

Harmonic map flow and variants

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Author(s):

Rupflin, Melanie

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Harmonic map flow and variants

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presented by
Melanie Rupflin
Dipl. Math. ETH Zurich
born March 27, 1982
citizen of Lommis TG, Switzerland

accepted on the recommendation of
Prof. Dr. Michael Struwe, examiner
Prof. Dr. Tristan Rivière, co-examiner
Prof. Dr. Peter Topping, co-examiner

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Abstract

We study weak solutions of the harmonic map heat flow and the related extrinsic polyharmonic map heat flows of arbitrary order.

In the first chapter of this thesis we consider the harmonic map heat flow in the critical dimension. We prove a sharp uniqueness criterion for weak solutions under a natural a priori assumption on the regularity of the energy function $t \mapsto E(u(t))$ and thus establish a conjecture of Topping. Furthermore we establish uniqueness for general weak solutions under the assumption that the energy does not instantaneously increase by more than a certain positive quantum. This theorem in particular improves a well known uniqueness result of Freire.

In the second chapter we generalise these uniqueness results for the harmonic map flow to the extrinsic polyharmonic map flows of arbitrary order in the critical dimension. Furthermore we describe the behaviour of all weak solutions of the polyharmonic flows that satisfy a natural a priori assumption; we prove that uniqueness can only be lost by reverse bubbling and obtain that the weak solutions under consideration are smooth away from a discrete set of time-slices.

In the third chapter we study the related questions of existence and uniqueness of (outgoing) selfsimilar solutions of the harmonic map heat flow in supercritical dimensions. We show that for settings with appropriate symmetry the issue of uniqueness of selfsimilar solutions is determined by the properties of the so called *equator maps*. On the one hand, we show that the harmonic map heat flow has a unique equivariant and selfsimilar weak solution for any admissible initial data whenever the equator map is energy-minimising. On the other hand, we obtain non-uniqueness results for selfsimilar solutions in settings with equator maps that are not energy-minimising; in fact, we prove that the number of (genuinely different) selfsimilar solutions of the harmonic map flow can be arbitrarily large, even infinite, for suitably chosen initial data. These non-uniqueness results extend earlier work of Angenent, Ilmanen and Velazquez. Our results yield examples of non-uniqueness of solutions to the harmonic map heat flow that respect the monotonicity formula of Struwe extending the work of Coron and Hong.

Zusammenfassung

Wir untersuchen schwache Lösungen des harmonischen Wärmeflusses und der verwandten extrinsischen polyharmonischen Wärmeflüsse.

Im ersten Kapitel beschäftigen wir uns mit der Frage der Eindeutigkeit von schwachen Lösungen des harmonischen Wärmeflusses in der kritischen Dimension. Wir beweisen ein optimales Eindeutigkeitskriterium für schwache Lösungen, deren Energiefunktional $t \mapsto E(u(t))$ einer natürlichen a priori Regularitätsbedingung genügt und zeigen damit eine Vermutung von Topping. Andererseits beweisen wir Eindeutigkeit für beliebige schwache Lösungen des harmonischen Flusses in Dimension zwei unter der Annahme, dass die Energie zu keinem Zeitpunkt um mehr als eine bestimmte positive Zahl anwächst. Dieses Theorem stellt eine Verbesserung des bekannten Eindeutigkeitsresultates von Freire dar.

Im zweiten Kapitel beweisen wir, dass sich diese Eindeutigkeitsresultate für den harmonischen Fluss auf die gesamte Familie der extrinsischen polyharmonischen Wärmeflüsse beliebiger Ordnung in der jeweiligen kritischen Dimension erweitern lassen. Wir beschreiben ausserdem das Verhalten von schwachen Lösungen, deren Energie eine natürliche a priori Voraussetzung erfüllt; wir zeigen, dass der Verlust der Eindeutigkeit nur durch *reverse bubbling* verursacht werden kann und erhalten eine Regularitätsaussage für die betrachteten schwachen Lösungen.

Im dritten Kapitel untersuchen wir die Fragen der Existenz und Eindeutigkeit von selbstähnlichen Lösungen des harmonischen Wärmeflusses in superkritischen Dimensionen. Wir zeigen, dass unter geeigneten Symmetrieannahmen die Frage der Eindeutigkeit durch die Eigenschaften der sogenannten *Äquatorabbildungen* bestimmt wird. Wir beweisen einerseits, dass der harmonische Wärmefluss zu jeder zulässigen Anfangsbedingung eine eindeutige equivariante, selbstähnliche schwache Lösung besitzt, falls die Äquatorabbildung energie-minimierend ist. Andererseits erhalten wir Nichteindeutigkeitsresultate für den harmonischen Fluss, falls die Äquatorabbildung nicht energie-minimierend ist; wir zeigen insbesondere, dass für geeignete Anfangsbedingungen die Zahl der (tatsächlich unterschiedlichen) selbstähnlichen Lösungen des harmonischen Flusses beliebig gross, ja sogar unendlich, sein kann. Diese Nichteindeutigkeitsresultate verallgemeinern eine frühere Arbeit von Angenent, Imanen und Velazquez. Unsere Resultate liefern Beispiele für Nichteindeutigkeit von Lösungen des harmonischen Wärmeflusses, welche die Monotonieformel von Struwe erfüllen, und erweitern damit die Resultate von Coron und Hong.

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Introduction

The main focus of this thesis lies on the issue of uniqueness of weak solutions of the harmonic map heat flow and the related extrinsic polyharmonic map heat flows.

Harmonic maps

Let (M, g) and (N, h) be two closed Riemannian manifolds. We assign to each map $u \in C^1(M, N)$ its Dirichlet energy

$$(0.1) \quad E(u) := \frac{1}{2} \int_M |\nabla u|^2 \, dvol_g$$

where $dvol_g$ denotes the volume element of the domain manifold (M, g) . The energy density $|\nabla u|^2 = |\nabla u|_{T^*M \otimes u^*TN}^2$ is given in local coordinates by

$$|\nabla u|^2 = g^{\alpha\beta} \cdot h_{ij} \circ u \cdot \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta},$$

where the usual summation convention is employed.

The critical points of E are called harmonic maps. In local coordinates they may be characterised by the equation

$$(0.2) \quad -\Delta u^k = (\Gamma(u)(\nabla u, \nabla u))^k := \Gamma_{ij}^k \circ u \cdot g^{\alpha\beta} \cdot \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta}, \quad k = 1, \dots, \dim(N).$$

Here Δ denotes the Laplace-Beltrami operator on (M, g) and Γ_{ij}^k are the Christoffel symbols of the target manifold (N, h) . For manifolds N that can be isometrically embedded in some Euclidean space, and thus by Nash's theorem in particular for sufficiently smooth targets, we may rewrite the harmonic map equation in the form

$$-\Delta u = A(u)(\nabla u, \nabla u),$$

where A denotes the second fundamental form of $N \hookrightarrow \mathbb{R}^N$.

The study of harmonic maps is a very rich field in geometric analysis with a wide range of applications. Special cases of harmonic maps include harmonic and holomorphic functions, geodesics, harmonic forms and minimal surfaces.

Of particular interest is the so called *homotopy problem* for harmonic maps: Can any smooth map $v : M \rightarrow N$ be deformed into a harmonic map that is homotopic to v ?

While the answer to this question is in general negative, positive results have been obtained for several different classes of target and domain manifolds; we refer to the surveys given in [15] and [16]. Since the direct methods of the calculus of

variations fail in general, various approaches have been developed to investigate the above problem, many of which lead to new interesting questions or even to whole new topics in geometric analysis.

Harmonic map heat flow

In their seminal work [18] Eells and Sampson introduced the negative gradient flow of the Dirichlet energy, the harmonic map heat flow, in order to study the homotopy problem. This flow is given by the equation

$$(0.3) \quad \partial_t u - \Delta u = A(u)(\nabla u, \nabla u).$$

It deforms maps in such a way that the energy is non-increasing in time. For target manifolds with non-positive sectional curvature Eells and Sampson proved that the harmonic map heat flow has a unique global smooth solution for any given initial data $u_0 \in C^\infty(M, N)$ and that this solution converges to a harmonic map homotopic to u_0 as $t \rightarrow \infty$ suitably. While short time existence and uniqueness of smooth solutions to equation (0.3) may be shown for arbitrary closed target manifolds and smooth initial data, the curvature bound on the target in [18] is essential for the global existence and behaviour of the solution. In fact, for general targets and domains of dimension $\dim(M) \geq 2$ solutions of (0.3) may blow up in finite or infinite time, see e.g. [13], [9] and [6].

In order to obtain global solutions of the harmonic map flow in general it is therefore necessary to relax the notion of solution. It is natural to call a function u a weak solution of the harmonic map heat flow on $M \times [0, T]$ if u is an element of the Sobolev space $H^1(M \times [0, T], N)$ which solves (0.3) in the sense of distributions.

In [50] Struwe proved the existence of a global weak solution of (0.3) to any initial data in $H^1(M, N)$ for manifolds of critical dimension $\dim(M) = 2$. This so-called Struwe solution has non-increasing energy, is smooth away from finitely many points in space-time where harmonic spheres *bubble off* and is unique in this class. The behaviour near singularities has been further analysed by various authors, see e.g. [44, 14, 58, 45, 56, 55, 3].

The existence of global weak solutions of the harmonic map flow in higher dimensions has been proven by Struwe [51] and Chen [8] in special cases, and, finally, by Chen-Struwe [11] in full generality. The solutions obtained in [11] have non-increasing energy and satisfy a monotonicity formula.

The question of regularity of weak solutions of the harmonic map heat flow was investigated in [21], [10] and [38].

The price to be paid for weakening the notion of solution is that we risk to lose the uniqueness property of the flow. Indeed, while smooth solutions are uniquely determined by their initial data, this cannot be expected of weak solutions. The first example of non-uniqueness for the harmonic map heat flow was given by Coron [12] in a setting in which equation (0.3) is supercritical. He gave examples of maps u_0 which are weakly harmonic but not stationary harmonic. The time-independent weak solution $u(x, t) = u_0(x)$ of (3.1) then violates the monotonicity formula in [11] and therefore must differ from the solution constructed in [8] or [11]. Based on the ideas of Coron, examples of non-uniqueness also satisfying the monotonicity inequality were constructed by Hong in [30].

The possibilities for non-uniqueness become more limited if the dimension is critical, i.e. if $\dim(M) = 2$. Indeed, the result of Freire [22] shows uniqueness for weak solutions of (0.3) with non-increasing energy. Furthermore, a uniqueness result for weak solutions of the harmonic map flow with small energy was proven by Rivière in [46].

On the other hand, the results of Topping [54] and of Bertsch, Dal Passo and Van der Hout [5] demonstrate that the energy of weak solutions may increase as time evolves. The mechanism which leads to this gain of energy in [54] and [5] is the so-called *reverse bubbling*. Contrary to the behaviour at a singularity of a Struwe solution, an infinitely concentrated harmonic sphere does not bubble off; instead it is inserted into the flow and then distributes its energy as time evolves. Such a reverse bubble causes an instant gain of energy of no less than the positive constant

$$(0.4) \quad \varepsilon^* := \min \left\{ \frac{1}{2} \int_{S^2} |\nabla u|^2 \, d\text{vol}_{S^2}, u : S^2 \rightarrow N \text{ non-constant, harmonic} \right\}.$$

The idea of reverse bubbling was first mentioned by Topping in [53]. In the same work Topping conjectured that uniqueness for the harmonic map flow in the critical dimension holds among all weak solutions whose energy does not instantaneously increase by ε^* or more at any time.

Polyharmonic map heat flow

Generalisations of the concept of harmonic maps involving higher order derivatives have likewise been studied, where the Dirichlet energy is replaced by either the intrinsic or extrinsic bi- or poly-energy.

The intrinsic bi-energy

$$(0.5) \quad E^{\text{int}}(u) := \frac{1}{2} \int_M |\Delta^{\text{int}} u|^2 \, d\text{vol}_g,$$

and its critical points and gradient flow have been studied in [34] and [39]. Here $\Delta^{\text{int}} u$ denotes the intrinsic laplacian, i.e. the trace of the second covariant derivative of u , often called the tension field of u . A different intrinsic bi-energy that takes into account the full second order covariant derivative was recently considered by Moser in [41] and [42]

If the target manifold $N \hookrightarrow \mathbb{R}^N$ is embedded into some fixed Euclidean space we can define an extrinsic bi-energy by

$$E^{(2)}(u) := \frac{1}{2} \int_M |\nabla^2 u|^2 \, d\text{vol}_g$$

where ∇ is the (extrinsic) derivative of maps from M into the Euclidean space \mathbb{R}^N in which the target is embedded.

Since the extrinsic bi-energy depends on the embedding $N \hookrightarrow \mathbb{R}^N$ chosen, the intrinsic bi-energy (0.5) is more natural from a geometric point of view. Conversely, from an analytic point of view, the extrinsic energy is preferable since we can bound the corresponding Sobolev norm, in this case the H^2 -norm, in terms of $E^{(2)}$ but not in terms of E^{int} . As we focus on the analytic aspects of the flow, and in particular on the properties of weak solutions, here we consider extrinsic generalisations of the harmonic map heat flow.

Given any $m \in \mathbb{N}$, we define the extrinsic polyenergy of order $m \in \mathbb{N}$ of maps $u : M \rightarrow N \hookrightarrow \mathbb{R}^N$ as

$$(0.6) \quad E^{(m)}(u) := \frac{1}{2} \int_M |\nabla^m u|^2 \, dvol_g.$$

Extrinsic polyharmonic maps are the critical points of (0.6). They satisfy an equation of the form

$$\Delta^m u = (-1)^m f[u] \in T_u^\perp N.$$

The non-linearity $f[u]$ depends on the derivatives of u up to order $2m - 1$ and is critical with respect to Sobolev's embedding theorem in dimension $\dim(M) = 2m$. It may be expressed in terms of the second fundamental form of N , its derivatives and the metric of M . Weakly (extrinsic) biharmonic maps have been studied in arbitrary dimensions, see [7, 1, 36, 52]. For general $m \in \mathbb{N}$ the regularity of weakly polyharmonic maps was studied in the critical dimension in [2], [26] and [37].

The m -th order extrinsic polyharmonic map heat flow is defined as the negative gradient flow of $E^{(m)}$,

$$(0.7) \quad \partial_t u + (-1)^m \Delta^m u = f[u] \in T_u^\perp N.$$

The existence of global weak solutions of this flow in its critical dimension has been shown by Lamm [33] and Wang [57] for $m = 2$ and for general $m \in \mathbb{N}$ by Gastel [25]. Once more the question of uniqueness of weak solutions is an issue. Up to now, the only known uniqueness result for weak solutions of these flows of higher order was given by Lamm and Rivière in [36] who extended the uniqueness result of Freire to the (extrinsic) biharmonic flow.

Selfsimilar solutions

For a large number of flows including the Ricci-flow, the mean curvature flow, wave maps and many more, the study of selfsimilar solutions is an important topic.

Selfsimilar weak solutions for the harmonic map flow were first considered by Angenent, Ilmanen and Velazquez in [31]. Since the harmonic map flow is not reversible in time, we need to distinguish between two basic classes of selfsimilar solutions: incoming selfsimilar solutions (or “shrinkers”), such as maps of the form

$$u(x, t) = v\left(\frac{x}{\sqrt{-t}}\right), \quad t < 0$$

and outgoing selfsimilar solutions (or “expanders”), like

$$u(x, t) = v\left(\frac{x}{\sqrt{t}}\right), \quad t > 0.$$

The existence and the properties of the first type of selfsimilar solutions are closely related to the formation of singularities of the first kind. An existence result for corotational incoming selfsimilar solutions of the harmonic map flow into spheres in dimension $3 \leq d \leq 6$ was presented in [31] and rigorously proved in [20]; see also [24] for a related result.

On the other hand, the properties of outgoing selfsimilar solutions are related to the question of uniqueness of weak solutions in supercritical dimensions. While similar problems have been studied for other flows, for example for the semi-linear heat equation $\partial_t u - \Delta u = u^p$, not much is known about outgoing selfsimilar solutions

of the harmonic map heat flow. The only known result in this context is due to Angenent, Imanen and Velazquez. This result, announced in [31], states the non-uniqueness of selfsimilar solutions in the corotational setting of maps from \mathbb{R}^d into the sphere S^d in dimensions $3 \leq d \leq 6$.

1. Main results

In chapter 1 we study the problem of uniqueness of weak solutions to the harmonic map heat flow in the critical dimension. Our first result, which we state in detail in theorem 1.1, is an improvement of the uniqueness result of Freire [22]. We prove the existence of a positive constant $\varepsilon_1(N) > 0$ depending only on the target manifold such that weak solutions of the harmonic map heat flow are unique if their energy does not instantaneously increase by $\varepsilon_1(N)$ or more at any time, i.e. if

$$\overline{\lim}_{s \searrow t} E(u(s)) < E(u(t)) + \varepsilon_1(N) \quad \text{for every } t.$$

The value of the constant $\varepsilon_1(N)$ we obtain in the proof of theorem 1.1 is in general smaller than the constant $\varepsilon^*(N)$ defined in (0.4). Nonetheless, theorem 1.1 represents a first step towards the proof of the conjecture of Topping, i.e. that the condition

$$(0.8) \quad \overline{\lim}_{s \searrow t} E(u(s)) < E(u(t)) + \varepsilon^* \quad \text{for every } t$$

is sufficient to guarantee uniqueness of weak solutions.

In the main result of the first chapter, theorem 1.2, we establish Topping's conjecture under the natural additional a priori assumption that the total variation of the energy function $t \mapsto E(u(t))$ is finite. Under this natural regularity assumption on the energy function we see in the proof of theorem 1.2 that a gain of energy, and thus non-uniqueness, can only be caused by reverse bubbling.

The second chapter deals with the study of the extrinsic polyharmonic map heat flows in their critical dimension $\dim(M) = 2m$. We focus once more on the question of uniqueness of weak solutions and prove that the uniqueness results of the first chapter can be generalised to these flows of arbitrary order.

In theorem 2.1 we show that a sufficient condition for uniqueness of weak solutions of (0.7) with energy $E^{(m)}(u(t)) \leq E_0 \in \mathbb{R}$ is given by

$$\overline{\lim}_{s \searrow t} E^{(m)}(u(s)) < E^{(m)}(u(t)) + \varepsilon(M, N, E_0) \quad \text{for every } t$$

for a suitable constant $\varepsilon(M, N, E_0) > 0$. In particular, weak solutions of (0.7) with non-increasing energy are unique; theorem 2.1 thus gives a generalisation of the results of Freire [22] and of Lamm and Rivière [36] for flows of arbitrary order.

As the main result of the second chapter, theorem 2.2, we prove the conjecture of Topping for the whole family of extrinsic polyharmonic map flows, again under the natural a priori assumption that the total variation of $t \mapsto E^{(m)}(u(t))$ is finite. In particular we find that in this class of weak solutions uniqueness can only be lost by reverse bubbling. Moreover, weak solutions in this class are smooth away from a discrete set of time-slices.

In the final chapter of this thesis we study (outgoing) selfsimilar weak solutions (0.9)

$$u(x, t) = v\left(\frac{x}{\sqrt{t}}\right)$$

of the harmonic map heat flow in supercritical dimensions.

We focus on target manifolds that are rotationally symmetric and on maps with equivariant symmetry. We prove that the properties of the so called *equator maps* $u^* : \mathbb{R}^d \rightarrow N$ are decisive for the issue of uniqueness of these weak solutions.

In theorem 3.3 we establish the unique solvability of (0.3) in the class of equivariant, selfsimilar maps for any admissible initial data under the main assumption that the equator maps minimise the Dirichlet energy in an appropriate class of functions.

Conversely, we prove non-uniqueness results for settings with equator maps that are not energy-minimising. In theorem 3.2 we show that multiple solutions of (0.2) exist for initial data given by an equator map that is not energy-minimising. This result applies not only to compact target manifolds but also to a large class of non-compact, rotationally symmetric manifolds.

In theorem 3.5 we then establish a stronger non-uniqueness result for compact rotationally symmetric manifolds under the main assumption that the equator map is not even locally energy-minimising. We show the existence of a continuous family of initial data for which the number of (genuinely different) selfsimilar solutions to the harmonic map heat flow is greater than any given number $n \in \mathbb{N}$. As a consequence we obtain that any equator map that is not even locally energy-minimising has infinitely many different evolutions of the form (0.9) under the harmonic map heat flow (0.3). This completes the analysis of Angenent, Ilmanen and Velazquez of [31].

In addition, theorem 3.2 and theorem 3.5 yield further examples of non-uniqueness of weak solutions to the harmonic map heat flow respecting the monotonicity formula of Struwe.

The results of the first chapter have been published in [48] while those of the second chapter will appear in [49]. The third chapter is partly based on joint work with Pierre Germain which will appear in [28].

CHAPTER 1

Uniqueness for the harmonic map flow in the critical dimension

1. Introduction

We consider the harmonic map heat flow for maps from a closed two-dimensional manifold M into a closed manifold $N \hookrightarrow \mathbb{R}^N$. A map $u \in H^1(M \times [0, T], N)$ is said to be a weak solution of the harmonic map heat flow, if

$$(1.1) \quad \partial_t u - \Delta u = A(u)(\nabla u, \nabla u)$$

is solved in the sense of distributions, where A is the second fundamental form of $N \hookrightarrow \mathbb{R}^N$.

We define the energy of u at time t as $E(u(t)) := \frac{1}{2} \int_M |\nabla u(t)|^2 dx$, where $u(t) = u(\cdot, t)$ is taken in the trace sense.

It was shown by Struwe [50] that for any initial data $u_0 \in H^1(M, N)$ there exists a global weak solution $v \in H^1(M \times [0, \infty), N)$ of (1.1) which is smooth away from finitely many points (x_i, t_i) in $M \times [0, \infty)$ and has non-increasing energy. This so called Struwe solution is unique in the class of functions with non-increasing energy such that $\nabla u \in L^4_{loc}(M \times ([0, \infty) \setminus S))$ where S is a discrete set of times.

For any *singular time* $t_i \in S$ harmonic spheres *bubble off* which causes a loss of energy of $\lim_{t \nearrow t_i} E(u(t)) \geq E(u(t_i)) + K \cdot \varepsilon^*$, where K is the number of singular points at time t_i and

$$(1.2) \quad \varepsilon^* = \min \left\{ \frac{1}{2} \int_{S^2} |\nabla u|^2 dvol_{S^2}, u : S^2 \rightarrow N \text{ non-constant, harmonic} \right\} > 0.$$

It was shown by Freire [22], [23] that any weak solution of (1.1) with non-increasing energy for initial data $u_0 \in H^1(M, N)$ is identical to the corresponding Struwe solution.

In this chapter we show that the condition $E(u(t)) \geq E(u(s))$ for all $t \leq s$ may be relaxed in the sense that we can allow the solutions to instantly gain a small amount of energy without losing the uniqueness property. In fact, we obtain

THEOREM 1.1. *Let M be a closed Riemannian surface and let N be a closed Riemannian manifold, isometrically embedded in \mathbb{R}^n .*

Let $u \in H^1(M \times [0, T], N)$ be any weak solution of equation (1.1) for initial data $u_0 \in H^1(M, N)$, such that the energy function $t \mapsto E(u(t))$ fulfils

$$\overline{\lim}_{s \searrow t} E(u(s)) < E(u(t)) + \varepsilon_1 \quad \text{for every } t \in [0, T]$$

for a constant $\varepsilon_1 = \varepsilon_1(N) > 0$ depending only on the target manifold N .

Then $u \equiv v$ on the whole domain $M \times [0, T]$, where v is the Struwe solution for initial data u_0 .

The question if non-uniqueness of weak solutions can occur at all in two dimensions was answered by Topping [54] and Bertsch et al. [5]. They have constructed examples of non-uniqueness for $M = B_1(0) \subset \mathbb{R}^2$ and $N = S^2$ which are based on the idea of attaching a *reverse bubble* at a certain point in space-time. The bubble corresponds to a harmonic map from the whole $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ to N and the energy function instantaneously increases by the energy of this map. In these examples the occurring gain of energy is exactly ε^* . Topping conjectured that the condition

$$\overline{\lim}_{s \searrow t} E(u(s)) < E(u(t)) + \varepsilon^*$$

is sufficient to prove uniqueness of weak solutions. In light of [54] and [5] this would provide a sharp criterion for uniqueness in the critical dimension.

Here we establish Topping's conjecture under a natural a priori assumption on the regularity of the energy function $t \mapsto E(u(t))$.

THEOREM 1.2. *Let M be a closed Riemannian surface and let N be a closed Riemannian manifold, isometrically embedded in \mathbb{R}^n .*

Let $u \in H^1(M \times [0, T], N)$ be any weak solution of (1.1) such that the energy-function $t \mapsto E(u(t))$ has finite total variation and

$$\overline{\lim}_{s \searrow t} E(u(s)) < E(u(t)) + \varepsilon^* \quad \text{for every } t \in [0, T].$$

Then $u \equiv v$ on the whole domain $M \times [0, T]$, where v is the Struwe solution for initial data $u(0) \in H^1(M, N)$.

We briefly sketch the main steps of the proofs.

Note that if $u \in H^1(M \times [0, T], N)$ solves (1.1) in the weak sense, then for almost every $t \in (0, T]$ the trace $u(t)$ weakly solves

$$(1.3) \quad -\Delta u(t) = A(u(t))(\nabla u(t), \nabla u(t)) + k \text{ on } M,$$

where $k = -\partial_t u(t) \in L^2(M)$.

Applying a regularity result due to Moser [38], we prove that in the two-dimensional case the H^2 -norm of a weak solution w of (1.3) may be estimated locally by its energy and the L^2 -norm of the inhomogeneity k , where k is an arbitrary function in $L^2(M)$. This estimate crucially depends on how small the *concentration radius* r has to be chosen to assure that the energy in a ball of radius r is at most a given quantum.

We apply these results to weak solutions of (1.1). The main step is then to establish that we can bound the concentration radius from below on small time intervals for weak solutions of (1.1) satisfying the assumptions of theorem 1.1 respectively of theorem 1.2.

To begin with we give an alternative proof of the regularity result in [38]. Our proof is based on the paper of Rivière and Struwe [47] about the regularity of harmonic maps which uses the fact that the second fundamental form $A(u)(\nabla u, \nabla u)$ may be written as an antisymmetric 1-form applied to the gradient of u .

Notations and remarks: The Dirichlet energy is conformally invariant in dimension two. Working in conformal charts, we may thus assume for *local* arguments that the domain is a subset of \mathbb{R}^2 .

We use the short hand notation $B_r = B_r(0)$ and $B = B_1$ for balls in \mathbb{R}^m . We define the energy on a subset $M' \subset M$ by

$$E(f, M') := \frac{1}{2} \int_{M'} |\nabla f|^2 \, d\text{vol}_g.$$

We write $u(t) := u(\cdot, t)$ for the trace of u on the time slice $M \times \{t\}$.

2. Regularity of almost harmonic maps

We consider maps that are almost harmonic maps in the sense that they solve the harmonic map equation up to an error term. More precisely we study weak solutions of the equation

$$(1.4) \quad -\Delta w = A(w)(\nabla w, \nabla w) + k$$

on an open set $U \subset \mathbb{R}^m$, $m \in \mathbb{N}$, where A is the second fundamental form of $N \hookrightarrow \mathbb{R}^n$ and k is a function in $L^s(U, \mathbb{R}^n)$, $s \geq 1$. Equations of this kind were first considered by Moser [38] who used them to prove partial regularity for the harmonic map flow in small dimensions. He proved Hölder continuity for any solution $w \in H^1(U)$ of (1.4) satisfying an appropriate Morrey estimate if $k \in L^s(U, \mathbb{R}^n)$ for some $s > \frac{m}{2}$. The proof in [38] is based on properties of a moving tangent frame field.

Following [47], we give an alternative proof of this result by rewriting the equation (1.4) in the form

$$(1.5) \quad -\Delta w = \Omega \cdot \nabla w + k,$$

where Ω is the antisymmetric 1-form

$$\Omega^{ij} = (\sigma_i^j \nabla \sigma_l^i - \sigma_l^j \nabla \sigma_i^l), \quad i, j = 1, \dots, n \text{ with } \sigma_l = \nu_l \circ w$$

for an orthonormal frame field $\nu_1, \dots, \nu_{n-k} : N \rightarrow S^{n-1}$ of $T^\perp N$.

We then obtain the following generalisation of Moser's Theorem 1 in [38] analogously to Theorem 1.1 of [47]:

PROPOSITION 1.3. *For every $m \in \mathbb{N}$ there exists a number $\varepsilon_0 = \varepsilon_0(m) > 0$ such that for every ball $B_R(x_0) \subset \mathbb{R}^m$, any antisymmetric 1-form $\Omega \in L^2(B_R(x_0), so(n) \otimes \Lambda^1 \mathbb{R}^m)$ and for every function $k \in L^s(B_R(x_0), \mathbb{R}^n)$ with*

$$s > \frac{m}{2}$$

the following statement holds.

Every weak solution $w \in H^1(B_R(x_0), N)$ of equation (1.5) on $B_R(x_0)$, satisfying the Morrey growth assumption

$$(1.6) \quad \sup_{\substack{x \in B_R(x_0) \\ r > 0}} \left(r^{2-m} \int_{B_r(x) \cap B_R(x_0)} |\nabla w|^2 + |\Omega|^2 \, dx \right) < \varepsilon_0^2$$

is Hölder continuous in $B_{R/2}(x_0)$. More precisely $w \in C^\alpha(B_{R/2}(x_0))$ for every $0 < \alpha < 1$ with $\alpha \leq 2 - \frac{m}{s}$ with estimate

$$R^\alpha [w]_{C^\alpha(B_{R/2}(x_0))} \leq C \cdot (\varepsilon_0 + R^{2-\frac{m}{s}} \|k\|_{L^s(B_R(x_0))}).$$

PROOF. Using a scaling argument, it suffices to consider the case $B = B_1(0)$. We may assume that $s < m$ and set $\alpha = 2 - \frac{m}{s} \in (0, 1)$.

By a translation of the function w and the manifold N we can assume without loss of generality that

$$(1.7) \quad \int_B w \, dx = 0.$$

We choose a number $1 < p < m/(m-1)$ with $p \cdot \alpha < 1$ and a cut-off function $\varphi \in C_c^\infty(B)$ with $\varphi \equiv 1$ on $B_{3/4}$ and set $\tilde{w} = \varphi \cdot w$.

We show

$$\sup_{x \in B_{1/2}} \sup_{r > 0} \left(r^{-(m-p+\alpha p)} \int_{B_r(x)} |\nabla \tilde{w}|^p \, dx \right) \leq C \cdot (\varepsilon_0 + \|k\|_{L^s})^p.$$

This claim implies the proposition by the use of Morrey's lemma.

For any given $\gamma > 0$, equation (1.7), equation (1.6) and the Poincaré inequality imply this estimate for any $r > \gamma$ and $x \in B_{1/2}$. In particular we can assume $r \leq 1/4$ and thus $w = \tilde{w}$.

