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# On quartic colour factors in splitting functions and the gluon cusp anomalous dimension

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## ABSTRACT

We have computed the contributions of the quartic Casimir invariants to the four-loop anomalous dimensions of twist-2 spin- $N$  operators at  $N \leq 16$ . The results provide new information on the structure of the next-to-next-to-next-to-leading order (N<sup>3</sup>LO) splitting functions  $P_{ik}^{(3)}(x)$  for the evolution of parton distributions, and facilitate approximate expressions which include the quartic-Casimir contributions to the (light-like) gluon cusp anomalous dimension. These quantities turn out to be closely related, by a generalization of the lower-order ‘Casimir scaling’, to the corresponding quark results. Using these findings, we present an approximate result for the four-loop gluon cusp anomalous dimension in QCD which is sufficient for phenomenological applications.

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## 1. Introduction

Over the past years, the next-to-next-to-leading order (NNLO, N<sup>2</sup>LO) of perturbative QCD has become the standard approximation for many hard-scattering processes at the LHC and other high-energy colliders. In certain cases, e.g., when a very high accuracy is required or when the NNLO corrections are rather large, it is useful to extend the analyses to the next order, N<sup>3</sup>LO. Coefficient functions (partonic cross sections) have been computed at N<sup>3</sup>LO for inclusive lepton-hadron deep-inelastic scattering (DIS) [1] and Higgs production in proton–proton collisions [2,3]. Very recently, first N<sup>3</sup>LO results have been presented for jet production in DIS [4].

In principle, N<sup>3</sup>LO analyses of processes involving initial-state hadrons require parton distribution functions (PDFs) evolved with the four-loop splitting functions. These functions also include quantities that are relevant beyond the evolution of PDFs. In particular, their leading behaviour for large momentum fractions  $x$  is given by important universal quantities, the (light-like) cusp anomalous dimensions of quarks and gluons [5]. The complete computation of the four-loop splitting functions is a formidable task. Until now, a phenomenologically relevant amount of partial

results has been published only for the (non-singlet) quark–quark splitting functions [6,7].

In this letter, we address a specific part of the four-loop flavour-singlet splitting functions, the terms with quartic Casimir invariants which occur at this order for the first time. As shown below, the present partial results for these terms provide structural and numerical information that is relevant for future research on N<sup>3</sup>LO corrections and for QCD phenomenology beyond PDFs.

## 2. Notations and general properties

The QCD evolution equations for the flavour-singlet quark and gluon distributions of hadrons,

$$q_s(x, \mu^2) = \sum_{i=1}^{n_f} [q_i(x, \mu^2) + \bar{q}_i(x, \mu^2)] \quad \text{and} \quad g(x, \mu^2), \quad (2.1)$$

can be written as

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix}. \quad (2.2)$$

Here  $q_i(x, \mu^2)$ ,  $\bar{q}_i(x, \mu^2)$  and  $g(x, \mu^2)$  denote the respective number distributions of quarks and antiquarks of flavour  $i$  and of the gluons in the fractional hadron momentum  $x$ ,  $\otimes$  stands for the Mellin convolution in the momentum variable, and  $\mu$  represents

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the factorization scale. In the present context, the renormalization scale can be identified with  $\mu$  without loss of information.

The quark–quark splitting function  $P_{\text{qq}}$  can be expressed as  $P_{\text{qq}} = P_{\text{ns}}^+ + P_{\text{ps}}$  in terms of the non-singlet splitting function  $P_{\text{ns}}^+$  for quark–antiquark sums and a pure-singlet contribution  $P_{\text{ps}}$ . The splitting functions can be expanded in powers of the strong coupling constant,

$$P_{\text{ik}}(x, \alpha_s) = \sum_{n=0} a_s^{n+1} P_{\text{ik}}^{(n)}(x) \quad \text{with} \quad a_s \equiv \frac{\alpha_s(\mu^2)}{4\pi}. \quad (2.3)$$

The off-diagonal quantities  $P_{\text{qg}}$  and  $P_{\text{gq}}$  include integrable logarithms up to  $a_s^n \ln^{2n-2}(1-x)$  in the threshold limit  $x \rightarrow 1$ , see refs. [8], while the diagonal quantities  $P_{\text{qq}}$  and  $P_{\text{gg}}$  have the form [9]

$$P_{\text{kk}}^{(n-1)}(x) = \frac{x A_{n,k}}{(1-x)_+} + B_{n,k} \delta(1-x) + C_{n,k} \ln(1-x) + D_{n,k} + \mathcal{O}((1-x) \ln^\ell(1-x)), \quad (2.4)$$

where  $A_{n,q}$  and  $A_{n,g}$  are the (light-like)  $n$ -loop quark and gluon cusp anomalous dimensions [5]. The coefficients  $C_{n,k}$  and  $D_{n,k}$  can be predicted from lower-order information [7,9]. In the small- $x$  (high-energy, BFKL [10]) limit, the splitting functions are single-logarithmic enhanced with terms up to  $x^{-1} \ln^n x$  for  $P_{\text{gk}}^{(n)}$  and  $x^{-1} \ln^{n-1} x$  for  $P_{\text{qk}}^{(n)}$  [11].

