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Analyticity of Nekrasov Partition Functions and Deformed Gaiotto States

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Abstract

In this thesis, we prove analyticity of the K-theoretic Nekrasov partition function on a suitable domain. To this end, we estimate the growth of its coefficients. We perform this estimate for a parameter range for which the expression of the coefficients as sums over partitions is badly behaved. We employ an integral representation of the coefficients. We prove the validity of an explicit contour description, necessary for the estimate. The estimate itself uses techniques from potential theory. We comment on the consequences of our results, via the AGT relation, for the norm of deformed Gaiotto states.

Kurzfassung

In dieser Arbeit beweisen wir die Analytizität der K-theoretischen Nekrasov Partitionsfunktion auf einem geeigneten Gebiet. Zu diesem Zweck schätzen wir das Wachstum ihrer Koeffizienten ab. Wir führen diese Abschätzung für einen Parameterbereich durch, in dem sich die Darstellung der Koeffizienten als Summe über Partitionen schlecht verhält. Wir nutzen eine Integraldarstellung der Koeffizienten. Wir beweisen die Gültigkeit einer expliziten Beschreibung der Integrationskontour, die für die Abschätzung notwendig ist. Die Abschätzung selbst beruht auf Techniken aus der Potentialtheorie. Wir kommentieren die Konsequenzen unserer Ergebnisse für die Norm deformierter Gaiotto-Zustände, welche auf Grund der AGT Beziehung bestehen.

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Contents

1. Introduction and Motivation

In this thesis, we estimate the radius of convergence of certain formal power series. In physics, those series appear in conformal field theory and supersymmetric gauge theory. In those theories, they represent so-called conformal blocks and Nekrasov partition functions, respectively. They are linked by a relation which goes under the name of AGT relation. They have numerous applications in mathematics. In this chapter, we will informally introduce those power series. We will try to describe a part of their meaning in physics and mathematics. We will motivate our studies of their convergence properties.

In section 1.1, we will introduce the notion of conformal blocks in two dimensional conformal field theory. They are defined in terms of the representation theory of the Virasoro algebra. Afterwards, in section 1.2, we will relate them to certain instanton partition functions defined for four-dimensional supersymmetric gauge theories via the AGT relation. Afterwards, in section 1.3, we describe a certain degeneration of conformal blocks. These are defined by certain states called Gaiotto states. Those states are motivated by their relation to different instanton partition functions via an extension of the AGT relation. In section 1.4, we then describe a deformation of the Virasoro algebra and of the notion of deformed Gaiotto states. On the gauge theory side, we also describe a certain deformation in section 1.5 and an extension of the AGT relation to this setting. In all those sections, the main objects of interest will be defined as formal power series. We motivate the study of their convergence properties. For the conformal blocks of section 1.1, this motivation partly comes from physics. For the various forms of Gaiotto states, this motivation partly comes from the study of isomonodromic deformations of ordinary differential equations in mathematics. We describe this application in section 1.6.

The discussion in this chapter is completely informal. None of its content is original work of the author. Most of the statements contained in this chapter are true only in an approximate sense or under additional assumptions we do not mention. Moreover, we do not present the theories in this chapter in their full generality. The sole purpose of this chapter is to tell a story that motivates the discussion in the following chapters. In those following chapters, we will then work with full mathematical rigor.

1.1. Conformal Blocks in Virasoro Conformal Field Theory

A map preserving angles and lengths is called conformal. A classical field theory is called conformal if the conformal maps are among the symmetries of the theory. A corresponding quantum field theory is then required to have among its symmetries a projective representation of the conformal maps. We consider such conformal field theories in two dimensions. On the level of Lie algebras this implies [9] an action of the Virasoro algebra on the Hilbert space of the quantum field theory. A typical example of such a theory would be bosonic string theory [41].

The Virasoro algebra is an infinite dimensional Lie algebra, whose generators $L_n, n \in \mathbf{Z}$ and C satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(n-1)n(n+1)\delta_{m+n,0}C, \tag{1.1}$$

and C is central. In a conformal field theory, one requires that C acts on the Hilbert space by multiplication with a complex number $c \in \mathbf{C}$. This number is called the central charge of the theory. A quantum field is an endomorphism of the Hilbert space, parametrized by a coordinate. The symmetry property of a quantum field $\phi(z)$ is then expressed by the property

$$[L_n, \phi(z)] = (z^{n+1}\partial_z + (n+1)hz^n)\phi(z). \tag{1.2}$$

Here, the parameter $h \in \mathbf{C}$ is called the conformal dimension of the field $\phi(z)$.

The aim of quantum field theory is the computation of all possible correlation functions of the fields. In conformal field theory, these correlation functions can be constructed from special kinds of correlation functions called conformal blocks [9, 16, 52, 29]. These are fixed by properties of the Virasoro algebra. In order to define them, we introduce the notion of Verma modules for the Virasoro algebra.

Given two complex numbers $h, c \in \mathbf{C}$, the corresponding Verma module $V_{h,c}$ is defined to be the representation of the Virasoro algebra generated by a vector $|h\rangle \in V_{h,c}$ satisfying

$$L_n|h\rangle = 0 \quad (n > 0) \qquad L_0|h\rangle = h|h\rangle \qquad C|h\rangle = c|h\rangle\,. \tag{1.3}$$

A basis of $V_{h,c}$ is given by

$$L_{-\lambda}|h\rangle = L_{-\lambda(1)} \cdots L_{-\lambda(l)}|h\rangle \tag{1.4}$$

where λ is a partition of an integer. See section A in the appendix for our conventions on partitions. We also set, for such a partition λ

$$L_\lambda = L_{\lambda(l)} \cdots L_{\lambda(1)}. \tag{1.5}$$

The module $V_{h,c}$ is non-negatively graded, where we assign the degree $-n$ to L_n. Its degree zero component is one dimensional and spanned by the vector $|h\rangle$. Denote by $\langle h|$ the projection onto $|h\rangle$ along all components of positive degree. The expression

$$\langle L_{-\lambda}h|L_{-\mu}h\rangle = \langle h|L_\lambda L_{-\mu}h\rangle \qquad \langle h|h\rangle = 1 \tag{1.6}$$

defines a bilinear form on the Verma module $V_{h,c}$. For each degree $n \geq 0$, the Gram matrix $K^{(n)}(h)$ is defined by its entries

$$K^{(n)}(h)_{\lambda,\mu} = \langle h|L_\lambda L_{-\mu}|h\rangle \qquad (|\lambda| = |\mu| = n). \tag{1.7}$$

Fix a complex number $c \in \mathbf{C}$. For a triple (h_1, h_2, h') of complex numbers, an intertwiner $\phi^{h_2}_{h'h_1}(z)$ between $V_{h_1,c}$ and $V_{h_2,c}$ is defined as a formal power series

$$\phi^{h_2}_{h'h_1}(z) = \sum_{n\in\mathbf{Z}} \phi_n z^{n+a} \qquad a = h_2 - h_1 - h \tag{1.8}$$

whose coefficients are linear maps $\phi_n : V_{h_1,c} \to V_{h_2,c}$. Moreover, one requires the intertwining property

$$[L_n, \phi_{h'h_1}^{h_2}(z)] = (z^{n+1}\partial_z + (n+1)h'z^n)\phi_{h'h_1}^{h_2}(z). \tag{1.9}$$

We also use the suggestive notation

$$\phi_{h'h_1}^{h_2}(z) : V_{h_1,c} \to V_{h_2,c}. \tag{1.10}$$

For generic parameters c, h_1, h_2 and h', the intertwining property characterizes an intertwiner uniquely up to normalization.

We consider conformal field theory on the Riemann sphere $\mathbf{CP}^1$. We think of the formal variable z as an element of $\mathbf{CP}^1$. The linear hull of the three elements L_{-1}, L_0, and L_1 realizes a copy of $\mathfrak{sl}_2(\mathbf{C})$ inside the Virasoro algebra. From the intertwining property, it follows, by exponentiating L_{-1}, L_0 and L_1, that the intertwiners $\phi_{h',h_1}^{h_2}(z)$ transform under Möbius transformations

$$f(z) = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2, \mathbf{C}) \tag{1.11}$$

as

$$\phi_{h',h_1}^{h_2}(f(z)) = f'(z)^{-h'}\phi_{h',h_1}^{h_2}(z). \tag{1.12}$$

Let $n \geq 0$ be an integer. Fix complex numbers

$$h_1, \ldots, h_{n+1} \in \mathbf{C} \qquad h'_1, \ldots, h'_n \in \mathbf{C} \tag{1.13}$$

and set

$$a_k = h_{k+1} - h_k - h'_k \qquad (k = 1, \ldots, n). \tag{1.14}$$

The corresponding conformal block is defined as

$$\langle\, h_{n+1}|\phi_{h'_n,h_n}^{h_{n+1}}(z_n)\phi_{h'_{n-1},h_{n-1}}^{h_n}(z_{n-1}) \cdots \phi_{h'_1,h_1}^{h_2}(z_1)|h_1\,\rangle \in \mathbf{C}[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]z^{a_1+\cdots+a_n}. \tag{1.15}$$

It is conjectured that this formal power series converges for

$$z_1, \ldots, z_n \in \mathbf{C} \qquad |z_1| < \cdots < |z_n|. \tag{1.16}$$

Consider an intertwiner

$$\phi_{h,0}^{h}(z) : V_{0,c} \to V_{h,c}. \tag{1.17}$$

From the intertwining property, we obtain

$$\phi_{h,0}^{h}(0)|0\,\rangle = \langle\, h|\phi_{h,0}^{h}(1)|0\,\rangle \quad |h\,\rangle\,. \tag{1.18}$$

Here, $\langle\, h|\phi^{h}_{h,0}(1)|0\,\rangle$ refers to the matrix element $\langle\, h|\phi^{h}_{h,0}(z)|0\,\rangle$ evaluated at $z=1$, which is well-defined in contrast to the expression $\phi^{h}_{h,0}(1)$. Similarly, for an intertwiner

$$\phi^{0}_{h,h}(z) : V_{h,c} \to V_{0,c}, \tag{1.19}$$

we have

$$\lim_{z\to\infty} z^{2h}\,\langle\, 0|\phi^{0}_{h,h}(z) = \langle\, 0|\phi^{0}_{h,h}(1)|h\,\rangle \quad \langle\, h|. \tag{1.20}$$

Hence, it suffices to consider $h_{n+1} = 0 = h_1$ in (1.15).

By insertion of a complete set of states

$$\sum_{n\geq 0}\sum_{|\lambda|=|\mu|=n} \left(K^{(n)}(h_k)^{-1}\right)_{\lambda\mu} L_{-\lambda}|h_k\,\rangle\,\langle\, h_k|L_\mu \tag{1.21}$$

in one of the intermediate Verma modules $V_{h_k,c}$, $k = 2,\ldots,n$, one can reduce the computation of the conformal block (1.15) to the computation of conformal blocks with fewer intertwiners.

By using the Möbius invariance (1.12) of intertwiners, one can assume $z_1 = 0, z_{n-1} = 1, z_n = \infty$. In particular, the general three point conformal block is determined by the constant

$$\langle\, 0|\phi^{0}_{-h_2,h_2}(\infty)\phi^{h_2}_{h,h_1}(1)\phi^{h_1}_{h_1,0}(0)|0\,\rangle = \langle\, h_2|\phi^{h_2}_{h,h_1}(1)|h_1\,\rangle \tag{1.22}$$

which can be computed explicitly. It determines all other conformal blocks.

We illustrate this statement with the four point conformal block. It is given by

$$\langle\, h_3|\phi^{h_3}_{h_2'h_2}(1)\phi^{h_2}_{h_1'h_1}(z)|h_1\,\rangle\,. \tag{1.23}$$

We insert a complete set of states to obtain

$$\sum_{n\geq 0}\sum_{|\lambda|=|\mu|=n} \left(K^{(n)}(h_2)^{-1}\right)_{\lambda\mu} \langle\, h_3|\phi^{h_3}_{h_2'h_2}(1)|L_{-\lambda}h_2\,\rangle\,\langle\, h_2|L_\mu\phi^{h_2}_{h_1'h_1}(z)|h_1\,\rangle\,. \tag{1.24}$$

Next, we want to determine the z dependence of the last factor. Fix a partition μ. We look at

$$\langle\, h_2|L_\mu L_0\phi^{h_2}_{h_1'h_1}(z)|h_1\,\rangle\,. \tag{1.25}$$

On the one hand, by commuting L_0 to the left, we obtain

$$(h_2+|\mu|)\,\langle\, h_2|L_\mu\phi^{h_2}_{h_1'h_1}(z)|h_1\,\rangle\,. \tag{1.26}$$

On the other hand, by commuting L_0 to the right, we obtain

$$(z\partial_z + h_1' + h_1)\,\langle\, h_2|L_\mu\phi^{h_2}_{h_1'h_1}(z)|h_1\,\rangle\,. \tag{1.27}$$

Equating both calculations, we obtain

$$\langle\, h_2|L_\mu\phi^{h_2}_{h_1'h_1}(z)|h_1\,\rangle = \langle\, h_2|L_\mu\phi^{h_2}_{h_1'h_1}(1)|h_1\,\rangle \quad z^{|\mu|+h_2-h_1-h_1'}. \tag{1.28}$$

Note that this also justifies the exponents in the formal power series (1.8). Our four point conformal block becomes the formal power series

$$\langle h_3 | \phi^{h_3}_{h_2' h_2}(1) \phi^{h_2}_{h_1' h_1}(z) | h_1 \rangle = \sum_{n \geq 0} z^{n+h_2-h_1-h_1'} \Bigg(\sum_{|\lambda|=|\mu|=n} \big(K^{(n)}(h_2)^{-1}\big)_{\lambda\mu} \tag{1.29}$$

$$\times \langle h_3 | \phi^{h_3}_{h_2' h_2}(1) | L_{-\lambda} h_2 \rangle \langle h_2 | L_\mu \phi^{h_2}_{h_1' h_1}(1) | h_1 \rangle \Bigg). \tag{1.30}$$

It is an element of $\mathbf{C}[[z]]z^{h_2-h_1-h_1'}$ whose coefficients are determined by the three point conformal blocks. It is conjectured to converge for $|z| < 1$. It is desirable to prove this convergence, since in the end the correlation functions in conformal field theory, which are composed of such conformal blocks, should yield well-defined functions.

1.2. Four-Dimensional Nekrasov Partition Functions and AGT Relation

The conformal blocks of Virasoro conformal field theory are related to supersymmetric Yang Mills theory. In this section, we sketch this relation. In part 1.2.1, we introduce the four-dimensional Nekrasov partition function and explain its meaning for supersymmetric gauge theory. In part 1.2.2 we relate it to the four point conformal block on $\mathbf{CP}^1$.

1.2.1. Four-Dimensional Nekrasov Partition Functions

Fix integers $r \geq 1$ and $N_f \geq 0$. Consider $\mathcal{N} = 2$ supersymmetric Yang Mills theory with gauge group $G = U(r)$ or $G = SU(r)$ on Euclidean $\mathbf{R}^4$ with N_f matter hypermultiplets in the fundamental representation. Let $a = (a_1, \dots, a_r)$ denote the vacuum expectation value of the massless vector multiplet and $m = (m_1, \dots, m_{N_f})$ the masses of the fundamental matter. If the gauge group is $SU(r)$, the parameter a satisfies $\sum_{\alpha=1}^r a_\alpha = 0$. In $\mathcal{N} = 2$ theory, the Lagrangian can be expressed in terms of a function $\mathcal{F}(a, m, \Lambda)$, the prepotential. Seiberg and Witten proposed [43, 44] an exact low energy effective action via the prepotential

$$\mathcal{F}(a, m, \Lambda) = \mathcal{F}_{\text{pert}}(a, m, \Lambda) + \mathcal{F}_{\text{inst}}(a, m, \Lambda). \tag{1.31}$$

The prepotential also depends on a dynamically generated scale Λ. It consists of perturbative and instanton corrections.

Nekrasov found [39] a new expression for the instanton corrections $\mathcal{F}_{\text{inst}}(a, m, \Lambda)$. The supersymmetric Yang Mills theory on $\mathbf{R}^4$ can be realized from dimensional reduction of a theory on the flat six dimensional space

$$\mathbf{R}^4 \times \mathbf{T}^2, \tag{1.32}$$

where $\mathbf{T}^2$ denotes the torus. Using this approach, Nekrasov introduced a deformation of the Yang Mills theory, parametrized by two parameters ϵ_1 and ϵ_2. He then sends the area of the torus to zero and performs instanton calculations for this deformation of Yang Mills theory. For an integer $n \geq 0$, let M_n denote a certain compactification of the moduli space of instantons with instanton number n. Both the gauge group G and a maximal torus T of the rotation group of $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$ act on those moduli spaces. Nekrasov considers $G \times T$-equivariant cohomology. For $n \geq 0$, let

$$p_{n*} : H^*_{G\times T}(M_n) \to H^*_{G\times T}(\mathrm{pt}) \tag{1.33}$$

denote the pushforward in equivariant cohomology defined by the map collapsing M_n to a point. Nekrasov constructs the generating series

$$\sum_{n\geq 0} \Lambda^{2rn} p_{n*}(1) \tag{1.34}$$

of the classes $1 \in H^*_{G\times T}(M_n)$ for $n \geq 0$. It is a formal power series. He expresses the Euclidean path integral of the deformed gauge theory as a product of perturbative and instanton contributions. He then identifies the instanton factor with the generating series (1.34). Nekrasov computes the coefficients of that generating series using localization techniques [34, 33]. In the limit $\epsilon_1, \epsilon_2 \to 0$, the deformation of the theory vanishes and we get the ordinary supersymmetric gauge theory on the flat space-time $\mathbf{R}^4$, which Seiberg and Witten considered. Nekrasov claims [39]

$$\mathcal{F}_{\mathrm{inst}}(a, m = \emptyset, \Lambda) = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log \sum_{n\geq 0} \Lambda^{2rn} p_{n*}(1) \tag{1.35}$$

for $N_f = 0$. In the presence of fundamental matter, a similar claim holds. One has to replace the pushforward of $1 \in H^*_{G\times T}(M_n)$ by the pushforward a certain equivariant Euler class $E(N_f)$. This representation of the prepotential was proved independently in [40] and [35].

1.2.2. Four-Dimensional AGT Relation

Many examples of four dimensional supersymmetric Yang Mills theory can be obtained [21] from dimensional reduction of products

$$\mathbf{R}^4 \times \Sigma \tag{1.36}$$

of space-time with a punctured Riemann surface Σ. This type of reduction is different from the one Nekrasov performed to deform ordinary Yang Mills theory. In the present case, one obtains a class of $\mathcal{N} = 2$ super*conformal* gauge theories on $\mathbf{R}^4$ which are indexed by two integers g and n, the genus g of the Riemann surface Σ and its number of punctures n. The parameter space of gauge couplings is given by the complex structure moduli space $\mathcal{M}_{g,n}$ of genus g Riemann surfaces with n punctures. Each decomposition of the Riemann surface Σ into $2g - 2 + n$ pairs of pants corresponds to a different Lagrangian description of the gauge theory.

On the Riemann surface Σ, one can consider two dimensional conformal field theory. When calculating vacuum expectation values, the punctures of the Riemann surface correspond to fields inserted at those points.

Consider, for example, the case $n = 4$, $g = 0$. Here, the Riemann surface Σ is a sphere with four punctures. The moduli space $\mathcal{M}_{0,4}$ is parametrized by their cross ratio $z \in \mathbf{CP}^1$. We can consider the punctures to be located at

$$z_1 = 0 \qquad z_2 = z \in \mathbf{C} \setminus \{0, 1\} \qquad z_3 = 1 \qquad z_4 = \infty. \tag{1.37}$$

On the one hand, we have already defined the four point conformal block (1.29) for the conformal field theory described by this punctured Riemann surface. On the other hand, dimensional reduction as in [21] produces from this Riemann surface a $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $SU(2)$ with $N_f = 4$ fundamental matter hypermultiplets. The parameter z is related to the ultraviolet coupling τ in the Lagrangian description of the gauge theory via

$$z = \exp(2\pi i \tau). \tag{1.38}$$

Alday, Gaiotto and Tachikawa conjectured [2] that the instanton part of the Nekrasov partition function (1.34), defined for the corresponding supersymmetric gauge theory, is proportional to the four point conformal block (1.29) under a suitable identification of parameters: The central charge c of the conformal field theory is parametrized in Liouville fashion as

$$c = 1 + 6Q^2 \qquad Q = b + b^{-1}. \tag{1.39}$$

The new parameter b is related to the gauge theory side via

$$\epsilon_1 = b \qquad \epsilon_2 = b^{-1} \tag{1.40}$$

The conformal dimensions $h_1, h_2, h_3, h'_1, h'_2$ in the four point block (1.29)

$$\langle\, h_3 | \phi^{h_3}_{h'_2 h_2}(1) \phi^{h_2}_{h'_1 h_1}(z) | h_1 \,\rangle \tag{1.41}$$

are related[1] to the masses m_1, m_2, m_3, m_4 and the parameter a on the gauge theory side as

$$h_1 = \frac{Q^2}{4} - \frac{1}{4}(m_3 - m_4)^2 \qquad h'_1 = \frac{1}{2}(m_3 + m_4)\big(Q - \frac{1}{2}(m_3 + m_4)\big) \tag{1.42}$$

$$h_2 = \frac{Q^2}{4} - a^2 \qquad h'_2 = \frac{1}{2}(m_1 + m_2)\big(Q - \frac{1}{2}(m_1 + m_2)\big) \tag{1.43}$$

$$h_3 = \frac{Q^2}{4} - \frac{1}{4}(m_2 - m_1)^2. \tag{1.44}$$

Then the authors of [2] claimed

$$\sum_{n \geq 0} z^n p_{n*}(E(N_f = 4)) = (1 - z)^{\left(Q - \frac{1}{2}(m_1 + m_2)\right)\left(Q - \frac{1}{2}(m_3 + m_4)\right)} \tag{1.45}$$

$$\langle\, h_3 | \phi^{h_3}_{h'_2 h_2}(1) \phi^{h_2}_{h'_1 h_1}(z) | h_1 \,\rangle \tag{1.46}$$

[1] In [2], two of the masses are considered in the anti-fundamental representation, hence there are some differences in the parameters.

as formal power series. This observation, and various extensions of it, are called the AGT conjecture. Certain instances of it, including the one described here, are proved [1].

From a practical perspective, this particular AGT relation defines new expressions for the coefficients of the conformal block (1.29). Fix $n \geq 0$ and set

$$(a_1, a_2) = (a, -a). \tag{1.47}$$

The application of the localization technique to the pushforwards $p_{n*}(E(N_f = 4))$ expresses [39] the respective coefficient of (1.34) as sums over pairs $\vec{Y} = (Y_1, Y_2)$ of Young diagrams of total size n. For $\alpha, \beta \in \{1, 2\}$ and a box $\Box$ in some Young diagram, define

$$E_{\alpha\beta}(\Box) = a_\alpha - a_\beta - l_{Y_\beta}(\Box)\epsilon_1 + (a_{Y_\alpha}(\Box) + 1)\epsilon_2. \tag{1.48}$$

We then have

$$p_{n*}(E(N_f = 4)) = \sum_{|Y_1|+|Y_2|=n} \frac{\prod_{\alpha=1}^{2} \prod_{\Box \in Y_\alpha} \prod_{j=1}^{4} \left(x_\Box^\alpha + \mu_j\right)}{\prod_{\alpha,\beta=1}^{2} \prod_{\Box \in Y_\alpha} E_{\alpha\beta}(\Box) \prod_{\boxtimes \in Y_\beta} \left(\epsilon_1 + \epsilon_2 - E_{\beta\alpha}(\boxtimes)\right)}. \tag{1.49}$$

Here, for a box $\Box = (x, y) \in Y_\alpha$ one sets

$$x_\Box^\alpha = a_\alpha + (x-1)\epsilon_1 + (y-1)\epsilon_2. \tag{1.50}$$

1.3. Gaiotto States and AGT Relation

The original AGT observation [2] only considers super-*conformal* gauge theories on $\mathbf{R}^4$. But, for example, $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang Mills theory with $N_f = 0$ matter hypermultiplets is not super-conformal but asymptotically free. Gaiotto extended [22] the AGT observation to such asymptotically free theories. To describe this extension, we have to express the original AGT relation in terms of Seiberg Witten curves.

The generating function

$$T(q) = \sum_{n \in \mathbf{Z}} L_n q^{-n-2} \tag{1.51}$$

of the subset $L_n, n \in \mathbf{Z}$ of generators of the Virasoro algebra acting on the Hilbert space of a conformal field theory is called the energy momentum tensor of the theory. Consider the conformal field theory side of the four dimensional AGT relation. The conformal field theory is defined on the Riemann sphere. The central object is the four point conformal block

$$\langle\, h_3 | \phi^{h_3}_{h_2' h_2}(1) \phi^{h_2}_{h_1' h_1}(z) | h_1 \,\rangle \tag{1.52}$$

Consider the insertion of $-T(q)$ into the four point conformal block. The resulting expression defines a function $\phi_2(q)$ on the Riemann surface Σ_z given by the Riemann sphere with four punctures located at $0, z, 1$ and ∞. This function defines a quadratic differential

$$\phi_2(q) dq \otimes dq \tag{1.53}$$

on the Riemann surface Σ_z. It has double poles at each of the punctures.

Now consider the gauge theory side. The central object is the Seiberg Witten prepotential $\mathcal{F}(a, m, \Lambda)$ from section 1.2. It is related [43, 44] to periods of a differential, the Seiberg Witten differential, on a Riemann surface, the Seiberg Witten curve as follows: Let Σ_z denote the Riemann sphere with four punctures located at $0, z, 1$, and ∞. The Seiberg Witten curve C can be thought of as the double cover

$$C \to \Sigma_z \qquad\qquad x^2 = \phi_2^{\mathrm{SW}}(q), \tag{1.54}$$

defined by the component $\phi_2^{\mathrm{SW}}(a)$ of the Seiberg Witten differential. One can now compute the Seiberg Witten prepotential from the set of equations

$$\frac{\partial \mathcal{F}}{\partial a_l} = a_l^D \qquad\qquad l = 1, 2, \tag{1.55}$$

where a_1, a_2, a_1^D, and a_2^D are defined as periods

$$\frac{1}{2\pi i} \oint x dq \tag{1.56}$$

around certain cycles on the Seiberg Witten curve C.

In this description, the AGT relation claims [2] that the differential $\phi_2(q)$ obtained from the insertion of $-T(q)$, in the limit of small ϵ_1, ϵ_2, yields the Seiberg Witten differential $\phi_2^{\mathrm{SW}}(q)$ from the gauge theory side of the AGT correspondence. Next, we consider Gaiotto's extension of this relation.

For the class of super-conformal field theories, which the original AGT paper considers, the corresponding quadratic differentials have poles of order ≤ 2. The asymptotically free gauge theories, to which Gaiotto extended the AGT correspondence, are still associated to Riemann surfaces Σ. However, the associated differentials are now allowed to have poles of higher order. Consider, for example, again the case of $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang Mills theory with $N_f = 0$. In this example, the Riemann surface Σ is a sphere with two punctures at zero and infinity. In the global coordinate q, the Seiberg Witten differential for this Riemann surface is defined by its component

$$\phi_2^{\mathrm{SW}}(q) = \frac{\Lambda^2}{q^3} + \frac{2u}{q^2} + \frac{\Lambda^2}{q}. \tag{1.57}$$

Here Λ is again the scale of the gauge theory and u is some parameter. Gaiotto now proposes [22], in analogy to the original AGT relation, to construct a state $|G\rangle$ on the conformal field theory side such that the expectation value of the energy momentum tensor reproduces the poles of the Seiberg Witten differential:

$$\frac{\langle G|T(q)|G\rangle}{\langle G|G\rangle} = \phi_2^{\mathrm{SW}}(q). \tag{1.58}$$

Parametrize the central charge c again in Liouville manner as

$$c = 1 + 6Q^2 \qquad\qquad Q = b + b^{-1} \tag{1.59}$$

with

$$\epsilon_1 = b \qquad\qquad \epsilon_2 = b^{-1}. \tag{1.60}$$

Also set

$$h = \frac{Q^2}{4} - a^2, \tag{1.61}$$

where $\pm a$ are the eigenvalues of the vector multiplet in the Nekrasov instanton partition function. The condition (1.58) is fulfilled if $|G\rangle$ is an element $|h, \Lambda^2\rangle$ of the Verma module $V_{h,c}$ satisfying

$$L_1|h, \Lambda^2\rangle = \Lambda^2|h, \Lambda^2\rangle \qquad\qquad L_n|h, \Lambda^2\rangle = 0 \quad (n \geq 2). \tag{1.62}$$

Such elements of the Verma modules $V_{h,c}$ are now called Gaiotto states. They are characterized by the above property up to scalar multiplication. One commonly requires that its coefficient in front of $|h\rangle$ equals one. We assume this in the following. Strictly speaking, the Gaiotto state $|h, \Lambda^2\rangle$ is an element of a certain completion of the Verma module $V_{h,c}$. Hence, after fixing the component along $|h\rangle$, its norm

$$\langle h, \Lambda^2|h, \Lambda^2\rangle \tag{1.63}$$

is defined only as a formal power series in Λ^2. We state the precise definition in chapter 2. Gaiotto's extension of the AGT conjecture now reads

$$\langle h, \Lambda^2|h, \Lambda^2\rangle = \sum_{n\geq 0} \Lambda^{2rn} p_{n*}(1) \qquad\qquad (r = 2) \tag{1.64}$$

as an equality between formal power series in Λ^2. A proof is proposed in [17]. It again yields expressions for the expansion coefficients of the norm of the Gaiotto states. The explicit form of the coefficients on the right hand side is [39]

$$p_{n*}(1) = \sum_{|Y_1|+|Y_2|=n} \prod_{\alpha,\beta=1}^{2} \prod_{\square\in Y_\alpha} \frac{1}{E_{\alpha\beta}(\square)} \prod_{\boxtimes\in Y_\beta} \frac{1}{\big(\epsilon_1 + \epsilon_2 - E_{\beta\alpha}(\boxtimes)\big)}. \tag{1.65}$$

Again, it would be desirable to know whether the Gaiotto state has a finite norm. In other words whether the formal power series on the left hand side of equation (1.64) is a well-defined function of Λ^2, i.e. whether it converges.

1.4. Deformed Virasoro Algebra and Gaiotto States

We have described how the Virasoro algebra acts as a symmetry algebra in two dimensional conformal field theory. It can be considered universal since it appears as a symmetry in all such theories. Conformal field theories are, for example, applied to the *critical phenomena* of two-dimensional statistical mechanics. The symmetry properties of *off-critical models* have also been studied in many examples [30, 32, 31, 3]. In particular, in analogy to the critical case, one is interested in finding a universal symmetry algebra which is present in all those models. In [45] a candidate for a universal symmetry algebra for off-critical models was discovered. It is called the deformed Virasoro algebra $\mathbf{Vir}_{q,t}$ and depends on two parameters q and t. It is universal in the sense that the symmetries of the off-critical models can be obtained as suitable limits for those parameters, see [6] for a review. The deformed Virasoro algebra is an associative algebra with generators $T_n, n \in \mathbf{Z}$ and relations

$$[T_m, T_n] = -\sum_{l \geq 1} f_l (T_{m-l} T_{n+l} - T_{n-l} T_{m+l}) \tag{1.66}$$

$$- \frac{(1-q)(1-t^{-1})}{1-q/t} \Big((q/t)^m - (q/t)^{-m} \Big) \delta_{m+n,0}, \tag{1.67}$$

for $m, n \in \mathbf{Z}$. Here, the coefficients $f_l, l \geq 0$ are determined by the expansion

$$\sum_{l \geq 0} f_l z^l = \exp\left(\sum_{n \geq 0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} z^n \right). \tag{1.68}$$

The infinite sum in the defining relations appears to be problematic. We defer the precise definition to chapter 2. There, we also explain the relation of the deformed Virasoro algebra to the ordinary Virasoro algebra.

In analogy to the Verma modules $V_{h,c}$ in the undeformed case, there exist modules M_h for the deformed Virasoro algebra with $h \in \mathbf{C}$ being the analog of the highest weight. They are generated by a vector $|h\,\rangle \in M_h$ which satisfies

$$T_n |h\,\rangle = 0 \quad (n > 0) \qquad\qquad T_0 |h\,\rangle = h |h\,\rangle\,. \tag{1.69}$$

The central charge is related to the parameters q and t of the deformed Virasoro algebra. They also carry a bilinear pairing

$$\langle\, T_{-\lambda} h | h T_{-\mu}\,\rangle = \langle\, h | T_\lambda T_{-\mu} h\,\rangle \qquad\qquad \langle\, h | h\,\rangle = 1 \tag{1.70}$$

as in the undeformed case. However, to the best of our knowledge, today there does not exist the notion of an intertwining property analogous to (1.2) in the undeformed case. However, the notion of a Gaiotto state has been generalized [7]. In analogy to the condition (1.62), one requires the deformed Gaiotto state $|h, \xi\,\rangle \in M_h$ to satisfy

$$T_1 |h, \xi\,\rangle = \xi |h, \xi\,\rangle \qquad\qquad T_n |h, \xi\,\rangle = 0 \quad (n \geq 2). \tag{1.71}$$

We can again, after fixing the degree zero component, define its norm

$$\langle\, h, \xi | h, \xi\,\rangle \tag{1.72}$$

as a formal power series in ξ. We defer the precise definition to chapter 2. Two questions naturally arise: Is this formal power series convergent? Is there a deformed AGT relation analogous to (1.64)?

1.5. Five-Dimensional Nekrasov Partition Function and AGT Relation

In section 1.1, we have introduced the notion of conformal blocks for the Virasoro algebra. We have seen in section 1.2, that those objects are related to an instanton partition functions in supersymmetric gauge theory on $\mathbf{R}^4$. In section 1.3, we have seen an analogous relation between the norm of the Gaiotto states in the Verma modules of the Virasoro algebra and another instanton partition function in supersymmetric gauge theory on $\mathbf{R}^4$. In section 1.4, we have discussed a deformation of the notion of Gaiotto states. In this chapter, we describe an extension of the AGT relation to the deformed setting. In section 1.5.1, we discuss a deformed version of the four-dimension Nekrasov partition version. In section 1.5.2, we relate this partition function and the deformed Gaiotto states.

