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Application of conditional random fields and sparse polynomial chaos expansions in structural reliability analysis

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Abstract: Random field theory is often used to describe spatial variability in engineering applications. Using discretization algorithms, realizations of random fields are generated, which may be used subsequently in e.g. structural reliability analysis. Furthermore, sparse polynomial chaos expansions can reduce the total computational costs and allow for an efficient estimation of failure probabilities. The proposed approach is illustrated on a geotechnical engineering problem.

1 Introduction

Mechanical properties are uncertain and they naturally vary within a geometrical body. In engineering applications, they are often inferred from a limited number of measurements taken in the region of interest. Between the measurements, however, the material properties remain uncertain. A common approach is though to use random fields to describe this uncertainty. Several approaches are available in the literature to describe unconditional random fields, such as the expansion optimal linear estimation method (EOLE) [11]. Accounting for available measurement data requires the modelling of conditional random fields [8, 14].

In modern engineering, analyses are often carried out with the help of finite element models (FEM), in particular when material properties vary spatially. In this context, analyses which require a large number of model evaluations, such as Monte Carlo-based reliability methods, may become intractable. To reduce the associated computational costs, Finite Element Method may be replaced by inexpensive-to-evaluate surrogate models such as sparse Polynomial Chaos Expansions (PCE) [2, 3]. The goal of this paper is to develop sparse PCE in the context of conditional random fields.

This paper is structured as follows. Section 2 introduces the basics of random field theory as well as a popular discretization method. Section 3 discusses meta-model-based estimation of failure probabilities. The combination of random fields and structural reliability analysis is described in Section 4 and illustrated on a slope reliability example in Section 5.

2 Spatial variability

2.1 Random field theory

Random field theory is a mathematical concept which allows the modelling of uncertainties of physical properties in continuous media with respect to randomness and spatial variability [16]. Basic definitions can be found in e.g. [17]. In this paper, a random field \( H(\mathbf{x}, \omega) \) is a collection of random variables, which are indexed by the continuous parameter \( \mathbf{x} \in \mathcal{X} \). The space \( \mathcal{X} \subset \mathbb{R}^{n_x} \) describes the geometry of the system or the geographical location of a variable. \( \omega \in \Omega \) denotes the outcomes in the space of elementary events.
A special case is Gaussian random fields where $H(x, \omega)$ is a Gaussian random variable at each location $x \in \mathcal{D}_X$. A Gaussian random field is defined completely by its mean value $\mu(x)$, variance $\sigma^2(x)$, and autocorrelation function $R(x, x'|\rho)$. If the random field is assumed stationary (a.k.a. homogeneous), this function only depends on $x - x'$. Due to some handy mathematical properties, Gaussian random fields are a popular choice in the literature and they are also the focus of this paper.

### 2.2 EOLE

Discretization procedures approximate the random field $H(x, \omega)$ by some function $\hat{H}(x, \mathbf{z})$, where $\mathbf{Z} = \{Z_1(\omega), \ldots, Z_M(\omega)\}$ is a finite set of random variables describing the randomness of the field. The explicit function $\hat{H}$ allows one to sample realizations (i.e. trajectories) of the random field $H$ in a straightforward way by sampling $\mathbf{Z}$.

The expansion optimal linear estimation method (OLE), which was published first by [11]. Consider the vector of nodal points $\mathbf{X} = (x^{(1)}, \ldots, x^{(n)})$ in the domain $\mathcal{D}_X$. The optimal linear estimation of the random field is then:

$$\hat{H}(x, \omega) = \mu(x) + \Sigma_{H(x)} Y \Sigma_{\bar{Y}}^{-1} (\bar{Y} - \mu_{\bar{Y}}),$$  

(1)

where $\bar{Y} = (Y^{(1)}, \ldots, Y^{(n)})$ is the set of correlated Gaussian variables associated to the points in $\mathbf{X}$, $\mu$ and $\Sigma$ are its mean value and covariance matrix, whose components are defined as:

$$\mu_{\bar{Y},k} = \mu(Y^{(k)}), \quad \Sigma_{\bar{Y},k,l} = \sigma(x^{(k)}) \sigma(x^{(l)}) R(x^{(k)} - x^{(l)}|\rho),$$

(2)

where $k = 1, \ldots, n$, $l = 1, \ldots, n$. Further, the correlation vector $\Sigma_{H(x)\bar{Y}}$ is defined by:

$$\Sigma_{H(x)\bar{Y}} = \sigma(x) \sigma(x^{(k)}) R(x - x^{(k)}|\rho).$$

(3)

