Non-decreasing Payment Rules in Combinatorial Auctions

Master Thesis

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Non-decreasing Payment Rules in Combinatorial Auctions

Master Thesis
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April 26, 2018

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Abstract

Combinatorial Auctions are used to allocate resources in domains where bidders have complex preferences over bundles of items. However, the behavior of bidders under different payment rules is not well understood, and there has been limited success in finding Bayes-Nash equilibria of such auctions due to the computational difficulties involved. In this master thesis, I introduce a series of payment rules, which are called non-decreasing payment rules. Under such a rule, a bidder can not decrease his payment by increasing his bids, which is a natural and intuitive property. However, the quadratic payment rule violates this property and even in a simple single-minded auction, bidders can manipulate their payments surprisingly. In contrast, I show that many other well-known payment rules have this property. Furthermore, I show that there exists an efficient algorithm which can find $\epsilon$-Bayesian Nash Equilibria in CAs with non-decreasing payment rules. I prove that the $\epsilon$ computed by this algorithm is the true $\epsilon$. This algorithm outperforms a recent state of the art algorithm by multiple orders of magnitude. Furthermore, I extend this algorithm to auctions with correlated valuations of bidders and introduce a polynomial approximation of bidders' utility function. Last but not least, I study over-bidding strategies in CAs and prove that an over-bidding strategy is never a best response in any single-minded CA with non-decreasing core-selecting payment rule.
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Combinatorial auctions (CAs) are widely used in the real world, where they are applied in multi-billion dollar domains [11, 25]. However, due to their complexity, a perfect mechanism for such auctions does not exist. The well-known Vickrey-Clarke-Groves (VCG) mechanism has been shown to be unsuitable for CAs [4]: even in auctions which are highly competitive, the VCG mechanism leads to a very low or even zero revenue with high probability, which does not meet the expectation of auctioneers.

Therefore, new mechanisms have been proposed for CAs and studied independently [12, 3, 13]. Nevertheless, these mechanisms are not strategy proof, resulting in dishonesty of bidders during auctions. Under such circumstances, intuitive actions cannot be guaranteed, making the auction’s outcome hard to predict. To analyse the consequences of such strategic behaviour, we have to study the equilibria in CAs.

1.1 Bayes Nash Equilibria in Combinatorial Auctions

In combinatorial auctions every bidder has a strategy, described as a function of his or her valuation to corresponding bids. Meanwhile, they want to keep their valuations of bundles in secret; accordingly, bids of others are considered unknown by each bidder. The only public information is value distributions of bidders and the strategy profile, with which bidders can compute their expected utilities. Due to limited information available to bidders, it is intractable to study any non-randomized strategies and Full information Nash equilibria. An alternative called Bayes-Nash Equilibria (BNE) is used to represent equilibria in CAs with imperfect information.

A BNE is a strategy profile $s^* = (s^*_1, s^*_2, ..., s^*_n)$, where no bidder can achieve a higher expected utility by changing his own strategy. Previous studies [12, 3, 13] have shown good analytical results of BNEs in small auctions,
yet analytically computing BNEs in complicated CAs is still a challenging problem[9]. Therefore, an algorithm that numerically computes BNEs in CAs is required.

1.2 Prior Algorithmic Work on Computing BNEs

In previous years, many algorithms for computing BNEs have been proposed. Due to the limitations of numerical solutions to continuous problems, however, many of the studies reduce action spaces and create algorithms under the restriction that bidders have only finite number of valuations and bids at their disposal [15, 16]. It is still computationally expensive to solve these constrained problems.

There are also attempts to deal with infinite auctions which look for ε-BNEs instead of BNEs. In ε-BNEs, bidders play a strategy that achieves at most ε less utility than the best response to the strategies of others, in which ε denotes the utility loss. One important class of these algorithms [20, 28, 19] is based on the iterated best response method [8]. These algorithms restrict settings in infinite CAs such as the type space, the action space, and the size of games, and the utility loss ε is computed under such restrictions. [6] propose a new algorithm that introduces a verification step to make sure that the estimated utility loss is not only in the restricted space but also in the whole space. My semester thesis [29] proposed a new algorithm based on utility plane method, but that algorithm only works for simple single-minded combinatorial auctions.

1.3 Overview of this Thesis

To study the payment rules in combinatorial auctions, I introduce a new property of payment rules, which is desirable and intuitive. I call payment rules with this property non-decreasing. I analyse many well-known and widely used payment rules. I show that the quadratic payment rule is not non-decreasing, which means that bidders can manipulate their payment surprisingly. More precisely, bidders can decrease their payments by increasing their bids and still win the same bundle of items. I also prove that many other payment rules are non-decreasing, which lead to less cheating in the auction.

It is computationally expensive to find the best response. Researchers[6] simplify the problem by computing the best response at fixed control points in the action space discretely and then approximating the strategy function with this point. However, this method cannot give a theoretical bound of its results. In my master thesis, I introduced a new method based on utility plane computation to solve this problem in my master thesis[29], but
that work only designed for simple single-minded auction. In this work, I extend the previous theoretical-proven algorithm into middle size multi-minded auctions and prove that there is only a finite number of potential best response strategies if every bidder has a piecewise constant strategy. Nevertheless, the action space is not restricted and the search of the best response is performed in the entire action space, which is totally different from [19] where fixed bid levels are applied. Our algorithm inherits the concept of a verification step in [6] and provides optimizations.

In previous work[19, 29], researchers used linear functions to present bidders’ utility in the valuation-utility space when the bids are fixed. I extend the utility plane method to compute approximate utility curves which makes the algorithm work even for correlated valuations of bidders. Compared to known analytical BNE [3], the output of this algorithm is relatively acceptable and this method is very easy to adjust to any correlated models.

Last but not least, I study over-bidding strategies in combinatorial auctions. I prove that in any single-minded CA with a non-decreasing core-selecting payment rule, over-bidding strategies are always dominated by the truthful strategy. I also analyse the complicated situation in multi-minded CAs, and give some intuition why it is more difficult to make a similar statement in the general case.

The thesis is organized as follows. In chapter 2, I come up with formal definitions. I introduce preliminary knowledge of payment rules in chapter 3. Chapter 4 defines non-decreasing payment rules and proves its existence in core-selecting payment rules. In chapter 5, I study over-bidding strategies in CAs with non-decreasing core-selecting payment rules. I present the algorithm for multi-minded CAs and compare the results of the algorithm in the LLLLGG auctions with different payment rules with previous algorithm in chapter 6. Chapter 7 discusses an extension of my approach to CAs with correlated valuation of bidders. Possible future work is discussed in the conclusion.
Chapter 2

Formal Model

In this chapter, I define the formal model which is used in this master thesis.

2.1 Auctions

2.1.1 Single-Item Sealed-Bid Auctions

Let $N = \{1, 2, \ldots, n\}$ denote a set of bidders. Each bidder $i \in N$ has valuation $v_i \in [0, v_{\text{max}}]$ for some $v_{\text{max}} > 0$. This valuation of bidder $i$ represents his or her willingness-to-pay for the item. For example, the valuation could represent his or her willingness-to-pay for a license of a computer game, his or her willingness-to-pay for a famous painting of a well-known artist, or his or her willingness-to-pay for a new version of iPhone.

In a (single-item) sealed-bid auction, every bidder submits a bid $b_i \in \mathbb{R}$ to the auctioneer or the mechanism without knowledge of other bidders’ bids. Let $b = (b_1, b_2, \ldots, b_n)$ denote the bid profile of bidders.

Definition 2.1 (Single-Item Sealed-Bid Auction) Given a bid profile $b$, a sealed-bid auction is defined with an allocation rule and a payment rule.

- An allocation rule $x(b) = (x_1(b), x_2(b), \ldots, x_n(b))$, where $x_i(b) \in \{0, 1\}$ for $i \in N$ indicates whether or not bidder $i$ is allocated the item.

- A payment rule $t(b) \in \mathbb{R}^n$, where $t_i(b)$ for $i \in N$ indicates the payment made by bidder $i$.

2.1.2 Combinatorial Auctions

In a combinatorial auction, bidders are allowed to bid on multiple items at the same time and state their willingness-to-pay for each distinct bundle of items.
Let $N = \{1, 2, ..., n\}$ denote a set of bidders, and $M = \{1, 2, ..., m\}$ denote a set of items. Each bidder $i \in N$ has a valuation $v_i \in \mathbb{R}^2$, with $v_{ij} \in [0, v_{\text{max}}]$ for some $v_{\text{max}} > 0$. This valuation of player $i$ represents his or her willingness-to-pay for each bundle of items.

The combinatorial auctions which are discussed in this master thesis are also sealed-bid. Every bidder submits a bid $b_i \in \mathbb{R}^2$ to the auctioneer without knowledge of other bidders’ bids. Let $b = (b_1, b_2, ..., b_n)$ denote the bid profile of bidders.

**Definition 2.2 (Combinatorial Auction)** Given a bid profile $b$, a combinatorial auction is defined with an allocation rule and a payment rule.

- An allocation rule $x(b) = (x_1(b), x_2(b), ..., x_n(b))$, where $x_i(b) \in \{0, 1\}^2$ for $i \in N$ indicates whether or not bidder $i$ is allocated each bundle of items. Each item can be given to at most one bidder. The bidding language of the auction is XOR, which means at most one bid made by a bidder may be accepted by the auctioneer.
- A payment rule $t(b) \in \mathbb{R}^n$, where $t_i(b)$ for $i \in N$ indicates the payment made by bidder $i$.

### 2.2 Allocation of Auctions

The auctions which are studied in this master thesis are all normalized, which means that the payment is zero if the bidder does not win anything. The auction are also standard, which means allocating items to bidders with the highest bids (breaking ties at random). Winners are determined by the standard allocation. There is no coalition of bidders which has higher bid than the sum of winners’ bid on their winning bundles.

There are also two ideal objectives for auctions[26],

- **Allocative efficiency**: Allocate items to bidders with the highest value. This is the efficient allocation, and auctions with this property are referred to as efficient auctions. This means the result of the auction leads to the maximum social welfare among all bidders.
- **Revenue maximization**: Maximize the expected revenue given bidders with values sampled from a known distribution. Auctions with this property are referred to as optimal auctions.

### 2.2.1 Core Selecting Payment Rules

In this master thesis, I assume all utility functions are quasilinear.
2.2. Allocation of Auctions

**Definition 2.3 (Quasilinear Utility)** A utility function is quasilinear if it is of the form $u(x_1, x_2, \ldots, x_L) = x_1 + \Theta(x_2, \ldots, x_L)$ and $\Theta(x_2, \ldots, x_L)$ is strictly concave. A quasilinear function in one-dimension can be represented as $u = v - p$.

For example, the utility function is denoted as $u_i = v_i(X_i) - p_i$ for allocation $X$ and payment $p_i$, which represents the payoff of bidder $i$. The auctioneer’s payoff is the revenue.

For a set $L \subseteq N$ of bidders, let $V(L)$ denote the coalitional value function, defined to be the value of the allocation that maximizes the total value of bidders in set (or coalition) $L$.

**Definition 2.4 (Core Selecting Payment Rule)** Given a bid profile $b$ and an individually rational (IR) allocation of items, a payment rule is core selecting if no coalition $L \in N$ can break away and trade with the auctioneer in a way that all bidders in $L$ and the auctioneer are strictly better off. Equivalently:

$$V(L) = \sum_{i \in N} p_i + \sum_{i \in L} u_i, \text{ for any } L \subseteq N \quad (2.1)$$

The following example describes the core in an auction.

**Example 2.5** Let $m = 2$ items, $A$ and $B$, $n = 3$ bidders, and let bids be as follows.

For bidder 1: $b_1(A) = 6, b_1(B) = 0, b_1(AB) = 0$;

For bidder 2: $b_2(A) = 0, b_2(B) = 4, b_2(AB) = 0$;

For bidder 3: $b_3(A) = 0, b_3(B) = 0, b_3(AB) = 9$.

