence diagram (obtained from a braid diagram by replacing the crossings by horizontal line segments) of the braid $a_1(a_2^2a_1^2)^2$. The tree induced by the distinguished homology generators exhibits $\widetilde{X}$ as incompressible subsurface of the fibre surface $\Sigma(a_1(a_2^2a_1^2)^2)$.

Example 11. Consider the torus knot $T(4,5)$. It is obtained as the closure of the positive braid $(a_1a_2a_3)^5$, which contains the subword $a_1a_2^2a_3a_1a_2^3a_3$, whose closure equals the closure of $a_1a_3a_2^2a_1a_3a_2^3$. In particular, the fibre surface $\Sigma(T(4,5))$ contains $\widetilde{X}$ as an incompressible subsurface. Together with the bound coming from the signature function $\sigma_{e^{\pi it}}(T(4,5)) = 10$ for $7/10 < t < 9/10$, this yields $g_4(T(4,5)) = 5$.

Remark 12. In Example 11, we used half the absolute value of a Levine-Tristram signature as a lower bound for the topological slice genus of a knot. While well-known to experts, until recently this lower bound had not been explicitly stated in the literature in the topological setting (compare [Tri69] for the smooth setting). This gap in the literature was closed by Powell with a new proof [Pow16].

Example 13. Consider the torus knot $T(3,7)$. It is obtained as the closure of the positive braid $(a_1a_2)^7$, which contains $a_1(a_2^2a_1^2)^2$ as a subword. On the other hand, $\Sigma(a_1(a_2^2a_1^2)^2)$ contains $\widetilde{X}$ as an incompressible subsurface. This is schematically depicted in Figure 3. Together with the bound coming from the signature function $\sigma_{e^{\pi it}}(T(3,7)) = 10$ for $16/21 < t < 20/21$, this yields $g_4(T(3,7)) = 5$.

Suppose α and β are braid words for non-split n-braids. Then $\Sigma(\alpha\beta)$ contains $\Sigma(\alpha) \sqcup \Sigma(\beta)$ as incompressible subsurface. So if Proposition 3 produces genus defects d_1 and d_2 in $\Sigma(\alpha)$ and $\Sigma(\beta)$, respectively, this will yield a defect of $d_1 + d_2$ in $\Sigma(\alpha\beta)$. The following lemma is a refinement of this strategy for constructing genus defect in the product of two braids. See Examples 15 and 17 for applications.

Lemma 14. *Let α, β be two braid words representing non-split n-braids. Let β' be the braid word of length $n-1$ obtained from β by deleting for all $i \in \{1,\dots,n-1\}$ all but the first occurrences of the generator a_i.*

Let $V \subset H_1(\Sigma(\alpha\beta'))$ and $V' \subset H_1(\Sigma(\beta))$ be Alexander-trivial subgroups. Let a basis of V be given with respect to which the Seifert form of $\Sigma(\alpha\beta')$ has a matrix of the following kind, built from four square blocks:[*]

$$\left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{matrix} & * \end{array}\right).$$

Suppose moreover that the first half of that basis is supported in $H_1(\Sigma(\alpha))$, which can be seen as a subgroup of $H_1(\Sigma(\alpha\beta'))$ by Lemma 10. Then there is an Alexander-trivial subgroup $V'' \subset H_1(\Sigma(\alpha\beta))$ of rank $\operatorname{rk} V + \operatorname{rk} V'$.