Then, consider the eigenvalue decomposition of the covariance matrix: $\Sigma_{\bar{Y}\bar{Y}} \phi = \lambda_i \phi$, where $\lambda_i$ are the eigenvalues and $\phi_i$ are the corresponding eigenfunctions. Note that the index $i$ refers to the $i$-th largest eigenvalue of the set of $n$ eigenvalues. Then, the EOLE approximation of the random field reads [11]:

$$\hat{H}(x, \mathbf{Z}(\omega)) = \mu(x) + \sum_{i=1}^{n_z} Z_i(\omega) \phi_i^T \Sigma_{H(x)\bar{Y}};$$

(4)

where $\mathbf{Z}$ is a vector of $n_z$ standard Gaussian variables. In practice, only the $n_z \leq n$ eigenvalues are used in the series expansion. The quality of EOLE approximations depends of the number of terms $n_z$. The variance of the approximation error in EOLE is [16]:

$$\text{Var} \left[ H(x) - \hat{H}(x) \right] = \sigma^2(x) - \sum_{i=1}^{n_z} \frac{1}{\lambda_i} \left( \phi_i^T \Sigma_{H(x)\bar{Y}} \right)^2 \equiv \text{Var}[H] - \text{Var}[\hat{H}],$$

(5)

where $\sigma^2(x)$ is the variance of the Gaussian random field. Due to the fact that variances are always positive, the variance of the approximation field $\hat{H}$ must be smaller or equal to the original $H$. Hence, EOLE always underestimates the true variability of the random field. Therefore, $n_z$ should be chosen large enough to ensure a good approximation of $H$. 

2.3 Conditional EOLE

Assume now that in the continuous domain \( \mathcal{X} \), there exists a set of locations \( \mathcal{X} = \{ \chi^{(1)}, \ldots, \chi^{(N)} \} \) where the random field value is deterministically known, i.e. \( h^{(i)} = H(\chi^{(i)}) \).

These points are called geographical evidence or observations. Let \( \mathbf{h} = \{ h^{(1)}, \ldots, h^{(N)} \} \) denote the vector of observed values.

The conditional random field can be discretized by a finite set of nodal points \( \mathcal{X} \), as introduced before. Then, it can be shown that the distributions of \( \hat{Y} \) conditional of the observations \( \mathbf{h} \) follows a Gaussian distribution, \( \hat{Y} | \mathbf{h} \sim N(\mu_{\hat{Y}|\mathbf{h}}, \Sigma_{\hat{Y}|\mathbf{h}}) \) whose mean and covariance are computed by:

\[
\mu_{\hat{Y}|\mathbf{h}} = \mu_{\hat{Y}} + \Sigma_{\hat{Y}\mathbf{h}}\Sigma_{\mathbf{h}\mathbf{h}}^{-1}(\mathbf{h} - \mu_{\mathbf{h}}) \quad (6)
\]

\[
\Sigma_{\hat{Y}|\mathbf{h}} = \Sigma_{\hat{Y}\hat{Y}} - \Sigma_{\hat{Y}\mathbf{h}}\Sigma_{\mathbf{h}\mathbf{h}}^{-1}\Sigma_{\mathbf{h}\hat{Y}}. \quad (7)
\]

In order to sample from this conditional Gaussian distribution, [8] proposed a simple two-step algorithm to sample. Here, a modified algorithm is shown which takes into account the EOLE approximation of the (unconditional) random field. In the first step, a realization of the unconditional random field is generated using EOLE (see Eq. (4)), i.e. ignoring the observations. The realization shall be denoted by \( \mathbf{z}^0 = (\mathbf{y}^0, \mathbf{h}^0)^T \), which is computed at the locations \( (\mathcal{X}, \mathcal{X})^T \).

In a second step, the corresponding realization of the conditional random field is computed by:

\[
\mathbf{y} = \mathbf{y}^0 + \Sigma_{\mathbf{y}\mathbf{h}}\Sigma_{\mathbf{h}\mathbf{h}}^{-1}(\mathbf{h} - \mathbf{h}^0). \quad (8)
\]

This is a simple deterministic transform including the unconditional realization as well as covariance matrices. Note that the support points \( \mathcal{X} \) for EOLE discretization do not coincide with observation points \( \mathcal{X} \).