It is easy to determine that the winners are bidder 1 and bidder 2. The core of this auction is shown in Figure 2.1. A core selecting payment rule gives a payment that lies in the grey triangle.
2.3 Bayesian Nash Equilibria (BNE)

**Definition 2.6 (Strategy)** Each bidder $i \in N$ has a valuation $v_i \in \mathbb{R}^{2M}$ with each $v_{ij} \in [0, v_{\text{max}}]$ for some $v_{\text{max}} > 0$. A strategy, $s_i(v_i) \in \mathbb{R}^{2M}$, defines bidder $i$’s bid for every possible value $v_i$.

**Definition 2.7 (Piecewise Constant Strategy)** A strategy is said to be piecewise constant if it is locally constant in connected regions separated by a possibly infinite number of lower-dimensional boundaries.

The Heaviside step function, rectangle function, and square wave are examples of one-dimensional piecewise constant functions.

**Definition 2.8 (Best Response Strategy)** The best response strategy $BR(s_{-i}, v_i) \in \mathbb{R}^{2M}$ is a strategy that achieves the highest expected utility for every possible value $v_i$ when other players’ strategies are $s_{-i}$ and value distributions of all other bidders are given.

**Definition 2.9 (Bayes-Nash Equilibrium)** A Bayes-Nash equilibrium (BNE) is a strategy profile $s^* = (s^*_1, ..., s^*_n)$ in an auction if and only if, for any bidder $i$, any value $v_i$, and any bid $b_i$, $E_{\theta_{-i}}[u_i(s^*_i(v_i), s^*_{-i}(v_{-i}))] \geq E_{\theta_{-i}}[u_i(b_i, s^*_{-i}(v_{-i}))]$, for all $v_i$. When value distributions of all bidders are given.

**Definition 2.10 (Utility Loss)** The utility loss $\epsilon$ of a strategy $s_i$ at value $v_i$ is defined as follows. Bidder $i$ knows the strategy profile $s = (s_1, ..., s_n)$ and computes his best response strategy $BR(s_{-i}, v_i)$.
2.3. Bayesian Nash Equilibria (BNE)

\[ \epsilon_i^{abs}(v_i) = \mathbb{E}_{v_{-i}} [u_i(BR(s_{-i}, v_i), v_{-i})] - \mathbb{E}_{v_{-i}} [u_i(v_i, v_{-i})] \]

**Definition 2.11 (\(\epsilon\)-Bayes-Nash Equilibrium)** An \(\epsilon\)-Bayes-Nash equilibrium (\(\epsilon\)-BNE) is a strategy profile \(s^* = (s^*_1, ..., s^*_n)\) in an auction if and only if, for any bidder \(i\), any value \(v_i\), and any bid \(b_i\),

\[ \mathbb{E}_{v_{-i}} [u_i(s^*_i(v_i), s^*_{-i}(v_{-i}))] \geq \epsilon + \mathbb{E}_{v_{-i}} [u_i(b_i, s^*_{-i}(v_{-i}))], \] it is an absolute \(\epsilon\)-BNE; for all \(v_i\) and all bids \(b_i\).
Chapter 3

Preliminaries

In this chapter, I introduce some preliminary knowledge for the following study.

3.1 Combinatorial Auctions

The following two auctions were studied as example of combinatorial auctions in this master thesis.

Definition 3.1 (LLG Auction[3]) There are 3 bidders in the auction. Two items are prepared for sale. Two of three bidders, 1 and 2, are local bidders, who are interested in only one item and they are interested in different item. There is no benefit for them to win another item which is not of interest to them. Their values are denoted as $v_1$ and $v_2$, respectively. These valuation are drawn from $\mathbb{R} [0, 1]$. The third bidder, 3, is interested in both items and has no utility from acquiring just one of them. Her value for the bundle of items is denoted as $v_g \in \mathbb{R} [0, 2]$.

Definition 3.2 (LLLLGG Auction[6]) There are 6 bidders in the auction. Eight items are prepared for sale. 4 of the bidders, $L_1$ to $L_4$, are local bidders and the rest of them, $G_1$ and $G_2$, are global bidders. Each of them is interested in two bundles. Their interest are as enumerated in Table 3.1. Each bidder $L_i$ draws its two bundle values from $\mathbb{R} [0, 1]$, while bidders $G_1$ and $G_2$ draw their two bundle values from $\mathbb{R} [0, 2]$; all draws are independent.
3. Preliminaries

Table 3.1: The Multi-Minded LLLLGG domain has 8 goods and 6 bidders. Each bidder is interested in exactly two bundles.

<table>
<thead>
<tr>
<th>Bidder</th>
<th>Bundle 1</th>
<th>Bundle 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>L₁</td>
<td>AB</td>
<td>BC</td>
</tr>
<tr>
<td>L₂</td>
<td>CD</td>
<td>DE</td>
</tr>
<tr>
<td>L₃</td>
<td>EF</td>
<td>EG</td>
</tr>
<tr>
<td>L₄</td>
<td>GH</td>
<td>HA</td>
</tr>
<tr>
<td>G₁</td>
<td>ABCD</td>
<td>EFGH</td>
</tr>
<tr>
<td>G₂</td>
<td>CDEF</td>
<td>GHAB</td>
</tr>
</tbody>
</table>

3.2 Payment Rules

After deciding the winners of the auction, payments should be made by each bidder who is allocated at least one item. In this master thesis, I studied seven payment rules, including Vickrey-Clarke-Groves (VCG), First-Price Package, Nearest-Bid, Nearest-VCG, Quadratic Rule (Nearest-VCG with Minimum Revenue of the Auctioneer), Proxy Auction and Proportional Pricing.

Definition 3.3 (Vickrey-Clarke-Groves (VCG)) Given a bid profile b and an allocation of items X, the VCG mechanism asks each winning bidder to pay $p_i = \sum_{k l \in X'} b_k(l) - \sum_{k l \in X, k \neq i} b_k(l)$, where $X'$ is the efficient allocation if bidder i is absent from this auction. $k l \in X$ represents that bidder k is allocated the l-th bundle of items that he or she is interested in. Other bidders’ payments are 0.

In Example 2.5, if the payment rule of the auction is Vickrey-Clarke-Groves (VCG), then the payment of bidder 1 is $p_1 = b_3(AB) - b_2(B) = 9 - 4 = 5$ and the payment of bidder 2 is $p_2 = b_3(AB) - b_1(A) = 9 - 6 = 3$. Bidder 3 does not win anything so his payment is $p_3 = 0$.

The following six payment rules are all core selecting payment rules.

Definition 3.4 (First Price Package) Given a bid profile b and an allocation of items X, the first price package asks each winning bidder to pay $p_i = b_{ij}$ if and only if $ij \in X$ where $ij$ represents the jth bundle of items that bidder i is interested in. Other bidders’ payments are 0.

In Example 2.5, if the payment rule of the auction is first price package, then the payment of bidder 1 is $p_1 = b_1(A) = 6$ and the payment of bidder 2 is $p_2 = b_2(B) = 4$. Bidder 3 does not win anything so his payment is $p_3 = 0$.

Definition 3.5 (Nearest-Bid) Given a bid profile b and an allocation of items X, the Nearest-Bid first calculates the First Price Package payments and then find the nearest point in the core to the VCG payments point which minimizes the revenue of the auctioneer. The payments of the auction are determined by this nearest point. Other bidders’ payments are 0.
3.2. Payment Rules

In Example 2.5, if the payment rule of the auction is Nearest-Bid, we calculate the First Price Package payments first. $p_{\text{FirstPrice}}^1 = b_1(A) = 6$ and $p_{\text{FirstPrice}}^2 = b_2(B) = 4$. Then we find the nearest point as in Figure 3.1. Thus, the payment of bidder 1 is $p_{\text{Nearest-Bid}}^1 = 5.5$ and the payment of bidder 2 is $p_{\text{Nearest-Bid}}^2 = 3.5$. Bidder 3 does not win anything so his payment is $p_3 = 0$.

![Figure 3.1: The payments space of the auction in Example 2.5 with the Nearest-Bid payment rule.](image)

**Definition 3.6 (Nearest-VCG)** Given a bid profile $b$ and an allocation of items $X$, the Nearest-VCG first calculates the VCG payment and then find the nearest point to the VCG payment point in the core. The payments of the auction are determined by this nearest point. Other bidders’ payments are 0.

In Example 2.5, if the payment rule of the auction is Nearest-VCG, we calculate the VCG payments first. $b_{\text{VCG}}^1 = 5$ and $b_{\text{VCG}}^2 = 3$. Then we find the nearest point in Figure 3.2. Thus, the payment of bidder 1 is $p_{\text{Nearest-VCG}}^1 = 5.5$ and the payment of bidder 2 is $p_{\text{Nearest-VCG}}^2 = 3.5$. Bidder 3 does not win anything so his payment is $p_3 = 0$. 

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Definition 3.7 (Quadratic Rule) Given a bid profile $b$ and an allocation of items $X$, the Quadratic Rule first calculates the VCG payments and then find the nearest point in the core to the VCG payments point with the minimum revenue of the auctioneer. The payments of the auction are determined by this nearest point. Other bidders’ payments are 0.

In Example 2.5, if the payment rule of the auction is Nearest-VCG, we calculate the VCG payments first. $b_1^{\text{VCG}} = 5$ and $b_2^{\text{VCG}} = 3$. Then we find the nearest point in Figure 3.2. Thus, the payment of bidder 1 is $p_1^{\text{Quadratic}} = 5.5$ and the payment of bidder 2 is $p_2^{\text{Quadratic}} = 3.5$. Bidder 3 does not win anything so his payment is $p_3 = 0$.

In some complicated auctions, the Nearest-VCG point does not have the minimum revenue property in the core. Let me give an example to show the difference between these two payment rules.

**Example 3.8** Let $m = 4$ items, $A$, $B$, $C$, $D$, $n = 7$ bidders, and let bids be as follows.

- For bidder 1: $b_1(A) = 5$ and $b_1(j) = 0$ for any other bundles of items.
- For bidder 2: $b_2(B) = 5$ and $b_2(j) = 0$ for any other bundles of items.
- For bidder 3: $b_3(C) = 3$ and $b_3(j) = 0$ for any other bundles of items.
- For bidder 4: $b_4(D) = 1$ and $b_4(j) = 0$ for any other bundles of items.
- For bidder 5: $b_5(AB) = 5$ and $b_5(j) = 0$ for any other bundles of items.
3.2. Payment Rules

For bidder 6: \( b_6(BC) = 4 \) and \( b_6(j) = 0 \) for any other bundles of items.

For bidder 7: \( b_7(ACD) = 7 \) and \( b_7(j) = 0 \) for any other bundles of items.

The winners of this auction are bidder 1, 2, 3 and 4. The VCG payments for winning bidders are

\[
p_{1VCG} = b_2(B) + b_7(ACD) - b_2(B) - b_3(C) - B_4(d)
= 5 + 7 - 5 - 3 - 1
= 3
\]

\[
p_{2VCG} = b_1(A) + b_4(D) + b_6(BC) - b_1(A) - b_3(C) - B_4(d)
= 5 + 1 + 4 - 5 - 3 - 1
= 1
\]

\[
p_{3VCG} = b_2(B) + b_7(ACD) - b_1(A) - b_2(B) - b_4(D)
= 5 + 7 - 5 - 5 - 1
= 1
\]

\[
p_{4VCG} = b_1(A) + b_2(B) + b_3(C) - b_1(A) - b_2(B) - b_5(C)
= 5 + 5 + 3 - 5 - 5 - 3
= 0
\]

Thus, the VCG point in this auction is \((3,1,1,0)\). The nearest point to the VCG point in the core is \((3.8,1.6,2.4,0.8)\), the distance between this two points is \(\sqrt{10}\). However, the revenue of the auctioneer at this point is \(3.8 + 1.6 + 2.4 + 0.8 = 8.6\), there exists another point \((4,1,3,0)\) in the core where the revenue of the auctioneer is minimal.