Proof. The idea is similar to the proof of Lemma 8. The surface $\Sigma(\alpha\beta)$ has incompressible subsurfaces $\Sigma(\alpha\beta')$ and $\Sigma(\beta)$, so we may treat $H_1(\Sigma(\alpha\beta'))$ and $H_1(\Sigma(\beta))$ as subgroups of $H_1(\Sigma(\alpha\beta))$. Their intersection is in fact trivial, and so we have $V \cap V' = \{0\}$ as well. Extend the given basis of V to a basis of $V + V'$. With respect to this basis, the restriction of the Seifert form of $\Sigma(\alpha\beta)$ to $V + V'$ is represented by the following matrix:

$$M' = \left(\begin{array}{c|c|c} 0 & \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline \begin{matrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{matrix} & * & * \\ \hline 0 & * & M \end{array}\right).$$

Here, M is the matrix of the Seifert form restricted to V', which has Alexander polynomial 1. After some basis changes, one sees that M' has Alexander polynomial 1 as well. ☐

Example 15. We have seen in Example 13 how the closure of

$$\beta = a_1(a_2^2 a_1^2)^2$$

has defect at least one, which comes from two vectors v, w restricted to which the Seifert form has the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & * \end{pmatrix}.$$

The vectors v and w are the homology classes of the red and blue curves drawn in Figure 1. As already discussed in the proof of Lemma 8, $v \in H_1(\Sigma(\alpha)) \subset H_1(\Sigma(\beta))$, where $\alpha = a_1 a_2^2 a_1^2 a_2^2 a_1$. Let $\beta' = a_1 a_2$ as in the previous lemma. Then $\alpha\beta'$ contains β as a subword, and so $\Sigma(\alpha\beta')$ also has defect at least 1. So the previous lemma implies that

[*]For example, all *trivial Alexander bases* [GT04] are of this kind (but not vice versa). In fact, one can prove that every Alexander-trivial subgroup V has such a basis; but we will not need this fact here.

$\alpha\beta = a_1(a_2^2 a_1^2)^4$ has defect at least 2. Continuing inductively, one finds a defect of at least i in the closure of the braid

$$\alpha^{i-1}\beta = a_1(a_2^2 a_1^2)^{2i}.$$

The same result may be obtained using Lemma 8, since $\widetilde{X} \subset \Sigma(\beta)$, as shown in Figure 3.

Remark 16. Proposition 3 shows how to construct slice surfaces using nothing but linear algebra. The following randomised algorithm exploits this. As input, it takes an arbitrary integral square matrix A, and returns as output the basis of a subgroup $V \subset \mathbb{Z}^{2g}$ with respect to which $A|_V$ has a matrix of the following kind:

$$\left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{matrix} & * \end{array}\right).$$

Note that such a matrix has Alexander polynomial 1. Here is a brief description of the algorithm:

(1) Randomly pick a primitive vector v with $v^\top A v = 0$, if such a vector exists. Otherwise, return the empty basis.
(2) Randomly pick a solution w of the following system of linear equations, if it is solvable:
$$v^\top A w = 1, \qquad w^\top A v = 0.$$
Otherwise, go back to (1), or eventually give up and return the empty basis.
(3) Let U be the subgroup of solutions of the following system of homogeneous linear equations:
$$v^\top A u = 0, \qquad u^\top A v = 0, \qquad u^\top A w = 0.$$
Let $(v_1, \ldots, v_k, w_1, \ldots, w_k)$ be the result of the recursive application of the algorithm to $A|_U$. Return
$$(v, v_1, \ldots, v_k, w, w_1, \ldots, w_k).$$

Implemented in pari/gp [PAR15], the algorithm performs quite well for small knots. See Table 1 for the results thus obtained for small torus knots, and Example 17 for the application to another positive braid. The bases of the respective subgroups V are available from ancillary files with the arXiv-version of this paper, which enables anybody to independently verify their correctness.

Example 17. Consider the positive braids $\omega = a_1 a_2 a_3 a_4, \widetilde{\omega} = a_4 a_3 a_2 a_1$. The algorithm described in Remark 16 returns an Alexander-trivial subgroup $V \subset H_1(\Sigma((\omega\widetilde{\omega})^4))$ of rank eight (we used [Col15] to obtain Seifert matrices). Moreover, the first half of the basis of V is supported in $H_1(\Sigma((\omega\widetilde{\omega})^3\omega))$. Similarly, there is an Alexander-trivial subgroup $V' \subset$

