A short review of computational methods for uncertainty quantification in engineering

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Publication Date:
2013-06-05

Permanent Link:
https://doi.org/10.3929/ethz-a-010060769

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A Short Review of Computational Methods for Uncertainty Quantification in Engineering

B. Sudret

Chair of Risk, Safety & Uncertainty Quantification

Common Challenges in Computationally-based Engineering Research
June 5th, 2013
Some common engineering structures

- Cattenom nuclear power plant (France)
- Cormet de Roselend dam (France)
- Airbus A380
- Military satellite
- Bladed disk
Computational models

Complex systems are designed using computational models that are based on:

- **A mathematical description** of the physics
- **Numerical algorithms** that solve the resulting set of (e.g. partial differential) equations, e.g. finite element models

**Computational models** are used:

- Together with experimental data for **calibration** purposes
- To explore the design space ("virtual prototypes")
- To **optimize** the system w.r.t cost constraints
- To assess its **robustness** w.r.t uncertainty and its **reliability**
Sources of uncertainty

- Differences between the **designed** and the **real** system:
  - Dimensions (tolerances in manufacturing)
  - Material properties (*e.g.* variability of the stiffness or resistance)

- Unforecast **exposures**: exceptional service loads, natural hazards (earthquakes, floods), climate loads (hurricanes, snow storms, etc.)
Global framework for managing uncertainties

**Step A**
Model(s) of the system
Assessment criteria

**Step B**
Quantification of sources of uncertainty

**Step C**
Uncertainty propagation

**Random variables**
Computational model
Moments
Probability of failure
Response PDF

**Step C’**
Sensitivity analysis
Global framework for managing uncertainties

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Random variables → Computational model → Moments  
Probability of failure  
Response PDF

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Global framework for managing uncertainties

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**Step C’**
Sensitivity analysis

Random variables ➔ Computational model ➔ Moments
Probability of failure
Response PDF
Step A: computational model(s)

- Vector of input parameters: \( \mathbf{x} \in \mathbb{R}^M \)
- Computational model: \( \mathcal{M} \)
- Model response: \( \mathbf{y} = \mathcal{M}(\mathbf{x}) \in \mathbb{R}^N \)

- geometry
- material properties
- loading

- analytical formula
- finite element model
- comput. workflow

- displacements
- strains, stresses
- temperature, etc.
Step A: computational model(s)

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Step C: uncertainty propagation methods

Computational model

Step A

Probabilistic model

Step B

Probabilistic-computational model

Central trend

Reliability analysis

Response PDF

Probabilistic-computational model

Introduction
Uncertainty quantification framework
Some examples
Conclusions

Step A
- Central trend: \( \mu \), \( \sigma \)

Step B
- Reliability analysis: \( P_f \)

Probabilistic-Computational model

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Comput. Methods in UQ
June 5th, 2013
Monte Carlo simulation allows the engineer to assess the performance of a large number of virtual systems featuring different realizations of the input parameters.

It uses a random number generator to compute a set of input parameters $\mathcal{X} = \{x_i, i = 1, \ldots, n\}$. The corresponding set of model responses $\mathcal{Y} = \{M(x_i), i = 1, \ldots, n\}$ is computed and post-processed.
Monte Carlo simulation

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Step C: Central trend

Given a computer model $\mathcal{M}$ and a probabilistic model of its input parameters $X \sim f_X$, what is the expected value / scattering of the output $Y = \mathcal{M}(X)$?

- mean value $\mu_Y$
- standard deviation $\sigma_Y$
- higher order moments (skewness / kurtosis of the output distribution)

Methods

- Perturbation method
- Monte Carlo simulation
- Quadrature methods
Step C: Reliability analysis / rare event simulation

Given a computer model $\mathcal{M}$, a probabilistic model of its input parameters $\mathbf{X} \sim f_X$ and a performance criterion (e.g. “$\mathcal{M}(\mathbf{X}) \leq y_{adm}$”), what is the probability of failure?

$$p_f = \mathbb{P}(\mathcal{M}(\mathbf{X}) \geq y_{adm})$$

Methods

- **Monte Carlo simulation**
- **FORM/SORM methods**: based on the assumption of a most probable failure point
- **Advanced simulation methods**: importance sampling, subset simulation
- **Surrogate-based methods**, e.g. using Kriging
Step C: Distribution analysis

Given a computer model $\mathcal{M}$ and a probabilistic model of its input parameters $X \sim f_X$, what are the characteristics of the output distribution of $Y = \mathcal{M}(X)$?

- range / shape (uni-/multi-modal?)
- quantiles (median, inter-quartile, 99%-quantile, etc.)

Methods

- Monte Carlo simulation + kernel smoothing (if large sample set available)
- Surrogate-based methods: polynomial chaos expansions, Kriging
Sensitivity analysis / Parametric study

What are the input parameters (or combinations thereof) that explain at best the variability of the output?