With the Nearest-VCG payment rule, the payments of winners are

\[
p_{1Nearest-VCG} = 3.8;
\]

\[
p_{2Nearest-VCG} = 1.6;
\]

\[
p_{3Nearest-VCG} = 2.4;
\]

\[
p_{4Nearest-VCG} = 0.8.
\]

With the Quadratic payment rule, the payments of winners are

\[
p_{1Quadratic} = 4;
\]

\[
p_{2Quadratic} = 1;
\]

\[
p_{3Quadratic} = 3;
\]

\[
p_{4Quadratic} = 0.
\]
Definition 3.9 (Proxy Auction[3]) Given a bid profile $b$ and an allocation of items $X$, the proxy auction payments reflect the outcome of “proxy agents” competing in a simultaneous ascending auction with package bidding and arbitrarily small bid increments, $\varepsilon$. The bids $b_{ij}$ are reinterpreted as limit prices that the bidders have given their respective proxy agents. Each proxy agent must bid in the “virtual auction” whenever it is not a provisionally-winning bidder, which means there is no other bidders have higher bids on a sub set of items, and the process repeats until all others proxy agents drop out of the auction. Other bidders’ payments are 0.

We can make sure that the payment of any subset of bidders is located in the core. Because their final bids are higher than their competitors’ maximum willingness to pay in the “virtual auction”, otherwise they cannot be provisionally-winning bidders.

Let me simulate Example 2.5 as a proxy auction. In the initial round, all three proxy agents submit bids of $\varepsilon$, making bidder 1 and bidder 2 provisional winners. In round two, the proxy agent for the bidder 3 raises its bid to $3\varepsilon$, making the bidder 3 provisional winner; in round three, the proxy agents for each of bidder 1 and bidder 2 raise their bids to $2\varepsilon$, making them provisional winners; the proxy agent for bidder 2 will stop increasing bid at 4 and until the proxy agent for bidder 3 drops out of the auction at a price of essentially $b_3(AB) = 9$. Thus, the payment of bidder 1 is $b_1^{\text{proxy}} = 5$ and the payment of bidder 2 is $b_2^{\text{proxy}} = 4$. Bidder 3 does not win anything so his payment is $p_3 = 0$.

![Figure 3.3: The payment space of the auction in Example 2.5 of the Proxy auction in the core.](image-url)
3.2. Payment Rules

The process of the proxy payment with Example 2.5 is demonstrated in Figure 3.3. The blue line represents the trajectory of bids for bidder 1 and bidder 2 in the proxy auction and the red point shows the final payment of the Proxy auction, which is also in the core.

Definition 3.10 (Proportional Pricing) Given a bid profile $b$ and an allocation of items $X$, the proportional pricing calculates the First Price Package payments and then the payments are determined such that the bids are scaled down, proportionally, until the boundary of the core is reached. Other bidders’ payments are 0.

In Example 2.5, if the payment rule of the auction is Proportional Pricing, we calculate the First Price Package payments first. $p_{First\text{Price}}^1 = b_1(A) = 6$ and $p_{First\text{Price}}^2 = b_2(B) = 4$. Then the payments are scaled down until the frontier of the core is reached. This process is shown in Figure 3.4, the red point is the final payments with the Proportional Pricing rule and the blue line represents the trajectory of scaling down. Thus, the payment of bidder 1 is $p_{Proportional}^1 = 5.4$ and the payment of bidder 2 is $p_{Proportional}^2 = 3.6$. Bidder 3 does not win anything so his payment is $p_3 = 0$.

![Figure 3.4: The payment space of the auction in Example 2.5 with the Proportional Pricing payment rule.](image-url)
In this chapter, I further discussed the non-decreasing payment rules in combinatorial auctions which were first proposed in the previous semester thesis[29]. I formally defined two different versions of Non-Decreasing Payment Rules in Combinatorial Auctions and proved their existence in different combinatorial auctions.

4.1 Vickrey-Clarke-Groves (VCG) Payment Rule

Truthfulness is always a desirable property in auction design. If an auction is truthful or incentive compatible, then each bidder prefers reporting their true valuation as strategy, regardless of how other bidders evaluate the bundles and regardless of their strategies. Truthful bidding is a dominant strategy for each bidder.

Definition 4.1 (Truthful Auctions) An auction \((A, P)\) is truthful if \(u_i(b_1, ..., b_{i-1}, t_i, b_{i+1}, ..., b_n) \geq u_i(b_1, ..., b_{i-1}, b_i, b_{i+1}, ..., b_n | t_i)\) for all \(i\), all \(t\) which are true valuation, for all \(b_1, ..., b_{i-1}, b_{i+1}, ..., b_n\) which are bids given by other bidders and for all \(b_i\).

It is known that a monotone allocation algorithm can lead to a truthful mechanism if the payment rule is suitable [17, 2, 14]. The well-known Vickrey-Clarke-Groves (VCG) mechanism is truthful. In other words, truth-telling is a dominant strategy. Besides, if every bidder is truth-telling in the auction, it implies an efficient outcome. However, the VCG mechanism has lots of drawbacks[5, 21, 23, 24]. Therefore, it is seldom used in real auctions. Consider the following example of a LLG auction.

Example 4.2 Let \(m = 2\) items, \(A\) and \(B\), \(n = 3\) bidders, and let bids be as follows.

For bidder 1: \(b_1(A) = 10, b_1(B) = 0, b_1(AB) = 0;\)
For bidder 2: \( b_1(A) = 0, b_1(B) = 10, b_1(AB) = 0; \)
For bidder 3: \( b_1(A) = 0, b_1(B) = 0, b_1(AB) = 10. \)

In example 4.2, it is easy to determine that the winners are bidder 1 and bidder 2. The payment of bidder 1 is \( p_1 = b_3(AB) - b_2(B) = 10 - 10 = 0 \) and the payment of bidder 2 is \( p_2 = b_3(AB) - b_1(A) = 10 - 10 = 0. \) Bidder 3 does not win anything so his payment is \( p_3 = 0. \) The total payment among bidders in this auction is 0, which is much lower than expected.

This payment is also not in the core. Even though bidder 3 would like to pay 10 for these two items, the winners in this auction pay 0 to the auctioneer. Thus, the bidder 3 and the auctioneer would like to transact outside the auction.

Another deficiency of the VCG mechanism is that its outcome is non-monotonic. Consider the following example modified from example 4.2. Bidder 1 and bidder 2 can form a coalition in this auction to decrease their payment by increasing their bids. This property is not ideal in real auctions.

**Example 4.3** Let \( m = 2 \) items, \( A \) and \( B, n = 3 \) bidders, and let bids be as follows.
For bidder 1: \( b_1(A) = 7, b_1(B) = 0, b_1(AB) = 0; \)
For bidder 2: \( b_1(A) = 0, b_1(B) = 7, b_1(AB) = 0; \)
For bidder 3: \( b_1(A) = 0, b_1(B) = 0, b_1(AB) = 10. \)

In example 4.3, the payment of bidder 1 is \( p_1 = b_3(AB) - b_2(B) = 10 - 7 = 3 \) and the payment of bidder 2 is \( p_2 = b_3(AB) - b_1(A) = 10 - 7 = 3. \) Bidder 3 does not win anything so his payment is \( p_3 = 0. \) The total payment among bidders in this auction is 6, which is much higher than the outcome in example 3.2.

The complexity of computing the payments is also relatively high in the VCG mechanism. It requires a winner determination for computing each winning bidder’s payment. The winner determination problem is well known to be \( NP \)-hard[22].

Based on the above reasons, other payment rules are needed for combinatorial auctions even though they are not *incentive compatible*.

### 4.2 Non-Decreasing Payment Rules

Core-selecting auctions are alternatives to the VCG auction as a response to these critiques. Roughly stated, a core-selecting payment rule gives an outcome such that no unsatisfied coalition of bidders can propose another outcome which is preferred by the auctioneer and all the members in the coalition.
As I discussed in Chapter 3, First-Price Package, Nearest-Bid, Nearest-VCG, Quadratic Rule (Nearest-VCG with Minimum Revenue of the Auctioneer), Proxy Auction and Proportional Pricing are all core-selecting payment rules.

Among all of these core-selecting payment rule, some of them have better properties than others. For example, the quadratic payment rule promises a minimum revenue payment point in the core. However, other payment rules do not guarantee such minimum payment. The minimum revenue payment point is a good property in economic side, which effects the willingness of bidders in the auction.\[12\]

I classify a series of payment rules which are called non-decreasing payment rules. General speaking, with those payment rules, bidders can not increase their utility by increasing their bids when the allocation of the auction is fixed.

This property is very intuitive in combinatorial auctions. If bidders can manipulate their payments so easily, without any coalition in the auction, they are more likely to deviate from their truthful valuation to some over-bidding strategies or other cheating strategies. Thus, I study this property in these core-selecting payment rule with single-minded CAs and multi-minded CAs.

First of all, let me give the formal definition of non-decreasing payment rules.

**Definition 4.4 (Non-decreasing Payment Rules)** For any \( b_{ij} > b'_{ij} \) and \( b_{-ij} \) for which the allocation of two profiles \( A(b_{ij}, b_{-ij}) \) and \( A(b'_{ij}, b_{-ij}) \) are the same. If the payment of bidder \( i \) is non-decreasing, which means \( p_i(b_{ij}, b_{-ij}) \geq p_i(b'_{ij}, b_{-ij}) \), then the payment rule \( p_i \) is non-decreasing.

### 4.2.1 First-Price Payment Rule

**Lemma 4.5** The first-price payment rule is a non-decreasing payment rule.

**Proof** Without loss of generality, consider the payment of bidder 1.

There are two different bids for bundle 1 of bidder 1, \( b_{11} \) and \( b'_{11} \) with \( b_{11} < b'_{11} \). Other bids are denoted as \( b_{-11} \). The allocation of two profiles \( A(b_{11}, b_{-11}) \) and \( A(b'_{11}, b_{-11}) \) are the same.

1. If bidder 1 does not win this auction, his payment is 0 under both profiles.

\[
p_1(b_{11}, b_{-11}) = p_1(b'_{11}, b_{-11}) = 0.
\]

2. If bidder 1 wins bundle \( j \neq 1 \), her payment is not relevant to the bid of bundle 1.
4. Non-Decreasing Payment Rules in Combinatorial Auctions

\[ p_1(b_{11}, b_{-11}) = p_1(b'_{11}, b_{-11}). \]

3. If bidder 1 wins bundle 1, her payment is her bid for bundle 1.
\[ p_1(b_{11}, b_{-11}) = b_{11} < p_1(b'_{11}, b_{-11}) = b'_{11}. \]

Thus, the first price payment rule is non-decreasing. \(\square\)

4.2.2 Proportional Payment Rule

**Lemma 4.6** The proportional payment rule is a non-decreasing payment rule.

**Proof** Without loss of generality, consider the payment of bidder 1.

There are two different bids for bundle 1 of bidder 1, \(b_{11}\) and \(b'_{11}\) with \(b_{11} < b'_{11}\). Other bids are denoted as \(b_{-11}\). The allocation of two profiles \(A(b_{11}, b_{-11})\) and \(A(b'_{11}, b_{-11})\) are the same.

1. If bidder 1 does not win this auction, his payment is 0 under both profiles.
\[ p_1(b_{11}, b_{-11}) = p_1(b'_{11}, b_{-11}) = 0. \]

2. If bidder 1 wins bundle \(j \neq 1\), her payment is proportional according to the winning bundle. Increasing the bid from \(b_{11}\) to \(b'_{11}\), the constraints of the core also weakly increase. Thus, the payment point \(p(b_{11}, b_{-11})\) is weakly lower than the core constraint of profile \(b'\). Thus, the payment of bidder 1 under the profile \(b'\) is weakly higher than the payment of bidder 1 under the profile \(b\).
\[ p_1(b_{11}, b_{-11}) \leq p_1(b'_{11}, b_{-11}). \]

3. If bidder 1 wins bundle 1, the core remains same with respect to \(b\) and \(b'\). Consider the payment of bidder 1 \(p_1(b)\) under the profile \(b\).