$H_1(\Sigma((\widetilde{\omega}\omega)^4))$ of rank eight with a basis whose first half is supported in $H_1(\Sigma((\widetilde{\omega}\omega)^3\widetilde{\omega}))$. Applying Lemma 14 to $(\omega\widetilde{\omega})^3\omega$ and $(\widetilde{\omega}\omega)^4$ gives a defect of eight in $\Sigma((\omega\widetilde{\omega})^7\omega)$. We may continue applying the lemma inductively, first to $(\omega\widetilde{\omega})^7$ and $(\omega\widetilde{\omega})^4$, producing a defect of twelve in $\Sigma((\omega\widetilde{\omega})^{11})$, then to $(\omega\widetilde{\omega})^{10}\omega$ and $(\widetilde{\omega}\omega)^4$ etc. In summary, we find for all $i \geq 0$ a defect of $4+8i$ for $\Sigma((\omega\widetilde{\omega})^{4+7i})$, and of $8i$ for $\Sigma((\omega\widetilde{\omega})^{7i}\omega)$.

3. Slice genus of large torus knots

The aim of this section is to prove the asymptotic bound for the genus defect of torus knots given by Theorem 2. As a start, we establish a weaker version of Theorem 2 with the benefit that its proof, unlike the proof of Theorem 2, does not require computer calculations. The strategies of both proofs are very much alike.

Proposition 18.

$$\lim_{n,m\to\infty} \frac{g_4(T(n,m))}{g(T(n,m))} \leq \frac{4}{5}.$$

The strategy of the proof of Proposition 18 is to establish that the fibre surface $\Sigma(T(n,n))$ of the torus link $T(n,n)$ contains as incompressible subsurface the split union of fibre surfaces of the form $\Sigma(a_1(a_1^2a_2^2)^{2i})$ such that this union takes up roughly four-fifths of the genus of $\Sigma(T(n,n))$. This yields Proposition 18 since the genus defect of the closure of $a_1(a_1^2a_2^2)^{2i}$ is at least i, which is about a quarter of the genus. Indeed, first conjugating by a_1 and then reading the braid word backwards (both of these operations preserve the closure up to changing the orientation of all components) turns $a_1(a_1^2a_2^2)^{2i}$ into $a_1(a_2^2a_1^2)^{2i}$, whose defect is discussed in Example 15. To make this strategy precise we use Lemma 19. Let Δ_n be the half twist on n strands, i.e.

$$\Delta_n = (a_1a_2\cdots a_{n-1})(a_1a_2\cdots a_{n-2})\cdots(a_1a_2)(a_1).$$

Furthermore, we define the positive braids Ω_i and Γ_j by

$$\Omega_i = a_1a_2\cdots a_{i-2}a_{i-1}^2a_{i-2}\cdots a_2a_1,$$
$$\Gamma_j = a_1a_2\cdots a_{j-2}a_{j-1}a_{j-2}\cdots a_2a_1.$$

Lemma 19. *Let $n \geq 2\ell$ be natural numbers. Then $\Sigma(\Delta_n)$ contains*

$$\Sigma(\Gamma_2\cdots\Gamma_\ell\Omega_\ell^{n-2\ell+1}\Gamma_\ell\cdots\Gamma_2) \sqcup \Sigma(\Delta_{n-2\ell+1})$$

as an incompressible subsurface.

Proof. We proceed by showing that one can delete generators and apply braid relations in the braid word Δ_n such that the resulting positive braid is the split union of the positive braids $\Gamma_2\cdots\Gamma_\ell\Omega_\ell^{n-2\ell+1}\Gamma_\ell\cdots\Gamma_2$ and $\Delta_{n-2\ell+1}$. This suffices to establish Lemma 19 since deleting a generator in a positive braid word corresponds to taking an incompressible