1.5.1. Five-Dimensional Nekrasov Partition Function

In section 1.2.1, we have introduced the Nekrasov partition function, which is defined for $\mathcal{N} = 2$ supersymmetric Yang Mills theory with gauge group $U(r)$ on $\mathbf{R}^4$ with N_f fundamental matter hypermultiplets. This gauge theory admits a deformation [38], where one compactifies the corresponding five-dimensional theory on a circle $\mathbf{S}^1_\lambda$ with circumference λ. More precisely, one replaces Euclidean $\mathbf{R}^4$ with a fiber bundle over $\mathbf{S}^1_\lambda$ whose typical fiber is $\mathbf{R}^4$. The resulting gauge theory is only $\mathcal{N} = 1$ supersymmetric.

The coefficients of the Nekrasov partition function, for this deformed theory, are computed using equivariant K-theory instead of equivariant cohomology. We refer to chapter 3 for the details. It can be computed [39] using localization techniques in equivariant K-theory. In the case of $N_f = 0$, and $r = 2$, the result is

$$\sum_{n\geq 0} \mathfrak{b}^n \sum_{|Y_1|+|Y_2|=n} \prod_{\alpha,\beta=1}^{2} \prod_{\square\in Y_\alpha} \frac{\lambda/2}{\sinh\left(\frac{\lambda}{2} E_{\alpha\beta}(\square)\right)} \prod_{\boxtimes\in Y_\beta} \frac{\lambda/2}{\sinh\left(\frac{\lambda}{2}(\epsilon_1+\epsilon_2 - E_{\beta\alpha}(\boxtimes))\right)}. \tag{1.73}$$

When one shrinks the circumference λ of the circle $\mathbf{S}^1_\lambda$ to zero, one obtains the $\mathcal{N} = 2$ supersymmetric gauge theory on $\mathbf{R}^4$ described in section 1.2.1. On the level of the coefficients of the respective partition functions, this corresponds to the property $\sinh(\lambda x)/\lambda \to x$ as $\lambda \to 0$.

1.5.2. Five-Dimensional AGT Relation

Now that we have discussed a deformed version of the Nekrasov partition function and a deformed version of the notion of a Gaiotto state, we can look for an analogue of relation (1.64). First, we have to relate the coefficients. We require

$$t = e^{-\lambda\epsilon_1} \qquad q = e^{\lambda\epsilon_2} \qquad h = e^{-\lambda a} + e^{\lambda a} \qquad \xi^2 = \lambda^4\, \mathfrak{b}\,. \tag{1.74}$$

Awata and Yamada conjectured that under those identifications, the norm [7]

$$\langle\, h, \xi | h, \xi\, \rangle \tag{1.75}$$

of the deformed Gaiotto state equals (1.73) as formal power series in ξ^2. This relation was proved in [50, 51]. We sketch this proof in section 4. It gives a practical definition of the coefficients of the expansion of the norm of the deformed Gaiotto state, with degree zero component scaled to one, in the parameter ξ in terms of the known coefficients of the Nekrasov partition function.

Again, it would be desirable to know if these objects define honest functions, i.e. if those formal power series converge. We prove this for a suitable range of parameters in this thesis. From a physical point of view, this problem is probably less interesting compared to the well-definedness of conformal blocks in ordinary Virasoro conformal field theory. Currently, we are not aware of a physical interpretation of the deformed Gaiotto states. However, those states and their well-definedness do play a role in the mathematics of isomonodromic deformations of ordinary differential equations. We will sketch this application in section 1.6

1.6. Monodromy Preserving Transformations

Let $N, n \geq 1$ be integers. Consider the ordinary differential equation

$$\partial_z \Phi(z) = \mathcal{A}(z)\Phi(z) \tag{1.76}$$

of rank N on the Riemann sphere with n regular singular points $a_1, \ldots, a_n \in \mathbf{CP}^1$:

$$\mathcal{A}(z) = \sum_{\nu=1}^{n} \frac{\mathcal{A}_\nu}{z - a_\nu}. \tag{1.77}$$

Here $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are traceless $N \times N$ matrices. Let

$$z_0 \in \mathbf{CP}^1 \setminus \{a_1, \ldots, a_n\} \tag{1.78}$$

be a point and $\Phi(z)$ be the fundamental solution to this differential equation with $\Phi(z_0) = 1_N$. The problem of isomonodromic deformation asks how one can deform the parameters

$$z_0, \qquad a_1, \ldots, a_n, \qquad \mathcal{A}_1, \ldots, \mathcal{A}_n \tag{1.79}$$

such that the monodromy of the differential equation remains constant. This requirement translates [42] to the Schlesinger equations

$$\partial_{a_\mu} \mathcal{A}_\nu = \frac{z_0 - a_\nu}{z_0 - a_\mu} \frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \qquad (\mu \neq \nu) \tag{1.80}$$

$$\partial_{a_\nu} \mathcal{A}_\nu = -\sum_{\mu \neq \nu} \frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{a_\mu - a_\nu}, \tag{1.81}$$

$$\partial_{z_0} \mathcal{A}_\nu = -\sum_{\mu \neq \nu} \frac{[\mathcal{A}_\mu, \mathcal{A}_\nu]}{z_0 - a_\mu},. \tag{1.82}$$

One then defines the isomonodromic tau function $\tau(a_1, \ldots, a_n)$ by

$$d \log \tau(a_1, \ldots, a_n) = \sum_{\mu < \nu} \operatorname{Tr} \mathcal{A}_\mu \, \mathcal{A}_\nu \; d \log(a_\mu - a_\nu). \tag{1.83}$$

It does not depend on the base point z_0. It was argued in [23] that this tau function can be expressed as a correlation function in Virasoro conformal field theory.

Consider the special case $N = 2$ and $n = 4$, i.e. second order ordinary differential equations with four regular singular points. By Möbius invariance, we can suppose that those points are located at $0, z, 1$ and ∞. The corresponding tau function is now a function of $z \in \mathbf{CP}^1$ and satisfies

$$z(z-1)\frac{d}{dz}\log\tau(z) = (z-1)\operatorname{Tr}\mathcal{A}_0\,\mathcal{A}_z + z\operatorname{Tr}\mathcal{A}_z\,\mathcal{A}_1\,. \tag{1.84}$$

An ordinary differential equation on the Riemann sphere of second order is said to have the Painlevé property if the location of the essential singularities of its solutions do not depend on the initial conditions. Such differential equations have been classified into six Painlevé equations. Among those is the sixth Painlevé equation. It is equivalent to the Schlesinger equations in this special case. It is solved by both sides of equation (1.84). The tau function $\tau(z)$ can be expressed [23, 25] as the correlation function of four fields in Virasoro conformal field theory of central charge $c = 1$. More precisely, there are local coordinates (p, t) on the Riemann sphere such that the n-th Fourier coefficient of $\tau(z)$, as a function of the coordinate t, is given[2] by the four point conformal block (1.29) with conformal dimensions

$$h_1 = \frac{1}{2}\operatorname{Tr}\mathcal{A}_0^2 \qquad h_2 = p + n \qquad h_3 = \frac{1}{2}\operatorname{Tr}\mathcal{A}_\infty^2 \tag{1.85}$$

$$h'_1 = \frac{1}{2}\operatorname{Tr}\mathcal{A}_t^2 \qquad h'_2 = \frac{1}{2}\operatorname{Tr}\mathcal{A}_1^2\,. \tag{1.86}$$

This application of conformal blocks to isomonodromy problems does have an extension to Gaiotto states. One can also associate a tau function to the third Painlevé equation. Similar to the case of Painlevé VI, described above, the tau function of Painlevé III has a Fourier expansion, whose coefficients are given [26] by the norm (1.64) of Gaiotto states for the Virasoro algebra with central charge

$$c = 1, \tag{1.87}$$

whose conformal dimension varies over the coefficients. The degree zero components of the Gaiotto states have to be fixed appropriately. In [26], the convergence of the formal power series for the norm of the Gaiotto state is also proved in the special case $c = 1$ using the representation (1.64) via Nekrasov coefficients.

There also exists a deformation of the Painlevé equations turning them into q-difference equations. For the q-difference equation of Painlevé III, one can construct a tau function [12] and express it as an infinite sum of norms of appropriately scaled deformed Gaiotto states (1.72) corresponding to varying conformal dimensions. The parameters q and t of the deformed Virasoro algebra are assumed to be equal. This corresponds to a central charge $c = 1$ in the limit, where the deformed Virasoro algebra reproduces the ordinary Virasoro algebra. In [12] convergence of the expansion (1.73) is proved for $\epsilon_1 + \epsilon_2 = 0$, which corresponds to $q = t$. This implies finiteness of the norm of deformed Gaiotto states via the AGT relation in that special case.

[2]Equation (6.27) in [25] has a different convention for the index placement on conformal blocks.

Very recently, this correspondence between deformed Gaiotto states and q-difference Painlevé equations has formally been extended [11, Conjecture 4.1] to a quantum version with arbitrary values of the parameters q and t. So it would be desirable to prove an extension of the analyticity of the norm of deformed Gaiotto states for $q = t$ to more general parameters. We will perform such an extension in this thesis.

1.7. Outline of the Thesis

In chapter 2, we will define the norm of deformed Gaiotto states as formal power series. In chapter 3, we then define the five dimensional Nekrasov partition function, defined for the five dimensional gauge theory compactified on the circle $\mathbf{S}^1_\lambda$. In the subsequent chapter 4 we sketch a proof of the relevant AGT relation, which relates the norm of deformed Gaiotto states to a particular instance of the five-dimensional Nekrasov partition function. In chapter 5, we define an integral representation for the coefficients of the Nekrasov partition function, prove the validity of an explicit description of the integration contours, and use it to estimate their growth. Our main tool is a theorem in potential theory, which we formulate and prove in chapter 6. In chapter 7, we apply our estimate from chapter 5 to bound the radius of convergence of the Nekrasov partition function from below. We will also prove analyticity in other parameters on a suitable domain. Using the AGT relation from chapter 4, we obtain convergence of the norm of deformed Gaiotto states as a special case of our analysis of the Nekrasov partition function. We close chapter 7 with a discussion of our results. We discuss shortcomings and possible extensions. In the following chapters we will work with full mathematical rigor. Our original contribution is contained in chapters 5, 6, and 7. Most of our findings here have been published electronically in

G. Felder and M. Müller-Lennert. Analyticity of Nekrasov Partition Functions. *ArXiv e-prints math-ph/1709.05232*, September 2017.

2. Deformed Gaiotto States

In this chapter, we define the notion of deformed Gaiotto states. Those states are elements of the completion of a certain representation of the deformed Virasoro algebra. In order to introduce them in section 2.3, we will first discuss the representation theory of the deformed Virasoro algebra in section 2.2. It is formulated using terms from the representation theory of Lie algebras. However, the deformed Virasoro algebra is not a Lie algebra. Hence standard techniques from the representation theory of Lie algebras have to be adapted to discuss its representations. We introduce those techniques in section 2.1 when we discuss the representation theory of the Virasoro algebra. The material in this chapter is standard material.

2.1. Representation Theory of the Virasoro Algebra

In this section, we will first introduce the Virasoro algebra in section 2.1.1. Then we will define positive energy representations in section 2.1.2, followed by Verma modules in section 2.1.3. We will close the discussion of the representation theory of the Virasoro algebra by defining the Shapovalov form and explaining its relation to irreducibility of Verma modules in section 2.1.4.

2.1.1. The Virasoro Algebra

The Virasoro algebra is a complex Lie algebra **Vir** with basis consisting of $L_n, n \in \mathbf{Z}$ and C satisfying the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m(m-1)(m+1)\delta_{m+n,0}C \qquad (m, n \in \mathbf{Z}), \tag{2.1}$$

$$[C, L_m] = 0 \qquad (m \in \mathbf{Z}). \tag{2.2}$$

It is the unique central extension of the Witt algebra. The Witt algebra is a Lie algebra with basis consisting of $l_n, n \in \mathbf{Z}$ and commutation relations

$$[l_m, l_n] = (m-n)l_{m+n} \qquad (m, n \in \mathbf{Z}). \tag{2.3}$$

The normalization constant $\frac{1}{12}$ in equation (2.1) is due to historic reasons. The cocycle defining the central extension was discovered by Gelfand and Fuks [24].

2.1.2. Positive Energy Representations

In this section, we define positive energy representations of the Virasoro algebra. Fix a complex number $c \in \mathbf{C}$. We will consider so-called positive energy representations with central charge $c \in \mathbf{C}$. We define $\mathcal{O}_c$ to be the full subcategory of the category of complex representations of **Vir**, consisting of those representations for which

- L_0 is diagonizable with finite-dimensional eigenspaces,
- the real part of the spectrum of L_0 is bounded from below, and
- the element $C \in \mathbf{Vir}$ acts by multiplication with $c \in \mathbf{C}$.

This definition is motivated by the observation that the Hamiltonian of a conformal field theory contains L_0.

Let $V \in \mathcal{O}_c$ be such a positive energy representation. Since

$$[L_0, L_n] = -nL_n \qquad (n \in \mathbf{Z}), \tag{2.4}$$

the elements $L_n, n \in \mathbf{Z}$, map eigenvectors of L_0 of eigenvalue λ in V to either zero or eigenvectors of L_0 of eigenvalue $\lambda - n$. Since the real part of the spectrum of L_0 is bounded below, we are led to consider nonzero vectors $v \in V$, such that

$$L_0 v = hv \qquad \text{(for some } h \in \mathbf{C}), \tag{2.5}$$
$$L_n v = 0 \qquad (n \geq 1). \tag{2.6}$$

Such vectors are called *singular vectors* of weight $h \in \mathbf{C}$. Since the spectrum of L_0 is bounded from below, each $V \in \mathcal{O}_c$ contains at least one singular vector. Moreover, any irreducible $V \in \mathcal{O}_c$ is generated by any singular vector. To see this, consider the sub-representation generated by a singular vector $v \in V$.

A positive energy representation $V \in \mathcal{O}_c$ is called a *highest weight representation* of highest weight $h \in \mathbf{C}$ and central charge $c \in \mathbf{C}$, if it is generated by a singular vector v of weight h. Each of those representations is spanned by the elements

$$L_{-\lambda} v := L_{-\lambda(1)} \cdots L_{-\lambda(l)} v \tag{2.7}$$

for which $l \geq 0$ is an integer and $\lambda(1) \geq \cdots \geq \lambda(l) \geq 1$ defines a partition λ of length l. If those vectors are linearly independent, the highest weight representation V is called a *Verma module*. In the next paragraph, we are going to construct all irreducible Verma modules corresponding to the highest weight h and central charge c.

2.1.3. Verma Modules

Consider the subspace $\mathbf{Vir}_{\geq} \subset \mathbf{Vir}$ spanned by all elements L_n with $n \geq 0$ and C. From the commutation relations (2.1), it follows that $\mathbf{Vir}_{\geq}$ is a Lie sub-algebra. For a pair $(h, c) \in \mathbf{C}^2$ of complex numbers, let $\mathbf{C}_{h,c}$ denote the one-dimensional complex $\mathbf{Vir}_{\geq 0}$ module where L_0 acts by multiplication with h, C acts by multiplication with c, and the other L_n, $n \geq 1$, act as zero. The module $V_{h,c}$ of $\mathbf{Vir}$, associated to the pair $(h, c) \in \mathbf{C}^2$, is defined as the corresponding induced representation:

$$V_{h,c} = \mathcal{U}(\mathbf{Vir}) \otimes_{\mathcal{U}(\mathbf{Vir}_{\geq})} \mathbf{C}_{h,c}. \tag{2.8}$$

Here $\mathcal{U}(\mathbf{X})$ denotes the universal enveloping algebra of a Lie algebra $\mathbf{X}$.

From the Poincaré-Birkhoff-Witt theorem, we know that the elements

$$L_{n_1} \cdots L_{n_l} C^s \in \mathcal{U}(\mathbf{Vir}) \qquad (n_1 \leq \cdots \leq n_l, l \geq 0, s \geq 0) \tag{2.9}$$

form a basis of $\mathcal{U}(\mathbf{Vir})$. It follows that, for $v = 1 \otimes 1 \in \mathcal{U}(\mathbf{Vir}) \otimes_{\mathcal{U}(\mathbf{Vir}_{\geq})} \mathbf{C}_{h,c}$, the elements

$$L_{-\lambda}v = L_{-\lambda(1)} \cdots L_{-\lambda(l)}v \in V_{h,c} \qquad (\lambda \text{ a partition}) \tag{2.10}$$

form a basis of $V_{h,c}$. From $L_0 v = hv$ and equation (2.4), it follows that $L_{-\lambda}v$ is an eigenvector of L_0 with eigenvalue $h+n$. We obtain the decomposition

$$V_{h,c} = \bigoplus_{n \geq 0} V_{h,c}^{(n)} \tag{2.11}$$

into finite-dimensional eigenspaces

$$V_{h,c}^{(n)} = \operatorname{span}\{L_{-\lambda}v : |\lambda| = n\} \tag{2.12}$$

of L_0, corresponding to eigenvalue $h+n$. Evidently, $V_{h,c}$ belongs to the category $\mathcal{O}_c$ of finite energy representations with central charge c. Moreover, $v \in V_{h,c}$ is a singular vector of weight h, rendering $V_{h,c}$ a Verma module. Note that the quotient

$$\mathcal{U}(\mathbf{Vir})/W_{h,c} \tag{2.13}$$

of $\mathcal{U}(\mathbf{Vir})$ by the left ideal $W_{h,c}$ generated by the elements $L_0 - h$, $C - c$, $L_n, n \geq 1$ of $\mathcal{U}(\mathbf{Vir})$ is isomorphic to $V_{h,c}$.

The representations $V_{h,c}$ have the following property: If $v \in V$ is a singular vector of weight h belonging to a representation $V \in \mathcal{O}_c$, then the map $\mathcal{U}(\mathbf{Vir}) \to V$ sending $x \mapsto xv$, defines a homomorphism

$$V_{h,c} \to V \tag{2.14}$$

sending the generator of $V_{h,c}$ to v. This homomorphism is the unique homomorphism with that property. If now V is irreducible, the map ϕ in (2.14) defines an isomorphism $V_{h,c}/\ker\phi \to V$ by Schur's lemma. Since $\ker\phi$ is a sub-representation of $V_{h,c}$ and V is irreducible, $\ker\phi$ has to be a maximal proper sub-representation. By using the grading (2.11), one can show that $V_{h,c}$ has at most one maximal proper sub-representation $N_{h,c}$.

We have seen that for each $(h,c) \in \mathbf{C}^2$, there exists, up to isomorphism, a unique irreducible highest weight representation $L_{h,c}$ of highest weight h and central charge c in $\mathcal{O}_c$ given by $V_{h,c}/N_{h,c}$. Moreover, each irreducible representation in $\mathcal{O}_c$ is of this type.

In case the representation $V_{h,c}$ contains, besides $v_{h,c}$, another singular vector, e.g. in $V_{h,c}^{(n)}$, we obtain a homomorphism $V_{h+n,c} \to V_{h,c}$. The structure of these mappings between different $V_{h,c}$ was explained by Feigin and Fuks, see [19] for a reference.

2.1.4. The Shapovalov Form

Consider the Verma module $V_{h,c} \in \mathcal{O}_c$. We want to consider its degree-wise dual

$$V_{h,c}^{*} := \bigoplus_{n \geq 0} \operatorname{Hom}(V_{h,c}^{(n)}, \mathbf{C}) \tag{2.15}$$

as an element of $\mathcal{O}_c$. To do so, we consider the map

$$A : \mathbf{Vir} \to \mathbf{Vir} \tag{2.16}$$

mapping $AC \mapsto C$ and $AL_n \mapsto L_{-n}$, $n \in \mathbf{Z}$. It is a Lie algebra anti-homomorphism. For $\lambda \in V_{h,c}^*$ and $L \in \mathbf{Vir}$, the formula $(L\lambda)(x) = \lambda(A(L)(x))$, $x \in V_{h,c}$, defines $V_{h,c}^*$ as a module of the Virasoro algebra. Since L_0 acts as $h+n$ on $\mathrm{Hom}(V_{h,c}^{(n)}, \mathbf{C})$, we obtain the decomposition

$$V_{h,c}^* = \bigoplus_{n\geq 0} (V_{h,c}^*)^{(n)} \qquad (V_{h,c}^*)^{(n)} = \mathrm{Hom}(V_{h,c}^{(n)}, \mathbf{C}) \tag{2.17}$$

into L_0-eigenspaces. Hence $V_{h,c}^*$ is an object in $\mathcal{O}_c$. Pick the element $v_{h,c}^*$ in $(V_{h,c}^*)^{(0)} = \mathrm{Hom}(\operatorname{span} v_{h,c}, \mathbf{C})$ sending $v_{h,c}$ to one. For degree reasons, $v_{h,c}^*$ is a singular vector of $V_{h,c}^*$ of weight h. By the universal property of $V_{h,c}$, there exists a unique homomorphism

$$V_{h,c} \to V_{h,c}^* \tag{2.18}$$

of Virasoro modules sending $v_{h,c}$ to $v_{h,c}^*$. This map defines a symmetric bilinear form on $V_{h,c}$, the *Shapovalov form*

$$S : V_{h,c} \otimes V_{h,c} \to \mathbf{C}. \tag{2.19}$$

It is contra-variant with respect to A, meaning $S(Lx, y) = S(x, A(L)y)$ for all $x, y \in V_{h,c}$ and all $L \in \mathbf{Vir}$.

Next, we will explain the relation between irreducibility of $V_{h,c}$ and the Shapovalov form. The kernel of S defines a sub-representation in $V_{h,c}$, which is maximal. Hence $\ker S = N_{h,c}$. Since $V_{h,c}$ is irreducible if and only if $N_{h,c} = \{0\}$, $V_{h,c}$ is irreducible if and only if the Shapovalov form is non-degenerate. For degree reasons, the decomposition (2.11) of $V_{h,c}$ is orthogonal with respect to S. Hence S decomposes as the sum of the restricted forms

$$S_n : V_{h,c}^{(n)} \otimes V_{h,c}^{(n)} \to \mathbf{C} \qquad (n \geq 0). \tag{2.20}$$

In particular, $V_{h,c}$ is irreducible if and only if all S_n are non-degenerate. Since each $V_{h,c}^{(n)}$ is finite-dimensional, one can express this property as the regularity of matrices. For each $n \geq 0$, the Gram matrix $K^{(n)}$ is defined by its entries

$$K_{\lambda\mu}^{(n)} = S_n(L_{-\lambda} v_{h,c}, L_{-\mu} v_{h,c}) \qquad (|\lambda| = |\mu| = n). \tag{2.21}$$

It is symmetric. Moreover, it is regular if and only if S_n is non-degenerate. Its determinant is given by the Kac determinant formula [28]

$$\det \left(K^{(n)}\right)^2 = \mathrm{const} \prod_{k=1}^{n} \prod_{j|k} \Phi_{j,k/j}(h,c)^{p(n-k)}, \tag{2.22}$$

where $p(j) = \dim V_{h,c}^{(j)}$ is the number of partitions of size j and

$$\Phi_{j_1,j_2}(h,c) = \prod_{i=1,2} \left(h + \frac{1}{24}(j_i^2 - 1)(c-13) + \frac{1}{2}(j_1 j_2 - 1) \right) + \frac{1}{16}(j_1^2 - j_2^2)^2. \tag{2.23}$$

A proof of this formula can be found in [18].

2.2. Representation Theory of the Deformed Virasoro Algebra

Now that we have an overview of the representation theory of the Virasoro algebra, we can develop the representation theory of the deformed Virasoro algebra. We proceed analogously to section 2.1. We follow the exposition in [51] and the original reference [45].

2.2.1. The Deformed Virasoro Algebra

Let $\mathbf{F} = \mathbf{Q}(q,t)$ denote the field of rational functions in the variables q and t. Define the elements $f_l \in \mathbf{F}$ by the expansion coefficients in the relation

$$\sum_{l\geq 0} f_l z^l = \exp\left(\sum_{n\geq 0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} z^n\right) \tag{2.24}$$

between formal power series in z with coefficients in $\mathbf{F}$. Let $F_0 = \mathbf{F}\left\langle T_n, n \in \mathbf{Z}\right\rangle$ denote the free associative algebra over $\mathbf{F}$ in the symbols $T_n, n \in \mathbf{Z}$. We associate the degree 0 to the element $1 \in F_0$ and the degree $-n$ to the element $T_n \in F_0$, where $n \in \mathbf{Z}$. We obtain a decomposition

$$F_0 = \bigoplus_{n\in\mathbf{Z}} F_0^{(n)}, \qquad F_0^{(n)} = \{x \in F_0 : \deg x = n\} \tag{2.25}$$

turning F_0 into a graded algebra. Let $F^{(n)}$ denote the completion of $F_0^{(n)}$ along the filtration of F_0 defined by the homogeneous, two-sided ideals $\mathcal{F}_j$ generated by all elements $x \in F_0$ of degree $\deg x \geq j$. Set

$$F = \bigoplus_{n\in\mathbf{Z}} F^{(n)} \tag{2.26}$$

It is a graded algebra over $\mathbf{F}$, whose homogeneous elements are infinite series $\sum_k p_k$ in expressions $p_k \in F_0^{(n)}$, which contain elements T_n of higher and higher degree n.

Let $I_{q,t} \subset F$ be the two-sided ideal generated by the relations

$$[T_m, T_n] = -\sum_{l\geq 1} f_l (T_{m-l}T_{n+l} - T_{n-l}T_{m+l}) \tag{2.27}$$

$$- \frac{(1-q)(1-t^{-1})}{1-q/t}\Big((q/t)^m - (q/t)^{-m}\Big)\delta_{m+n,0} \tag{2.28}$$

where $m, n \in \mathbf{Z}$. Since those relations are homogeneous, $I_{q,t}$ is a homogeneous ideal. Define[1] the deformed Virasoro algebra as

$$\mathbf{Vir}_{q,t} = F/I_{q,t}. \tag{2.29}$$

[1] In the existing literature, the deformed Virasoro algebra is introduced without constructing a completion. We have introduced the completion to make the defining relations rigorous.

Since the ideal I is homogeneous, the degree map is still well defined, and we obtain the degree decomposition

$$\mathbf{Vir}_{q,t} = \bigoplus_{n\in\mathbf{Z}} \mathbf{Vir}_{q,t}^{(n)} \tag{2.30}$$

Next, we clarify the relation to the (undeformed) Virasoro algebra. Note that in contrast to $\mathcal{U}(\mathbf{Vir})$, the deformed Virasoro algebra $\mathbf{Vir}_{q,t}$ is not the universal enveloping algebra of a Lie algebra. When we write

$$q = e^{\hbar\kappa_1}, \quad t = e^{\hbar\kappa_2}, \tag{2.31}$$

the element $T_n \in \mathbf{Vir}_{q,t}$ has [51, Lemma 1.2] the formal expansion

$$T_n = \kappa_1\kappa_2 L_n\hbar^2 + O(\hbar^4) \qquad (n \neq 0) \tag{2.32}$$

$$T_0 = 2 + \kappa_1\kappa_2\left(L_0 + \frac{1}{4}(\kappa_1 - \kappa_2)^2\right)\hbar^2 + O(\hbar^4) \tag{2.33}$$

in $\hbar$, where the elements L_n share the same commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m-1)(m+1)\delta_{m+n,0} \qquad (m, n \in \mathbf{Z}) \tag{2.34}$$

as the images of the generators of the Virasoro algebra in $\mathcal{U}(\mathbf{Vir})/(C-c)$, where

$$c = 13 - 6(\beta + \beta^{-1}), \quad \beta = \frac{\kappa_2}{\kappa_1}. \tag{2.35}$$

2.2.2. Verma Modules for the Deformed Virasoro Algebra

Note that in contrast to the relation (2.4) in the undeformed case, we now only have

$$[T_0, T_n] = -\sum_{l\geq 1} f_l(T_{-l}T_{n+l} - T_{n-l}T_l). \tag{2.36}$$

In particular, we cannot directly relate the T_0 eigenvalue and the grading of $\mathbf{Vir}_{q,t}$. Hence, we do not define positive energy representations as in the undeformed case. To develop the representation theory of $\mathbf{Vir}_{q,t}$, we follow [51].

We consider modules of the $\mathbf{Vir}_{q,t}$ over the field $\mathbf{Q}$. We say that a $\mathbf{Vir}_{q,t}$ module M is graded, if

$$M = \bigoplus_{z\in\mathbf{Z}} M^{(n)} \tag{2.37}$$

as $\mathbf{Q}$ vector spaces and

$$\mathbf{Vir}_{q,t}^{(n)} M^{(m)} \subset M^{(n+m)} \qquad (m, n \in \mathbf{Z}) \tag{2.38}$$

Let $\mathcal{C}$ denote the category whose objects are graded $\mathbf{Vir}_{q,t}$ modules and the morphisms are homomorphisms of $\mathbf{Vir}_{q,t}$ modules that respect the grading (2.37).

In analogy to the undeformed case, we call a representation $M \in \mathcal{C}$ a *highest weight module* with highest weight $h \in \mathbf{Q}$, if there exists a nonzero vector $v \in M$ such that $T_0 v = hv$, $T_n v = 0$ for all $n \geq 1$ and the set

$$\{T_{-\lambda} v = T_{-\lambda(1)} \cdots T_{-\lambda(l)} v : \lambda \text{ a partition}\} \tag{2.39}$$

spans M. If this set is linearly independent, we call M a Verma module.

Next, we want to define an analogue of the Verma modules $L_{h,c}$ for the deformed Virasoro algebra. Since the commutation relations (2.27) do not map the commutator of two elements T_n, T_m with $m, n \geq 0$ to an element of $\mathbf{Vir}_{q,t}$ defined by only T_k with $k \geq 0$, we cannot directly adapt definition (2.8). However, we can adapt the description in equation (2.13): For $h \in \mathbf{Q}$, we define the Verma module $M(h)$ as the quotient

$$M_h = \mathbf{Vir}_{q,t} / K_h \tag{2.40}$$

of $\mathbf{Vir}_{q,t}$ by the left ideal generated by all elements $T_0 - h \in \mathbf{Vir}_{q,t}$ and $T_n \in \mathbf{Vir}_{q,t}, n \geq 1$.

Since the associative algebra $\mathbf{Vir}_{q,t}$ is not the universal enveloping algebra of a Lie algebra, the Poincaré-Birkhoff-Witt theorem does not apply. However, the action of $[T_n, T_m]$ in M_h is expressed as a finite sum by (2.27), since the degree of M_h is bounded below. One can show [51, Lemma 1.1] that the elements

$$T_{n_1} \cdots T_{n_l} \qquad\qquad n_1 \leq \cdots \leq n_l, l \geq 0, \sum_{i=1}^{l} n_l = n \tag{2.41}$$

are linearly independent and each element of $\mathbf{Vir}_{q,t}^{(n)}$ can be written as a series $\sum_k \alpha_k p_k$ with $\alpha_k \in \mathbf{F}$ and p_k as in 2.41. Hence we obtain the decomposition

$$M_h = \bigoplus_{n \geq 0} M_h^{(n)} \qquad\qquad M_h^{(n)} = \operatorname{span}\{T_{-\lambda} 1_h : |\lambda| = n\}, \tag{2.42}$$

where 1_h is the image of $1 \in \mathbf{Vir}_{q,t}$ in M_h. This grading establishes M_h as a highest weight module of $\mathbf{Vir}_{q,t}$ of highest weight h.

As in the undeformed case, the Verma module M_h has [51, Lemma 2.7] the following universal property: For any highest weight module M with highest weight h, there exists a homomorphism $M_h \to M$ sending 1_h to the generating vector of M. Moreover, the Verma module M_h has a unique maximal proper graded submodule N_h, which is also maximal among the proper submodules. The quotient M_h/N_h is an irreducible highest weight module of highest weight h.

For a graded $\mathbf{Vir}_{q,t}$ module $M = \bigoplus_n M^{(n)}$, we call a nonzero vector $v \in M$ *singular vector* of weight d, if $v \in M^{(d)}$ and $T_n v = 0$ for all $n \geq 1$. The relation to highest weight modules is more subtle as in the undeformed case, since T_0 is not related to the grading. We say that the singular vector v has *energy* h if $T_0 v = hv$.

Let M be a graded $\mathbf{Vir}_{q,t}$ module with singular vector v of weight d and energy h. Consider the homomorphism

$$\mathbf{Vir}_{q,t} \to M \tag{2.43}$$

sending $1 \in \mathbf{Vir}_{q,t}$ to v. Since v is a singular vector and $T_0 v = hv$, its kernel contains the left ideal K_h. Hence we obtain a homomorphism

$$M_h \to M \tag{2.44}$$

sending 1_h to v.

2.2.3. The Shapovalov Form in the Deformed Case

In this section, we define the Shapovalov form for the deformed Virasoro algebra. Fix $h \in \mathbf{Q}$ and consider the Verma module M_h. First, we want to consider the degree wise dual

$$M_h^* = \bigoplus_{n \geq 0} \mathrm{Hom}(M_h^{(n)}, \mathbf{F}) \tag{2.45}$$

as a $\mathbf{Vir}_{q,t}$ module. Following the undeformed case, consider the anti-homomorphism

$$B : \mathbf{Vir}_{q,t} \to \mathbf{Vir}_{q,t} \tag{2.46}$$

induced by the map

$$F \to F \tag{2.47}$$

$$T_{m_1} \cdots T_{m_l} \mapsto T_{-m_l} \cdots T_{-m_1} \qquad (m_1, \ldots, m_l \in \mathbf{Z}, l \in \mathbf{N}). \tag{2.48}$$

The formula $T\lambda(x) = \lambda(B(T)x)$, $x \in M_h$, $\lambda \in M_h^*$, $T \in \mathbf{Vir}_{q,t}$ defines an action of $\mathbf{Vir}_{q,t}$ on M_h^*. By construction, it respects the grading and establishes $M_h^* \in \mathcal{C}$ with grading given by the decomposition (2.45). Since $1_h \in M_h$ spans $M_h^{(0)}$, we can pick

$$1_h^* \in \mathrm{Hom}(M_h^{(0)}, \mathbf{F}) \tag{2.49}$$

mapping 1_h to 1. For degree reasons, it is a singular vector of degree 0. Since $BT_0 = T_0$ it has energy h. We obtain a homomorphism

$$M_h \to M_h^* \tag{2.50}$$

mapping 1_h to 1_h^*. We obtain a bilinear form

$$\langle -, - \rangle : M_h \otimes M_h \to \mathbf{F}. \tag{2.51}$$

In analogy to the undeformed case, it is called the Shapovalov form. It is symmetric and contra-variant with respect to B. For degree reasons, the decomposition (2.42) is orthogonal and the Shapovalov form decomposes as the sum of its restrictions

$$\langle -, - \rangle_n : M_h^{(n)} \otimes M_h^{(n)} \to \mathbf{F} \qquad (n \geq 0). \tag{2.52}$$

Since its kernel is a maximal submodule of M_h [51, Lemma 2.12], the non-degeneracy of all those restrictions $\langle -, - \rangle_n$ is a characterization of irreducibility of M_h. The Gram matrix $W^{(n)}$ is defined by its entries

$$W_{\lambda\mu}^{(n)} = \langle T_{-\lambda} 1_h, T_{-\mu} 1_h \rangle \qquad (|\lambda| = |\mu| = n) \tag{2.53}$$

as in the undeformed case. The analog of the Kac determinant formula is given by

$$\det W^{(n)} = \text{const} \prod_{1 \le rs \le n} (h^2 - h_{r,s}^2)^{p(n-rs)} \left(\frac{(1-q^r)(1-t^r)}{q^r + t^r} \right)^{p(n-rs)}. \tag{2.54}$$

Here $p(j) = \dim M_h^{(j)}$ again denotes the number of partitions of the integer j and

$$h_{r,s} = t^{r/2} q^{-s/2} + t^{-r/2} q^{s/2}. \tag{2.55}$$

Formula (2.54) was conjectured in [45] and proved in [14].