The splitting functions in eq. (2.2) are related to the anomalous dimensions of twist-2 spin- $N$  operators with  $N = 2, 4, 6, \dots$  by a Mellin transformation,

$$\gamma_{\text{ik}}^{(n)}(N, \alpha_s) = - \int_0^1 dx x^{N-1} P_{\text{ik}}^{(n)}(x, \alpha_s), \quad (2.5)$$

where the negative sign is a standard convention. The splitting functions  $P_{\text{ik}}^{(n)}(x)$  are known to NNLO, i.e., at  $n \leq 2$  in eq. (2.3) [12]. The N<sup>3</sup>LO contributions to eq. (2.5) have been obtained at  $N \leq 6$  [13]; the results for  $N = 2$  and  $N = 4$  have been presented in numerical form in ref. [14]. Much more is known about their non-singlet counterparts [6,7], see below.

Here we are interested in contributions with quartic colour factors which we abbreviate as

$$d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}, \quad (2.6)$$

where  $x, y$  labels the representations with generators  $T_r^a$  and

$$d_r^{abcd} = \frac{1}{6} \text{Tr}(T_r^a T_r^b T_r^c T_r^d + \text{five } bcd \text{ permutations}). \quad (2.7)$$

In  $SU(n_c)$ , for fermions in the fundamental representation (trace-normalized with  $T_F = \frac{1}{2}$ ),

$$d_{AA}^{(4)}/n_A = \frac{1}{24} n_c^2 (n_c^2 + 36), \quad (2.8)$$

$$d_{FA}^{(4)}/n_A = \frac{1}{48} n_c (n_c^2 + 6), \quad (2.9)$$

$$d_{FF}^{(4)}/n_A = \frac{1}{96} (n_c^2 - 6 + 18n_c^{-2}). \quad (2.10)$$

The dimension of the adjoint representation is related to  $n_F = n_c = C_A$  by  $n_A = (n_c^2 - 1) = 2 n_c C_F$ .

Terms with quartic Casimir invariants occur for the first time at four loops, the order considered here, in splitting functions, coefficient functions and the beta function of QCD [15]. This effective

‘leading-order’ situation implies particular relations and facilitates calculational simplifications.

The  $d_{xy}^{(4)}$  terms at four loops are scheme-independent. They should therefore fulfil the relation

$$\gamma_{\text{qq}}^{(3)}(N) + \gamma_{\text{gq}}^{(3)}(N) - \gamma_{\text{qg}}^{(3)}(N) - \gamma_{\text{gg}}^{(3)}(N) \stackrel{Q}{=} 0 \quad (2.11)$$

for the colour-factor substitutions [14]

$$(2n_f)^2 \frac{d_{FF}^{(4)}}{n_A} = 2n_f \frac{d_{FA}^{(4)}}{n_A} = 2n_f \frac{d_{FF}^{(4)}}{n_c} = \frac{d_{FA}^{(4)}}{n_c} = \frac{d_{AA}^{(4)}}{n_A} \quad (2.12)$$

that lead to an  $\mathcal{N} = 1$  supersymmetric theory, for lower-order discussions see refs. [16]. Here and below  $\stackrel{Q}{=}$  denotes equality for the quartic Casimir contributions. The factor of two for each power of  $n_f$  is due to the transition from QCD and its  $SU(n_c)$  generalization to  $n_f = 1$  Majorana fermions. We have verified eq. (2.11) at a sufficient number of  $N$ -values. At higher values of  $N$  this relation can be used to avoid the hardest diagram computations, those of the  $d_{AA}^{(4)}$  contributions to  $\gamma_{\text{gg}}^{(3)}$ .