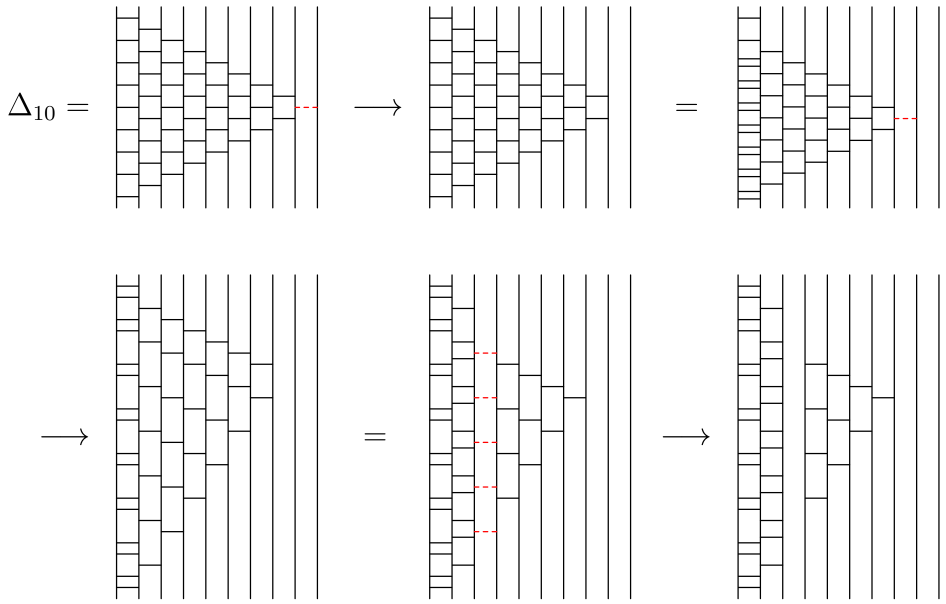

FIGURE 4. By applying braid relations and deleting generators, the 10-strand braid word Δ_{10} is transformed into $\Gamma_2\Gamma_3\Gamma_4 a_4 a_5 a_6 a_7 \Gamma_4 a_4 a_5 a_6 \Gamma_4 a_4 a_5 \Gamma_4 a_4 \Gamma_4 \Gamma_3 \Gamma_2$. In the final step, deleting generators produces the disjoint union of $\Gamma_2\Gamma_3\Omega_3^5\Gamma_3\Gamma_2$ and Δ_5. Arrows indicate the deletion of generators drawn red and dashed.

subsurface of the associated fibre surface (see Lemma 10). We start by considering the positive braid word

$$\Delta_n = \Gamma_2(a_2 \cdots a_{n-1})\Gamma_2(a_2 \cdots a_{n-2}) \cdots \Gamma_2(a_2 a_3)\Gamma_2(a_2)\Gamma_2.$$

We delete the single occurrence of the generator a_{n-1} in Δ_n and then apply braid relations to obtain the positive braid word

$$\Gamma_2\Gamma_3(a_3 \cdots a_{n-2})\Gamma_3(a_3 \cdots a_{n-3}) \cdots \Gamma_3(a_3 a_4)\Gamma_3(a_3)\Gamma_3\Gamma_2.$$

This can be achieved by multiple substitutions of the form

$$(a_i \cdots a_j)\Gamma_i(a_i \cdots a_j) \to \Gamma_{i+1}(a_{i+1} \cdots a_j)(a_i \cdots a_{j-1})$$

for $i \leq j$, which can in turn be realised by braid relations. To see the realisation of this substitution by braid relations, commute generators to rewrite the positive braid word

$$(a_i \cdots a_j)\Gamma_i(a_i \cdots a_j)$$

as

$$a_i\Gamma_i a_{i+1} a_i a_{i+2} a_{i+1} \cdots a_{j-1} a_{j-2} a_j a_{j-1} a_j.$$