3 PCE-based structural reliability analysis

3.1 Computational model

Consider a system whose behaviour is represented by a computational model \( \mathcal{M} \) which maps the \( M \)-dimensional input space to the one-dimensional output space:

\[
\mathcal{M} : \mathbf{u} \in \mathcal{D}_U \subset \mathbb{R}^M \mapsto \mathbf{v} = \mathcal{M}(\mathbf{u}) \in \mathbb{R}, \quad (9)
\]

where \( \mathbf{u} = (u_1, \ldots, u_M)^T \). The computational model is considered a deterministic black box mapping, such as a finite element model (FEM). As the input vector \( \mathbf{u} \) can be affected by uncertainty, its components are modelled by probability distributions. In particular, \( U_i \) is modelled by a probability density distribution (PDF) \( f_{U_i} \) and the corresponding cumulative distribution function (CDF) \( F_{U_i} \). For the sake of simplicity, the components of \( U \) are assumed to be statistically independent in this paper. Then, the joint PDF of \( U \) can be written as the product of the marginal PDFs. Note that in the case of dependent input variables, an isoprobabilistic transform, such as the Nataf or Rosenblatt transform [9, 10], can produce an input vector of independent components.

3.2 Reliability analysis

Reliability analysis is a framework to estimate the probability that the system of interest does not comply with a performance criterion. In the simplest case, the failure probability is defined as \( P_f = \mathbb{P}(V \leq v_0) \), where \( V = \mathcal{M}(U) \). The failure probability can be recast as the following integral:

\[
P_f = \int_{\mathcal{D}_f} f_{U}(\mathbf{u}) \, d\mathbf{u}, \quad (10)
\]
where $\mathcal{D}_f = \{ \mathbf{u} \in \mathcal{D}_U : \mathcal{M}(\mathbf{u}) \leq v_0 \}$ is the failure domain. Due to the generally complex shape of the failure domain and the black box computational model, the integration cannot be solved analytically. A numerical estimation of $P_f$ can be obtained by sampling-based approaches such as Monte Carlo simulation. Assuming a large number of samples of $\mathbf{U}$, the failure probability can be estimated by $P_f = n_f/n$, where $n_f$ is the number of samples belonging to the failure domain and $n$ is the total number of samples.

In engineering applications, the failure probability typically ranges from $10^{-3}$ to $10^{-6}$, which can be interpreted as rare events. In order to estimate accurately the failure probability, a large number of samples is required in the Monte Carlo simulation. When the computational model is an expensive-to-evaluate function, the reliability analysis becomes intractable. In this situation, a popular strategy consists of approximating the expensive-to-evaluate computational model with a meta-model, such as polynomial chaos expansions, which are described in the following.

### 3.3 Polynomial chaos expansions

A popular meta-modelling method is polynomial chaos expansions (PCE) which approximate the computational model $\mathcal{M}$ with a sum of polynomials orthogonal with respect to the distributions of the input variables [6, 15]:

$$V \approx \mathcal{M}(\mathbf{U}) = \sum_{\alpha \in \mathcal{A}} a_{\alpha} \psi_{\alpha}(\mathbf{U}),$$

(11)

where $a_{\alpha} \in \mathbb{R}$ are polynomial coefficients corresponding to indices $\alpha$ in the truncated set $\mathcal{A} \subset \mathbb{N}^M$ and $\psi_{\alpha}(\mathbf{U})$ are multivariate orthonormal polynomials.

The efficiency of the meta-modelling algorithm greatly depends on the choice of the truncated index set $\mathcal{A}$ and the subsequent computation of the coefficients $a_{\alpha}$. [1] introduced a non-intrusive least-squares minimization approach. Consider a set of $N$ samples of the input vector, denoted by $\mathcal{U} = \{ \mathbf{u}^{(i)} , i = 1, \ldots , N \}$, and the corresponding response values of the exact computational model, denoted by $\mathcal{V} = \{ v^{(i)} = \mathcal{M}(\mathbf{u}^{(i)}) , i = 1, \ldots , N \}$. The coefficients $a_{\alpha}$ can be computed by the solution of the least-squares problem:

$$\hat{a} = \arg \min_{\mathbf{a} \in \mathbb{R}^{|\mathcal{A}|}} \frac{1}{N} \sum_{i = 1}^{N} \left( v^{(i)} - \sum_{\alpha \in \mathcal{A}} a_{\alpha} \psi_{\alpha}(\mathbf{u}^{(i)}) \right)^2.$$ 

(12)

To further increase the efficiency of the meta-modelling algorithm, in particular when $M$ is large, a number of algorithms have been developed to select out of a candidate set of polynomials the ones that are most influential to the system response. Following [5], [3] introduced the least angle regression selection algorithm (LARS) for this purpose. LARS determines a sparse set of polynomials that best describes the behaviour of the exact computational model $\mathcal{M}$ based on the experimental design $\mathcal{U}$, hence the name sparse PCE.