The following calculations are completely analogous to those in [47] except that we get an additional term involving k but have no error term.

We use the gauge transformation introduced in Lemma 3.1 in [47] to rewrite equation (1.5). If $\varepsilon_0 = \varepsilon_0(m)$ is small enough there exists $P \in H^1(B, SO(n))$ and $\xi \in H^1(B, so(n) \otimes \Lambda^{m-2}\mathbb{R}^m)$ such that

$$P^{-1}dP + P^{-1}\Omega P = \star d\xi$$

with

$$\sup_{x \in B, r > 0} \left(r^{2-m} \int_{B_r(x) \cap B} |dP|^2 + |d\xi|^2 \, dx \right) < C \cdot \varepsilon_0^2.$$

Equation (1.5) is then equivalent to

$$(1.8) \quad \begin{aligned} -\operatorname{div}(P^{-1}\nabla w) &= -d(P^{-1})\nabla w - P^{-1}\Delta w \\ &= (P^{-1}dP + P^{-1}\Omega P)P^{-1}\nabla w + P^{-1}k \\ &= \star d\xi \cdot P^{-1}dw + P^{-1}k. \end{aligned}$$

We fix a ball $B_R(x_0)$ with $R \leq 1/4$ and $x_0 \in B_{1/2}$ and use the Hodge decomposition to write

$$P^{-1}dw = df + \star dg + h,$$

for a function $f \in H_0^1(B_R(x_0))$, a co-closed $(m-2)$ -form g of class $H^1(B_R(x_0))$ whose restriction to the boundary $\partial B_R(x_0)$ vanishes, and a harmonic 1-form $h \in L^2(B_R(x_0))$.

Using (1.8) for f and g we obtain the equations

$$(1.9) \quad \begin{aligned} -\Delta f &= -\operatorname{div}(P^{-1}\nabla w) = \star d\xi \cdot P^{-1}dw + P^{-1}k, \\ -\Delta g &= \star d(P^{-1}dw) = \star(dP^{-1} \wedge dw). \end{aligned}$$

We fix now $1 < p < \frac{m}{m-1}$ such that $\alpha \cdot p < 1$ and let $q > m$ be the conjugate exponent of p .

Since $f = 0$ on $\partial B_R(x_0)$ we can use duality between L^p and L^q to bound

$$\|df\|_{L^p} \leq C \cdot \sup_{\{\varphi \in W_0^{1,q}(B_R(x_0)), \|\varphi\|_{W^{1,q}} \leq 1\}} \int df \cdot d\varphi dx.$$

Here and in the following all the norms are computed with respect to $B_R(x_0)$ unless specified otherwise. Let $\varphi \in W_0^{1,q}(B_R(x_0))$ with $\|\varphi\|_{W^{1,q}} \leq 1$ be fixed. As $q > m$, we have $\varphi \in L^\infty$ with

$$\|\varphi\|_{L^\infty} \leq CR^{1-m/q} \|\varphi\|_{W^{1,q}} \leq CR^{1-m/q}.$$

Integrating by parts, we obtain

$$\begin{aligned} \int_{B_R(x_0)} df \cdot d\varphi dx &= - \int_{B_R(x_0)} \Delta f \cdot \varphi dx \\ &= \int_{B_R(x_0)} d\xi \wedge P^{-1}\varphi dw + \int_{B_R(x_0)} P^{-1}k \cdot \varphi dx = I + II. \end{aligned}$$

Exactly as in [47] we can estimate

$$|I| \leq R^{m/p-1} \varepsilon_0 [w]_{BMO(B_R(x_0))}.$$

Using $\frac{m}{s} = 2 - \alpha$, the second term may be estimated as

$$\begin{aligned} |II| &= \left| \int_{B_R(x_0)} P^{-1}k \cdot \varphi dx \right| \leq \|\varphi\|_{L^\infty} \cdot \|k\|_{L^1} \\ &\leq CR^{1-m/q} \cdot \|k\|_{L^s} \cdot (R^m)^{1-1/s} = C \cdot \|k\|_{L^s} \cdot R^{m/p-1+\alpha}. \end{aligned}$$

The equation for g in (1.9) is identical to the one for g in [47] so we have the same estimate

$$\|dg\|_{L^p} \leq C \cdot R^{m/p-1} \cdot \varepsilon_0 \cdot [w]_{BMO(B_R(x_0))}.$$

Using the Campanato estimates for harmonic functions to estimate h we can thus conclude that for $r < R < \frac{1}{4}$

$$\begin{aligned} \int_{B_r(x_0)} |dw|^p dx &\leq C \int_{B_r(x_0)} |h|^p dx + C \int_{B_r(x_0)} |df|^p + |dg|^p dx \\ &\leq C \cdot \left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |h|^p dx + C \int_{B_r(x_0)} |df|^p + |dg|^p dx \\ &\leq C \cdot \left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |dw|^p dx + C \int_{B_R(x_0)} |df|^p + |dg|^p dx \\ &\leq C \cdot \left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |dw|^p dx + C \cdot R^{m-p+p\alpha} \cdot \|k\|_{L^s}^p \\ (1.10) \quad &+ C \cdot R^{m-p} \varepsilon_0^p \cdot [w]_{BMO(B_R(x_0))}^p. \end{aligned}$$

Following [47] we define

$$\Phi(x_0, r) := r^{p-m} \int_{B_r(x_0)} |dw|^p dx$$

and

$$\psi(R) := \sup_{\substack{x_0 \in B_{1/2} \\ 0 < r < R}} \Phi(x_0, r).$$

The Poincaré inequality implies

$$\sup_{x_0 \in B_{1/2}} [w]_{BMO(B_R(x_0))}^p \leq C \cdot \psi(R).$$

According to (1.10) we thus find for every $x_0 \in B_{1/2}$ and $0 < r < R < \frac{1}{4}$

$$\Phi(x_0, r) \leq C \left(\frac{r}{R}\right)^p \Phi(x_0, R) + C \left(\frac{r}{R}\right)^{p-m} \cdot \varepsilon_0^p \cdot \psi(R) + C \left(\frac{r}{R}\right)^{p-m} R^{p-\alpha} \cdot \|k\|_{L^s}^p.$$

For a fixed ratio $0 < \gamma = r/R < 1$, this gives

$$\Phi(x_0, \gamma R) \leq C \gamma^p \Phi(x_0, R) + C \gamma^{p-m} \varepsilon_0^p \psi(R) + C \gamma^{p-m} R^{p-\alpha} \|k\|_{L^s}^p.$$

Since ψ is non-decreasing, we find for any $R_0 < \frac{1}{4}$ and $0 < R < R_0$

$$\Phi(x_0, \gamma R) \leq C_1 \gamma^p (1 + \varepsilon_0^p \gamma^{-m}) \psi(R_0) + C \gamma^{p-m} \cdot R_0^{p-\alpha} \|k\|_{L^s}^p$$

for universal constants $C_1, C \in \mathbb{R}$. Passing to the supremum with respect to x_0 and $R < R_0$ we find

$$\psi(\gamma R_0) \leq C_1 \gamma^p (1 + \varepsilon_0^p \gamma^{-m}) \psi(R_0) + C \gamma^{p-m} \cdot R_0^{p-\alpha} \|k\|_{L^s(B_1)}^p.$$

Thus if we choose $\gamma > 0$ such that

$$C_1 \gamma^{(p-1)/2} \leq 1/2$$

and $\varepsilon_0 = \varepsilon_0(m)$ with $\varepsilon_0^p \leq \gamma^m$ we obtain (writing again R instead of R_0)

$$(1.11) \quad \psi(\gamma R) \leq \gamma^{(p+1)/2} \psi(R) + C \cdot R^{p-\alpha} \|k\|_{L^s(B_1)}^p.$$

This estimate allows us now to conclude that $\psi(r) \leq C \cdot r^{p\alpha}$. Indeed, given $r \in (0, \gamma)$ we choose $l \in \mathbb{N}$ such that $\gamma^{l+1} < r \leq \gamma^l$. Iterating (1.11) we get the estimate

$$\begin{aligned} \psi(r) &\leq \psi(\gamma^l) \leq \gamma^{(p+1)/2} \psi(\gamma^{l-1}) + C \cdot (\gamma^l)^{p\alpha} \|k\|_{L^s(B_1)}^p \\ &\leq (\gamma^{(p+1)/2})^{l-1} \cdot \psi(\gamma) + C \cdot \|k\|_{L^s(B_1)}^p \sum_{j=1}^l (\gamma^j)^{p\alpha} (\gamma^{(p+1)/2})^{l-j} \\ &\leq C \cdot r \cdot \psi(\gamma) + C \cdot r^{p\alpha} \cdot \|k\|_{L^s(B_1)}^p \leq C \cdot (\varepsilon_0^p + \|k\|_{L^s(B_1)}^p) r^{p\alpha}, \end{aligned}$$

as $\psi(\gamma) < C \cdot \varepsilon_0^p$ and $p\alpha < 1$. □

REMARK 1.4. *Let $w \in H^1(U)$ be a weak solution of (1.4) for $k \in L^2(U)$ and assume that w is continuous. Then $w \in H_{loc}^2(U) \cap W_{loc}^{1,4}(U)$.*

For the proof of this statement we refer to section 4 of the next chapter.

We will now show that for $m = 2$ the H^2 -norm of a solution $w \in H^1$ of (1.4) may be estimated locally by quantities depending only on the concentration radius of w and the L^2 -norm of k .

We use the following Sobolev interpolation inequality which is originally due to Gagliardo-Nirenberg and Ladyzhenskaya.

PROPOSITION 1.5. *For any function $g \in H_{loc}^1(\mathbb{R}^2)$, any $R > 0$ and any function $\varphi \in C_c^\infty(B_R)$ with $0 \leq \varphi \leq 1$ and $|\nabla\varphi| \leq \frac{c_1}{R}$ there holds*

$$\int_{B_R} |g|^4 \varphi^2 dx \leq c_0 \left(\int_{B_R} |g|^2 dx \right) \cdot \left(\int_{B_R} |\nabla g|^2 \varphi^2 dx + c_1^2 R^{-2} \int_{B_R} |g|^2 dx \right)$$

for a universal constant c_0 .

PROOF OF PROPOSITION 1.5. This interpolation inequality follows from the fact that $W_0^{1,1}$ embeds continuously into L^2 for two-dimensional domains. Indeed, let g and φ be as in proposition 1.5. Then

$$\int (\varphi |g|^2)^2 dx \leq C \left(\int |\nabla(\varphi |g|^2)| dx \right)^2 \leq C \left(\int \varphi |g| \cdot |\nabla g| dx + \frac{c_1}{R} \int |g|^2 dx \right)^2$$

which implies the claim by Hölder's inequality. \square

We apply this inequality to show the following H^2 -estimate.

PROPOSITION 1.6. *Let $U \subset \mathbb{R}^2$ be an open set, let $k \in L^2(U, \mathbb{R}^n)$ and let $w \in H^2(U, \mathbb{R}^2)$ be a solution of the equation*

$$(1.12) \quad -\Delta w = B(w)(\nabla w, \nabla w) + k \quad \text{on } U$$

for a bounded bilinear form B .

Then there exist constants $\varepsilon_2 = \varepsilon_2(B) > 0$ and $C = C(B)$ with the following property.

If the energy of w on $B_{2r}(x_0) \subset U$ is small in the sense that $E(w, B_{2r}(x_0)) \leq \varepsilon_2$, then the estimate

$$\int_{B_r(x_0)} |\nabla^2 w|^2 dx + \int_{B_r(x_0)} |\nabla w|^4 dx \leq C \cdot \left[\frac{E(w, B_{2r}(x_0))}{r^2} + \|k\|_{L^2(B_{2r}(x_0))}^2 \right]$$

holds true.

PROOF. Let $\varphi \in C_0^\infty(B_{2r}(x_0))$ be a cut-off function with $\varphi \equiv 1$ on $B_r(x_0)$ and $|\nabla\varphi|^2 + |\nabla^2\varphi| \leq \frac{C}{r^2}$.

Since we have assumed that $w \in H^2$ we may multiply (1.12) with φ and take the square to obtain

$$(1.13) \quad \begin{aligned} \int \varphi^2 |\Delta w|^2 dx &\leq C \cdot \int \varphi^2 |B(w)(\nabla w, \nabla w)|^2 dx + C \cdot \int \varphi^2 |k|^2 dx \\ &\leq C \cdot \int \varphi^2 |\nabla w|^4 dx + C \cdot \|k\|_{L^2(B_{2r}(x_0))}^2, \end{aligned}$$

where $C = C(B)$ depends on the bilinear form B . Applying proposition 1.5 to the first term on the right hand side we find

$$(1.14) \quad \begin{aligned} \int \varphi^2 |\nabla w|^4 dx &\leq C \int_{B_{2r}(x_0)} |\nabla w|^2 dx \cdot \left(\int \varphi^2 |\nabla^2 w|^2 dx + \frac{1}{r^2} \int_{B_{2r}(x_0)} |\nabla w|^2 dx \right) \\ &\leq C \cdot E(w, B_{2r}(x_0)) \left(\int \varphi^2 |\nabla^2 w|^2 dx + \frac{E(w, B_{2r}(x_0))}{r^2} \right). \end{aligned}$$

Integrating by parts twice we can easily see that

$$(1.15) \quad \int \varphi^2 |\nabla^2 w|^2 dx \leq 2 \int \varphi^2 |\Delta w|^2 dx + \frac{C}{r^2} E(w, B_{2r}(x_0)).$$

Inserting (1.14) and (1.15) into (1.13) and using $E(w, B_{2r}(x_0)) < \varepsilon_2$ we get

$$\begin{aligned} \int \varphi^2 |\Delta w|^2 dx &\leq C \cdot \int \varphi^2 |\nabla w|^4 dx + C \cdot \|k\|_{L^2(B_{2r}(x_0))}^2 \\ &\leq C\varepsilon_2 \left(\int \varphi^2 |\Delta w|^2 dx + \frac{E(w, B_{2r}(x_0))}{r^2} \right) + C \cdot \|k\|_{L^2(B_{2r}(x_0))}^2. \end{aligned}$$

Choosing $\varepsilon_2 = \varepsilon_2(B)$ small enough we can absorb the first term on the right hand side into the left hand side and the proposition follows. \square

Combining the results of proposition 1.3 and proposition 1.6 we can prove a global version of the above estimate.

COROLLARY 1.7. *Let M be a closed two-dimensional manifold and let $w \in H^1(M, N)$ be a weak solution of equation (1.4) for a function $k \in L^2(M)$. Let $r > 0$ be such that*

$$\sup_{x \in M} E(w, B_r(x)) \leq \varepsilon_2,$$

where $\varepsilon_2 = \varepsilon_2(A) > 0$ is the constant of proposition 1.6 for the second fundamental form A of $N \hookrightarrow \mathbb{R}^N$.

Then $w \in H^2(M)$ and

$$\int_M |\nabla^2 w|^2 dvol_g + \int_M |\nabla w|^4 dvol_g \leq C \cdot \left(\frac{E(w)}{r^2} + \|k\|_{L^2(M)}^2 \right).$$

PROOF. Covering M with sufficiently small balls we see from proposition 1.3 that w is continuous and thus, by remark 1.4, $w \in H^2(M)$. We now cover M by balls $B_r(x_i)$ of the radius $r > 0$ specified in corollary 1.7. Since M is compact we may choose this cover in such a way that no more than K balls $B_{2r}(x_i)$ overlap at any given point of M , where $K = K(M) < \infty$ is a constant independent of r (see [50], lemma 3.3). Applying proposition 1.6 to this cover of balls we obtain the claim of corollary 1.7. \square

3. Proof of the main results

We use the following statement about uniqueness of weak solutions to (1.1) with additional regularity which was proven in [50].

LEMMA 1.8. *Let $u, v \in L^2([0, T], H^2(M, N)) \cap H^1(M \times [0, T], N)$ be weak solutions of (1.1) to the same initial data $u_0 \in H^1(M, N)$. Then u and v are identical.*

The key idea for the proofs of the main results is to show that the assumptions of theorem 1.1 respectively 1.2 allow us to cover M by balls $B_r(x_i)$ such that $E(u(t), B_{2r}(x_i)) < \varepsilon_2$ uniformly on a small time interval. We can then conclude uniqueness from the statements of proposition 1.6 and lemma 1.8.

We use the following lemma about the behaviour of the energy on subsets of M .

LEMMA 1.9. *Let $M' \subset M$ be open and let $u \in H^1(M \times [0, T])$ be a weak solution of (1.1).*

Then for every $t_0 \in [0, T]$ and every sequence $t_m \rightarrow t_0$ we have

$$\underline{\lim}_{m \rightarrow \infty} E(u(t_m), M') \geq E(u(t_0), M').$$

PROOF. Choosing a subsequence we may assume that $E(u(t_m), M')$ converges. Since N is compact there exists a subsequence (denoted again by t_m) and a function $u_\infty \in H^1(M')$ such that

$$\begin{aligned} u(t_m) &\rightarrow u_\infty && \text{strongly in } L^2(M') \\ \nabla u(t_m) &\rightharpoonup \nabla u_\infty && \text{weakly in } L^2(M') \end{aligned}$$

as $m \rightarrow \infty$. The weak convergence implies

$$\|\nabla u_\infty\|_{L^2(M')}^2 \leq \underline{\lim}_{m \rightarrow \infty} \|\nabla u(t_m)\|_{L^2(M')}^2.$$

However, by the trace theorem we know that $u(t_m) \rightarrow u(t_0)$ in $L^2(M)$ for every sequence $t_m \rightarrow t_0$ which implies $u_\infty = u(t_0)|_{M'}$ and thus the claim of lemma 1.9. \square

PROOF OF THEOREM 1.1. Let $\varepsilon_2 = \varepsilon_2(A) > 0$ be the constant of proposition 1.6 for the second fundamental form A of $N \hookrightarrow \mathbb{R}^N$. We set $\varepsilon_1(N) := \varepsilon_2$ and show

LEMMA 1.10. *Let $u \in H^1(M \times [0, T])$ be as in theorem 1.1 for $\varepsilon_1 > 0$ as above.*

Then for every time $t_0 \in [0, T]$ there exist finitely many balls $B_r(x_i)$, $i = 1, \dots, k$ covering M and a number $\delta = \delta(t_0) > 0$ such that

$$E(u(t), B_{2r}(x_i)) \leq \varepsilon_1 \text{ for all } t \in [t_0, t_0 + \delta], i = 1, \dots, k.$$

PROOF. Let $t_0 \in [0, T]$ be fixed and let $\rho > 0$ be such that

$$(1.16) \quad \overline{\lim}_{t \searrow t_0} E(u(t)) \leq E(u(t_0)) + \varepsilon_1 - \rho.$$

We choose a radius $r > 0$ and points x_i , $i = 1, \dots, k$, such that the balls $B_r(x_i)$ cover M and

$$E(u(t_0), B_{2r}(x_i)) \leq \rho/2.$$

We claim that the lemma holds true for this choice of balls if $\delta > 0$ is chosen small enough. Indeed, suppose there is a sequence $t_m \searrow t_0$ such that for every m there exists a number i with

$$E(u(t_m), B_{2r}(x_i)) > \varepsilon_1.$$

As the number of balls is finite we may assume (after choosing a subsequence) that the index i is always the same, say $i = 1$.

We set $M' = M \setminus \overline{B_{2r}(x_1)}$. Then lemma 1.9 and estimate (1.16) imply

$$\begin{aligned} \varepsilon_1 &\leq \overline{\lim}_{m \rightarrow \infty} E(u(t_m), B_{2r}(x_1)) = \overline{\lim}_{m \rightarrow \infty} E(u(t_m), M \setminus M') \\ &\leq \overline{\lim}_{m \rightarrow \infty} E(u(t_m), M) - \underline{\lim}_{m \rightarrow \infty} E(u(t_m), M') \\ &\leq E(u(t_0)) + \varepsilon_1 - \rho - E(u(t_0), M') \leq \varepsilon_1 - \rho/2 \end{aligned}$$

which is a contradiction since $\rho > 0$. \square

Consider now a weak solution $u \in H^1(M \times [0, T])$ of (1.1) satisfying the conditions of theorem 1.1 for ε_1 as above. Given any $t_0 \in [0, T]$ let $\delta > 0$, $r > 0$ and x_i , $i = 1, \dots, k$, be as in lemma 1.10. Since the trace $u(t)$ solves an equation of the form (1.4) for almost every t we may apply proposition 1.6 to obtain

$$\int_M |\nabla u(t)|^4 \, d\text{vol}_g + \int_M |\nabla^2 u(t)|^2 \, d\text{vol}_g \leq C \cdot k \left(\frac{E(u(t))}{r^2} + \|\partial_t u(t)\|_{L^2(M)}^2 \right)$$

for almost every $t \in [t_0, t_0 + \delta]$.

This implies that $\nabla^2 u \in L^2(M \times [t_0, t_0 + \delta(t_0)])$ as the energy $E(u(t))$ is bounded uniformly by $E(u(t_0)) + 2\varepsilon_1$ for $\delta(t_0)$ small enough.

By lemma 1.8 we therefore find that $u = v_{t_0}$ on $[t_0, t_0 + \delta(t_0)]$ where v_{t_0} is the Struwe solution with initial data $v(t_0) = u(t_0)$.

The above argument shows that the non-empty set

$$K := \{t \in [0, T] : u = v \text{ on } M \times [0, t]\}$$

is relatively open in $[0, T]$.

On the other hand if $u = v$ on an interval $[0, t_0)$ we have by the trace theorem

$$u(t_0) = \lim_{t \nearrow t_0} u(t) = \lim_{t \nearrow t_0} v(t) = v(t_0)$$

where the limits are taken in $L^2(M)$.

This shows that K is also closed implying $K = [0, T]$ and thus

$$u = v \text{ on } M \times [0, T]$$

as claimed in theorem 1.1. □

REMARK 1.11. *If the constant $\varepsilon_1(N)$ of theorem 1.1 is no less than the number $\varepsilon^*(N)$ defined by (1.2) then theorem 1.2 is a direct consequence of theorem 1.1. However the constant ε_1 we obtain in the proof of theorem 1.1 is in general smaller than ε^* . Carefully tracking the constants in the proof of proposition 1.6 we obtain for example only the lower bound*

$$\varepsilon_1(S^2) \geq \frac{\pi}{2} = \frac{\varepsilon^*(S^2)}{8}.$$

PROOF OF THEOREM 1.2. Let $\varepsilon_1 > 0$ be as above and let ε^* be defined by (1.2). We assume that $\varepsilon_1 < \varepsilon^*$ and set

$$S := \{t \in [0, T] : \text{osc}_{[t-\delta, t+\delta] \cap [0, T]} E(u(\cdot)) \geq \varepsilon_1\}$$

where $\text{osc}_I(f) := \sup_I f - \inf_I f$ denotes the oscillation of a function over an interval I . Since the total variation of the energy functional is finite the set S contains only finitely many points, $|S| \leq \frac{TV(E(u))}{\varepsilon_1}$.

On every closed interval $[t_0, t_1] \subset [0, T] \setminus S$ the assumptions of theorem 1.1 are satisfied and thus $u = v_{t_0}$ on $[t_0, t_1]$ where v_{t_0} is the Struwe solution with initial data $v_{t_0}(t_0) = u(t_0)$. Since any possible singularity of v_{t_0} on $[t_0, t_1]$ would cause an instant loss of energy of no less than $\varepsilon^* > \varepsilon_1$, the function v_{t_0} is smooth on $[t_0, t_1]$. Therefore u is smooth away from the set of singular times S .

We use this information about the regularity of u to discuss the behaviour of u at times that are near to S . Let $s_1 = \min S$. If $s_1 > 0$ we have $u = v$ on $[0, s_1]$. We may thus assume $s_1 = 0$ and we finish the proof of theorem 1.2 by showing

LEMMA 1.12. *Let $u \in H^1(M \times [0, T])$ be a weak solution of (1.1) with initial data $u_0 \in H^1(M, N)$ with*

$$(1.17) \quad \overline{\lim}_{t \searrow 0} E(u(t)) < E(u_0) + \varepsilon^*,$$

and assume that u is smooth on $M \times (0, T_1]$ for a number $0 < T_1 < T$.

Then there exists a number $\delta > 0$ and a radius $r > 0$ such that

$$E(u(t), B_r(x_0)) \leq \varepsilon_1 \quad \text{for all } x_0 \in M, \quad t \in [0, \delta]$$

for ε_1 as in corollary 1.7.

In particular $\nabla^2 u \in L^2(M \times [0, \delta])$.

PROOF. We argue by contradiction and show that if the claim were false we would have a reverse bubble at $t = 0$ carrying an energy quantum of at least ε^* which will lead to a contradiction to (1.17).

Assume there exist sequences $t_m \searrow 0$, $\tilde{r}_m \rightarrow 0$ such that

$$\sup_{x \in M} E(u(t_m), B_{\tilde{r}_m}(x)) > \varepsilon_1 \quad \text{for every } m.$$

As M is compact and $\nabla u(t_m) \in L^2(M)$ we may choose slightly smaller radii r_m and a sequence x_m such that

$$E(u(t_m), B_{r_m}(x_m)) = \sup_{x \in M} E(u(t_m), B_{r_m}(x)) = \frac{\varepsilon_1}{2}.$$

Restricting ourselves to a subsequence we may assume that $x_m \rightarrow x_0$ for a point $x_0 \in M$.

We use the following lemma of [50].

LEMMA 1.13. *There exists a constant $c_1 = c_1(N)$ such that for every solution $u \in H^1(M \times [0, T], N)$ with $\nabla u \in L^4(M \times [0, T]) \cap L^\infty([0, T], L^2(M))$ of (1.1) and every $R > 0$, $(x, t) \in M \times [0, T]$ we have*

$$E(u(t), B_R(x)) \leq E(u(0), B_{2R}(x)) + c_1 \cdot \frac{t}{R^2} E(u(0)).$$

Applying this lemma to balls of radius $\frac{r_m}{2}$ and times $t \in [t_m, t_m + c_2 r_m^2]$ we get

$$E(u(t), B_{r_m/2}(x)) \leq E(u(t_m), B_{r_m}(x)) + c_1 c_2 \cdot E(u(t_m)) \leq \varepsilon_1$$

for any $x \in M$ if we choose $c_2 \leq \frac{\varepsilon_1}{2E_0 \cdot c_1}$. Here we denote by $E_0 = E(u(0)) = \max_{t \in [0, T]} E(u(t))$.

We now apply corollary 1.7 to estimate

$$\int_M \int_{t_m}^{t_m + c_2 r_m^2} |\nabla^2 u|^2 dt d\text{vol}_g \leq C \left(\frac{E_0}{r_m^2} c_2 r_m^2 + \int_M \int_{t_m}^{t_m + c_2 r_m^2} |\partial_t u|^2 dt d\text{vol}_g \right) \leq c_3$$

uniformly in m .

Finally we wish to estimate the energies $E(u(t), B_{2r_m}(x_m))$ from below. We use

LEMMA 1.14. *There exists a constant $c_4 > 0$ such that for every solution $u \in H^1(M \times [0, T], N)$ of (1.1) with $\nabla u \in L^4(M \times [0, T]) \cap L^\infty([0, T], L^2(M))$ and every $R > 0$, $(x, t) \in M \times [0, T]$ the following estimate holds*

$$E(u(t), B_{2R}(x)) \geq E(u(0), B_R(x)) - 2 \|\partial_t u\|_{L^2(B_{2R}(x) \times [0, t])}^2 - c_4 \cdot \frac{t}{R^2} E(u(0)).$$

PROOF. We multiply equation (1.1) with $\varphi^2 \partial_t u \in T_u N$ where $\varphi \in C_c^\infty(B_{2R}(x))$ is a cut-off function with $\varphi \equiv 1$ on $B_R(x)$ and $|\nabla \varphi| \leq \frac{C}{R}$. Recall that the right hand side of (1.1) is in the normal space $T_u^\perp N$ so we obtain

$$\begin{aligned} 0 &= \int_0^t \int_M \varphi^2 |\partial_t u|^2 dx dt - \int_0^t \int_M \varphi^2 \Delta u \cdot \partial_t u dx dt \\ &= \int_0^t \int_M \varphi^2 |\partial_t u|^2 dx dt + \int_0^t \int_M \frac{1}{2} \frac{d}{dt} (\varphi^2 |\nabla u|^2) + \partial_t u \cdot \nabla u \cdot \varphi \nabla \varphi dx dt \\ &= \int_0^t \int_M \varphi^2 |\partial_t u|^2 dx dt + \frac{1}{2} \int_M \varphi^2 \cdot |\nabla u(t)|^2 dx \\ &\quad - \frac{1}{2} \int_M \varphi^2 \cdot |\nabla u(0)|^2 dx + \int_0^t \int_M \partial_t u \cdot \nabla u \cdot \varphi \nabla \varphi dx dt. \end{aligned}$$

Hölder's inequality and the fact that $E(u(0)) = \max_{t \in [0, T]} (E(u(t)))$ lead to the claim. \square

We turn back to the proof of theorem 1.2. Choosing $T_1 > 0$ such that

$$\int_0^{T_1} \int_M |\partial_t u|^2 d\text{vol}_g dt \leq \frac{\varepsilon_1}{16}$$

and applying lemma 1.14 we have for m large enough

$$E(u(t), B_{2r_m}(x_m)) \geq E(u(t_m), B_{r_m}(x_m)) - 2 \cdot \frac{\varepsilon_1}{16} - \frac{c_4}{r_m^2} \cdot c_5 r_m^2 E_0 \geq \frac{\varepsilon_1}{4}$$

for every $t \in [t_m, t_m + c_5 r_m^2]$, provided $c_5 \leq \frac{\varepsilon_1}{8c_4 E_0}$.

We set $c_6 = \min(c_2, c_5)$ and proceed by a standard blow-up argument. We choose $\rho > 0$ such that

$$\overline{\lim}_{t \searrow 0} E(u(t)) \leq E(u(0)) + \varepsilon^* - \rho$$

and fix a radius $R_0 > 0$ with $E(u_0, B_{R_0}(x_0)) \leq \rho/2$. We rescale u around the points (x_m, t_m) and define

$$u_m(x, t) := u(x_m + r_m x, t_m + r_m^2 t)$$

for $t \in [0, c_6]$ and $x \in D_m = \{x : x_m + r_m x \in B_{R_0}(x_0)\}$.