Let \(\tilde{p}\) be the unique point with \(\tilde{p}_i = p_1(b)\) and payments proportional according to \(b'\), which is \(\tilde{p} = (b'_{11} \cdot \frac{p_1(b)}{p_{11}}, \ldots, b_j \cdot \frac{p_1(b)}{p_{11}}, \ldots).\)

\[ \therefore p_1(b) \leq b_{11} < b'_{11} \]

\[ \therefore \tilde{p} \text{ is strictly lower than the core constraint.} \]

\[ \therefore \tilde{p} < p(b'). \]

\[ \therefore p_1(b) = \tilde{p}_1(b') < p_1(b'). \]

In summary, \(p_1(b_{11}, b_{-11}) < p_1(b'_{11}, b_{-11}).\)

Thus, the proportional payment rule is non-decreasing. \(\square\)
4.3 Not Non-Decreasing Payment Rules

4.2.3 Proxy Payment Rule

Lemma 4.7 The proxy payment rule is a non-decreasing payment rule.

Proof Without loss of generality, consider the payment of bidder 1.

There are two different bids for bundle 1 of bidder 1, \( b_{11} \) and \( b'_{11} \) with \( b_{11} < b'_{11} \). Other bids are denoted as \( b_{-11} \). The allocation of two profiles \( A(b_{11}, b_{-11}) \) and \( A(b'_{11}, b_{-11}) \) are the same.

1. If bidder 1 does not win this auction, his payment is 0 under both profiles.
\[
p_1(b_{11}, b_{-11}) = p_1(b'_{11}, b_{-11}) = 0.
\]

2. If bidder 1 wins bundle \( j \neq 1 \), increasing the bid from \( b_{11} \) to \( b'_{11} \), the constraints of the core also weakly increase. Thus, the payment point \( p(b_{11}, b_{-11}) \) is weakly lower than the core constraint of profile \( b'_{11}, b_{-11} \). Thus, the payment of bidder 1 under the profile \( b'_{11}, b_{-11} \) is weakly higher than the payment of bidder 1 under the profile \( b_{11}, b_{-11} \).
\[
p_1(b_{11}, b_{-11}) \leq p_1(b'_{11}, b_{-11}).
\]

3. If bidder 1 wins bundle 1, the core remains same with respect to \( b \) and \( b' \). If \( p_1(b) < b_{11} \), bidder 1 increases her bid in the proxy auction and does not quit the auction.
\[
\therefore p_1(b') = p_1(b).
\]

Otherwise, if \( p_1(b) = b_{11} \), bidder 1 quits the proxy auction because the current bid is higher than her preference. However, if the profile is \( b' \), bidder 1 has a higher preference in bundle 1.
\[
\therefore p_1(b') \geq p_1(b) = b_{11}.
\]

In summary, \( p_1(b_{11}, b_{-11}) \leq p_1(b'_{11}, b_{-11}) \).

Thus, the proxy payment rule is non-decreasing. \( \square \)

4.3 Not Non-Decreasing Payment Rules

4.3.1 Quadratic Payment Rule

Lemma 4.8 The quadratic payment rule is not a non-decreasing payment rule.

To prove this lemma, I show two different kinds of combinatorial auctions with quadratic rule, multi-minded auctions and single-minded auctions. In the multi-minded auction, each bidder is interested in multiple bundles. They can use an over-bidding strategy to manipulate their payment and utility. In other words, bidders can increase their bids on the bundles which
do not belong to winning bundles to change the shape of the core and the position of the VCG payment point. Based on this operation, the nearest point changes as well.

**Example 4.9** Let \( m = 4 \) items, \( A, B, C \) and \( D \). \( n = 4 \) bidders, and let bids be as follows.

For bidder 1: \( b_1(A) = 1 \) and \( b_1(j) = 0 \) for any other bundles of items.

For bidder 2: \( b_1(B) = 1 \) and \( b_2(j) = 0 \) for any other bundles of items.

For bidder 3: \( b_1(C) = 1 \) and \( b_3(j) = 0 \) for any other bundles of items.

For bidder 4: \( b_4(D) = 1, \) \( b_4(ABCD) = 3.5 \) and \( b_4(j) = 0 \) for any other bundles of items.

The winners of this auction are bidder 1, 2, 3 and 4. And the VCG payments for winning bidders are

\[
p_{1^{\text{VCG}}} = b_4(ABCD) + b_2(B) - b_3(C) - b_4(D) \\
= 3.5 - 1 - 1 - 1 \\
= 0.5
\]

\[
p_{2^{\text{VCG}}} = b_4(ABCD) + b_1(A) - b_3(C) - b_4(D) \\
= 3.5 - 1 - 1 - 1 \\
= 0.5
\]

\[
p_{3^{\text{VCG}}} = b_4(ABCD) + b_1(A) - b_2(B) - b_3(C) \\
= 3.5 - 1 - 1 - 1 \\
= 0.5
\]

\[
p_{4^{\text{VCG}}} = b_1(A) + b_2(B) + b_3(C) - b_1(A) - b_2(B) - b_3(C) \\
= 1 + 1 + 1 - 1 - 1 - 1 \\
= 0
\]

Thus, the VCG point in this auction is \( (0.5, 0.5, 0.5, 0) \). The nearest point to the VCG point in the core is \( (1, 1, 1, 0.5) \), the distance between these two points is 1. This point is also the minimum revenue point in the core.

Let us consider another auction. The only difference between the previous one and the following one is that bidder 4 increases her bid of bundle \( ABCD \) from 3.5 to 3.6.

Let \( m = 4 \) items, \( A, B, C \) and \( D \). \( n = 4 \) bidders, and let bids be as follows.

For bidder 1: \( b_1(A) = 1 \) and \( b_1(j) = 0 \) for any other bundles of items.
4.3. Not Non-Decreasing Payment Rules

For bidder 2: \( b_2(B) = 1 \) and \( b_2(j) = 0 \) for any other bundles of items.

For bidder 3: \( b_3(C) = 1 \) and \( b_3(j) = 0 \) for any other bundles of items.

For bidder 4: \( b_4(D) = 1 \), \( b_4(ABCD) = 3.6 \) and \( b_4(j) = 0 \) for any other bundles of items.

The winners of this auction are bidder 1, 2, 3 and 4. And the VCG payments for winning bidders are

\[
p_{1VCG} = b_4(ABCD) + b_2(B) - b_3(C) - b_4(D)
= 3.6 - 1 - 1 - 1
= 0.6
\]

\[
p_{2VCG} = b_4(ABCD) + b_1(A) - b_3(C) - b_4(D)
= 3.6 - 1 - 1 - 1
= 0.6
\]

\[
p_{3VCG} = b_4(ABCD) + b_1(A) - b_2(B) - b_3(C)
= 3.6 - 1 - 1 - 1
= 0.6
\]

\[
p_{4VCG} = b_1(A) + b_2(B) + b_3(C) - b_1(A) - b_2(B) - b_3(C)
= 1 + 1 + 1 - 1 - 1 - 1
= 0
\]

Thus, the VCG point in this auction is \((0.6, 0.6, 0.6, 0)\). The nearest point to the VCG point in the core is \((1.05, 1.05, 1.05, 0.45)\), the distance between these two points is 0.9. This point is also the minimum revenue point in the core.

By increasing the bid of bundle \((ABCD)\) from 3.5 to 3.6, bidder 4 manipulates her payment and gets 0.05 extra utility without any coalition.

Until now, I have shown that the quadratic rule is not a non-decreasing payment rule with multi-minded bidders.

The next step is to show that even in the single-minded auction, the quadratic rule is still not non-decreasing. In the single-minded auction, if bidder \( i \) increases her bid, the core does not change. Furthermore, she cannot manipulate her VCG payment. Consider other bidders’ VCG payments, \( p_j = \sum_{k \in X_j} b_k(l) - \sum_{k \in X, k \neq j} b_k(l) \). Bidder \( i \) can only increase the second term of the equation. Thus, if bidder \( i \) increases her bid, the VCG point weakly decreases.
4. Non-Decreasing Payment Rules in Combinatorial Auctions

Example 4.10 Let $m = 6$ items, $A, B, C, D, E, F$. $n = 11$ bidders, and let bids be as follows.

For bidder 1: $b_1(A) = 5$ and $b_1(j) = 0$ for any other bundles of items.
For bidder 2: $b_2(B) = 5$ and $b_2(j) = 0$ for any other bundles of items.
For bidder 3: $b_3(C) = 4$ and $b_3(j) = 0$ for any other bundles of items.
For bidder 4: $b_4(D) = 1$ and $b_4(j) = 0$ for any other bundles of items.
For bidder 5: $b_5(E) = 1$ and $b_5(j) = 0$ for any other bundles of items.
For bidder 6: $b_6(F) = 1$ and $b_6(j) = 0$ for any other bundles of items.
For bidder 7: $b_7(ABD) = 5$ and $b_7(j) = 0$ for any other bundles of items.
For bidder 8: $b_8(BCE) = 5$ and $b_8(j) = 0$ for any other bundles of items.
For bidder 9: $b_9(ACF) = 7$ and $b_9(j) = 0$ for any other bundles of items.
For bidder 10: $b_{10}(DEF) = 2$ and $b_{10}(j) = 0$ for any other bundles of items.
For bidder 11: $b_{11}(BCD) = 5$ and $b_{11}(j) = 0$ for any other bundles of items.

The winners of this auction are bidder 1, 2, 3, 4, 5 and 6. And the VCG payments for winning bidders are

\[
p_1^{VCG} = b_2(B) + b_4(D) + b_5(E) + b_9(ACF) - b_2(B) - b_3(C) - b_4(D) - b_5(E) - b_6(F) \\
= 5 + 1 + 1 + 7 - 5 - 4 - 1 - 1 - 1 \\
= 2
\]

\[
p_2^{VCG} = b_1(A) + b_5(E) + b_6(F) + b_{11}(BCD) - b_1(A) - b_3(C) - b_4(D) - b_5(E) - b_6(F) \\
= 5 + 1 + 1 + 5 - 5 - 4 - 1 - 1 - 1 \\
= 0
\]

\[
p_3^{VCG} = b_2(B) + b_4(D) + b_5(E) + b_9(ACF) - b_1(A) - b_2(B) - b_4(D) - b_5(E) - b_6(F) \\
= 5 + 1 + 1 + 7 - 5 - 5 - 1 - 1 - 1 \\
= 1
\]

\[
p_4^{VCG} = b_1(A) + b_2(B) + b_3(C) + b_5(E) + b_6(F) - b_1(A) - b_2(B) - b_3(C) - b_5(E) - b_6(F) \\
= 5 + 5 + 4 + 1 + 1 - 5 - 5 - 4 - 1 - 1 \\
= 0
\]

\[
p_5^{VCG} = b_1(A) + b_2(B) + b_3(C) + b_4(D) + b_6(F) - b_1(A) - b_2(B) - b_3(C) - b_4(D) - b_6(F) \\
= 5 + 5 + 4 + 1 + 1 - 5 - 5 - 4 - 1 - 1 \\
= 0
\]
Thus, the VCG point in this auction is \( (2, 0, 1, 0, 0, 0) \). The minimum revenue in the core is 9.5 and the nearest point to the VCG point in the core with minimum revenue is \( (\frac{37}{12}, \frac{4}{3}, \frac{37}{12}, \frac{7}{12}, \frac{7}{12}, \frac{5}{3}) \), the distance between these two points is \( \sqrt{\frac{78}{3}} \).

Let us consider another auction. The only difference between the previous one and the following one is that bidder 3 increases her bid of bundle C from 4 to 5.

Let \( m = 6 \) items, A, B, C, D, E, F. \( n = 11 \) bidders, and let bids be as follows.

For bidder 1: \( b_1(A) = 5 \) and \( b_1(j) = 0 \) for any other bundles of items.

For bidder 2: \( b_2(B) = 5 \) and \( b_2(j) = 0 \) for any other bundles of items.

For bidder 3: \( b_3(C) = 5 \) and \( b_3(j) = 0 \) for any other bundles of items.

For bidder 4: \( b_4(D) = 1 \) and \( b_4(j) = 0 \) for any other bundles of items.

For bidder 5: \( b_5(E) = 1 \) and \( b_5(j) = 0 \) for any other bundles of items.