As in the non-singlet cases, the splitting functions are (conjectured to be) constrained by a conformal symmetry of QCD at some non-integer space-time dimension  $D = 4 - 2\varepsilon$  [17]. We find that the moments of the off-diagonal splitting functions are consistent with the resulting prediction in terms of reciprocity-respecting sums (see below), but fulfil the stronger, newly discovered condition

$$\gamma_{\text{qg}}^{(0)}(N) \gamma_{\text{gq}}^{(3)}(N) \stackrel{Q}{=} \gamma_{\text{qg}}^{(3)}(N) \gamma_{\text{gq}}^{(0)}(N). \quad (2.13)$$

This result provides a stringent check of our very challenging high- $N$  computations. Other features resulting from the special status of quartic Casimir contributions at four loops are discussed below.

### 3. Diagram computations and $N$ -space results

Computations of four-loop inclusive DIS have been performed at  $N \leq 6$  for all colour factors in a manner analogous those at three loops in refs. [18], for sample results see ref. [14]. The ensuing moments of the four-loop splitting functions provide crucial reference results for validating the present calculation performed in the framework of the operator-product expansion (OPE).

Our OPE diagram computations have been performed analogously to those presented in ref. [7]. The Feynman diagrams for the anomalous dimensions of the flavour-singlet twist-2 spin- $N$  operators have been generated using QGRAF [19], and then processed by a FORM [20] program, see ref. [21], that collects self-energy insertions, determines the colour factors and finds the topologies in the notation of the FORCER package [22] that performs the integral reduction after the harmonic projection [23] to the desired value of  $N$ . For computational efficiency, diagrams with the same colour factor and topology are merged into meta-diagrams.

The main issue in these covariant-gauge calculations in a massless off-shell case is the correct treatment of the gluon operators, see refs. [24]. Since, as discussed above, we are dealing here with an effective lowest-order case, we are not confronted with the full complexity of this issue. We expect to return to this point in a future publication of all four-loop contributions to the singlet anomalous dimensions. For three-loop on-shell OPE calculations with heavy quarks see ref. [25].

We now present our results for the quartic-Casimir contributions to the anomalous dimensions (2.5) at  $N = 2, 4, \dots, 16$ . These include fractions and the values  $\zeta_3$  and  $\zeta_5$  of Riemann’s  $\zeta$ -function,

but, as factorization-scheme independent ‘leading-order’ contributions, do not include terms with even- $n$  values  $\zeta_n$ , see refs. [26]. For brevity the results are written down in a numerical form.

The non-singlet and pure-singlet quark-quark anomalous dimensions include

$$\begin{aligned} \gamma_{\text{ns}}^{(3)+}(N) \Big|_{d_{FA}^{(4)}/n_F} &= +773.10566 \delta_{N,2} + 69.385963 \delta_{N,4} - 186.61376 \delta_{N,6} \\ &- 346.75182 \delta_{N,8} - 465.07282 \delta_{N,10} - 559.57588 \delta_{N,12} \\ &- 638.52578 \delta_{N,14} - 706.44946 \delta_{N,16} - \dots, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \gamma_{\text{ns}}^{(3)}(N) \Big|_{n_f d_{FF}^{(4)}/n_F} &= -65.736531 \delta_{N,2} - 135.95246 \delta_{N,4} - 176.15626 \delta_{N,6} \\ &- 205.54604 \delta_{N,8} - 229.07719 \delta_{N,10} - 248.80626 \delta_{N,12} \\ &- 265.82674 \delta_{N,14} - 280.80532 \delta_{N,16} - \dots, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \gamma_{\text{ps}}^{(3)}(N) \Big|_{n_f d_{FF}^{(4)}/n_F} &= -146.97872 \delta_{N,2} - 88.852325 \delta_{N,4} - 37.651992 \delta_{N,6} \\ &- 20.290180 \delta_{N,8} - 12.702121 \delta_{N,10} - 8.7568928 \delta_{N,12} \\ &- 6.4465398 \delta_{N,14} - 4.9735061 \delta_{N,16} - \dots. \end{aligned} \quad (3.3)$$