Then, applying the braid relation

$$a_k a_{k-1} a_k \to a_{k-1} a_k a_{k-1}$$

once for each k starting at j and descending down to i yields the positive braid word

$$\Gamma_{i+1}a_{i+1}a_ia_{i+2}\cdots a_{j-3}a_{j-1}a_{j-2}a_ja_{j-1},$$

for which generators can again be commuted to finally result in

$$\Gamma_{i+1}(a_{i+1}\cdots a_j)(a_i\cdots a_{j-1}).$$

In the next step, we delete the single occurrence of the generator a_{n-2} in the positive braid word

$$\Gamma_2\Gamma_3(a_3\cdots a_{n-2})\Gamma_3(a_3\cdots a_{n-3})\cdots\Gamma_3(a_3a_4)\Gamma_3(a_3)\Gamma_3\Gamma_2$$

and, again using substitutions of the form

$$(a_i\cdots a_j)\Gamma_i(a_i\cdots a_j)\to\Gamma_{i+1}(a_{i+1}\cdots a_j)(a_i\cdots a_{j-1}),$$

obtain the positive braid word

$$\Gamma_2\Gamma_3\Gamma_4(a_4\cdots a_{n-3})\Gamma_4(a_4\cdots a_{n-4})\cdots\Gamma_4(a_4a_5)\Gamma_4(a_4)\Gamma_4\Gamma_3\Gamma_2.$$

We continue in the same way until we arrive at the positive braid word

$$\Gamma_2\cdots\Gamma_{\ell+1}(a_{\ell+1}\cdots a_{n-\ell})\Gamma_{\ell+1}(a_{\ell+1}\cdots a_{n-\ell-1})\cdots \\ \Gamma_{\ell+1}(a_{\ell+1}a_{\ell+2})\Gamma_{\ell+1}(a_{\ell+1})\Gamma_{\ell+1}\cdots\Gamma_2.$$

Finally, we delete all occurrences of a_ℓ. The closure of the positive braid obtained in this way is the split union of the closures of the braids $\Gamma_2\cdots\Gamma_\ell\Omega_\ell^{n-2\ell+1}\Gamma_\ell\cdots\Gamma_2$ and $\Delta_{n-2\ell+1}$. This procedure is illustrated in Figure 4 for $n=10$ and $\ell=3$. □

Proof of Proposition 18. Consider the positive braid word Δ_{5n}. By Lemma 19 with $\ell=3$, $\Sigma(\Delta_{5n})$ contains

$$\Sigma(\Gamma_2\Gamma_3\Omega_3^{5n-5}\Gamma_3\Gamma_2)\sqcup\Sigma(\Delta_{5n-5})$$

as an incompressible subsurface. Using Lemma 19 with $\ell=3$ inductively on the last split summand, we obtain that $\Sigma(\Delta_{5n})$ contains

$$\Sigma(\Gamma_2\Gamma_3\Omega_3^{5n-5}\Gamma_3\Gamma_2)\sqcup\cdots\sqcup\Sigma(\Gamma_2\Gamma_3\Omega_3^{5}\Gamma_3\Gamma_2)$$

as an incompressible subsurface. The same argument gives

$$\Sigma(\Gamma_2\Gamma_3\Omega_3^{10n-10}\Gamma_3\Gamma_2)\sqcup\cdots\sqcup\Sigma(\Gamma_2\Gamma_3\Omega_3^{10}\Gamma_3\Gamma_2)$$

as an incompressible subsurface of the fibre surface $\Sigma(\Delta_{5n}^2)$. By the definitions of Ω_3, Γ_2 and Γ_3, the positive braid $\Gamma_2\Gamma_3\Omega_3^{10i}\Gamma_3\Gamma_2$ contains $a_1(a_1^2a_2^2)^{10i}$ as a subword. Furthermore the closure of the braid $a_1(a_1^2a_2^2)^{10i}$ has genus defect at least $5i$ (see Example 15). In this way, using all the surfaces of the split union, we can produce a genus defect of at least

$$\sum_{i=1}^{n-1}5i=\frac{5n^2-5n}{2}.$$

From this we obtain

$$\frac{\Delta g(T(5n,5n))}{g(T(5n,5n))} \geq \frac{5n^2-5n}{25n^2-15n+2} \xrightarrow{n\to\infty} \frac{1}{5},$$