2.3. Deformed Gaiotto States

We now define the notion of Gaiotto states. First we describe the construction in the undeformed case in section 2.3.1. Then we define the Gaiotto states for the deformed Virasoro algebra in section 2.3.2.

2.3.1. Gaiotto States for the Virasoro Algebra

In this section, we will describe the notion of Gaiotto states in the Verma module $V_{h,c}$ of the undeformed Virasoro algebra **Vir**.

For a finite-dimensional Lie algebra $\mathfrak{g}$, let $\chi : \mathfrak{n} \to \mathbf{C}$ be a character of some maximal nilpotent Lie sub-algebra $\mathfrak{n} \subset \mathfrak{g}$. An element $v \in V$ of a representation V of $\mathfrak{g}$ is called a *Whittaker vector* for χ if $xv = \chi(x)v$ for all $x \in \mathfrak{n}$.

Gaiotto states [22] are an analogue of this concept. The Virasoro algebra is not finite-dimensional. However, the linear hull $\mathbf{Vir}_> \subset \mathbf{Vir}$ of all L_n, with $n \geq 1$ forms a Lie sub-algebra. Since the elements L_1 and L_2 generate $\mathbf{Vir}_>$, any character $\chi : \mathbf{Vir}_> \to \mathbf{C}$ is determined by the images $\chi(L_1)$ and $\chi(L_2)$ of L_1 and L_1, and satisfies $\chi(L_n) = 0$ for $n \geq 3$. A Whittaker vector v for the Verma module $V_{h,c} = \bigoplus_{n \geq 0} V_{h,c}^{(n)}$ is then defined to be an element of the completion

$$\hat{V}_{h,c} = \prod_{n \geq 0} V_{h,c}^{(n)} \tag{2.56}$$

satisfying

$$L_1 v = \chi(L_1) v \qquad L_2 v = \chi(L_2) v \qquad L_n v = 0 \quad (n \geq 3). \tag{2.57}$$

We call v a Gaiotto state from now on.

2.3.2. Gaiotto States for the Deformed Virasoro Algebra

In the section, we define Gaiotto states for the deformed Virasoro algebra $\mathbf{Vir}_{q,t}$, which are called deformed Gaiotto states. Let ξ be an indeterminate. In analogy to equations (2.57), we consider elements $w(\xi)$ of the completion

$$\hat{M}_h := \prod_{n \geq 0} \left(M_h^{(n)} \otimes_{\mathbf{Q}} \mathbf{Q}(\xi) \right) \tag{2.58}$$

of the Verma module M_h for which

$$T_1 w(\xi) = \xi w(\xi) \qquad\qquad T_n w(\xi) = 0 \quad (n \geq 2). \tag{2.59}$$

When we expand

$$w(\xi) = \sum_{n\geq 0} \xi^n w_n(\xi) \in \prod_{n\geq 0} \left(M_h^{(h)} \otimes_{\mathbf{Q}} \mathbf{Q}(\xi) \right), \tag{2.60}$$

the conditions (2.59) are equivalent to

$$T_1 w_{n+1}(\xi) = w_n(\xi) \quad (n \geq 0), \qquad T_k w_n(\xi) = 0 \quad (k \geq 2, n \geq 0). \tag{2.61}$$

We assume the following normalization: $w_n(\xi) = w_n \in M_h^{(n)}$ and $w_0 = 1_h \in M_h^{(0)}$. Following [7], we denote an element

$$w(\xi) = \sum_{n\geq 0} \xi^n w_n \in \prod_{n\geq 0} \left(M_h^{(n)} \otimes_{\mathbf{Q}} \mathbf{Q}(\xi) \right) \tag{2.62}$$

a *deformed Gaiotto state* if the expansion coefficients $w_n \in M_h^{(n)}$ satisfy

$$T_1 w_{n+1} = w_n \quad (n \geq 0), \qquad T_k w_n = 0 \quad (n \geq 0, k \geq 2), \qquad w_0 = 1_h. \tag{2.63}$$

Let (1^n) denote the partition $(1, \ldots, 1)$ of length n. Expanding the homogeneous components of a deformed Gaiotto state as

$$w_n = \sum_{|\mu|=n} c_\mu^{(n)} T_{-\mu} 1_h, \tag{2.64}$$

the relations (2.63) imply

$$\sum_{|\mu|=n} W_{\lambda\mu}^{(n)} c_\mu^{(n)} = \delta_{\lambda,(1^n)} \qquad\qquad (|\lambda| = n). \tag{2.65}$$

Hence for generic parameters q and t the deformed Gaiotto state exists uniquely, since the Kac determinant is invertible in $\mathbf{F} = \mathbf{Q}(q, t)$. Its norm is given by

$$\langle\, w(\xi) | w(\xi) \,\rangle = \sum_{n\geq 0} \xi^{2n} \big(W_{(n)}^{-1} \big)_{(1^n)(1^n)} \tag{2.66}$$

as a formal power series in $\mathbf{F}[[\xi^2]]$, where $W_{(n)}^{-1} \in \mathbf{F}^{n\times n}$ is the inverse of the Gram matrix $W^{(n)} \in \mathbf{F}^{n\times n}$. It is customary to call $\langle\, w(\xi), w(\xi) \,\rangle$ a norm, although the Shapovalov form is just a bilinear form over the field of rational functions with rational coefficients.

3. Nekrasov Partition Functions

In this part, we will define the Nekrasov partition function as the generating function of the weighted Euler characteristics of certain K-theory classes. Those classes live in the equivariant K-theory of a particular example of so-called Nakajima quiver varieties. We will introduce those varieties in section 3.1 and define weighted Euler characteristics. Afterwards, in section 3.2, we will then define the Nekrasov partition function. This chapter is largely based on the treatment in [35] and [36].

3.1. A Nakajima Quiver Variety and its Equivariant K-Theory

In section 3.1.1, we define a particular type of Nakajima quiver varieties. Then, in section 3.1.2 we will discuss their K-theory. Afterwards, we will introduce the notion of a weighted Euler characteristic in section 3.1.3.

3.1.1. A Nakajima Quiver Variety

A quiver consists of a set of vertices and a set of directed edges. One can associate vector spaces to the vertices of a quiver. Fix two integers r and n with $r \geq 1$ and $n \geq 0$. An example would be the quiver consisting of two vertices, to which we associate the vector spaces $V := \mathbf{C}^n$ and $W := \mathbf{C}^r$, respectively, and one edge from W to V. Schematically, the quiver is depicted in figure 3.1, where the edge is depicted as a dashed arrow. Consider the space

$$\mathrm{End}(V) \oplus \mathrm{Hom}(W, V). \tag{3.1}$$

With respect to the standard bases on V and W, it parametrizes pairs of linear maps $V \to V$ and $W \to V$. The cotangent bundle to the linear space (3.1) is given by the space

$$N(r, n) = \mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W). \tag{3.2}$$

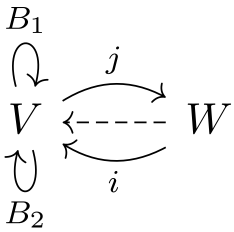


Figure 3.1.: The linear maps parametrized by the space $N(r, n)$.

The quadruples of linear maps it parametrizes are also depicted in figure 3.1 as solid arrows.

The group $GL_n(\mathbf{C})$ acts on the space $N(r,n)$ by changing the basis in V. For an invertible matrix $g \in GL_n(\mathbf{C})$ and an element $(B_1, B_2, i, j) \in N(r,n)$ we have

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}). \tag{3.3}$$

Being a cotangent bundle, the space $N(r,n)$ carries a canonical symplectic form. The action (3.3) turns out to be Hamiltonian, with moment map given by

$$\mu(B_1, B_2, i, j) = [B_1, B_2] + ij \in \mathrm{End}(V) \cong \mathrm{Lie}(GL_n(\mathbf{C})). \tag{3.4}$$

We want to perform a Hamiltonian reduction, i.e. we want to quotient the zero set $\mu^{-1}(0)$ by the $GL_n(\mathbf{C})$ action in an appropriate way. The naive quotient is not a smooth algebraic variety. Various conditions are commonly introduced to deal with this. We say, that an element $(B_1, B_2, i, j) \in N(r,n)$ is *stable*, if

$$\nexists\, S \subset V \text{ proper subspace with } B_1(S) \subset S, B_2(S) \subset S, i(W) \subset S. \tag{3.5}$$

Both the condition $\mu(B_1, B_2, i, j) = 0$ and the stability condition are compatible with the $GL_n(\mathbf{C})$ action. We define

$$M(r,n) = \{X \in N(r,n) : \mu(X) = 0 \text{ and } X \text{ is stable}\}/GL_n(\mathbf{C}). \tag{3.6}$$

This space is an example of a GIT quotient. It is a smooth algebraic variety of dimension $2rn$. Smoothness follows from the fact that the differential of μ is surjective and the action of $GL_n(\mathbf{C})$ on set of stable points is free. The variety $M(r,n)$ is an example what is called a Nakajima quiver variety.

There are different approaches to performing the Hamiltonian reduction. One can for example also consider the quotient

$$M_0^{\mathrm{reg}}(r,n) \tag{3.7}$$
$$= \{X \in N(r,n) : \mu(X) = 0 \text{ and both } X \text{ and } X^T \text{ are stable}\}/GL_n(\mathbf{C}). \tag{3.8}$$

Here, $(B_1, B_2, i, j)^T = (B_2^T, B_1^T, j^T, i^T)$ is obtained by transposition. By the ADHM construction [4], the space $M_0^{\mathrm{reg}}(r,n)$ parametrizes anti-self-dual connections of instanton number n in $SU(r)$ gauge theory. One also considers its Uhlenbeck partial compactification $M_0(r,n)$. As a set, we have

$$M_0(r,n) = \bigsqcup_{n'=0}^{n} M_0^{\mathrm{reg}}(r,n') \times S^{n-n'}\mathbf{C}^2. \tag{3.9}$$

Here $S^k\mathbf{C}^2$ denotes the k-fold symmetric product of $\mathbf{C}^2$. The algebraic structure of $M_0(r,n)$ is given by its definition as an algebro-geometric quotient

$$M_0(r,n) = \mathrm{Spec}\left(\mathbf{C}[\mu^{-1}(0)]^{GL_n(\mathbf{C})}\right). \tag{3.10}$$

For us, the main object of interest is $M(r,n)$. The space $M_0(r,n)$ will serve as a tool to define the Nekrasov partition function.

3.1.2. Equivariant K-Theory

Let T denote the torus

$$T = \mathbf{C}^* \times \mathbf{C}^* \times \left(\mathbf{C}^*\right)^r. \tag{3.11}$$

It acts on the space $N(r,n)$ from the right[1] For an element $(t_1, t_2, e_1, \ldots, e_r) \in T$ and an element $(B_1, B_2, i, j) \in N(r,n)$, we set

$$(B_1, B_2, i, j) \cdot (t_1, t_2, e_1, \ldots, e_r) = (t_1 B_1, t_2 B_2, ie, t_1 t_2 e^{-1} j), \tag{3.12}$$

where e denotes the diagonal matrix with diagonal entries $e_1, \ldots, e_r$. This torus action $N(r,n)$ commutes with the action of $GL_n(\mathbf{C})$. Moreover, it is compatible with the moment map and the stability condition. Hence it defines a right T action on the spaces $M(r,n)$ and $M_0(r,n)$.

Let $\mathbf{Coh}_T(M(r,n))$ denote the category of T-equivariant, coherent sheaves on $M(r,n)$. Let

$$\mathbf{K}_T(M(r,n)) \tag{3.13}$$

denote the Grothendieck group of $\mathbf{Coh}_T(M(r,n))$. It is called the equivariant K-theory group of $M(r,n)$. Its elements are called equivariant K-theory classes. Introduce the same objects $\mathbf{Coh}_T(M_0(r,n))$ and $\mathbf{K}_T(M_0(r,n))$ for $M_0(r,n)$. In case of a single point, we obtain

$$\mathbf{K}_T(\mathrm{pt}) = R(T), \tag{3.14}$$

the representation ring of the torus T. Equivariant K-theory groups are modules over the ring $R(T)$. Since $M(r,n)$ is nonsingular, we can say more: It is also given by the Grothendieck group of equivariant, locally free sheaves on $M(r,n)$. Moreover, it is generated by the isomorphism classes of equivariant vector bundles over $M(r,n)$. It carries a ring structure. On the level of equivariant vector bundles, the multiplication is given by the tensor product of vector bundles. In particular, the $R(T)$ module structure is given by the tensor product with trivial vector bundles that carry the appropriate representation of T.

In equivariant K-theory, it is important to understand the fixed points of the torus action. In our case the fixed points

$$M(r,n)^T = \{I_{\vec{Y}} : |\vec{Y}| = n\} \tag{3.15}$$

of $M(r,n)$ are indexed [35] by r-tuples $\vec{Y}$ of partitions with total size n. Let

$$\iota : M(r,n)^T \to M(r,n), \qquad \iota_{\vec{Y}} : \{I_{\vec{Y}}\} \to M(r,n) \qquad (|\vec{Y}| = n) \tag{3.16}$$

denote the corresponding inclusion maps. The space $M_0(r,n)$ has only one fixed point

$$M_0(r,n)^T = \{p_0\}. \tag{3.17}$$

Again denote its inclusion map by

$$\iota_0 : \{p_0\} \to M_0(r,n). \tag{3.18}$$

[1] Here, we use a different convention compared to [35, 36].

3.1.3. Weighted Euler Characteristic

First, we are going to recall the notion of a character of a not necessarily finite dimensional representation. Let X be a complex representation of some torus G. Let

$$X = \bigoplus X_\chi \tag{3.19}$$

be the corresponding weight space decomposition. Here χ runs over the (multiplicative) character group

$$\mathrm{Hom}_{\mathrm{alg}}(T, \mathbf{C}^*) \tag{3.20}$$

of X. For a character $\chi \in \mathrm{Hom}_{\mathrm{alg}}(T, \mathbf{C}^*)$, the weight space is given by

$$X_\chi = \{x \in X : gx = \chi(g)x \;\forall\; g \in G\}. \tag{3.21}$$

In case all weight spaces are finite dimensional, one defines the character of X as the formal sum

$$\mathrm{ch}\, X = \sum_\chi \dim X_\chi\, \chi \tag{3.22}$$

in the completed group algebra of the character group of G. Note that the group algebra of the character group equals

$$\mathbf{Z}[\mathrm{Hom}_{\mathrm{alg}}(T, \mathbf{C}^*)] = R(T). \tag{3.23}$$

In case V is finite dimensional itself, we thus have $\mathrm{ch}\, X \in R(T)$.

Let

$$E \in \mathbf{Coh}_T(M(r,n)) \tag{3.24}$$

be an equivariant, coherent sheaf on $M(r,n)$. We want to define the weighted Euler characteristic of E as

$$Z_n(E) = \sum_{i=0}^{2rn} (-1)^i\, \mathrm{ch}\, H^i(M(r,n), E). \tag{3.25}$$

It will later be used in the definition of the Nekrasov partition function. Since the cohomology groups $H^i(M(r,n), E)$ need not be finite-dimensional as complex vector spaces, we have to show that the definition makes sense and determine in which space is is supposed to live. Let

$$\mathcal{R}(T) = \mathrm{Quot}(R(T)) \tag{3.26}$$

denote the field of fractions of the ring $R(T)$. We want to establish [35, Proposition 4.1]

$$Z_n(E) = \sum_{|\vec{Y}|=n} \frac{\iota_{\vec{Y}}^* E}{\Lambda_{-1} T_{\vec{Y}}^* M(r,n)} \in \mathcal{R}(T). \tag{3.27}$$

Some comments are in order: On the right hand side, E is considered as an element of $\mathbf{K}_T(M(r,n))$. In the numerator, the pullback $\iota^*_{\vec{Y}}(E)$ of $E \in \mathbf{K}_T(M(r,n))$ to $\mathbf{K}_T(I_{\vec{Y}}) = R(T)$ uses the description of $\mathbf{K}_T(M(r,n))$ via locally free sheaves. In the denominator, $\Lambda_{-1}T^*_{\vec{Y}}M(r,n)$ denotes the character, as a T-module, of the alternating exterior power of the cotangent space to $M(r,n)$ at the fixed point $I_{\vec{Y}}$. It is an element of $R(T)$.

The reader in a hurry can take equation (3.27) as a practical definition and proceed to the next section. In the rest of this section, we will relate the practical definition to the conceptual definition in equation (3.25). We sketch the argument given in [35]. It uses the space $M_0(r,n)$.

From the respective descriptions of $M(r,n)$ and $M_0(r,n)$ as quotients, we obtain a projective morphism

$$\pi : M(r,n) \to M_0(r,n). \tag{3.28}$$

It is equivariant with respect to the torus action. Any proper, equivariant morphism $f : M \to M'$ between algebraic varieties M and M' induces a homomorphism

$$f_* : \mathbf{K}_T(M) \to \mathbf{K}_T(M') \tag{3.29}$$

between the corresponding equivariant K-theory groups by the formula

$$f_*F = \sum_{i\geq 0}(-1)^i R^i f_* F \in \mathbf{K}_T(M') \qquad (F \in \mathbf{Coh}_T(M)). \tag{3.30}$$

Here, $R^i f_* F \in \mathbf{Coh}_T(M')$, $i \geq 0$, denote the higher direct image sheaves. We call f_* the pushforward in K-theory corresponding to f. In particular, we have a homomorphism

$$\pi_* : \mathbf{K}_T(M(r,n)) \to \mathbf{K}_T(M_0(r,n)). \tag{3.31}$$

Since, for the coherent sheaf $E \in \mathbf{Coh}(M(r,n))$ and $i \geq 0$, the sheaf $R^i\pi_*E$ on $M_0(r,n)$ is associated to the pre-sheaf $U \mapsto H^i(\pi^{-1}(U), E)$, we obtain for $U = M_0(r,n)$,

$$H^i(M(r,n), E) = H^0(M_0(r,n), R^i\pi_*E). \tag{3.32}$$

Now, by [35, Lemma 4.2], the right hand side of this equation is a representation of T with weight spaces of finite complex dimension. So

$$\operatorname{ch} R^i\pi_*E := \operatorname{ch} H^0(M_0(r,n), R^i\pi_*E) \tag{3.33}$$

is well-defined. We extend this definition from coherent sheafs to K-theory elements by setting

$$\operatorname{ch} \pi_*E := \sum_{i\geq 0}(-1)^i \operatorname{ch} R^i\pi_*E. \tag{3.34}$$

We obtain

$$Z_n(E) = \operatorname{ch} \pi_*E. \tag{3.35}$$

The inclusion map ι_0, defined in equation (3.18), yields the pushforward homomorphism

$$\iota_{0*} : \mathbf{K}_T(M_0(r,n)^T) \to \mathbf{K}_T(M_0(r,n)). \tag{3.36}$$

Since $M_0(r,n)$ only has one fixed point, the space on the left hand side is given by $R(T)$. If now $r \in R(T)$ is a finite dimensional representation of T, its pushforward $\iota_{0*}r$ is a skyscraper sheaf located at the fixed point $\iota_0(p_0) \in M_0(r,n)$. Moreover, its space of global sections is a T module with character $r \in R(T)$. In other words,

$$\operatorname{ch} \iota_{0*} r = \operatorname{ch} H^0(M_0(r,n), \iota_{0*}r) = r. \tag{3.37}$$

Analogously to the inclusion map ι_0, the inclusion map ι, defined in equation (3.16), yields the pushforward homomorphism

$$\iota_* : \mathbf{K}_T(M(r,n)^T) \to \mathbf{K}_T(M(r,n)). \tag{3.38}$$

Since the fixed points are indexed by r-tuples of partitions with total size n, the space on the right hand side is given by $\bigoplus_{|\vec{Y}|=n} R(T)$. The Thomason localization theorem [48] says, that both inclusion maps define isomorphisms after tensoring with $\mathcal{R}(T)$. We obtain the following commutative diagram

$$\begin{array}{ccc}
\mathbf{K}_T(M(r,n)) \otimes_{R(T)} \mathcal{R}(T) & \xrightarrow{\iota_*^{-1}} & \bigoplus_{|\vec{Y}|=n} \mathcal{R}(T) \\
\Big\downarrow \pi_* & & \Big\downarrow \sum_{|\vec{Y}|=n} \\
\mathbf{K}_T(M_0(r,n)) \otimes_{R(T)} \mathcal{R}(T) & \xrightarrow{\iota_{0*}^{-1}} & \mathcal{R}(T).
\end{array}$$

By equation (3.37), the bottom arrow equals ch. If we apply the commutativity of the diagram to $Z_n(E) = \operatorname{ch} \pi_* E$, we obtain equation (3.27).

In the next section, we are going to apply the definition $Z_n(E)$ to a special class of equivariant, coherent sheaves E to define the Nekrasov partition function.

3.2. The Nekrasov Partition Function as a Generating Function

In this section, we define the Nekrasov partition function. It computes the weighted Euler characteristic of certain K-theory classes, called tautological classes. In section 3.2.1, we define those classes. In the subsequent section 3.2.2, we define the Nekrasov partition function as a generating function of tautological classes. We describe an explicit description of its coefficients in terms of a generating set of the representation ring $R(T)$.

3.2.1. The Weighted Euler Characteristic of Tautological Classes

In the last section, we have established formula (3.27) for the weighted Euler characteristic $Z_n(E)$ of a equivariant, coherent sheaf on $M(r,n)$. The right hand side makes sense for K-theory classes as well. Hence we extend the construction to $\mathbf{K}_T(M(r,n))$:

$$Z_n(E) = \sum_{|\vec{Y}|=n} \frac{\iota_{\vec{Y}}^* E}{\Lambda_{-1} T_{\vec{Y}}^* M(r,n)} \in \mathcal{R}(T) \qquad (E \in \mathbf{K}_T(M(r,n))). \tag{3.39}$$

We now want to construct certain K-theory classes which are called *tautological classes*. We follow the description in [37]. They are constructed using the tautological bundle $\mathcal{V} \to M(r,n)$ on $M(r,n)$. It is constructed as an associated bundle:

$$\mathcal{V} = \{X \in N(r,n) : \mu(X) = 0, X \text{ stable}\} \times_{GL_n(\mathbf{C})} V, \tag{3.40}$$

with respect to the standard action of $GL_n(\mathbf{C})$ on $V = \mathbf{C}^n$. By construction its typical fiber is V. The vector bundle $\mathcal{V}$ and all its exterior powers $\Lambda^i \mathcal{V}, i = 0, \dots, n$ define classes in $\mathbf{K}_T(M(r,n))$. More generally a polynomial in such classes is called a tautological class. They can also be constructed from the map

$$\tau : \mathbf{K}_T(\mathrm{pt})[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathrm{Sym}} \to \mathbf{K}_T(M(r,n)) \tag{3.41}$$

which takes a symmetric polynomial with coefficients in $\mathbf{K}_T(\mathrm{pt}) = R(T)$ and evaluates it at the K-theoretic Chern roots of the tautological bundle $\mathcal{V}$. It is conjectured [37] that this map is surjective. In the next section, we will use this map to define the Nekrasov partition function.

3.2.2. The Nekrasov Partition Function

Fix two monic polynomials

$$f(x), g(x) \in \mathbf{K}_T(\mathrm{pt})[x] \tag{3.42}$$

with coefficients in $\mathbf{K}_T(\mathrm{pt}) = R(T)$. Fix a positive integer r. For each nonnegative integer n, construct the K-theory class

$$E_n = \tau\Bigg(\prod_{j=1}^{n} f(x_j) g(x_j^{-1}) \Bigg) \in \mathbf{K}_T(M(r,n)). \tag{3.43}$$

The Nekrasov partition function is the generating function of the weighted Euler characteristics $Z_n(E_n)$ of the classes $E_n \in \mathbf{K}_T(M(r,n))$, $n \geq 0$. It is a formal power series with coefficients $Z_n(E_n) \in \mathcal{R}(T)$.

For the choice $f = g = 1$ and each n, the class E_n is given by the K-theory class defined by the structure sheaf on $M(r,n)$. The corresponding Nekrasov partition function agrees with the one studied in [35, 36]. Physically, it corresponds to pure $SU(r)$ Yang Mills theory. More general polynomials f, g allow for the inclusion of matter fields.

In the remainder of the chapter, we make the coefficient $Z_n(E_n) \in \mathcal{R}(T)$ more explicit. Recall that an element of the torus T is given by

$$(t_1, t_2, e_1, \dots, e_r) \in \mathbf{C}^* \times \mathbf{C}^* \times (\mathbf{C}^*)^r. \tag{3.44}$$

We use the same symbols $t_1, t_2, e_1, \dots, e_r$ for the corresponding T characters:

$$t_i : (t_1, t_2, e_1, \dots, e_r) \mapsto t_i \in \mathbf{C}^* \qquad (i = 1, 2) \tag{3.45}$$

$$e_\alpha : (t_1, t_2, e_1, \dots, e_r) \mapsto e_\alpha \in \mathbf{C}^* \qquad (\alpha = 1, \dots, r). \tag{3.46}$$

Hence

$$R(T) = \mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, e_1^{\pm 1}, \dots, e_r^{\pm 1}] \tag{3.47}$$

and

$$\mathcal{R}(T) = \mathbf{Q}(t_1, t_2, e_1, \dots, e_r). \tag{3.48}$$

We will now describe $Z_n(E_n)$ as an element of this field.

First, we turn to the denominators in (3.39). Fix an r tuple $\vec{Y}$ of partitions with total size n. Suppose the tangent space $T_{\vec{Y}}M(r,n)$ to $M(r,n)$ at the fixed point $I_{\vec{Y}}$ is given, as a T module, by

$$T_{\vec{Y}}M(r,n) = t_1 + \cdots + t_d \in R(T). \tag{3.49}$$

Then the alternating sum of exterior powers of its dual is given by

$$\Lambda_{-1}T^*_{\vec{Y}}M(r,n) = (1 - t_1^{-1}) \cdots (1 - t_d^{-1}) \in R(T) \tag{3.50}$$

as a T module. The T module structure of $T_{\vec{Y}}M(r,n)$ is given [35, Theorem 2.11] by

$$T_{\vec{Y}}M(r,n) = \sum_{\alpha,\beta=1}^{r} e_\alpha e_\beta^{-1} \left(\sum_{\Box \in Y_\alpha} t_1^{-l_{Y_\beta}(\Box)} t_2^{a_{Y_\alpha}(\Box)+1} + \sum_{\boxtimes \in Y_\beta} t_1^{l_{Y_\alpha}(\boxtimes)+1} t_2^{-a_{Y_\beta}(\boxtimes)} \right). \tag{3.51}$$

Hence the denominator in (3.39) is given by

$$\Lambda_{-1}T^*_{\vec{Y}}M(r,n) = \prod_{\alpha,\beta=1}^{r} \left(\prod_{\Box \in Y_\alpha} \left(1 - e_\alpha^{-1} e_\beta \ t_1^{l_{Y_\beta}(\Box)} t_2^{-a_{Y_\alpha}(\Box)-1}\right) \right. \tag{3.52}$$

$$\left. \prod_{\boxtimes \in Y_\beta} \left(1 - e_\alpha^{-1} e_\beta \ t_1^{-l_{Y_\alpha}(\boxtimes)-1} t_2^{a_{Y_\beta}(\boxtimes)}\right) \right). \tag{3.53}$$

Next, we turn to the numerators in (3.39). Fix a r tuple of partitions $\vec{Y}$ of total size n. The pullback $\iota^*_{\vec{Y}}(E_n)$ of the tautological class

$$E_n = \tau\left(\prod_{j=1}^{n} f(x_j) g(x_j^{-1}) \right) \tag{3.54}$$

is given [37] by

$$\iota^*_{\vec{Y}}(E_n) = \prod_{\alpha=1}^{r} \prod_{(x,y)\in Y_\alpha} f\big(e_\alpha t_1^{x-1} t_2^{y-1}\big) g\big(e_\alpha^{-1} t_1^{-x+1} t_2^{-y+1}\big). \tag{3.55}$$

All together, we obtain the coefficient $Z_n(E_n)$ as an element of

$$\mathbf{Q}(t_1, t_2, e_1, \dots, e_r) \tag{3.56}$$

from the expression

$$Z_n(t_1, t_2, e_1, \ldots, e_r; f, g) \tag{3.57}$$

$$= \sum_{|\vec{Y}|=n} \frac{\prod_{\alpha=1}^{r} \prod_{(x,y)\in Y_\alpha} f\big(e_\alpha t_1^{x-1} t_2^{y-1}\big) g\big(e_\alpha^{-1} t_1^{-x+1} t_2^{-y+1}\big)}{\prod_{\alpha,\beta=1}^{r} K_{\alpha,\beta}^{\vec{Y}}(t_1, t_2, e_1, \ldots, e_r)}, \tag{3.58}$$

where

$$K_{\alpha,\beta}^{\vec{Y}}(t_1, t_2, e_1, \ldots, e_r) = \prod_{\square\in Y_\alpha} \Big(1 - e_\alpha^{-1} e_\beta \; t_1^{l_{Y_\beta}(\square)} t_2^{-a_{Y_\alpha}(\square)-1}\Big) \tag{3.59}$$

$$\prod_{\boxtimes\in Y_\beta} \Big(1 - e_\alpha^{-1} e_\beta \; t_1^{-l_{Y_\alpha}(\boxtimes)-1} t_2^{a_{Y_\beta}(\boxtimes)}\Big). \tag{3.60}$$

We write the Nekrasov partition function as the formal power series

$$Z(t_1, t_2, e_1, \ldots, e_r; f, g; \mathfrak{q}) = \sum_{n\geq 0} \Big(\mathfrak{q}\; (t_1 t_2)^{-r/2}\Big)^n \tag{3.61}$$

$$\times Z_n(t_1, t_2, e_1, \ldots, e_r; f, g) \tag{3.62}$$

in the variable $\mathfrak{q}$. The additional scaling factor will be convenient for the AGT relation, described in chapter 4. It is not essential in any sense. For r odd, the square root is purely formal.

It is customary [36] to formally write

$$t_1 = e^{\lambda\epsilon_1} \qquad t_2 = e^{\lambda\epsilon_2} \qquad e_\alpha = e^{\lambda a_\alpha} \quad (\alpha = 1, \ldots, r). \tag{3.63}$$

Using these symbols, the Nekrasov partition function is written [36] as the formal power series

$$Z(\epsilon_1, \epsilon_2, \vec{a}, \lambda; f, g; \mathfrak{b}) = \sum_{n\geq 0} \Big(\mathfrak{b}\; \lambda^{2r-\deg f+\deg g} e^{-r\lambda(\epsilon_1+\epsilon_2)/2}\Big)^n Z_n(\epsilon_1, \epsilon_2, \vec{a}, \lambda; f, g), \tag{3.64}$$

in the variable $\mathfrak{b}$. The coefficients are given by

$$Z_n(\epsilon_1, \epsilon_2, \vec{a}, \lambda; f, g) \tag{3.65}$$

$$= \sum_{|\vec{Y}|=n} \frac{\prod_{\alpha=1}^{r} \prod_{(x,y)\in Y_\alpha} f\big(e_\alpha t_1^{x-1} t_2^{y-1}\big) g\big(e_\alpha^{-1} t_1^{-x+1} t_2^{-y+1}\big)}{\prod_{\alpha,\beta=1}^{r} K_{\alpha,\beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}; \lambda)} \qquad (n \geq 0), \tag{3.66}$$

where

$$K_{\alpha,\beta}^{\vec{Y}}(\epsilon_1, \epsilon_2, \vec{a}; \lambda) = \prod_{\square\in Y_\alpha} \Big(1 - e^{-\lambda\big(-l_{Y_\beta}(\square)\epsilon_1 + (a_{Y_\alpha}(\square)+1)\epsilon_2 + a_\alpha - a_\beta\big)}\Big) \tag{3.67}$$

$$\prod_{\boxtimes\in Y_\beta} \Big(1 - e^{-\lambda\big((l_{Y_\alpha}(\boxtimes)+1)\epsilon_1 - a_{Y_\beta}(\boxtimes)\epsilon_2 + a_\alpha - a_\beta\big)}\Big). \tag{3.68}$$

The variable $\mathfrak{q}$ in the expression (3.64) is scaled by two factors. The factor

$$\lambda^{2r-\deg f+\deg g} \tag{3.69}$$

ensures that the coefficients have a limit for $\lambda \to 0$. We will comment on this limit in the discussion of our results in chapter 7. The other factor

$$e^{-r\lambda(\epsilon_1+\epsilon_2)/2} \tag{3.70}$$

is related to the fact that the coefficients in (3.67) are sometimes defined using hyperbolic functions instead of exponentials, c.f. formula (1.73) in chapter 1 or the original reference [39]. We have included this second factor also in the definition (3.61) without the exponentials since it is the correct scaling for the AGT relation.