The corresponding results for the quark-gluon and gluon-quark quantities read

$$\begin{aligned} \gamma_{\text{gq}}^{(3)}(N) \Big|_{d_{FA}^{(4)}/n_F} &= -773.10566 \delta_{N,2} - 154.99156 \delta_{N,4} - 33.190677 \delta_{N,6} \\ &- 3.6877393 \delta_{N,8} + 5.6280884 \delta_{N,10} + 8.8432407 \delta_{N,12} \\ &+ 9.8467770 \delta_{N,14} + 9.9636194 \delta_{N,16} + \dots, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \gamma_{\text{gq}}^{(3)}(N) \Big|_{n_f d_{FF}^{(4)}/n_F} &= +212.71525 \delta_{N,2} + 40.812981 \delta_{N,4} + 20.540955 \delta_{N,6} \\ &+ 13.623478 \delta_{N,8} + 10.207939 \delta_{N,10} + 8.1771347 \delta_{N,12} \\ &+ 6.8293658 \delta_{N,14} + 5.8681866 \delta_{N,16} + \dots \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \gamma_{\text{qg}}^{(3)}(N) \Big|_{n_f d_{FA}^{(4)}/n_A} &= -386.55283 \delta_{N,2} - 154.99156 \delta_{N,4} - 41.488346 \delta_{N,6} \\ &- 5.1628350 \delta_{N,8} + 8.4421327 \delta_{N,10} + 13.896521 \delta_{N,12} \\ &+ 16.001012 \delta_{N,14} + 16.606032 \delta_{N,16} + \dots, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \gamma_{\text{qg}}^{(3)}(N) \Big|_{n_f^2 d_{FF}^{(4)}/n_A} &= +106.35762 \delta_{N,2} + 40.812981 \delta_{N,4} + 25.676194 \delta_{N,6} \\ &+ 19.072869 \delta_{N,8} + 15.311908 \delta_{N,10} + 12.849783 \delta_{N,12} \\ &+ 11.097719 \delta_{N,14} + 9.7803110 \delta_{N,16} + \dots. \end{aligned} \quad (3.7)$$

Finally the quartic-Casimir contributions to the four-loop gluon-gluon anomalous dimension are found to be

$$\begin{aligned} \gamma_{\text{gg}}^{(3)}(N) \Big|_{d_{AA}^{(4)}/n_A} &= +139.70415 \delta_{N,4} - 42.404324 \delta_{N,6} \\ &- 196.38527 \delta_{N,8} - 317.16583 \delta_{N,10} - 414.93934 \delta_{N,12} \\ &- 496.71624 \delta_{N,14} - 566.90638 \delta_{N,16} - \dots, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \gamma_{\text{gg}}^{(3)}(N) \Big|_{n_f d_{FA}^{(4)}/n_A} &= +386.55283 \delta_{N,2} - 441.47043 \delta_{N,4} - 458.54303 \delta_{N,6} \\ &- 462.79602 \delta_{N,8} - 469.79385 \delta_{N,10} - 478.54929 \delta_{N,12} \\ &- 487.97876 \delta_{N,14} - 497.49491 \delta_{N,16} - \dots, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \gamma_{\text{gg}}^{(3)}(N) \Big|_{n_f^2 d_{FF}^{(4)}/n_A} &= -106.35762 \delta_{N,2} - 117.11160 \delta_{N,4} - 121.74849 \delta_{N,6} \\ &- 123.79780 \delta_{N,8} - 124.86680 \delta_{N,10} - 125.48946 \delta_{N,12} \\ &- 125.88108 \delta_{N,14} - 126.14172 \delta_{N,16} - \dots. \end{aligned} \quad (3.10)$$

The absence of a  $\delta_{N,2}$  term, i.e., the vanishing of the  $N=2$  contribution in eq. (3.8) is required by the momentum sum rule. The implications of eqs. (3.8)–(3.10) for the large- $x$  limit (2.4) are addressed in section 4 below.

As for the non-singlet case in ref. [7], these fixed- $N$  results are sufficient to deduce the all- $N$  form of the  $\zeta_5$  contributions. The quark-quark anomalous dimensions can be expressed as

$$\begin{aligned} \gamma_{\text{ns}}^{(3)+}(N) \Big|_{\zeta_5 d_{FA}^{(4)}/n_F} &= \frac{320}{3} \left( S_1(24\eta - 24S_1 + 58) - 69\eta^2 + \frac{63}{2}\eta - 37 \right), \end{aligned} \quad (3.11)$$

$$\gamma_{\text{ns}}^{(3)}(N) \Big|_{\zeta_5 n_f d_{FF}^{(4)}/n_F} = \frac{1280}{3} \left( 6\eta^2 - 2S_1 - 5\eta + 3 \right), \quad (3.12)$$

$$\gamma_{\text{ps}}^{(3)}(N) \Big|_{\zeta_5 n_f d_{FF}^{(4)}/n_F} = \frac{1280}{3} \left( 9\eta^2 + 14\eta - 4\nu - \frac{1}{4} \right) \quad (3.13)$$

in terms of the quantities

$$\eta \equiv \frac{1}{N} - \frac{1}{N+1} \equiv D_0 - D_1 = \frac{1}{N(N+1)}, \quad (3.14)$$

$$\nu \equiv \frac{1}{N-1} - \frac{1}{N+2} \equiv D_{-1} - D_2 = \frac{3}{(N-1)(N+2)} \quad (3.15)$$

and the harmonic sum  $S_1 \equiv S_1(N) = \sum_{k=1}^N k^{-1}$  which are reciprocity-respecting (RR), i.e., invariant under the replacement  $N \rightarrow 1-N$  corresponding to  $f(x) \rightarrow -xf(x^{-1})$  in  $x$ -space.