For bidder 6: \( b_6(F) = 1 \) and \( b_6(j) = 0 \) for any other bundles of items.

For bidder 7: \( b_7(ABD) = 5 \) and \( b_7(j) = 0 \) for any other bundles of items.

For bidder 8: \( b_8(BCE) = 5 \) and \( b_8(j) = 0 \) for any other bundles of items.

For bidder 9: \( b_9(ACF) = 7 \) and \( b_9(j) = 0 \) for any other bundles of items.

For bidder 10: \( b_{10}(DEF) = 2 \) and \( b_{10}(j) = 0 \) for any other bundles of items.

For bidder 11: \( b_{11}(BCD) = 5 \) and \( b_{11}(j) = 0 \) for any other bundles of items.

The winners of this auction are bidder 1, 2, 3, 4, 5 and 6. And the VCG payments for winning bidders are

\[
p_{1}^{VCG} = b_1(B) + b_4(D) + b_5(E) + b_9(ACF) - b_1(A) - b_2(B) - b_3(C) - b_4(D) - b_5(E) - b_6(F)
= 5 + 1 + 7 - 5 - 5 - 1 - 1 - 1
= 1
\]

\[
p_{2}^{VCG} = b_1(A) + b_3(C) + b_4(D) + b_5(E) + b_6(F) - b_1(A) - b_2(B) - b_3(C) - b_4(D) - b_5(E) - b_6(F)
= 5 + 5 + 1 + 1 - 5 - 5 - 1 - 1 - 1
= 0
\]

\[
p_{3}^{VCG} = b_2(B) + b_4(D) + b_5(E) + b_9(ACF) - b_1(A) - b_2(B) - b_4(D) - b_5(E) - b_6(F)
= 5 + 1 + 7 - 5 - 5 - 1 - 1 - 1
= 1
\]
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\[ p_4^{\text{VCG}} = b_1(A) + b_2(B) + b_3(C) + b_5(E) + b_6(F) - b_1(A) - b_2(B) - b_3(C) - b_5(E) - b_6(F) \]
\[ = 5 + 5 + 5 + 1 + 1 - 5 - 5 - 5 - 1 - 1 \]
\[ = 0 \]

\[ p_5^{\text{VCG}} = b_1(A) + b_2(B) + b_3(C) + b_4(D) + b_6(F) - b_1(A) - b_2(B) - b_3(C) - b_4(D) - b_6(F) \]
\[ = 5 + 5 + 5 + 1 + 1 - 5 - 5 - 5 - 1 - 1 \]
\[ = 0 \]

\[ p_6^{\text{VCG}} = b_1(A) + b_2(B) + b_3(C) + b_4(D) + b_5(E) - b_1(A) - b_2(B) - b_3(C) - b_4(D) - b_5(E) \]
\[ = 5 + 5 + 5 + 1 + 1 - 5 - 5 - 5 - 1 - 1 \]
\[ = 0 \]

Thus, the VCG point in this auction is \((1, 0, 1, 0, 0, 0)\). The core does not change so the minimum revenue in the core is still 9.5. The nearest point to the VCG point in the core with the minimum revenue is \((3, 1.5, 3, 0.5, 0.5, 1)\), the distance between these two points is \(\sqrt{47}/2\).

By increasing bid of bundle \((C)\) from 4 to 5, bidder 3 manipulates her payment and gets \(\frac{1}{12}\) extra utility without any coalition.

Theses two examples show that the quadratic payment rule is not ideal because bidders can manipulate their payment without any penalty. This property of the quadratic payment rule introduce more incentives of lying to auctions.

4.3.2 Nearest-VCG Payment Rule

Lemma 4.11 The nearest VCG payment rule(without minimum revenue) is not a non-decreasing payment rule.

The nearest VCG payment rule has the same property as the quadratic payment rule, which is not a non-decreasing payment rule. Example 4.9 implies that the nearest-VCG payment rule is not locally monotone in multi-minded auctions. The following example demonstrates that it is also not non-decreasing in single-minded auctions.

Example 4.12 Let \(m = 4\) items, \(A, B, C\) and \(D\), \(n = 7\) bidders, and let bids be as follows.

For bidder 1: \(b_1(A) = 5\) and \(b_1(j) = 0\) for any other bundles of items.

For bidder 2: \(b_2(B) = 5\) and \(b_2(j) = 0\) for any other bundles of items.

For bidder 3: \(b_3(C) = 4\) and \(b_3(j) = 0\) for any other bundles of items.
4.3. Not Non-Decreasing Payment Rules

For bidder 4: \( b_4(D) = 1 \) and \( b_4(j) = 0 \) for any other bundles of items.

For bidder 5: \( b_5(AB) = 5 \) and \( b_5(j) = 0 \) for any other bundles of items.

For bidder 6: \( b_6(BC) = 4 \) and \( b_6(j) = 0 \) for any other bundles of items.

For bidder 7: \( b_7(ACD) = 7 \) and \( b_7(j) = 0 \) for any other bundles of items.

The winners of this auction are bidder 1, 2, 3 and 4. And the VCG payments for winning bidders are

\[
p_1^{VCG} = b_2(B) + b_7(ACD) - b_2(B) - b_3(C) - b_4(D)
\]
\[
= 5 + 7 - 5 - 4 - 1
\]
\[
= 2
\]

\[
p_2^{VCG} = b_1(A) + b_3(C) + b_4(D) - b_1(A) - b_3(C) - b_4(D)
\]
\[
= 5 + 4 + 1 - 5 - 4 - 1
\]
\[
= 0
\]

\[
p_3^{VCG} = b_2(B) + b_7(ACD) - b_2(A) - b_3(B) - b_4(D)
\]
\[
= 5 + 7 - 5 - 5 - 1
\]
\[
= 1
\]

\[
p_4^{VCG} = b_1(A) + b_2(B) + b_3(C) - b_1(A) - b_2(B) - b_3(C)
\]
\[
= 5 + 5 + 4 - 5 - 5 - 4
\]
\[
= 0
\]

Thus, the VCG point in this auction is \((2, 0, 1, 0)\). The nearest point to the VCG point in the core is \((\frac{15}{7}, \frac{10}{7}, \frac{18}{7}, \frac{6}{7})\), the distance between these two points is \(3\sqrt{\frac{42}{7}}\).

Let us consider another auction. The only difference between the previous one and the following one is that bidder 3 increases her bid of bundle C from 4 to 5.

Let \( m = 4 \) items, A, B, C and D, \( n = 7 \) bidders, and let bids be as follows.

For bidder 1: \( b_1(A) = 5 \) and \( b_1(j) = 0 \) for any other bundles of items.

For bidder 2: \( b_2(B) = 5 \) and \( b_2(j) = 0 \) for any other bundles of items.

For bidder 3: \( b_3(C) = 5 \) and \( b_3(j) = 0 \) for any other bundles of items.

For bidder 4: \( b_4(D) = 1 \) and \( b_4(j) = 0 \) for any other bundles of items.

For bidder 5: \( b_5(AB) = 5 \) and \( b_5(j) = 0 \) for any other bundles of items.

For bidder 6: \( b_6(BC) = 4 \) and \( b_6(j) = 0 \) for any other bundles of items.
4. Non-Decreasing Payment Rules in Combinatorial Auctions

For bidder 7: \( b_7(ACD) = 7 \) and \( b_7(j) = 0 \) for any other bundles of items.

The winners of this auction are bidder 1, 2, 3 and 4. And the VCG payments for winning bidders are

\[
p_1^{VCG} = b_2(B) + b_7(ACD) - b_2(B) - b_3(C) - b_4(D)
= 5 + 7 - 5 - 5 - 1
= 1
\]

\[
p_2^{VCG} = b_1(A) + b_3(C) + b_4(D) - b_1(A) - b_3(C) - b_4(D)
= 5 + 5 + 1 - 5 - 5 - 1
= 0
\]

\[
p_3^{VCG} = b_2(B) + b_7(ACD) - b_2(A) - b_3(B) - b_4(D)
= 5 + 7 - 5 - 5 - 1
= 1
\]

\[
p_4^{VCG} = b_1(A) + b_2(B) + b_3(C) - b_1(A) - b_2(B) - b_3(C)
= 5 + 5 + 5 - 5 - 5 - 5
= 0
\]

Thus, the VCG point in this auction is (1, 0, 1, 0). The nearest point to the VCG point in the core is (3.5, 1.5, 2.5, 1), the distance between these two points is \( \sqrt{47} \).

By increasing the bid of bundle (C) from 4 to 5, bidder 3 manipulates her payment and gets \( \frac{1}{12} \) extra utility without any coalition.

4.4 Discussion

I have claimed that non-decreasing payment rules are intuitively good payment rules in combinatorial auctions. It does not only imply that bidders cannot manipulate their payment in an auction under such a rule, but also contributes to the algorithm design of computing BNEs in combinatorial auctions and the understanding of over-bidding strategies in combinatorial auctions. I will discuss these two applications in detail in the following chapters.
In Chapter 3, I studied the non-decreasing payment rules. It is not surprising that bidders might like to play over-bidding strategy if the auction is under decreasing payment rule because bidders can manipulate their payment by increasing their bids. In this chapter, I study over-bidding strategies in core-selecting non-decreasing payment rules.

I start from single-minded CAs with core-selecting non-decreasing payment rule. It is trivial to have the following theorem.

**Theorem 5.1** In single-minded CAs with core-selecting non-decreasing payment rules, if bidders have non-correlated valuation distribution, over-bidding will never be the best response for any bidder to any strategy profile.

**Proof** Consider bidder $i$.

If the best response $b_i^{\text{best}}$ is over-bidding, which means $b_i^{\text{best}} > v_i$, where $v_i$ is the true valuation to bidder $i$. Then we have

$$
\mathbb{E} \left[ b_i^{\text{best}}, s_{-i}(v_{-i}) \right] = \text{Pr} \left[ X_i(b_i^{\text{best}}, s_{-i}(v_{-i})) = 1 \right] \cdot (v_i - p_i(b_i^{\text{best}}, s_{-i}(v_{-i}))) > \\
\mathbb{E} \left[ v_i, s_{-i}(v_{-i}) \right] = \text{Pr} \left[ X_i(v_i, s_{-i}(v_{-i})) = 1 \right] \cdot (v_i - p_i(v_i, s_{-i}(v_{-i}))).
$$

Because $b_i^{\text{best}} > v_i$, for any deterministic $s_{-i}(v_{-i})$, there are only three possible winning state.

1) $X_i(b_i^{\text{best}}, s_{-i}(v_{-i})) = 1, X_i(v_i, s_{-i}(v_{-i})) = 1$
2) $X_i(b_i^{\text{best}}, s_{-i}(v_{-i})) = 1, X_i(v_i, s_{-i}(v_{-i})) = 0$
3) $X_i(b_i^{\text{best}}, s_{-i}(v_{-i})) = 0, X_i(v_i, s_{-i}(v_{-i})) = 0$
5. Over-bidding Strategies in Core-Selecting Non-decreasing Payment Rules

If \( X_i(b^\text{best}_i, s_{-i}(v_{-i})) = 1, X_i(v_i, s_{-i}(v_{-i})) = 1 \), because of non-decreasing property, \( p_i(b^\text{best}_i, s_{-i}(v_{-i})) \geq p_i(v_i, s_{-i}(v_{-i})) \).

If \( X_i(b^\text{best}_i, s_{-i}(v_{-i})) = 1, X_i(v_i, s_{-i}(v_{-i})) = 0 \), because of core-selecting property, \( p_i(b^\text{best}_i, s_{-i}(v_{-i})) \geq v_i \).

Thus, \( Pr[X_i(b^\text{best}_i, s_{-i}(v_{-i})) = 1] \cdot (v_i - p_i(b^\text{best}_i, s_{-i}(v_{-i}))) \leq Pr[X_i(v_i, s_{-i}(v_{-i})) = 1] \cdot (v_i - p_i(v_i, s_{-i}(v_{-i}))). \)

Truthful strategy is always weakly dominate over-bidding strategy. \( \Box \)

However, this argument does not generalize to multi-minded auctions. Let us consider the following example in the action space of a two-minded bidder.