Observe that

$$(1.18) \quad \int_{D_m} \int_0^{c_6} |\partial_t u_m|^2 dt dx = \int_{B_{R_0}(x_0)} \int_{t_m}^{t_m + c_6 r_m^2} |\partial_t u|^2 dt dx \xrightarrow{m \rightarrow \infty} 0$$

while for every $m \in \mathbb{N}$

$$\int_{D_m} \int_0^{c_6} |\nabla^2 u_m|^2 dt dx \leq c_3.$$

We may thus construct a sequence $\tau_m \in [0, c_6]$ such that the functions $v_m(x) := u_m(x, \tau_m)$ satisfy

- v_m converges to a function v_∞ weakly in $H_{loc}^2(\mathbb{R}^2)$ and strongly in $H_{loc}^1(\mathbb{R}^2)$.
- The time derivatives $\partial_t u_m(\tau_m)$ converge to zero in $L_{loc}^2(\mathbb{R}^2)$.
- The energies of v_m are bounded uniformly in m by $E(v_m, D_m) \leq E_0$ as well as $E(v_m, B_2) \geq \frac{\varepsilon_1}{4}$.

As the functions u_m solve equation (1.1) and as the time derivatives $\partial_t u_m(\tau_m)$ converge to zero, the limit v_∞ is a non-constant harmonic map with finite energy and can thus be extended to S^2 . By definition of ε^* we have

$$\lim_{m \rightarrow \infty} E(u(t_m + r_m^2 \tau_m), B_{R_0}(x_0)) = \frac{1}{2} \lim_{m \rightarrow \infty} \int_{D_m} |\nabla v_m|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_\infty|^2 dx \geq \varepsilon^*.$$

But then lemma 1.9 leads to a contradiction since

$$\begin{aligned} E(u_0) &\leq E(u_0, M \setminus B_{R_0}(x_0)) + \rho/2 \\ &\leq \liminf_{m \rightarrow \infty} E(u(t_m + r_m^2 \tau_m), M \setminus B_{R_0}(x_0)) + \rho/2 \\ &= \liminf_{m \rightarrow \infty} [E(u(t_m + r_m^2 \tau_m)) - E(u(t_m + r_m^2 \tau_m), B_{R_0}(x_0))] + \rho/2 \\ &\leq E(u_0) + \varepsilon^* - \varepsilon^* - \rho/2. \end{aligned}$$

The final claim of lemma 1.12, i.e. $\nabla^2 u \in L^2(M \times [0, \delta])$ for $\delta > 0$ small enough follows by the use of the H^2 -estimate for almost harmonic maps as in the proof of theorem 1.1. \square

Finally, the same argument about the set where u is identical to the corresponding Struwe solution as in the proof of theorem 1.1 concludes the proof of the main result, theorem 1.2, of this chapter. \square

CHAPTER 2

Uniqueness for the heat flow for extrinsic polyharmonic maps in the critical dimension

1. Introduction

Let M be a closed Riemannian manifold and let $N \hookrightarrow \mathbb{R}^N$ be a closed submanifold of Euclidean space. We assume that both M and N are smooth.

For maps $u : M \rightarrow N \hookrightarrow \mathbb{R}^N$ the m -th order extrinsic polyenergy is defined by

$$E^{(m)}(u) := \frac{1}{2} \int_M |\nabla^m u|^2 \, d\text{vol}_g.$$

Here ∇ is to be understood as extrinsic derivative of maps with values in \mathbb{R}^N . This energy is a higher order generalisation of the Dirichlet energy $E(\cdot)$ we considered in the first chapter. Contrary to the Dirichlet energy it is an extrinsic quantity that depends on the way N is embedded into Euclidean space.

The negative gradient flow of $E^{(m)}$ is called the *extrinsic polyharmonic map heat flow*. It is given by the equation

$$(2.1) \quad \partial_t u + (-1)^m \Delta^m u = f[u] \in T_u^\perp N$$

for a function f depending on u and its derivatives up to order $2m - 1$, see (2.6) below. We study this flow in its critical dimension $\dim(M) = 2m$.

We consider weak solutions

$$u \in H^1(M \times [0, T], N) \cap L^2([0, T], H^m(M, N))$$

that solve (2.1) in the sense of distributions as specified in section 2.

As remarked in the previous chapter, the existence of a global weak solution for any initial data $u_0 \in H^m(M, N)$ was shown by Struwe [50] for $m = 1$. Based on his methods later the result was extended to $m = 2$ by the work of Lamm [35] and Wang [57] and, finally, to any $m \in \mathbb{N}$ by Gastel [25]. These special solutions are smooth away from finitely many time-slices and their energy is non-increasing in time. Weak solutions with these two properties are uniquely determined by their initial data [25].

The first result of this chapter shows that uniqueness holds in a much broader class of weak solutions and indeed generalises the result of theorem 1.1 from the previous chapter to the whole family of extrinsic polyharmonic map flows.

THEOREM 2.1. *Let $m \in \mathbb{N}$, let M be a closed $2m$ -dimensional manifold, let $N \hookrightarrow \mathbb{R}^N$ be a closed submanifold and let $E_0 \in \mathbb{R}$.*

Then there exists a constant $\varepsilon_1 > 0$ such that weak solutions of equation (2.1) on $M \times [0, T)$ with $E^{(m)}(u(\cdot)) \leq E_0$ on $[0, T)$ and

$$(2.2) \quad \overline{\lim}_{t \searrow t_0} E^{(m)}(u(t)) < E^{(m)}(u(t_0)) + \varepsilon_1, \quad \text{for all } t_0 \in [0, T)$$

are uniquely determined by their initial data $u_0 \in H^m(M)$. Here $u(t)$ denotes the trace of u on the time-slice $M \times \{t\}$ and $0 < T \leq \infty$ is an arbitrary number.

This theorem in particular implies that weak solutions with non-increasing energy are unique and thus generalises the result of Lamm and Rivière [36] for the biharmonic flow. For $m > 2$, theorem 2.1 is new and significantly improves upon Gastel's uniqueness result for almost regular solutions mentioned earlier.

As seen in the last chapter for the harmonic map heat flow an even stronger uniqueness result is given by theorem 1.2. Our main theorem of this chapter shows that this result can be extended to polyharmonic flows of arbitrary order.

In order to state the result in detail we introduce the following notations.

For a function $f : [a, b] \rightarrow \mathbb{R}$ we denote by

$$TV(f) := \sup \left\{ \sum_i |f(t_{i+1}) - f(t_i)|, a \leq t_1 < \dots < t_k \leq b \right\}$$

the total variation of f .

We shall say that for a weak solution u of (2.1) reverse bubbling occurs at the point (x_0, t_0) if there are sequences $x_i \rightarrow x_0$, $t_i \searrow t_0$ and $r_i \rightarrow 0$ such that the rescaled maps

$$u_i(x) = u(t_i, \exp_{x_i}(r_i x))$$

converge to a non-constant polyharmonic map $u^* \in W^{1,2m}(\mathbb{R}^{2m}, N)$ in the sense of

$$u_i \xrightarrow{i \rightarrow \infty} u^* \quad \text{weakly in } H_{loc}^{2m}(\mathbb{R}^{2m}).$$

Furthermore, let

$$\varepsilon^* = \varepsilon^*(m, N) := \inf \{ E^{(m)}(u), u \in W^{1,2m}(\mathbb{R}^{2m}, N) \text{ non-constant, } m\text{-polyharmonic} \}.$$

Note that $\varepsilon^* > 0$, see [35] resp. [25].

THEOREM 2.2. *Let M, N be as in theorem 2.1 and let $0 < T \leq \infty$. Then for any weak solution u of (2.1) on $M \times [0, T)$ satisfying $TV(E^{(m)}(u(\cdot))) < \infty$ we have either*

- (1) *u coincides with the almost regular solution with initial data $u(0)$,*
- or*
- (2) *there is at least one point $(x_0, t_0) \in M \times [0, T)$ where reverse bubbling occurs.*

As a consequence of theorem 2.2 we obtain the analogue of the uniqueness criterion for weak solutions given by theorem 1.1.

COROLLARY 2.3. For M, N as above, let u be a weak solution of (2.1) with $TV(E^{(m)}(u(\cdot))) < \infty$ satisfying the condition

$$(2.3) \quad \overline{\lim}_{t \searrow t_0} E^{(m)}(u(t)) < E^{(m)}(u(t_0)) + \varepsilon^*, \quad \text{for all } t_0 \in [0, T].$$

Then $u \equiv u_{reg}$, the almost regular solution for initial data $u(0)$.

This corollary proves that under the above natural a priori assumption Topping's conjecture is valid not only for the harmonic map heat flow but for extrinsic polyharmonic flows of arbitrary order. While condition (2.3) is sharp in the harmonic case it is so far unclear if the same can be said for general polyharmonic flows since there are no known examples of reverse bubbling for any $m \geq 2$; in fact, not even examples of finite time blow-up are known for these flows of higher order. The lack of a maximum principle and the more complicated structure of the equation cause these problems to be much more challenging if $m > 1$. Nonetheless, our result shows that uniqueness in the critical dimension can only be lost due to reverse bubbling.

Another consequence of theorem 2.2 and corollary 2.3 is that each weak solution whose energy function has locally finite total variation is smooth away from a discrete set of time-slices. Indeed, away from the set of times at which reverse bubbling occurs, the solution coincides with an appropriate *almost regular* solution. Since any singularity of such a solution causes a loss of energy of no less than ε^* , see e.g. [25], while any reverse bubble leads to a gain of energy of at least ε^* , we can bound the total number of singular times of u on an interval $I \subset\subset [0, \infty)$ by $\frac{TV_I(E^{(m)}(u(\cdot)))}{\varepsilon^*}$.

As remarked, here we consider only closed domain manifolds M . It is however to be expected that uniqueness criteria of the same kind as theorems 2.1 and 2.2 may be shown by similar methods for domain manifolds with boundary. Of greater interest is the question if it is possible at all to have an energy concentration at the boundary at some point in time. So far no examples of singularities or reverse bubbles forming at the boundary are known, not even for the harmonic map flow. Indeed, the known examples of reverse bubbling have been obtained in highly symmetric settings, compare [54] and [5].

This chapter is organised as follows. In the first section we introduce polyharmonic maps and define the notion of weak solutions. We then give the proof of theorems 2.1 and 2.2. The general structure is similar to the proof of the corresponding results for the harmonic map flow of the previous chapter. Several key steps of the proof however require substantial modification due to the higher order nature of the problem. The main focus of this chapter lies on these new aspects. As in the case $m = 1$, a basic concept is the notion of *almost polyharmonic maps* whose regularity properties we investigate in section 4. This analysis requires the use of certain interpolation inequalities that we derive in appendix A. Section 5 will provide the necessary tools for the proof of theorem 2.1. Finally in section 6 we study the evolution of the local energy for smooth solutions of the flow. Note that there is an important difference between the second order harmonic map heat flow

and general polyharmonic flows. Due to the higher order of these flows, we can only control the evolution of the local energy on sets where the energy is small.

Furthermore, the presence of lower order derivatives with critical exponent in the non-linearity $f[\cdot]$ forces us to work with the energy quantity

$$E(w) := \frac{1}{2} \sum_{k=1}^m \int_M |\nabla^k w|^{\frac{2m}{k}} \, d\text{vol}_g$$

instead of $E^{(m)}$ and the corresponding local energies. Obviously the polyenergy is bounded by this *total energy* which takes into account all derivatives up to order m . On the other hand, Sobolev's embedding theorem and interpolation arguments allow us to bound the global total energy $E(w)$ in terms of the polyenergy $E^{(m)}(w)$.

Contrary, we cannot derive bounds on the local quantities of the total energy

$$(2.4) \quad E(u, M') := \sum_{k=1}^m \int_{M'} |\nabla^k w|^{\frac{2m}{k}} \, d\text{vol}_g$$

on subsets $M' \subset M$ using the weaker polyenergy $E^{(m)}(u, M')$. Of course for $m = 1$ these two notions of energy coincide.

Notations: A cut-off function on a ball $B_R(x_0)$ is a function $\varphi \in C_c^\infty(B_R(x_0))$ such that $0 \leq \varphi \leq 1$ with $\varphi \equiv 1$ on $B_{R/2}(x_0)$ and $|\nabla^k \varphi| \leq \frac{C}{R^k}$ for $k = 1, \dots, 2m$.

We denote by α and β multi-indices $\alpha, \beta \in \mathbb{N}_0^k$ for $k \in \mathbb{N}$ an arbitrary number.

2. Weakly polyharmonic maps

Smooth extrinsic polyharmonic maps are the critical points of $E^{(m)}$ with respect to variations on the target and are thus characterized by the property that

$$\Delta^m u \perp T_u N.$$

We now choose a smooth local orthonormal frame field $\{\nu_i\}_{i=\dim(N)+1}^N$ for the normal bundle near a point $P = u(x)$. Then near x we can write the polyharmonic map equation as

$$\Delta^m u = \sum_i \lambda_i \cdot \nu_i \circ u,$$

with functions λ_i determined by

$$\begin{aligned} \lambda_i &= \Delta^m u \cdot \nu_i \circ u = \text{div}^m(\nabla^m u) \cdot \nu_i \circ u \\ &= \text{div}^m(\nabla^{m-1}(\nabla u \cdot \nu_i \circ u)) - \sum_{k=0}^{m-1} \text{div}^k(\text{div}^{m-k-1} \nabla^m u \cdot \nabla(\nu_i \circ u)) \\ &\quad - \sum_{l=0}^{m-2} \text{div}^m(\nabla^l(\nabla^{m-l-1} u \cdot \nabla(\nu_i \circ u))). \end{aligned}$$

Since u maps into N and thus $\nabla u \perp \nu_i \circ u$, the polyharmonic map equation is given locally by

$$\begin{aligned}
\Delta^m u &= - \sum_i \sum_{k=0}^{m-1} \operatorname{div}^k (\langle (II_N \circ u)(\operatorname{div}^{m-k-1} \nabla^m u, \nabla u), \nu_i \circ u \rangle) \cdot \nu_i \circ u \\
&\quad - \sum_i \sum_{l=0}^{m-2} \operatorname{div}^m (\nabla^l \langle (II_N \circ u)(\nabla^{m-l-1} u, \nabla u), \nu_i \circ u \rangle) \cdot \nu_i \circ u \\
(2.5) \quad &=: (-1)^m f[u].
\end{aligned}$$

Here II_N denotes the second fundamental form of $N \hookrightarrow \mathbb{R}^N$. Indeed, we may write $f[\cdot]$ solely in terms of the second fundamental form of N and its derivatives without the explicit use of a normal frame. These computations shed no additional light on the form of the equation. Indeed, they make the decisive feature of the polyharmonic map equation, i.e. that $\Delta^m u$ lies in the normal space, less apparent; so we refrain from going into details. We may however write equation (2.5) in coordinate charts in the form

$$(2.6) \quad (-1)^m \Delta^m u = \sum_{\substack{|\alpha| \leq 2m \\ 0 < \alpha_j \leq 2m-1}} (T^\alpha \circ u)(\nabla^{\alpha_1} u, \nabla^{\alpha_2} u, \dots),$$

for multilinear forms T^α that may be expressed in terms of the second fundamental form of N and the metric of M . All lower order forms T^α , $|\alpha| < 2m$, in (2.6) are identically zero if the manifold M is locally Euclidean since they result from derivatives falling onto the metric of M .

To define the notion of weakly polyharmonic maps, remark first of all that

$$\nabla^k u \in L^{\frac{2m}{k}}(M), \quad 1 \leq k \leq m$$

for each $u \in L^\infty(M, \mathbb{R}^N) \cap H^m(M, \mathbb{R}^N)$, in particular for functions in $H^m(M, N)$. This allows us to define $f[u]$ in the sense of distributions for all functions $u \in H^m(M, N)$ as follows.

We consider test functions $v \in H^m(M, \mathbb{R}^N) \cap L^\infty(M, \mathbb{R}^N)$. We can assume without loss of generality that their support is contained in a coordinate neighbourhood of M . Formally integrating by parts, we may write

$$(2.7) \quad \int v \cdot f[u] \, d\operatorname{vol}_g = \sum_{k=0}^{m-1} \sum_{\substack{|\beta| \leq 2m-k \\ 0 < \beta_j \leq m}} \int (\tilde{T}^{k,\beta} \circ u)(\nabla^k v, \nabla^{\beta_1} u, \dots) \, d\operatorname{vol}_g$$

for suitable multilinear forms $\tilde{T}^{k,\beta}$. By the above remark the quantity on the right hand side is well defined. We define a *weakly polyharmonic map* as a function $u \in H^m(M, N)$ satisfying

$$\int (-1)^m \Delta^m u \cdot v \, d\operatorname{vol}_g = \int f[u] \cdot v \, d\operatorname{vol}_g$$

for all $v \in H^m(M, \mathbb{R}^N) \cap L^\infty(M, \mathbb{R}^N)$ in the sense of distributions as specified above. Here and in the following we integrate over the whole manifold M unless specified otherwise.

Weak solutions of (2.1) and later on of perturbed polyharmonic map equations are defined accordingly.

Remark that no generality is lost if we assume that the domain is locally Euclidean. Indeed, all global arguments remain unchanged for general closed domain manifolds. Computations in local coordinate charts may change slightly due to the addition of lower order terms coming from derivatives of the metric. Since the target is compact and has no boundary, all these lower order terms can be easily controlled and lead only to some additional constants in the corresponding estimates. For the sake of simplicity we thus assume from now on that the domain is locally Euclidean.

3. Proof of the main results

We show in this section how theorem 2.1 and theorem 2.2 may be derived from several key properties of the polyharmonic map flow and of *almost polyharmonic maps* which are proven in the following sections. We begin by giving the

PROOF OF THEOREM 2.1. We use

LEMMA 2.4. *If two weak solutions u_1 and u_2 of (2.1) on $M \times [0, T]$ with initial data $u_1(0) = u_2(0) \in H^m(M, N)$ satisfy the additional regularity assumption*

$$\nabla^k u_i \in L^{\frac{4m}{k}}(M \times [0, T]) \text{ for } k = 1, \dots, 2m, \quad i = 1, 2,$$

then they are identical.

As remarked before, the uniqueness of smooth solutions of (2.1) was proven in [25]. In fact, the very same proof may be applied also for weak solutions if they satisfy the regularity assumptions of lemma 2.4.

Now let $u \in H^1(M \times [0, T], N) \cap L^2([0, T], H^m(M, N))$ be any weak solution of equation (2.1) and let u_{reg} be the corresponding almost regular solution of (2.1). Consider the set

$$K := \{t \in [0, T] : u = u_{reg} \text{ on } M \times [0, t]\}$$

which obviously contains $t = 0$ and is closed by the trace theorem. If we can prove that for any $t_0 \in [0, T]$ there exists $\delta = \delta(t_0) > 0$ with

$$(2.8) \quad \nabla^k u \in L^{\frac{4m}{k}}(M \times [t_0, t_0 + \delta]) \text{ for all } k = 1, \dots, 2m,$$

then lemma 2.4 implies that K is also relatively open and thus that u and u_{reg} coincide.

As a first step in this direction we investigate how the spatial derivatives of u can be estimated on fixed time-slices $M \times \{t\}$.

Remark that for any weak solution u of (2.1) the trace on almost every time-slice solves a perturbed polyharmonic map equation in local coordinates. In fact, for almost every $t \in [0, T]$, the function $w = u(t)$ is a weak solution of

$$(2.9) \quad (-1)^m \Delta^m w = f[w] + \chi \quad \text{in } \Omega \subset \mathbb{R}^{2m}$$

where $\chi = -\partial_t u(t) \in L^2(M)$.

In section 4 we prove

PROPOSITION 2.5. *Let N be a closed submanifold of \mathbb{R}^N , let Ω be an open subset of \mathbb{R}^{2m} , $m \in \mathbb{N}$, and let $\chi \in L^2(\Omega)$.*

Then any weak solution $w \in H^m(\Omega, N)$ of (2.9) lies in $H_{loc}^{2m}(\Omega, N)$.

Furthermore, there exist constants $\varepsilon_2 > 0$ and $C \in \mathbb{R}$ depending only on N and m such that the condition

$$(2.10) \quad E(w, B_{2R}(x_0)) \leq \varepsilon_2$$

implies

$$(2.11) \quad \sum_{k=1}^{2m} \int \varphi^{4m} |\nabla^k w|^{\frac{4m}{k}} dx \leq C \cdot \frac{E(w, B_{2R}(x_0))}{R^{2m}} + C \cdot \int \varphi^{4m} |\chi|^2 dx.$$

Here $\varphi \in C_c^\infty(B_{2R}(x_0))$ denotes an arbitrary cut-off function and $E(\cdot)$ is the total energy introduced in (2.4).

Proposition 2.5 applied to weak solutions of the flow provides bounds on the integrals

$$\int_M |\nabla^k u(t)|^{\frac{4m}{k}} dx, \quad k = 1, \dots, 2m$$

on almost every time-slice. These bounds crucially depend on the radius R for which (2.10) is fulfilled at time t .

However, in section 5, we establish the following result.

LEMMA 2.6. *For every $\varepsilon_2 > 0$ and every $E_0 \in \mathbb{R}$ there exists a constant $\varepsilon_1 > 0$ such that the following holds true.*

Let u be a weak solution of (2.1) satisfying the assumptions of theorem 2.1 for these choices of E_0 and ε_1 . Then for every $t_0 \in [0, T)$ there exist finitely many balls $B_R(x_i), i = 1, \dots, L$ covering M and a number $\delta > 0$ such that the energy is uniformly small in the sense that

$$E(u(t), B_{2R}(x_i)) \leq \varepsilon_2 \text{ for all } t \in [t_0, t_0 + \delta], i = 1, \dots, L.$$

Let now $\varepsilon_2 > 0$ be the constant of proposition 2.5, let $E_0 \in \mathbb{R}$ and let ε_1 be the corresponding constant determined in lemma 2.6. Consider any weak solution u of (2.1) satisfying the assumptions of theorem 2.1 for E_0 and ε_1 . Let $t_0 \in [0, T]$ be any given number and let $\delta = \delta(t_0)$, $L = L(t_0)$ be as in lemma 2.6. Then proposition 2.5 implies

$$\sum_{k=1}^{2m} \int_M |\nabla^k u(t)|^{\frac{4m}{k}} dx \leq L \cdot \left[\frac{C\varepsilon_2}{R^{2m}} + C \int_M |\partial_t u(t)|^2 dx \right] \text{ on } [t_0, t_0 + \delta].$$

Since the right hand side is integrable in time, (2.8) follows and $u \equiv u_{reg}$ on all of $M \times [0, T]$. \square

The proof of the second result is based on theorem 2.1.

PROOF OF THEOREM 2.2.

Let u be a weak solution of (2.1) such that the total variation of $t \mapsto E^{(m)}(u(t))$ is finite. In particular, the energy is bounded, say $E^{(m)}(u(t)) \leq E_0$. We can apply theorem 2.1 away from the (finite!) set of time-slices where the energy instantaneously increases by at least $\varepsilon_1 = \varepsilon_1(E_0) > 0$, the constant of theorem 2.1. We thus find that u is smooth away from finitely many time-slices. In order to prove theorem 2.2 it is therefore enough to consider weak solutions of (2.1) on $M \times [0, T]$ that are smooth on $M \times (0, T]$. Thanks to this a priori information on the regularity, we can apply local energy estimates away from time $t = 0$. Instead of considering the total energy we work with equivalent energy quantities.

REMARK 2.7. *By Sobolev's embedding theorem we can replace assumption (2.10) in proposition 2.5 by the condition*

$$\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x_0)} |\nabla^k w|^2 dx \leq \varepsilon_3$$

for a constant $\varepsilon_3 = \varepsilon_3(N, m) > 0$.

For smooth solutions of (2.1) the evolution of these energy quantities is controlled by

PROPOSITION 2.8. *Given any $E_0 \in \mathbb{R}$ there exist constants $\varepsilon_4 > 0$ and $c_0 > 0$ such that the following holds true.*

Let $u \in C^\infty(M \times [0, T], N)$ be any smooth solution of (2.1) with initial energy $E^{(m)}(u(0)) \leq E_0$ and let $R > 0$ be an arbitrary number.

If

$$\sup_{x \in M} \left(\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x)} |\nabla^k u(0)|^2 dx \right) \leq \varepsilon_4$$

then for all $t \in I := [0, \min\{T, c_0 R^{2m}\}]$

$$\sup_{x \in M} \left(\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x)} |\nabla^k u(t)|^2 dx \right) \leq \varepsilon_3$$

for the constant ε_3 of remark 2.7.

In addition, the evolution of the local energy is controlled by

$$(2.12) \quad \left| \partial_t E_\varphi(t) + 2 \int \varphi^{4m} |\partial_t u|^2 dx \right| \leq \frac{C(E_0, M, N)}{R^{2m}} + \int \varphi^{4m} |\partial_t u|^2 dx$$

for all $t \in I$. Here

$$(2.13) \quad E_\varphi(t) := \sum_{k=1}^m R^{2k-2m} \int \varphi^{2m+2k} |\nabla^k u(t)|^2 dx,$$

for an arbitrary cut-off function $\varphi \in C_c^\infty(B_{2R}(x))$, $x \in M$.

The proof of this proposition makes use of proposition 2.5 and is presented in section 6.

Based on this local energy estimate we can now prepare the setting for the blow-up argument which will lead to the proof of theorem 2.2.

Let u be a weak solution of (2.1) on $M \times [0, T]$ that is smooth away from $t = 0$. Let us first assume that there is some radius $r > 0$ such that the energies are uniformly small in the sense that

$$\sup_{t \in [0, \delta]} \sup_{x \in M} E(u(t), B_r(x)) \leq \varepsilon_2$$

for some $\delta > 0$. Applying proposition 2.5 immediately implies (2.8) and thus $u \equiv u_{reg}$.

It remains to consider the case where we have no such uniform control on the energy. We can then choose sequences $t_i \searrow 0$, $r_i \rightarrow 0$ and $x_i \in M$ such that

$$\sum_{k=1}^m r_i^{2k-2m} \int_{B_{2r_i}(x_i)} |\nabla^k u(t_i)|^2 dx = \sup_{x \in M} \left(\sum_{k=1}^m r_i^{2k-2m} \int_{B_{2r_i}(x)} |\nabla^k u(t_i)|^2 dx \right) = \varepsilon_4$$

for the positive constant ε_4 of proposition 2.8. By compactness of M we may also assume that the points x_i converge to some $x_0 \in M$.

Since the energy distribution at time t_i satisfies the assumption of proposition 2.8 for $R = r_i$, we find that for i sufficiently large

$$\sup_{x \in M} \left(\sum_{k=1}^m r_i^{2k-2m} \int_{B_{2r_i}(x)} |\nabla^k u(t)|^2 dx \right) \leq \varepsilon_3 \text{ for all } t \in [t_i, t_i + c_0 r_i^{2m}].$$

A global version of proposition 2.5 given later in corollary 2.14 then implies a uniform bound

$$(2.14) \quad \sum_{k=1}^m \int_{t_i}^{t_i + c_0 r_i^{2m}} \int_M |\nabla^k u|^{\frac{4m}{k}} dx dt \leq C.$$

On the other hand, let us cover each ball $B_{2r_i}(x_i)$ by a fixed number K_M of balls of radius r_i , say $B_{r_i}(x_i^{(j)})$, $j = 1, \dots, K_M$. We may then apply the lower bound on the evolution of the energy given by (2.12). For i sufficiently large, we find

$$\begin{aligned} \sum_{k=1}^m r_i^{2k-2m} \int_{B_{2r_i}(x_i)} |\nabla^k u(t)|^2 dx &\geq \sum_{k=1}^m r_i^{2k-2m} \int_{B_{r_i}(x_i^{(j)})} |\nabla^k u(t_i)|^2 dx \\ &\quad - \frac{C}{r_i^{2m}} (t - t_i) - 3 \int_{t_i}^t \int_M |\partial_t u|^2 dx dt \\ &\geq \sum_{k=1}^m r_i^{2k-2m} \int_{B_{r_i}(x_i^{(j)})} |\nabla^k u(t_i)|^2 dx - \frac{\varepsilon_4}{2K_M} \end{aligned}$$

on $[t_i, t_i + \bar{c} \cdot r_i^{2m}]$ for an appropriate constant $0 < \bar{c} \leq c_0$ independent of i .

Recall that the balls $B_{r_i}(x_i^{(j)})$ cover the original ball $B_{2r_i}(x_i)$ on which the energy at time t_i was assumed to be ε_4 . Adding up the above estimates therefore implies a

uniform lower bound on the energies

$$(2.15) \quad K_M \cdot \sum_{k=1}^m r_i^{2k-2m} \int_{B_{4r_i}(x_i)} |\nabla^k u(t)|^2 dx \geq \varepsilon_4 - K_M \cdot \frac{\varepsilon_4}{2K_M} = \frac{\varepsilon_4}{2}$$

for $t \in [t_i, t_i + \bar{c} \cdot r_i^{2m}]$.

We can now proceed by a standard blow-up argument similar to the proof of lemma 1.12 in the previous chapter. Estimate (2.14) allows us to prove that suitably rescaled maps converge weakly in H_{loc}^{2m} to a polyharmonic map which is forced to be non-constant due to (2.15). All in all we find that reverse bubbling occurs at the point $(x_0, 0)$. \square

We conclude this section with a sketch of the

PROOF OF COROLLARY 2.3. Let u be a weak solution of equation (2.1) with $TV(E(u(\cdot))) < \infty$ which satisfies (2.3). If u is not identical to the corresponding almost regular solution of (2.1) reverse bubbling must occur at some point (x_0, t_0) according to theorem 2.2. So let $x_i \rightarrow x_0$, $t_i \searrow t_0$ and $r_i \rightarrow 0$ be such that

$$u(t_i, \exp_{x_i}(r_i x)) \xrightarrow{i \rightarrow \infty} u_0 \quad \text{weakly in } H_{loc}^{2m}(\mathbb{R}^{2m})$$

for a non-constant polyharmonic map $u_0 \in W^{1,2m}(\mathbb{R}^{2m}, N)$.

We may then proceed exactly as in the proof of lemma 1.12 to show that the formation of this reverse bubble contradicts the assumption (2.3) at t_0 since $E^{(m)}(u_0) \geq \varepsilon^*$. \square

4. Almost polyharmonic maps

We consider functions that are *almost polyharmonic maps* in the sense that they weakly solve equation (2.5) up to an L^2 -error term. That is we study weak solutions of the equation

$$(-1)^m \Delta^m w = f[w] + \chi \quad \text{in } \Omega \subset \mathbb{R}^{2m}$$

for arbitrary perturbations $\chi \in L^2(\Omega)$ and for the non-linearity $f[\cdot]$ given by (2.5).

First of all, by a slight variation of the regularity proof of Gastel and Scheven [26] for polyharmonic maps in the critical dimension we obtain

PROPOSITION 2.9. *Let $\Omega \subset \mathbb{R}^{2m}$ be open and let $\chi \in L^2(\Omega)$. Then any weak solution $w \in H^m(\Omega)$ of equation (2.9) is in $C^{m-1}(\Omega)$.*

Indeed, the arguments of [26] may be carried over without change to the perturbed setting. The presence of an L^2 -function on the right hand side of (2.9) prevents us from getting full regularity of w , but we may still conclude that $w \in C^{m-1,\alpha}$ for each $0 \leq \alpha < 1$.

Next we show the following proposition which holds true for more general equations with non-linearities satisfying similar growth conditions as $f[\cdot]$ and in arbitrary dimensions.