The all- $N$  results for the  $\zeta_5 d_{xy}^{(4)}$  contributions to the four-loop quark-gluon and gluon-quark anomalous dimensions read

$$\begin{aligned} \gamma_{\text{gq}}^{(3)}(N) \Big|_{\zeta_5 d_{FA}^{(4)}/n_F} &= \frac{320}{3} \left( 24(2D_{-1} - 2D_0 + D_1)S_1 - 24D_{-1}^2 + 126D_0^2 + 63D_1^2 \right. \\ &\left. - 30D_{-1} - 202D_0 + \frac{391}{2}D_1 - 8D_2 \right), \end{aligned} \quad (3.16)$$

$$\begin{aligned} \gamma_{\text{gq}}^{(3)}(N) \Big|_{\zeta_5 n_f d_{FF}^{(4)}/n_F} &= \frac{1280}{3} \left( -24D_0^2 - 12D_1^2 + 4D_{-1} + 32D_0 - 34D_1 \right) \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \gamma_{\text{qg}}^{(3)}(N) \Big|_{\zeta_5 n_f d_{FA}^{(4)}/n_A} &= \frac{640}{3} \left( 24(D_0 - 2D_1 + 2D_2)S_1 - 63D_0^2 - 126D_1^2 + 24D_2^2 \right. \\ &\left. - 8D_{-1} + \frac{391}{2}D_0 - 202D_1 - 30D_2 \right), \end{aligned} \quad (3.18)$$

$$\begin{aligned} \mathcal{Y}_{\text{qg}}^{(3)}(N) \Big|_{\zeta_5 n_f^2 d_{FF}^{(4)}/n_A} \\ = \frac{2560}{3} \left( 12D_0^2 + 24D_1^2 - 34D_0 + 32D_1 + 4D_2 \right). \end{aligned} \quad (3.19)$$

By multiplying eqs. (3.16) and (3.17) with

$$\mathcal{Y}_{\text{qg}}^{(0)}(N) = -2 C_F (D_0 - 2 D_1 + 2 D_2), \quad (3.20)$$

and eqs. (3.18) and (3.19) with

$$\mathcal{Y}_{\text{gq}}^{(0)}(N) = -2 n_f (2 D_{-1} - 2 D_0 + D_1), \quad (3.21)$$

one arrives at two RR expressions that fulfil eq. (2.13) at all  $N$ ; the required relation between  $n_A$ ,  $n_F$  and  $C_F$  has been given below eq. (2.10).

The corresponding gluon–gluon anomalous dimension are given by the RR expressions

$$\begin{aligned} \mathcal{Y}_{\text{gg}}^{(3)}(N) \Big|_{\zeta_5 d_{AA}^{(4)}/n_A} \\ = \frac{64}{3} \left( 30 \left( 12\eta^2 - 4\nu^2 - S_1(4S_1 + 8\eta - 8\nu - 11) - 7\nu \right) \right. \\ \left. + 188\eta - \frac{751}{3} - \frac{1}{6}N(N+1) \right), \end{aligned} \quad (3.22)$$

$$\begin{aligned} \mathcal{Y}_{\text{gg}}^{(3)}(N) \Big|_{\zeta_5 n_f d_{FA}^{(4)}/n_A} \\ = \frac{128}{3} \left( 10 \left( 15\eta^2 - 6S_1 + 2\nu \right) - 121\eta \right. \\ \left. + \frac{287}{3} + \frac{1}{3}N(N+1) \right), \end{aligned} \quad (3.23)$$

$$\begin{aligned} \mathcal{Y}_{\text{gg}}^{(3)}(N) \Big|_{\zeta_5 n_f^2 d_{FF}^{(4)}/n_A} \\ = \frac{256}{3} \left( -120\eta^2 + 23\eta - \frac{17}{6} - \frac{1}{6}N(N+1) \right). \end{aligned} \quad (3.24)$$

These results exhibit interesting features in the large- $N$  threshold limit and the  $N \rightarrow 1$  BFKL limit.