![Figure 5.1: An action space of a two-minded bidder.](image)

Even though the blue point is over-bidding for bundle 2 in this auction, but it can not be eliminated from consideration as a best response. Because the bidder will win bundle 2 in one of these two situation. However, a core-selecting non-decreasing payment rule only promises that the payment is larger than \( k_1 \), but there is no deterministic relationship between the value
of bundle 2 at the blue point and the red point, which represents the true valuation of the bidder.

Here is an example of this situation.

**Example 5.2** Let \( m = 3 \) items, \( A, B \) and \( C \), \( n = 3 \) bidders, this auction is under proportional payment rule.

**Bidder 1** is interested in two bundles: \( \{AB\} \) and \( \{BC\} \).

**Bidder 3** is interested in two bundles: \( \{C\} \) and \( \{A\} \).

**Bidder 2** is interested in the bundle \( \{ABC\} \).

The strategy of **bidder 2** is bidding \( b_{21}^C = 1004 \) on \( \{C\} \) and \( b_{21}^A = 5 \) on \( \{A\} \) with 90% probability; bidding \( b_{22}^C = 1003 \) on \( \{C\} \) and \( b_{22}^A = 3 \) on \( \{A\} \) with 10% probability.

The strategy of **bidder 3** is bidding \( b_2 = 1005 \) on \( \{ABC\} \) with 100% probability.

If **bidder 1** has value \( v_{1}^{AB} = 2.5 \) on \( \{AB\} \) and \( v_1^{DE} = 1000.5 \) on \( \{BC\} \). Then his best response is bidding \( b_1^{AB} = 2 + \epsilon \) on \( \{AB\} \) and \( b_1^{BC} = 1001 + 2\epsilon \) on \( \{BC\} \).

**Bidder 1** wins \( \{AB\} \) with 10% probability and \( \{BC\} \) with 90% probability. His expected utility is

\[
\mathbb{E} [b_1, s_{-i}(v_{-i})] = 10\% \cdot (v_1^{AB} - p_1^{AB}(b_3 = 1005, b_{21}^C = 1003, b_1^{AB} = 2 + \epsilon)) \\
+ 90\% \cdot (v_1^{BC} - p_1^{BC}(b_3 = 1005, b_2^A = 5, b_1^{BC} = 1001 + 2\epsilon)) \\
= 0.1 \cdot (2.5 - 2) + 0.9 \cdot (1000.5 - \frac{(1001 + 2\epsilon) \cdot 1005}{1001 + 2\epsilon + 5}) \\
= 0.5
\]

Even though **bidder 1** over-bidding on bundle \( \{DE\} \), this is still the best response strategy to the strategy profile of other bidders.

However, we can still make a claim of over-bidding strategy with multi-minded bidders.

Until now, I assumed that bidders are not free to bid on any bundle, which means that a bidder can only bid on those bundles that are of interest to him.

Now I will prove that even if bidders are free to bid on any bundles, they will never bid on a bundle that includes only items which are not of interest to them.

**Theorem 5.3** In CAs with core-selecting non-decreasing payment rules, bidders will never bid on a bundle \( j \) which is consist of items that are not interested to her. They will never bid on bundles for which they have zero value.
5. Over-bidding Strategies in Core-Selecting Non-decreasing Payment Rules

Proof If bidder $i$ wins bundle $j$, for which her valuation is 0, then her utility is always negative.

If bidder $i$ wins another bundle. Because the auction is using non-decreasing payment rule, bidding a positive bid on bundle $j$ is always dominated by bidding 0 when bundle $j$ is not assigned to the bidder $i$.

Thus, bidding on bundle $j$ will never happen. □

However, this does not exclude bidding on bundles containing items of 0 marginal value over a subset of the items. Because bidding on other bundles which contain some items with no marginal value gives bidders an opportunity to represent a mixed strategy in a deterministic auctions.

Here is an example of how bidders increase his own utility by bidding on bundles containing items of 0 marginal value.

Example 5.4 Let $m = 4$ items, $A$, $B$, $C$ and $D$, $n = 3$ bidders, this auction is under the first-price payment rule.

Bidder 1 is interested in the bundle $\{A\}$.

Bidder 2 is interested in two bundles: $\{BC\}$ and $\{CD\}$.

Bidder 3 is interested in the bundle $\{ABCD\}$.

The strategy of bidder 2 is bidding $b_{CD}^{21} = 8$ on $\{CD\}$ and $b_{BC}^{21} = 1$ on $\{BC\}$ with 50% probability; bidding $b_{CD}^{22} = 1$ on $\{CD\}$ and $b_{BC}^{22} = 9$ on $\{BC\}$ with 50% probability.

The strategy of bidder 3 is bidding $b_3 = 10$ on $\{ABCD\}$ with 100% probability.

If bidder 1 has value $v_1^A = 10$ on $\{A\}$ and his valuation $v_1^{AB} = 10$ on $\{AB\}$, which means item $\{B\}$ has zero marginal value to bidder 1. Then his best response is bidding $b_1^A = 1 + \epsilon$ on $\{A\}$ and also bidding $b_1^{AB} = 2 + \epsilon$ on $\{AB\}$.

Bidder 1 wins $\{AB\}$ with 50% probability and $\{A\}$ with 50% probability. His expected utility is

$$E[b_1, s_{-i}(v_{-i})] = 50\% \cdot (v_1^A - p_1^{AB} (b_3 = 10, b_{CD}^{31} = 8, b_1^{AB} = 2 + \epsilon))$$

$$+ 50\% \cdot (v_1^A - p_1^A (b_3 = 10, b_{BC}^{31} = 9, b_1^A = 1 + \epsilon))$$

$$= 0.5 \cdot (10 - 2) + 0.5 \cdot (10 - 1)$$

$$= 8.5$$

Even though item $\{B\}$ has no marginal value to bidder 1, bidding on bundles $\{A\}$ and $\{AB\}$ together will be more profitable than only bidding on the bundle $\{A\}$.

Thus, in multi-minded auction with non-decreasing core-selecting payment rules, over-bidding strategy still exists. This is still an open problem.
In Chapter 4, I introduced the definition of non-decreasing payment rules and studied their existence in core-selecting payment rules. In this chapter, I will use this property of payment rules to design an efficient algorithm for computing Bayesian Nash Equilibria (BNE) in Combinatorial Auctions (CAs).

My semester thesis[29] showed that there exists an efficient algorithm for computing BNE in single-minded auctions, such as the LLG auction. However, I extend this method to multi-minded auctions and show experimental results in the LLLLGG auction. Compared to previous work[6], the new algorithm performs 100 times faster while reacting the same absolute-epsilon BNE.

6.1 Efficient Algorithm for Computing BNE in Single-minded Combinatorial Auctions

In my semester thesis[29], I introduced an efficient algorithm for computing BNEs in single-minded combinatorial auctions based on the iterative best response algorithm. In this algorithm, the epsilon which is computed by the algorithm is the true epsilon. There is no requirement for the verification step. I also proved that in single-minded auctions, the $\epsilon$ can be bounded by
what I called critical points.

**Algorithm 1:** Iterated Best Response For single-minded CAs with Locally Monotone Payment Rule (without Verification)

**Input:** Mechanism \( M' \) with locally monotone payment rule, Distribution \( V \) of bidders’ value

**Output:** strategy profile \( s \) and utility loss \( \epsilon \)

\[
\begin{align*}
&1 \quad s := \text{Truthful Strategies}; \\
&2 \quad \text{repeat} \\
&3 \quad \quad \text{foreach bidder } i \text{ do} \\
&4 \quad \quad \quad s_i' := \text{BestResponseStrategy}(M; V; s_{-i}); \\
&5 \quad \quad \text{end} \\
&6 \quad \quad \epsilon := \text{UtilityLoss}(s; s'); \\
&7 \quad \quad s := \text{Update}(s; s'); \\
&8 \quad \text{until Converged } (\epsilon); \\
&9 \quad \text{return } (s, \epsilon)
\end{align*}
\]

This proof is given by the following lemmas.

**Lemma 6.1**  In an efficient single-minded auction, if every bidder has a piecewise constant strategy, the best response strategy profile is also piecewise constant.

Because the utility loss of a bidder is determined by his strategy and the best response strategy of other bidders’ strategy profile, it is very important to analyse the relationship between these two strategies of the bidder. Assume these two strategies are piecewise constant, we separate these two strategies into some small constant cells.

**Definition 6.2 (Cell)** Consider any two univariate piecewise constant functions \( f_1 \) and \( f_2 \). A cell is a maximum interval \((v_1, v_2)\) where for any \( w_1 \in (v_1, v_2) \), \( f_1(w_1) \) is constant and \( f_2(w_2) \) is constant.

**Lemma 6.3** In a single-minded auction with an efficient allocation rule, if every bidder has a piecewise constant strategy, the utility loss of the strategy is bounded by the boundary points of cells between the piecewise constant function strategy and the piecewise constant best response strategy.
6.2 Best Response Strategy in General Combinatorial Auctions

Previous lemma works for single-minded combinatorial auctions. However, considering the multi-minded case, the proof of Lemma 6.1 does not work anymore. I study auction with continuous action space in this work. Even though other bidders' strategies are piecewise constant, which means there only exists finite number of bids combination of other bidders. The critical points of each bidder are not discrete. Consider the following example.

Figure 6.1 shows that for a two-minded auction, the critical points of changing the winning state (e.g., changing from winning bundle 1 to winning bundle 2) of a bidder are not discrete anymore. However, this is a requirement in the proof of Lemma 6.3[29]. Thus, a new proof is needed. I start from the proof for two-minded combinatorial auctions.

6.2.1 Best Response in Two-Minded Combinatorial Auctions

**Theorem 6.4** In a two-minded combinatorial auction with a locally-monotone payment rule, if all players have piecewise constant strategies and all of their possible bids and 1 have lowest common divisor k, then the best response of these strategies is a multiple of k at every valuation.

We need a lemma to prove Theorem 6.4.

**Lemma 6.5** There is only one Pareto Point in each polytope of the winning state space.

**Proof (Lemma 6.5)** Assume there are two Pareto Points in a polytope, \((x_1, y_1)\) and \((x_2, y_2)\).
w.l.o.g, we assume that $x_1 < x_2$ and $y_1 > y_2$.

Consider the point $(x_1, y_2)$ which is not in the cell. Compare $(x_1, y_1)$ and $(x_1, y_2)$. Because $y_1 > y_2$, the probability of winning bundle $y$ at point $(x_1, y_1)$ is larger or equal to the probability of winning bundle $y$ at point $(x_1, y_2)$.

Compare $(x_2, y_2)$ and $(x_1, y_2)$. Because $x_1 < x_2$, the probability of winning bundle $x$ at point $(x_2, y_2)$ is larger or equal to the probability of winning bundle $y$ at point $(x_1, y_2)$.

We know that the probability of winning bundle $x$ and bundle $y$ are the same at the points $(x_1, y_1)$ and $(x_2, y_2)$. At least one of the two bundles has different winning probability at the point $(x_1, y_2)$, w.l.o.g, we assume it is the bundle $x$.

Now we have that $P_x(x_1, y_1) < P_x(x_2, y_1), P_y(x_1, y_1) \leq P_y(x_2, y_1)$, which contradicts to the efficient allocation assumption.

Thus, there is only one Pareto Point in each polytope. □

Proof (Theorem 6.4) w.l.o.g, we consider the best response strategy of bidder $i$.

Every bidder has piecewise constant strategy, so the number of $s_{-i}$ is finite. For each possible realization of $s_{-i}$, the action space of bidder $i$ is divided into 3 continuous cells. The outcome of the auction inside each polytope is fixed. The structure of this split is shown in Figure 6.1.

We can compute these splits of all deterministic strategies in one single action space of bidder $i$. Then the action space will be divided into $m$ cells. No matter which $s_{-i}$ is realized, the outcome of auctions in each polytope is the same.