PROPOSITION 2.10. *Let $d \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be an open set and consider a weak solution*

$$w \in C^{m-1}(\Omega) \cap H^m(\Omega)$$

of the equation

$$(2.16) \quad \Delta^m w = \sum_{\substack{|\alpha| \leq 2m \\ 0 < \alpha_i < 2m}} (T^\alpha \circ w)(\nabla^{\alpha_1} w, \nabla^{\alpha_2} w, \dots, \nabla^{\alpha_j} w) + \chi \quad \text{in } \Omega,$$

for a function $\chi \in L^2(\Omega)$ and arbitrary smooth multilinear forms T^α .

Then $w \in H_{loc}^{2m}(\Omega)$.

Remark that the proof of proposition 2.10 will already give some bound on the H^{2m} -norm of w . However, contrary to what we claimed in proposition 2.5, this bound will not only depend on $\|\chi\|_{L^2}$ and local bounds on the energy of w but also on the modulus of continuity of the derivatives of w up to order $m-1$. We will therefore focus here on proving finiteness of the local H^{2m} -norms and derive estimate (2.11) later with different methods.

For a function $v \in L^1(\Omega)$ we denote by $D_h v = (D_{h,i} v)_i$, $h > 0$ the central difference quotient of v whose components are given by

$$D_{h,i} v = \frac{v(x + h \cdot e_i) - v(x - h \cdot e_i)}{2h},$$

e_i being the i -th coordinate vector.

Analogous to partial integration, for functions $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ with $\text{supp}(g) \subset\subset \Omega$ and p, q conjugated exponents we find

$$(2.17) \quad \int_{\Omega} D_{h,i} f \cdot g \, dx = - \int_{\Omega} f \cdot D_{h,i} g \, dx$$

for any $h < \text{dist}(\partial\Omega, \text{supp}(g))$ and $i = 1, \dots, d$.

Note that for each open set U , each $V \subset\subset U$ and each $1 < p < \infty$, there exists a constant $C \in \mathbb{R}$ such that for all $u \in W^{1,p}(U)$ (see e.g. [19])

$$(2.18) \quad \sup_{h \in (0, d]} \int_V |D_h u|^p \, dx \leq C \cdot \int_U |\nabla u|^p \, dx, \quad d = \frac{1}{2} \text{dist}(\partial U, V).$$

PROOF OF PROPOSITION 2.10. Let $w \in C^{m-1}(\Omega) \cap H^m(\Omega)$ be a weak solution of (2.16) and let $x_0 \in \Omega$ be fixed. For $\varepsilon > 0$ to be determined later, we choose $0 < r < 1$ in such a way that for each $0 \leq j \leq m-1$

$$(2.19) \quad \sup_{y \in B_r(x_0)} |\nabla^j w(y) - \nabla^j w(x_0)| + r \cdot |\nabla^j w(y)| \leq \varepsilon.$$

We claim that for a suitable choice of $\varepsilon > 0$, the integrals

$$I = I(h) := \int \varphi^{4m} |\nabla^m D_h^m w|^2 \, dx$$

are bounded uniformly in $h \in (0, r/4]$ for any cut-off function $\varphi \in C_c^\infty(B_{r/2}(x_0))$.

By partial integration and (2.17)

$$I = \int \varphi^{4m} \Delta^m w \cdot \text{div}_h^m D_h^m w \, dx + l.o.t.,$$

where *l.o.t.* contains all lower order terms that occur when a difference quotient or a derivative falls onto the cut-off function.

As w weakly solves (2.16), we have

$$(2.20) \quad I = \sum_{\substack{|\alpha| \leq 2m \\ 0 < \alpha_i \leq 2m-1}} \int \varphi^{4m} (T^\alpha \circ w) (\nabla^{\alpha_1} w, \dots, \nabla^{\alpha_j} w) \cdot \operatorname{div}_h^m D_h^m w \, dx \\ + \int \varphi^{4m} \chi \cdot \operatorname{div}_h^m D_h^m w \, dx + l.o.t.,$$

where the integrals containing derivatives of order greater than m are to be understood in the sense of distributions as described in section 2.

The last term of (2.20) is for any $\delta > 0$ bounded by

$$\begin{aligned} \int \varphi^{4m} \chi \cdot \operatorname{div}_h^m D_h^m w \, dx &\leq \delta \int \varphi^{4m} |D_h^{2m} w|^2 \, dx + C_\delta \|\chi\|_{L^2(B_r(x_0))}^2 \\ &= \delta \int |D_h^m (\varphi^{2m} D_h^m w)|^2 \, dx + l.o.t. + C_\delta \\ &\leq C\delta \int \varphi^{4m} |\nabla^m D_h^m w|^2 \, dx + l.o.t. + C_\delta \\ &= C\delta I + l.o.t. + C_\delta, \end{aligned}$$

where we applied (2.18) in the third step. Here and in the following C_δ denotes a generic constant depending on δ and quantities such as $\|w\|_{H^m(B_r(x_0))}$, $\|w\|_{C^{m-1}(B_r(x_0))}$, $\|\chi\|_{L^2(B_r(x_0))}$ and $r > 0$ independent of h . As remarked there is no need to keep track of the explicit form of these constants.

Going back to (2.20), our main task then is to estimate the integrals

$$I^\alpha := \int \varphi^{4m} (T^\alpha \circ w) (\nabla^{\alpha_1} w, \dots, \nabla^{\alpha_j} w) \cdot \operatorname{div}_h^m D_h^m w \, dx$$

for all multi-indices α with $|\alpha| \leq 2m$ and $0 < \alpha_j < 2m$.

If all components α_i are smaller than m we find

$$I^\alpha \leq C \int \varphi^{4m} |D_h^{2m} w| \, dx \leq \delta I + C_\delta + l.o.t.$$

since $u \in C^{m-1}$. If α contains components greater or equal to m , we need to proceed with more care. As an example we treat the case $\alpha_1 = \alpha_2 = m$ which we would like to remark is the only one that has to be considered for $m = 1$.

Transferring m difference quotients from the last factor in

$$I^{(m,m)} = \int \varphi^{4m} (T \circ w) (\nabla^m w, \nabla^m w) \cdot \operatorname{div}_h^m D_h^m u \, dx$$

onto the other factors using (2.17), we obtain

$$\begin{aligned}
I^{(m,m)} &= (-1)^m \int D_h^m \left(\varphi^{4m} (T \circ w) (\nabla^m w, \nabla^m w) \right) \cdot D_h^m w \, dx \\
&= (-1)^m \int \varphi^{4m} (D_h^m (T \circ w)) (\nabla^m w, \nabla^m w) \cdot D_h^m w \, dx \\
&\quad + (-1)^m \int 2\varphi^{4m} (T \circ w) (\nabla^m D_h^m w, \nabla^m w) \cdot D_h^m w \, dx \\
&\quad + \sum_{\substack{|\beta| \leq m \\ \beta_i < m}} C_\beta \int \varphi^{3m+|\beta|} (D_h^{\beta_1} (T \circ w)) (\nabla^m D_h^{\beta_2} w, \nabla^m D_h^{\beta_3} w) \cdot D_h^m w \, dx.
\end{aligned}$$

By Hölder's and Young's inequality the second term is obviously bounded by $\delta \cdot I + C_\delta \mathcal{J}$ where \mathcal{J} denotes the integral

$$\mathcal{J} = \int \varphi^{4m} |D_h^m w|^2 |\nabla^m w|^2 \, dx.$$

Since $w \in C^{m-1}$, clearly also the first term may be bounded by $C\mathcal{J} + C$ with constant C depending on r , $\|w\|_{H^m}$ and $\|w\|_{C^{m-1}}$.

Transferring one more difference quotient away from the last factor in the remaining terms, we can exploit the pointwise bound of $|D_h^{m-1} w| \leq C$ to obtain

$$\begin{aligned}
&\sum_{\substack{|\beta| \leq m \\ \beta_i < m}} C_\beta \int \varphi^{3m+|\beta|} (D_h^{\beta_1} (T \circ w)) (\nabla^m D_h^{\beta_2} w, \nabla^m D_h^{\beta_3} w) D_h^m w \, dx \\
&\leq C \cdot \int \varphi^{4m-1} |D_h^m w| (|\nabla^m D_h w| + |\nabla^m w|) |\nabla^m w| \, dx \\
&\quad + C \cdot \sum_{\substack{|\beta| \leq m+1 \\ \beta_i \leq m}} \int \varphi^{2m+|\beta|} |\nabla^m D_h^{\beta_2} w| \cdot |\nabla^m D_h^{\beta_3} w| \, dx \\
&\leq \delta \cdot I + \mathcal{J} + C_\delta \sum_{k=1}^{m-1} \int \varphi^{2m+2k} |\nabla^m D_h^k w|^2 \, dx + C_\delta.
\end{aligned}$$

Altogether we find

$$I^{(m,m)} \leq \delta I + C_\delta \left[\mathcal{J} + \sum_{k=1}^{m-1} \int \varphi^{2m+2k} |\nabla^m D_h^k w|^2 \, dx + 1 \right].$$

Similar arguments lead to bounds for all integrals I^α as well as the lower order terms contained in *l.o.t.*

In total we find for every $\delta > 0$

$$I \leq \delta \cdot I + C_\delta (\mathcal{J} + S + 1)$$

for S given by

$$S = \sum_{k=1}^{m-1} \int \varphi^{2m+2k} |\nabla^m D_h^k w|^2 \, dx + \sum_{l=1}^m \sum_{k=0}^{m-1} \int \varphi^{4m} |\nabla^k D_h^l w|^{\frac{4m}{k+l}} \, dx.$$

By interpolation we can bound

$$(2.21) \quad S \leq \delta \cdot I + C_\delta.$$

Similar arguments will be presented in detail in appendix A in a less standard situation and therefore need not be repeated here.

It remains to find an appropriate estimate for \mathcal{J} . Let P_w be the Taylor polynomial of w of order $m - 1$ with base point x_0 . Since $\nabla^k P_w(x_0) = \nabla^k w(x_0)$ for $k = 0, \dots, m - 1$, for these values of k assumption (2.19) implies

$$(2.22) \quad \begin{aligned} |\nabla^k w(y) - \nabla^k P_w(y)| &\leq |\nabla^k w(y) - \nabla^k w(x_0)| + |\nabla^k P_w(x_0) - \nabla^k P_w(y)| \\ &\leq \varepsilon + C \cdot \sum_{l=k+1}^{m-1} |\nabla^l w(x_0)| \cdot |y - x_0|^{l-k} \leq C \cdot \varepsilon \end{aligned}$$

for any $y \in B_r(x_0)$.

Using that $\nabla^m(P_w) \equiv 0$, we can rewrite \mathcal{J} as

$$\mathcal{J} = \int \varphi^{4m} \nabla^m w \cdot \nabla^m(w - P_w) |D_h^m w|^2 dx = \mathcal{J}_0 - \sum_{k=1}^m \binom{m}{k} R_k$$

where

$$\mathcal{J}_0 = (-1)^m \int \varphi^{4m} \Delta^m w \cdot (w - P_w) |D_h^m w|^2 dx$$

and

$$R_k := \int \nabla^m w \cdot \nabla^{m-k}(w - P_w) \nabla^k(\varphi^{4m} |D_h^m w|^2) dx.$$

Since w is a weak solution of (2.16), we may write \mathcal{J}_0 in the form

$$(2.23) \quad \mathcal{J}_0 = \sum_{j=0}^{m-1} \sum_{\substack{|\beta| \leq 2m-j \\ 0 < \beta_i \leq m}} I_j^\beta + \int \varphi^{4m} \chi \cdot (w - P_w) |D_h^m w|^2 dx$$

where

$$I_j^\beta = \int (\tilde{T}^{j,\beta} \circ w) \left(\nabla^j \left(\varphi^{4m} (w - P_w) |D_h^m w|^2 \right), \nabla^{\beta_1} w, \dots \right) dx$$

for suitable multilinear forms $\tilde{T}^{j,\beta}$ as discussed in section 2.

The term involving χ in (2.23) is obviously bounded by

$$C\varepsilon \cdot \|\chi\|_{L^2} \cdot \left(\int \varphi^{4m} |D_h^m w|^4 dx \right)^{1/2} \leq C\varepsilon(S + 1).$$

For $j = 0$ estimate (2.22) implies

$$I_0^\beta \leq C\varepsilon \int \varphi^{4m} (|\nabla^m w|^2 + 1) |D_h^m w|^2 dx \leq C\varepsilon(\mathcal{J} + 1).$$

Since for multi-indices with length $|\beta| \leq 2m - 1$ no more than one component may be equal to m , we find for $j = 1$

$$I_1^\beta \leq C\varepsilon \int \varphi^{4m-1} \left(|\nabla D_h^m w| + |D_h^m w| \right) \cdot |D_h^m w| \cdot \left(|\nabla^m w| + 1 \right) dx \leq C\varepsilon(\mathcal{J} + S + 1).$$

For $2 \leq j \leq m-1$ all considered multi-indices have length $|\beta| \leq 2m-2$. They either consist only of components strictly less than m , in which case

$$I_j^\beta \leq \int \left| \nabla^j \left(\varphi^{4m}(w - P_w) |D_h^m w|^2 \right) \right| dx \leq C\varepsilon(S+1),$$

or they contain one $\beta_j = m$ while all other components of β are strictly less than $m-1$. In this last case we integrate by parts one final time and reach

$$\begin{aligned} I_j^\beta &\leq C \sum_{k=1}^m \int \left| \nabla^k \left(\varphi^{4m}(w - P_w) |D_h^m w|^2 \right) \right| dx \\ &\leq C \int \varphi^{4m} |\nabla^m w| \cdot |D_h^m w|^2 dx + C\varepsilon \int \varphi^{4m} |D_h^m w| \cdot |\nabla^m D_h^m w| dx + C\varepsilon(S+1) \\ &\leq \delta \cdot \mathcal{J} + C\varepsilon(S+I+1) + C_\delta. \end{aligned}$$

All in all we have shown

$$\mathcal{J}_0 \leq C\varepsilon(\mathcal{J} + I + S + 1) + \delta\mathcal{J} + C_\delta.$$

In fact, the same estimate is also true for all R_k , $1 \leq k \leq m$. Using (2.21) we conclude

$$\mathcal{J} \leq \left(\frac{1}{2} + C\varepsilon\right)\mathcal{J} + C\varepsilon I + C$$

and thus for $\varepsilon > 0$ small enough

$$\mathcal{J} \leq C\varepsilon I + C.$$

Combined with the knowledge that $I \leq C\mathcal{J} + C$, we find as claimed a uniform bound for the integrals $I = I(h)$ if $\varepsilon > 0$ is chosen sufficiently small. By (2.21) also all terms contained in S are bounded uniformly in $h \in (0, \frac{r}{4}]$.

We conclude that $w \in H^{2m}(B_{r/4}(x_0))$. Indeed, let $v := \nabla^m(\varphi^{2m}w)$ and let $1 \leq k \leq m$. The uniform bounds for $S = S(h)$ and $I = I(h)$ imply

$$\begin{aligned} \sup_{h \in (0, r/4]} \int |D_h^k v|^2 dx &\leq C \sup_{h \in (0, r/4]} \sum_{j=0}^k \sum_{l=0}^m \int_{\text{supp}(\varphi)} \varphi^{2(j+l)} |\nabla^l D_h^j w|^2 dx \\ &\leq C \sup_{h \in (0, r/4]} I(h) + S(h) + C < \infty. \end{aligned}$$

Thus $(D_{h_j}^k v)$ is bounded in $L^2(\mathbb{R}^d)$ for any sequence $h_j \searrow 0$ and we can extract a weakly converging subsequence

$$D_{h_j}^k v \rightharpoonup f_k \quad \text{in } L^2(\mathbb{R}^d).$$

Taking any test function $\psi \in C_c^\infty(\mathbb{R}^d)$, we find by the dominated convergence theorem

$$\begin{aligned} \int f_k \cdot \psi dx &= \lim_{j \rightarrow \infty} \int D_{h_j}^k v \cdot \psi dx = \lim_{j \rightarrow \infty} (-1)^k \int v \cdot D_{h_j}^k \psi dx \\ &= (-1)^k \int v \cdot \nabla^k \psi dx. \end{aligned}$$

For any $1 \leq k \leq m$ the function $f_k \in L^2(\mathbb{R}^d)$ therefore represents the k -th weak derivative of v and thus $v \in H^m(\mathbb{R}^d)$. As $\varphi \equiv 1$ on $B_{r/4}(x_0)$, this implies that $w \in H^{2m}(B_{r/4}(x_0))$.

Since this holds true for an arbitrary point $x_0 \in \Omega$ and with $r = r(x_0) > 0$, we have completed the proof of proposition 2.10. \square

Let now $B_{2R}(x_0) \subset \mathbb{R}^{2m}$ be fixed and let $\varphi \in C_c^\infty(B_{2R}(x_0))$ be a cut-off function. We define

$$I_k = I_k(w) := \int \varphi^{4m} |\nabla^k w|^{\frac{4m}{k}} dx, \quad k = 1, \dots, 2m$$

for functions $w \in H^{2m}(B_{2R}(x_0))$. These integrals satisfy the following useful interpolation inequality which is proved in appendix A.

PROPOSITION 2.11. *For every $m \in \mathbb{N}$ there exist constants $\varepsilon_0 > 0$ and C such that the following holds true.*

For any $B_{2R}(x_0) \subset \mathbb{R}^{2m}$ and any $w \in H^{2m}(B_{2R}(x_0))$ with energy

$$E(w, B_{2R}(x_0)) \leq \varepsilon \leq \varepsilon_0$$

the estimate

$$I_k \leq C \varepsilon^{\frac{1}{k} \cdot \frac{2m-k}{m}} I_{2m} + C \varepsilon^{-2 + \frac{2}{k}} \cdot \frac{E(w, B_{2R}(x_0))}{R^{2m}}$$

holds true for each $k = 1, \dots, 2m - 1$.

It is decisive for this interpolation inequality that we work with the total energy quantity $E(\cdot)$ introduced in (2.4) containing all derivatives up to order m instead of the extrinsic energy.

REMARK 2.12. *While proposition 2.11 is only valid in dimension $2m$, the above result may be generalised to higher dimensions. In fact, for every $m \in \mathbb{N}$ and every $d \in \mathbb{N}$, there exists a constant $\varepsilon_0 = \varepsilon_0(m, d) > 0$ such that for functions $w \in H^{2m}(B_{3R}(x_0)) \cap W^{1,4m}(B_{3R}(x_0))$ on $B_{3R}(x_0) \subset \mathbb{R}^d$ satisfying*

$$[w]_{BMO(B_{3R}(x_0))} \leq \varepsilon \leq \varepsilon_0,$$

an inequality of the form

$$I_k \leq C \varepsilon^{\gamma_1} I_{2m} + C \varepsilon^{-\gamma_2} \frac{E(w, B_{2R}(x_0))}{R^{2m}}$$

holds true for positive exponents $\gamma_1, \gamma_2 > 0$ depending on d, k and m .

The proof of this generalised result is based on a local Gagliardo-Nirenberg type inequality which is similar to proposition 2.2 of [40].

We are now in the position to give the

PROOF OF PROPOSITION 2.5. The first part of proposition 2.5 concerning the regularity of almost polyharmonic maps is an immediate consequence of proposition 2.9 and proposition 2.10.

So let $w \in H^{2m}(B_{2R}(x_0))$ be any weak solution of (2.9) on a ball $B_{2R}(x_0) \subset \mathbb{R}^{2m}$ satisfying (2.10) for $\varepsilon_2 > 0$ to be determined later.

Remark that

$$I_{2m} = \int \varphi^{4m} |\nabla^{2m} w|^2 dx = \int \varphi^{4m} |\Delta^m w|^2 dx + R(w)$$

with error term

$$\begin{aligned} |R(w)| &\leq C \int |\nabla(\varphi^{4m})| |\nabla^{2m-1} w| \cdot |\nabla^{2m} w| dx \\ &\leq \frac{1}{4} \int \varphi^{4m} |\nabla^{2m} w|^2 dx + CR^{-2} \int \varphi^{4m-2} |\nabla^{2m-1} w|^2 dx. \end{aligned}$$

We easily find (compare (A.4))

$$R^{-2} \int \varphi^{4m-2} |\nabla^{2m-1} w|^2 dx \leq \delta I_{2m} + C_\delta \cdot \kappa(w)$$

where $\kappa(w)$ is shorthand for the rescaled energy $\kappa(w) = 2R^{-2m} E(w, B_{2R}(x_0))$.

In particular

$$I_{2m} \leq 2 \int \varphi^{4m} |\Delta^m w|^2 dx + C \cdot \kappa(w).$$

Recall that since $w \in H^{2m}(B_{2R}(x_0))$ is a solution of (2.9), it satisfies

$$|\Delta^m w| \leq |f[w]| + |\chi| \leq C \cdot \sum_{k=1}^{2m-1} |\nabla^k w|^{\frac{2m}{k}} + |\chi|$$

for almost every point and thus we find

$$(2.24) \quad I_{2m} \leq C \sum_{k=1}^{2m-1} I_k + C \int \varphi^{4m} |\chi|^2 dx + C\kappa(w).$$

On the other hand, choosing $0 < \varepsilon_2 < 1$ no larger than the constant ε_0 of proposition 2.11, we may estimate the integrals I_k for $1 \leq k \leq 2m - 1$ by

$$(2.25) \quad I_k \leq C \cdot \varepsilon_2^{\frac{1}{k} \frac{2m-k}{m}} I_{2m} + C\varepsilon_2^{-2+\frac{2}{k}} \kappa(w) \leq C\varepsilon_2^{\frac{1}{m(2m-1)}} I_{2m} + C\varepsilon_2^{-2} \kappa(w).$$

Combined with (2.24) we find

$$I_{2m} \leq C\varepsilon_2^{\frac{1}{m(2m-1)}} I_{2m} + C\varepsilon_2^{-2} \kappa(w) + C \int \varphi^{4m} |\chi|^2 dx,$$

which implies the desired estimate for I_{2m} if ε_2 is chosen small enough.

The estimates for I_k , $1 \leq k \leq 2m - 1$ are now a consequence of (2.25). \square

REMARK 2.13. *We have indeed shown a slightly better estimate than (2.11) for I_k if k is strictly less than $2m$. Assuming $E(w, B_{2R}(x_0)) \leq \varepsilon \leq \varepsilon_2$, we have seen that for $k = 1, \dots, 2m - 1$*

$$I_k \leq C(\varepsilon) \cdot \frac{E(w, B_{2R}(x_0))}{R^{2m}} + C\varepsilon^{\gamma_k} \cdot \int \varphi^{4m} |\chi|^2 dx$$

where $\gamma_k = \frac{2m-k}{mk} > 0$. We will apply this improved version of (2.11) in section 6 to derive bounds for the evolution of the local energy of smooth solutions of the polyharmonic map flow.

Using the set-additivity of the energy we may show a global version of estimate (2.11) by a simple covering argument similar to the proof of corollary 1.7. We state this estimate using the equivalent energy quantities introduced in remark 2.7.

COROLLARY 2.14. *Let $w \in H^{2m}(M, N)$ be a weak solution of (2.9) and let $\varepsilon_3 > 0$ be defined as in remark 2.7. If for some $R > 0$*

$$\sup_{x \in M} \left(\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x)} |\nabla^k w|^2 dx \right) \leq \varepsilon_3$$

then

$$\sum_{k=1}^{2m} \int_M |\nabla w|^{\frac{4m}{k}} dx \leq C \cdot \frac{E(w)}{R^{2m}} + C \cdot \int_M |\chi|^2 dx.$$

5. Proof of lemma 2.6

We show in this section how the assumptions of theorem 2.1 lead to the uniform smallness of the energy claimed in lemma 2.6.

We have already pointed out that it is essential to have bounds on the local quantities of the total energy $E(\cdot)$ that contain all derivatives up to order m . We therefore demonstrate first how assumptions about the polyenergy $E^{(m)}$ translate to similar restrictions on E .

LEMMA 2.15. *For every $\varepsilon_2 > 0$ and every $E_0 \in \mathbb{R}$ there exists $\varepsilon_1 > 0$ such that the estimate*

$$\overline{\lim}_{n \rightarrow \infty} E(v_n) < E(v) + \varepsilon_2$$

holds true for all sequences $v_n \in H^m(M, N)$ and functions $v \in H^m(M, N)$ with $E^{(m)}(v) \leq E_0$ such that

$$v_n \rightarrow v \text{ in } L^2(M) \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty} E^{(m)}(v_n) < E^{(m)}(v) + \varepsilon_1.$$

PROOF. We first remark that as M is closed, by standard interpolation

$$\int_M |\nabla^k v_n|^2 dx \leq \left(\int_M |v_n|^2 dx \right)^{\frac{m-k}{m}} \cdot \left(\int_M |\nabla^m v_n|^2 dx \right)^{\frac{k}{m}}$$

for each $w \in H^m(M, N)$ and $1 \leq k \leq m-1$. Thus the H^m -norm and, by Sobolev's embedding theorem, also the total energy can be uniformly bounded by

$$E(w) \leq E_\infty = E_\infty(E_0, M, N)$$

for all functions $w \in H^m(M, N)$ with $E^{(m)}(w) \leq E_0$.

Let now (v_n) and v be as in lemma 2.15 for $0 < \varepsilon_1 < \varepsilon_2$ to be determined later. By the above remark we can assume without loss of generality that the energies $E(v_n)$ converge. Furthermore, by weak compactness and Rellich's theorem there exists a subsequence (still denoted by v_n) and a limit function $v_\infty \in H^m(M)$ with

$$\begin{aligned} v_n &\rightarrow v_\infty && \text{in } H^{m-1}(M), \\ v_n &\overset{w}{\rightharpoonup} v_\infty && \text{weakly in } H^m(M). \end{aligned}$$

The assumption $v_n \rightarrow v$ in $L^2(M)$ forces the limit function v_∞ to be v . So

$$\begin{aligned} \int_M |\nabla^m v_n - \nabla^m v|^2 dx &= \int |\nabla^m v_n|^2 - |\nabla^m v|^2 dx - 2 \int_M \nabla^m(v_n - v) \cdot \nabla^m v dx \\ &= E^{(m)}(v_n) - E^{(m)}(v) + o(1) < \varepsilon_1 + o(1) \end{aligned}$$

and thus

$$\|v_n - v\|_{H^m(M)}^2 = \|\nabla^m(v_n - v)\|_{L^2(M)}^2 + \|v_n - v\|_{H^{m-1}}^2 < \varepsilon_1 + o(1),$$

where $o(1) \rightarrow 0$, as $n \rightarrow \infty$. Sobolev's embedding theorem then implies for all $1 \leq k \leq m$

$$\|\nabla^k(v_n - v)\|_{L^{\frac{2m}{k}}(M)} < C\sqrt{\varepsilon_1} + o(1).$$

Combining the estimate

$$(a + b)^{\frac{2m}{k}} \leq a^{\frac{2m}{k}} + \frac{2m}{k}(a + b)^{\frac{2m}{k}-1} \cdot b, \quad a, b > 0$$

with the triangle inequality we find

$$\|\nabla^k v_n\|_{L^{\frac{2m}{k}}(M)} \leq \|\nabla^k v\|_{L^{\frac{2m}{k}}(M)} + C(E_\infty + 1)^{\frac{2m}{k}-1} \sqrt{\varepsilon_1} + o(1).$$

Adding up these estimates for $1 \leq k \leq m$ implies the claim of lemma 2.15 for ε_1 suitably small. \square

LEMMA 2.16. *For any weak solution u of (2.1) on $M \times [0, T)$ and any open set $M' \subset M$ we have*

$$E(u(t_0), M') \leq \varliminf_{t \rightarrow t_0} E(u(t), M')$$

for each time $t_0 \in [0, T)$.

This lemma can be shown based on the weak lower semi-continuity of L^p norms and Rellich's theorem by arguments as presented in the proof of lemma 1.9.

We now turn to the only remaining step of the proof of theorem 2.1.

PROOF OF LEMMA 2.6. Let $E_0 \in \mathbb{R}$ and $\varepsilon_2 > 0$ be given and let $\varepsilon_1 > 0$ be the corresponding constant of lemma 2.15.

Consider a weak solution u of (2.1) on $M \times [0, T)$ satisfying the assumptions of theorem 2.1 for E_0 and ε_1 . The trace theorem tells us that $u(t_n) \rightarrow u(t_0)$ in $L^2(M)$ for any sequence $t_n \rightarrow t_0$, $t_0 \in [0, T)$. We may therefore apply lemma 2.15 and conclude that the behaviour of the total energy is constrained by

$$\overline{\lim}_{t \searrow t_0} E(u(t)) < E(u(t_0)) + \varepsilon_2 \text{ for every } t_0 \in [0, T).$$

Let now $t_0 \in [0, T)$ be fixed and $B \subset M$ be any ball. Applying lemma 2.16 on the complement of \overline{B} we find that the local energy on B may not instantaneously increase by more than

$$\overline{\lim}_{t \searrow t_0} E(u(t)) - E(u(t_0)) < \varepsilon_2 - \rho$$

for a number $\rho = \rho(t_0) > 0$. Choosing finitely many balls $B_R(x_i)$ covering M such that

$$E(u(t_0), B_{2R}(x_i)) \leq \rho/2$$

we find that the claim of lemma 2.6 holds true for $\delta = \delta(t_0) > 0$ small enough. \square

6. Local energy estimates for the extrinsic polyharmonic flow

In order to study the behaviour of a solution, we need to know how local energy quantities evolve in time. Instead of trying to control the evolution of the total energy, we work with the equivalent energy quantities of remark 2.7.

In this section we derive the estimate for the evolution of the local energy of smooth solutions of (2.1) given by proposition 2.8. It is unclear if similar local energy estimates hold true for general weak solutions.

We first show that if the local energy is sufficiently small, its momentary rate of change is controlled.

LEMMA 2.17. *There exist constants $C_1 > 0$ and $\varepsilon_5 > 0$ such that for any smooth solution $u \in C^\infty(M \times [0, T], N)$ of equation (2.1) the following local energy estimate holds true. If*

$$(2.26) \quad \sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x_0)} |\nabla^k u(t_0)|^2 dx \leq \varepsilon_5$$

for some $(x_0, t_0) \in M \times [0, T]$ and some $R > 0$, then

$$\left| \partial_t E_\varphi(t_0) + 2 \int \varphi^{4m} |\partial_t u(t_0)|^2 dx \right| \leq C_1 \frac{E(u(t_0), B_{2R}(x_0)) + 1}{R^{2m}} + \int \varphi^{4m} |\partial_t u(t_0)|^2 dx$$

for $E_\varphi(t)$ defined by (2.13) and $\varphi \in C_c^\infty(B_{2R}(x_0))$ a cut-off function.

PROOF. Let u and φ be as in lemma 2.17.