which establishes Proposition 18. □

Proof of Theorem 2. We proceed as in the proof of Proposition 18. However, instead of $\ell = 3$ we use $\ell = 5$ when applying Lemma 19 and obtain that $\Sigma(\Delta_{9n})$ contains

$$\Sigma(\Omega_5^{9(n-1)}) \sqcup \Sigma(\Omega_5^{9(n-2)}) \sqcup \cdots \sqcup \Sigma(\Omega_5^{9})$$

as an incompressible subsurface. As seen in Example 17, the closure of the braid Ω_5^{4+7j} has genus defect at least $4+8j$. For every split summand Ω_5^{9i}, we consider the largest subword of the form Ω_5^{4+7j} and produce genus defect accordingly. In this way, we produce at least $4 + 8\left\lfloor\frac{9i-4}{7}\right\rfloor \geq \frac{72i}{7} - \frac{60}{7}$ genus defect per summand. In total, this amounts to a genus defect of at least

$$\sum_{i=1}^{n-1} \frac{72}{7}i - \frac{60}{7} = \frac{72n^2}{14} + O(n).$$

On the other hand, we have

$$g(\Sigma(\Delta_{9n})) = \frac{81n^2}{4} + O(n).$$

From this we obtain

$$g_4(\Sigma(\Delta_{9n})) \leq \frac{81n^2}{4} + O(n) - \frac{72n^2}{14} - O(n) = \frac{423n^2}{28} + O(n),$$

which finally yields

$$\frac{g_4(T(9n,9n))}{g(T(9n,9n))} \leq \frac{\frac{423n^2}{28} + O(n)}{\frac{81n^2}{4} + O(n)} \xrightarrow{n\to\infty} \frac{47}{63} < \frac{3}{4}$$

and establishes Theorem 2. □

4. Slice genus of small torus knots

This section is devoted to the proof of Theorem 1. In fact, we will prove a generalisation to links. For links, the topological slice genus is bounded by the signature and nullity (denoted by μ) as follows [KT76]:

$$|\sigma(L)| - \#L + 1 + \mu(L) \leq 2g_4(L).$$

Proposition 20. *Let $L = T(p,q)$ be a torus link with non-maximal signature and nullity bound, i.e. $L \neq T(2,n)$, $T(3,3)$, $T(3,4)$, $T(3,5)$, $T(3,6)$, $T(4,4)$. Then*

$$g_4(L) \leq \frac{6}{7}g(L).$$

According to Theorem 2, most torus links satisfy $\frac{g_4}{g} < \frac{3}{4}$. The bulk of the proof of Proposition 20 is thus an investigation of small torus links. Their genus defects can often be found by computer calculation (see Remark 16), or are inherited by incompressible subsurfaces, e.g. using the following construction:

Lemma 21 ([Baa12, Proposition 1])**.** *Let $p, q, r \in \mathbb{N}$ with $p \leq r$. Then $\Sigma(T(pq, r))$ contains $\Sigma(T(p, qr))$ as incompressible subsurface.* □

The following lemma helps us dealing with the exceptional cases in the proof of Proposition 20:

Lemma 22. *The following lower bounds hold for the quotient $\frac{2\Delta g(T(p,q))}{b_1(T(p,q))}$:*

(i) *For $3|p$ and $q \geq 10$, the quotient is greater or equal to $8/51$.*
(ii) *For $4|p$ and $q \geq 7$, the quotient is greater or equal to $2/11$.*
(iii) *For $5|p$ and $q \geq 6$, the quotient is greater or equal to $1/5$.*