4. Five-dimensional AGT Relation

In this chapter, we discuss a particular instance of the AGT relation. It is a relation between the Nekrasov partition function defined in chapter 3 and the deformed Gaiotto states defined in chapter 2. More precisely, it relates the generating function of the weighted Euler characteristics of the structure sheaves of the Nakajima quiver varieties $M(2,n), n \geq 0$, to the norm of the deformed Gaiotto state under a certain identification of parameters. In section 4.1, we recall the definition of those objects, relate their parameters and state the AGT relation. In the subsequent section 4.2, we sketch a proof of the equality of both objects due to Yanagida [50, 51].

4.1. Nekrasov Partition Functions and Deformed Gaiotto States

4.1.1. The Norm of the Deformed Gaiotto State

Recall that we have defined the norm of the deformed Gaiotto state as a formal power series in equation (2.66). The deformed Gaiotto state is an element of the completion (2.58) of the Verma module M_h, defined for a rational number $h \in \mathbf{Q}$. Suppose h is of the form

$$h = Q^{1/2} + Q^{-1/2}. \tag{4.1}$$

By the Kac determinant formula (2.54), the expansion (2.66) defines $\langle\, w(\xi) | w(\xi)\,\rangle$ as an element

$$\langle\, w(\xi) | w(\xi)\,\rangle = \sum_{n \geq 0} (\xi^2 t/q)^n F_n(q,t,Q) \in \mathbf{Q}(q,t,Q)[[\xi^2]], \tag{4.2}$$

where

$$F_n(q,t,Q) = q^n t^{-n} \big(W_{(n)}^{-1}\big)_{(1^n)(1^n)} \in \mathbf{Q}(q,t,Q) \qquad (n \geq 0). \tag{4.3}$$

Recall that $W_{(n)}^{-1}$ is simply the inverse of the Gram matrix $W^{(n)}$ with entries

$$W^{(n)}_{\lambda\mu} = \langle\, T_{-\lambda} 1_h, T_{-\mu} 1_h \,\rangle \qquad (|\lambda| = |\mu| = n). \tag{4.4}$$

4.1.2. A Special Case of the Nekrasov partition function

In section 3.2.2, we have defined, for $n \geq 0$, the n-th coefficient $Z_n(E_n) \in \mathcal{R}(T)$ of the Nekrasov partition function corresponding to the K-theory class $E_n \in \mathbf{K}_T(M(r,n))$ defined by two symmetric polynomials f, g in n variables with coefficients in $R(T)$. In this section, we specialize to the case $r = 2$ and $f = g = 1$. In this case, E_n is the K-theory class defined by the structure sheaf $\mathcal{O}$ on $M(r, n)$ and the explicit description in (3.57) yields

$$Z_n(\mathcal{O}) = \sum_{|Y_1|+|Y_2|=n} \frac{1}{\prod_{\alpha,\beta=1}^{2} N_{\alpha,\beta}(t_1, t_2, e_1, e_2)} \in \mathbf{Q}(t_1, t_2, e_1, e_2), \tag{4.5}$$

where

$$N_{\alpha,\beta}(t_1, t_2, e_1, e_2) = \prod_{\square \in Y_\alpha} \left(1 - e_\alpha^{-1} e_\beta \; t_1^{l_{Y_\beta}(\square)} t_2^{-a_{Y_\alpha}(\square)-1}\right) \tag{4.6}$$

$$\prod_{\boxtimes \in Y_\beta} \left(1 - e_\alpha^{-1} e_\beta \; t_1^{-l_{Y_\alpha}(\boxtimes)-1} t_2^{a_{Y_\beta}(\boxtimes)}\right). \tag{4.7}$$

Note that the rational function $Z_n(\mathcal{O})$, only depends on t_1, t_2 and the ratio e_2/e_1 and hence defines an element in $\mathbf{Q}(t_1, t_2, e_2/e_1)$. We want to relate these coefficients to the expansion coefficients of the norm of the deformed Gaiotto state given in equation (4.2). We identify

$$t = t_1^{-1} \qquad q = t_2 \qquad Q = e_2/e_1 \qquad \mathfrak{q} = \xi^2. \tag{4.8}$$

Hence the definition of the Nekrasov partition function in equation (3.61) defines the formal power series

$$Z(\xi) = \sum_{n \geq 0} (\xi^2 t/q)^n Z_n(q, t, Q) \in \mathbf{Q}(q, t, Q)[[\xi^2]], \tag{4.9}$$

where we can express $Z_n(q, t, Q) = Z_n(\mathcal{O})$ as

$$Z_n(q, t, Q) = \sum_{|Y|+|W|=n} \frac{1}{N_{Y,Y}(1) N_{Y,W}(Q) N_{W,Y}(Q^{-1}) N_{W,W}(1).} \tag{4.10}$$

Here, for two Young diagrams Y and W,

$$N_{Y,W}(Q) = \prod_{\square \in W} (1 - Q q^{a_W(\square)} t^{l_Y(\square)+1}) \prod_{\boxtimes \in Y} (1 - Q q^{-a_Y(\boxtimes)-1} t^{-l_W(\boxtimes)}). \tag{4.11}$$

The scaling factor t/q in the definition of the power series is again due to the fact that one often defines the coefficients using hyperbolic functions. It corresponds to the exponential scaling factor in (3.64).

One of the relations that are known under the name of AGT relations states that the two formal power series (4.2) and (4.9), defining the norm of the deformed Gaiotto state and the Nekrasov partition function respectively, agree. This version of the AGT relation was conjectured in [7] and proved in [50, 51]. We will sketch the argument in the next section.

4.2. Proof of the five-dimensional AGT Relation

Zamolodchikov has observed [53] that the expansion coefficients of conformal blocks of the Virasoro algebra satisfy a recursion relation, which is derived from the poles of those coefficients as a function of the central charge and conformal dimensions. One strategy to prove the AGT relations is to derive similar recursion relations for both the norm of the deformed Gaiotto state (4.2) and the Nekrasov partition function (4.9). We will sketch the argument in this section.

4.2.1. Recursion Relation for the Nekrasov Partition Function

In order to derive a recursion relations for the coefficients $Z_n(q,t,Q)$, given in equation (4.10), Yanagida proposes an integral representation for $Z_n(q,t,Q)$ in [50]. This integral representation is similar to the one we discuss in chapter 5. The main difference is, that in [50], no explicit description of the integration contour is given.

With the help of the integral representation, Yanagida describes the pole structure of $Z_n(q,t,Q)$ as a rational function of Q. Yanagida finds [50, Proposition 4.2] that there is one pole at $Q = q^r t^{-t}$ for each pair (r,s) of integers $r,s \in \mathbf{Z}$ with $1 \leq rs \leq n$ and no others. Moreover, all poles are simple. He then proceeds with the calculation of the residues of $Z_n(q,t,Q)$ at each of those simple poles. The integral representation is used again. The result is

$$\operatorname{Res}_{Q=q^r t^{-s}} Z_n(q,t,Q) = A_{r,s}(q,t) Z_{n-rs}(q^r t^s, q, t), \tag{4.12}$$

where $r,s \in \mathbf{Z}$, $1 \leq rs \leq n$. The coefficient $A_{r,s}(q,t)$ is given by

$$A_{r,s}(q,t) = -\operatorname{sign}(r) q^r t^{-s} \prod_{ij} \frac{1}{1 - q^i t^{-j}}, \tag{4.13}$$

where the product runs over all pairs $(i,j) \in \mathbf{Z}^2$ with $(i,j) \neq (0,0)$, $-|r| \leq i \leq |r| - 1$, and $-|s| \leq j \leq |s| - 1$.

Yanagida concludes [50, Theorem 4.1]

$$Z_n(Q,q,t) = \delta_{n,0} + \sum_{\substack{r,s\in\mathbf{Z} \\ 1\leq rs\leq n}} \frac{A_{r,s}(q,t) Z_{n-rs}(q^r t^s, q, t)}{Q - q^r t^{-s}} \qquad (n \geq 0). \tag{4.14}$$

Indeed, the difference of both sides define an entire function in Q. It is bounded in Q since $Z_n(Q,q,t)$ is invariant under inversion of Q. Moreover, for $Q \to \infty$ it converges to zero.

4.2.2. Recursion Relation for Deformed Gaiotto States

In a subsequent paper [51], Yanagida analyzes the coefficients $F_n(q,t,Q)$ in the expansion (4.2) of the norm of the deformed Gaiotto state. By analyzing the poles and zeros of the Kac determinant, Yanagida proves [51, Theorem 4.3] that the expansion coefficients

$F_n(Q,q,t) \in \mathbf{Q}(Q,q,t)$ satisfy the recursive relation

$$F_n(Q,q,t) = \delta_{n,0} + \sum_{\substack{r,s\in\mathbf{Z} \\ 1\leq rs\leq n}} \frac{A_{r,s}(q,t)F_{n-rs}(q^r t^s,q,t)}{Q - q^r t^{-s}} \qquad (n \geq 0). \tag{4.15}$$

Here, the coefficient $A_{r,s}(q,t)$ is also given by equation (4.13).

4.2.3. Comparison and Conclusion

Comparing the two recursion relations (4.14) and (4.15) for the coefficients in the formal power series (4.2) and (4.10) for the norm of the deformed Gaiotto state and the Nekrasov partition function, we obtain

$$\langle\, w(\xi) | w(\xi)\,\rangle = Z(t_1,t_2,e_1,e_2;f,g,\mathfrak{q}) \in \mathbf{Q}(q,t,Q)[[\xi^2]]. \tag{4.16}$$

under the identifications

$$t_1 = t^{-1} \qquad t_2 = q \qquad e_2/e_1 = Q \qquad f=1 \qquad g=1 \qquad \mathfrak{q} = \xi^2. \tag{4.17}$$

5. Integral Representation

In this chapter, we will define an integral representation for the coefficients of the Nekrasov partition function defined in chapter 3. It is a sequence of multiple contour integrals, depending on complex parameters in a certain range. We define it in section 5.1. The integrand and the contour description are known. However, to the best of our knowledge, a rigorous proof for the validity of the contour description has not been published. We provide it in section 5.2. In the subsequent section 5.3, we estimate the growth of the sequence of integrals.

5.1. Definition of the Integral and Parameter Ranges

In this section, we define a sequence of multiple contour integrals. Let q_1 and q_2 denote two complex numbers with

$$|q_1| < 1 \qquad\qquad |q_2| < 1. \tag{5.1}$$

Assume they are complex conjugate or both positive, i.e.

$$q_1 = \overline{q_2} \qquad\qquad \text{or} \qquad\qquad q_1, q_2 \in (0,1). \tag{5.2}$$

These two complex numbers will be related to the formal parameters t_1, t_2 from chapter 3. Note that in either case $q_1 q_2 = |q_1 q_2| \in (0,1)$.

Fix an integer $r \geq 1$. Let $\vec{u} = (u_1, \ldots, u_r)$ be an r-tuple of complex numbers. Assume

$$|q_i| \max_{\alpha=1,\ldots,r} |u_\alpha| < \min_{\alpha=1,\ldots,r} |u_\alpha|, \qquad \forall\, i = 1,2. \tag{5.3}$$

The complex numbers $u_1, \ldots, u_r$ will be related to the formal parameters $e_1, \ldots, e_r$ from chapter 3.

Fix two integers $s, s' \geq 0$. Let $\vec{F} = (F_1, \ldots, F_s)$ and $\vec{G} = (G_1, \ldots, G_{s'})$ denote two tuples of complex numbers. They define the two monic polynomials

$$F(z) = \prod_{k=1}^{s} (z - F_k) \qquad\qquad G(z) = \prod_{k=1}^{s'} (z - G_k). \tag{5.4}$$

They will be related to the polynomials $f(z), g(z)$ from chapter 3. Assume that the degree of G is bounded as

$$s' = \deg G(z) \leq r - 1. \tag{5.5}$$

By condition (5.3), we can pick a positive real number $\rho > 0$ with

$$|u_\alpha| < \rho < |q_i|^{-1} |u_\alpha| \qquad\qquad \forall\, \alpha = 1, \ldots, r, \forall\, i = 1,2. \tag{5.6}$$

For each integer $n \geq 0$, we now define the symmetric function

$$\mathcal{I}(z_1, \ldots, z_n; q_1, q_2, \vec{u}) = \prod_{j=1}^{n} \prod_{\alpha=1}^{r} \frac{-u_\alpha z_j}{(z_j - u_\alpha)(q_1 q_2 z_j - u_\alpha)} \tag{5.7}$$

$$\prod_{1 \leq j \neq k \leq n} \frac{(z_j - z_k)(z_j - q_1 q_2 z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)}. \tag{5.8}$$

Also define the symmetric function

$$\mathcal{J}(z_1, \ldots, z_n; \vec{F}, \vec{G}) = \prod_{j=1}^{n} F(z_j) G(z_j^{-1}). \tag{5.9}$$

Again, for each integer $n \geq 0$, we construct the integral

$$\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) = \frac{1}{n!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n \tag{5.10}$$

$$\int_{C_n(\rho)} \prod_{j=1}^{n} \frac{dz_j}{2\pi i z_j} \, \mathcal{J}(z_1, \ldots, z_n; \vec{F}, \vec{G}) \tag{5.11}$$

$$\mathcal{I}(z_1, \ldots, z_n; q_1, q_2, \vec{u}) \tag{5.12}$$

over the multiple contour

$$C_n(\rho) = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_j| = \rho, j = 1, \ldots, n\}. \tag{5.13}$$

The assumption (5.5) ensures that the integrand is regular at zero. The factorial in the definition is related to the permutation invariance of the integrand. The other pre-factor corresponds to the absent diagonal product in line (5.8). Similar integrals appeared in the literature [50, 37].

In the subsequent section 5.2, we will evaluate the integral using iterated residues. Afterwards, in section 5.3, we will estimate the large n behavior of the coefficients $\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})$.

5.2. Evaluation of the Integral

Fix an integer $n \geq 0$. We evaluate the integral (5.10) using iterated residues. The residues will be indexed by r-tuples $\vec{Y}$ of Young diagrams with total size $|\vec{Y}| = n$. For a box

$$\square = (x, y) \in Y_\alpha \tag{5.14}$$

in some Young diagram Y_α, we define

$$z_\square^\alpha = z_{x,y}^\alpha = u_\alpha q_1^{x-1} q_2^{y-2}. \tag{5.15}$$

Theorem 5.2.1. *Assume the complex numbers* q_1 *and* q_2 *satisfy*

$$q_1^x \neq q_2^{y+1}, \quad q_1^{x+1} \neq q_2^y, \qquad \forall x, y \in \{0, \ldots, n-1\}. \tag{5.16}$$

Moreover, assume the complex numbers $u_1, \ldots, u_\alpha$ *satisfy*

$$u_\alpha u_\beta^{-1} \neq q_1^x q_2^y, \qquad \forall x, y \in \{-n, \ldots, n\}, \quad \forall \alpha \neq \beta \in \{1, \ldots, r\}. \tag{5.17}$$

The value of $\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})$ *is given as a sum over* r*-tuples of partitions of total size* n *in two equivalent ways:*

$$\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) = \sum_{|\vec{Y}|=n} \frac{\prod_{\alpha=1}^r \prod_{(x,y)\in Y_\alpha} F\big(u_\alpha q_1^{x-1} q_2^{y-1}\big) G\big(u_\alpha^{-1} q_1^{-x+1} q_2^{-y+1}\big)}{\prod_{\alpha,\beta=1}^r N_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u})}, \tag{5.18}$$

where, for $\alpha, \beta \in \{1, \ldots, r\}$,

$$N_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u}) = \prod_{\square \in Y_\alpha} \Big(1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y_\alpha}(\square)+1} q_2^{-a_{Y_\beta}(\square)}\Big) \tag{5.19}$$

$$\prod_{\boxtimes \in Y_\beta} \Big(1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y_\beta}(\boxtimes)} q_2^{a_{Y_\alpha}(\boxtimes)+1}\Big). \tag{5.20}$$

Alternatively,

$$\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) = \sum_{|\vec{Y}|=n} \frac{\prod_{\alpha=1}^r \prod_{(x,y)\in Y_\alpha} F\big(u_\alpha q_1^{x-1} q_2^{y-1}\big) G\big(u_\alpha^{-1} q_1^{-x+1} q_2^{-y+1}\big)}{\prod_{\alpha,\beta=1}^r M_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u})}, \tag{5.21}$$

where, for $\alpha, \beta \in \{1, \ldots, r\}$,

$$M_{\alpha,\beta}^{\vec{Y}}(q_1, q_2, \vec{u}) = \prod_{\square \in Y_\alpha} \Big(1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y_\beta}(\square)} q_2^{a_{Y_\alpha}(\square)+1}\Big) \tag{5.22}$$

$$\prod_{\boxtimes \in Y_\beta} \Big(1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y_\alpha}(\boxtimes)+1} q_2^{-a_{Y_\beta}(\boxtimes)}\Big). \tag{5.23}$$

The assumptions (5.16) on q_1, q_2 and (5.17) on $u_1, \ldots, u_r$ are in addition to the standing assumptions (5.1), (5.2) and (5.3). They are necessary to ensure that all terms in (5.18) and (5.21) are well-defined. If they are violated, some residues might not be simple residues anymore and consequently some of those terms might be infinite. However, their sum $\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})$ is still well-defined, as the integral in equation (5.10) is.

Proof of Theorem 5.2.1. We evaluate the integral defined in (5.10) by iteratively taking residues. In a first step, we show that the iterated residues appearing in the evaluation of the n fold integral are parametrized, up to permutation of the variables, by r tuples $\vec{Y}$ of Young diagrams with total size n. In a second step, we calculate the iterated residues.

In order to avoid collisions of the integration contour and poles of the integrand, we perturb the modulus of q_1 and q_2 to

$$|q_1| = |q_2| + \delta < 1 \tag{5.24}$$

where $\delta > 0$ is small enough such that

$$|q_1| > |q_2| > |q_1|^2 > |q_1|^3 > \cdots . \tag{5.25}$$

The general case follows from analytic continuation.

We start with a slightly more general integral. Let U and W be two finite sets of complex numbers. Fix $n \geq 0$. Let

$$f(x_1, \ldots, x_n) \tag{5.26}$$

denote an arbitrary meromorphic, symmetric function, whose poles in each variable z_j do not depend on the other variables and lie outside the integration contour $|z_j| = \rho$. Define the symmetric function

$$I(z_1, \ldots, z_n) = f(z_1, \ldots, z_n) \tag{5.27}$$

$$\times \prod_{j=1}^{n} \left(\frac{\prod_{w \in W}(z_j - w)}{\prod_{u \in U}(z_j - u)} \prod_{j<k} D(z_j, z_k) \right), \tag{5.28}$$

where

$$D(z_j, z_k) = \frac{(z_j - z_k)^2 (z_j - q_1 q_2 z_k)(z_j - q_1^{-1} q_2^{-1} z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)(z_j - q_1^{-1} z_k)(z_j - q_2^{-1} z_k)}. \tag{5.29}$$

Our integral is of the form

$$\int_{|z_n|=\rho} \frac{dz_n}{2\pi i} \cdots \int_{|z_1|=\rho} \frac{dz_1}{2\pi i} \quad I(z_1, \ldots, z_n). \tag{5.30}$$

The evaluation happens by iteratively taking residues. In particular the integral equals

$$\sum_{(\hat{z}_n, \ldots, \hat{z}_1) \in R} \operatorname{Res}_{z_n = \hat{z}_n} \cdots \operatorname{Res}_{z_1 = \hat{z}_1} I(z_1, \ldots, z_n), \tag{5.31}$$

for a suitable finite set $R \subset \mathbf{C}^n$.

From line (5.28), we see that we can either pick up residues at poles determined by the set U or at poles of $D(z_j, z_k)$ determined by variables we have not yet integrated over. In the first case, we say that we pick up a residue *at the front*. In the second case, we say that we pick up a residue *at the back*.

The evaluation of the integral (5.30) happens in stages. Our first claim is

Claim 1. *We can assume each element of the residue set R is partitioned as*

$$(\hat{z}_n, \ldots, \hat{z}_1) = (\hat{z}_{J_1 + \cdots + J_k}, \ldots, \hat{z}_{J_1 + \cdots + J_{k-1} + 1}, \tag{5.32}$$

$$\ldots, \tag{5.33}$$

$$\hat{z}_{J_1 + J_2}, \ldots, \hat{z}_{J_1 + 1}, \tag{5.34}$$

$$\hat{z}_{J_1}, \ldots, \hat{z}_1), \tag{5.35}$$

defining k stages *of the evaluation procedure with respective* sizes *J_l, $l = 1, \ldots, k$, such that the following holds:*

After each stage, the remaining integrand is of the same form (5.30) as the original integral, with different sets U, V and different symmetric function f.

Let

$$U_l, W_l, f_l \qquad (l \in \{1, \ldots, k\}) \tag{5.36}$$

denote the sets U_l, W_l and symmetric function f_l before evaluation of the l-th stage. In particular, $U_1 = U$, $W_1 = W$, and $f_1 = f$.

For each stage

$$\hat{z}_{J_1+\cdots+J_l}, \ldots, \hat{z}_{J_1+\cdots+J_{l-1}+1} \qquad (l \in \{1, \ldots, k\}) \tag{5.37}$$

there exist indices

$$\{i_l, s_l\} = \{1, 2\} \qquad d_l \in \{0, \ldots, J_l - 1\}, \tag{5.38}$$

such that

1. *The first $J_l - 1$ residues are picked at the back at the poles*

$$\hat{z}_j = q_{i_l} z_{j+1} \qquad (j = J_1 + \cdots + J_{l-1} + 1, \ldots, J_1 + \cdots + J_l). \tag{5.39}$$

2. *The last residue is picked up at the front:*

$$\hat{z}_{J_1+\cdots+J_l} = q_{i_l}^{-d_l} u_l \qquad (u_l \in U_l, d_l \in \{0, \ldots, J_l - 1\}). \tag{5.40}$$

3. *The residues are all simple.*

4. *The sets change as*

$$U_{l+1} = U_l \cup \{q_{i_l}^{-1} \hat{z}_{b_l}, q_{s_l} \hat{z}_{b_l}, q_{s_l}^{-1} \hat{z}_{a_l}, q_{i_l} \hat{z}_{a_l}\} \tag{5.41}$$

$$W_{l+1} = W_l \cup \{\hat{z}_{b_l}, q_{i_l}^{-1} q_{s_l}^{-1} \hat{z}_{b_l}, \hat{z}_{a_l}, q_{i_l} q_{s_l} \hat{z}_{a_l}\}, \tag{5.42}$$

   where

$$a_l = J_1 + \cdots + J_{l-1} + 1 \qquad b_l = J_1 + \cdots + J_l \tag{5.43}$$

   are the first and the last index in stage l.

5. *The symmetric function changes as*

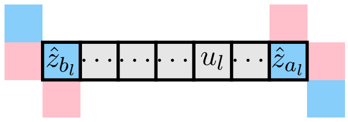
$$f_{l+1}(z_{b_l+1}, \ldots, z_n) = f_l(q_{i_l}^{J_l-1} q_{i_l}^{-d_l} u_l, \ldots, q_{i_l}^{-d_l} u_l, z_{b_l+1}, \ldots, z_n) \tag{5.44}$$

$$\times \prod_{j=0}^{J_l-1} \frac{\prod_{w \in W_l}(q_{i_l}^{j-d_l} u_l - w)}{\prod_{u \in U \setminus \{u_l\}}(q_{i_l}^{j-d_l} u_l - u)} \prod_{\substack{j=0 \\ j \neq d_l}}^{J_l-1} \frac{1}{(q_{i_l}^{j-d_l} - 1)} \tag{5.45}$$

$$\times \frac{(q_{i_l} - 1)^{J_l}(1 - q_{s_l})^{J_l-1}}{(q_{i_l} - q_{s_l}^{-1})^{J_l-1}(q_{i_l}^{J_l} - 1)} \prod_{j=1}^{J_l-1} \frac{(q_{i_l}^j - q_{i_l}^{-1} q_{s_l}^{-1})}{(q_{i_l}^j - q_{s_l})} \tag{5.46}$$

$$\times q_{i_l}^{\frac{1}{2}(J_l+2)(J_l-1)-d_l J_l}. \tag{5.47}$$

We think of the positions of the residues (5.37) in stage l as a strip. For $i_l = 2$, we draw the strip horizontally:



(5.48)

The positive q_{i_l} direction goes from west to east. The order of evaluation of the residues (5.39) and (5.40) goes from east to west over the residue strip. Here we have also indicated the poles in red and zeros in blue the strip adds according to (5.41) to the sets U_l and W_l to obtain U_{l+1} and W_{l+1}. For $i_l = 1$ the strip is drawn vertically, with positive q_{i_l} direction from north to south.

Let us prove claim 1. We write the integral (5.30) as

$$\int_{|z_n|=1} \frac{dz_n}{2\pi i} \frac{\prod_{w\in W}(z_n - w)}{\prod_{u\in U}(z_n - u)} \prod_{n<k} D(z_n, z_k) \tag{5.49}$$

$$\vdots \tag{5.50}$$

$$\int_{|z_2|=1} \frac{dz_2}{2\pi i} \frac{\prod_{w\in W}(z_2 - w)}{\prod_{u\in U}(z_2 - u)} \prod_{2<k} D(z_2, z_k) \tag{5.51}$$

$$\int_{|z_1|=1} \frac{dz_1}{2\pi i} \frac{\prod_{w\in W}(z_1 - w)}{\prod_{u\in U}(z_1 - u)} \prod_{1<k} D(z_1, z_k) \quad f(z_1, \ldots, z_n). \tag{5.52}$$

When integrating over z_1, we can either pick up residues at the back at $q_1 z_j$ or $q_2 z_j$ for some $j \in \{2, \ldots, n\}$. Or we can pick up a residue at the front at some $u \in U$ inside the integration contour. Assume, we pick a residue at the back at

$$\hat{z}_1 = q_i z_j \tag{5.53}$$

for some $i \in \{1, 2\}$ and some $j \in \{2, \ldots, n\}$. Denote by s the other index, i.e. $\{i, s\} = \{1, 2\}$. The residue is simple. The evaluation of the residue yields

$$\int_{|z_n|=1} \frac{dz_n}{2\pi i} \frac{\prod_{w\in W}(z_n - w)}{\prod_{u\in U}(z_n - u)} \prod_{n<k} D(z_n, z_k) \tag{5.54}$$

$$\vdots \tag{5.55}$$

$$\int_{|z_2|=1} \frac{dz_2}{2\pi i} \frac{\prod_{w\in W}(z_2 - w)}{\prod_{u\in U}(z_2 - u)} \prod_{2<k} D(z_2, z_k) \tag{5.56}$$

$$\frac{\prod_{w\in W}(q_i z_j - w)}{\prod_{u\in U}(q_i z_j - u)} \operatorname{Res}_{z_1 = q_i z_j} \prod_{1<k} (D(z_1, z_k)) \quad f(q_i z_j, z_2, \ldots, z_n). \tag{5.57}$$

We now use Fubini's theorem to permute the integration over z_2 and z_j. This is possible due to our perturbation (5.24). Additionally, we rename $z_j \leftrightarrow z_2$. Using the symmetry of the function f, we obtain

$$\int_{|z_n|=1} \frac{dz_n}{2\pi i} \frac{\prod_{w\in W}(z_n - w)}{\prod_{u\in U}(z_n - u)} \prod_{n<k} D_1(z_n, z_k) \tag{5.58}$$

$$\vdots \tag{5.59}$$

$$\int_{|z_3|=1} \frac{dz_3}{2\pi i} \frac{\prod_{w\in W}(z_3 - w)}{\prod_{u\in U}(z_3 - u)} \prod_{3<k} D_1(z_3, z_k) \tag{5.60}$$

$$\int_{|z_2|=1} \frac{dz_2}{2\pi i} \frac{\prod_{w\in W}(z_2 - w)(q_i z_2 - w)}{\prod_{u\in U}(z_2 - u)(q_i z_2 - u)} \prod_{2<k} D_2(z_2, z_k) f(q_i z_2, z_2, \dots, z_n) \tag{5.61}$$

$$\times\, q_i z_2 \frac{(q_i - 1)^2(1 - q_s)(q_i - q_i^{-1} q_s^{-1})}{(q_i - q_s)(q_i - q_i^{-1})(q_i - q_s^{-1})}. \tag{5.62}$$

where we have extended the definition of $D(z_j, z_k)$ to

$$D_m(z_j, z_k) = \frac{(z_j - z_k)(z_j - q_i q_s z_k)(z_j q_i^{m-1} - z_k)(z_j q_i^{m-1} - q_i^{-1} q_s^{-1} z_k)}{(z_j - q_i z_k)(z_j - q_s^{-1} z_k)(z_j q_i^{m-1} - q_s z_k)(z_j q_i^{m-1} - q_i^{-1} z_k)} \tag{5.63}$$

for $m \geq 1$. Note that $D_1(z_j, z_k) = D(z_j, z_k)$. Moreover, we could have supposed $j = 2$ in (5.53).

For the integration over z_2, we again can either pick up a residue at the front at u or uq_i^{-1} for some $u \in U$ such that the pole lies inside the integration contour. Or we can pick up a residue at the back from a pole of

$$D_2(z_2, z_k) = \frac{(z_2 - z_k)(z_2 - q_i q_s z_k)(z_2 q_i - z_k)(z_2 q_i - q_i^{-1} q_s^{-1} z_k)}{(z_2 - q_i z_k)(z_2 - q_s^{-1} z_k)(z_2 q_i - q_s z_k)(z_2 q_i - q_i^{-1} z_k)}. \tag{5.64}$$

The poles inside the integration contour are located at $\hat{z}_2 = q_i z_j$ with $j \in \{3, \dots, n\}$. Note that the index i is one we fixed by choosing the residue of the previous integration. If $i = 1$, we have

$$|q_s|/|q_i| = |q_2|/|q_1| < 1 \tag{5.65}$$

by (5.25) and it appears that there is also a residue coming from the factor

$$(z_2 q_i - q_s z_k). \tag{5.66}$$

However, the residues coming from these poles cancel in the sum of all residues. By direct calculation,

$$\lim_{z_k \to q_1^{-1} q_2 z_l} (z_k - q_1^{-1} q_2 z_l) \sum_{i=1,2} \sum_{j=2}^{n} \mathrm{Res}_{\hat{z}_1 = q_i z_j} I(z_1, \dots, z_n) = 0. \tag{5.67}$$

Assume, we again pick a residue at the back at

$$\hat{z}_2 = q_i z_j \tag{5.68}$$

where $j \in \{3, \dots, n\}$. By the same argument as above, we may suppose without loss of generality $j = 3$. The residue is simple and we obtain

$$\int_{|z_n|=1} \frac{dz_n}{2\pi i} \frac{\prod_{w\in W}(z_n - w)}{\prod_{u\in U}(z_n - u)} \prod_{n<k} D_1(z_n, z_k) \tag{5.69}$$

$$\vdots \tag{5.70}$$

$$\int_{|z_4|=1} \frac{dz_4}{2\pi i} \frac{\prod_{w\in W}(z_4 - w)}{\prod_{u\in U}(z_4 - u)} \prod_{4<k} D_1(z_4, z_k) \tag{5.71}$$

$$\int_{|z_3|=1} \frac{dz_3}{2\pi i} \frac{\prod_{w\in W}(z_3 - w)(q_i z_3 - w)(q_i^2 z_3 - w)}{\prod_{u\in U}(z_3 - u)(q_i z_3 - u)(q_i^2 z_3 - u)} \prod_{3<k} D_3(z_3, z_k) \tag{5.72}$$

$$\times q_i^3 z_3^2 \frac{(q_i - 1)^3(1 - q_s)^2(q_i^2 - q_s^{-1})(q_i^3 - q_s^{-1})}{(q_i - q_s)(q_i - q_s^{-1})^2(q_i^2 - q_s)(q_i^3 - 1)} f(q_i^2 z_3, q_i z_3, z_3, \dots, z_n). \tag{5.73}$$

In the next step, we can again pick up residues at the front at u, uq_i^{-1}, uq_i^{-2} with $u \in U$ such that the residues are located inside the integration contour. Or we can pick up a residue at the back at $q_i z_j$ for some $j \in \{4, \dots, n\}$, for which we may suppose $j = 4$. From the definition (5.63) of $D_m(z_3, z_k)$ and condition (5.25) we see that the poles from the factor

$$(z_2 q_i^{m-1} - q_s z_k) \tag{5.74}$$

lie outside the integration contour for all $m \geq 2$.

We continue this evaluation procedure. At some point, we have to pick up a residue at the front. Assume that we have picked and evaluated simple residues at

$$\hat{z}_1, \dots, \hat{z}_{J-1} \tag{5.75}$$

during the first $J - 1$ integrations with

$$\hat{z}_j = q_i z_{j+1} \qquad j = 1, \dots, J - 1. \tag{5.76}$$

Here, the choice of indices $\{i, s\} = \{1, 2\}$ is necessarily the same for all residues. The remaining integral equals

$$\int_{|z_n|=1} \frac{dz_n}{2\pi i} \frac{\prod_{w\in W}(z_n - w)}{\prod_{u\in U}(z_n - u)} \prod_{n<k} D_1(z_n, z_k) \tag{5.77}$$

$$\vdots \tag{5.78}$$

$$\int_{|z_{J+1}|=1} \frac{dz_{J+1}}{2\pi i} \frac{\prod_{w\in W}(z_{J+1} - w)}{\prod_{u\in U}(z_{J+1} - u)} \prod_{J+1<k} D_1(z_{J+1}, z_k) \tag{5.79}$$

$$\int_{|z_J|=1} \frac{dz_J}{2\pi i} \prod_{l=0}^{J-1} \frac{\prod_{w\in W}(z_J q_i^l - w)}{\prod_{u\in U}(z_J q_i^l - u)} \prod_{J<k} D_J(z_J, z_k) \tag{5.80}$$

$$\times z_J^{J-1} q_i^{\frac{1}{2}(J+2)(J-1)} \frac{(q_i - 1)^J (1 - q_s)^{J-1}}{(q_i - q_s^{-1})^{J-1}(q_i^J - 1)} \prod_{j=1}^{J-1} \frac{(q_i^j - q_i^{-1} q_s^{-1})}{(q_i^j - q_s)} \tag{5.81}$$

$$\times f(q_i^{J-1} z_J, \dots, z_J, \dots, z_n), \tag{5.82}$$

Now pick up a pole at the front at

$$\hat{z}_J = q_i^{-d} u_0 \tag{5.83}$$

for some $d \in \{0, \ldots, J-1\}$ and some $u_0 \in U$ such that the pole lies inside the integration contour. We obtain the integral

$$\int_{|z_n|=1} \frac{dz_n}{2\pi i} \frac{\prod_{w\in W\cup\{\hat{z}_J, q_i^{-1}q_s^{-1}\hat{z}_J, \hat{z}_1, q_i q_s \hat{z}_a\}}(z_n - w)}{\prod_{u\in U\cup\{q_i^{-1}\hat{z}_J, q_s\hat{z}_J, q_s^{-1}\hat{z}_1, q_i\hat{z}_1\}}(z_n - u)} \prod_{n<k} D_1(z_n, z_k) \tag{5.84}$$

$$\vdots \tag{5.85}$$

$$\int_{|z_{J+1}|=1} \frac{dz_{J+1}}{2\pi i} \frac{\prod_{w\in W\cup\{\hat{z}_J, q_i^{-1}q_s^{-1}\hat{z}_J, \hat{z}_1, q_i q_s \hat{z}_a\}}(z_{J+1} - w)}{\prod_{u\in U\cup\{q_i^{-1}\hat{z}_J, q_s\hat{z}_J, q_s^{-1}\hat{z}_1, q_i\hat{z}_1\}}(z_{J+1} - u)} \prod_{J+1<k} D_1(z_{J+1}, z_k) \tag{5.86}$$

$$\times \prod_{j=0}^{J-1} \frac{\prod_{w\in W}(q_i^{j-d}u_0 - w)}{\prod_{u\in U\setminus\{u\}}(q_i^{j-d}u_0 - u)} \prod_{\substack{j=0\\ j\neq d}}^{J-1} \frac{1}{(q_i^{j-d} - 1)} q_i^{\frac{1}{2}(J+2)(J-1)-dJ} \tag{5.87}$$

$$\times \frac{(q_i - 1)^J (1 - q_s)^{J-1}}{(q_i - q_s^{-1})^{J-1}(q_i^J - 1)} \prod_{j=1}^{J-1} \frac{(q_i^j - q_i^{-1}q_s^{-1})}{(q_i^j - q_s)} \tag{5.88}$$

$$f(q_i^{J-1}q_i^{-d}u_0, \ldots, q_i^{-d}u_0, z_{J+1}, \ldots, z_n). \tag{5.89}$$

This integral is of the same form as the one (5.30) we started with. Hence Claim 1 is proved.