We multiply equation (2.1) with $(-1)^k \operatorname{div}^k(\varphi^{2m+2k} \nabla^k u(t))$ for $k = 1, \dots, m-1$ and integrate by parts to obtain

$$\begin{aligned} & \int \varphi^{2m+2k} \nabla^k u(t) \cdot \nabla^k \partial_t u(t) dx + \int \nabla^{k+m} u(t) \cdot \nabla^m (\varphi^{2m+2k} \nabla^k u(t)) dx \\ &= (-1)^k \int f[u(t)] \cdot \operatorname{div}^k(\varphi^{2m+2k} \nabla^k u(t)) dx \\ &=: R_1^{(k)}(u). \end{aligned}$$

This can be written in the form

$$\frac{1}{2} \partial_t \int \varphi^{2m+2k} |\nabla^k u(t)|^2 dx = R_1^{(k)}(u) + R_2^{(k)}(u),$$

where

$$R_2^{(k)}(u) = - \sum_{l=0}^m \binom{m}{l} \int \nabla^{m+k} u(t) \cdot \nabla^{m+k-l} u(t) \nabla^l (\varphi^{2m+2k}) dx.$$

All terms contained in $R_1^{(k)}(u)$ and $R_2^{(k)}(u)$ may be bounded using interpolation arguments similar to what is done in appendix A. In fact, using the notation $I_l = I_l(u(t))$ of section 4 we find

$$(2.27) \quad |R_1^{(k)}(u)| + |R_2^{(k)}(u)| \leq R^{2m-2k} \left[\delta I_{2m} + C \sum_{l=1}^{2m-1} I_l + C_\delta \frac{E(u, B_{2R}(x_0)) + 1}{R^{2m}} \right]$$

for each $1 \leq k \leq m-1$.

In order to discuss the highest order term of the total energy, we multiply equation (2.1) with $\varphi^{4m} \partial_t u \in T_u N$ and use that $f[u] \in T_u^\perp N$. We find that

$$\frac{1}{2} \partial_t \int \varphi^{4m} |\nabla^m u|^2 dx + \int \varphi^{4m} |\partial_t u|^2 dx = R_1^{(m)}(u)$$

where

$$\begin{aligned} |R_1^{(m)}(u)| &\leq \sum_{l=1}^m \binom{m}{l} \left| \int \nabla^m u \cdot \nabla^l (\varphi^{4m}) \nabla^{m-l} \partial_t u dx \right| \\ &= \sum_{l=1}^m \binom{m}{l} \left| \int \operatorname{div}^{m-l} (\nabla^m u \cdot \nabla^l (\varphi^{4m})) \cdot \partial_t u dx \right| \\ &\leq C \cdot \sum_{l=1}^m R^{-l} \int \varphi^{4m-l} |\nabla^{2m-l} u| \cdot |\partial_t u| dx \\ &\leq \frac{1}{4} \int \varphi^{4m} |\partial_t u|^2 dx + \delta I_{2m} + C_\delta \frac{E(u, B_{2R}(x_0))}{R^{2m}} \end{aligned}$$

again by interpolation arguments.

We have shown so far that

$$(2.28) \quad \left| \partial_t E_\varphi(t) + 2 \int \varphi^{4m} |\partial_t u|^2 dx \right| \leq \delta I_{2m}(u(t)) + C_\delta \cdot \frac{E(u(t), B_{2R}(x_0)) + 1}{R^{2m}} + C \sum_{k=1}^{2m-1} I_k(u(t)) + \frac{1}{2} \int \varphi^{4m} |\partial_t u|^2 dx.$$

for arbitrary smooth solutions u , times t and points $x_0 \in M$.

Only now we use the smallness assumption (2.26) to bound the integrals I_k . Choosing ε_5 to be no more than the constant ε_3 of remark 2.7 we have

$$I_{2m}(u(t_0)) \leq C \cdot \frac{E(u(t_0), B_{2R}(x_0))}{R^{2m}} + C \int \varphi^{4m} |\partial_t u(t_0)|^2 dx$$

according to proposition 2.5 and remark 2.7.

While the same bound holds true for the integrals $I_k = I_k(u(t_0))$, $1 \leq k \leq 2m-1$, we need to apply the more refined version of this estimate discussed in remark 2.13. We thus find

$$I_k \leq C(\varepsilon_5) \cdot \frac{E(u(t_0), B_{2R}(x_0))}{R^{2m}} + C \varepsilon_5^{\gamma_k} \int \varphi^{4m} |\partial_t u(t_0)|^2 dx$$

for $\gamma_k = \frac{2m-k}{mk} > 0$. Combined with (2.28) this implies lemma 2.17 for ε_5 sufficiently small. \square

We can now establish the local energy estimate we used in section 3.

PROOF OF PROPOSITION 2.8. Let $E_0 \in \mathbb{R}$ and let $\varepsilon_3 > 0$ be the constant of remark 2.7 and let $0 < \varepsilon_5 \leq \varepsilon_3$ be as in lemma 2.17.

Consider any smooth solution u of (2.1) on $M \times [0, T]$ with $E^{(m)}(u(0)) \leq E_0$ and let $R > 0$ be such that

$$\sup_{x \in M} \left(\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x)} |\nabla^k u(0)|^2 dx \right) \leq \varepsilon_4$$

for $\varepsilon_4 > 0$ to be determined below. Set

$$T_1 := \max \left\{ t \in [0, T] : \sup_{s \in [0, t]} \sup_{x \in M} \left(\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x)} |\nabla^k u(s)|^2 dx \right) \leq \varepsilon_5 \right\}.$$

Since u is smooth, the polyenergy of u is non-increasing in time, in particular $E^{(m)}(u(t)) \leq E_0$ for every $t \in [0, T]$. As remarked in section 2.6, the total energy $E(u(t))$ is thus uniformly bounded by $E_\infty = E_\infty(M, N, E_0) < \infty$ for all times $t \in [0, T]$.

We claim that if $\varepsilon_4 > 0$ is chosen small enough then

$$T_1 \geq \min(T, c_0 R^{2m}) \text{ for } c_0 := \frac{\varepsilon_4}{C_1(E_\infty + 1)}.$$

Here C_1 denotes the constant of lemma 2.17.

Indeed, let us assume that $T_1 < \min(T, c_0 R^{2m})$. On the one hand, the smallness of the energy up to time T_1 allows us to control the evolution of the local energy on the interval $[0, T_1]$ according to lemma 2.17. We find that for every $x \in M$

$$\begin{aligned} \sum_{k=1}^m R^{2k-2m} \int_{B_R(x)} |\nabla^k u(T_1)|^2 dx &\leq \sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x)} |\nabla^k u(0)|^2 dx \\ &\quad + C_1 R^{-2m} (E_\infty + 1) \cdot T_1 \\ &\leq 2\varepsilon_4. \end{aligned}$$

On the other hand, there is at least one point $x_0 \in M$ with

$$\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x_0)} |\nabla^k u(T_1)|^2 dx = \varepsilon_5$$

by maximality of $T_1 < T$.

The compactness of M allows us to cover the ball $B_{2R}(x_0)$ by a fixed number of balls $B_R(x_i)$, $i = 1, \dots, K = K(M)$ with half the radius. But then the above estimate implies that

$$\sum_{k=1}^m R^{2k-2m} \int_{B_{2R}(x_0)} |\nabla^k u(T_1)|^2 dx \leq 2K\varepsilon_4 < \varepsilon_5$$

if $\varepsilon_4 > 0$ is chosen small enough contradicting the assumption. We thus obtain that $T_1 \geq \min(T, c_0 R^{2m})$. Estimate (2.12) is now a consequence of lemma 2.17 and of the uniform bound E_∞ on the total energy. \square

CHAPTER 3

Selfsimilar solutions of the harmonic map heat flow

1. Introduction

We consider selfsimilar weak solutions which are locally of class H^1 of the harmonic map heat flow

$$(3.1) \quad \partial_t u - \Delta u = \Gamma(u)(\nabla u, \nabla u) \quad \text{on } \mathbb{R}^d \times [0, \infty)$$

from Euclidean space \mathbb{R}^d into a smooth target manifold N . Such solutions $u \in H_{loc}^1(\mathbb{R}^d \times [0, \infty))$ of equation (3.1) exist only in supercritical dimensions $d \geq 3$. We focus here on the class of outgoing selfsimilar solutions which are of the form

$$u(x, y) = v\left(\frac{x}{\sqrt{t}}\right), \quad t > 0, x \in \mathbb{R}^d$$

for a suitable map $v : \mathbb{R}^d \rightarrow N$. By the translation invariance of (3.1) these maps represent all solutions of (3.1) which are selfsimilar in forward time up to translations in space-time.

We study the questions of existence and uniqueness of weak solutions to (3.1) with initial data

$$(3.2) \quad u(0) = u_0 \in H_{loc}^1(\mathbb{R}^d)$$

in this special class of maps. The desired form of the solution u restricts the class of admissible initial data to homogeneous maps

$$u_0(x) = u_0\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^d \setminus \{0\}.$$

We first consider the model case of corotational maps from \mathbb{R}^d into the unit sphere S^d . We introduce non-standard coordinates on the target S^d by $(s, \omega) \in \mathbb{R} \times S^{d-1}$ (periodic in s) where s represents the length of a geodesic connecting a given point to the north pole while ω denotes the angular coordinate of \mathbb{R}^d . We consider corotational maps from \mathbb{R}^d into S^d , i.e. maps of the form

$$v(x) = (h(|x|), \frac{x}{|x|}) \in \mathbb{R} \times S^{d-1}.$$

The only maps that are homogeneous and corotational are those induced by constant functions $h \equiv s$, $s \in \mathbb{R}$. Each admissible initial map u_0 thus projects the points of \mathbb{R}^d onto a fixed lateral sphere $C_s = \{(s, \omega) : \omega \in S^{d-1}\}$. Of special interest for us is the *equator map* $x \mapsto (\pi/2, \frac{x}{|x|})$ which is a homogeneous, weakly harmonic map and thus a time-independent selfsimilar weak solution of the harmonic map heat flow.

We will see that the properties of this equator map alone determine whether selfsimilar solutions to (3.1), (3.2) are unique. In fact, we prove that selfsimilar solutions of (3.1) are unique for all admissible initial data (3.2) if and only if the equator map minimises the energy for an appropriate class of functions. For the

model case of corotational maps from \mathbb{R}^d into the sphere, this is the case in dimensions $d \geq 7$.

As a corollary of the main results of this chapter, theorem 3.3 and theorem 3.5, we obtain

THEOREM 3.1. *Let $d \geq 3$ and let $N = S^d$ be the d -dimensional unit sphere on which we introduce rotationally symmetric coordinates as specified above.*

- (i) *Let $d \geq 7$. Then the initial value problem (3.1), (3.2) has a unique self-similar and corotational weak solution for any initial data $u_0(x) = (s, \frac{x}{|x|})$, $s \in \mathbb{R}$.*
- (ii) *Let $3 \leq d \leq 6$. Then there exists a selfsimilar and corotational weak solution of the harmonic map heat flow for each initial data $u_0(x) = (s, \frac{x}{|x|})$, $s \in \mathbb{R}$, which is, however, in general not unique. In fact, given any $K \in \mathbb{N}$ there exists a number $\varepsilon_K > 0$ such that the initial value problem (3.1), (3.2) for initial data $u_0(x) = (s, \frac{x}{|x|})$, $s \in [\pi/2 - \varepsilon_K, \pi/2 + \varepsilon_K]$, has at least K different solutions which are selfsimilar and corotational.*

This theorem in particular implies the non-uniqueness result for corotational, selfsimilar solutions of (3.1) into spheres in dimensions $3 \leq d \leq 6$ of Angenent, Ilmanen and Velazquez [31].

A different class of selfsimilar solutions has been introduced in the same work [31]. The properties of these incoming selfsimilar solutions $u(x, t) = v(\frac{x}{\sqrt{-t}})$, $t < 0$, are vastly different from those of the (outgoing) selfsimilar solutions we consider here. The existence of this second type of selfsimilar solutions is connected with the question of regularity and the formation of singularities of the first kind rather than the issue of uniqueness of weak solutions. For the model case in dimension $3 \leq d \leq 6$ the existence of a countable family (u_n) of incoming selfsimilar solutions of the harmonic map flow was shown in [31] and [20], see also [25]. The weak limit of these solutions at the singular time $t = 0$ is of the form $u_n(x, 0) = (s_n, \frac{x}{|x|})$, for numbers $s_n \in \mathbb{R}$ that converge to the coordinate $\frac{\pi}{2}$ of the equator as $n \rightarrow \infty$. Theorem 3.1 thus implies that the selfsimilar continuation of the solutions u_n after the singularity is non-unique at least for n large enough.

The presented results for outgoing selfsimilar solutions are not restricted to the case of corotational maps into spheres but extend to a greater class of rotationally symmetric target manifolds and a larger class of equivariant functions.

1.1. Geometric setting. We consider maps from a fixed Euclidean space \mathbb{R}^d , $d \geq 3$, into a smooth rotationally symmetric target manifold N^n without boundary. We introduce coordinates $(s, \omega) \in \mathbb{R} \times S^{n-1}$ on N in which the metric is given by

$$ds^2 + g^2(s)d\omega_{n-1}^2.$$

Here $d\omega_{n-1}^2$ denotes the standard metric of the sphere S^{n-1} and g shall be a smooth function, symmetric with respect to each point p where $g(p) = 0$. For these special values p of the lateral coordinate which represent the poles of N , it is necessary to

assume that $|g'(p)| = 1$ in order to obtain a smooth manifold. The coordinate s and the function g are of course periodic if N is compact.

Observe that the (intrinsic) diameter of the *lateral sphere*

$$C_s := \{(s, \omega) : \omega \in S^{d-1}\}, \quad s \in \mathbb{R}$$

is equal to $\pi |g(s)|$. In analogy to the model case, we therefore call C_{s^*} an *equator* of N if s^* is a local maximum of g^2 . Similarly, we call a lateral sphere whose diameter is locally minimal but positive a *minimal sphere*.

We consider for the moment both compact and non-compact target manifolds N , but we want to assume throughout this work that

$$(3.3) \quad \sup_{s \in \mathbb{R}} (|g(s)| + |g'(s)| + |g''(s)|) < \infty$$

for the function g representing the metric of N . For simplicity, we also exclude targets for which g' has roots with multiplicity greater than one or for which the function $s \mapsto \frac{d^2}{ds^2}(g^2)(s)$ is constant on an interval of positive length.

We consider maps from \mathbb{R}^d to N with the following type of symmetry.

DEFINITION 1. Let $d, m \in \mathbb{N}$.

- (i) We call a map $\chi : S^{d-1} \rightarrow S^{n-1}$ a (k) -*eigenmap*, if χ is an eigenmap of the negative laplacian $-\Delta_{S^{d-1}}$ with constant energy density

$$|\nabla \chi|^2 = k.$$

- (ii) Let N^n be a rotationally symmetric manifold and let $\chi : S^{d-1} \rightarrow S^{n-1}$ be an eigenmap. We say that a map $u : \mathbb{R}^d \rightarrow N^n$ is χ -*equivariant* if there exists a function $h : [0, \infty) \rightarrow \mathbb{R}$ such that

$$u(x) = R_\chi h(x) := (h(|x|), \chi(\frac{x}{|x|}))$$

with respect to the rotationally symmetric coordinates introduced on N .

Let us remark that the spectrum of the negative laplacian on the sphere S^{d-1}

$$\{l(d-2+l) : l \in \mathbb{N}\}$$

contains no eigenvalues smaller than $d-1$. An eigenmap to eigenvalue $d-1$ is given by the identity $id : S^{d-1} \rightarrow S^{d-1}$. The corresponding equivariant maps are the corotational maps $x \mapsto (h(|x|), \frac{x}{|x|})$ introduced above. The components of general eigenmaps to eigenvalue $\lambda_l = l(d-2+l)$ are given by the restriction of l -homogeneous, harmonic polynomials to the sphere, see [17], chapter VIII.

Given any equator C_{s^*} of N and any eigenfunction χ as above, we define the corresponding *equator map* by

$$u^* = u_{\chi, s^*}^* := R_\chi h^*$$

for the constant function $h^* \equiv s^*$. Note that this equator map and its properties depend both on the eigenmap χ and on the value of s^* , compare remark 3.6 below.

DEFINITION 2. Let C_{s^*} be an equator of a rotationally symmetric manifold N and let χ be an eigenmap. We say that the equator map u_{χ, s^*}^* is χ -energy-minimising if it minimises the Dirichlet energy

$$E(u, B_1(0)) = \frac{1}{2} \int_{B_1(0)} |\nabla u|^2 dx$$

in the set

$$\mathcal{F}_{\chi, s^*} := \{R_\chi h : h : [0, 1] \rightarrow \mathbb{R} \text{ with } h(1) = s^*\}$$

of χ -equivariant functions with the same boundary data.

Let us remark that we do not demand that the equator map u_{χ, s^*}^* is energy-minimising in the larger class of maps

$$F = \{u \in H^1(B_1, N) : u|_{\partial B_1} = u_{\chi, s^*}^*|_{\partial B_1}\}.$$

It is an open question if there exist rotationally symmetric manifolds N and eigenmaps χ for which

$$\inf_{v \in F} E(v, B_1) < \inf_{v \in \mathcal{F}_{\chi, s^*}} E(v, B_1).$$

For certain manifolds such as spheres and ellipsoids such a breaking of symmetries can be excluded, see e.g. [4].

The following non-uniqueness result holds true for generic settings (N, χ) with an equator map that is not χ -energy-minimising.

THEOREM 3.2. *Let $d \geq 3$, let N^n be a rotationally symmetric manifold such that (3.3) is satisfied and let $\chi : S^{d-1} \rightarrow S^{n-1}$ be a fixed eigenmap. Assume that N has an equator map u_{χ, s^*}^* that is not χ -energy-minimising.*

Then there exists a selfsimilar and χ -equivariant weak solution $u \in H_{loc}^1(\mathbb{R}^d \times [0, \infty))$ of the initial value problem (3.1), (3.2) for initial data $u_0 = u_{\chi, s^}^*$ which is not constant in time.*

Conversely, we prove the unique solvability of (3.1), (3.2) in the class of selfsimilar, equivariant weak solutions under the main assumption that all equator maps of N are χ -energy-minimising.

We impose the following mild restrictions on the function g representing the metric of N .

CONDITION (C1). *Let C_{s^*} be an equator of a compact, rotationally symmetric manifold N and let $s_1 < s^* < s_2$ be the local minima of g^2 to the left and to the right of s^* , i.e. the local minima of g^2 such that $g^2|_{[s_1, s^*]}$ is increasing while $g^2|_{[s^*, s_2]}$ is decreasing.*

We then demand that

$$G'(s^*) = \min_{s \in [s_1, s_2]} G'(s)$$

for $G(s) := g'(s)g(s) = \frac{1}{2} \frac{d}{ds}(g^2)(s)$.

For manifolds that contain a minimal sphere C_{s_0} we furthermore impose

CONDITION (C2). *Let k be any given eigenvalue of $-\Delta_{S^{d-1}}$. We say that a rotationally symmetric manifold N fulfils condition (C2) (for k) if for each minimal sphere C_{s_0} of N at least one of the following conditions holds*

- (i) $G'(s_0) \geq \frac{d-1}{k}$
- (ii) s_0 is a (positive) local maximum of G' .

Conditions (C1) and (C2) are fulfilled for a wide variety of rotationally symmetric manifolds, in particular for round spheres and for rotationally symmetric ellipsoids.

Our main uniqueness result for selfsimilar solutions of the harmonic map heat flow is given by

THEOREM 3.3. *Let $d \geq 3$ and let N^n be a compact, rotationally symmetric manifold satisfying condition (C1) and (C2) for an eigenvalue $k \in \mathbb{N}$ of $-\Delta_{S^{d-1}}$. Let $\chi : S^{d-1} \rightarrow S^{n-1}$ be a k -eigenmap and assume that the equator maps u_{χ, s^*}^* are χ -energy-minimising for every equator C_{s^*} of N .*

Then there is a unique χ -equivariant and selfsimilar weak solution $u \in H_{loc}^1(\mathbb{R}^d \times \mathbb{R}_0^+)$ to the initial value problem (3.1), (3.2) for each initial data $u_0(x) = (s, \chi(\frac{x}{|x|}))$, $s \in \mathbb{R}$.

The results of theorem 3.2 and theorem 3.3 demonstrate that the issue of uniqueness for selfsimilar solutions of (3.1) is determined by the properties of the equator maps. It is therefore of interest to have a simple criterion to test whether or not a given equator map is energy-minimising.

Extending a result from [27] we show

PROPOSITION 3.4. *Let $d \geq 3$, let N^n be a smooth, rotationally symmetric manifold and let $\chi : S^{d-1} \rightarrow S^{n-1}$ be a k -eigenmap. Let C_{s^*} be an equator of N and let $G := g \cdot g'$.*

- (i) *If*

$$(3.4) \quad -4kG'(s^*) > (d-2)^2.$$

then the equator map u_{χ, s^}^* is not χ -energy-minimising.*

- (ii) *Suppose that*

$$-4kG'(s) \leq (d-2)^2 \text{ for } s \in [s^* - S, s^* + S]$$

where $S := \frac{2\sqrt{k}}{d-2} \cdot \|g\|_\infty$. Then u_{χ, s^}^* is χ -energy-minimising.*

An equator map u_{χ, s^*} satisfying condition (3.4) is not even locally χ -energy-minimising and thus certainly not χ -energy-minimising in the sense of definition 2, see [27]. For settings (N, χ) with an equator map satisfying condition (3.4) the following theorem gives a much stronger non-uniqueness statement than theorem 3.2.

THEOREM 3.5. *Let $d \geq 3$, let N be a compact, rotationally symmetric manifold, let χ be an eigenmap and let C_{s^*} be an equator of N . Assume that $-4kG'(s^*) > (d-2)^2$.*

Then given any number $K \in \mathbb{N}$, there exists a neighbourhood U_K of s^ such that the initial value problem (3.1), (3.2) has at least K different weak solutions that are χ -equivariant and selfsimilar for each initial data $u_0(x) = (s, \frac{x}{|x|})$ with $s \in U_K$.*

Theorem 3.5 implies the existence of a large number of solutions to the harmonic map heat flow to the same initial data which are genuinely different; they do not result by time-translations of only one non-trivial solution of (3.1) like the examples of Coron [12] and of Hong [30]. In addition, theorems 3.2 and 3.5 yield examples of non-uniqueness for the harmonic map flow that respect the monotonicity formula of Struwe [51] extending the work of Coron [12] and Hong [30], see section 9.

REMARK 3.6. *Since the metric of the unit sphere S^n is described by the function $g(r) = \sin(r)$ in rotationally symmetric coordinates, the χ -equivariant equator map, χ a k -eigenmap, is energy minimising if and only if*

$$4k \leq (d-2)^2,$$

see proposition 3.4. In the corotational case theorem 3.3 thus implies uniqueness for selfsimilar weak solutions in dimensions $d \geq 7$.

On the other hand, also for $d \geq 7$ we can choose an eigenmap χ to an eigenvalue $k > \frac{(d-2)^2}{4}$. Theorem 3.5 thus gives a non-uniqueness statement for selfsimilar, χ -equivariant solutions of (3.1) for the harmonic map flow into the sphere S^d .

This example illustrates that the issue of uniqueness for selfsimilar solutions of the harmonic map flow depends not only on the target manifold and the dimension of the domain but also on the considered equivariance class of maps.

The present chapter is organised as follows. We begin with a short discussion of equation (3.1) in the equivariant setting. We then give a proof of proposition 3.4. In section 4 we establish the first non-uniqueness result, theorem 3.2, using variational methods. We analyse the differential equation corresponding to the problem of equivariant, selfsimilar weak solutions of the harmonic map flow in section 5. A key tool for the proof of theorem 3.3 is the comparison principle presented in proposition 3.19 which is applicable only if the equator maps of the setting (N, χ) are χ -energy-minimising. Another important step for the proof of theorem 3.3 is given by the classification of functions which induce selfsimilar weak solutions to (3.1) stated in proposition 3.22. Section 8 deals with the proof of the second non-uniqueness result, theorem 3.5. The monotonicity properties of selfsimilar solutions of the harmonic map heat flow are discussed in section 9. We conclude with examples of manifolds for which the phenomena of uniqueness and of non-uniqueness described in theorem 3.3 and theorem 3.5 can be observed simultaneously in the corotational setting.

The results of this chapter, in particular sections 3, 4 and 9, are based on joint work with Pierre Germain which will appear in [28].

2. Weak solutions of the harmonic map flow in the equivariant setting

Let $d \geq 3$ be any fixed natural number. Let N^n be a rotationally symmetric manifold, let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the function describing the metric of N and let $\chi : S^{d-1} \rightarrow S^{n-1}$ be an eigenmap to eigenvalue $k \in \mathbb{N}$. A short calculation shows that the Dirichlet energy of an equivariant map $v = R_\chi h$ is given by

$$E(v, B_R(0)) = \frac{1}{2} \int_{B_R(0)} |\nabla v|^2 dx = \frac{c_d}{2} \int_0^R [|h'|^2 + \frac{k}{r^2} g^2(h)] r^{d-1} dr$$

for $c_d = |S^{d-1}|$, the Hausdorff-measure of the $d - 1$ dimensional unit sphere.

In view of condition (3.3) the set of functions h that induce equivariant maps with locally finite energy can be described by

DEFINITION 3. Given $d \in \mathbb{N}$ and a ball $B_R = B_R(0) \subset \mathbb{R}^d$ we define

$$H_{rad}^1(B_R) := \{h : [0, R] \rightarrow \mathbb{R} : \int_0^R |h'|^2 r^{d-1} dr < \infty\},$$

and set

$$H_{rad}^1(\mathbb{R}^d) := \bigcap_{R>0} H_{rad}^1(B_R).$$

Observe that the equivariant function $R_\chi h : \mathbb{R}^d \rightarrow N$ is an element of $H_{loc}^1(\mathbb{R}^d)$ but not necessarily of $H^1(\mathbb{R}^d)$ if $h \in H_{rad}^1(\mathbb{R}^d)$. Let us also remark that the global energies $E(u(t), \mathbb{R}^d)$ of solutions of the harmonic map heat flow (3.1) are in general infinite.

Direct computations (see e.g. [27]) lead to the following characterisation of equivariant weak solutions of the harmonic map heat flow.

LEMMA 3.7. Consider a rotationally symmetric manifold N^n with metric described by $g \in C^1(\mathbb{R})$ and let $\chi : S^{d-1} \rightarrow S^{n-1}$ be a k -eigenmap.

- (i) Let $u \in H_{loc}^1(\mathbb{R}^d \times \mathbb{R}_0^+)$ be of the form $u = R_\chi h$ for a function $h : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}$. Then u is a weak solution of (3.1) if and only if h solves the scalar partial differential equation

$$(3.5) \quad h_t - [h_{rr} + \frac{d-1}{r} h_r - \frac{k}{r^2} g(h)g'(h)] = 0 \text{ on } \mathbb{R}_0^+ \times \mathbb{R}_0^+$$

in the sense of distributions.

- (ii) Let $u \in H_{loc}^1(\mathbb{R}^d \times \mathbb{R}_0^+)$ be of the form $u(x, t) = R_\chi h(\frac{x}{\sqrt{t}})$, $t > 0$, for some $h : \mathbb{R}_0^+ \rightarrow \mathbb{R}$.

Then u is a weak solution of (3.1) if and only if h solves the differential equation

$$(3.6) \quad h'' + \left(\frac{d-1}{r} + \frac{r}{2}\right)h' - \frac{k}{r^2}g(h)g'(h) = 0 \text{ on } (0, \infty).$$

Remark that we can rewrite equation (3.6) in divergence-form as

$$(3.7) \quad \frac{d}{dr} \left(r^{d-1} e^{r^2/4} h'(r) \right) = k r^{d-3} e^{r^2/4} g(h)g'(h).$$

Note that lemma 3.7 establishes the equivalence of (3.5) and (3.1), respectively for selfsimilar maps of (3.6) and (3.1), only under the assumption that $u \in H_{loc}^1(\mathbb{R}^d \times \mathbb{R}_0^+)$. We thus need to investigate what further conditions need to be satisfied by a solution of (3.6) such that the resulting map is in H_{loc}^1 .

For any selfsimilar map $u(x, t) = R_\chi h(\frac{x}{\sqrt{t}})$ and any $R > 0, t > 0$

$$\begin{aligned} E(u(t), B_R(0)) &= \frac{1}{2}(\sqrt{t})^{d-2} \int_{B_{R/\sqrt{t}}(0)} |\nabla R_\chi h|^2 dx \\ &= \frac{c_d}{2}(\sqrt{t})^{d-2} \int_0^{R/\sqrt{t}} [|h'|^2 + \frac{k}{r^2}g^2(h)]r^{d-1} dr \end{aligned}$$

and thus $\nabla u \in L^2(B_R(0) \times [0, T])$ if and only if

$$\int_0^T (\sqrt{t})^{d-2} \int_0^{R/\sqrt{t}} [|h'|^2 + \frac{k}{r^2}g^2(h)]r^{d-1} dr dt < \infty.$$

Using (3.3) and $d \geq 3$, we can split the above condition into the restrictions

$$(3.8) \quad \int_0^R |h'|^2 r^{d-1} dr < \infty \text{ for all } R > 0, \quad \text{i.e. } h \in H_{rad}^1(\mathbb{R}^d)$$

and

$$(3.9) \quad \int_0^1 (\sqrt{t})^{d-2} \int_1^{R/\sqrt{t}} |h'|^2 r^{d-1} dr dt < \infty.$$

On the other hand, $\partial_t u \in L^2(B_R(0) \times [0, T])$ if and only if

$$(3.10) \quad \int_0^R |h'|^2 r^{d+1} dr < \infty \text{ for all } R > 0$$

and

$$(3.11) \quad \int_0^1 (\sqrt{t})^{d-4} \int_1^{R/\sqrt{t}} |h'|^2 r^{d+1} dr dt < \infty.$$

Obviously condition (3.10) holds true for all elements of $H_{rad}^1(\mathbb{R}^d)$ whereas (3.11) imposes a stronger restriction on h than (3.9).

At first glance the assumption $h \in H_{rad}^1(\mathbb{R}^d)$ imposes only a mild constraint on the behaviour of h near $r = 0$ while the condition (3.11) seems to seriously restrict the allowed behaviour at infinity. We will see later that the converse is true for solutions of equation (3.6). Indeed, the first derivative of each solution of (3.6) decays sufficiently fast for (3.11) to be fulfilled, but most solutions of (3.6) blow up as $r \rightarrow 0$ in such a way that (3.8) is violated.

Let us finally remark that the L^2 -trace of a selfsimilar map $u(x, t) = R_\chi h(\frac{x}{\sqrt{t}})$ on the time slice $\mathbb{R}^d \times \{0\}$ is given by $u(x, 0) = (s, \chi(\frac{x}{|x|}))$ if h converges to $s \in \mathbb{R}$ as $r \rightarrow \infty$.