Unlike all  $N$ -space expressions for QCD splitting functions calculated up to now, eqs. (3.22)–(3.24) include terms of the form  $\zeta_5 N(N+1)$ . In the complete results, these terms have to be compensated by contributions that develop  $\zeta_5$ -terms in the limit  $N \rightarrow \infty$ , since the overall leading large- $N$  behaviour is given by  $\ln N$  multiplied by the cusp anomalous dimension [5] due to the Mellin transform of eq. (2.4). This compensation has occurred before, in the three-loop coefficient functions for inclusive DIS [1], where  $\zeta_5$  enters with positive powers of  $N$  in the combination

$$\begin{aligned} f(N) = 5\zeta_5 - 2S_{-5} + 4\zeta_3 S_{-2} - 4S_{-2,-3} + 8S_{-2,-2,1} \\ + 4S_{3,-2} - 4S_{4,1} + 2S_5 \end{aligned} \quad (3.25)$$

of  $\zeta$ -values and harmonic sums [27] that ensures the correct large- $N$  behaviour. It may be worthwhile to note that the  $N(N+1)$  terms in eqs. (3.22)–(3.24) cancel in the SUSY limit (2.12).

In addition, both eq. (3.11) and (3.22) include terms of the form  $\zeta_5 [S_1(N)]^2$  – with the same coefficients, as required in view of eqs. (2.11) and (2.12) – that also need to be compensated in the large- $N$  limit. A natural possibility is that the diagonal QCD splitting function include terms with  $[S_1(N)]^2 f(N)$ , i.e., the same structure as the ‘wrapping correction’ in the anomalous dimensions in  $\mathcal{N} = 4$  maximally supersymmetric Yang–Mills theory [28]. Unfortunately we are not (yet) in a position to derive the all- $N$  structure of the  $\zeta_3$ -terms, which could provide further evidence for

**Table 1**

Fourth-order coefficients of the quark and gluon cusp anomalous dimensions determined from the large- $x$  limit (2.4) of the quark–quark and gluon–gluon splitting functions. The errors in the quark case are correlated due to the exactly known large- $n_c$  limit. Our numerical value of  $-31.00 \pm 0.4$  [7] for the coefficient of  $n_f C_F^3$  in  $A_{4,q}$  has been replaced by recent exact result of ref. [32]. This and the exact values for the  $n_f^2$  and  $n_f^3$  coefficients have been rounded to seven digits.

Quark	Gluon	$A_{4,q}$	$A_{4,g}$
$C_F^4$	–	0	–
$C_F^3 C_A$	–	0	–
$C_F^2 C_A^2$	–	0	–
$C_F C_A^3$	$C_A^4$	610.25 ± 0.1	–
$d_{FA}^{(4)}/N_F$	$d_{AA}^{(4)}/N_A$	–507.0 ± 2.0	–507.0 ± 5.0
$n_f C_F^3$	$n_f C_F^2 C_A$	–31.00554	–
$n_f C_F^2 C_A$	$n_f C_F C_A^2$	38.75 ± 0.2	–
$n_f C_F C_A^2$	$n_f C_A^3$	–440.65 ± 0.2	–
$n_f d_{FF}^{(4)}/N_F$	$n_f d_{FA}^{(4)}/N_A$	–123.90 ± 0.2	–124.0 ± 0.6
$n_f^2 C_F^2$	$n_f^2 C_F C_A$	–21.31439	–
$n_f^2 C_F C_A$	$n_f^2 C_A^2$	58.36737	–
–	$n_f^2 d_{FF}^{(4)}/N_A$	–	0.0 ± 0.1
$n_f^3 C_F$	$n_f^3 C_A$	2.454258	2.454258

(or exclude) the occurrence of the function (3.25) in the four-loop anomalous dimensions in QCD.

In the limit  $N \rightarrow 1$ , eqs. (3.16) and (3.22) include terms with  $1/(N-1)^2$ . Since the leading terms at four-loop are proportional to  $1/(N-1)^4$  [11], these represent next-to-next-to-leading logarithmic (NNLL) contributions in this high-energy (small- $x$ ) limit. Unless these terms are compensated by contributions that develop  $\zeta_5$ -terms in the limit  $N \rightarrow 1$ , the complete NNLL four-loop contributions in QCD cannot possibly be obtained by resumming lower-order information, as such information cannot predict coefficients of quartic Casimir invariants.

#### 4. $x$ -space results and cusp anomalous dimensions

The fixed- $N$  moments (3.1)–(3.9) of the quartic-Casimir contributions to the four-loop splitting functions can be employed to obtain  $x$ -space approximations which small uncertainties at least at  $x \gtrsim 0.1$ . In the quark–quark and gluon–gluon cases, these approximations involve only two unknown coefficients of terms that do not vanish for  $x \rightarrow 1$ , i.e., the coefficients  $A_4$  and  $B_4$  in eq. (2.4). The predictable coefficients  $C_4$  and  $D_4$  vanish for the  $d_{xy}^{(4)}$  terms, since there are no lower-order quantities with these colour factors. Consequently, the coefficients of the leading large- $x$  terms, i.e., the cusp anomalous dimensions, can be determined with a rather high accuracy.