In a next step we show

Claim 2. *In the evaluation process described in Claim 1, only residue strips* (5.37) *with* $d_l = 0$ *as in* (5.40) *contribute in the sum of all residues.*

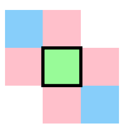
In order to prove this claim, we fix a stage l_0 as described in Claim 1. In stage l_0, we have to pick residues for the variables

$$z_{a_{l_0}}, \ldots, z_{b_{l_0}}. \tag{5.90}$$

We suppose $i_{l_0} = 2$. The other case is treated identically. The procedure (5.39), (5.40) in our stage l_0 yields the residue strip (5.48) with $l = l_0$. *After* the evaluation, the residues lie at

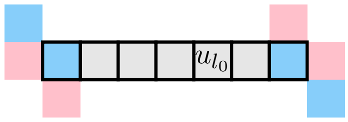
$$q_{i_{l_0}}^{-d_{l_0}} (q_{i_{l_0}}^{J_{l_0}-1} u_{l_0}, \ldots, u_{l_0}). \tag{5.91}$$

We have to compare different procedures to pick residues that yield the same final positions 5.91. We can, for example, achieve this by iteratively picking residues at the front, i.e. we choose $J = 1$ in the above calculation and repeat it. We call a residue strip of length 1 a residue box. We draw a residue box as



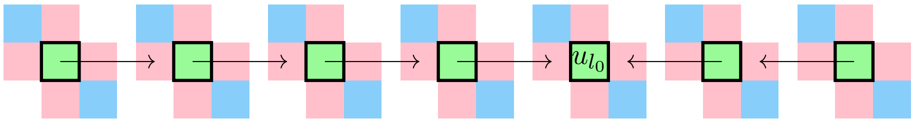
(5.92)

The green color signifies a zero of order two. Graphically, we want to compare, for instance,



(5.93)

to



(5.94)

All formulas in Claim 1 remain valid for evaluating residue boxes. Applying the formulas repeatedly, we see that, regardless of the order in which we pick residues on the west or on the east in (5.94), the two procedures (5.93) and (5.94) yield the same final sets U_{l_0+1}, W_{l_0+1}. This can also be seen from the two pictures using the graphical rule

(5.95)

However, the respective final symmetric functions, which the two procedures yield, only agree up to a factor of

$$(-1)^{d_{l_0}}. \tag{5.96}$$

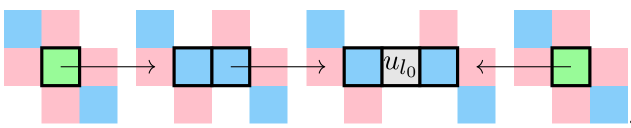
Graphically, the integer d_{l_0} equals the number of boxes westwards of u_{l_0}. In other words, there are cancellations possible between the different procedures in which one can pick residues that produce the same final residues (5.91) as the procedure defined by the strip (5.93).

Up to now, we can conclude the following:

Claim 3. *In the case $d_{l_0} = 0$, the procedure* (5.39), (5.40) *leads to the same result compared to evaluating residues at the front repeatedly. Graphically, in this case, the whole residue strip lies to the east of u_{l_0}.*

In the other extreme case $d_{l_0} = J_{l_0} - 1$, both procedures lead to the same result up to a factor of $(-1)^{J_{l_0}+1}$. Graphically, in this case, the whole residue strip lies to the west of u_{l_0}. It has the length J_{l_0}.

Of course, there are more procedures, by which we can obtain the final residues (5.91) for the variables (5.90). For instance, compared to the procedure (5.93) there is also the diagram



. (5.97)

Moreover, to characterize a procedure uniquely, we have to add to a diagram like (5.97) the specification of the order, in which to add the pieces to the west or east. This order was irrelevant in the above examples (5.93) (because there was only one piece) and (5.94) (because the result was independent of the order). From Claim 3 we see that the relative sign of the final symmetric function does only depend on the diagram associated to a procedure to pick residues, and not on the order in which we place pieces to the east or west. By what we have shown, the relative sign is equal to the product

$$\prod_{\mathcal{L}} (-1)^{\mathcal{L}-1} \tag{5.98}$$

where $\mathcal{L}$ runs over the lengths of the sub-strips in the diagram to the west of u_{l_0}, after cutting the sub-strip containing u_{l_0} directly to the east of u_0. For instance, the relative sign corresponding to any procedure associated to the diagram (5.97) is given by

$$(-1)^{1-1}(-1)^{2-1}(-1)^{(3-1)-1} = 1. \tag{5.99}$$

We want to show that the sum of all the final symmetric functions produced by the procedures of picking residues for the variables (5.90) yielding the final residues (5.91) equals zero *if* $d_{l_0} > 0$. So suppose $d_{l_0} > 0$. We want to show

$$\sum_x \operatorname{sign}(x) = 0 \tag{5.100}$$

where the sum goes over all procedures and the sign is the relative sign of the final function, computed in equation (5.98). To formalize the treatment of the cancellations, we introduce the set

$$\Gamma(J_{l_0}) = \{(\gamma_1, \ldots, \gamma_N) : N \in \mathbf{N}, \gamma_1, \ldots, \gamma_N \in \mathbf{N},\ \gamma_0 + \cdots + \gamma_N = J_{l_0}\} \tag{5.101}$$

of ordered partitions of the integer J_{l_0}. The integer J_{l_0} equals the length of the residue strip (5.93) and each element $\vec{\gamma} \in \Gamma(J_{l_0})$ corresponds to a unique way to cut the strip (5.93) into sub-strips: The N sub-strips have length $\gamma_1, \ldots, \gamma_N$, from west to east. For instance, the cutting (5.97) corresponds to the element

$$\vec{\gamma} = (1, 2, 3, 1) \in \Gamma(7). \tag{5.102}$$

Since the sign (5.98) only depends on the cutting of the residue strip, we obtain

$$\sum_x \operatorname{sign}(x) = \sum_{\vec{\gamma} \in \Gamma(J_{l_0})} \operatorname{wgt}(\vec{\gamma}) \operatorname{sign}(\vec{\gamma}). \tag{5.103}$$

Here, the sign $\operatorname{sign}(\vec{\gamma})$ can be calculated from equation (5.98) and the weight $\operatorname{wgt}(\vec{\gamma})$ is derived from the number of procedures corresponding to the diagram defined by $\vec{\gamma}$. We are going to formalize both now.

Firstly, we formalize the sign. Define the cutting map

$$\operatorname{cut} : \Gamma(J_{l_0}) \to \Gamma(d_0 + 1) \tag{5.104}$$

by demanding that, for each

$$\vec{\gamma} = (\gamma_1, \ldots, \gamma_N) \in \Gamma(J_{l_0}), \tag{5.105}$$

the element

$$\mathrm{cut}(\vec{\gamma}) = (\gamma'_1, \dots, \gamma'_{N'}) \in \Gamma(d_0 + 1) \tag{5.106}$$

satisfies

$$\gamma'_a = \gamma_a \qquad\qquad (a = 1, \dots, N' - 1). \tag{5.107}$$

Hence $\mathrm{cut}(\vec{\gamma})$ contains the lengths of the sub-strips to the west of u_{l_0} after cutting directly to the east of u_{l_0}. The sign corresponding to the cutting $\vec{\gamma}$, calculated via equation (5.98), is given by

$$\prod_a (-1)^{\left(\mathrm{cut}(\vec{\gamma})\right)_a - 1}. \tag{5.108}$$

Hence we have formalized the sign in (5.103).

It remains to formalize the weight appearing in (5.103). A procedure to pick residues associated to a diagram $\vec{\gamma}$ is fixed by specifying an order in which we place the sub-strips. The sub-strip containing u_{l_0} has to be placed first. Then one can alternate between placing sub-strips westwards or eastwards. We define the reordering bijection

$$\mathrm{reo} : \Gamma(J_{l_0}) \to \Gamma(J_{l_0}) \tag{5.109}$$

by the following condition. Fix a cutting

$$\vec{\gamma} \in \Gamma(J_{l_0}). \tag{5.110}$$

Let $b = b(\vec{\gamma})$ denote the number of sub-strips strictly westwards of the one containing u_{l_0}. Let $c = c(\vec{\gamma})$ denote the number of sub-strips strictly eastwards of the one containing u_{l_0}. We demand

$$\mathrm{reo}(\vec{\gamma}) = (B_0, B_1, \dots, B_b, C_1, \dots, C_c), \tag{5.111}$$

where B_0 is the length of the sub-strip containing u_{l_0}. The integers $B_1, \dots, B_b$ are equal to the respective lengths of the sub-strips strictly westwards of the one containing u_{l_0}, with increasing index in westward direction. The integers $C_1, \dots, C_c$ are equal to the respective lengths of the sub-strips strictly eastwards of the one containing u_{l_0}, with increasing index in eastward direction. For example, the element

$$\vec{\gamma} = (1, 2, 3, 1), \tag{5.112}$$

corresponding to the cutting depicted in (5.97), yields

$$\mathrm{reo}(\vec{\gamma}) = (3, 2, 1, 1) \qquad\qquad b = 2 \qquad\qquad c = 1. \tag{5.113}$$

The order of the components will be interpreted as the order in which we place the residue strips. All possible such orderings are given by permutations of the components of $\mathrm{reo}(\vec{\gamma})$ such that B_0 remains fixed and also the respective orderings of the elements in $(B_1, \dots, B_b)$ and $(C_1, \dots, C_c)$ remain fixed. Define the set of all (b, c)-shuffles by

$$S_{b,c} = \{\sigma \in S_{b+c} : \sigma(1) < \dots < \sigma(b),\ \sigma(b+1) < \dots < \sigma(b+c)\}. \tag{5.114}$$

An element $\sigma \in S_{b,c}$ acts on

$$\mathrm{reo}(\vec{\gamma}) = (B_0, B_1, \ldots, B_b, C_1, \ldots, C_c) \tag{5.115}$$

by

$$\sigma\ \mathrm{reo}(\vec{\gamma}) = \big(B_0, \sigma(B_1, \ldots, B_b, C_1, \ldots, C_c)\big), \tag{5.116}$$

where the action on the right is the usual one of the permutation $\sigma \in S_{b+c}$. The set

$$\{\sigma\, \mathrm{reo}(\vec{\gamma}) : \sigma \in S_{b,c}\} \tag{5.117}$$

parametrizes all possible procedures to pick residues that lead to a diagram specified by $\vec{\gamma} \in \Gamma(J_{l_0})$.

We still have to account for the fact that we can pick any of the remaining variables in (5.90), when picking up residues at the back as done in (5.39). When we place the k-th sub-strip with length

$$\big(\sigma\, \mathrm{reo}(\vec{\gamma})\big)_k > 1 \tag{5.118}$$

out of a tuple $\sigma\, \mathrm{reo}(\vec{\gamma})$ of N sub-strips, we have to choose

$$\big(\sigma\, \mathrm{reo}(\vec{\gamma})\big)_k - 1 \tag{5.119}$$

times a variable at the back (5.39). The pole can come from any of the variables we have not integrated out so far. The remaining sub-strips, that have not yet been placed, correspond to a number of

$$\big(\sigma\, \mathrm{reo}(\vec{\gamma})\big)_{k+1} + \cdots + \big(\sigma\, \mathrm{reo}(\vec{\gamma})\big)_N \tag{5.120}$$

variables from (5.90). Hence we have to account for

$$\prod_{h=1}^{\sigma(\mathrm{reo}\,\vec{\gamma})_k - 1} \Big(\sum_{j=k+1}^{N} \sigma(\mathrm{reo}\,\vec{\gamma})_j + \sigma(\mathrm{reo}\,\vec{\gamma})_k - h \Big) \tag{5.121}$$

choices when placing the sub-strip corresponding to (5.118). This formula extends to the case

$$\big(\sigma\, \mathrm{reo}(\vec{\gamma})\big)_k = 1, \tag{5.122}$$

in which we pick up a residue at the front, which does not involve an additional choice. We obtain the formula

$$\mathrm{wgt}(\vec{\gamma}) = \sum_{\sigma \in S_{b,c}} \prod_{k=1}^{N} \Bigg(\prod_{h=1}^{\sigma(\mathrm{reo}\,\vec{\gamma})_k - 1} \Big(\sum_{j=k}^{N} \sigma(\mathrm{reo}\,\vec{\gamma})_j - h \Big) \Bigg), \tag{5.123}$$

where $b = b(\vec{\gamma})$, $c = c(\vec{\gamma})$ and $\vec{\gamma} \in \Gamma(J_{l_0})$ consists of $N = N(\vec{\gamma})$ components.

We have formalized both the weight and the relative sign of a diagram $\vec{\gamma} \in \Gamma(J_{l_0})$. In order to prove Claim 2, we thus have to show that

$$\sum_{\vec{\gamma}\in\Gamma(J_{l_0})} \operatorname{wgt}(\vec{\gamma})\operatorname{sign}(\vec{\gamma}) = \sum_{\vec{\gamma}\in\Gamma(J_{l_0})} \Bigg(\prod_a (-1)^{\left(\operatorname{cut}(\vec{\gamma})\right)_a - 1} \tag{5.124}$$

$$\sum_{\sigma\in S_{b,c}} \prod_{k=1}^{N} \Bigg(\prod_{h=1}^{\sigma(\operatorname{reo}\vec{\gamma})_k - 1} \Big(\sum_{j=k}^{N} \sigma(\operatorname{reo}\vec{\gamma})_j - h \Big) \Bigg) \Bigg) \tag{5.125}$$

vanishes. We do so, by splitting the sum in pairs of terms that cancel each other. Fix one $\vec{\gamma} \in \Gamma(J_{l_0})$ with $\gamma_1 > 1$. Define $\vec{\gamma}' \in \Gamma(J_{l_0})$ by

$$\gamma_1' = 1 \qquad \gamma_2' = \gamma_1 - 1 \qquad \gamma_k' = \gamma_k \quad (k \geq 2) \tag{5.126}$$

The map

$$\{\vec{\gamma} \in \Gamma_{J_{l_0}} : \gamma_1 > 1\} \to \{\vec{\gamma} \in \Gamma_{J_{l_0}} : \gamma_1 = 1\} \tag{5.127}$$

$$\vec{\gamma} \mapsto \vec{\gamma}' \tag{5.128}$$

is a bijection. Its inverse sums the two first components. The sub-strips described by $\vec{\gamma}'$ are obtained from the sub-strips described by $\vec{\gamma}$ by cutting off the westernmost box. To prove that (5.124) vanishes, it suffices to show

$$\operatorname{wgt}(\vec{\gamma})\operatorname{sign}(\vec{\gamma}) = -\operatorname{wgt}(\vec{\gamma}')\operatorname{sign}(\vec{\gamma}'). \tag{5.129}$$

By our assumption $d_{l_0} > 0$, the westernmost box, we cut away, lies to the west of u_{l_0}. From formula (5.108), we thus see that the sign of $\vec{\gamma}$ and $\vec{\gamma}'$ are opposite. We are left to show that

$$\operatorname{wgt}(\vec{\gamma}) = \operatorname{wgt}(\vec{\gamma}'). \tag{5.130}$$

This equation reads

$$\sum_{\sigma\in S_{b,c}} \prod_{k=1}^{N} \Bigg(\prod_{h=1}^{\sigma(\operatorname{reo}\vec{\gamma})_k - 1} \Big(\sum_{j=k}^{N} \sigma(\operatorname{reo}\vec{\gamma})_j - h \Big) \Bigg) \tag{5.131}$$

$$= \sum_{\sigma\in S_{b+1,c}} \prod_{k=1}^{N+1} \Bigg(\prod_{h=1}^{\sigma(\operatorname{reo}\vec{\gamma}')_k - 1} \Big(\sum_{j=k}^{N+1} \sigma(\operatorname{reo}\vec{\gamma}')_j - h \Big) \Bigg), \tag{5.132}$$

We want to split the sum on the right hand side. Write

$$\operatorname{reo}(\vec{\gamma}) = (B_0, B_1, \ldots, B_b, C_1, \ldots, C_c) \qquad b + c + 1 = N \tag{5.133}$$

We decompose

$$S_{b+1,c} = \bigcup_{\sigma\in S_{b,c}} \{\sigma_\lambda : \lambda = 1, \ldots, N + 1 - \sigma(r)\}, \tag{5.134}$$

where, for $j = 1, \dots, N+1$,

$$\sigma_\lambda(j) = \begin{cases} \sigma(j), & j = 1, \dots, b \\ \sigma(b) + \lambda, & j = b+1 \\ \sigma(j-1), & j = b+2, \dots, \sigma(b) + \lambda \\ \sigma(j-1)+1, & j > \sigma(b) + \lambda + 1. \end{cases} \tag{5.135}$$

Fix $\sigma \in S_{b,c}$. It suffices to show

$$\prod_{k=1}^{N} \Big(\prod_{h=1}^{\sigma(\operatorname{reo}\vec{\gamma})_k - 1} \Big(\sum_{j=k}^{N} \sigma(\operatorname{reo}\vec{\gamma})_j - h \Big) \Big) \tag{5.136}$$

$$= \sum_{\lambda=1}^{N+1-\sigma(r)} \prod_{k=1}^{N+1} \Big(\prod_{h=1}^{\sigma_\lambda(\operatorname{reo}\vec{\gamma}')_k - 1} \Big(\sum_{j=k}^{N+1} \sigma_\lambda(\operatorname{reo}\vec{\gamma}')_j - h \Big) \Big), \tag{5.137}$$

In (5.133), we have $B_b > 1$ and

$$\operatorname{reo}(\vec{\gamma}') = (B'_0, B'_1, \dots, B'_{b+1}, C'_1, \dots, C'_c) \tag{5.138}$$

$$= (B_0, B_1, \dots, B_{b-1}, B_b - 1, 1, C_1, \dots, C_c). \tag{5.139}$$

Set $B = \sigma(b)$, which is the step in which we place the strip B'_b. From the explicit description of σ_λ in equation (5.135), we obtain that the right hand side of equation (5.136) is given by

$$\sum_{\lambda=1}^{N+1-B} \prod_{k=1}^{B-1} \Big(\prod_{h=1}^{\sigma(\operatorname{reo}(\vec{\gamma}))_k - 1} \Big(\sum_{j=k}^{N} \sigma(\operatorname{reo}(\vec{\gamma}))_j - h \Big) \Big) \tag{5.140}$$

$$\Big(\prod_{h=1}^{\sigma(\operatorname{reo}(\vec{\gamma}))_B - 2} \Big(\sum_{j=B}^{N} \sigma(\operatorname{reo}(\vec{\gamma}))_j - h \Big) \Big) \tag{5.141}$$

$$\prod_{k=B+1}^{B+\lambda-1} \Big(\prod_{h=0}^{\sigma(\operatorname{reo}(\vec{\gamma}))_k - 2} \Big(\sum_{j=k}^{N} \sigma(\operatorname{reo}(\vec{\gamma}))_j - h \Big) \Big) \tag{5.142}$$

$$\prod_{k=B+\lambda}^{N} \Big(\prod_{h=1}^{\sigma(\operatorname{reo}(\vec{\gamma}))_k - 1} \Big(\sum_{j=k}^{N} \sigma(\operatorname{reo}(\vec{\gamma}))_j - h \Big) \Big). \tag{5.143}$$

This equals the left hand side of equation (5.136) up to a factor of

$$\Big(\sum_{j=B}^{N} \sigma(\operatorname{reo}(\vec{\gamma}))_j - \sigma(\operatorname{reo}(\vec{\gamma}))_B + 1 \Big)^{-1} \tag{5.144}$$

$$\times \sum_{\lambda=1}^{N+1-B} \prod_{k=B+1}^{B+\lambda-1} \frac{\sum_{j=k}^{N} \sigma(\operatorname{reo}(\vec{\gamma}))_j}{\sum_{j=k}^{N} \sigma(\operatorname{reo}(\vec{\gamma}))_j - \sigma(\operatorname{reo}(\vec{\gamma}))_k + 1}. \tag{5.145}$$

We are left to show that this factor equals 1. Define

$$x_j = \sigma(\mathrm{reo}(\vec{\gamma}))_j \qquad (j = 1, \dots, M) \qquad M = N - B. \tag{5.146}$$

We have to show

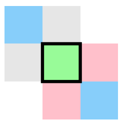
$$\sum_{j=1}^{M} x_j = \sum_{\lambda=1}^{M} \prod_{k=1}^{\lambda} \frac{\sum_{j=k}^{M} x_j}{\sum_{j=k+1}^{M} x_j + 1}. \tag{5.147}$$

This equation is true. Its right hand side equals

$$\frac{x_1 + \dots + x_M}{x_2 + \dots + x_M + 1}\left(1 + \frac{x_2 + \dots + x_M}{x_3 + \dots + x_M + 1}\left(\cdots\left(1 + \frac{x_{M-1} + x_M}{x_M + 1}\left(1 + \frac{x_M}{1}\right)\right)\right)\right) \tag{5.148}$$

which equals the left hand side. We have proved Claim 2.

So far, we can conclude the following: All the iterated residues appearing in (5.31) can be taken as described in Claim 1. Moreover, we can always suppose $d_l = 0$ in equation (5.40). In combination with the first part of Claim 3, we see that we also may suppose $J_l = 1$, i.e. it suffices to consider residues picked up at the front. Graphically, we obtain all relevant residues by iteratively placing boxes.



(5.149)

Moreover, we can discard any residues coming from poles of the form $q_i^{-1}u_0$ since they either lie outside the integration contour, or are part of a zero sum (for $J_l = 2$ and $d_l = 1$) described above. Accordingly, we have colored the corresponding boxes in (5.149) in gray.

Now we prove by induction, that all residues

$$(\hat{z}_n, \dots, \hat{z}_1) \in R \tag{5.150}$$

in equation (5.31) are simple residues and can be taken of the form

$$\{\hat{z}_n, \dots, \hat{z}_1\} = \{z_\square^\alpha : \square \in Y_\alpha, \alpha = 1, \dots, r\} \tag{5.151}$$

where $\vec{Y}$ is an r-tuple of Young diagrams of total size n. Here, $z_\square^\alpha$ was defined in equation 5.15. We start our residue evaluation with the sets

$$U = \{u_1, \dots, u_r\} \qquad W = \emptyset. \tag{5.152}$$

We start with the symmetric function

$$f(z_1, \dots, z_n) = \prod_{j=1}^{n}\left(\frac{1}{z_j}F(z_j)G(z_j^{-1})\prod_{\alpha=1}^{r}\frac{-u_\alpha z_j}{q_1 q_2 z_j - u_\alpha}\right) \tag{5.153}$$

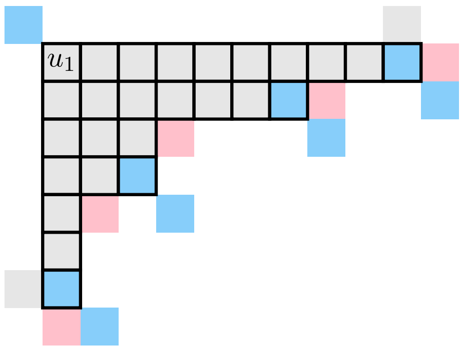
By condition (5.5), it is regular at zero. Its poles in z_j lie outside the integration contour for the z_j integration and do not depend on the other variables. Because of condition (5.17), the poles from U do not interact and we may suppose $r = 1$. Depict the pole at u_1 as

(5.154)

By what we have proved so far, we obtain all iterated residues, by placing boxes (5.149) at poles of the diagram we have obtained so far. All the possibilities for the next two steps are for instance

(5.155)

The double zero at the center of a box (5.149) cancels with the pole we place the box at to give a simple zero. It is clear, that the resulting diagram has the form of a Young diagram. Indeed, if we have for example evaluated residues according to



(5.156)

then placing a box (5.149) at one of the red poles in (5.156) yields again a Young diagram with the same structure of the poles and zeros as the one in (5.156).

We have shown that all residues appearing in the sum (5.31) are of the form (5.151). By permutation invariance of the integrand, all possible assignments of variables to boxes in a given Young diagram appear. This cancels the factorial in front of the integral. By condition (5.17), all residues are simple.

We obtain that $\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})$ is a sum

$$\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) = \sum_{|\vec{Y}|=n} \mathcal{Z}_{\vec{Y}}(q_1, q_2, \vec{u}; \vec{F}, \vec{G}), \tag{5.157}$$

of iterated, simple residues

$$\mathcal{Z}_{\vec{Y}}(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) := \left(\frac{1 - q_1 q_2}{(1-q_1)(1-q_2)}\right)^n \mathcal{J}(\hat{z}_1, \ldots, \hat{z}_n; \vec{F}, \vec{G}) \tag{5.158}$$

$$\lim_{\substack{z_j \to \hat{z}_j \\ j=1,\ldots,n}} \left(\prod_{j=1}^{n} (z_j - \hat{z}_j) \quad \frac{\mathcal{I}(z_1, \ldots, z_n; \vec{u})}{z_1 \cdots z_n} \right), \tag{5.159}$$

where

$$\{\hat{z}_1, \ldots, \hat{z}_n\} = \{z^{\alpha}_{\Box} : \Box \in Y_\alpha, \alpha = 1, \ldots, r\}, \tag{5.160}$$

in any order. The evaluation of these simple residues is performed in chapter B in the appendix. There, we show that line (5.159) equals

$$\prod_{\alpha,\beta=1}^{r} \left(\prod_{\Box \in Y_\alpha} \frac{1}{1 - u_\alpha u_\beta^{-1} q_1^{l_{Y_\alpha}(\Box)+1} q_2^{-a_{Y_\beta}(\Box)}} \right. \tag{5.161}$$

$$\left. \prod_{\boxtimes \in Y_\beta} \frac{1}{1 - u_\alpha u_\beta^{-1} q_1^{-l_{Y_\beta}(\boxtimes)} q_2^{a_{Y_\alpha}(\boxtimes)+1}} \right). \tag{5.162}$$

This proves equation (5.18) in Theorem 5.2.1.

The integral (5.10) remains invariant under exchange of q_1 and q_2. The set of residues (5.160) remains invariant, if we exchange q_1 and q_2 and transpose all diagrams Y_α, $\alpha = 1, \ldots, r$. We obtain

$$\sum_{|\vec{Y}|=n} \mathcal{Z}_{Y_1,\ldots,Y_r}(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) = \sum_{|\vec{Y}|=n} \mathcal{Z}_{Y_1^T,\ldots Y_r^T}(q_2, q_1, \vec{u}; \vec{F}, \vec{G}). \tag{5.163}$$

We have $a_Y(x, y) = l_{Y^T}(y, x)$ for all $x, y \in \mathbf{N}$ and hence equation (5.21) follows from equation (5.18). The proof of Theorem 5.2.1 is complete. □

5.3. Estimate of the Integral

In this section, we want to analyze the growth of the coefficients $\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})$, defined in equation (5.10), in the limit $n \to \infty$.

Note that the double product in line (5.8) is nonnegative. This motivates the language of probability theory. Set $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Define

$$f : \mathbf{T} \to \mathbf{R} \cup \{\infty\} \qquad f(\theta) = -\log \frac{|e^{i\theta} - 1|\,|e^{i\theta} - q_1 q_2|}{|e^{i\theta} - q_1|\,|e^{i\theta} - q_2|}. \tag{5.164}$$

For each integer $n \geq 1$, let $\mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$ denote the torus. We want to change variables in the integral (5.10) as

$$z_j = \rho e^{i\theta_j} \qquad (j \in \{1, \ldots, n\}) \qquad \theta = (\theta_1, \ldots, \theta_n) \in \mathbf{T}^n. \tag{5.165}$$

In the new variables, the double product in line (5.8) can be written as

$$e^{-\sum_{j \neq k} f(\theta_k - \theta_j)}. \tag{5.166}$$

Hence, for each $n \in \mathbf{N}$, we define a probability measure P_n on $\mathbf{T}^n$ by its density

$$p_n(\theta) = \frac{1}{Z_n} e^{-\sum_{j \neq k} f(\theta_k - \theta_j)}, \tag{5.167}$$

with respect to the Lebesgue measure. Here the normalization factor Z_n is given by

$$Z_n = \int_{\mathbf{T}^n} d\theta e^{-\sum_{j\neq k} f(\theta_k - \theta_j)}. \tag{5.168}$$

Denote, for each $n \geq 1$, the expectation functional of the probability measure P_n by $\mathbf{E}_n$. Moreover, define

$$g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta) = F(\rho e^{i\theta}) G(\rho^{-1} e^{-i\theta}) \prod_{\alpha=1}^{r} \frac{-u_\alpha \rho e^{i\theta}}{(\rho e^{i\theta} - u_\alpha)(q_1 q_2 \rho e^{i\theta} - u_\alpha)}. \tag{5.169}$$

We can now write the integral (5.10) as

$$\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) = \mathbf{E}_n \left[\prod_{j=1}^{n} g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta_j) \right] \tag{5.170}$$

$$\times \frac{1}{n!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n \frac{Z_n}{(2\pi)^n}. \tag{5.171}$$

The factor in line 5.171 is positive. Denote it by

$$a_n = \frac{1}{n!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n \frac{Z_n}{(2\pi)^n}. \tag{5.172}$$

By the triangle inequality,

$$|\,\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})| \leq a_n \, \mathbf{E}_n \left[\prod_{j=1}^{n} \left| g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta_j) \right| \right]. \tag{5.173}$$

By changing variables from $\theta \in \mathbf{T}$ to

$$w_j = e^{i\theta_j} \qquad (j \in \{1, \ldots, n\}) \qquad \theta = (\theta_1, \ldots, \theta_n) \in \mathbf{T}^n \tag{5.174}$$

in the integral in the definition (5.168) of Z_n, the last line (5.171) can be written as

$$a_n = \frac{1}{n!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n \int_{C_n(1)} \prod_{j=1}^{n} \frac{dw_j}{2\pi i w_j} \prod_{j\neq k} \frac{(w_j - w_k)(w_j - q_1 q_2 w_k)}{(w_j - q_1 w_k)(w_j - q_2 w_k)}. \tag{5.175}$$

The growth of those coefficients is known. By [10] the coefficients a_n, $n \geq 0$, are the coefficients of the power series

$$\sum_{n\geq 0} a_n z^n = \exp\left(\sum_{n\geq 1} \frac{1 - q_1^n q_2^n}{(1 - q_1^n)(1 - q_2^n)} \frac{z^n}{n} \right). \tag{5.176}$$

By our assumptions (5.1) and (5.2) on q_1 and q_2, we have

$$0 \leq \frac{1 - q_1^n q_2^n}{(1 - q_1^n)(1 - q_2^n)} \to 1 \qquad (n \to \infty). \tag{5.177}$$

Hence the power series (5.176) has radius of convergence equal to one and

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1. \tag{5.178}$$

It remains to study the large n behavior of the expectation values

$$\mathbf{E}_n\left[\prod_{j=1}^{n}\left|g(q_1,q_2,\vec{u};\vec{F},\vec{G};\rho,\theta_j)\right|\right]. \tag{5.179}$$

To this end, we claim

Theorem 5.3.1. *Let h be a continuous, real-valued function on the torus $\mathbb{T}$. We have*

$$\frac{1}{n}\log\mathbf{E}_n\left[e^{\sum_j h(\theta_j)}\right] \to \frac{1}{2\pi}\int_{\mathbf{T}} h(\theta)d\theta \qquad (n\to\infty). \tag{5.180}$$

The proof of this theorem uses ideas from potential theory and is postponed to chapter 6. If we apply this theorem to the continuous, real-valued function

$$h(q_1,q_2,\vec{u};\vec{F},\vec{G};\rho,\theta) = \log|g(q_1,q_2,\vec{u};\vec{F},\vec{G};\rho,\theta)| \qquad (\theta\in\mathbf{T}), \tag{5.181}$$

we obtain from (5.173), with the help of (5.178), the estimate

$$\limsup_{n\to\infty} |\mathcal{Z}_n(q_1,q_2,\vec{u};\vec{F},\vec{G})|^{\frac{1}{n}} \le \exp\left(\frac{1}{2\pi}\int_{\mathbf{T}} h(q_1,q_2,\vec{u};\vec{F},\vec{G};\rho,\theta)d\theta\right). \tag{5.182}$$

The integral on the right hand side can be evaluated explicitly. By equation (6.19) in chapter 6, we have

$$\frac{1}{2\pi}\int_{\mathbf{T}} h(q_1,q_2,\vec{u};\vec{F},\vec{G};\rho,\theta)d\theta = \sum_{k=1}^{\deg F}\max\{\log\rho,\log|F_k|\} \tag{5.183}$$

$$+\sum_{k=1}^{\deg G}\max\{\log\rho^{-1},\log|G_k|\} \tag{5.184}$$

$$+\sum_{\alpha=1}^{r}\Bigg(\log|u_\alpha|+\log\rho-\max\{\log\rho,\log|u_\alpha|\} \tag{5.185}$$

$$-\max\{\log q_1q_2\rho,\log|u_\alpha|\}\Bigg). \tag{5.186}$$

By conditions (5.6) on ρ and (5.1) on q_1,q_2, we have

$$|u_\alpha|<\rho \qquad |u_\alpha|>\rho q_1q_2 \qquad (\alpha\in\{1,\dots,r\}). \tag{5.187}$$

Hence the sum over $\alpha=1,\dots,r$ in lines (5.185) and (5.186) equals zero. We have proved

Theorem 5.3.2. *Let*

$$\rho_- = \max_{\alpha=1,\dots,r} |u_\alpha| \qquad \rho_+ = \min\{|q_1|^{-1}, |q_2|^{-1}\} \min_{\alpha=1,\dots,r} |u_\alpha|. \tag{5.188}$$

denote the bounds in condition (5.6). *We have*

$$\limsup_{n\to\infty} |\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})|^{\frac{1}{n}} \tag{5.189}$$

$$\leq \inf_{\rho\in(\rho_-,\rho_+)} \left(\prod_{k=1}^{s} \max\{\rho, |F_k|\} \prod_{k=1}^{s'} \max\{\rho^{-1}, |G_k|\} \right) \tag{5.190}$$

In case one of the polynomials F or G is constant, one can make the optimization in ρ more explicit by letting ρ tend to the bounds ρ_+ or ρ_-, respectively: The bound in equation (5.190) specializes to

$$\begin{cases} \prod_{k=1}^{s'} \max\{|q_1||u_1|^{-1}, \dots, |q_1||u_r|^{-1}, |q_2||u_1|^{-1}, \dots, |q_2||u_r|^{-1}, |G_k|\} & \text{if } F = 1 \\ \prod_{k=1}^{s} \max\{|u_1|, \dots, |u_r|, |F_k|\} & \text{if } G = 1 \\ 1 & \text{if } F = G = 1 \end{cases} \tag{5.191}$$

We conclude this section, by applying our results to the power series

$$\mathcal{Z}(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; z) = \sum_{n\geq 0} \mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) z^n. \tag{5.192}$$

We have

Theorem 5.3.3. *Let q_1 and q_2 be two complex numbers with $|q_1| < 1$ and $|q_2| < 1$. Assume either $q_1 = \overline{q_2}$ or $q_1, q_2 \in (0,1)$. Fix three integers r, s, s' with*

$$1 \leq r \qquad 0 \leq s \qquad 0 \leq s' \leq r-1. \tag{5.193}$$

Define the functions

$$\rho_-(\vec{u}) = \max_{\alpha=1,\dots,r} |u_\alpha| \tag{5.194}$$

$$\rho_+(\vec{u}) = \min\{|q_1|^{-1}, |q_2|^{-1}\} \min_{\alpha=1,\dots,r} |u_\alpha| \qquad (\vec{u} \in \mathbf{C}^r). \tag{5.195}$$

The power series

$$\mathcal{Z}(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; z) = \sum_{n\geq 0} \mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) z^n \tag{5.196}$$

with coefficients defined in equation (5.10) *converges to an analytic function in the variables $(\vec{u}, \vec{F}, \vec{G}, z) \in \mathbf{C}^r \times \mathbf{C}^s \times \mathbf{C}^{s'} \times \mathbf{C}$ on the domain*

$$\mathcal{D} \subset \mathbf{C}^r \times \mathbf{C}^s \times \mathbf{C}^{s'} \times \mathbf{C} \tag{5.197}$$

defined by the conditions

$$\rho_-(\vec{u}) < \rho_+(\vec{u}) \tag{5.198}$$

$$|z| < \sup_{\rho \in (\rho_-(\vec{u}), \rho_+(\vec{u}))} \left(\prod_{k=1}^{s} \min\{\rho^{-1}, |F_k|^{-1}\} \prod_{k=1}^{s'} \min\{\rho, |G_k|^{-1}\} \right) \tag{5.199}$$

The latter bound on $|z|$ has the three special cases

$$\begin{cases} \prod_{k=1}^{s'} \min\{|q_1|^{-1}|u_1|, \dots, |q_1|^{-1}|u_r|, |q_2|^{-1}|u_1|, \dots, |q_2|^{-1}|u_r|, |G_k|^{-1}\} & \text{if } s = 0 \\ \prod_{k=1}^{s} \min\{|u_1|^{-1}, \dots, |u_r|^{-1}, |F_k|^{-1}\} & \text{if } s' = 0 \\ 1 & \text{if } s = s' = 0. \end{cases} \tag{5.200}$$

Proof. For fixed $\vec{u}$ such that condition (5.198) holds, and fixed $(\vec{F}, \vec{G}) \in \mathbf{C}^s \times \mathbf{C}^{s'}$, convergence and analyticity in z on (5.199) follows from Theorem 5.3.2.