3. Classification of settings with energy-minimising equator maps

As remarked in [27], the criterion given in proposition 3.4 is closely connected with the value of the optimal constant in the following well known Hardy-inequality.

LEMMA 3.8. *Let $d \geq 3$. Then $C_H = \frac{4}{(d-2)^2}$ is the optimal constant such that the Hardy-inequality*

$$(3.12) \quad \int_0^1 w^2 r^{d-3} dr \leq C_H \int_0^1 |w'|^2 r^{d-1} dr$$

holds true for all $w \in H_{rad}^1(B_1)$ with $w(1) = 0$.

PROOF OF LEMMA 3.8. We remark first of all that (3.12) holds true with $C_H = \frac{4}{(d-2)^2}$ for the dense set of bounded maps $w \in H_{rad}^1(\mathbb{R}^d) \cap C^1((0, 1])$ and thus for all $w \in H_{rad}^1(B_1)$. Indeed, by partial integration and Hölder's inequality, we obtain

$$\begin{aligned} (d-2) \int_0^1 w^2 r^{d-3} dr &= \int_0^1 w^2 \cdot (r^{d-2})' dr = -2 \int_0^1 (r^{-1}w) \cdot w' r^{d-1} dr \\ &\leq 2 \left(\int_0^1 w^2 \cdot r^{d-3} dr \right)^{1/2} \left(\int_0^1 |w'|^2 \cdot r^{d-1} dr \right)^{1/2}. \end{aligned}$$

Given any $\varepsilon > 0$, we now define $f_\varepsilon \in H_{rad}^1(B_1)$ by

$$f_\varepsilon(r) = \begin{cases} \varepsilon^{-d/2+1} & \text{for } 0 \leq r \leq \varepsilon \\ r^{-d/2+1} & \text{for } \varepsilon < r \leq 1/2 \\ 2^{d/2}(1-r) & \text{for } 1/2 < r \leq 1. \end{cases}$$

We find that

$$\int_0^1 |f_\varepsilon|^2 r^{d-3} dr \geq \int_\varepsilon^{1/2} r^{-1} dr = -\log(\varepsilon) - \log(2)$$

while

$$\int_0^1 |f'_\varepsilon|^2 r^{d-1} dr = \frac{(d-2)^2}{4} \int_\varepsilon^{1/2} r^{-1} dr + \int_{1/2}^1 2^d r^{d-1} dr = -\frac{(d-2)^2}{4} \log(\varepsilon) + C$$

for a constant C independent of $\varepsilon > 0$. Given any factor $\lambda > 1$, we may therefore choose $\varepsilon = \varepsilon(\lambda) > 0$ so small that

$$\lambda \cdot \frac{(d-2)^2}{4} \int |f_\varepsilon|^2 r^{d-3} dr > \int_0^1 |f'_\varepsilon|^2 r^{d-1} dr.$$

This implies that $C_H = \frac{4}{(d-2)^2}$ is indeed the optimal Hardy-constant. \square

PROOF OF PROPOSITION 3.4. Let N be rotationally symmetric, let χ be a k -eigenmap and let C_{s^*} be an equator of N . For the proof of statement (i) we follow the ideas of [27]. Suppose there exists $\lambda > 1$ with

$$k\theta C_H > \lambda^2, \quad \text{for } \theta := -G'(s^*)$$

where C_H is the optimal Hardy-constant of lemma 3.8. We need to show that there exists a function h with $h(1) = s^*$ for which

$$\int_0^1 \left[|h'|^2 + \frac{k}{r^2} (g^2(h) - g^2(s^*)) \right] r^{d-1} dr < 0.$$

We let $f = f_{\varepsilon(\lambda)}$ be the bounded function from the proof of lemma 3.8 for which $f(1) = 0$ and

$$\lambda \cdot \int_0^1 |f|^2 r^{d-3} dr > C_H \int_0^1 |f'|^2 r^{d-1} dr.$$

In addition, we choose $\delta > 0$ such that

$$\lambda \cdot G'(s) \leq G'(s^*) = -\theta \text{ for all } s \in [s^* - \delta, s^* + \delta].$$

Applying lemma 3.8 we find that for $h := s^* - \delta \frac{f}{\|f\|_\infty}$

$$\begin{aligned} \delta^{-2} \|f\|_\infty^2 \int_0^1 \left[|h'|^2 + \frac{k}{r^2} (g^2(h) - g^2(s^*)) \right] r^{d-1} dr &\leq \int_0^1 \left[|f'|^2 - \frac{k\theta}{\lambda r^2} f^2 \right] r^{d-1} dr \\ &\leq \left(1 - \frac{k\theta C_H}{\lambda^2} \right) \int_0^1 |f'|^2 r^{d-1} dr \\ &< 0. \end{aligned}$$

The equator map u_{χ, s^*}^* is therefore not χ -energy-minimising as claimed.

For the proof of the second statement, we first remark that $E(R_\chi h, B_1) = \infty$ if $h \notin H_{rad}^1(B_1)$. We know by assumption that

$$kG'(s) \geq -C_H^{-1} \text{ for all } s \in [s^* - S, s^* + S]$$

where $S := \sqrt{kC_H} \cdot \|g\|_\infty = \frac{2\sqrt{k}}{d-2} \cdot \|g\|_\infty$. Using the quadratic Taylor expansion of g^2 around s^* , we find for these values of s

$$k[g^2(s) - g^2(s^*)] = kG'(\xi) \cdot (s - s^*)^2 \geq -C_H^{-1}(s - s^*)^2$$

where ξ is an appropriate function taking values between s^* and s . This estimate is trivially true if $|s - s^*| \geq S$.

The Hardy-inequality of lemma 3.8 now implies that for all $h \in H_{rad}^1(B_1)$ with $h(1) = s^*$

$$\begin{aligned} \int_0^1 \left[|h'|^2 + \frac{k}{r^2} (g^2(h) - g^2(s^*)) \right] r^{d-1} dr &\geq \int_0^1 \left[|(h - s^*)'|^2 - \frac{C_H^{-1}}{r^2} (h - s^*)^2 \right] r^{d-1} dr \\ &\geq 0. \end{aligned}$$

This completes the proof of proposition 3.4. \square

4. Proof of theorem 3.2

This section covers the proof of the first non-uniqueness result, theorem 3.2. Contrary to the arguments used for the proofs of theorem 3.3 and theorem 3.5, we do not require any restrictions on the manifold N other than the general assumption (3.3). Theorem 3.2 is thus valid also for a large class of non-compact rotationally symmetric target manifolds.

So let N be a rotationally symmetric manifold, χ an eigenmap and let C_{s^*} be an equator of N . Assume that the equator map u_{χ, s^*}^* is not χ -energy-minimising. According to the discussion in section 2 we need to establish the existence of a non-constant solution $h \in H_{rad}^1(\mathbb{R}^d)$ to equation (3.6) with $\lim_{r \rightarrow \infty} h(r) = s^*$ which satisfies condition (3.11).

We consider the set

$$\mathcal{F} := \{f \in H_{rad}^1(\mathbb{R}^d) : \text{supp}(f) \subset\subset [0, \infty)\}$$

and take its closure $\overline{\mathcal{F}}$ with respect to the norm

$$\|f\|^2 := \int (|f'|^2 + \frac{|f|^2}{r^2}) r^{d-1} e^{r^2/4} dr.$$

Let us remark that condition (3.11) is trivially fulfilled for elements of $\overline{\mathcal{F}}$. Furthermore, functions in $\overline{\mathcal{F}}$ converge to zero as $r \rightarrow \infty$; indeed, given any $\varepsilon > 0$ we let $R > 1$ be such that $\|f\| e^{-R^2/8} < \varepsilon$. We may then choose $r_0 \in [R, R+1]$ with $|f(r_0)| \leq \varepsilon$ and obtain that for $r > R+1$

$$|f(r)| \leq \varepsilon + \int_{r_0}^r |f'| dr \leq \varepsilon + \|f\| \cdot e^{-R^2/8} \left(\int_1^\infty r^{1-d} dr \right) \leq 2\varepsilon.$$

In view of the divergence form (3.7) of equation (3.6) we formally introduce the functional

$$\mathcal{E}(f) = \int_0^\infty \left[|f'|^2 + \frac{k}{r^2} g^2(s^* + f) \right] r^{d-1} e^{r^2/4} dr$$

representing the global energy of the map $R_\chi(s^* + f)$ with respect to the measure $e^{|x|^2/4} dx$ on \mathbb{R}^d . Since $\mathcal{E}(f) = \infty$ for each function $f \in \overline{\mathcal{F}}$, we renormalize \mathcal{E} and consider the variational integral

$$\overline{E}(f) := \int_0^\infty \left[|f'|^2 + \frac{k}{r^2} (g^2(s^* + f) - g^2(s^*)) \right] r^{d-1} e^{r^2/4} dr$$

on the reflexive space $(\overline{\mathcal{F}}, \|\cdot\|)$. We prove

Claim 1. If the equator map $u_{s^*, \chi}^*$ is not χ -energy-minimising, then

$$\inf_{f \in \overline{\mathcal{F}}} \overline{E}(f) < 0 = \overline{E}(0).$$

Claim 2. The functional \overline{E} is weakly lower semi-continuous and bounded from below on $(\overline{\mathcal{F}}, \|\cdot\|)$.

These claims imply that the global minimum of \bar{E} is achieved for a function $f \in \bar{\mathcal{F}}$ that is not identically zero. Consequently $s^* + f$ is a non-constant solution of (3.6). By the discussion of section 2 and the above remarks, we thus find that $u(x, t) = R_\chi(s^* + f)(\frac{x}{\sqrt{t}})$ is solution of (3.1), (3.2) for initial data $u_0 = u_{\chi, s^*}^*$ which is not constant in time. This establishes our first non-uniqueness result, theorem 3.2.

PROOF OF CLAIM 1. We define a family of weighted energies $(E_\lambda)_{\lambda \in [0, 1]}$ on the space $\bar{\mathcal{F}}$ by

$$E_\lambda(f) := \int_0^\infty [|f'|^2 + \frac{k}{r^2} (g^2(s^* + f) - g^2(s^*))] r^{d-1} e^{\lambda r^2/4} dr.$$

Note the scaling

$$E_\lambda(h) = \lambda^{-\frac{d-2}{2}} \cdot E_1(h(\frac{\cdot}{\sqrt{\lambda}})) = \lambda^{-\frac{d-2}{2}} \cdot \bar{E}(h(\frac{\cdot}{\sqrt{\lambda}})).$$

Since the equator map $u^* = u_{s^*, \chi}^*$ is by assumption not energy-minimising, there exists a function $h \in H_{rad}^1(B_1)$ with $h(1) = 0$ and

$$E(R_\chi(s^* + h), B_1) - E(u^*, B_1) = \frac{c_d}{2} \int_0^1 [|h'|^2 + \frac{k}{r^2} (g^2(s^* + h) - g^2(s^*))] r^{d-1} dr < 0.$$

Extending h by zero on $[1, \infty)$, we thus obtain that $E_0(h) < 0$ and by continuity of $\lambda \mapsto E_\lambda(h)$ also $E_\lambda(h) < 0$ for $\lambda > 0$ small. Consequently

$$\inf_{f \in \bar{\mathcal{F}}} \bar{E}(f) = \lambda^{\frac{d-2}{2}} \inf_{f \in \bar{\mathcal{F}}} E_\lambda(h) < 0$$

as claimed. □

For the proof of the second claim we use

LEMMA 3.9. *There exists a constant $C_1 > 0$ such that the following estimate holds true for all $f \in \bar{\mathcal{F}}$*

$$\int_0^\infty f^2 (1 + r^2) r^{d-3} e^{r^2/4} dr \leq C_1 \int_0^\infty |f'|^2 r^{d-1} e^{r^2/4} dr.$$

PROOF. For $f \in \mathcal{F}$ we find

$$\begin{aligned} (d-2) \int_0^\infty |f|^2 r^{d-3} e^{r^2/4} dr &= \int_0^\infty \frac{d}{dr} (r^{d-2}) |f|^2 e^{r^2/4} dr \\ &= -\frac{1}{2} \int_0^\infty r^{d-1} |f|^2 e^{r^2/4} dr - 2 \int f \cdot f' r^{d-2} e^{r^2/4} dr \end{aligned}$$

and the estimate of lemma 3.9 follows by Hölder's and Young's inequality.

For general elements of $\bar{\mathcal{F}}$ the claim follows by approximation and Fatou's lemma since convergence in the sense of $\|\cdot\|$ in particular implies pointwise convergence almost everywhere. □

PROOF OF CLAIM 2. We can estimate the non-linear term of \overline{E} by

$$g^2(s^* + f) - g^2(s^*) \geq -\min(\|G'\|_\infty f^2, \|g\|_\infty^2).$$

For any given $R > 0$ and any $f \in \overline{\mathcal{F}}$, we thus obtain

$$\begin{aligned} \overline{E}(f) &= \int_0^\infty \left[|f'|^2 + \frac{k}{r^2} (g^2(s^* + f) - g^2(s^*)) \right] r^{d-1} e^{r^2/4} dr \\ &\geq \int_0^\infty |f'|^2 r^{d-1} e^{r^2/4} dr - k \|G'\|_\infty \int_R^\infty f^2 r^{d-3} e^{r^2/4} dr \\ &\quad - k \|g\|_\infty^2 \int_0^R r^{d-3} e^{r^2/4} dr \\ &\geq \int_0^\infty |f'|^2 r^{d-1} e^{r^2/4} dr - CR^{-2} \int_R^\infty f^2 r^{d-1} e^{r^2/4} dr - C(R) \end{aligned}$$

for a constant $C(R)$ independent of f . Lemma 3.9 thus implies a uniform lower bound for \overline{E} on $\overline{\mathcal{F}}$ if $R > 0$ is chosen large enough.

On the other hand, lemma 3.9 shows that an equivalent norm to $\|\cdot\|$ on $\overline{\mathcal{F}}$ is given by $\|f\|^2 := \int |f'|^2 r^{d-1} e^{r^2/4} dr$. The weak lower semi-continuity of \overline{E} then follows from the estimate

$$\int_R^\infty f^2 r^{d-3} e^{r^2/8} \leq \frac{C}{R^2} \|f\|^2$$

and the dominated convergence theorem applied on finite intervals $[0, R]$. \square

5. General properties of the associated differential equations

We analyse the differential equation (3.6) describing selfsimilar solutions in the equivariant setting.

5.1. Asymptotic behaviour.

The behaviour of arbitrary solutions h of (3.6) for $r \rightarrow \infty$ can be described by

LEMMA 3.10.

- (i) *Let h be any solution of (3.6). Then there exists a constant $C = C(h)$ such that*

$$|h'(r)| \leq \frac{C}{r^3} \text{ for all } r \geq 1.$$

- (ii) *The estimate of (i) holds true with a universal constant $\overline{C} = \overline{C}(g, k)$ for all solutions h of (3.6) with $\lim_{r \rightarrow 0} r \cdot h'(r) = 0$.*

PROOF OF LEMMA 3.10. The quantity

$$(3.13) \quad V(r) = V(h)(r) := r^2 |h'(r)|^2 - kg^2(h(r))$$

is decreasing for any non-constant solution h of (3.6) with

$$(3.14) \quad V'(r) = 2h'(r)r^2 \cdot [h''(r) + \frac{h'(r)}{r} - \frac{k}{r^2}G(h)] = -r^2 |h'(r)|^2 \left[\frac{2(d-2)}{r} + r \right].$$

The possible behaviour of $F(r) := V(r) + kg^2(h(r)) = r^2 |h'(r)|^2$ is thus constrained by

$$F'(r) + rF(r) \leq 2kG(h)h'(r) \leq Cr^{-1}(F(r))^{1/2} \leq \frac{r}{2}F(r) + \frac{C_1}{r^3}$$

for a constant $C_1 = C_1(\|G\|_\infty, k)$ independent of h . Consequently

$$(e^{r^2/4}F(r))' \leq \frac{C_1}{r^3}e^{r^2/4}$$

and thus for $r \geq 1$

$$F(r) \leq e^{-r^2/4}(e^{1/4}F(1) + C_1 \int_1^r e^{s^2/4}s^{-3} ds).$$

We estimate

$$\begin{aligned} I &= \int_1^r e^{s^2/4}s^{-3} ds = \int_1^r (e^{s^2/4})' \cdot 2s^{-4} ds \\ &\leq \frac{2}{r^4}e^{r^2/4} + 8 \int_1^r e^{s^2/4}s^{-5} ds \leq \frac{2}{r^4}e^{r^2/4} + \frac{1}{2} \int_4^r e^{s^2/4}s^{-3} ds + C \end{aligned}$$

and thus find that $I \leq \frac{4}{r^4}e^{r^2/4} + C$. The resulting estimate

$$(3.15) \quad F(r) \leq (e^{1/4}F(1) + C) \cdot e^{-r^2/4} + \frac{C}{r^4}$$

implies statement (i) since h is smooth on $(0, \infty)$ and thus $F(1) < \infty$.

Finally, let h be a solution of (3.6) with $\lim_{r \rightarrow 0} rh'(r) = 0$. Since V is non-increasing with $\lim_{r \rightarrow 0} V(r) \leq 0$, we find that $F(1) = V(1) + kg^2(h(1)) \leq k \|g\|_\infty^2$. The uniform bound claimed in statement (ii) thus follows from (3.15). \square

REMARK 3.11. *An important consequence of lemma 3.10 is that each solution h of (3.6) converges as $r \rightarrow \infty$ in such a way that condition (3.11) is satisfied. In order to find selfsimilar solutions of the harmonic map heat flow we can therefore concentrate on finding solutions of (3.6) that are elements of $H_{rad}^1(\mathbb{R}^d)$.*

REMARK 3.12. *The fact that $V(h)$ is non-increasing implies that non-constant solutions of (3.6) cannot be tangential to the horizontal line $s = s^*$ for finite values of $r > 0$ if C_{s^*} is an equator of N .*

Furthermore, if $h \in C([0, \infty))$ is a non-constant solution of (3.6) for which $\lim_{r \rightarrow 0} rh'(r) = 0$ then $V(h) < -kg^2(h(0))$ on $(0, \infty)$. Consequently h cannot reach points s with $g^2(s) \leq g^2(h(0))$ for $r > 0$. In addition, the derivative of h satisfies the inequality $r |h'(r)| \leq k \|g\|_\infty$ on $(0, \infty)$.

The above remark applies also for solutions of equation (3.16) considered below.

5.2. Existence of solutions to (3.6) in $H_{rad}^1(\mathbb{R}^d)$.

Let N be a rotationally symmetric manifold whose metric is described by g and let s_0 be an arbitrary local minimum of g^2 . Let $k \in \mathbb{N}$ be an eigenvalue of $-\Delta_{S^{d-1}}$.

We prove the existence of a one-parameter family $(h_a) \subset H_{rad}^1(\mathbb{R}^d)$ of solutions to (3.6) with $h_a(0) = s_0$. If s_0 represents a pole, i.e. if $g(s_0) = 0$, the induced family

of selfsimilar solutions $u(x, t) = R_\chi h_a(\frac{x}{\sqrt{t}})$ to the harmonic map heat flow is smooth on $(\mathbb{R}^d \times [0, \infty)) \setminus \{(0, 0)\}$. On the other hand, solutions of (3.1) constructed in that way remain singular at the origin for all times if C_{s_0} is a minimal sphere.

We first consider the corresponding stationary problem, i.e. the harmonic map equation (0.2) in the equivariant setting, given by

$$(3.16) \quad f'' + \frac{d-1}{r} f' - \frac{k}{r^2} G(f) = 0.$$

LEMMA 3.13. *Let s_0 be a local minimum of g^2 and assume that condition (C2) is fulfilled if C_{s_0} is a minimal sphere. We set*

$$\gamma := \frac{1}{2}(\sqrt{(d-2)^2 + 4kG'(s_0)} - (d-2)) > 0.$$

Then there is a unique solution $\bar{h} \in C^2((0, \infty))$ of the harmonic map equation (3.16) such that

$$\bar{h}(0) = s_0 \text{ and } \lim_{r \rightarrow 0} r^{-\gamma}(\bar{h}(r) - s_0) = 1.$$

Furthermore $\lim_{r \rightarrow 0} \bar{h}'(r)r^{1-\gamma} = \gamma$.

The proof of lemma 3.13 is presented in appendix B.1. Observe that equation (3.16) is invariant under scaling. The unique solution of (3.16) satisfying $h(0) = s_0$ and $r^{-\gamma}(h(r) - s_0) \rightarrow a > 0$ as $r \rightarrow 0$ is therefore given by $r \mapsto \bar{h}(a^{1/\gamma}r)$.

We employ the solutions of the harmonic map equation (3.16) given by lemma 3.13 to derive the existence of solutions to (3.6) for the same type of initial data.

PROPOSITION 3.14. *Let s_0 be a local minimum of g^2 for which condition (C2) is fulfilled and let $a \in \mathbb{R}$. Then there exists a unique solution $h_a \in C^2((0, \infty))$ of equation (3.6) such that*

$$h_a(0) = s_0 \text{ and } \lim_{r \rightarrow 0} r^{-\gamma}(h_a(r) - s_0) = a,$$

where $\gamma = \frac{1}{2}(\sqrt{(d-2)^2 + 4kG'(s_0)} - (d-2))$. Furthermore, $r^{1-\gamma}h'_a(r) \rightarrow \gamma a$ as $r \rightarrow 0$, and $h_a \in H_{rad}^1(\mathbb{R}^d)$.

REMARK 3.15. *The solutions (h_a) of (3.6) constructed in proposition 3.14 induce a one-parameter family of selfsimilar weak solutions of the harmonic map flow. In fact, as we will prove in section 7, the only other solutions of (3.6) which induce selfsimilar weak solutions of (3.1) are the constant functions $h \equiv s^*$, for C_{s^*} an equator of N .*

PROOF OF PROPOSITION 3.14. Let s_0 and γ be as in proposition 3.14 and let a be any given number. If $a = 0$ we let $h_0 \equiv s_0$ be the constant solution of (3.6). So let $a \neq 0$. By symmetry, we may assume that $a > 0$. We associate to each function h_a with $h_a(0) = s_0$ and $\lim_{r \rightarrow 0} r^{-\gamma}(h_a(r) - s_0) = a$ the rescaled map

$$\tilde{h}_a(r) := h_a(a^{-1/\gamma}r).$$

Obviously $\tilde{h}_a(0) = s_0$ and $r^{-\gamma}(\tilde{h}_a(r) - s_0) \rightarrow 1$ as $r \rightarrow 0$. Equation (3.6) can be rewritten in terms of \tilde{h}_a as

$$(3.17) \quad \tilde{h}_a'' + \left(\frac{d-1}{r} + \frac{r}{2a^{2/\gamma}} \right) \tilde{h}_a' - \frac{k}{r^2} G(\tilde{h}_a) = 0.$$

A function \tilde{h}_a with $\tilde{h}_a(0) = s_0$ is a solution of the differential equation (3.17) if and only if $(\tilde{h}_a, \tilde{h}_a')$ is a fixed point of the map

$$\Phi_a(p, q)(r) := \left(s_0 + \int_0^r q dt \quad , \quad r^{1-d} \int_0^r \left[\frac{k}{t^2} G(p) - \frac{t}{2a^{2/\gamma}} q \right] \cdot t^{d-1} dt \right).$$

Comparing equation (3.17) with equation (3.16), we expect that the solution \tilde{h}_a of (3.17) is close to \bar{h} for $r > 0$ small, where \bar{h} is the solution of (3.16) constructed in lemma 3.13. We thus consider the map Φ_a on the affine Banachspace

$$X_a := \{ (p, q) \in L^\infty([0, r_0]) \times L^\infty([0, r_0]), \| (p, q) - (\bar{h}, \bar{h}') \| < \infty \}$$

defined by the norm

$$\| (p, q) \| := \max \left(\| r^{-(\gamma+2)} p \|_{L^\infty([0, r_0])}, \| (\gamma+1)^{-1} r^{-(\gamma+1)} q \|_{L^\infty([0, r_0])} \right).$$

Here we set $r_0 = r_0(a) := \bar{c} \cdot \min\{1, a^{1/\gamma}\}$. The constant $\bar{c} > 0$ is independent of a and determined by

Claim 3. There exist constants $\bar{c}, C_0 > 0$ and $\lambda < 1$ independent of a such that the map

$$\Phi_a : D(C_0, a) \rightarrow D(C_0, a)$$

is a contraction with Lipschitz constant smaller than λ on the closed ball

$$D(C_0, a) := \{ (p, q) \in X_a, \| (p, q) - (\bar{h}, \bar{h}') \| \leq C_0 a^{-2/\gamma} \} \subset X_a.$$

PROOF OF CLAIM 3. Let C_0 be any given number. A direct computation implies that if $\bar{c} = \bar{c}(C_0) > 0$ is chosen small enough, then

$$\| \Phi_a(p_1, q_1) - \Phi_a(p_2, q_2) \| \leq \lambda \| (p_1, q_1) - (p_2, q_2) \| \quad \text{for all } (p, q) \in D(C_0, a)$$

for a constant $\lambda < 1$ independent of C_0 and \bar{c} .

In addition we claim that $\Phi_a(D(C_0, a)) \subset D(\frac{1+\lambda}{2}C_0, a)$ for C_0 sufficiently large. In fact, since \bar{h} is a solution of (3.16), we find for every $a > 0$

$$\| \Phi_a(\bar{h}, \bar{h}') - (\bar{h}, \bar{h}') \| \leq 2a^{-2/\gamma}$$

if \bar{c} is chosen small enough. Thus for any $(p, q) \in D(C_0, a)$

$$\begin{aligned} \| \Phi_a(p, q) - (\bar{h}, \bar{h}') \| &\leq \| \Phi_a(p, q) - \Phi_a(\bar{h}, \bar{h}') \| + 2a^{-2/\gamma} \\ &\leq (\lambda C_0 + 2)a^{-2/\gamma} \leq \frac{1+\lambda}{2} C_0 a^{-2/\gamma} \end{aligned}$$

for C_0 large as claimed. \square

By the fixed point theorem of Banach and claim 3 there exists a unique fixed point $P_a = (\tilde{h}_a, \tilde{h}_a')$ of Φ_a in the ball $D(\frac{1+\lambda}{2}C_0, a) \subset D(C_0, a)$. Observe that

$$\lim_{r \rightarrow 0} r^{-\gamma}(\tilde{h}_a(r) - s_0) = \lim_{r \rightarrow 0} r^{-\gamma}(\tilde{h}_a(r) - \bar{h}(r)) + \lim_{r \rightarrow 0} r^{-\gamma}(\bar{h}(r) - s_0) = 1.$$

Furthermore, $\lim_{r \rightarrow 0} r^{1-\gamma} \tilde{h}'_a(r) = \lim_{r \rightarrow 0} r^{1-\gamma} \bar{h}'(r) = \gamma$.

Rescaling \tilde{h}_a as described above we obtain the desired solution h_a of (3.6) on a small interval. Since equation (3.6) is regular away from $r = 0$, we can extend h_a to a solution of (3.6) on all of $[0, \infty)$.

For the proof of the uniqueness statement we refer to the corresponding part of the proof of lemma 3.13 given in appendix B.1. \square

5.3. Continuous dependence.

LEMMA 3.16. *Let (h_a) be the family of solutions to equation (3.6) constructed in proposition 3.14 and let \bar{h} and γ be as in lemma 3.13.*

(i) *Given any numbers $R_0 > 0$ and $\varepsilon > 0$, there exists $a_0 > 0$ such that*

$$\sup_{a \geq a_0} \sup_{r \in [0, R_0]} |h_a(a^{-1/\gamma} r) - \bar{h}(r)| < \varepsilon.$$

(ii) *The map $\mathbb{R} \ni a \mapsto h_a(R)$ is continuous for every $R \in [0, \infty]$.*

PROOF. Let \bar{c} , C_0 and $\lambda < 1$ be the constants of claim 3 and let \bar{h} and h_a be as in the above lemma. Recall that the corresponding fixed point $P_a = (\tilde{h}_a, \tilde{h}'_a)$ of Φ_a is an element of the ball $D(a, C_0) \subset X_a$. The resulting estimate

$$(3.18) \quad \sup_{r \in [0, \bar{c}]} \left| \tilde{h}_a(r) - \bar{h}(r) \right| + \left| \tilde{h}'_a(r) - \bar{h}'(r) \right| \leq \frac{C}{a^{2/\gamma}}$$

for $a \geq 1$ implies statement (i) for $R_0 \leq c_0$. For general values of $R_0 > 0$ claim (i) follows from (3.18) and the fact that the coefficients of the differential equations (3.17) and (3.16) are regular away from $r = 0$ and differ only by the term $\frac{r}{2a^{2/\gamma}}$.

We can check that the maps h_a converge to s_0 uniformly on any bounded interval as $a \rightarrow 0$. Consequently, the map $a \mapsto h_a(R)$ is continuous in $a = 0$ for any finite number $R > 0$.

On the other hand, let $a_0 \neq 0$ be fixed, say $a_0 > 0$. Let $(X_{a_0}, \|\cdot\|)$ be the affine Banach space defined in the proof of lemma 3.13. Remark that the maps $(\Phi_a)_{a > 0}$ are well defined on X_{a_0} . In addition, the fixed points $P_a \in D(\frac{1+\lambda}{2}C_0, a) \subset X_a$ of Φ_a are contained in the ball $D(C_0, a_0) \subset X_{a_0}$ if $|a - a_0|$ is small. Using claim 3 and the definition of Φ_a we thus find

$$\begin{aligned} \|P_{a_0} - P_a\| &= \|\Phi_{a_0}(P_{a_0}) - \Phi_a(P_a)\| \leq \|\Phi_{a_0}(P_{a_0}) - \Phi_{a_0}(P_a)\| + \|\Phi_{a_0}(P_a) - \Phi_a(P_a)\| \\ &\leq \lambda \|P_{a_0} - P_a\| + C \left| a_0^{-2/\gamma} - a^{2/\gamma} \right|. \end{aligned}$$

Since $\lambda < 1$ the fixed points P_a converge to P_{a_0} as $a \rightarrow a_0$ in the sense of $(X_{a_0}, \|\cdot\|)$. The same argument as for statement (i) then implies that the map $a \mapsto h_a(R)$ is continuous in a_0 for any finite number $R > 0$.

Finally, we observe that $\lim_{r \rightarrow 0} h'_a(r) \cdot r = 0$ for every $a \in \mathbb{R}$. We can therefore control the asymptotic behaviour of the whole family (h_a) by $|h'_a(r)| \leq \frac{\bar{C}}{r^3}$ for the constant \bar{C} of lemma 3.10. The continuity of $a \mapsto h_a(\infty) = \lim_{r \rightarrow \infty} h_a(r)$ follows from statement (ii) for finite values of R . \square

5.4. Properties of equivariant harmonic maps.

The qualitative behaviour of corotational harmonic maps from \mathbb{R}^d to S^d was described by Jäger and Kaul in [32]. Based on their methods we obtain the following result which we prove in appendix B.2.