### 3.4 PCE-based failure probability

The sparse PCE model is an inexpensive-to-evaluate function due to its polynomial format. Hence, a plain Monte Carlo simulation with an appropriate number of samples is tractable. Therein, the exact failure domain in Eq. (10) is replaced by its PCE-based estimate denoted by $\mathcal{D}_U^{(p)} = \{ \mathbf{u} \in \mathcal{D}_U : \mathcal{M}^{(p)}(\mathbf{u}) \leq v_0 \}$. In this context, the estimation of failure probabilities by simple Monte Carlo-based methods becomes tractable. In particular, the failure probability is estimated as:

$$\hat{P}_f^{(p)} = n_f \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{ \mathbf{u} \in \mathcal{D}_U^{(p)} \}}(\mathbf{u}^{(i)}),$$

(13)
where \( n \) is the number of samples of a sample set \( \mathcal{S} = \{\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}\} \) sampled from the input vector \( \mathbf{U} \), \( n_f \) is the number of samples in the PCE-based failure domain \( \mathcal{D}^{(P)}_U \), and \( \mathbb{I} \) is the indicator function which reads \( \mathbb{I} = 1 \) for \( \mathbf{u} \in \mathcal{D}^{(P)}_U \) and \( \mathbb{I} = 0 \) otherwise.

4 Failure probability in the presence of conditional random fields

In the previous sections, the discretization of random fields as well as the PCE-based reliability analysis are discussed. When the problem at hand includes both, the analysis is formulated as follows. As an example, consider a geotechnical slope stability analysis, where the mechanical properties of the soil are modelled by random fields (see also Section 5). The conditional EOEL algorithm presented in Section 2.3 allows one to model the spatial variability with a finite number of Gaussian variables, denoted by the random vector \( \mathbf{Z} \). Therefore, we can formulate a random vector \( \tilde{\mathbf{X}} \) that includes \( \mathbf{Z} \) as well as other random parameters, summarized in \( \mathbf{X} \):

\[
\tilde{\mathbf{X}} = (\mathbf{X}, \mathbf{Z})^T. \tag{14}
\]

Accordingly, a computational model can be defined for the following workflow. Given a realization of \( \tilde{\mathbf{X}} \) (denoted by \( \tilde{\mathbf{x}} \)), the corresponding realization \( \mathbf{z} \) is used to assign the spatially-distributed mechanical properties to the soil. Subsequently, a solver, such as the finite element method, is used to estimate the behaviour of the soil taking into account the realization of the remaining random variables \( \mathbf{x} \) corresponding to \( \tilde{\mathbf{x}} \).

In this context, a meta-model-based reliability analysis is possible, due to the purely probabilistic description of the variability in the input vector. Hence, the Monte Carlo simulation in Eq. (13) is applicable for the vector \( \tilde{\mathbf{X}} \).

5 Slope reliability

5.1 Problem statement

Consider the slope sketched in Figure 1(a). The soil mass consists of two layers connected by a spatially varying soil interface \( I(x) \). The interface between the two soil layers is modelled by a one-dimensional Gaussian random field with mean value \( \mu_I = -9 \) meters measured from the top of the embankment, standard deviation \( \sigma_I = 2 \) meters, and an Gaussian autocorrelation function with correlation length \( l_I = 20 \) meters. In the following analysis, two cases are distinguished: (i) no information of the interface is known (unconditional random field) and (ii) the location of the interface at the surface is identified at coordinates \( \{ x = 40 \text{ m}, y = 5 \text{ m}, \} \), \( \{ x = 60 \text{ m}, y = 0 \text{ m}, \} \), indicated in Figure 2(b) by diamond markers. The latter figure also shows a number of realizations of the conditional random field.

![Geometry sketch and FEM mesh](image_url)

*(Figure 1: Slope reliability model)*

Further, it is assumed that the soil behaves undrained, i.e. its failure is modelled by the Tresca failure criterion, which is characterized by the undrained shear strength \( c_u \). In order to account
for uncertainties, \( c_u \) is modelled by a random variable in each layer of the soil model, \( i.e. c_{u,u} \) and \( c_{u,l} \) for the upper and lower layer, respectively. Moreover, the linear elastic behaviour of the soil is modelled by Young’s modulus \( E \) and Poisson’s ratio \( v \). The random variables are summarized in Table 1, where \( \rho \) denotes the density of the soil mass.

\[
\begin{array}{|c|c|c|c|}
\hline
U_i & Description & Distribution & Mean & CoV \\
\hline
E & Young’s modulus & Lognormal & 50 MPa & 5 \% \\
\nu & Poisson’s ratio & Constant & 0.3 & - \\
\rho & Soil density & Constant & 2000 kg/m^3 & - \\
c_{u,u} & Undrained shear strength upper layer & Lognormal & 40 kPa & 20 \% \\
c_{u,l} & Undrained shear strength lower layer & Lognormal & 60 kPa & 20 \% \\
\hline
\end{array}
\]