Now fix $(\vec{u}_0, \vec{F}_0, \vec{G}_0) \in \mathbf{C}^r \times \mathbf{C}^s \times C^{s'}$ with

$$\rho_-(\vec{u}_0) < \rho_+(\vec{u}_0). \tag{5.201}$$

Fix an open set U and a compact set K with

$$(\vec{u}_0, \vec{F}_0, \vec{G}_0) \in U \subset K \subset \{(\vec{u}, \vec{F}, \vec{G}) \in \mathbf{C}^r \times \mathbf{C}^s \times \mathbf{C}^{s'} : \rho_-(\vec{u}) < \rho_+(\vec{u})\} \tag{5.202}$$

By continuity of the functions $\rho_-(\vec{u})$ and $\rho_+(\vec{u})$, we can shrink the sets $U \subset K$ until there is a $\rho > 0$ for which

$$\rho_-(\vec{u}) < \rho < \rho_+(\vec{u}) \qquad \forall\ (\vec{u}, \vec{F}, \vec{G}) \in K. \tag{5.203}$$

Fix such a ρ. Recall the function $g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta)$ defined in equation (5.169). We estimate

$$|g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta)| \le C(K) := \sup_{(\vec{u}, \vec{F}, \vec{G}, \theta) \in K \times \mathbf{T}} |g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta)|. \tag{5.204}$$

By continuity of g and compactness of $K \times \mathbf{T}$, the bound $C(K)$ is finite. Starting again from equation (5.173), we can estimate

$$|\,\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G})| \le a_n C(K)^n \qquad (n \ge 0) \tag{5.205}$$

uniformly in $(\vec{u}, \vec{F}, \vec{G}) \in K$. By (5.178), the power series (5.196) now converges for

$$|z| < \lambda := C(K)^{-1} \tag{5.206}$$

uniformly in the neighborhood U of $(\vec{u}_0, \vec{F}_0, \vec{G}_0)$. Hence

$$(\vec{u}, \vec{F}, \vec{G}, z) \mapsto \mathcal{Z}(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; z) \tag{5.207}$$

is analytic on

$$U \times \{z \in \mathbf{C} : |z| < \lambda\}. \tag{5.208}$$

Set

$$\Lambda = \inf_{(\vec{u},\vec{F},\vec{G}) \in U} \sup_{\rho \in (\rho_-(\vec{u}),\rho_+(\vec{u}))} \left(\prod_{k=1}^{s} \min\{\rho^{-1}, |F_k|^{-1}\} \prod_{k=1}^{s'} \min\{\rho, |G_k|^{-1}\} \right). \tag{5.209}$$

We already know, that for each fixed $(\vec{u}, \vec{F}, \vec{G}) \in U$, the map

$$z \to \mathcal{Z}(q_1, q_2; \vec{u}; \vec{F}, \vec{G}; z) \tag{5.210}$$

is analytic for $|z| < \Lambda$. By Hartog's Lemma [13, Theorem 2], the map (5.207) is analytic on

$$U \times \{z \in \mathbf{C} : |z| < \Lambda\}. \tag{5.211}$$

The maps $\rho_-(\vec{u})$ and $\rho_+(\vec{u})$ are both continuous. We have

$$\Lambda \to \sup_{\rho \in (\rho_-(\vec{u}_0),\rho_+(\vec{u}_0))} \left(\prod_{k=1}^{s} \min\{\rho^{-1}, |(\vec{F}_0)_k|^{-1}\} \prod_{k=1}^{s'} \min\{\rho, |(\vec{G}_0)_k|^{-1}\} \right) \tag{5.212}$$

as we shrink the neighborhood U of $(\vec{u}_0, \vec{F}_0, \vec{G}_0)$. Hence analyticity on the whole domain $\mathcal{D}$ follows. □

6. Random Matrices and Potential Theory

In this chapter, we prove Theorem 5.3.1 using methods from potential theory. In section 6.1, we describe the historical context of random matrix theory. In section 6.2, we pose a problem from potential theory. In section 6.3, we solve the problem from the preceding section. Finally, in section 6.4, we apply our findings to prove Theorem 6.2.1.

In this chapter, the variables we consider are not related to the variables of the preceding body of the text.

6.1. Random Matrices and Toeplitz Determinants

In this section, we provide the historical context of our analysis. We follow the exposition in [27].

Consider the set $U(n)$ of unitary $n \times n$ matrices. The normalized Haar measure dU on $U(n)$ allows one to consider random unitary matrices. Let $\mathbf{T}^n = \mathbf{R}^n/(2\pi\mathbf{Z})^n$ denote the torus. The eigenvalues of a unitary matrix are of the form $(e^{i\theta_1}, \ldots, e^{i\theta_n})$, where $\theta \in \mathbf{T}^n$. Let $h : \mathbf{T} \to \mathbf{C}$ be a continuous function on $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. The Weyl integral formula says [49]

$$\int_{U(n)} e^{\operatorname{Tr} h(U)} dU = \frac{1}{n!} \int_{\mathbf{T}^n} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi} \, e^{\sum_{j=1}^{n} h(\theta_j)} \prod_{j \neq k} |e^{i\theta_j} - e^{i\theta_k}|. \tag{6.1}$$

A similar integral appears in the theory of Toeplitz determinants. Let $f \in L^1(\mathbf{T})$ be an integrable function on the torus. Let

$$f_k = \frac{1}{2\pi} \int_T f(\theta) e^{-ik\theta} d\theta \qquad (k \in \mathbf{Z}) \tag{6.2}$$

denote its Fourier coefficients. The corresponding Toeplitz determinant $D_n(f)$ is given by [46]

$$D_n(f) = \det(f_{j-k})_{0 \leq j,k \leq n-1} \tag{6.3}$$

$$= \frac{1}{n!} \int_{\mathbf{T}^n} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi} \prod_{j=1}^{n} f(\theta_j) \prod_{j \neq k} |e^{i\theta_j} - e^{i\theta_k}|. \tag{6.4}$$

For $f(\theta) = e^{h(\theta)}$, where $h : \mathbf{T} \to \mathbf{R}$ is a continuous, real-valued function, whose Fourier coefficients $h_k, k \in \mathbf{Z}$ satisfy

$$\sum_{k \in \mathbf{Z}} |k| \, |h_k|^2 < \infty, \tag{6.5}$$

Szegö proved [47] the asymptotic formula

$$\log D_n(f) = n \int_{\mathbf{T}} h(\theta) \frac{d\theta}{2\pi} + \sum_{k=1}^{\infty} k h_k h_{-k} + o(1) \qquad (n \to \infty). \tag{6.6}$$

This result looks like an improved version of our Theorem 6.2.1. However, the double product in line (6.4) is different from ours. In the following sections, we will use techniques from potential theory to prove our result. We adapt methods from [27], where random Hermitian matrices were considered.

6.2. Potential Theory

Let q_1 and q_2 be two complex numbers with

$$|q_1| < 1 \qquad |q_2| < 1. \tag{6.7}$$

Assume either that q_1 and q_2 are complex conjugate or that both are real numbers in $(0, 1)$. In either case $q_1 q_2 = |q_1 q_2| \in (0, 1)$. Define the function

$$f : \mathbf{T} \to \mathbf{R} \cup \{\infty\} \qquad f(\theta) = -\log \frac{|e^{i\theta} - 1|\,|e^{i\theta} - q_1 q_2|}{|e^{i\theta} - q_1|\,|e^{i\theta} - q_2|}. \tag{6.8}$$

For $n \geq 0$, define

$$Z_n = \int_{\mathbf{T}^n} d\theta \; e^{-\sum_{j \neq k} f(\theta_k - \theta_j)}. \tag{6.9}$$

On the torus $\mathbf{T}^n$, we define a probability measure P_n by its density function p_n with respect to the Lebesgue measure:

$$p_n(\theta) = \frac{1}{Z_n} e^{-\sum_{j \neq k} f(\theta_k - \theta_j)} \qquad (\theta \in \mathbf{T}^n). \tag{6.10}$$

Denote the corresponding expectation functional by $\mathbf{E}_n$. We claim the following.

Theorem 6.2.1. *Let*

$$h : \mathbf{T} \to \mathbf{R} \tag{6.11}$$

be a continuous, real-valued function on the torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. We have

$$\frac{1}{n} \log \mathbf{E}_n \left[e^{\sum_{j=1}^{n} h(\theta_j)} \right] \to \frac{1}{2\pi} \int_T h(\theta) d\theta \qquad (n \to \infty). \tag{6.12}$$

Before we prove this theorem, we discuss its interpretation. The measure defined by

$$p_n(\theta) = \frac{1}{Z_n} e^{-2\sum_{j<k} f(\theta_k - \theta_j)} \qquad (\theta \in \mathbf{T}^n) \tag{6.13}$$

is the canonical measure, at inverse temperature $\beta = 2$, of an ensemble of n particles living on $\mathbf{T}$ and interacting through the logarithmic potential given by the function

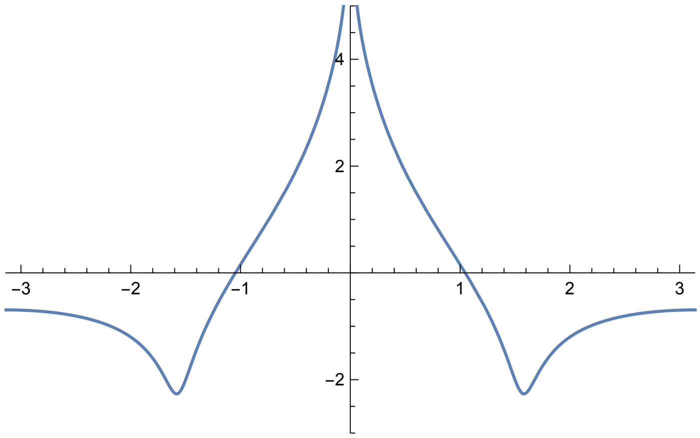


Figure 6.1.: The function $f(\theta)$ for $\theta \in [-\pi, \pi]$ and the choices $|q_1| = |q_2| = 0.9$ and $\arg q_1 = \pi/2$. The energetically favorable distance θ_0 corresponds to the position of the local minimum on the right.

f. For some parameters q_1 and q_2 which are complex conjugate, it is plotted in figure 6.2. For the case $q_1, q_2 \in (0, 1)$, the function looks similar. We see that the interaction potential is repulsive at short distances and defines an energetically preferred distance θ_0 between the particles. Hence, we expect the particles to align with that distance when the number of particles is small. For large number n of particles, however, the torus $\mathbf{T}$ does not offer enough space and the particles move closer together. Then, the repulsive part dominates and forces the particles to equidistribute in the limit $n \to \infty$. The leading contribution to the integral on the left hand side of statement (6.12) will then, in the limit $n \to \infty$, come from equidistributed $(\theta_1, \ldots, \theta_n) \in \mathbf{T}^n$. Evaluation of the left hand side at those equidistributed points defines a Riemann sum, approximating the integral on the right hand side of (6.12).

In the next section 6.3, we make the minimization of the potential energy between the particles a rigorous statement. We use this statement in section 6.4 to prove Theorem 6.2.1. In both those chapters, we adapt strategies used in [27].

6.3. Equilibrium Measures

The function f is continuous and has a single pole at $\theta = 0$. It is bounded from below. Let $M(\mathbf{T})$ denote the set of all Borel probability measures on $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Since f is continuous and bounded from below, we can define a functional

$$I : M(\mathbf{T}) \to \mathbf{R} \cup \{\infty\} \tag{6.14}$$

by

$$I[\mu] = \iint_{\theta \neq \phi} f(\theta - \phi) d\mu(\theta) d\mu(\phi) \qquad (\mu \in M(\mathbf{T})). \tag{6.15}$$

Since f is bounded from below, I is bounded from below. We set

$$I_0 = \inf I[\mu] \tag{6.16}$$

where the infimum is taken over the subset $M_0(\mathbf{T}) \subset M(\mathbf{T})$ of Borel probability measures which do not contain point masses. We have $I_0 > -\infty$. A measure $\mu \in M_0(\mathbf{T})$ that realizes the infimum in (6.16) is called an *equilibrium measure*.

For the analysis of equilibrium measures, we utilize the representation of f by Fourier series. Let

$$f_k = \frac{1}{2\pi}\int_{\mathbf{T}} f(\theta)e^{-ik\theta}d\theta \qquad (k \in \mathbf{Z}) \tag{6.17}$$

denote the Fourier coefficients of f. The Fourier coefficients of

$$g_\sigma(\theta) = \log|e^{i\theta} - \sigma| = \frac{1}{2}\log(1 - 2\sigma\cos\theta + \sigma^2) \qquad (\theta \in \mathbf{T}), \tag{6.18}$$

where $\sigma > 0$, are well-known [15] to be

$$g_k(\sigma) = \begin{cases} -\frac{1}{2|k|}\min\{\sigma^{|k|}, \sigma^{-|k|}\}, & \text{if } k \neq 0 \\ \max\{0, \log\sigma\}, & \text{if } k = 0 \end{cases} \qquad (k \in \mathbf{Z}). \tag{6.19}$$

A direct calculation now shows $f_0 = 0$ and, for $k \neq 0$,

$$f_k = \frac{1}{2|k|}\Big(1 + |q_1q_2|^{|k|} - |q_1|^{|k|}e^{-ik\arg q_1} - |q_2|^{|k|}e^{-ik\arg q_2}\Big). \tag{6.20}$$

We obtain the estimate, for $k \neq 0$,

$$f_k \geq \begin{cases} \frac{1}{2|k|}\Big(1 - |q_1q_2|^{|k|/2}\Big)^2, & \text{if } q_1 = \overline{q_2} \\ \frac{1}{2|k|}\Big(1 - |q_1|^{|k|}\Big)\Big(1 - |q_2|^{k}\Big), & \text{if } q_1, q_2 \in (0,1). \end{cases} \tag{6.21}$$

Hence we have established

Lemma 6.3.1. *The Fourier coefficients $(f_k)_{k\in\mathbf{Z}}$ of the function f defined in* (6.8) *satisfy $f_0 = 0$ and $f_k > 0$ for $k \neq 0$.*

Let $\lambda \in M_0(\mathbf{T})$ denote the normalized Lebesgue measure on $\mathbf{T}$. From $f_0 = 0$, we directly obtain $I[\lambda] = 0$. We have

Theorem 6.3.2. *The normalized Lebesgue measure λ is the unique equilibrium measure for the functional $I[\mu]$ defined in* (6.15)*. In particular, $I_0 = 0$.*

Proof. We first prove $I_0 = 0$. From $I[\lambda] = 0$ we obtain $I_0 \leq 0$. Let $\mu \in M_0(\mathbf{T})$. Since μ does not contain point masses, we obtain

$$I[\mu] = \iint f(\theta - \mu)d\mu(\theta)d\mu(\phi). \tag{6.22}$$

We apply Tonelli's theorem to obtain

$$I[\mu] = \sum_{k \neq 0} f_k \iint e^{ik(\theta - \phi)} d\mu(\theta) d\mu(\phi). \tag{6.23}$$

Now, for each $k \neq 0$, the double integral gives the squared absolute value of the k-th Fourier coefficient

$$\mu_k = \frac{1}{2\pi} \int_{\mathbf{T}} e^{-ik\theta} d\mu(\theta) \tag{6.24}$$

of the measure μ. In particular, each double integral is nonnegative. Together with Lemma 6.3.1, we obtain $I[\mu] \geq 0$. We have established $I_0 = 0$.

Next, we let $\mu, \nu \in M_0(\mathbf{T})$ be two Borel probability measures without point masses realizing the infimum, i.e. $I[\mu] = I_0 = I[\nu]$. We claim $\mu = \nu$. The signed measure $\mu - \nu$ is not in $M_0(\mathbf{T})$. However,

$$I[\mu - \nu] := \iint f(\theta - \phi) d\mu(\theta) d\mu(\phi) - \iint f(\theta - \phi) d\nu(\theta) d\mu(\phi) \tag{6.25}$$

$$- \iint f(\theta - \phi) d\mu(\theta) d\nu(\phi) + \iint f(\theta - \phi) d\nu(\theta) d\nu(\phi) \tag{6.26}$$

makes sense in $[-\infty, \infty)$, since $I[\mu] = I[\nu] = 0$ is a finite number. We apply Tonelli's theorem to each of the four double integrals to obtain

$$I[\mu - \nu] = \sum_{k \neq 0} f_k |\mu_k - \nu_k|^2 \geq 0. \tag{6.27}$$

In particular, $I[\mu - \nu]$ and all four double integrals in its definition are finite real numbers. Next, consider the convex combination $\nu + t(\mu - \nu) \in M_0(\mathbf{T})$, where $t \in [0, 1]$. We have

$$I[\nu + t(\mu - \nu)] = I[\nu] + tA + t^2 B, \tag{6.28}$$

where both terms

$$A = \iint f(\theta - \phi) d\nu(\theta) d\mu(\phi) - \iint f(\theta - \phi) d\nu(\theta) d\nu(\phi) \tag{6.29}$$

$$+ \iint f(\theta - \phi) d\mu(\theta) d\nu(\phi) - \iint f(\theta - \phi) d\nu(\theta) d\nu(\phi) \tag{6.30}$$

and $B = I[\mu - \nu]$ are finite. We obtain

$$0 = I_0 \leq I[\nu + t(\mu - \nu)] = I[\nu] + tA + t^2 B = tA + t^2 B \tag{6.31}$$

for all $t \in [0, 1]$. Hence the right hand side of the inequality defines a polynomial in t, which is nonnegative on $[0, 1]$ and vanishes at both $t = 0$ and $t = 1$. Hence $I[\mu - \nu] = B \leq 0$. Together with equation (6.27), we obtain

$$\sum_{k \neq 0} f_k |\mu_k - \nu_k|^2 = 0. \tag{6.32}$$

By Lemma (6.3.1), we obtain $\mu_k = \nu_k$ for all $k \neq 0$. Since μ and ν are both probability measures, we also have $\mu_0 = 1 = \nu_0$. Since all Fourier coefficients of the measures μ and ν agree, we have $\mu = \nu$. This finishes the proof of uniqueness. □

6.4. Limit of Large Number of Particles

For a point $\theta \in \mathbf{T}^n$ we define

$$\delta_\theta = \frac{1}{n}\sum_{j=1}^{n} \delta_{\theta_j} \in M(\mathbf{T}) \tag{6.33}$$

as a convex combination of Dirac measures. Set

$$\mathbf{T}_0^n = \{\theta \in \mathbf{T}^n : \theta_j \neq \theta_k \ (j \neq k)\}. \tag{6.34}$$

For $\theta \in \mathbf{T}_0^n$, we have

$$I[\delta_\theta] = \frac{1}{n^2}\sum_{j\neq k} f(\theta_j - \theta_k), \tag{6.35}$$

which relates the functional I to the interpretation of Theorem 6.2.1 given at the end of section 6.2.

We now prove that the integral on the left hand side of equation (6.12) will, for large n, be dominated by those $\theta \in \mathbf{T}^n$ for which $I[\delta_\theta]$ is close to $I_0 = 0$. To this end, we define, for $\eta > 0$,

$$\begin{aligned} A_{n,\eta} =& \{\theta \in \mathbf{T}_0^n : I_0 \leq I[\delta_\theta] \leq I_0 + \eta\} && (6.36)\\ =& \{\theta \in \mathbf{T}^n : \sum_{j\neq k} f(\theta_k - \theta_j) \leq \eta\, n^2\}. && (6.37) \end{aligned}$$

We have

Lemma 6.4.1. *For any integer $n \geq 2$ and real number $\eta > 0$, we have*

$$P_n[A_{n,\eta}] \geq 1 - e^{-\eta\, n^2}. \tag{6.38}$$

Proof. By construction of the set $A_{n,\eta}$, we have

$$P_n[\mathbf{T}^n \setminus A_{n,\eta}] = \frac{1}{Z_n}\int_{\mathbf{T}^n\setminus A_{n,\eta}} e^{-\sum_{j\neq k} f(\theta_j-\theta_k)}\, d\theta \leq \frac{(2\pi)^n}{Z_n} e^{-\eta\, n^2}. \tag{6.39}$$

By Jensen's inequality,

$$\begin{aligned} \frac{Z_n}{(2\pi)^n} &= \int_{\mathbf{T}_0^n} \frac{d\theta}{(2\pi)^n} e^{-\sum_{j\neq k} f(\theta_j-\theta_k)} && (6.40)\\ &\geq \exp\left(-\sum_{j\neq k}\int_{\mathbf{T}_0^n} \frac{d\theta}{(2\pi)^n} f(\theta_j - \theta_k)\right) && (6.41)\\ &= \exp\left(-n(n-1)\iint f(\theta_1-\theta_2)\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\right) = 1. && (6.42) \end{aligned}$$

In the last step, we have used that the normalized Lebesgue measure is the equilibrium measure. □

Next we show, that those points $\theta \in A_{n,\eta}$ that dominate our integral equidistribute in the large n limit. To describe the equidistribution process, we use the notion of weak convergence of measures. A sequence (μ_n) of measures $\mu_n \in M(\mathbf{T})$ is said to *converge weakly* to a measure $\mu \in M(\mathbf{T})$ if

$$\int_{\mathbf{T}} g d\mu_n \to \int_{\mathbf{T}} g d\mu \qquad (n \to \infty) \tag{6.43}$$

for all continuous, bounded functions $g : \mathbf{T} \to \mathbf{C}$. The space $M(\mathbf{T})$ with the notion of weak convergence is sequentially compact.

Lemma 6.4.2. *1. Fix $\eta > 0$ and consider any sequence $(\theta^{n,\eta})_n$ of points $\theta^{n,\eta} \in A_{n,\eta}$. Let $\nu_{n,\eta} \in M(\mathbf{T})$ define the corresponding sequence of Dirac measures. Let $(n_k)_{k\in\mathbf{N}}$ define a convergent subsequence*

$$\nu_{n_k,\eta} \to \nu_\eta \in M(\mathbf{T}) \qquad (k \to \infty). \tag{6.44}$$

Then, $\nu_\eta \in M_0(\mathbf{T})$, i.e. it does not contain point masses. Moreover,

$$I[\nu_\eta] \leq \eta. \tag{6.45}$$

2. Consider, for each $\eta > 0$, the limit measures $\nu_\eta \in M_0(\mathbf{T})$ with $I[\nu_\eta] \leq \eta$ from the first part of the lemma. If $(\eta_k)_{k\in\mathbf{N}}$ defines a convergent sequence

$$\nu_{\eta_k} \to \nu \qquad (k \to \infty) \tag{6.46}$$

with

$$\eta_k \to 0 \qquad (k \to \infty) \tag{6.47}$$

then ν is the normalized Lebesgue measure.

Proof. 1. We introduce a cutoff $L \in \mathbf{R}$ to make the function f bounded and continuous. We estimate, for each n,

$$\eta \geq I[\nu_{n,\eta}] \geq \iint_{\theta\neq\phi} \min\{f(\theta-\phi), L\} d\nu_{n,\eta}(\theta) d\nu_{n,\eta}(\phi) \tag{6.48}$$

Next, we separate the part with $\theta = \phi$. Since $\tau^{n,\eta} \in \mathbf{T}_0^n$, it corresponds to the $j \neq k$ term in the following sum:

$$\iint_{\theta\neq\phi} \min\{f(\theta-\phi), L\} d\nu_{n,\eta}(\theta) d\nu_{n,\eta}(\phi) \tag{6.49}$$

$$= \frac{1}{n^2} \sum_{j\neq k} \min\{f(\tau_j^{n,\eta} - \tau_k^{n,\eta}), L\} + \frac{L}{n} - \frac{L}{n} \tag{6.50}$$

$$= \iint \min\{f(\theta-\phi), L\} d\nu_{n,\eta}(\theta) d\nu_{n,\eta}(\phi) - \frac{L}{n}. \tag{6.51}$$

Next, we want to factorize the integrand to take the limit of the subsequence defined by $(n_k)_{k\in\mathbf{N}}$. Fix $\epsilon > 0$. By the Weierstrass approximation theorem, there exists a polynomial $p(\theta,\phi)$ such that

$$\sup_{(\theta,\phi)\in\mathbf{T}^2} |p(\theta,\phi) - \min\{f(\theta-\phi), L\}| \leq \epsilon. \tag{6.52}$$

So far, we obtain

$$\eta \geq \iint p(\theta,\phi) d\nu_{n,\eta}(\theta) d\nu_{n,\eta}(\phi) - \epsilon - \frac{L}{n}. \tag{6.53}$$

The integrand is a sum of functions that are factorized in functions depending on θ and functions depending on ϕ. Hence, we can look at the inequality at $n = n_k$ and take the limit $k \to \infty$:

$$\eta \geq \iint p(\theta,\phi) d\nu_\eta(\theta) d\nu_\eta(\phi) - \epsilon \tag{6.54}$$

$$\geq \iint \min\{f(\theta,\phi), L\} d\nu_\eta(\theta) d\nu_\eta(\phi) - 2\epsilon. \tag{6.55}$$

We send $\epsilon \to 0$ and obtain

$$\eta \geq \iint \min\{f(\theta,\phi), L\} d\nu_\eta(\theta) d\nu_\eta(\phi). \tag{6.56}$$

The integrand is monotone in L. Hence we can remove the cutoff by taking the limit $L \to \infty$ and obtain

$$\eta \geq \iint f(\theta,\phi) d\nu_\eta(\theta) d\nu_\eta(\phi). \tag{6.57}$$

In particular, $\nu_\eta \in M_0(\mathbf{T})$ and thus we may introduce the condition $\theta \neq \phi$ to obtain

$$\eta \geq \iint_{\theta\neq\phi} f(\theta,\phi) d\nu_\eta(\theta) d\nu_\eta(\phi) = I[\nu_\eta]. \tag{6.58}$$

2. The first part of the lemma is proved. For the second part, use $\nu_{\eta_k} \in M_0(\mathbf{T})$ to write

$$\eta_k \geq I[\nu_{\eta_k}] = \iint f(\theta-\phi) d\nu_{\eta_k}(\theta) d\nu_{\eta_k}(\phi). \tag{6.59}$$

We again introduce a cutoff $L \in \mathbf{R}$ to estimate

$$\eta_k \geq \iint \min\{f(\theta-\phi), L\} d\nu_{\eta_k}(\theta) d\nu_{\eta_k}(\phi). \tag{6.60}$$

As in the proof of the first part of the lemma, given $\epsilon > 0$, we estimate the continuous integrand by a polynomial $p(\theta,\phi)$ and get

$$\eta_k \geq \iint p(\theta,\phi) d\nu_{\eta_k}(\theta) d\nu_{\eta_k}(\phi) - \epsilon. \tag{6.61}$$

Now, we can take the limit $k \to \infty$ to get

$$0 \geq \iint p(\theta, \phi) d\nu(\theta) d\nu(\phi) - \epsilon \tag{6.62}$$

$$\geq \iint \min\{f(\theta - \phi), L\} d\nu(\theta) d\nu(\phi) - 2\epsilon. \tag{6.63}$$

Sending first $\epsilon \to 0$ and then $L \to \infty$, we obtain

$$0 \geq \iint f(\theta - \phi) d\nu(\theta) d\nu(\phi). \tag{6.64}$$

In particular, $\nu \in M_0(\mathbf{T})$ and thus we can introduce the condition $\theta \neq \phi$ to obtain $I[\nu] \leq 0$. By Theorem 6.3.2, we find that ν is the normalized Lebesgue measure.

□

We are now in a position to prove Theorem 6.2.1. Let $h : \mathbf{T} \to \mathbf{R}$ be a real-valued, continuous function. We are going to estimate the limes inferior and the limes superior of the sequence defined by

$$\frac{1}{n} \log \mathbf{E}_n \left[e^{\sum_{j=1}^n h(\theta_j)} \right] \qquad (n \geq 0) \tag{6.65}$$

separately. Fix $\eta > 0$. By Lemma 6.4.1, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{E}_n \left[e^{\sum_{j=1}^n h(\theta_j)} \right] = \limsup_{n\to\infty} \frac{1}{n} \log \int_{A_{n,\eta}} e^{\sum_{j=1}^n h(\theta_j)} p_n(\theta) d\theta \tag{6.66}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{E}_n \left[e^{\sum_{j=1}^n h(\theta_j)} \right] = \liminf_{n\to\infty} \frac{1}{n} \log \int_{A_{n,\eta}} e^{\sum_{j=1}^n h(\theta_j)} p_n(\theta) d\theta \tag{6.67}$$

$$\tag{6.68}$$

For each n, the set $A_{n,\eta}$ is compact. Hence the continuous function $e^{\sum_{j=1}^n h(\theta_j)}$ attains both its maximum and its minimum on $A_{n,\eta}$. Let $\tau^{n,\eta,+}$ and $\tau^{n,\eta,-}$ denote the respective points in $A_{n,\eta}$. Denote the corresponding Dirac measures by $\nu_{n,\eta}^+$ and $\nu_{n,\eta}^-$. We have

$$\frac{1}{n} \log \int_{A_{n,\eta}} e^{\sum_{j=1}^n h(\theta_j)} p_n(\theta) d\theta \leq \frac{1}{n} \log P_n[A_{n,\eta}] e^{\sum_{j=1}^n h(\tau_j^{n,\eta,+})} \tag{6.69}$$

$$\leq \frac{1}{n} \sum_{j=1}^n h(\tau_j^{n,\eta,+}) = \int_{\mathbf{T}} h(\theta) d\nu_{n,\eta,+}(\theta). \tag{6.70}$$

On the other hand

$$\frac{1}{n} \log \int_{A_{n,\eta}} e^{\sum_{j=1}^n h(\theta_j)} p_n(\theta) d\theta \geq \frac{1}{n} \log P_n[A_{n,\eta}] e^{\sum_{j=1}^n h(\tau_j^{n,\eta,-})} \tag{6.71}$$

$$= \frac{1}{n} \log P_n[A_{n,\eta}] + \int_{\mathbf{T}} h(\theta) d\nu_{n,\eta,-}(\theta). \tag{6.72}$$

By Lemma 6.4.1, we have

$$\frac{1}{n} \log P_n[A_{n,\eta}] \geq \frac{1}{n} \log(1 - e^{-\eta n^2}) \to 1 \qquad (n \to \infty). \tag{6.73}$$

We have isolated the dominant contribution to the integrals. Now we want to apply our equidistribution property in Lemma 6.4.2.