Now we can claim, because the payment rule is non-decreasing, there is a potential best response in each cell and these potential best responses are the Pareto points of the polytopes.

According to Lemma 6.5, there is only one Pareto point in each polytope. Polytopes in the winning state space are surrounded by lines with 0 degree, 90 degree and 45 degree. No matter it is the intersection of any two kinds of lines, its coordinate must be the span of the lowest common divisor $k$. □

6.2.2 Best Response in Multi-Minded Combinatorial Auctions

Theorem 6.6 In a multi-minded combinatorial auction with a locally-monotone payment rule, if all players have piecewise constant strategies and all of their possible bids and 1 have lowest common divisor $k$, then the best response of these strategies is a multiple of $k$ at every valuation.
To prove this theorem, we need two lemmas.

**Lemma 6.7** There is only one Pareto Point in each high-dimension polytope.

**Lemma 6.8** Every Pareto point is a vertex of a polytope.

**Proof (Lemma 6.7)** Assume there are two Pareto Points in a polytope, they differ in \( k \) coordinates.

w.l.o.g. We denote them as \( p_1 = (x_{11}, x_{21}, \ldots, x_{k1}, x_{k+1}, \ldots) \) and \( p_2 = (x_{12}, x_{22}, \ldots, x_{k2}, x_{k+1}, \ldots) \). Let \( x_i' = \min(x_{i1}, x_{i2}) \) for any \( 1 \leq i \leq k \).

Consider the point \( p' = (x_1', x_2', \ldots, x_k', x_{k+1}, \ldots) \).

Compare the probability of winning with bundle \( j \in J \) at point \( p_1 \) and \( p' \), where \( x_{j1} = x_j' \), the probability at \( p' \) is weakly larger than the probability at \( p_1 \).

Compare the probability of winning with bundle \( k \in K \) at point \( p_2 \) and \( p' \), where \( x_{j2} = x_j' \), the probability at \( p' \) is weakly larger than the probability at \( p_2 \).

We know that the probability of winning every bundle is the same at the points \( p_1 \) and \( p_2 \). At least one of these bundles has different winning probability at the point \( p' \), w.l.o.g. we assume it is the bundle 1.

Now we have that \( P_1(p_1) < P_1(p'), P_i(p_1) \leq P_i(p') \) for any \( i \neq 1 \), which contradicts to the efficient allocation assumption.

Thus, there is only one Pareto Point in each high-dimensional polytope. □

**Proof (Lemma 6.8)** If the Pareto point in a polytope is not a vertex, there are two possible locations of this point: inside the polytope or lies on a boundary of this polytope.

If this Pareto point \( p = (b_1, b_2, \ldots, b_n) \) is inside the polytope, then we can find a new point \( p' = (b_1 - \delta, b_2 - \delta, \ldots, b_n - \delta) \) which is strictly smaller than \( p \) in every dimension. This means that \( p \) is not a Pareto point of this polytope.

If this Pareto point \( p = (b_1, b_2, \ldots, b_n) \) lies on a boundary of this polytope, then there is at least one free dimension to move from \( p \) along this boundary. w.l.o.g. assume it is dimension 1. Thus, we can find another point \( p' = (b_1 - \delta, b_2', \ldots, b_n') \) which also lies on the same boundary.

According to Lemma 6.7, we know that there is only one Pareto point in each polytope. However, \( p' \) is strictly smaller than \( p \) in dimension 1, which means \( p' \) is also a Pareto point in this polytope. This contradicts to Lemma 6.7, thus the Pareto point cannot lie on a boundary of the polytope.

To sum up, the Pareto point in a polytope must be a vertex. □
6. Efficient Algorithm for Computing BNE in Combinatorial Auctions

**Proof (Theorem 6.6)** w.l.o.g. We consider the best response strategy of bidder $i$ and he has $n$-bundles of interest.

Every bidder has a piecewise constant strategy, so the domain of $s_{-i}$ is finite. For each possible realization of $s_{-i}$, the action space of bidder $i$ is divided into $n + 1$ continuous polytopes. The outcome of the auction inside each polytope is fixed.

We can compute these splits of all deterministic strategies in one single action space of bidder $i$. Then the action space will be divided into $m$ polytopes. No matter which $s_{-i}$ is realized, the outcome of auctions in each polytope is the same.

Now we can claim, because of the locally-monotone property of the payment rule, there are potential best response strategies in each polytope and these potential best response strategies are the Pareto points of the polytope.

According to Lemma 6.7 and Lemma 6.8, there is only one Pareto point in each polytope and it is a vertex of the polytope.

We can project these polytopes into any $i - j$ planes and use the same argument in the proof of theorem 6.4 to show that the Pareto Point is lying on the span of the lowest common divisor $k$. □
6.2. Best Response Strategy in General Combinatorial Auctions
6. Efficient Algorithm for Computing BNE in Combinatorial Auctions

6.3 Efficient Algorithm for Computing BNE in General Combinatorial Auctions

6.3.1 Algorithm Structure

With the proof in the previous section, we can modify Algorithm 4.1 to make it generally applicable to multi-minded CAs.

Algorithm 2: Iterated Best Response For multi-minded CAs with Locally Monotone Payment Rule (without Verification)

**Input:** Mechanism $M$, Distribution $V$ of bidders’ value

**Output:** strategy profile $s$ and utility loss $\epsilon$

1. $s :=$ Piece Wise Constant Strategy;

2. repeat
3.  foreach bidder $i$ do
4.     foreach Pareto Point $b_i$ in bidder $i$’s action space do
5.         Compute Utility Plane of $b_i$ with fair tie breaking rule.
6.     end
7.     $env_i :=$ upper envelop with fair tie breaking rule.
8.  endforeach
9.  foreach Pareto Point $b_i$ in bidder $i$’s action space do
10.     Compute Utility Plane of $b_i$ with optimal tie breaking rule.
11. end
12. $env_i^{OPT} :=$ upper envelop with optimal tie breaking rule.
13. $s_i' :=$ strategy induced by $env_i$.
14. $s_i^{OPT} :=$ strategy induced by $env_i$.
15. $\epsilon_i := \max_{v_i, \text{vertex of } s_i \text{ and } s_i^{OPT}} [env_i^{OPT}(v_i) - env_i(v_i)].$
16. end
17. $\epsilon = \max_i \left[ \epsilon_i \right].$
18. $s = s'$.
19. if $\epsilon < \epsilon^{BEST}$ then
20.     $\epsilon^{BEST} = \epsilon;$
21.     $s^{BEST} = s'$
22. end
23. until $\epsilon^{BEST} < \bar{\epsilon}$ or enough iteration;

24. return $(s^{BEST}, \epsilon^{BEST})$
6.3. Efficient Algorithm for Computing BNE in General Combinatorial Auctions

With the same argument as Lemma 6.3, the verification step is not needed anymore. Because we compute the true best response and the tight bound of absolute epsilon in the utility space.

In the following subsections, I will discuss some technical details of implementation of this algorithm and show how to parallelize this algorithm, in order to run it on a computer cluster.

6.3.2 Quasi-Random Sequences Generator

To avoid the discrepancy caused by random sequence, I introduce quasi-random number to generate Monte-Carlo samples for computing utility planes with different bids of bidders.

Quasi-random sequences are also called low-discrepancy sequences because for any values of $N$, its subsequence $x_1, ..., x_N$ has a low discrepancy. The discrepancy represents the difference between the proportion of points in the sequence falling into an arbitrary set and the proportion of the measure of that set.

Quasi-random sequences provide numbers that fill the number space more uniformly than the pseudo random sequence. Here I show two different random sequence in 2-dimension.

(a) High-discrepancy sequence in 2-dimension. (b) Low-discrepancy sequence in 2-dimension.

Figure 6.2: An example of randomized sequence distribution[27].

Comparing sub-figures in Figure 6.2, we can observe that the quasi-random sequence fills the space more uniformly than the pseudo random sequence. Thus, a algorithm which can generate quasi-random sequence in high-dimension space is needed for the implementation.

```python
for(int i = 0; i < max_iter; i++)
```
6. Efficient Algorithm for Computing BNE in Combinatorial Auctions

```cpp
for (int j = 0; j < 10; j++)
{
    ran_num[i][j] = 0;
    int index = coun;
    double f = 1.0/(double) pri[j];
    while (index > 0)
    {
        ran_num[i][j] += f * (double) (index % pri[j]);
        index = index / pri[j];
        f = f / (double) pri[j];
    }
    coun++;
}
```

The array `pri` stores a series of primes. The prime seed for each direction is not the same, thus the samples generated by this generator are not correlated between different coordinates.

### 6.3.3 Algorithm for Point Location

In each iteration, the algorithm maps the valuation of the bundle to the bids of the bundles. Thus, we need an algorithm to find the corresponding points in the action space.

The action space is divided into many polytopes. This leads to a fundamental topic of computational geometry, called the point location problem.

The trivial algorithm iterates over every polygon and determines whether the point is inside the polygon or outside the polygon. The complexity of this algorithm is $O(n)$, where $n$ is the sum of edges of all polygons.

However, this is not efficient enough to meet our requirement. An algorithm which achieves $O(\log n)$ time complexity is used in our implementation.

The Slab decomposition was discovered by Dobkin and Lipton in 1976. The space is subdivided by vertical lines that pass through each vertex of polygons. Every region between two adjacent vertical lines is called a slab. Each slab is divided by non-intersecting line segments and all of these line segments cross the slab from left to right. Thus, we can use binary search to locate the correct slab, then use binary search again without the slab.
6.4. Experimental Results

(a) The action space of bidders with two-minded preferences.
(b) The slab decomposition of the action space.

Figure 6.3: An example of slab decomposition.

Figures 6.3 shows an example of slab decomposition. Even though it consumes more space to store the slab data structure, it still has better performance than the trivial point location algorithm.

6.3.4 Parallel Version of the Algorithm

The algorithm can be divided into two parts. First is computing the utility plane of each bid profile. Second is computing the upper envelope of utility planes generated from the first part.

The only thing that is needed for computing utility plans is the strategy profiles from the previous iteration and Monte-Carlo samples. Therefore, each utility plane can be computed separately from others. Based on this property, I enhanced the efficiency by paralleling the algorithm on the cluster.

For each iteration, the cluster computes utility planes synchronously. Then the main core collect these results and finding the first upper envelope for the next iteration.

6.4 Experimental Results

In this section, I compare the performance of the BNE algorithm and the algorithm from previous work[6] in the LLLLGG domain. In my semester thesis[29], I proposed that the new BNE algorithm is more efficient than the previous one and have a tight bound of epsilon. This is not only true in the LLG domain which has been demonstrated already, but also in the LLLLLGG domain.
6.4.1 Performance Comparison

To compute $\epsilon$-BNEs for the LLLLGG auction, I used a grid resolution of $\frac{1}{64}$. Since local bidders have values in the $[0,1]$ range and global bidders in the $[0,2]$ range, I compute utility lines corresponding to $65^2$ and $129^2$ equally spaced bids, respectively.

In Table 6.1 I compare the runtime of the two algorithms for reaching a target $\epsilon$ of 0.02 for three non-decreasing payment rules: First-price, proxy and proportional. In this two dimensional setting, the new algorithm finds a BNE over 60 times faster than the baseline for each payment rule.

<table>
<thead>
<tr>
<th>Domain and Rule</th>
<th>Runtime (Baseline)</th>
<th>Runtime (Utility Planes)</th>
<th>Speed-up Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>LLLLGG First Price</td>
<td>165.34</td>
<td>1.59</td>
<td>103.98</td>
</tr>
<tr>
<td>LLLLGG Proxy</td>
<td>233.14</td>
<td>3.78</td>
<td>61.77</td>
</tr>
<tr>
<td>LLLLGG Proportional</td>
<td>174.68</td>
<td>2.09</td>
<td>83.57</td>
</tr>
</tbody>
</table>

6.4.2 Structure of Strategy Profile

In this subsection, I show the BNEs found by the new algorithm.
6.4. Experimental Results

(a) The local bidders’ strategies for bundle 1

(b) The local bidders’ strategies for bundle 2

(c) The global bidders’ strategies for bundle 1

**Figure 6.4:** The bidders’ strategies in the ε-BNE computed by the new algorithm for the LLLLGG setting with First Price rule. The second bundle’s strategy of the global bidder is symmetric to the first one.
Figure 6.5: The bidders’ strategies in the $\varepsilon$-BNE computed by the new algorithm for the LLLLGG setting with Proportional rule. The second bundle’s strategy of the global bidder is symmetric to the first one.
6.4. Experimental Results

Figure 6.6: The bidders’ strategies in the ε-BNE computed by the new algorithm for the LLLLGG setting with Proxy rule. The second bundle’s strategy of the global bidder is symmetric to the first one.
6.5 Locally Linearity of Payment Rules

After introducing locally monotonicity of payment rules, I state another property of payment rules which is also important in CAs.