Proof. To prove (i), let $p = 3a$. By Lemma 21, we have

$$\Delta g(T(p, q)) \geq \Delta g(T(3, aq)).$$

Let $aq = 17k + r$ with $0 \leq r \leq 16$. Applying the computed defects shown in Table 1 of the knots $T(3, 7), T(3, 10), T(3, 13)$ and $T(3, 17)$ yields

$$\Delta g(T(3, aq)) \geq 4k + s(r),$$

where $s(r) = 0, 1, 2, 3$ for r in $[0, 6], [7, 9], [10, 12], [13, 16]$, respectively. For

$$\frac{2\Delta g(T(3a, q))}{b_1(T(3a, q))} \geq \frac{8}{51},$$

it suffices that

$$\begin{aligned} 4k + s(r) &\geq \frac{4}{51}(3a - 1)(q - 1) \quad \Leftrightarrow \\ 51k + 51s(r)/4 &\geq 3aq - 3a - q + 1 \quad \Leftrightarrow \\ 51k + 51s(r)/4 &\geq 51k + 3r - 3a - q + 1 \quad \Leftrightarrow \\ 3a + q &\geq 1 + 3r - 51s(r)/4 \quad \Leftarrow \end{aligned}$$

(to find the maximum of the right-hand side, which is at $r = 6$, it suffices to check the cases $r = 6, 9, 12, 16$)

$$\begin{aligned} 3a + q &\geq 19 \quad \Leftrightarrow \\ \frac{(3a - 1) + (q - 1)}{2} &> 8. \end{aligned}$$

b_1	(p,q)	Δg	Lower bound	b_1	(p,q)	Δg	Lower bound
4	(3,3)	0		40	(5,11)	[5,6]	Remark 16
6	(3,4)	0		40	(6,9)	[4,6]	
8	(3,5)	0		42	(3,22)	[4,6]	
9	(4,4)	0		42	(4,15)	[4,6]	
10	(3,6)	0		42	(7,8)	[5,6]	Remark 16
12	(3,7)	1	Example 13	44	(3,23)	[5,6]	$\Sigma(3,10) \sqcup \Sigma(3,13)$
12	(4,5)	1	Example 11	44	(5,12)	[5,7]	
14	(3,8)	1		45	(4,16)	[5,6]	$\Sigma(4,5) \sqcup \Sigma(4,11)$
15	(4,6)	[1,2]		45	(6,10)	[5,8]	$\Sigma(6 \cdot 2, 5)$
16	(3,9)	1		46	(3,24)	[5,6]	
16	(5,5)	1		48	(3,25)	[5,7]	
18	(3,10)	2	Remark 16	48	(4,17)	[5,7]	
18	(4,7)	2	Remark 16	48	(5,13)	[5,8]	
20	(3,11)	2		48	(7,9)	[5,8]	
20	(5,6)	2	$\Sigma(5 \cdot 2, 3)$	49	(8,8)	[5,6]	
21	(4,8)	2		50	(3,26)	[6,7]	$\Sigma(3,13) \sqcup \Sigma(3,13)$
22	(3,12)	2		50	(6,11)	[5,8]	
24	(3,13)	3	Remark 16	51	(4,18)	[6,8]	$\Sigma(4,7) \sqcup \Sigma(4,11)$
24	(4,9)	3	Remark 16	52	(3,27)	[6,7]	
24	(5,7)	3	Remark 16	52	(5,14)	[6,8]	$\Sigma(5,6) \sqcup \Sigma(5,8)$
25	(6,6)	2		54	(3,28)	[6,8]	
26	(3,14)	3		54	(4,19)	[6,8]	
27	(4,10)	[3,4]		54	(7,10)	[6,9]	$\Sigma(4,7) \sqcup \Sigma(6,7)$
28	(3,15)	3		55	(6,12)	[6,8]	$\Sigma(5,6) \sqcup \Sigma(6,7)$
28	(5,8)	4	Remark 16	56	(3,29)	[6,8]	
30	(3,16)	[3,4]		56	(5,15)	[7,10]	$\Sigma(5,7) \sqcup \Sigma(5,8)$
30	(4,11)	4	Remark 16	56	(8,9)	[6,9]	$\Sigma(4,9) \sqcup \Sigma(4,9)$
30	(6,7)	4	Remark 16	57	(4,20)	[7,8]	$\Sigma(4,9) \sqcup \Sigma(4,11)$
32	(3,17)	4	Remark 16	58	(3,30)	[7,8]	$\Sigma(3,13) \sqcup \Sigma(3,17)$
32	(5,9)	4		60	(3,31)	[7,9]	
33	(4,12)	4		60	(4,21)	[7,9]	
34	(3,18)	4		60	(5,16)	[8,10]	$\Sigma(5,8) \sqcup \Sigma(5,8)$
35	(6,8)	[4,6]		60	(6,13)	[6,10]	$\Sigma(3,13) \sqcup \Sigma(3,13)$
36	(3,19)	[4,5]		60	(7,11)	[7,10]	$\Sigma(5,7) \sqcup \Sigma(6,7)$
36	(4,13)	[4,5]		62	(3,32)	[7,9]	
36	(5,10)	4		63	(4,22)	[8,10]	$\Sigma(4,11) \sqcup \Sigma(4,11)$
36	(7,7)	[4,6]		63	(8,10)	[8,12]	$\Sigma(5,8) \sqcup \Sigma(5,8)$
38	(3,20)	[4,5]		64	(3,33)	[7,9]	
39	(4,14)	[4,6]		64	(5,17)	[8,11]	
40	(3,21)	[4,5]		64	(9,9)	[7,9]	$\Sigma(4,9) \sqcup \Sigma(5,9)$