Let $(n_{k,+})_{k\in\mathbf{N}}$ and $(n_{k,-})_{k\in\mathbf{N}}$ define subsequences which realize the respective limit points on the respective right hand sides in equation (6.66). We get

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{E}_n\left[e^{\sum_{j=1}^n h(\theta_j)}\right]=\lim_{k\to\infty}\frac{1}{n_{k,+}}\log\int_{A_{n_{k,+},\eta}}e^{\sum_{j=1}^{n_{k,+}} h(\theta_j)}p_{n_{k,+}}(\theta)d\theta \tag{6.74}$$

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbf{E}_n\left[e^{\sum_{j=1}^n h(\theta_j)}\right]=\lim_{k\to\infty}\frac{1}{n_{k,-}}\log\int_{A_{n_{k,-},\eta}}e^{\sum_{j=1}^{n_{k,-}} h(\theta_j)}p_{n_{k,-}}(\theta)d\theta. \tag{6.75}$$

By passing to respective subsequences, we may suppose that the

$$\nu_{n_{k,+},\eta,+}\to\nu_{\eta,+}\qquad\qquad \nu_{n_{k,-},\eta,-}\to\nu_{\eta,-}\qquad\qquad (k\to\infty) \tag{6.76}$$

since the space $M(\mathbf{T})$ is sequentially compact. We obtain

$$\int_{\mathbf{T}}h(\theta)d\nu_{\eta,-}(\theta)\le\liminf_{n\to\infty}\frac{1}{n}\log\mathbf{E}_n\left[e^{\sum_{j=1}^n h(\theta_j)}\right] \tag{6.77}$$

$$\le\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{E}_n\left[e^{\sum_{j=1}^n h(\theta_j)}\right]\le\int_{\mathbf{T}}h(\theta)d\nu_{\eta,+}(\theta). \tag{6.78}$$

We use sequential compactness again: Let $(\eta_{k,+})_{k\in\mathbf{N}}$ and $(\eta_{k,-})_{k\in\mathbf{N}}$ define sequences $(\nu_{\eta_k,+})_{k\in\mathbf{N}}$ and $(\nu_{\eta_k,-})_{k\in\mathbf{N}}$ with $\eta_{k,+}\to 0$ and $\eta_{k,-}\to 0$ for $k\to\infty$ such that

$$\nu_{\eta_k,+}\to\nu_+\qquad\qquad \nu_{\eta_k,-}\to\nu_-(k\to\infty) \tag{6.79}$$

for $\nu_+,\nu_-\in M(\mathbf{T})$. By the second part of Lemma 6.4.2, we know that both measures ν_+ and ν_- are given by the normalized Lebesgue measure. Hence

$$\frac{1}{2\pi}\int_{\mathbf{T}}h(\theta)d\theta\le\liminf_{n\to\infty}\frac{1}{n}\log\mathbf{E}_n\left[e^{\sum_{j=1}^n h(\theta_j)}\right] \tag{6.80}$$

$$\le\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{E}_n\left[e^{\sum_{j=1}^n h(\theta_j)}\right]\le\frac{1}{2\pi}\int_{\mathbf{T}}h(\theta)d\theta. \tag{6.81}$$

The proof of Theorem 6.2.1 is complete.

7. Conclusion and Discussion

In this chapter, we conclude this thesis. In section 7.1, we summarize our setup from chapters 2, 3 and 4. We apply our findings from chapter 5. Afterwards, we discuss our results in section 7.2.

7.1. Summary and Conclusion

Fix an integer $r \geq 1$. In chapter 3, we have defined the Nekrasov partition function as the generating function of the weighted Euler characteristic of certain equivariant K-theory classes for a sequence of certain Nakajima quiver varieties $M(r,n), n \in \mathbf{N}$. Equivariance has been defined with respect to a torus action, whose representation ring in a certain set of characters is given by

$$R(T) = \mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}, e_1^{\pm 1}, \dots, e_r^{\pm 1}] \tag{7.1}$$

The equivariant K-theory classes have been defined by monic polynomials f and g with coefficients in $R(T)$. The Nekrasov partition function is a formal power series whose coefficients live in the field of fractions of $R(T)$. In equation (3.61), we have defined it as the formal power series

$$Z(t_1, t_2, e_1, \dots, e_r; f, g; \mathfrak{q}) = \sum_{n \geq 0} \Big(\mathfrak{q} \ (t_1 t_2)^{-r/2} \Big)^n \tag{7.2}$$

$$\times Z_n(t_1, t_2, e_1, \dots, e_r; f, g) \tag{7.3}$$

The coefficients are given by (3.57):

$$Z_n(t_1, t_2, e_1, \dots, e_r; f, g) \tag{7.4}$$

$$= \sum_{|\vec{Y}|=n} \frac{\prod_{\alpha=1}^r \prod_{(x,y)\in Y_\alpha} f\big(e_\alpha t_1^{x-1} t_2^{y-1}\big) g\big(e_\alpha^{-1} t_1^{-x+1} t_2^{-y+1}\big)}{\prod_{\alpha,\beta=1}^r K_{\alpha,\beta}^{\vec{Y}}(t_1, t_2, e_1, \dots, e_r)}, \tag{7.5}$$

where

$$K_{\alpha,\beta}^{\vec{Y}}(t_1, t_2, e_1, \dots, e_r) = \prod_{\square \in Y_\alpha} \Big(1 - e_\alpha^{-1} e_\beta \ t_1^{l_{Y_\beta}(\square)} t_2^{-a_{Y_\alpha}(\square)-1}\Big) \tag{7.6}$$

$$\prod_{\boxtimes \in Y_\beta} \Big(1 - e_\alpha^{-1} e_\beta \ t_1^{-l_{Y_\alpha}(\boxtimes)-1} t_2^{a_{Y_\beta}(\boxtimes)}\Big). \tag{7.7}$$

We now evaluate the formal parameters at complex numbers. We want to apply our integral estimate from chapter 5. We identify

$$q_1 = t_1^{-1} \qquad q_2 = t_2^{-1} \qquad u_\alpha = e_\alpha^{-1} \quad (\alpha = 1, \dots, r) \tag{7.8}$$

and assume equation (5.1), (5.2), and (5.3). Hence t_1, t_2 are two complex numbers outside the closed unit disc which are either complex conjugate or real numbers strictly bigger than 1. In particular, the scaling factor in the definition of the Nekrasov partition function is well-defined. Next, we allow arbitrary complex coefficients for the monic polynomials f and g. We parametrize them via their roots as

$$f(z) = \prod_{k=1}^{s} (z - F_k) \qquad\qquad g(z) = \prod_{k=1}^{s'} (z - G_k) \tag{7.9}$$

and identify them with the polynomials $F(z)$ and $G(z)$ from chapter 5. According to equation (5.5), we assume that the degree s' of $g(z)$ is bounded by $r - 1$. Instead of the polynomials f and g, we refer to the respective tuples $\vec{F}$, $\vec{G}$ in the definition of the Nekrasov partition function from now on. We obtain, for each $n \geq 0$ the integral

$$\mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}) = \frac{1}{n!} \left(\frac{1 - q_1 q_2}{(1 - q_1)(1 - q_2)} \right)^n \tag{7.10}$$

$$\int_{C_n(\rho)} \prod_{j=1}^{n} \frac{dz_j}{2\pi i z_j} \; \mathcal{J}(z_1, \ldots, z_n; \vec{F}, \vec{G}) \tag{7.11}$$

$$\mathcal{I}(z_1, \ldots, z_n; q_1, q_2, \vec{u}) \tag{7.12}$$

from chapter 5.

Fix $n \geq 0$. Theorem 5.2.1 now states that we can express the n-th coefficient of the Nekrasov partition function as an integral

$$Z_n(t_1, t_2, e_1, \ldots, e_r; \vec{F}, \vec{G}) = \mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}), \tag{7.13}$$

provided

$$t_1^x \neq t_2^{y+1}, \;\; t_1^{x+1} \neq t_2^y, \qquad \forall x, y \in \{0, \ldots, n-1\}. \tag{7.14}$$

$$e_\alpha e_\beta^{-1} \neq t_1^x t_2^y, \qquad \forall x, y \in \{-n, \ldots, n\}, \quad \forall \alpha \neq \beta \in \{1, \ldots, r\}. \tag{7.15}$$

If this condition fails, some terms in the sum (7.5) are infinite. However, since the integral (7.10) is still well-defined, we see that the whole sum (7.4) remains well-defined and is given by our integral formula. Higher order residues now appear in the integration process, rendering individual terms in the sum (7.5) infinite since they correspond to simple residues. The infinite terms combine to higher order derivatives of the integrand and the infinites cancel. A simple analogue is the computation

$$f'(0) = \oint \frac{dz}{2\pi i} \frac{f(z)}{z^2} = \lim_{\delta \to 0} \oint \frac{dz}{2\pi i} \frac{f(z)}{(z - \delta) z} = \lim_{\delta \to 0} \left(\frac{f(\delta)}{\delta} + \frac{f(0)}{-\delta} \right). \tag{7.16}$$

Hence we do not have to impose conditions (7.14). Note that in the case $t_1, t_2 > 1$, the denominators in the sum (7.5) can get arbitrarily small, even if the above condition holds. Nevertheless, we can estimate the growth of the coefficients of the Nekrasov partition function in any case using the integral representation and Theorem 5.3.2. This allows us to determine the analyticity properties of the Nekrasov partition function. From Theorem 5.3.3, we obtain our main result:

Theorem 7.1.1. *Let t_1 and t_2 be two complex numbers with $|t_1| > 1$ and $|t_2| > 1$. Assume either $t_1 = \overline{t_2}$ or $t_1, t_2 > 1$. Fix three integers r, s, s' with*

$$1 \le r \qquad 0 \le s \qquad 0 \le s' \le r-1. \tag{7.17}$$

Define the functions

$$\rho_-(\vec{e}) = \max_{\alpha=1,\dots,r} |e_\alpha|^{-1} \tag{7.18}$$

$$\rho_+(\vec{e}) = \min\{|t_1|, |t_2|\} \min_{\alpha=1,\dots,r} |e_\alpha|^{-1} \qquad (\vec{e} \in \mathbf{C}^r). \tag{7.19}$$

The Nekrasov partition function

$$Z(t_1, t_2, e_1, \dots, e_r; \vec{F}, \vec{G}; \mathfrak{q}) = \sum_{n \ge 0} \left(\mathfrak{q}(t_1 t_2)^{-r/2} \right)^n Z_n(t_1, t_2, e_1, \dots, e_r; \vec{F}, \vec{G}) \tag{7.20}$$

is an analytic function in the variables

$$(\vec{e}, \vec{F}, \vec{G}, \mathfrak{q}) \in \mathbf{C}^r \times \mathbf{C}^s \times \mathbf{C}^{s'} \times \mathbf{C} \tag{7.21}$$

on the domain defined by the conditions

$$\rho_-(\vec{e}) < \rho_+(\vec{e}) \tag{7.22}$$

$$|\,\mathfrak{q}\,| < (t_1 t_2)^{r/2} \sup_{\rho \in (\rho_-(\vec{e}), \rho_+(\vec{e}))} \left(\prod_{k=1}^{s} \min\{\rho^{-1}, |F_k|^{-1}\} \prod_{k=1}^{s'} \min\{\rho, |G_k|^{-1}\} \right) \tag{7.23}$$

The supremum in the last equation can be simplified to

$$\begin{cases} \prod_{k=1}^{s'} \min\{|t_1||e_1|^{-1}, \dots, |t_1||e_r|^{-1}, |t_2||e_1|^{-1}, \dots, |t_2||e_r|^{-1}, |G_k|^{-1}\} & \text{if } s = 0 \\ \prod_{k=1}^{s} \min\{|e_1|, \dots, |e_r|, |F_k|^{-1}\} & \text{if } s' = 0 \\ 1 & \text{if } s = s' = 0. \end{cases} \tag{7.24}$$

From the definition of the coefficients (3.57) of the Nekrasov partition function, we immediately obtain

$$Z(t_1, t_2, e_1, \dots, e_r; f, g; \mathfrak{q}) = Z(t_1^{-1}, t_2^{-1}, e_1^{-1}, \dots, e_r^{-1}; g, f; \mathfrak{q}) \tag{7.25}$$

as formal power series. Hence we also have

Theorem 7.1.2. *Let t_1 and t_2 be two complex numbers with $|t_1| < 1$ and $|t_2| < 1$. Assume either $t_1 = \overline{t_2}$ or $t_1, t_2 \in (0, 1)$. Fix three integers r, s, s' with*

$$1 \le r \qquad 0 \le s \le r-1 \qquad 0 \le s'. \tag{7.26}$$

Define the functions

$$\rho'_-(\vec{e}) = \max\{|t_1|, |t_2|\} \max_{\alpha=1,\dots,r} |e_\alpha|^{-1} \tag{7.27}$$

$$\rho'_+(\vec{e}) = \min_{\alpha=1,\dots,r} |e_\alpha|^{-1} \qquad (\vec{e} \in \mathbf{C}^r). \tag{7.28}$$

The Nekrasov partition function

$$Z(t_1,t_2,e_1,\dots,e_r;\vec{F},\vec{G};\mathfrak{q}) = \sum_{n\geq 0}\Big(\mathfrak{q}(t_1t_2)^{-r/2}\Big)^n Z_n(t_1,t_2,e_1,\dots,e_r;\vec{F},\vec{G}) \tag{7.29}$$

is an analytic function in the variables

$$(\vec{e},\vec{F},\vec{G},\mathfrak{q}) \in \mathbf{C}^r\times\mathbf{C}^s\times\mathbf{C}^{s'}\times\mathbf{C} \tag{7.30}$$

on the domain defined by the conditions

$$\rho'_-(\vec{e}) < \rho'_+(\vec{e}) \tag{7.31}$$

$$|\,\mathfrak{q}\,| < (t_1t_2)^{-r/2} \sup_{\rho\in(\rho'_-(\vec{e}),\rho'_+(\vec{e}))}\left(\prod_{k=1}^{s}\min\{\rho^{-1},|F_k|^{-1}\}\prod_{k=1}^{s'}\min\{\rho,|G_k|^{-1}\}\right) \tag{7.32}$$

The supremum in the last equation can be simplified to

$$\begin{cases}\prod_{k=1}^{s'}\min\{|e_1|^{-1},\dots,|e_r|^{-1},|G_k|^{-1}\} & \text{if } s=0\\ \prod_{k=1}^{s}\min\{|t_1|^{-1}|e_1|,\dots,|t_1|^{-1}|e_r|,|t_2|^{-1}|e_1|,\dots,|t_2|^{-1}|e_r|,|F_k|^{-1}\} & \text{if } s'=0\\ 1 & \text{if } s=s'=0.\end{cases} \tag{7.33}$$

For completeness, we also translate this result for the Nekrasov partition function (3.64) defined for the formal symbols

$$t_1 = e^{\lambda\epsilon_1} \qquad t_2 = e^{\lambda\epsilon_2} \qquad e_\alpha = e^{\lambda a_\alpha} \quad (\alpha=1,\dots,r). \tag{7.34}$$

We have defined it as

$$Z(\epsilon_1,\epsilon_2,\vec{a},\lambda;f,g;\mathfrak{b}) = \sum_{n\geq 0}\Big(\mathfrak{b}\ \lambda^{2r-\deg f+\deg g}e^{-r\lambda(\epsilon_1+\epsilon_2)/2}\Big)^n Z_n(\epsilon_1,\epsilon_2,\vec{a},\lambda;f,g), \tag{7.35}$$

where the coefficients are given in equation (3.65). We now evaluate the formal parameters at complex numbers. We assume λ to be a positive real number and set

$$q_1 = t_1^{-1} = e^{-\lambda\epsilon_1} \qquad q_2 = t_2^{-1} = e^{-\lambda\epsilon_2} \qquad u_\alpha = e_\alpha^{-1} = e^{-\lambda a_\alpha} \quad (\alpha=1,\dots,r). \tag{7.36}$$

We therefore assume that the complex numbers ϵ_1 and ϵ_2 have positive real part and are either complex conjugate or both real and positive. Hence assumptions (5.1) and (5.2) are satisfied. Moreover, we assume

$$\max_{\alpha=1,\dots,r}\mathrm{Re}(a_\alpha) - \min_{\alpha=1,\dots,r}\mathrm{Re}(a_\alpha) < \min_{i=1,2}\mathrm{Re}(\epsilon_i). \tag{7.37}$$

Hence also assumption (5.3) is satisfied. We again allow general complex coefficients in the polynomials f and g. We identify them with the respective polynomials F and G from chapter 5 and parametrize their roots $\vec{F}$ and $\vec{G}$ as

$$F_k = e^{-\lambda\phi_k} \qquad (k=1,\dots,s) \tag{7.38}$$

$$G_k = e^{-\lambda\psi_k} \qquad (k=1,\dots,s') \tag{7.39}$$

for tuples of complex numbers $\vec{\phi}$ and $\vec{\psi}$. We refer to those tuples in the arguments of the Nekrasov partition function rather than to the polynomials f and g. We again assume the degree bound (5.5). For $n \geq 0$, Theorem 5.2.1 now states that we can express the n-th coefficient of the Nekrasov partition function as an integral

$$Z_n(q_1, q_2, \vec{a}; \vec{\phi}, \vec{\psi}) = \mathcal{Z}_n(q_1, q_2, \vec{u}; \vec{F}, \vec{G}), \tag{7.40}$$

provided

$$a_\alpha - a_\beta \not\equiv x\epsilon_1 + y\epsilon_2, \qquad \forall \alpha \neq \beta \in \{1, \ldots, r\} \quad \forall x, y \in \{-n, \ldots, n\}, \tag{7.41}$$

$$x\epsilon_1 \not\equiv (y+1)\epsilon_2,\ (x+1)\epsilon_1 \not\equiv y\epsilon_2, \qquad \forall x, y \in \{0, \ldots, n-1\}, \tag{7.42}$$

where the inequalities are modulo $\frac{2\pi i}{\lambda}\mathbf{Z}$. As in the above case, we do not have to impose those inequalities since the integral expression takes care of the cancellations of infinities in the sum (3.65). Theorem 7.1.1 now translates to

Theorem 7.1.3. *Let $\lambda > 0$ be a positive real number. Let ϵ_1 and ϵ_2 be complex numbers with $\operatorname{Re}\epsilon_1 > 0$ and $\operatorname{Re}\epsilon_2 > 0$. Assume either $\epsilon_1 = \overline{\epsilon_2}$ or both $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Let r, s, s' be three integers with*

$$1 \leq r \qquad 0 \leq s \qquad 0 \leq s' \leq r-1. \tag{7.43}$$

Define the functions

$$A(\vec{a}) = \max_{\alpha=1,\ldots,r} \operatorname{Re} a_\alpha - \min_{i=1,2} \operatorname{Re}\epsilon_i \qquad (a \in \mathbf{C}^r) \tag{7.44}$$

$$B(\vec{a}) = \min_{\alpha=1,\ldots,r} \operatorname{Re} a_\alpha \qquad (a \in \mathbf{C}^r) \tag{7.45}$$

and set

$$C(\vec{\phi}, \vec{\psi}, P) = \sum_{k=1}^{s} \min\{P, \operatorname{Re}\phi_k\} + \sum_{k=1}^{s'} \min\{-P, \operatorname{Re}\psi_k\} \tag{7.46}$$

for $\vec{\phi} \in \mathbf{C}^s, \vec{\psi} \in \mathbf{C}^{s'}$ and $P \in \mathbf{R}$. The Nekrasov partition function

$$Z(\epsilon_1, \epsilon_2, \vec{a}, \lambda; \vec{\phi}, \vec{\psi}; \mathfrak{b}) = \sum_{n \geq 0} \Big(\mathfrak{b}\ \lambda^{2r-s+s'} e^{-r\lambda(\epsilon_1+\epsilon_2)/2} \Big)^n Z_n(\epsilon_1, \epsilon_2, \vec{a}, \lambda; \vec{\phi}, \vec{\psi}) \tag{7.47}$$

is an analytic function in the variables

$$(\vec{a}, \vec{\phi}, \vec{\psi}, \mathfrak{b}) \in \mathbf{C}^r \times \mathbf{C}^s \times \mathbf{C}^{s'} \times \mathbf{C} \tag{7.48}$$

on the domain defined by the conditions

$$A(\vec{a}) < B(\vec{a}) \tag{7.49}$$

$$|\,\mathfrak{b}\,| < \lambda^{-2r+s-s'} \exp\lambda\Big(\frac{r}{2}\big(\operatorname{Re}\epsilon_1 + \operatorname{Re}\epsilon_2\big) + \sup_{A(\vec{a})<P<B(\vec{a})} C(\vec{\phi}, \vec{\psi}, P)\Big). \tag{7.50}$$

The supremum in the last equation can be simplified to

$$\begin{cases} \sum_{k=1}^{s'} \min\{\operatorname{Re}\epsilon_1 - \operatorname{Re} a_1, \dots, \operatorname{Re}\epsilon_1 - \operatorname{Re} a_r, \\ \qquad \operatorname{Re}\epsilon_2 - \operatorname{Re} a_1, \dots, \operatorname{Re}\epsilon_2 - \operatorname{Re} a_r, \operatorname{Re}\psi_k\} & \text{if } s = 0 \\ \sum_{k=1}^{s} \min\{\operatorname{Re} a_1, \dots, \operatorname{Re} a_r, \operatorname{Re}\phi_k\} & \text{if } s' = 0 \\ 0 & \text{if } s = s' = 0. \end{cases} \tag{7.51}$$

Analogous to the symmetry relation (7.25), we have

$$Z(\epsilon_1, \epsilon_2, a_1, \dots, a_r; f, g; \mathfrak{b}) = Z(-\epsilon_1, -\epsilon_2, -a_1, \dots, -a_r; g, f; \mathfrak{b}) \tag{7.52}$$

as formal power series. We obtain

Theorem 7.1.4. *Let $\lambda > 0$ be a positive real number. Let ϵ_1 and ϵ_2 be complex numbers with $\operatorname{Re}\epsilon_1 < 0$ and $\operatorname{Re}\epsilon_2 < 0$. Assume either $\epsilon_1 = \overline{\epsilon_2}$ or both $\epsilon_1 < 0$ and $\epsilon_2 < 0$. Let r, s, s' be three integers with*

$$1 \le r \qquad 0 \le s \le r-1 \qquad 0 \le s'. \tag{7.53}$$

Define the functions

$$A'(\vec{a}) = \max_{\alpha=1,\dots,r} \operatorname{Re} a_\alpha \qquad (a \in \mathbf{C}^r) \tag{7.54}$$

$$B'(\vec{a}) = \min_{\alpha=1,\dots,r} \operatorname{Re} a_\alpha + \min_{i=1,2} |\operatorname{Re}\epsilon_i| \qquad (a \in \mathbf{C}^r) \tag{7.55}$$

and set

$$C(\vec{\phi}, \vec{\psi}, P) = \sum_{k=1}^{s} \min\{P, \operatorname{Re}\phi_k\} + \sum_{k=1}^{s'} \min\{-P, \operatorname{Re}\psi_k\} \tag{7.56}$$

for $\vec{\phi} \in \mathbf{C}^s, \vec{\psi} \in \mathbf{C}^{s'}$ and $P \in \mathbf{R}$. The Nekrasov partition function

$$Z(\epsilon_1, \epsilon_2, \vec{a}, \lambda; \vec{\phi}, \vec{\psi}; \mathfrak{b}) = \sum_{n \ge 0} \left(\mathfrak{b}\ \lambda^{2r-s+s'} e^{-r\lambda(\epsilon_1+\epsilon_2)/2} \right)^n Z_n(\epsilon_1, \epsilon_2, \vec{a}, \lambda; \vec{\phi}, \vec{\psi}) \tag{7.57}$$

is an analytic function in the variables

$$(\vec{a}, \vec{\phi}, \vec{\psi}, \mathfrak{b}) \in \mathbf{C}^r \times \mathbf{C}^s \times \mathbf{C}^{s'} \times \mathbf{C} \tag{7.58}$$

on the domain defined by the conditions

$$A'(\vec{a}) < B'(\vec{a}) \tag{7.59}$$

$$|\,\mathfrak{b}\,| < \lambda^{-2r+s-s'} \exp \lambda \left(\frac{r}{2} \left| \operatorname{Re}\epsilon_1 + \operatorname{Re}\epsilon_2 \right| + \sup_{A'(\vec{a}) < P < B'(\vec{a})} C(\vec{\phi}, \vec{\psi}, P) \right). \tag{7.60}$$

The supremum in the last equation can be simplified to

$$\begin{cases} \sum_{k=1}^{s'} \min\{-\operatorname{Re} a_1, \dots, -\operatorname{Re} a_r, \operatorname{Re}\psi_k\} & \text{if } s = 0 \\ \sum_{k=1}^{s} \min\{\operatorname{Re} a_1 + |\operatorname{Re}\epsilon_2|, \dots, \operatorname{Re} a_r + |\operatorname{Re}\epsilon_2|, \\ \qquad \operatorname{Re} a_1 + |\operatorname{Re}\epsilon_2|, \dots, \operatorname{Re} a_r + |\operatorname{Re}\epsilon_2|, \operatorname{Re}\phi_k\} & \text{if } s' = 0 \\ 0 & \text{if } s = s' = 0. \end{cases} \tag{7.61}$$

In chapter 2, we have defined the norm of the deformed Gaiotto state, with degree zero component equal to one, as a formal power series, whose coefficients are determined by the Gram matrix of the deformed Virasoro algebra. Given two formal parameters t, q and a rational number h, we have obtained this norm via the expression

$$\langle\, w(\xi)|w(\xi)\,\rangle = \sum_{n\geq 0} \xi^{2n} \big(W^{-1}_{(n)}\big)_{(1^n)(1^n)} \tag{7.62}$$

as a formal power series in ξ^2 with coefficients in $\mathbf{Q}(q,t)$. In chapter 4, we have introduced the relation

$$h = Q^{\frac{1}{2}} + Q^{-\frac{1}{2}} \tag{7.63}$$

and reinterpreted these coefficients as elements of the field $\mathbf{Q}(q,t,Q)$. Under the identifications

$$t_1 = t^{-1} \qquad t_2 = q \qquad e_2/e_1 = Q \qquad f = 1 \qquad g = 1 \qquad \mathfrak{q} = \xi^2. \tag{7.64}$$

we have obtained the norm of the deformed Gaiotto state as a Nekrasov partition function:

$$\langle\, w(\xi)|w(\xi)\,\rangle = Z(t_1,t_2,e_1,e_2;f,g;\mathfrak{q}). \tag{7.65}$$

We again want to evaluate q, t, Q and ξ at complex numbers. From the defining relations (2.27) of the deformed Virasoro algebra, we see that we have to assume that q and t are nonzero and q/t is not a root of -1. Moreover, the existence of the deformed Gaiotto state depends on the non-degeneracy of the Gram matrix. For formal parameters q, t and Q, the inverse of the Gram matrix exists in the sense of rational functions in those parameters. For complex values, we have to look at the Kac determinant formula (2.54). From it, we see that the deformed Gaiotto state is well-defined as long as

$$\{Q, -Q\} \cap \{q^x t^y : x, y \in \mathbf{Z}\} = \emptyset. \tag{7.66}$$

Note the similarity to condition (5.17) from Theorem (5.2.1). Given this assumption for complex numbers q, t and Q, the norm of the deformed Gaiotto state is a well-defined formal power series with complex coefficients.

From Theorem 7.1.1, we immediately obtain

Corollary 7.1.5. *Let q and t be two complex numbers with $|q| > 1$ and $|t| < 1$. Assume either $q\bar{t} = 1$ or both $q > 1$ and $t \in (0,1)$. The norm*

$$\langle\, w(\xi)|w(\xi)\,\rangle \tag{7.67}$$

of the deformed Gaiotto state, with degree zero component equal to one, is an analytic function in the variables

$$(Q,\xi) \in \mathbf{C} \times \mathbf{C} \tag{7.68}$$

on the domain defined by the four conditions

$$\{Q, -Q\} \cap \{q^x t^y : x, y \in \mathbf{Z}\} = \emptyset \tag{7.69}$$
$$|t| < |Q| < |t|^{-1} \tag{7.70}$$
$$|q|^{-1} < |Q| < |q| \tag{7.71}$$
$$|\xi| < (q/t)^{1/2}. \tag{7.72}$$

From Theorem 7.1.2 we get the following analog:

Corollary 7.1.6. *Let q and t be two complex numbers with $|q| < 1$ and $|t| > 1$. Assume either $q\bar{t} = 1$ or both $q \in (0, 1)$ and $t > 1$. The norm*

$$\langle\, w(\xi) | w(\xi)\, \rangle \tag{7.73}$$

of the deformed Gaiotto state, with degree zero component equal to one, is an analytic function in the variables

$$(Q, \xi) \in \mathbf{C} \times \mathbf{C} \tag{7.74}$$

on the domain defined by the four conditions

$$\{Q, -Q\} \cap \{q^x t^y : x, y \in \mathbf{Z}\} = \emptyset \tag{7.75}$$
$$|t|^{-1} < |Q| < |t| \tag{7.76}$$
$$|q| < |Q| < |q|^{-1} \tag{7.77}$$
$$|\xi| < (t/q)^{1/2}. \tag{7.78}$$

7.2. Discussion

In the previous section, we have summarized our estimates for the radius of convergence of Nekrasov partition functions and the related norm of Gaiotto states. In this section, we discuss our findings. In section 7.2.1, we discuss the sharpness of our estimates. In section 7.2.2, we discuss the bound on one of the polynomials appearing in the integral representation. In section 7.2.3, we discuss the relation of our findings to the convergence of four-dimensional Nekrasov partition functions. In the final section 7.2.4, we discuss generalization to partition functions of other types of supersymmetric Yang Mills theory.

7.2.1. Sharpness of Estimate

In this thesis we have estimated the radius of convergence of the Nekrasov partition function. One can further ask whether our estimate is sharp. The only place where we have used an inequality in our estimate is in equation (5.173) in chapter 5 where we have used the triangle inequality for integrals. Our motivation was to obtain a *real-valued* function h in equation (5.181) to which we can apply our Theorem 6.2.1 from potential theory. The real-valuedness of the function h was used in the proof of Theorem 6.2.1 on page 73, where we have estimated the integrand by its maximum and minimum value on a compact set. However, the physical intuition behind Theorem 6.2.1, which we have formulated right after the statement of the theorem on page 66, also holds true for complex valued h. We therefore believe that Theorem 6.2.1 also holds for complex valued, continuous functions h. Consequently, we believe that our estimate is sharp: Indeed, assume Theorem 6.2.1 is true for complex valued, continuous $h : \mathbf{T} \to \mathbf{C}$ and reconsider the argument in section 5.3. Write each of the $s + s' + 3r$ factors

$$\rho^{\pm 1} e^{\pm i\theta} - g_k \qquad (k = 1, \dots, s + s' + 3r) \tag{7.79}$$

appearing in the function g in equation (5.169) as the exponential of a complex valued function h_k. The real parts of the functions h_k are continuous and given by

$$\operatorname{Re} h_k(\theta) = \log |\rho^{\pm 1} e^{\pm i\theta} - g_k| \qquad (\theta \in \mathbf{T}, \quad k = 1, \dots, s + s' + 3r). \tag{7.80}$$

The imaginary parts may have jumps of height $2\pi i$. We can now write

$$g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta) = e^{h(\theta)} \qquad h(\theta) = \sum_{k=1}^{s+s'+3r} h_k(\theta). \tag{7.81}$$

We want to apply Theorem 6.2.1 with our new assumptions to conclude

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}_n[g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta)] = \frac{1}{2\pi} \int_{\mathbf{T}} h(\theta) d\theta. \tag{7.82}$$

The coefficients $\mathcal{Z}_n(q_1, q_2, \vec{u}, \vec{F}, \vec{G})$ are all real valued, as can be seen from taking their complex conjugate. Hence for large n, if our intuition about the physics behind Theorem 6.2.1 is true, their sign will be either positive or negative. In the latter case, one has to multiply all coefficients and hence the expectation value in equation (7.82) by minus one to get a well-defined logarithm. Hence we may assume that the sign in the large n case is positive. The possible jumps in the imaginary part of the function h play no role: We use the continuity of h twice in the proof of Theorem 6.2.1. First, when we take maxima and minima on a compact set. Here the function we extremize is $e^{h(\theta)}$ which is continuous in our case. Then, we use continuity when taking limits of measures in equations (6.76) and (6.79). However, we are already in the large n case, where we have a positive sign of the integral and therefore no jumps in the imaginary part of $h(\theta)$ play a role. Moreover, the right hand side of equation (7.82) will be real valued since the left hand side is. Hence one can calculate the limit as

$$\operatorname{Re} \sum_{k=1}^{s+s'+3r} \frac{1}{2\pi} \int_{\mathbf{T}} h_k(\theta) d\theta = \frac{1}{2\pi} \int_{\mathbf{T}} \log |g(q_1, q_2, \vec{u}; \vec{F}, \vec{G}; \rho, \theta)| d\theta \tag{7.83}$$

which yields exactly the same limit as in our analysis in section 5.3.