This programme has been carried out in ref. [7] for the complete non-leading large- $n_c$  ( $Nn_c$ )  $n_f^0$  and  $n_f^1$  parts of  $P_{\text{ns}}^{(3)}(x)$  in QCD as well as for all individual colour factors. The leading large- $n_c$  contributions and the  $n_f^2$  and  $n_f^3$  terms are completely known [6,7,29]. The results for  $A_{4,q}$  are collected in Table 1, where the  $n_f^0$  part has been improved upon using the  $Nn_c$  results for QCD. The coefficients of  $A_{4,q}$  which are known exactly have also been determined from the quark form factor [30,31], the results are in complete agreement. Very recently, the exact coefficient of  $C_F^3 n_f$  has been obtained in ref. [32]. For numerical results in  $\mathcal{N} = 4$  maximally supersymmetric Yang–Mills theory see ref. [33].

We have now performed analogous determinations of the quartic-Casimir coefficients of  $A_{4,g}$ . The results are also shown in Table 1, together with the only piece known exactly so far, the  $C_A n_f^3$  contribution [6,34]. We see that, as up to the third order



[12], the corresponding quark and gluon entries have the same coefficients (for now: as far as they have been computed, and within numerical errors). We refer to this (for now: conjectured) relation as *generalized Casimir scaling*.

Unlike to three loops, this relation does not have the consequence that the values of  $A_{4,g}$  and  $A_{4,q}$  are related by a simple numerical Casimir scaling in QCD, i.e., a factor of  $C_A/C_F = 9/4$ . However, this numerical Casimir scaling is restored in the large- $n_c$  limit of the quartic colour factors, and therefore also in the overall large- $n_c$  limit, see also ref. [35].

The results of refs. [6,7] and the present paper lead to the following results for the four-loop quark and gluon cusp anomalous dimensions, expanded in powers of  $\alpha_s/(4\pi)$ , recall eq. (2.3),

$$A_{4,q} = 20702(2) - 5171.9(2)n_f + 195.5772n_f^2 + 3.272344n_f^3, \quad (4.1)$$

$$A_{4,g} = 40880(30) - 11714(2)n_f + 440.0488n_f^2 + 7.362774n_f^3. \quad (4.2)$$

For comparison, the large- $n_c$  coefficients of  $A_{4,q}$  (not changing the overall factor of  $C_F$ ) read 21209.0, 5179.37 and 190.841, respectively, for the  $n_f^0$ ,  $n_f^1$  and  $n_f^2$  contributions. The numerical Casimir scaling between  $A_{4,g}$  and  $A_{4,q}$  is broken by almost 15% in the  $n_f^0$  terms. This breaking is due to the non-leading large- $n_c$  ( $Nn_c$ ) part of the quartic-Casimir term, which is larger by a factor of 6 in  $A_{4,g}$  than in  $A_{4,q}$  due to ‘36’ and ‘6’ in eqs. (2.8) and (2.9). This much larger size of the  $Nn_c$  contribution in the gluon case also leads to the much larger uncertainty of its  $n_f^0$  coefficient.

Combining the above with the lower-order coefficients, we arrive at the very benign expansions

$$\begin{aligned} A_q(\alpha_s, n_f = 3) &= 0.42441\alpha_s[1 + 0.72657\alpha_s + 0.73405\alpha_s^2 \\ &\quad + 0.6647(2)\alpha_s^3 + \dots], \\ A_q(\alpha_s, n_f = 4) &= 0.42441\alpha_s[1 + 0.63815\alpha_s + 0.50998\alpha_s^2 \\ &\quad + 0.3168(2)\alpha_s^3 + \dots], \\ A_q(\alpha_s, n_f = 5) &= 0.42441\alpha_s[1 + 0.54973\alpha_s + 0.28403\alpha_s^2 \\ &\quad + 0.0133(3)\alpha_s^3 + \dots] \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} A_g(\alpha_s, n_f = 3) &= 0.95493\alpha_s[1 + 0.72657\alpha_s + 0.73405\alpha_s^2 \\ &\quad + 0.415(2)\alpha_s^3 + \dots], \\ A_g(\alpha_s, n_f = 4) &= 0.95493\alpha_s[1 + 0.63815\alpha_s + 0.50998\alpha_s^2 \\ &\quad + 0.064(2)\alpha_s^3 + \dots], \\ A_g(\alpha_s, n_f = 5) &= 0.95493\alpha_s[1 + 0.54973\alpha_s + 0.28403\alpha_s^2 \\ &\quad - 0.243(2)\alpha_s^3 + \dots] \end{aligned} \quad (4.4)$$

in terms of  $\alpha_s$  for the physically relevant values of the number  $n_f$  of light flavours. Due to the additional cancellations between the terms without and with  $n_f$  in eq. (4.1) and (4.2), the numerical Casimir scaling is completely broken in fourth-order contributions.