TABLE 1. All (p,q)-torus links with $p, q \geq 3$ up to Betti number $b_1 \leq 64$, including all links of genus $g \leq 28$. The upper bounds for the genus defect Δg are induced by the signature and nullity functions. For the lower bounds, there is either a reference given, or an incompressible subsurface from which the defect is inherited (see Lemmas 6 and 21). Subsurfaces of the kind $\Sigma(p-r,q) \subset \Sigma(p,q)$ are left out.

Since the arithmetic dominates the geometric mean, this is implied by

$$\begin{aligned}\sqrt{(3a-1)(q-1)} &> 8 \quad \Leftrightarrow\\ \sqrt{b_1(T(3a,q))} &> 8 \quad \Leftrightarrow\\ b_1(T(3a,q)) &> 64.\end{aligned}$$

The case $b_1(T(3a,q)) \leq 64$ is dealt with by Table 1. The proofs of (ii) and (iii) proceed in the same way. For (ii), let $aq = 11k + r$, and use the computed defects of $T(4,5), T(4,7), T(4,9)$ and $T(4,11)$. This covers the case $b_1(T(4a,q)) > 49$. For (iii), setting $aq = 8k + r$ and using $T(5,4), T(5,6), T(5,7), T(5,8)$ covers the case $b_1(T(5a,q)) > 36$. □

Proof of Proposition 20. The cases $p, q \leq 9$ are all contained in Table 1. So let us assume $q \geq 10$. We will prove that in this case we even have $2\Delta g/b_1 \geq 1/7$, which suffices since $b_1 \geq 2g$. If p is divisible by 3, 4 or 5, then the statement follows from Lemma 22. All other p can be written as $p = 3a + 4b$ with $a, b \geq 1$. By Lemma 22,

$$\begin{aligned}2\Delta g(T(3a+4b,q)) &\geq 2\Delta g(T(3a,q)) + 2\Delta g(T(4b,q))\\ &\geq \frac{8(3a-1)(q-1)}{51} + \frac{2(4b-1)(q-1)}{11}.\end{aligned}$$

So now it suffices to show

$$\begin{aligned}\frac{8(3a-1)(q-1)}{51} + \frac{2(4b-1)(q-1)}{11} \geq \frac{(3a+4b-1)(q-1)}{7} \quad &\Leftrightarrow\\ 616(3a-1) + 714(4b-1) \geq 561(3a+4b-1) \quad &\Leftrightarrow\\ 1848a - 616 + 2856b - 714 \geq 1683a + 2244b - 561 \quad &\Leftrightarrow\\ 165a + 612b \geq 769,&\end{aligned}$$

which follows from $a, b \geq 1$. □

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