An apparent contradiction to our believe that our estimate is sharp is [12, Proposition 3.1], where the norm of the deformed Gaiotto state is proved to be an entire function for the choice of parameters $q = t$, which is different from our setting. Moreover, this is a very special case in the following sense: The coefficients of the Nekrasov partition function (3.57) contain a double product in the denominator of each summand indexed by a tuple $\vec{Y}$ of Young diagrams. Each diagonal factor $\alpha = \beta$ in the double product yields, for $t_1 = t_2^{-1}$, a product of all hook lengths in the Young diagram Y_α. The hook length formula leads to the appearance of the Plancherel measure on partitions times an additional factor of $1/|Y_\alpha|!$. See [12] for the details. This additional factor ensures the convergence of the power series on the whole complex plane. The appearance of the Plancherel measure was already noted in [40] and the same technique was applied to prove convergence of the norm of the (undeformed) Gaiotto state in [26, Proposition 1] for the corresponding special case $\epsilon_1 = -\epsilon_2$. Also note that in the case where t_1 and t_2 lie on opposite sides of the unit circle in $\mathbf{C}$, which includes the special case $t_1 = t_2^{-1}$, all the individual summands in all the coefficients (3.57) of the Nekrasov partition function are well-defined for complex values of the parameters as soon as we require

$$e_\alpha e_\beta^{-1} \notin \{t_1^x t_2^y : x, y \in \mathbf{Z}\} \qquad \forall\, \alpha \neq \beta \in \{1, \ldots, r\}. \tag{7.84}$$

In order to achieve the same well-definedness of all summands for all coefficients in our case, where the parameters t_1 and t_2 lie on the same side of the unit circle in $\mathbf{C}$, we had to additionally assume

$$t_1^x \neq t_2^y, \qquad \forall\, (x, y) \in \mathbf{N}^2 \setminus \{(0,0)\}, \tag{7.85}$$

see equation (7.14). This is a strong condition if t_1 and t_2 lie on the same side of the unit circle and are, for example, both positive real numbers. However, it is automatically satisfied if t_1, t_2 lie on opposite sides of the unit circle. We see, in particular, that the coefficients of the Nekrasov partition function are much better behaved in the case $t_1 = t_2^{-1}$ considered in [12].

7.2.2. General Polynomials

In our integral representation from chapter 5, we have assumed a bound (5.5) on the degree of the polynomial $G(z)$ which we evaluate at the variables z_j^{-1}, $j = 1, \ldots, n$. We have required

$$\deg G(z) \leq r - 1. \tag{7.86}$$

We have done so, to ensure regularity of the integrand at zero. This was necessary to ensure we pick up the right poles in our residue calculation in Theorem 5.2.1 to obtain the coefficients of the Nekrasov partition function. These coefficients are defined for arbitrary polynomials g, see chapter 3. If one wants to allow arbitrary inverse powers of the integration variables in the integrand, one has to add a prescription to the integral which tells one to disregard all residues coming from poles at zero, see for example [37, Proposition II.7]. Such kind of prescriptions are common in the field but spoil any attempt at estimating the integral. Such an estimate requires precise knowledge of the location of the integration contours.

It would still be desirable to find a way to be able to perform an estimate of the integral in the presence of arbitrary polynomials $G(z)$, evaluated at the inverses of the integration variables. A motivation for this comes from conjectures [5, 8] about a generalization of the notion of deformed Gaiotto states and a generalization of the AGT relation relating the norm of those states to the five-dimensional Nekrasov partition functions for gauge groups $SU(r)$ with $r \geq 3$. However, the relevant Nekrasov partition functions are defined [8] for polynomials f and g with

$$\deg f = \deg g = r. \tag{7.87}$$

Hence our technique does not apply.

7.2.3. Four-Dimensional Nekrasov Partition Function

In this thesis, we have considered the K-theoretic Nekrasov partition function. We have described its mathematical definition in chapter 3. Physically, it corresponds to the partition function of the five-dimensional super Yang Mills theory described in section 1.5. The partition function of its four-dimensional counterpart, described in section 1.2, also has a precise mathematical definition [35]. One has to replace equivariant K-theory with equivariant cohomology [39, 35]. Localization techniques, similar to the one described in chapter 3, apply defining the four-dimensional Nekrasov partition function as a formal power series. The relation between the coefficients of the K-theoretic version and the four-dimensional version is simple: One takes definition (3.64) of the K-theoretic Nekrasov partition function, which reads

$$Z(\epsilon_1, \epsilon_2, \vec{a}, \lambda; f, g; \mathfrak{b}) = \sum_{n \geq 0} \left(\mathfrak{b} \; \lambda^{2r - \deg f + \deg g} e^{-r\lambda(\epsilon_1+\epsilon_2)/2} \right)^n Z_n(\epsilon_1, \epsilon_2, \vec{a}, \lambda; f, g), \tag{7.88}$$

and sends [36], in each term separately, $\lambda \to 0$. The roots of the polynomials f and g have to be formally expressed as exponentials as the other parameters (3.63). One obtains a formal power series in the variable $\mathfrak{b}$. The scaling factor $\lambda^{2r-\deg f+\deg g}$ ensures that this coefficient-wise limit exists. Assume the real part of the parameters ϵ_1 and ϵ_2 have the same sign. If we take the naive limit $\lambda \to 0$ in our bounds of the radius of convergence formulated in Theorems 7.1.3 and 7.1.4, we then expect the four-dimensional Nekrasov partition function to have radius of convergence bounded from below by one. This expectation agrees with the conjecture that the four-point conformal block (1.29) is analytic in the open unit ball in the complex plane: The four-dimensional Nekrasov partition function is related to this conformal block via the AGT relation [1], as described in chapter 1. The necessary identification of parameters implies, via equation (1.40), that the sign of the real part of both ϵ_1 and ϵ_2 agree, which is what we have assumed in our estimate. However, the convergence of the limits

$$\lim_{\lambda \to 0} \left(\lambda^{2r - \deg f + \deg g} e^{-r\lambda(\epsilon_1+\epsilon_2)/2} \right)^n Z_n(\epsilon_1, \epsilon_2, \vec{a}, \lambda; f, g) \qquad n \geq 0 \tag{7.89}$$

is not uniform in n. In case one considers the representation of $Z_n(\epsilon_1, \epsilon_2, \vec{a}, \lambda; f, g)$ as a sum over partitions, the situation is even worse: One has to impose the inequalities formulated in equation (7.41), which are supposed to hold modulo $\frac{2\pi i}{\lambda}$. Hence the limit $\lambda \to 0$ is ill-defined for those individual terms. Moreover, the estimate in Theorem 5.3.2 of the growth of the integral representation relies on our Theorem 5.3.1 from potential theory. The limit statement in this latter statement is not evidently uniform in the parameter λ, since the function $f(\theta)$ in equation (5.164), defining the probability measures (5.167), depends on this parameter. Hence our estimate of the growth of the coefficients of the five-dimensional Nekrasov partition function is not uniform in the parameter λ. Hence we cannot conclude from our estimate that the radius of convergence of the four-dimensional Nekrasov partition function is bounded from below by one.

In order to prove convergence of the four-dimensional Nekrasov partition function, one could try and use a corresponding integral representation. Such representations exist [39]. However, a prerequisite for an estimate of the integral would be explicit integration contours. As far as we know, one would have to close the contours for each integration variable at infinity to be able to get the correct residues. Considering the explicit form of the integrand [39, Equation (3.10)], one sees that the degrees of both the polynomials f and g have to be bounded as

$$\deg g(z) + \deg f(z) \leq 2(r-1). \tag{7.90}$$

7.2.4. Generalizations

In this thesis, we have considered the Nekrasov partition function computed for the compactification of $\mathcal{N} = 2$ supersymmetric Yang Mills theory on a circle. Other types of gauge theory include $\mathcal{N} = 2^*$ supersymmetric Yang Mills theory, i.e. with massive matter in the adjoint representation. For this theory, we have to modify our integral from chapter 5 in the following way: Let $m > 0$ denote the mass parameter of the adjoint matter. Assume

$$m \in (0, q_1 q_2) \cup (1, \infty). \tag{7.91}$$

Firstly, we have to multiply the pre-factor in line 5.10 by

$$\left(\frac{(1 - q_1 m^{-1})(1 - q_2 m^{-1})}{(1 - m^{-1})(1 - q_1 q_2 m^{-1})} \right)^n. \tag{7.92}$$

Secondly, we have to multiply the double product in line (5.8) by the term

$$\prod_{j \neq k} \frac{(z_j - q_1 m^{-1} z_k)(z_j - q_2 m^{-1} z_k)}{(z_j - m^{-1} z_k)(z_j - q_1 q_2 m^{-1} z_k)}. \tag{7.93}$$

Finally, one has to specialize the polynomials $F(z)$ and $G(z)$ in line (5.9) to

$$F(z) = \prod_{m=1}^{r} (z - u_\alpha m^{-1}) \qquad G(z) = \prod_{m=1}^{r} (z - u_\alpha^{-1} q_1 q_2 m^{-1}). \tag{7.94}$$

Now assume this integral representation is correct [39, equation (3.25)]. In particular, we ignore the bounds on the coefficients of $G(z)$. Provided that we can estimate the normalization coefficients, analogous to the a_n from line (5.171), as in equation (5.178), we can estimate the growth of the coefficients defined by our modified integral as in chapter 5. Indeed, both the pre-factor (7.92) and the double product (7.93) are positive, so we can use our language of potential theory. The double product (5.8), multiplied by the double product (7.93), in coordinates $z_j = \rho e^{i\theta_j}$, is still of the form

$$\exp\left(-\sum_{j \neq k} \tilde{f}(\theta_k - \theta_k) \right), \tag{7.95}$$

where $\tilde{f} : \mathbf{T} \to \mathbf{R} \cup \{+\infty\}$ is continuous, bounded from below, has a single pole at $\theta = 0$. Its Fourier coefficients $\tilde{f}_k$ are given by

$$\tilde{f}_k = \begin{cases} \left(1 - (mq_1^{-1}q_2^{-1})^{|k|}\right) f_k & \text{if } m \in (0, q_1 q_2) \\ \left(1 - m^{-|k|}\right) f_k & \text{if } m > 1 \end{cases} \qquad (k \in \mathbf{Z}), \tag{7.96}$$

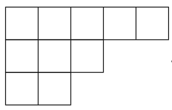
where f_k are the Fourier coefficients of the original function f given in equation (6.20). Hence $\tilde{f}_k > 0$ for $k \neq 0$ and $\tilde{f}_0 = 0$. Here, the condition (7.91) is necessary. The positivity $\tilde{f}_k > 0$ for $k \neq 0$ and $\tilde{f}_0 = 0$ are the only requirements for the proof of Theorem 6.2.1 from chapter 6. Hence Theorem 5.3.3 is valid for our modified coefficients without any modification of the bounds stated in equations (5.198) and (5.199), provided the integral representation is correct and we can estimate the normalization coefficients analogous to the a_n in line (5.171).

A. Conventions for Partitions and Young Diagrams

We use the following conventions regarding partitions of nonnegative integers n: A partition $Y = (Y(1), Y(2), \ldots, Y(l))$ of size n is an nonincreasing sequence of nonnegative integers that sum to n. We write $l(Y) = l$ for the length of the partition $Y = (Y(1), \ldots, Y(l))$ and $|Y|$ for its size $Y(1) + \cdots + Y(l)$. To a partition Y one associates its Young diagram

$$\{(x, y) : 1 \leq x \leq l(Y), 1 \leq y \leq Y(x)\}. \tag{A.1}$$

We use the English convention to draw the Young diagram corresponding to the partitions. For example the partition $Y = (5, 3, 2)$ of size 10 has length $l(Y) = 3$ and its Young diagram is given by



. (A.2)

In the Young diagram, the row index x increases as we go south and the column index y increases as we go east. We identify the partition $Y = (Y(1), \ldots, Y(l))$ with its Young diagram. In particular, we write

$$Y = \{(x, y) : 1 \leq x \leq l(Y), 1 \leq y \leq Y(x)\}. \tag{A.3}$$

We use the notation $Y(j)$ to refer to the j-th element of a partition Y and not a subscript since we often encounter tuples $\vec{Y} = (Y_1, \ldots, Y_r)$ of partitions $Y_1, \ldots, Y_r$. The total size of such an r tuple of partitions is defined as $|Y_1| + \cdots + |Y_r|$.

B. Evaluation of Iterated Residues

In this appendix, we conclude the proof of Theorem 5.2.1. Fix an r tuple $\vec{Y}$ of partitions of total size n. Fix complex numbers $\hat{z}_1, \ldots, \hat{z}_n$ such that

$$\{\hat{z}_1, \ldots, \hat{z}_n\} = \{z^{\alpha}_{\square} : \square \in Y_\alpha, \alpha = 1, \ldots, r\}. \tag{B.1}$$

We evaluate the iterated residues

$$\mathcal{R}_{\vec{Y}}(q_1, q_2, \vec{u}) := \lim_{\substack{z_j \to \hat{z}_j \\ j=1,\ldots,n}} \left(\prod_{j=1}^{n} (z_j - \hat{z}_j) \quad \frac{\mathcal{I}(z_1, \ldots, z_n; \vec{u})}{z_1 \cdots z_n} \right), \tag{B.2}$$

where

$$\mathcal{I}(z_1, \ldots, z_n; q_1, q_2, \vec{u}) = \prod_{j=1}^{n} \prod_{\alpha=1}^{r} \frac{-u_\alpha z_j}{(z_j - u_\alpha)(q_1 q_2 z_j - u_\alpha)} \tag{B.3}$$

$$\prod_{1 \le j \ne k \le n} \frac{(z_j - z_k)(z_j - q_1 q_2 z_k)}{(z_j - q_1 z_k)(z_j - q_2 z_k)}. \tag{B.4}$$

In particular, we claim they are given by

$$\mathcal{S}_{\vec{Y}}(q_1, q_2, \vec{u}) := \prod_{\alpha,\beta=1}^{r} \left(\prod_{\square \in Y_\alpha} \frac{1}{1 - u_\alpha u_\beta^{-1} q_1^{l_{Y_\alpha}(\square)+1} q_2^{-a_{Y_\beta}(\square)}} \right. \tag{B.5}$$

$$\left. \prod_{\boxtimes \in Y_\beta} \frac{1}{1 - u_\alpha u_\beta^{-1} q_1^{-l_{Y_\beta}(\boxtimes)} q_2^{a_{Y_\alpha}(\boxtimes)+1}} \right). \tag{B.6}$$

The calculation in this appendix is adapted from [50], where it was performed for the special case $r = 2$.

For any r-tuple $\vec{Y}$ of partitions with $Y_r \neq \emptyset$, define the r-tuple $\vec{Y}'$ of partitions by removing the last box from the last partition in $\vec{Y}$, i.e., we set $Y'_\alpha = Y_\alpha$ for $\alpha = 1, \ldots, r-1$ and $Y'_r = (Y_1, \ldots, Y_{l-1}, Y_l - 1)$, where l is the length of Y_r. In terms of Young diagrams, we go from $\vec{Y}$ to $\vec{Y}'$ by removing the box

$$(l, w) := (l(Y_r), Y_r(l)) \tag{B.7}$$

from the last Young diagram Y_r in $\vec{Y}$. We will prove

$$\frac{\mathcal{R}_{\vec{Y}}(q_1, q_2, \vec{u})}{\mathcal{R}_{\vec{Y}'}(q_1, q_2, \vec{u})} = \frac{\mathcal{S}_{\vec{Y}}(q_1, q_2, \vec{u})}{\mathcal{S}_{\vec{Y}'}(q_1, q_2, \vec{u})}. \tag{B.8}$$

This already suffices: Using equation (B.8), we can reduce the statement

$$\mathcal{R}_{\vec{Y}}(q_1, q_2, \vec{u}) = \mathcal{S}_{\vec{Y}}(q_1, q_2, \vec{u}) \tag{B.9}$$

to the case $\vec{Y} = (Y_1, \ldots, Y_{r-1}, \emptyset)$. Both $\mathcal{R}_{\vec{Y}}(q_1, q_2, \vec{u})$ and $\mathcal{S}_{\vec{Y}}(q_1, q_2, \vec{u})$ are invariant under simultaneous permutation of the components of $\vec{Y} = (Y_1, \ldots, Y_r)$ and $u = (u_1, \ldots, u_r)$. This follows directly from the respective definitions. Hence, we can reduce the statement of the theorem to the case $\vec{Y} = (Y_1, \ldots, Y_{r-2}, \emptyset, Y_r)$ and, again using equation (B.8), to the case $\vec{Y} = (Y_1, \ldots, Y_{r-2}, \emptyset, \emptyset)$. Continuing in this fashion, we can reduce the statement to the case $\vec{Y} = (\emptyset, \ldots, \emptyset)$, in which it holds trivially.

In the calculation of both sides of equation (B.8) we have to evaluate telescopic products. In order to group the factors for such evaluations, we will have to keep track when $Y_\alpha(x)$ and $Y_\alpha^T(y)$ remain constant as we vary the row and column indices. Fix $\alpha \in \{1, \ldots, r\}$. We write

$$Y_\alpha = (Y_\alpha(1), \ldots, Y_\alpha(l(Y_\alpha))) = (\underbrace{F_\alpha(1), \ldots, F_\alpha(1)}_{G_\alpha(1) \text{ times}}, \ldots, \underbrace{F_\alpha(m_\alpha), \ldots, F_\alpha(m_\alpha)}_{G_\alpha(m_\alpha) \text{ times}}) \tag{B.10}$$

We set $F_\alpha(m_\alpha + 1) = 0$. Note that for any $x, y \in \mathbf{N}$,

$$Y_\alpha(y) \in \{F_\alpha(j) : j = 1, \ldots, m_\alpha + 1\} \tag{B.11}$$

$$Y_\alpha^T(x) \in \{G_\alpha(1) + \cdots + G_\alpha(j) : j = 0, \ldots, m_\alpha\}. \tag{B.12}$$

Recall that l denotes the length of the last Young diagram Y_r. Define the index j_α by

$$G_\alpha(1) + \cdots + G_\alpha(j_\alpha - 1) < l \leq G_\alpha(1) + \cdots G_\alpha(j_\alpha) \tag{B.13}$$

when this condition can be satisfied and $j_\alpha = m_\alpha + 1$ otherwise. We also introduce the notation $H_\alpha(j) = G_\alpha(1) + \cdots + G_\alpha(j)$. We will split certain products over rows of Young diagrams as

$$\prod_{x=1}^{l}(-) = \prod_{j=1}^{j_\alpha - 1} \prod_{x=H_\alpha(j-1)+1}^{H_\alpha(j)} (-) \times \prod_{x=H_\alpha(j_\alpha-1)+1}^{l} (-) \tag{B.14}$$

When x comes from the product with index $j \in \{1, \ldots, j_\alpha - 1\}$, we have $Y_\alpha(y) = F_\alpha(j)$. In the remaining product, we have $Y_\alpha(y) = F_\alpha(j_\alpha)$. Products over columns of Young diagrams will be grouped as

$$\prod_{y=1}^{Y_\alpha(l)} (-) = \prod_{y=1}^{F_\alpha(j_\alpha)} (-) = \prod_{j=j_\alpha}^{m_\alpha} \prod_{y=F_\alpha(j+1)+1}^{F_\alpha(j)} (-), \tag{B.15}$$

where we have $Y_\alpha^T(y) = H_\alpha(j)$ if y comes from the factor with value j.

The right hand side of equation (B.8) equals

$$\frac{\mathcal{S}_{\vec{Y}}(q_1, q_2, \vec{u})}{\mathcal{S}_{\vec{Y}'}(q_1, q_2, \vec{u})} \tag{B.16}$$

$$= \prod_{\alpha,\beta=1}^{r} \frac{\prod_{\square \in Y'_\alpha} 1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y'_\alpha}(\square)+1} q_2^{-a_{Y'_\beta}(\square)} \prod_{\boxtimes \in Y'_\beta} 1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y'_\beta}(\boxtimes)} q_2^{a_{Y'_\alpha}(\boxtimes)+1}}{\prod_{\square \in Y_\alpha} 1 - \frac{u_\alpha}{u_\beta} q_1^{l_{Y_\alpha}(\square)+1} q_2^{-a_{Y_\beta}(\square)} \prod_{\boxtimes \in Y_\beta} 1 - \frac{u_\alpha}{u_\beta} q_1^{-l_{Y_\beta}(\boxtimes)} q_2^{a_{Y_\alpha}(\boxtimes)+1}}. \tag{B.17}$$

We introduce a variable ξ to be able to ignore poles during the calculation. Regrouping we get

$$\frac{\mathcal{S}_{\vec{Y}}(q_1,q_2,\vec{u})}{\mathcal{S}_{\vec{Y}'}(q_1,q_2,\vec{u})} = \lim_{\xi\to 1} S(\xi)\prod_{\alpha=1}^{r-1} T_\alpha(\xi)U_\alpha(\xi), \tag{B.18}$$

where

$$S(\xi) = \frac{1}{(\xi-q_1)(\xi-q_2)}\prod_{\boxtimes\in Y'_r}\frac{(\xi-q_1^{l_{Y'_r}(\boxtimes)+1}q_2^{-a_{Y'_r}(\boxtimes)})(\xi-q_1^{-l_{Y'_r}(\boxtimes)}q_2^{a_{Y'_r}(\boxtimes)+1})}{(\xi-q_1^{l_{Y_r}(\boxtimes)+1}q_2^{-a_{Y_r}(\boxtimes)})(\xi-q_1^{-l_{Y_r}(\boxtimes)}q_2^{a_{Y_r}(\boxtimes)+1})} \tag{B.19}$$

and, for each $\alpha\in\{1,\dots,r-1\}$,

$$T_\alpha(\xi) = \frac{1}{\xi-u_\alpha u_r^{-1}q_1q_2^{a_{Y_\alpha}(l,w)+1}}\prod_{\boxtimes\in Y_r}\frac{\xi-u_\alpha u_r^{-1}q_1^{-l_{Y'_r}(\boxtimes)}q_2^{a_{Y_\alpha}(\boxtimes)+1}}{\xi-u_\alpha u_r^{-1}q_1^{-l_{Y_r}(\boxtimes)}q_2^{a_{Y_\alpha}(\boxtimes)+1}} \tag{B.20}$$

$$\times\prod_{\square\in Y_\alpha}\frac{\xi-u_\alpha u_r^{-1}q_1^{l_{Y_\alpha}(\square)+1}q_2^{-a_{Y'_r}(\square)}}{\xi-u_\alpha u_r^{-1}q_1^{l_{Y_\alpha}(\square)+1}q_2^{-a_{Y_r}(\square)}} \tag{B.21}$$

$$U_\alpha(\xi) = \frac{1}{\xi-u_ru_\alpha^{-1}q_2^{-a_{Y_\alpha}(l,w)}}\prod_{\boxtimes\in Y_r}\frac{\xi-u_ru_\alpha^{-1}q_1^{l_{Y'_r}(\boxtimes)+1}q_2^{-a_{Y_\alpha}(\boxtimes)}}{\xi-u_ru_\alpha^{-1}q_1^{l_{Y_r}(\boxtimes)+1}q_2^{-a_{Y_\alpha}(\boxtimes)}} \tag{B.22}$$

$$\times\prod_{\square\in Y_\alpha}\frac{\xi-u_ru_\alpha^{-1}q_1^{-l_{Y_\alpha}(\square)}q_2^{a_{Y'_r}(\square)+1}}{\xi-u_ru_\alpha^{-1}q_1^{-l_{Y_\alpha}(\square)}q_2^{a_{Y_r}(\square)+1}}. \tag{B.23}$$

Using

$$a_{Y'_\alpha}(x,y) = \begin{cases} Y_\alpha(x)-y-1, & x=l,\alpha=r \\ Y_\alpha(x)-y, & \text{otherwise}\end{cases}, \tag{B.24}$$

$$l_{Y'_\alpha}(x,y) = \begin{cases} Y_\alpha^T(y)-x-1, & y=w,\alpha=r \\ Y_\alpha^T(y)-x, & \text{otherwise}\end{cases} \qquad (\alpha\in\{1,\dots,r\}) \tag{B.25}$$

and the splitting described in equations (B.14) and (B.15) we get

$$S(\xi) = \frac{1}{(\xi-q_1q_2^{-w+1})(\xi-q_2^w)}\frac{(\xi-q_1)(\xi-q_2)}{(\xi-1)(\xi-q_1q_2)} \tag{B.26}$$

$$\prod_{j=1}^{m_r}\frac{(\xi-q_1^{l-H_r(j)}q_2^{-F_r(j)+w})(\xi-q_1^{-l+H_r(j)+1}q_2^{F_r(j)-w+1})}{(\xi-q_1^{l-H_r(j-1)}q_2^{-F_r(j)+w})(\xi-q_1^{-l+H_r(j-1)+1}q_2^{F_r(j)-w+1})}, \tag{B.27}$$

and, for each $\alpha\in\{1,\dots,r-1\}$,

$$T_\alpha(\xi) = \frac{1}{\xi-u_\alpha u_r^{-1}q_1^{l(Y_\alpha)-l+1}q_2^{-w+1}}\prod_{j=1}^{m_\alpha}\frac{\xi-u_\alpha u_r^{-1}q_1^{-l+H_\alpha(j)+1}q_2^{F_\alpha(j)-w+1}}{\xi-u_\alpha u_r^{-1}q_1^{-l+H_\alpha(j-1)+1}q_2^{F_\alpha(j)-w+1}}, \tag{B.28}$$

$$U_\alpha(\xi) = \frac{1}{\xi-u_ru_\alpha^{-1}q_1^{-l(Y_\alpha)+l}q_2^w}\prod_{j=1}^{m_\alpha}\frac{\xi-u_ru_\alpha^{-1}q_1^{l-H_\alpha(j)}q_2^{-F_\alpha(j)+w}}{\xi-u_ru_\alpha^{-1}q_1^{l-H_\alpha(j-1)}q_2^{-F_\alpha(j)+w}}. \tag{B.29}$$

Together

$$\frac{\mathcal{S}_{\vec{Y}}(q_1,q_2,\vec{u})}{\mathcal{S}_{\vec{Y}'}(q_1,q_2,\vec{u})} = \lim_{\xi\to 1}\frac{(\xi-q_1)(\xi-q_2)}{(\xi-1)(\xi-q_1q_2)} \tag{B.30}$$

$$\times\prod_{\alpha=1}^{r}\left(\frac{1}{(\xi-u_\alpha u_r^{-1}q_1^{l(Y_\alpha)-l+1}q_2^{-w+1})(\xi-u_ru_\alpha^{-1}q_1^{-l(Y_\alpha)+l}q_2^{w})}\right. \tag{B.31}$$

$$\left.\times\prod_{j=1}^{m_\alpha}\frac{(\xi-u_ru_\alpha^{-1}q_1^{l-H_\alpha(j)}q_2^{-F_\alpha(j)+w})(\xi-u_\alpha u_r^{-1}q_1^{-l+H_\alpha(j)+1}q_2^{F_\alpha(j)-w+1})}{(\xi-u_ru_\alpha^{-1}q_1^{l-H_\alpha(j-1)}q_2^{-F_\alpha(j)+w})(\xi-u_\alpha u_r^{-1}q_1^{-l+H_\alpha(j-1)+1}q_2^{F_\alpha(j)-w+1})}\right). \tag{B.32}$$

For the residue calculation, fix the order of the variables such that the integration over z_n picks up the residue $z^r_{l,w}$ coming from the box $(l,w)\in Y_r$ we remove from the last partition in $\vec{Y}$ to get $\vec{Y}'$. The left hand side of equation (B.8) equals

$$\frac{\mathcal{R}_{\vec{Y}}(q_1,q_2,\vec{u})}{\mathcal{R}_{\vec{Y}'}(q_1,q_2,\vec{u})} = \lim_{z_j\to\hat{z}_j, j=1,\ldots,n}(1-\frac{z^r_{l,w}}{z_n})\frac{\mathcal{I}(z_1,\ldots,z_n;q_1,q_2,\vec{u})}{\mathcal{I}(z_1,\ldots,z_{n-1};q_1,q_2,\vec{u})}. \tag{B.33}$$

We take the first $(n-1)$ limits separately: The quotient $\frac{\mathcal{I}(z_1,\ldots,z_n;q_1,q_2,\vec{u})}{\mathcal{I}(z_1,\ldots,z_{n-1};q_1,q_2,\vec{u})}$ converges to

$$\prod_{\alpha=1}^{r}\left(\frac{-u_\alpha z_n}{(z_n-u_\alpha)(q_1q_2z_n-u_\alpha)}\right. \tag{B.34}$$

$$\left.\times\prod_{\square\in Y'_\alpha}\frac{(z_n-z^\alpha_\square)^2(z_n-q_1q_2z^\alpha_\square)(z_n-q_1^{-1}q_2^{-1}z^\alpha_\square)}{(z_n-q_1z^\alpha_\square)(z_n-q_2z^\alpha_\square)(z_n-q_1^{-1}z^\alpha_\square)(z_n-q_2^{-1}z^\alpha_\square)}\right) \tag{B.35}$$

for $z_j\to\hat{z}_j, j=1,\ldots,n-1$. The factors with $\alpha\neq r$ do not have poles for $z_n\to z^r_{l,w}$ since $u_\alpha/u_r\notin\{q_1^xq_2^y: x,y\in\mathbf{Z}\}$. We define $\xi=\frac{z_n}{z^r_{l,w}}$ and set

$$A_\alpha(\xi) := \prod_{\square\in Y_\alpha}\frac{(\xi-\frac{z^\alpha_\square}{z^r_{l,w}})(\xi-q_1q_2\frac{z^\alpha_\square}{z^r_{l,w}})(\xi-\frac{z^\alpha_\square}{z^r_{l,w}})(\xi-q_1^{-1}q_2^{-1}\frac{z^\alpha_\square}{z^r_{l,w}})}{(\xi-q_1\frac{z^\alpha_\square}{z^r_{l,w}})(\xi-q_2\frac{z^\alpha_\square}{z^r_{l,w}})(\xi-q_1^{-1}\frac{z^\alpha_\square}{z^r_{l,w}})(\xi-q_2^{-1}\frac{z^\alpha_\square}{z^r_{l,w}})}. \tag{B.36}$$

Hence, the remaining limit $z_n\to z^r_{l,w}$ is given by

$$\frac{\mathcal{R}_{\vec{Y}}(q_1,q_2,\vec{u})}{\mathcal{R}_{\vec{Y}'}(q_1,q_2,\vec{u})} \tag{B.37}$$

$$=\lim_{\xi\to1}\frac{(\xi-q_1^{-1})(\xi-q_2^{-1})}{(\xi-1)(\xi-q_1^{-1}q_2^{-1})}\prod_{\alpha=1}^{r}\left(\frac{-\xi u_\alpha z^r_{l,w}}{(\xi z^r_{l,w}-u_\alpha)(q_1q_2\xi z^r_{l,w}-u_\alpha)}A_\alpha(\xi)\right), \tag{B.38}$$

Using the splitting described in (B.14) and (B.15) we get

$$A_\alpha(\xi) = \frac{(\xi-\frac{u_\alpha}{u_r}q_1^{1-l}q_2^{1-w})(\xi-\frac{u_\alpha}{u_r}q_1^{-l}q_2^{-w})}{(\xi-\frac{u_\alpha}{u_r}q_1^{l(Y_\alpha)-l+1}q_2^{1-w})(\xi-\frac{u_\alpha}{u_r}q_1^{l(Y_\alpha)-l}q_2^{-w})} \tag{B.39}$$

$$\times\prod_{j=1}^{m_\alpha}\frac{(\xi-\frac{u_\alpha}{u_r}q_1^{H_\alpha(j)-l+1}q_2^{F_\alpha(j)-w+1})(\xi-\frac{u_\alpha}{u_r}q_1^{H_\alpha(j)-l}q_2^{F_\alpha(j)-w})}{(\xi-\frac{u_\alpha}{u_r}q_1^{H_\alpha(j-1)-l+1}q_2^{F_\alpha(j)-w+1})(\xi-\frac{u_\alpha}{u_r}q_1^{H_\alpha(j-1)-l}q_2^{F_\alpha(j)-w})}. \tag{B.40}$$

Finally, we use the step

$$(\rho - \sigma) = -\rho\sigma(\rho^{-1} - \sigma^{-1}) \tag{B.41}$$

repeatedly to conclude

$$\frac{\mathcal{R}_{\vec{Y}}(q_1, q_2, \vec{u})}{\mathcal{R}_{\vec{Y}'}(q_1, q_2, \vec{u})} \tag{B.42}$$

$$= \lim_{\xi \to 1} \frac{(\xi - q_1^{-1})(\xi - q_2^{-1})}{(\xi - 1)(\xi - q_1^{-1} q_2^{-1})} \tag{B.43}$$

$$\prod_{\alpha=1}^{r} \left(\frac{-\xi u_\alpha u_r^{-1} q_1^{-l} q_2^{-w}}{(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l+1} q_2^{1-w})(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l} q_2^{-w})} \right. \tag{B.44}$$

$$\left. \times \prod_{j=1}^{m_\alpha} \frac{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l+1} q_2^{F_\alpha(j)-w+1})(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l} q_2^{F_\alpha(j)-w})}{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l+1} q_2^{F_\alpha(j)-w+1})(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l} q_2^{F_\alpha(j)-w})} \right) \tag{B.45}$$

$$= \lim_{\xi \to 1} \frac{(\xi^{-1} - q_1)(\xi^{-1} - q_2)}{(\xi^{-1} - 1)(\xi^{-1} - q_1 q_2)} \tag{B.46}$$

$$\prod_{\alpha=1}^{r} \left(\frac{1}{(\xi - \frac{u_\alpha}{u_r} q_1^{l(Y_\alpha)-l+1} q_2^{1-w})(\xi^{-1} - \frac{u_r}{u_\alpha} q_1^{-l(Y_\alpha)+l} q_2^{w})} \right. \tag{B.47}$$

$$\left. \times \prod_{j=1}^{m_\alpha} \frac{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j)-l+1} q_2^{F_\alpha(j)-w+1})(\xi^{-1} - \frac{u_r}{u_\alpha} q_1^{-H_\alpha(j)+l} q_2^{-F_\alpha(j)+w})}{(\xi - \frac{u_\alpha}{u_r} q_1^{H_\alpha(j-1)-l+1} q_2^{F_\alpha(j)-w+1})(\xi^{-1} - \frac{u_r}{u_\alpha} q_1^{-H_\alpha(j-1)+l} q_2^{-F_\alpha(j)+w})} \right) \tag{B.48}$$

$$= \frac{\mathcal{S}_{\vec{Y}}(q_1, q_2, \vec{u})}{\mathcal{S}_{\vec{Y}'}(q_1, q_2, \vec{u})} \tag{B.49}$$

since no factor $(\xi^{+1} - \cdots)$ vanishes in the limit.

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