- **Deconstruction** of the model structure: detection of **non-linearities**, interactions between input parameters, dummy variables

- **Variance decomposition**: **Sobol’ indices**

$$\text{Var} [Y] = \sum_{i=1}^{M} D_i + \sum_{1 \leq i < j \leq M} D_{ij} + \cdots + D_{1\,2\,\ldots\,M}$$

**Methods**

- **Sobol’ indices** using Monte Carlo simulation (if large sample set available)

- **Polynomial chaos expansions**
Truss structure

Problem statement

Input: 10 independent random variables
- Bars properties (2 cross-sections, 2 Young’s moduli)
- Loads (6 parameters)

Output: maximal deflection

Uncertainty quantification

- Distribution of the maximal deflection?
- Mean value and standard deviation?
- Reliability analysis: Prob\[v \geq \frac{L}{200} = 12 \text{ cm}\]?

## Statistical moments

<table>
<thead>
<tr>
<th></th>
<th>Reference</th>
<th>Monte Carlo</th>
<th>Polynomial chaos</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean (cm)</strong></td>
<td>7.94</td>
<td>8.02 ± 0.49</td>
<td>7.98</td>
</tr>
<tr>
<td><strong>Std. dev. (cm)</strong></td>
<td>1.11</td>
<td>1.36 ± 0.10</td>
<td>1.10</td>
</tr>
</tbody>
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### Maximum deflection

![PDF distribution](image_url)

- **Reference**
- **Learning set**
- **PC expansion**

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**Truss structure**

**Statistical moments**

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**Truss structure**

**Reliability analysis**

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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100,000 runs</td>
<td>500 runs</td>
</tr>
<tr>
<td>10 cm</td>
<td>4.39e-02 ± 3.0%</td>
<td>4.30e-02 ± 0.9%</td>
</tr>
<tr>
<td>11 cm</td>
<td>8.61e-03 ± 6.7%</td>
<td>8.71e-03 ± 2.1%</td>
</tr>
<tr>
<td>12 cm</td>
<td>1.62e-03 ± 15.4%</td>
<td>1.51e-03 ± 5.1%</td>
</tr>
<tr>
<td>13 cm</td>
<td>2.20e-04 ± 41.8%</td>
<td>2.03e-04 ± 13.8%</td>
</tr>
</tbody>
</table>
Truss structure

Sensitivity analysis

<table>
<thead>
<tr>
<th>Variable</th>
<th>Reference</th>
<th>QMC</th>
<th>Smolyak</th>
<th>Regression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.388</td>
<td>0.366</td>
<td>0.372</td>
<td>0.367</td>
</tr>
<tr>
<td>$E_1$</td>
<td>0.367</td>
<td>0.373</td>
<td>0.372</td>
<td>0.367</td>
</tr>
<tr>
<td>$P_3$</td>
<td>0.075</td>
<td>0.077</td>
<td>0.077</td>
<td>0.080</td>
</tr>
<tr>
<td>$P_4$</td>
<td>0.079</td>
<td>0.077</td>
<td>0.077</td>
<td>0.080</td>
</tr>
<tr>
<td>$P_5$</td>
<td>0.035</td>
<td>0.046</td>
<td>0.037</td>
<td>0.039</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.031</td>
<td>0.039</td>
<td>0.037</td>
<td>0.039</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.014</td>
<td>0.014</td>
<td>0.013</td>
<td>0.012</td>
</tr>
<tr>
<td>$E_2$</td>
<td>0.010</td>
<td>0.013</td>
<td>0.013</td>
<td>0.012</td>
</tr>
<tr>
<td>$P_6$</td>
<td>0.005</td>
<td>0.014</td>
<td>0.005</td>
<td>0.005</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.004</td>
<td>0.005</td>
<td>0.005</td>
<td>0.005</td>
</tr>
</tbody>
</table>

# FE runs | 5 500 000 | 10 000 | 231     | 66         |
Earthquake engineering: performance-based design

Question

What is the probability of collapse of a building as a function of the “intensity” of a potential earthquake?

Uncertainties

- Properties of the structure (material strength, stiffness of the connections, etc.)
- Earthquake magnitude, duration, peak ground acceleration

Non linear transient finite element analysis of the structure for different synthetic earthquakes
Earthquake engineering: performance-based design

- The vulnerability is represented by a **fragility curve** (probability of attaining some state of damage conditionally on the PGA)
- Seismologists provide models for the PGA w.r.t. the local seismicity (occurrence / magnitude)
- Damage-related costs may be incorporated towards a global risk assessment

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Conclusions

- **Uncertainty quantification** has become a hot topic in many (if not all) domains of applied science and engineering.

- It is a **transdisciplinary field** which takes advantage from research progress in the mathematical- (statistics, PDEs), engineering- (civil, mechanical, chemical, etc.) and computer science communities.

- **Non intrusive** approaches allow for applications in various fields of the same algorithms.

- Generic analysis tools may be developed and disseminated towards the community.

  “The UQLab platform”

Thank you very much for your attention!