PROPOSITION 3.17. *Let N be a compact, rotationally symmetric manifold and let χ be a k -eigenmap. Assume that conditions (C1) and (C2) are satisfied. Given any local minimum s_0 of g^2 we let $s^* > s_0$ be the local maximum of g^2 to the right of s_0 . Then the behaviour of the solution \bar{h} of (3.16) with $\bar{h}(0) = s_0$ given by lemma 3.13 can be described as follows.*

- (i) *If $-4kG'(s^*) \leq (d-2)^2$, then \bar{h} is increasing and converges to s^* as $r \rightarrow \infty$.*
- (ii) *Otherwise \bar{h} still converges to a local extremum \tilde{s} of g^2 (not necessarily equal to s^*). The convergence is monotone if $-4kG'(\tilde{s}) \leq (d-2)^2$, while \bar{h} oscillates around the level $s = \tilde{s}$ infinitely many times if $-4kG'(\tilde{s}) > (d-2)^2$.*

6. Comparison principles

Comparison principles and maximum principles are very valuable tools to analyse the behaviour of solutions of differential equations. To study the properties of solutions of equation (3.6) for general settings, we use

LEMMA 3.18. *Let $G \in C^1((0, \infty))$ and $\varphi \in C((0, \infty))$ be arbitrary fixed functions. We consider the differential operator*

$$(3.19) \quad T_\varphi(f) := f'' + \left(\frac{d-1}{r} + \varphi\right)f' - \frac{k}{r^2} \cdot G(f)$$

on an interval $I = [r_1, r_2] \subset (0, \infty)$.

- (i) *Suppose that $G|_{(a,b)} > 0$ on some interval $(a, b) \subset \mathbb{R}$. Then a non-constant function $f \in C^2(I, (a, b))$ with $T_\varphi(f) \geq 0$ cannot achieve a local maximum in the interior of I .*
- (ii) *Suppose that $G'|_{(c,d)} > 0$ on some interval $(c, d) \subset \mathbb{R}$. Let $f_1 \neq f_2$ be two functions in $C^2(I, (c, d))$ with*

$$T_\varphi(f_2) \leq T_\varphi(f_1) \text{ on } I.$$

Assume

$$f_2(r_1) \leq f_1(r_1) \text{ and } f_2'(r_1) \leq f_1'(r_1).$$

Then

$$f_2(r) < f_1(r) \text{ and } f_2'(r) < f_1'(r)$$

for all $r \in I$.

This lemma can be easily reduced to the classical maximum principle by the use of Taylor expansion, see the proof of proposition 3.19 below.

We remark that the condition $G' > 0$ is violated for the non-linearity $G = g \cdot g'$ of the equations (3.16) and (3.6) in a neighbourhood of s^* if C_{s^*} is an equator of N . Using the above lemma, we can thus compare solutions of these equations only as long as they map into an appropriate neighbourhood of a pole or a minimal sphere. In contrast, the following comparison principle applies to general solutions of (3.6) if the considered setting satisfies the assumptions of theorem 3.3.

PROPOSITION 3.19 (comparison principle). *Let $k \in \mathbb{N}$, let $s_1 < s_2$ and let $G \in C^1(\mathbb{R})$ be any given function. Assume that*

$$4k\theta \leq (d-2)^2$$

for $\theta := \max\{-G'(s) : s \in [s_1, s_2]\}$. Then the following comparison principle holds true for the operator T_φ defined by (3.19) if φ is such that $\varphi(r) \geq c \cdot r$ for a constant $c = c(\varphi) > 0$.

Let h_1 and h_2 be two functions in $C^2((0, \infty), [s_1, s_2])$ such that

$$T_\varphi(h_1) \geq T_\varphi(h_2)$$

and assume that

$$(3.20) \quad h_1(r_0) \geq h_2(r_0) \quad \text{and} \quad h_1'(r_0) \geq h_2'(r_0)$$

for some $r_0 > 0$. Then either h_1 and h_2 coincide or

$$(i) \quad h_1(r) > h_2(r) \text{ for all } r > r_0$$

and

$$(ii) \quad \lim_{r \rightarrow \infty} h_1(r) > \lim_{r \rightarrow \infty} h_2(r).$$

REMARK 3.20. *Let N be rotationally symmetric, χ a k -eigenmap and let C_{s^*} be an equator of N with $-4kG'(s^*) \leq (d-2)^2$. Assume that condition (C1) is satisfied and let $s_1 < s^* < s_2$ be the local minima of g^2 to the left and to the right of s^* . Then the comparison principle applies to the operator $T_{r/2}$ corresponding to equation (3.6). By the characterisation of settings with energy-minimising equator maps given in proposition 3.4 the comparison principle in particular is applicable for settings (N, χ) that satisfy the assumptions of theorem 3.3.*

PROOF OF PROPOSITION 3.19. Let h_1 and h_2 be as in proposition 3.19 and assume that $h_1 \neq h_2$. In order to prove statement (i), we consider the rescaled difference

$$f_1(r) := r^\eta \cdot (h_1(r) - h_2(r))$$

for $\eta > 0$ to be determined later. Observe that f_1 satisfies the linear differential inequality

$$f_1'' + \left(\frac{d-1-2\eta}{r} + \varphi\right)f_1' + a_\eta(r)f_1 \geq 0$$

for

$$a_\eta(r) = \frac{\eta(\eta+1)}{r^2} - \frac{\eta}{r} \left(\frac{d-1}{r} + \varphi\right) - \frac{k \cdot G'(\xi)}{r^2} < \frac{1}{r^2}[\eta^2 - (d-2)\eta + k\theta].$$

Choosing $\eta = \frac{d-2}{2}$ in view of our assumption that $4k\theta \leq (d-2)^2$ we have $a_\eta < 0$. Thus, if we assume that f_1 achieves a positive local maximum at a point $r_1 \geq r_0$ a contradiction results; hence f_1 is an increasing, positive function on $[r_0, \infty)$ and statement (i) follows.

For the second part of the proof we let $\eta = \frac{d-2}{2}$ be as above and consider

$$f_2(r) := \left(\frac{r}{C+r}\right)^\eta \cdot (h_1(r) - h_2(r))$$

for a (large) constant C which is chosen later on.

The first part of the proof implies that if $r_0 < 1$ then

$$f_1'(1) = \eta(h_1 - h_2)(1) + (h_1 - h_2)'(1) \geq \delta$$

for some $\delta = \delta(h_1, h_2) > 0$. For f_2 defined as above, we thus find not only that $f_2(1) \geq 0$, but also that

$$\begin{aligned} f_2'(1) &= \left(\frac{1}{C+1}\right)^\eta \cdot \left(\frac{C\eta}{C+1}(h_1 - h_2)(1) - (h_1 - h_2)'(1)\right) \\ &\geq \left(\frac{1}{C+1}\right)^\eta \cdot \left(\frac{C\delta}{C+1} - \frac{1}{C+1}\eta(h_1 - h_2)(1)\right) \geq 0 \end{aligned}$$

for C sufficiently large. We can thus assume that $f_2(r_0) \geq 0$ and $f_2'(r_0) \geq 0$ for some $r_0 \geq 1$. The function f_2 satisfies the inequality

$$f_2'' + \left(\frac{d-1-2\eta}{r} + \frac{2\eta}{C+r} + \varphi\right)f_2' + \tilde{a}_C(r)f_2 \geq 0$$

where the coefficient $\tilde{a}_C(r)$ may be estimated as

$$\tilde{a}_C(r) \leq \frac{(d-3)\eta}{r(C+r)} + (\eta - \eta^2) \frac{2C+r}{(C+r)^2 r} - \frac{\varphi(r)}{r} \frac{C\eta}{C+r}.$$

On the interval $[1, \infty)$ the dominating term in the above bound is $-\frac{\varphi(r)}{r} \frac{C\eta}{C+r} < 0$ and thus $\tilde{a}_C(r) < 0$ if C is large enough. The same argument as above implies that f_2 is increasing and positive on $[r_0, \infty)$. Therefore

$$\lim_{r \rightarrow \infty} h_1(r) - h_2(r) = \lim_{r \rightarrow \infty} f_2(r) > 0$$

as claimed. \square

7. Proof of theorem 3.3

One of the key steps for the proof of both the existence and the uniqueness statement of theorem 3.3 is given by

LEMMA 3.21. *Let N be rotationally symmetric and let χ be a k -eigenmap. Let s_0 be a local minimum of g^2 for which condition (C2) holds true and let (h_a) be the family of solutions to (3.6) with $h_a(0) = s_0$ constructed in proposition 3.14. Assume that condition (C1) holds true for the local maximum $s^* > s_0$ of g^2 to the right of s_0 and that $-4kG'(s^*) \leq (d-2)^2$. Then the map*

$$L : a \mapsto \lim_{r \rightarrow \infty} h_a(r)$$

is a continuous bijection from $[0, \infty)$ to $[s_0, s^)$.*

PROOF OF LEMMA 3.21. We first apply the comparison principle to prove that all solutions $(h_a)_{a \geq 0}$ take values in $[s_0, s^*)$. On the one hand, the constant solution $h_0 \equiv s_0$ of (3.6) is a uniform lower bound for the solutions $(h_a)_{a \geq 0}$. On the other hand, let us recall that the solution \bar{h} of the harmonic map equation (3.16) given by lemma 3.13 is increasing on $[0, \infty)$ with $\lim_{r \rightarrow \infty} \bar{h}(r) = s^*$, see proposition 3.17.

The rescaled functions $\bar{h}_a(r) := \bar{h}(ra^{1/\gamma})$ are thus supersolutions of (3.6) and $h_a \leq \bar{h}_a < s^*$ by the comparison principle as claimed. An important consequence is that the functions $(h_a)_{a>0}$ are increasing on all of $[0, \infty)$ according to lemma 3.18 and remark 3.12.

We may also apply the comparison principle to conclude that two different solutions h_a and $h_{\bar{a}}$ do not intersect at any finite $r > 0$ nor converge to the same limit as $r \rightarrow \infty$. The map L defined in lemma 3.21 is thus increasing. Since L is continuous according to lemma 3.16 and since $L(0) = s_0$ it remains to prove that

$$\lim_{a \rightarrow \infty} L(a) = s^*.$$

So let $\varepsilon > 0$ be any given number. We choose $R = R(\varepsilon) > 0$ with $\bar{h}(R) > s^* - \varepsilon$. According to lemma 3.16 there exists $a_0 = a_0(\varepsilon, R) > 0$ such that

$$|h_a(a^{-1/\gamma}R) - \bar{h}(R)| < \varepsilon, \text{ for all } a \geq a_0.$$

Consequently

$$s^* > L(a) \geq h_a(a^{-1/\gamma}R) > s^* - 2\varepsilon$$

for all $a > a_0$. The claim follows since $\varepsilon > 0$ was an arbitrary number. \square

PROOF OF THEOREM 3.3. Let N be any compact, rotationally symmetric manifold and let χ be a k -eigenmap. We assume that conditions (C1) and (C2) are fulfilled, and that all equator maps u_{χ, s^*}^* of N are χ -energy-minimising. Thus, according to proposition 3.4, we have $-4kG'(s^*) \leq (d-2)^2$ for every equator C_{s^*} of N . Given any $s \in \mathbb{R}$, we need to prove that the initial value problem (3.1), (3.2) for $u_0(x) = (s, \chi(\frac{x}{|x|}))$ has a unique equivariant and selfsimilar weak solution.

Let $s_1^* < s \leq s_2^*$ be the two local maxima of g^2 to the left and to the right of s and let s_0 be the local minimum of g^2 in (s_1^*, s_2^*) . We let $(h_a) \subset H_{rad}^1(\mathbb{R}^d)$ be the family of solutions to (3.6) with $h_a(0) = s_0$ given by proposition 3.14.

Then a solution of the initial value problem (3.1), (3.2) is given by

$$u_s(x, t) := (R_\chi h_{L^{-1}(s)})(\frac{x}{\sqrt{t}}).$$

for the bijection $L : \mathbb{R} \cup \{\pm\infty\} \rightarrow [s_1^*, s_2^*]$ of lemma 3.21 which we extend by $L(-\infty) := s_1^*$ and $L(\infty) := s_2^*$. Here and in the following, we use the notations $h_{-\infty} \equiv s_1^*$ and $h_\infty \equiv s_2^*$ for the constant solutions of (3.6) which induce the equator maps.

Since *all* equator maps are χ -energy-minimising lemma 3.21 implies that the above solution u_s is unique among all solutions to (3.1), (3.2) induced by elements of the families (h_a) , $h_a(0)$ any local minimum of g^2 , of proposition 3.14. To conclude the proof of theorem 3.3, we therefore only need to show that there are no selfsimilar, equivariant solutions to (3.1), (3.2) other than those induced by these families (h_a) of solutions to (3.6). This is achieved in the following proposition.

PROPOSITION 3.22. *Let N be any compact, rotationally symmetric manifold, χ a k -eigenmap and assume that condition (C2) is valid. Then every solution $h \in H_{rad}^1(\mathbb{R}^d)$ to (3.6) is a member of one of the families $(h_a)_{-\infty \leq a \leq \infty}$ given in proposition 3.14 corresponding to the local minima of g^2 .*

This result might be surprising since the condition imposed by $h \in H_{rad}^1(\mathbb{R}^d)$ is relatively mild. A priori, it does not exclude functions with singularities at $r = 0$, but merely restricts the allowed blow-up rates.

As we will see below, most solutions of equation (3.6) are unbounded and can thus be described by

LEMMA 3.23. *Let N be compact, $k \in \mathbb{N}$ and let h be an unbounded solution of equation (3.6). Then there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$|h'(r)| > \frac{\delta}{r^{d-1}} \text{ for } r \in (0, \varepsilon).$$

In particular $h \notin H_{rad}^1(\mathbb{R}^d)$.

PROOF. Let h be any unbounded solution of (3.6). Since N is compact, h must reach the level of a pole for some $r_0 > 0$, i.e. $g(h(r_0)) = 0$. Let now

$$\tilde{V}(r) = \tilde{V}(h)(r) := r^{2(d-1)} \cdot \left[|h'|^2 - \frac{k}{r^2} g^2(h) \right].$$

Obviously $\tilde{V}(r_0) \geq 0$ and a short calculation shows that \tilde{V} is decreasing for any non-constant solution of (3.6). Given any $0 < \varepsilon < r_0$, we can thus choose $\delta > 0$ such that $\tilde{V}|_{[0, \varepsilon]} \geq \delta^2 > 0$ and the claim follows. \square

The behaviour of general solutions to (3.6) is furthermore restricted by

LEMMA 3.24. *Let N and χ be as in proposition 3.22. Then for any solution h of (3.6) there exists $\varepsilon = \varepsilon(h) > 0$ such that $h|_{(0, \varepsilon)}$ is monotonous.*

PROOF. For unbounded solutions of (3.6) the claim of lemma 3.24 is a direct consequence of lemma 3.23.

So let h be any bounded solution of (3.6). According to the proof of lemma 3.23 the quantity $\tilde{V}(h)$ is negative on $(0, \infty)$. To avoid unnecessarily complicated notations, we only present the proof for targets which have merely one equator. The general case follows from a very similar argument.

So let N be compact with poles given by $s = p_1$ and $s = p_2$ and let $s^* \in (p_1, p_2)$ be the coordinate of the equator of N . If the solution h of (3.6) is not monotonous on any interval $(0, \varepsilon)$ it must have infinitely many local extrema in the interval $(0, 1]$. According to remark 3.12 and lemma 3.18 we thus find an infinite set

$$\{R_1 > R_2 > \dots\} \subset (0, 1]$$

of points with $h(R_i) = s^*$.

Let now $V(r) = r^{-2(d-2)} \tilde{V}(r) < 0$ be the decreasing quantity introduced in (3.13).

We prove that the difference $V(R_{i+1}) - V(R_i)$ is bounded from below by a positive quantum $\Delta = \Delta(h) > 0$ independent of i and conclude that

$$V(R_{i+1}) \geq V(R_1) + i \cdot \Delta \xrightarrow{i \rightarrow \infty} \infty.$$

This obviously contradicts the assumption that V is negative on $(0, \infty)$.

The key observation is given by

Claim 4. Let (N, χ) be as above and let h be a non-constant solution of (3.6). Then for any $0 < r_1 < r_2 \leq 1$ with $h(r_1) = h(r_2) = s^*$ the estimate

$$V(r_1) - V(r_2) \geq c \cdot (r_1 h'(r_1))^2$$

holds true for a universal constant $c = c(d, k, g) > 0$.

The above claim leads to the conclusion of the proof of lemma 3.24 as follows. For any $i \in \mathbb{N}$ we find that

$$(R_i h'(R_i))^2 = V(R_i) + kg^2(s^*) \geq V(R_1) + kg^2(s^*) = (R_1 h'(R_1))^2$$

since V is decreasing and $R_i \leq R_1$. According to remark 3.12 we have $h'(R_1) \neq 0$. Thus

$$V(R_{i+1}) - V(R_i) \geq c \cdot (R_1 h'(R_1))^2 =: \Delta > 0$$

for all $i \in \mathbb{N}$ as claimed.

PROOF OF CLAIM 4. Let h be a non-constant solution of equation (3.6) and let $0 < r_1 < r_2 \leq 1$ be such that $h(r_1) = h(r_2) = s^*$. By symmetry we can restrict our attention to the case $h'(r_1) > 0$. We choose $\bar{r} \in (r_1, r_2)$ such that

$$h'(r) \geq \frac{1}{2} h'(r_1) = h'(\bar{r}) \text{ for all } r \in [r_1, \bar{r}]$$

and observe that on this interval

$$(3.21) \quad V'(r) \leq -2(d-2)r |h'(r)|^2 \leq -\frac{d-2}{2} r_1 |h'(r_1)|^2,$$

according to (3.14). It is therefore enough to establish an appropriate lower bound for the length of the interval $[r_1, \bar{r}]$.

We can easily verify that $h'(r) \leq h'(r_1)$ on $[r_1, \bar{r}]$ and thus find that for all $r \in [r_1, \bar{r}]$

$$G(h(r)) = G(h(r_1) + h'(\xi)(r - r_1)) \geq -\theta h'(r_1)(r - r_1) \geq -\theta h'(r_1)r$$

where $\theta := \max_{s \in [p_1, p_2]} -G'(s)$. The decay of the first derivative of h is thus bounded from below on the interval $[r_1, \bar{r}] \subset (0, 1]$ by

$$h''(r) \geq -\left(\frac{d-1}{r} + \frac{r}{2}\right) h'(r_1) - \frac{\theta k}{r} h'(r_1) \geq -c_1 \cdot \frac{h'(r_1)}{r}$$

for a constant $c_1 = c_1(d, k, \theta) > 0$ independent of h . Consequently

$$-\frac{1}{2} h'(r_1) = h'(\bar{r}) - h'(r_1) = \int_{r_1}^{\bar{r}} h'' dr \geq -c_1 h'(r_1) \log\left(\frac{\bar{r}}{r_1}\right),$$

and the length of the interval $[r_1, \bar{r}]$ can be no less than a fixed positive multiple of r_1 . Together with (3.21) we obtain claim 4 and thus lemma 3.24. \square

REMARK 3.25. For settings (N, χ) satisfying the assumptions of theorem 3.3, a much shorter proof of lemma 3.24 based on the comparison principle can be given. In fact, we obtain that solutions of (3.6) are not only monotonous near $r = 0$ but have at most one local extremum on all of $(0, \infty)$.

Finally, we conclude the proof of our main uniqueness result for selfsimilar solutions, theorem 3.3, by giving the

PROOF OF PROPOSITION 3.22. Let $h \in H_{rad}^1(\mathbb{R}^d)$ be any solution of (3.6). By lemma 3.23 the function h is bounded. It can therefore be extended continuously up to $r = 0$ according to lemma 3.24. We analyse the properties of h based on the value $h(0) = \lim_{r \rightarrow 0} h(r)$. We begin with

Case 1. $h(0)$ is a local minimum of g^2 .

Let s_0 be any local minimum of g^2 and let $\gamma > 0$ and $(h_a)_{a \in \mathbb{R}}$ be as in proposition 3.14. We know that any solution h of (3.6) with $h(0) = s_0$ and $\lim_{r \rightarrow 0} r^{-\gamma}(h(r) - s_0) = a \in \mathbb{R}$ coincides with the map h_a by the uniqueness statement of proposition 3.14.

So let us assume that there exists a solution h of (3.6) with $h(0) = s_0$ for which $r^{-\gamma}(h(r) - s_0)$ diverges as $r \rightarrow 0$. According to lemma 3.24 and by symmetry, we may assume that h is increasing on a small interval $(0, \varepsilon)$. We chose $b > s_0$ such that $G'|_{[s_0, b]} > 0$ and fix $r_0 \in (0, \varepsilon)$ with $h(r_0) < b$. Following the arguments of the proof of lemma 3.21, we then find $a_0 > 0$ with $h(r_0) < h_{a_0}(r_0)$ and with $h_{a_0}|_{[0, r_0]} \leq b$. According to lemma 3.18 the function h_{a_0} is an upper bound for h on $[0, r_0]$ and thus $\overline{\lim}_{r \rightarrow 0} r^{-\gamma}(h(r) - s_0) \leq a_0 < \infty$. Since this quantity by assumption diverges, there exists a number $a > 0$ with

$$0 \leq \underline{\lim}_{r \rightarrow 0} r^{-\gamma}(h(r) - s_0) < a < \overline{\lim}_{r \rightarrow 0} r^{-\gamma}(h(r) - s_0).$$

But then h has to intersect the corresponding solution h_a of (3.6) in points arbitrarily close to $r = 0$ in contradiction to lemma 3.18.

We conclude that the only solutions of (3.6) with $h(0) = s_0$ are those of the family $(h_a)_{a \in \mathbb{R}}$.

Case 2. $h(0)$ is a local maximum of g^2 .

Let C_{s^*} be an equator of N . We claim that the only solution of (3.6) with $h(0) = s^*$ is the constant map $h_\infty \equiv s^*$.

Indeed, let us assume that h is a non-constant solution of (3.6) with $h(0) = s^*$ and let $r_1 > 0$ be such that $g^2(h(r_1)) < g^2(s^*)$. We set $\delta := g^2(s^*) - g^2(h(r_1)) > 0$ and choose $r_0 \in (0, r_1)$ such that $g^2(h(r)) \geq g^2(s^*) - \delta/2$ for all $r \in [0, r_0]$. Since the quantity $V(r)$ given by (3.13) is non-increasing we obtain that on $(0, r_0)$

$$(rh')^2 - kg^2(s^*) + k\delta/2 \geq V(r) \geq V(r_1) \geq -kg^2(s^*) + k\delta.$$

Consequently

$$|h'(r)| \geq \frac{\sqrt{k\delta/2}}{r}$$

on $(0, r_0)$ and h cannot converge as $r \rightarrow 0$, in contradiction to the assumption $h(0) = s^*$.

Finally, we need to consider

Case 3. $h(0)$ is no local extremum of g^2 .

We have assumed from the very beginning that g' has no roots of multiplicity greater than one and thus find that $G(h(0)) \neq 0$. By symmetry we can focus on solutions h of (3.6) with $G(h(0)) > 0$.

Suppose h is decreasing on some interval $(0, \varepsilon)$. We can then bound the second derivative of h on a small interval $(0, r_0] \subset (0, \varepsilon)$ by

$$h'' = k \cdot \frac{G(h(0)) + o(1)}{r^2} - \left(\frac{d-1}{r} + \frac{r}{2}\right)h' \geq \frac{c}{r^2}$$

for a constant $c > 0$ independent of r and for $o(1) \rightarrow 0$ as $r \rightarrow 0$.

Integrating the obtained inequality from r to r_0 gives

$$h'(r) \leq -\frac{c}{r} + h'(r_0) + \frac{c}{r_0} = -\frac{c}{r} + C(r_0)$$

for every $r \in (0, r_0)$, which is obviously wrong for bounded functions.

According to lemma 3.24, we obtain that h is increasing on some interval $(0, \varepsilon)$. Using the divergence form of (3.6) given in (3.7) we then find for $r \in (0, r_0)$

$$(e^{r^2/4}r^{d-1}h')' \geq cr^{d-3}$$

for a constant $c > 0$ and for $r_0 > 0$ small enough.

Integrating from $r/2$ to $r < r_0$ we find

$$e^{r^2/4}r^{d-1}h'(r) \geq \left(\frac{r}{2}\right)^{d-1}e^{r^2/16}h'\left(\frac{r}{2}\right) + c\frac{1-2^{2-d}}{d-2}r^{d-2} \geq \tilde{c}r^{d-2} > 0,$$

since h is increasing on $(0, \varepsilon)$. The resulting lower bound of $h'(r) \geq \frac{\tilde{c}}{r}$ on $(0, r_0)$ once more stands in contrast to the assumption that h is continuous up to $r = 0$.

We conclude that $h(0)$ is a local extremum of g^2 for each bounded solution h of (3.6). Combined with cases 1 and 2 and the description of unbounded solutions of lemma 3.23, we obtain proposition 3.22. \square

8. Proof of theorem 3.5

Let N , χ and s^* be as in theorem 3.5 and let $s_0 < s^* < s_1$ be the local minima of g^2 to the left and to the right of s^* . We can assume without loss of generality that $g^2(s_0) \geq g^2(s_1)$.

Let $(h_a)_{a \geq 0}$ be the family of solutions to (3.6) with $h_a(0) = s_0$ constructed in proposition 3.14 and let \bar{h} be the solution to equation (3.16) with $\bar{h}(0) = s_0$ from lemma 3.13. According to remark 3.12 we have $s_0 < h_a, \bar{h} < s_1$ on $(0, \infty)$ for each $a > 0$. The crucial difference to settings (N, χ) as in theorem 3.3 is that the solutions (h_a) are able to cross the level $s = s^*$ of the equator.

We consider the function

$$(3.22) \quad [0, \infty) \ni a \mapsto I(a) := \#\{r > 0 : h_a(r) = s^*\}$$

counting the number of intersection points of h_a with the level $s = s^*$ of the equator C_{s^*} .

Lemma 3.16 together with lemma 3.10 implies that $I(a) = I(0) = 0$ for $a > 0$ small enough. Conversely, the solution \bar{h} of (3.16) reaches the level $s = s^*$ infinitely many times according to proposition 3.17. Applying lemma 3.16, we thus find that $I(a) \rightarrow \infty$ as $a \rightarrow \infty$.

The number $I(a)$ is however finite for each $a \in [0, \infty)$; in fact, we prove

LEMMA 3.26. *For any rotationally symmetric manifold N , any equator C_{s^*} of N and for any $k \in \mathbb{N}$ there exists a number $R > 0$ such that the following holds true.*

- (i) *No solution h of (3.6) intersects the level $s = s^*$ more than once on the interval $[R, \infty)$.*
- (ii) *If $h(r) = s^*$ for some $r > R$, then h cannot converge to s^* as $r \rightarrow \infty$.*

PROOF. The key idea is to compare a given solution h of (3.6) with supersolutions of an appropriate differential equation for which the comparison principle is valid. So let N be any rotationally symmetric manifold, let C_{s^*} be an equator of N and let $k \in \mathbb{N}$.

We set $\Theta := \max_{s \in \mathbb{R}} -G'(s)$ for the function $G = g \cdot g'$ and choose $D \geq d$ such that

$$4k\Theta \leq (D - 2)^2.$$

We claim that lemma 3.26 holds true for $R := 2\sqrt{D - d}$.

So let h be a solution of (3.6) with $h(r) = s^*$ for some $r \geq R$. By symmetry and remark 3.12 we can assume that $h'(r) < 0$. The claim is obviously true if h is decreasing on all of $[r, \infty)$. Suppose therefore that h achieves a local minimum at some point $(r_0, h(r_0))$, $r_0 > R$.

We now consider the solution f of

$$(3.23) \quad f'' + \frac{D-1}{r}f' - \frac{k}{r^2}G(h) = 0$$

with $f(0) = s_0$ and $\lim_{r \rightarrow 0} r^{-\Gamma}(f(r) - s_0) = 1$, for $\Gamma := \frac{1}{2}(\sqrt{(D-2)^2 + 4kG'(s_0)} - (D-2)) > 0$. As usual, $s_0 < s^*$ denotes the local minimum of g^2 to the left of s^* .

We should remark here that (3.23) does not necessarily represent the harmonic map equation of a new setting since k is in general no eigenvalue of $\Delta_{S^{D-1}}$. The proof of the existence statement of lemma 3.13 and the characterisation of solutions given by proposition 3.17 remain however valid for equation (3.23). The solutions $f_a(r) = f(a^{1/\Gamma}r)$, $a > 0$ of (3.23) are thus increasing on $(0, \infty)$ and converge to s^* as $r \rightarrow \infty$. Since $h(r_0) < s^*$ we find

$$h(r_0) < f_a(r_0) \quad \text{and} \quad h'(r_0) = 0 < f'_a(r_0)$$

for a large enough.

Because f_a is an increasing solution of (3.23), it satisfies $\tilde{T}_{r/4}(f_a) \geq 0$ on all of $(0, \infty)$ for the operator

$$\tilde{T}_{r/4}(f) := f'' + \left(\frac{D-1}{r} + \frac{r}{4}\right)f' - \frac{k}{r^2}G(f).$$

On the other hand, let $r_1 \in (r_0, \infty]$ be the maximal number such that h is increasing on (r_0, r_1) . Then $\tilde{T}_{r/4}(h) \leq 0$ on this interval because $r_0 > R$. Since the operator $\tilde{T}_{r/4}$ satisfies the assumptions of the comparison principle, we find

$$h \leq f_a < s^* \quad \text{on} \quad (r_0, r_1).$$

However, according to lemma 3.18 the function h cannot achieve a local maximum at r_1 unless $h(r_1) > s^*$. Therefore $r_1 = \infty$ and $h < s^*$ on (r_0, ∞) . Finally, the comparison principle implies $\lim_{r \rightarrow \infty} h(r) < \lim_{r \rightarrow \infty} f_a(r) = s^*$. \square

The connection between the properties of the function $I(\cdot)$ defined in (3.22) and the existence of multiple solutions to the initial value problem (3.1), (3.2) is given by

LEMMA 3.27. *The function $I : [0, \infty) \rightarrow \mathbb{N}_0$ defined in (3.22) has the following properties if N , χ and C_{s^*} satisfy the assumptions of theorem 3.5.*

(i) *I is subcontinuous and if a_0 is a point of discontinuity of $I(\cdot)$ then*

$$\overline{\lim}_{a \rightarrow a_0} I(a) = I(a_0) + 1$$

and

$$\lim_{r \rightarrow \infty} h_{a_0}(r) = s^*.$$

(ii) *For any $n \in \mathbb{N}_0$ there is number $A_n > 0$ with $I(A_n) = n$ such that the corresponding solution h_{A_n} of (3.6) converges to s^* as $r \rightarrow \infty$.*

(iii) *The union $S_{2k} \cup S_{2k+1}$ of the sets*

$$S_n := \{ \lim_{r \rightarrow \infty} h_a(r) : I(a) = n \}, \quad n \in \mathbb{N}_0,$$

is a neighbourhood of s^ for every $k \in \mathbb{N}_0$.*

As an immediate consequence of this lemma, we obtain the statement of theorem 3.5 for the neighbourhoods U_K of s^* given by

$$U_K := \bigcap_{n=0}^{K-1} (S_{2n} \cup S_{2n+1}).$$

REMARK 3.28. *As a consequence of the above lemma and by the continuity statement of lemma 3.16 we obtain that $S_0 = [s_0, s^*]$. The existence of a selfsimilar and equivariant weak solution of the initial value problem (3.1), (3.2) is thus guaranteed for any admissible initial data also for settings with an equator map that is not energy-minimising.*

PROOF OF LEMMA 3.27. We need to understand how the number of intersection points of the continuous family of maps (h_a) with the level $s = s^*$ can change as we vary the parameter a . So let $a_0 \in [0, \infty)$ be any given number.