**Definition 6.9 (Locally Linear Payment Rules)** If every bidder has a piecewise constant strategy profile, for any \( b_{ij}, b'_{ij} \) which are in the same grid cell of the strategy space and \( b_{-ij} \) that the allocation of two profiles \( A(b_{ij}, b_{-ij}) \) and \( A(b'_{ij}, b_{-ij}) \) are same. If the payment of bidder \( i \) is linear between \( b_{ij} \) and \( b'_{ij} \), then the payment rule \( p_i \) is locally linear.

The algorithm I state in previous in this chapter also works for auctions with locally linear payment rules.

**Conjecture 6.10** The quadratic payment rule is locally linear.

Until now, I have tested some middle size auctions, the quadratic payment rule has locally linearity. However, there is no proof for the locally linearity of the quadratic rule in general. On the other hand, no counter example has been observed. Thus, this is still an open problem for future study.
In my semester thesis [29] and previous chapters of this thesis, I assumed that the bidders' valuation are independent and their utility functions are quasi-linear. On the other hand, another method for computing the BNE with correlated bidders was first shown in [29]. However, this method only works for special kinds of correlations, as defined in [3]. In this chapter, I introduce a fully general method for computing BNE in CAs with any kind of correlations.

7.1 Utility Function

Let us recall the definition of utility lines when bidders' valuations are independent.

Lemma 7.1 In a single-item auction where every bidder is only interested in a single bundle with independent value distribution. if we fix the bid of bidder \(i\), the expected utility of bidder \(i\) is a linear function of bidder \(i\)'s values.

Proof The expected utility is the difference between the expected gains and the expected payment. For bidder \(i\) with strategy \(s_i(v_i)\), it is

\[
\mathbb{E}[u_i(s_i(v_i), s_{-i}(v_{-i}))] = \mathbb{E}[v_i \cdot X_i(b_i, s_{-i}(v_{-i})) - p_i(b_i, s_{-i}(v_{-i}))] = \mathbb{E}[v_i \cdot X_i(b_i, s_{-i}(v_{-i}))] - \mathbb{E}[p_i(b_i, s_{-i}(v_{-i}))]
\]

(7.1)

where \(X_i\) is an indicator variable that represents if bidder \(i\) wins the bundle and \(p_i\) is the payment of bidder \(i\). If we fix the bid of bidder \(i\), \(X_i(b_i, s_{-i}(v_{-i}))\) and \(\mathbb{E}[p_i(b_i, s_{-i}(v_{-i}))\) do not change with \(v_i\). Hence, the expected utility of
bidder $i$ with a fixed bid $b_i$ can be represented as a linear function of $v_i$. $k$ and $t$ are slope and intercept of the function, respectively:

$$E[u_i(s_i^*(v_i), s_{-i}^*(v_{-i}))]$$

$$= E[v_i \cdot X_i(b_i, s_{-i}(v_{-i}))] - E[p_i(b_i, s_{-i}(v_{-i}))]$$

$$= Pr[X_i(b_i, s_{-i}(v_{-i})) = 1] \cdot v_i - E[p_i(b_i, s_{-i}(v_{-i}))]$$

$$= k \cdot v_i + t \hspace{1cm} (7.2)$$

Notice that with correlations present, the first part of this equation $Pr[X_i(b_i, s_{-i}(v_{-i})) = 1]\cdot v_i$, not independent of $v_i$ anymore. If the probability distribution of bidders are dependent, the $s_{-i}(v_{-i})$ will not be the same for all $v_i$ and this utility function need to be modified. In this chapter, I demonstrated a new method that approximate the utility function as polynomial function instead of linear function.

### 7.2 Copula Probability Distribution

To study the shape of the utility function in CAs, I first introduce a general way of describing probability distribution function, which is called a copula.

Copula is a definition in probability theory and statistics. It describes a multivariate probability distribution. Each variable has uniform marginal probability distribution and there are dependence between each pair of random variables.

**Definition 7.2 (Copula[18])** In probabilistic terms, $C : [0, 1]^d \rightarrow [0, 1]$ is a d-dimensional copula if $C$ is a joint cumulative distribution function of a d-dimensional random vector on the unit cube $[0, 1]^d$ with uniform marginals.

### 7.3 Utility Curve with Correlated Bidders in LLG Auction

I start with a LLG auction with correlated value distribution. The joint probability distribution of bidders’ valuation is Gaussian Copula.

**Definition 7.3 (Gaussian Copula[7, 1])** The Gaussian copula is a distribution over the unit cube $[0, 1]^d$. It is constructed from a multivariate normal distribution over $\mathbb{R}^d$ by using the probability integral transform.

For a given correlation matrix $R \in [-1, 1]^{d \times d}$, the Gaussian copula with parameter matrix $R$ can be written as $C_R^{\text{Gauss}}(u) = \Phi_R \left( \Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d) \right)$, where $\Phi^{-1}$ is the inverse cumulative distribution function of a standard normal and $\Phi_R$ is the joint cumulative distribution function of a multivariate
normal distribution with mean vector zero and covariance matrix equal to the correlation matrix $R$. While there is no simple analytical formula for the copula function, $C^\text{Gauss}_R(u)$, it can be upper or lower bounded, and approximated using numerical integration.

The density can be written as

$$c^\text{Gauss}_R(u) = \frac{1}{\sqrt{\det R}} \exp \left( -\frac{1}{2} \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix}^T \left( R^{-1} - I \right) \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix}, \right)$$

where $I$ is the identity matrix.

An example of Gaussian Copula structure is shown below.

![Gaussian Copula Density](a) Gaussian copula probability density function for two marginal variables with correlation coefficient $\rho = 0.5$

![Gaussian Copula Cumulative Distribution](b) Gaussian copula cumulative distribution function for two marginal variables with correlation coefficient $\rho = 0.5$

**Figure 7.1:** An example of Gaussian Copula structure[10].
We can observe that the utility function are smooth curve. To make sure that this property also holds for other correlated auctions, I also test another model for general correlated auction.

Assume that the value of each bundle consists of two parts. The first part is a common knowledge and is a random variable following a uniform distribution in $[0, \alpha]$. The second part is private for each bidder and is a random variable following a uniform distribution in $[0, 1 - \alpha]$. Each bidder also has an additive valuation on the bundle, which is an independent random variable. The utility function is shown in Figure 7.2.

**Figure 7.2:** The utility curve of a local bidders when bidding $b_i = 0.25$ and other bidders’ strategies are truthful in LLG auction with Nearest-Bid Payment and correlated valuation distribution is Gaussian Copula. $\gamma$ is the correlation between valuations of local bidders, with $\gamma = 0$ for independent valuations; $\gamma = 1$ for perfectly correlated valuations.
7.4 Experimental Results

First I show the runtime and the number of iteration for finding $\varepsilon$-BNEs with Guassian Copula distribution under Nearest-Bid Payment, Proportional Payment, Proxy Payment and Quadratic Payment.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Number of Iteration</th>
<th>Runtime(ms)</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadratic</td>
<td>4</td>
<td>4676</td>
<td>$2.58 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Nearest-Bid</td>
<td>3</td>
<td>3304</td>
<td>$1.86 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Proxy</td>
<td>3</td>
<td>3389</td>
<td>$3.02 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Proportional</td>
<td>3</td>
<td>3365</td>
<td>$2.24 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Average</td>
<td>3.25</td>
<td>3683.5</td>
<td>$2.42 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

The following experimental setup is equivalent to that used in [6]. Because the only available analytical BNEs is given by [5], I also plot the utility function with those 8 correlated settings and compare its numerical BNEs with the analytical one.
7. **Approximate Utility Surfaces in Combinatorial Auctions**

(a) Under Nearest-Bid Payment and local bidders’ valuation marginal distribution is \( p(v) = v \).

(b) Under Nearest-Bid Payment and local bidders’ valuation marginal distribution is \( p(v) = v^2 \).

(c) Under Proportional Payment and local bidders’ valuation marginal distribution is \( p(v) = v \).

(d) Under Proportional Payment and local bidders’ valuation marginal distribution is \( p(v) = v^2 \).

(e) Under Proxy Payment and local bidders’ valuation marginal distribution is \( p(v) = v \).

(f) Under Proxy Payment and local bidders’ valuation marginal distribution is \( p(v) = v^2 \).

(g) Under Quadratic Payment and local bidders’ valuation marginal distribution is \( p(v) = v \).

(h) Under Quadratic Payment and local bidders’ valuation marginal distribution is \( p(v) = v^2 \).

Figure 7.4: Utility curves of a local bidders when bidding \( b_l = 0.25 \) under different payment rules and different correlated valuation marginal distributions in LLG auction[5]. \( \gamma \) is the correlation.
7.4. Experimental Results

Table 7.2: Runtimes for finding $\varepsilon$-BNEs of LLG with correlated distribution valuations in [5].

<table>
<thead>
<tr>
<th>Rule</th>
<th>$a$</th>
<th>Number of Iteration</th>
<th>Runtime(ms)</th>
<th>$\varepsilon$</th>
<th>Distance to Analytical BNEs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadratic</td>
<td>1</td>
<td>6</td>
<td>29846</td>
<td>2.38E-04</td>
<td>0.0195551</td>
</tr>
<tr>
<td>Quadratic</td>
<td>2</td>
<td>8</td>
<td>48805</td>
<td>2.06E-04</td>
<td>0.0248784</td>
</tr>
<tr>
<td>Nearest-Bid</td>
<td>1</td>
<td>4</td>
<td>18045</td>
<td>2.76E-04</td>
<td>0.0278128</td>
</tr>
<tr>
<td>Nearest-Bid</td>
<td>2</td>
<td>5</td>
<td>26915</td>
<td>1.62E-04</td>
<td>0.0223726</td>
</tr>
<tr>
<td>Proxy</td>
<td>1</td>
<td>8</td>
<td>41588</td>
<td>1.62E-04</td>
<td>0.0498522</td>
</tr>
<tr>
<td>Proxy</td>
<td>2</td>
<td>7</td>
<td>40941</td>
<td>3.70E-04</td>
<td>0.0714377</td>
</tr>
<tr>
<td>Proportional</td>
<td>1</td>
<td>7</td>
<td>33524</td>
<td>1.64E-04</td>
<td>0.0235466</td>
</tr>
<tr>
<td>Proportional</td>
<td>2</td>
<td>7</td>
<td>38268</td>
<td>1.28E-04</td>
<td>0.0220128</td>
</tr>
</tbody>
</table>

These results show that the utility curve method is general and can be adapted to any correlated auctions.
In this master thesis, I have introduced non-decreasing payment rules and shown that this property has important consequences for incentives and algorithm design. On the negative side, I have shown that in the absence of a non-decreasing rule, bidders can manipulate the CA easily. On the positive side, I have shown that there exists an efficient algorithm to compute $\varepsilon$-BNE in multi-minded CAs with non-decreasing payment rules. I have developed the theory to create utility plane method to compute the BNE in CAs and empirically found it to be highly performant, beating a recent state-of-the-art algorithm by multiple orders of magnitudes. I also showed that this method can be extended to utility curve method for computing BNE in more complicated settings with correlated bidders. The relationship between non-decreasing payment rules and over-bidding strategy is also very important to understand the truthfulness in CAs. This is a promising direction of further study.
Bibliography


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