The remaining uncertainties are practically irrelevant for all phenomenological applications, which include (but are by no means exhausted by) calculations of the soft-gluon exponentiation at next-to-next-to-next-to-leading logarithmic ( $N^3LL$ ) [36] and higher accuracy, see, e.g., ref. [37], and similar calculations in other frameworks such as soft-collinear effective theory.

Another application is the absolute ratio  $|\mathcal{F}^g(q^2)/\mathcal{F}^g(-q^2)|$  of the renormalized time-like and space-like Higgs-gluon-gluon form

factors in the heavy-top limit. This quantity is infrared finite and directly enters the cross section for Higgs boson production in hadronic collisions. Using eq. (4.4) for the small  $A_{4,g}$  contribution, we can update the result of ref. [38] which used a value based on a Padé estimate for  $A_{4,q}$  and numerical Casimir scaling. The new result for  $n_f = 5$  reads

$$\begin{aligned} \left| \frac{\mathcal{F}^g(q^2)}{\mathcal{F}^g(-q^2)} \right|^2 &= 1 + 4.7124\alpha_s + 13.694\alpha_s^2 + 25.935\alpha_s^3 \\ &\quad + (34.82 \pm 0.01)\alpha_s^4 + \dots \end{aligned} \quad (4.5)$$

While the coefficient of  $\alpha_s^4$  is noticeably smaller than in ref. [38], the general pattern is unchanged: large coefficients, but definitely no sign of a runaway growth – on the contrary. The numerical  $\alpha_s^4$  effect in eq. (4.5) is a fraction of a percent at scales close to the mass of the Higgs boson.

## 5. Summary

We have presented the first calculations of a substantial number of moments of contributions to the four-loop ( $N^3LO$ ) flavour-singlet splitting functions  $P_{ik}^{(3)}$  outside the large- $n_f$  limit. Specifically, we have obtained the even moments  $N \leq 16$  of all terms with quartic Casimir invariants. The calculations have been performed in the framework of the operator-product expansion; the results at  $N \leq 6$  (and partly at  $N = 8$ ) have been checked against those of conceptually much simpler, but computationally much harder determinations via structure functions in deep-inelastic scattering.

Our results show features expected for these effectively lowest-order contributions, such as the supersymmetric relation, and properties not predicted before, in particular a simple relation between the quartic-Casimir parts of the off-diagonal splitting functions  $P_{qg}^{(3)}$  and  $P_{gq}^{(3)}$ . We have obtained the all- $N$  expressions for the  $\zeta_5$  parts. The diagonal quantities  $P_{qq}^{(3)}$  and  $P_{gg}^{(3)}$  include contributions which have the structure of the wrapping corrections found in  $\mathcal{N} = 4$  maximally supersymmetric Yang-Mills theory. The all- $N$  expressions for  $P_{gg}^{(3)}$  include numerator- $N$  terms. Such terms are not entirely new, but have not been encountered in splitting functions before.

The calculated moments of  $P_{gg}^{(3)}$  enable a numerical determination of a quantity that is important in a much wider context, the (light-like) four-loop gluon cusp anomalous dimension  $A_{4,g}$ . We find for the quartic-Casimir parts, and conjecture for all other terms, that the coefficients for  $A_{4,g}$  are related to those of its quark counterpart  $A_{4,q}$  by a direct generalization of the Casimir scaling found at lower orders. This allows us to present numerical results for  $A_{4,g}$  in QCD that are sufficiently accurate for all phenomenological purposes. Due to differences in (the contributions that are non-leading in the large- $n_c$  limit to) the quartic colour factors, there is no simple relation between the numerical values of  $A_{4,g}$  and  $A_{4,q}$  for physical values of the number of flavours  $n_f$ .

FORM files of our fixed- $N$  and all- $N$  moments of the four-loop splitting functions, including the analytic expressions for the former quantities not shown in section 3, can be obtained from the preprint server <https://arXiv.org> by downloading the source of this article. They are also available from the authors upon request.

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