### Table 1: Input parameters

#### 5.2 Analysis

The random field is discretised with the EOLE algorithm and \( n_z = 15 \) standard Gaussian variables, which accounts for more than 99.99 \% of the variability of the random field. Figure 2 shows a number of realizations of the random field, in particular the case of unconditional (Figure 2(a)) and conditional random fields (Figure 2(b)). The large variability of the interface \( I(x) \) is clearly visible by the large scatter of the different curves. Moreover, the realizations of the conditional random field nicely interpolate the two observation points, seen in Figure 2(b).

![Figure 2: Random field realizations](image)

The discretization of \( I(x) \) is then used in a finite element model to distinguish the two layers of soil by assigning different mechanical properties to the elements. The resulting soil model implemented in the free finite-element software slope64 [7], which is designed for slope stability analyses. The mesh implemented for the current example consists of 900 elements and is shown in Figure 1(b).

Given the discretization of the random field, the performance of the system is assessed by the factor of safety (FOS), which is obtained through the so-called shear strength reduction method (see details in \( e.g. \) [13]). In the present case of an undrained analysis, the reduced shear strength is defined by:

\[
c_{u,f} = \frac{c_u}{\text{FOS}}.
\]

At the beginning of the analysis, gravity is applied to the soil model. Then, FOS is determined as the value that brings the slope to the point of failure. In other words, FOS is the largest value where the FEM converges to a stable solution. Finally, failure occurs when FOS \( \leq 1 \). Hence, the corresponding limit-state function reads \( g_{\text{FOS}} = \text{FOS}(X) - 1 \) and the corresponding failure
probability is $P_f = \mathbb{P}(\text{FOS}(X) \leq 1)$. Note that other than $c_u$, random parameters of the input vector are not affected by the shear strength reduction method.

Then, a number of runs of $slopes64$ are used to calibrate a sparse PCE model of the limit-state function $g_{\text{FOS}}$. Last but not least, the sparse PCE model is used to estimate the failure probability of the embankment by Monte Carlo simulation with $n_{MC} = 10^6$ samples.

### 5.3 Results

Table 2 summarizes the results of the PCE-based structural reliability problem. In particular, the relative generalization error describes the mean squared error between meta-model prediction value and exact response values relative to the variance of the response values, typically estimated based on a validation set (which consists here of $n_{MC} = 1000$ Monte Carlo samples). In this example, the relative generalization error is small, indicating a good approximation of the computational model in both random field cases.

<table>
<thead>
<tr>
<th>Method</th>
<th>Unconditional RF</th>
<th>Conditional RF</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCE</td>
<td>$N$</td>
<td>$\hat{P}_f$</td>
</tr>
<tr>
<td></td>
<td>300</td>
<td>$7.33 \cdot 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\text{err}}_{\text{gen}}$</td>
<td>$5.85 \cdot 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{P}_f$</td>
<td>$1.90 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>AK-MCS</td>
<td>$\hat{P}_f$</td>
<td>$5.73 \cdot 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\beta}_{HL}$</td>
<td>2.53</td>
</tr>
</tbody>
</table>

Table 2 shows the PCE-based estimation of the failure probability as well as the corresponding reliability index, computed by $\hat{\beta}_{HL} = -\Phi^{-1}(-\hat{P}_f)$, where $\Phi$ is the CDF of a standard normal variable. The resulting failure probability is rather large for the unconditional random field, whereas it is smaller by a factor of three for the conditional random field.

In order to estimate the accuracy of the proposed PCE-based approach, the failure probability is also computed by AK-MCS [4] implemented in UQLAB [12]. The accuracy of the failure probability estimate is higher for the conditional random field when comparing to the AK-MCS solution. This is in agreement with the estimation of the relative generalization error.

### 6 Conclusions

Spatial variability occurs often in engineering applications and can be modelled by random field theory. Using the EOLE algorithm, realizations of the random field can be generated conveniently for further sampling-based algorithms. Moreover, the expansion optimal linear estimation (EOLE) method discretizes the random field by a finite set of independent Gaussian variables. Then, the combination of EOLE and polynomial chaos expansions (PCE) allows for an efficient approach to meta-model quantities of interest as a function of spatially varying input quantities. In this paper, the proposed approach is applied to structural reliability analysis, allowing for an efficient estimation of the failure probability, as illustrated in the slope stability example.

### References