Since h_{a_0} is not tangential to $s = s^*$ at any finite point and since $h_{a_0}(0) \neq s^*$ we find a neighbourhood of $a_0 > 0$ on which $I(\cdot) \geq I(a_0)$, i.e. I is subcontinuous at a_0 .

We now fix a point of discontinuity a_0 of $I(\cdot)$ and let $a_i \rightarrow a_0$ be such that $\lim_{i \rightarrow \infty} I(a_i) = \overline{\lim}_{a \rightarrow a_0} I(a) > I(a_0)$. Let $R > 0$ be the number determined in lemma 3.26 and recall that at most one of the roots of $h_{a_i} - s^*$ can be larger than R . In addition

$$\|h_{a_0} - h_{a_i}\|_{C^1([0, 2R])} \xrightarrow{i \rightarrow \infty} 0$$

by the arguments of the proof of lemma 3.16. If the distance between two distinct roots of h_{a_i} were to converge to zero as $i \rightarrow \infty$ we would therefore find a point $0 \leq r < R$ with $h_{a_0}(r) = s^*$ and $h'_{a_0}(r) = 0$. This is a contradiction to remark 3.12.

The discontinuity of I at a_0 must therefore be caused by roots of $h_{a_i} - s^*$ escaping to infinity in the sense that $h_{a_i}(r_i) = s^*$ for a sequence $r_i \rightarrow \infty$.

By lemma 3.26 all roots of $h_{a_i} - s^*$ different from r_i must be strictly less than the constant R for i large enough. Consequently $I(a_i) \leq I(a_0) + 1$ for i large.

Furthermore, lemma 3.10 implies that

$$\left| \lim_{r \rightarrow \infty} h_{a_i}(r) - s^* \right| = \left| \lim_{r \rightarrow \infty} h_{a_i}(r) - h_{a_i}(r_i) \right| \leq \frac{\bar{C}}{2r_i^2} \xrightarrow{i \rightarrow \infty} 0.$$

Applying lemma 3.16 we find that h_{a_0} converges to s^* as $r \rightarrow \infty$ as claimed in (i).

A first consequence of statement (i) and the fact that $I(a) \rightarrow \infty$ as $a \rightarrow \infty$ is that $I : [0, \infty) \rightarrow \mathbb{N}_0$ is surjective. Given any number $n \in \mathbb{N}_0$ we define

$$A_n := \max\{a : I(a) = n\} \in (0, \infty).$$

The function I is obviously discontinuous at A_n and we conclude that h_{A_n} tends to s^* as $r \rightarrow \infty$ by statement (i).

Finally, according to the first part of the proof, we can choose $\varepsilon_n > 0$ so small that the solutions h_a intersect the level $s = s^*$ at a point $r_a > R$ for all $a \in (A_n, A_n + \varepsilon_n)$. Lemma 3.26 thus implies that $\lim_{r \rightarrow \infty} h_a(r) \neq s^*$ for all $a \in (A_n, A_n + \varepsilon_n)$. But of course

$$\lim_{r \rightarrow \infty} h_a(r) \xrightarrow{a \rightarrow A_n} s^* = \lim_{r \rightarrow \infty} h_{A_n}$$

again by lemma 3.16.

The connected subset

$$\left\{ \lim_{r \rightarrow \infty} h_a(r) : a \in (A_{n-1}, A_{n-1} + \varepsilon_{n-1}) \right\} \subset I_n, \quad n \in \mathbb{N}$$

therefore contains an open interval of the form $(s^* - \delta_n, s^*)$ (for n even) respectively $(s^*, s^* + \delta_n)$ (for n odd). Since $I_0 = [s_0, s^*]$ the final claim of lemma 3.27 follows. \square

9. Monotonicity properties of selfsimilar solutions

An important feature of classical solutions of the harmonic map heat flow is that they satisfy a monotonicity inequality. In fact, for smooth solution $u : \mathbb{R}^d \times [0, \infty) \rightarrow N$ of (3.1) it was shown by Struwe [51] that

$$(3.24) \quad r^2 \int_{\mathbb{R}^d \times \{t_0 - r^2\}} |\nabla u|^2 G_{(x_0, t_0)} dx \leq R^2 \int_{\mathbb{R}^d \times \{t_0 - R^2\}} |\nabla u|^2 G_{(x_0, t_0)} dx$$

for every $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$ and every $0 < r < R \leq \sqrt{t_0}$. Here $G_{(x_0, t_0)}$ denotes the backwards heat kernel centred at (x_0, t_0) ,

$$G_{(x_0, t_0)}(x, t) = \frac{1}{(4\pi(t_0 - t))^{d/2}} \cdot \exp\left(-\frac{|x_0 - x|^2}{4(t_0 - t)}\right), \quad t < t_0.$$

Similar monotonicity inequalities involving local energy quantities hold true for smooth solutions of the harmonic map heat flow from compact domain manifolds, see [11].

Chen and Struwe [11] furthermore proved that these monotonicity inequalities are also satisfied for the global weak solutions of the harmonic map flow constructed in [8] and [11].

In fact, the non-uniqueness result of Coron [12] is based on the monotonicity property of these global weak solutions. Coron proves the existence of maps $u_0 : B_1 \subset \mathbb{R}^3 \rightarrow S^2$ that are weakly harmonic but not stationary harmonic. The corresponding time-independent solutions $u_1(x, t) = u_0(x)$ of (3.1), (3.2) violate the monotonicity formula and must thus differ from the global weak solution u_2 of [11].

Arbitrary time-translations of u_2 provide of course an infinite family of solutions to (3.1), (3.2). Similar examples of non-uniqueness that satisfy the monotonicity formula were given by Hong in [30].

Our main non-uniqueness result, theorem 3.5, implies the existence of infinitely many solutions to (3.1), (3.2) which are genuinely different and which satisfy the monotonicity inequality (3.24) of Struwe. For simplicity we focus here on manifolds without minimal spheres.

PROPOSITION 3.29. *Let N be a compact manifold that has no minimal sphere and let C_{s^*} be the equator of N .*

Then all weak solutions of the harmonic map heat flow that are equivariant and self-similar satisfy the monotonicity inequality (3.24).

PROOF. Let us first assume that $u(x, t) = v(x)$ is a time-independent solution of the harmonic map flow. Condition (3.24) then translates into the restriction that

$$R \mapsto R^{2-d} \int_{\mathbb{R}^d} |\nabla v|^2 \exp\left(-\frac{|x_0 - x|^2}{4R^2}\right) dx$$

in non-decreasing for each $x_0 \in \mathbb{R}^d$. It can be verified, see e.g. [12] and [30], that the above condition is satisfied for weakly harmonic maps v that are stationary harmonic; i.e. that are critical points of the Dirichlet energy both with respect to variations on the domain and the target manifold.

Let s_1 be any local extremum of g^2 and let χ be a k -eigenmap. We claim that that the weakly harmonic map

$$v(x) = (s_1, \chi(\frac{x}{|x|}))$$

is stationary harmonic.

The map v is stationary harmonic if and only if the distributional divergence of the stress-energy tensor $S \in L^1_{loc}(\mathbb{R}^d)$ defined by

$$S_{ij} := \frac{1}{2} \delta_{ij} |\nabla v|^2 - \partial_i v \partial_j v$$

vanishes, see [29].

Since v is smooth on $\mathbb{R}^d \setminus \{0\}$ and thus stationary harmonic on this set, the divergence of

$$S_{ij} = \frac{g^2(s_1)}{|x|^2} \cdot \left[\frac{1}{2} |\nabla \chi|^2 - \partial_i \chi(\frac{x}{|x|}) \partial_j \chi(\frac{x}{|x|}) \right]$$

is zero away from $x = 0$. In addition, the coefficients of the tensor S are even functions since the components of χ are homogeneous polynomials of degree l where $k = l(d - 2 + l)$, see [17].

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be any test function, let $j \in \{1, \dots, d\}$ and let $\varepsilon > 0$. Using the standard summation convention and the above remarks we obtain

$$\begin{aligned}
\left| \int \partial_i(S_{ij})\varphi \, dx \right| &\leq \left| \int_{\mathbb{R}^d \setminus B_\varepsilon(0)} S_{ij} \partial_i \varphi \, dx \right| + \left| \int_{B_\varepsilon(0)} S_{ij} \partial_i \varphi \, dx \right| \\
&\leq \left| \int_{\partial B_\varepsilon(0)} S_{ij}(x) \frac{x_i}{|x|} \cdot \varphi(x) \, d\sigma \right| + \|\varphi\|_{C^1} \cdot \|S\|_{L^1(B_\varepsilon(0))} \\
&= \left| \int_{\partial B_\varepsilon(0)} S_{ij}(x) \cdot \frac{x_i}{|x|} (\varphi(x) - \varphi(0)) \, d\sigma \right| + \|\varphi\|_{C^1} \cdot \|S\|_{L^1(B_\varepsilon(0))} \\
(3.25) \quad &\leq C \|\varphi\|_{C^1} (\varepsilon + \|S\|_{L^1(B_\varepsilon(0))}).
\end{aligned}$$

Remark that $\|S\|_{L^1(B_\varepsilon(0))}$ tends to zero as $\varepsilon \rightarrow 0$ since $S \in L^1_{loc}(\mathbb{R}^d)$. Since estimate (3.25) holds true for $\varepsilon > 0$ arbitrarily small and any given test function φ , we find that the divergence of the stress-energy tensor vanishes in the sense of distributions. Therefore v is stationary harmonic and the monotonicity formula (3.24) is satisfied for the time-independent solution $u(x, t) = v(x)$ of the harmonic map flow.

Let now $u(x, t) = R_\chi h(\frac{x}{\sqrt{t}})$ be an equivariant, selfsimilar weak solution of (3.1) that is induced by a non-constant function h . Then h must be an element of $H^1_{rad}(\mathbb{R}^d)$ which solves (3.6). Since our manifold has no minimal spheres, proposition 3.22 implies that $h(0)$ is a pole of N and that $h = h_a$ for a member of the family (h_a) of solutions to (3.6) given by proposition 3.14. Furthermore, $h'(r) = O(r^{\gamma-1})$ as $r \rightarrow 0$ for the number $\gamma \geq 1$ specified in proposition 3.14. The first derivatives $\partial_t u$ and ∇u of the selfsimilar weak solution u of (3.1) are thus bounded uniformly on compact subsets $\Omega \subset\subset \mathbb{R}^d \times (0, \infty)$. A parabolic bootstrapping argument then implies that u is smooth on $\mathbb{R}^d \times (0, \infty)$. The monotonicity inequality (3.24) is therefore valid away from $t = 0$.

It remains to verify that

$$t_0 \int_{\mathbb{R}^d} |\nabla u_0|^2 G_{(x_0, t_0)}(x, 0) \, dx = \lim_{t \nearrow t_0} t \int_{\mathbb{R}^d \times \{t_0 - t\}} |\nabla u|^2 G_{(x_0, t_0)} \, dx$$

for each $(x_0, t_0) \in \mathbb{R}^d \times (0, \infty)$. Here we denote by $u_0 := (h(\infty), \chi(\frac{x}{|x|}))$ the trace of u on $\mathbb{R}^d \times \{0\}$ for the limit $h(\infty) := \lim_{r \rightarrow \infty} h(r)$ which exists according to lemma 3.10.

Let t_0 and x_0 be fixed. Since $G_{(x_0, t_0)}$ is smooth away from the point (x_0, t_0) and since $G_{(x_0, t_0)}(x, 0) \leq C(t_0, x_0) \cdot e^{-c_0|x|^2}$ for $c_0 := \frac{1}{4t_0}$, the above claim easily follows if we show that

$$\Delta(t) := \int_{\mathbb{R}^d} [|\nabla u_0|^2 - |\nabla u(t)|^2] e^{-c_0|x|^2} \, dx \rightarrow 0 \text{ as } t \searrow 0.$$

Let us recall that according to lemma 3.10

$$|h'(r)| \leq \frac{\bar{C}}{r^3} \text{ and thus } |h(\infty) - h(r)| \leq \frac{\bar{C}}{2r^2} \text{ for } r \geq 1.$$

We can therefore estimate

$$\begin{aligned}
|\Delta(t)| &\leq C \int_0^\infty |g^2(h(\infty)) - g^2(h(\frac{r}{\sqrt{t}}))| r^{d-3} e^{-c_0 r^2} dr + \frac{C}{t} \int_0^\infty |h'(\frac{r}{\sqrt{t}})|^2 r^{d-1} e^{-c_0 r^2} dr \\
&\leq C \cdot \int_0^{\sqrt{t}} r^{d-3} dr + C \int_{\sqrt{t}}^\infty \frac{t}{r^2} \cdot r^{d-3} e^{-c_0 r^2} dr \\
&\quad + \frac{C}{t} \int_0^{\sqrt{t}} r^{d-1} dr + \frac{C}{t} \int_{\sqrt{t}}^\infty \frac{t^3}{r^6} \cdot r^{d-1} e^{-c_0 r^2} dr \\
&\leq C(t^{\frac{d-2}{2}} + t).
\end{aligned}$$

Since $d \geq 3$, we find that $\Delta(t) \rightarrow 0$ for $t \rightarrow 0$ as claimed. \square

10. An example

We have observed in remark 3.6 that the properties of the equator maps of a setting (N, χ) , and thus the issue of uniqueness for selfsimilar solutions, depend not only on the domain and the target manifold but also on the chosen eigenmap χ .

In this final section we now present examples of manifolds N for which uniqueness and non-uniqueness occur for a fixed setting (N, χ) . Indeed, we construct compact manifolds N of dimension $d \geq 3$ with two equators $C_{s_1^*}$ and $C_{s_2^*}$ and one minimal sphere C_{s_0} with the following properties.

- (i) There exists a unique selfsimilar and corotational weak solution of the harmonic map flow for each initial data $u_0(x) = (s, \frac{x}{|x|})$ with $s \in [p_1, s_0]$. Here $p_1 < s_1^*$ denotes the coordinate of a pole of N .
- (ii) Given any $K \in \mathbb{N}$ there exists a neighbourhood U_K of s_2^* such that the initial value problem (3.1), (3.2) has at least K different selfsimilar and corotational weak solutions for each initial data $u_0(x) = (s, \frac{x}{|x|})$, $s \in U_K$.

Before we proceed with the explicit construction, we need one more property of solutions to (3.6).

LEMMA 3.30. *Let N^n be any rotationally symmetric target and let $s_0 \neq s_1$ be two local minima of g^2 with $g^2(s_0) \geq g^2(s_1)$. Let $\chi : S^{d-1} \rightarrow S^{n-1}$ be an eigenmap and assume that conditions (C1) and (C2) hold true.*

Then the set of solutions $(h_a)_{a \in \mathbb{R}}$ to (3.6) with $h_a(0) = s_0$ of proposition 3.14 is bounded away from s_1 in the sense that

$$\varepsilon(s_0, N, \chi) := \inf_{a \in \mathbb{R}, r \geq 0} |s_1 - h_a(r)| > 0.$$

PROOF. We can assume without loss of generality that $s_0 < s_1$ and that there is only one equator C_{s^*} with lateral coordinate $s^* \in (s_0, s_1)$. If $4k\theta \leq (d-2)^2$ for $\theta := -G'(s^*)$, the comparison principle and proposition 3.17 imply that the solutions h_a do not even reach the level $s = s^*$ of the equator and thus that $\varepsilon(s_0, N, \chi) \geq s^* - s_0 > 0$.

So let us assume that $4k\theta > (d-2)^2$. We first analyse the behaviour of the solution \bar{h} of (3.16) with $\bar{h}(0) = s_0$ provided by lemma 3.13. According to proposition 3.17 and remark 3.12 the function \bar{h} oscillates around the level $s = s^*$ infinitely many

times without reaching the level $s = s_1$. We denote the first local maximum of \bar{h} by $\bar{r} > 0$ and claim that

$$\bar{h}(\bar{r}) = \max_{r \geq 0} \bar{h}(r).$$

Indeed, the quantity $V(\bar{h})$ introduced in (3.13) is decreasing and thus

$$-g^2(\bar{h}(\bar{r})) = V(\bar{r}) > V(r) \geq -g^2(\bar{h}(r))$$

for all $r > \bar{r}$. The claim follows since $\frac{d}{ds}(g^2)(\bar{h}(\bar{r})) = 2G'(\bar{h}(\bar{r})) < 0$ according to lemma 3.18 and remark 3.12

By the same reasoning

$$M(a) := \sup_{r \in [0, \infty)} h_a(r) = h_a(r_a)$$

if the solution h_a of (3.6) achieves a first local maximum at the point r_a . Lemma 3.16 implies that the functions h_a achieve such a local maximum at least for $a \gg 1$ and that $h_a(r_a) \rightarrow \bar{h}(\bar{r})$ as $a \rightarrow \infty$.

Consequently, $\lim_{a \rightarrow \infty} M(a) = \bar{h}(\bar{r}) < s_1$. The observation that $M : \mathbb{R} \rightarrow [s_0, s_1)$ is a continuous function with $M(a) = s_0$ for $a \leq 0$, see lemma 3.16 and remark 3.12, concludes the proof of lemma 3.30. \square

We now turn to the construction of the manifolds mentioned above. The basic idea is to attach a compact manifold whose corotational equator map is energy-minimising to another manifold of the same dimension for which the corotational equator map is not even locally energy-minimising, i.e. for which condition (3.4) is satisfied. The explicit form of these rotationally symmetric manifolds is not of great importance as long as the functions representing the metrics satisfy condition (C1). To fix ideas, we focus here on the case of two ellipsoids with different half-axis.

So let $d \geq 3$ be any given number and let $b > 0$. We denote by E_b the d -dimensional rotationally symmetric ellipsoid which has one half-axis of length b while all other half-axis have length 1. We can check that condition (C1) is satisfied for E_b for any choice of $b > 0$. We may thus apply the criterion presented in proposition 3.4 to characterise settings with energy-minimising equator maps. We obtain that the corotational equator map is energy-minimising if and only if

$$4(d-1) \leq (d-2)^2 b^2.$$

We choose numbers $b_1 \geq \frac{2\sqrt{d-1}}{d-2} > b_2 > 0$ and consider the ellipsoids $E_1 = E_{b_1}$ and $E_2 = E_{b_2}$ on which we introduce rotationally symmetric coordinates (s, ω) . We assume that one of the poles of E_i has lateral coordinate $s = 0$ and denote the coordinate of the second pole of E_i by $p_i > 0$, $i = 1, 2$.

For $0 < \varepsilon_i < p_i/2$, $i = 1, 2$, and $\delta > 0$ to be chosen later, we join the subsets

$$\begin{aligned} \hat{E}_1 &:= \{(s, \omega) : 0 \leq s \leq p_1 - \varepsilon_1\} \subset E_1 \\ \hat{E}_2 &:= \{(s, \omega) : \varepsilon_2 \leq s \leq p_2\} \subset E_2 \end{aligned}$$

by a neck of length $\delta > 0$, in such a way that the resulting manifold is smooth.

We introduce rotationally symmetric coordinates (s, ω) on the new manifold N . We denote by $s = 0$ and $s = p$ the lateral coordinates of the poles of N . Furthermore we let $0 < s_1^* < s_0 < s_2^* < p$ be the coordinates of the equators of N corresponding

to the original equators on E_i , $i = 1, 2$, respectively of the resulting minimal sphere. As usual g denotes the function representing the metric on N and $G = g \cdot g'$.

Choosing $\delta > 0$ small we may assume that

$$G'(s_0) = g(s_0)g''(s_0) \geq 1$$

and thus that condition (C2) is satisfied.

To understand the issue of uniqueness for selfsimilar solutions of the harmonic map flow from \mathbb{R}^d to N we need to analyse the behaviour of the solutions (h_a) of (3.6) of proposition 3.14.

Since the corotational equator map of the ellipsoid E_2 is not even locally energy-minimizing, i.e. since condition (3.4) is satisfied, the same holds true also for the corresponding equator map $u_{id, s_2^*}^*$ of our new manifold N . Let now $(h_a)_{a \geq 0}$ be the family of solutions of (3.6) with $h_a(0) = s_0$ from proposition 3.14. The values of these functions are contained in the interval $[s_0, p_1]$ according to remark 3.12. Furthermore, we may apply lemma 3.27 to describe the behaviour of the solutions h_a . We obtain that for $a \gg 1$, the function h_a oscillates around the level $s = s_2^*$ a large number of times and finally converges to a limit which is contained in a small neighbourhood of s_2^* . Thus the non-uniqueness property (ii) is valid for arbitrary choices of $\varepsilon_i < p_i$.

On the other hand, for ε_i close to $p_i/2$, $i = 1, 2$, solutions of (3.6) starting at the second pole p_2 may reach the part of N corresponding to points of \hat{E}_1 . The induced solutions of the harmonic map flow destroy the uniqueness property (i) that was valid for the original manifold E_1 .

However, choosing ε_2 less than the constant $\varepsilon(0, E_2, id) > 0$ of lemma 3.30 we find that the values of all solution of (3.6) with $h(0) = p_2$ are greater than s_0 .

In addition, for ε_1 and ε_2 sufficiently small the function G' is positive at the points $s = p_1 - \varepsilon_1$ and $s = p_1 - \varepsilon_1 + \delta$ where the neck is glued to \hat{E}_1 and \hat{E}_2 . Since $G(p_1 - \varepsilon_1) < 0 < G(p_1 - \varepsilon_1 + \delta)$ we can therefore construct our manifold N in such a way that $G'(s) \geq 0$ for points on the neck. We find that condition (C1) is fulfilled for N .

The arguments of the proof of theorem 3.3 now guarantee uniqueness for corotational, selfsimilar weak solutions of the harmonic map flow for all initial data $u_0(x) = (s, \frac{x}{|x|})$ with $p_1 \leq s \leq s_0$. Indeed, for $s \in [p_1, s_1^*)$ the unique solution $h \in H_{rad}^1(\mathbb{R}^d)$ of equation (3.6) with $\lim_{r \rightarrow \infty} h(r) = s$ is given by the element $h_{L^{-1}(s)}$ of the family (h_a) of proposition 3.14 with $h_a(0) = p_1$. Here L denotes the bijection of lemma 3.21. Similarly, the only solutions of (3.6) which induce selfsimilar, corotational weak solutions of (3.1) to initial data $u_0(x) = (s, \frac{x}{|x|})$ with $s_1^* < s \leq s_0$ are the members of the family $(h_a)_{a \leq 0}$ with $h_a(0) = s_0$.

Similar constructions lead to examples of manifolds with several equators where the regions with properties (i) and (ii) alternate.

Appendix

A. Interpolation inequalities

Let $d, m \in \mathbb{N}$, let $B_{2R}(x_0) \subset \mathbb{R}^d$ be a ball and let $\varphi \in C_c^\infty(B_{2R}(x_0))$ be a cut-off function. For functions $w \in W^{2m,1}(B_{2R}(x_0))$ we consider the (possibly infinite) integrals

$$I_k = I_k(w) = \int \varphi^{4m} |\nabla^k w|^{\frac{4m}{k}} dx, \quad k = 1, \dots, 2m.$$

Remark that these integrals are defined in such a way that the power of the cut-off function is the same in each integral. This feature of the interpolation inequalities derived below is useful in applications, as seen in sections 4 and 6 of the second chapter.

We first prove an interpolation inequality for these integrals which is valid in every dimension $d \in \mathbb{N}$. It allows us to bound all integrals I_k in terms of I_1 and I_{2m} as well as the rescaled total energy

$$\kappa(w) := 2R^{-2m} E(w, B_{2R}(x_0)) = R^{-2m} \sum_{k=1}^m \int_{B_{2R}(x_0)} |\nabla^k w|^{\frac{2m}{k}} dx.$$

PROPOSITION 3.31. *Let $d, m \in \mathbb{N}$, $B_{2R}(x_0) \subset \mathbb{R}^d$ and let $\varphi \in C_c^\infty(B_{2R}(x_0))$ be a cut-off function. Then for any function $w \in W^{2m,1}(B_{2R}(x_0))$ satisfying*

$$I_1(w) + I_{2m}(w) + \kappa(w) < \infty$$

all integrals $I_k = I_k(w)$, $1 \leq k \leq 2m$, are finite.

In addition, the interpolation inequality

$$(A.1) \quad I_k \leq C(I_1 + \kappa(w))^{\frac{1}{k} \cdot \frac{2m-k}{2m-1}} (I_{2m})^{\frac{2m}{k} \cdot \frac{k-1}{2m-1}} + C\kappa(w) + CI_1$$

holds true for $1 \leq k \leq 2m$ and a universal constant $C = C(m, d)$.

REMARK 3.32. *There is a similar interpolation inequality derived by Gastel in [25] which involves terms of the form $\frac{C}{R^{2m}}$ instead of $\kappa(w)$. As we should think of $\kappa(w)$ as a rescaled (small) local energy, our estimate is stronger. The main difference is that $\kappa(w)$ is a (rescaled) set-additive quantity which allows us to apply the above interpolation inequality in conjunction with covering arguments as in the proof of corollary 2.14. In turn, we are forced to be more careful in the computations in particular in the way we estimate lower order terms.*

PROOF OF PROPOSITION 3.31. We can assume without loss of generality that w is smooth. The general case then follows by approximating w by smooth functions and the density of $C^\infty(B_{2R}(x_0))$ in all Sobolev spaces $W^{k,p}(B_{2R}(x_0))$, $1 \leq p < \infty$, $k \in \mathbb{N}$.

So let $w \in C^\infty(B_{2R}(x_0))$ and let $\varphi \in C_c^\infty(B_{2R}(x_0))$ be a cut off function. We prove (A.1) using careful applications of partial integration and Hölder's and Young's inequality together with an induction argument.

For $2 \leq k \leq 2m - 1$

$$\begin{aligned} I_k &= - \int \nabla^{k-1} w \cdot \operatorname{div}(\varphi^{4m} |\nabla^k w|^{\frac{4m}{k}-2} \nabla^k w) dx \\ &\leq C \cdot (I_{k-1})^{\frac{k-1}{4m}} (I_k)^{\frac{4m-2k}{4m}} \cdot [(I_{k+1})^{\frac{k+1}{4m}} + \frac{C}{R} (L_k)^{\frac{k+1}{4m}}] \end{aligned}$$

where the lower order term L_k is given by

$$L_k := \int (\varphi^k |\nabla^k w|)^{\frac{4m}{k+1}} dx.$$

Since $\operatorname{supp}(\varphi) \subset\subset B_{2R}(x_0)$ and thus $I_k = I_k(w) < \infty$, we find

$$(A.2) \quad I_k \leq C \cdot (I_{k-1})^{\frac{k-1}{2k}} \cdot (I_{k+1})^{\frac{k+1}{2k}} + CR^{-\frac{2m}{k}} \cdot (I_{k-1})^{\frac{k-1}{2k}} \cdot (L_{k+1})^{\frac{k+1}{2k}}$$

by Young's inequality.

If $2 \leq k \leq m$, we directly estimate L_k in terms of the norms $\|\varphi^k \nabla^k w\|_{L^{\frac{2m}{k}}} \leq (R^{2m} \kappa(w))^{\frac{k}{2m}}$ and $\|\varphi^k \nabla^k w\|_{L^{\frac{4m}{k}}} = (I_k)^{\frac{k}{4m}}$ using the interpolation inequality of L^p spaces. Combined with estimate (A.2) we find

$$(A.3) \quad I_k \leq C \cdot (I_{k-1})^{\frac{k-1}{2k}} (I_{k+1})^{\frac{k+1}{2k}} + C \cdot (I_{k-1})^{\frac{k-1}{k+1}} \kappa(w)^{\frac{2}{k}} \text{ for } 2 \leq k \leq m.$$

For larger values of k it is no longer possible to estimate lower order terms directly by $\kappa(w)$ since no derivatives of order greater than m are contained in the total energy. Here lies the difference to the interpolation inequality of Gastel in that we do not directly apply Hölder's inequality to estimate L_k by I_k and a constant. Instead, we rewrite and estimate L_k until we are able to bound it using only the integrals I_l , $1 \leq l \leq 2m$ and the rescaled energy $\kappa(w)$.

First of all, we estimate L_k in terms of I_k and the corresponding L^2 -integrals

$$J_k := \int \varphi^{2k} |\nabla^k w|^2 dx$$

using the interpolation inequality of L^p spaces. The integrals J_k in turn satisfy

$$J_k = - \int \nabla^{k-1} w \cdot \operatorname{div}(\varphi^{2k} \cdot \nabla^k w) dx \leq (J_{k-1})^{\frac{1}{2}} \cdot [(J_{k+1})^{\frac{1}{2}} + \frac{C}{R} (J_k)^{\frac{1}{2}}]$$

and thus

$$J_k \leq C (J_{k-1})^{\frac{1}{2}} (J_{k+1})^{\frac{1}{2}} + CR^{-2} J_{k-1}.$$

Using induction we can easily prove

$$J_k \leq C \cdot (J_{k-h})^{\frac{j}{j+h}} (J_{k+j})^{\frac{h}{j+h}} + CR^{-2h} J_{k-h}$$

for all $k \in \mathbb{N}$ and $j, h \in \mathbb{N}$ with $1 \leq h \leq k-1$. In particular

$$\begin{aligned} J_k &\leq C(J_m)^{\frac{2m-k}{m}} (J_{2m})^{\frac{k-m}{m}} + C \cdot R^{4m-2k} J_m \\ (A.4) \quad &\leq C \cdot R^{4m-2k} \left[\kappa(w)^{\frac{2m-k}{m}} \cdot I_{2m}^{\frac{k-m}{m}} + \kappa(w) \right]. \end{aligned}$$

Altogether, we find for $m < k < 2m$

$$(A.5) \quad I_k \leq C \cdot (I_{k-1})^{\frac{k-1}{2k}} \cdot (I_{k+1})^{\frac{k+1}{2k}} + C \cdot (I_{k-1})^{\frac{k-1}{k} \cdot \frac{2m-k}{2m-k+1}} \cdot (I_{2m} + \kappa(w))^{\frac{2m}{k} \cdot \frac{1}{2m-k+1}}.$$

Combining (A.3) and (A.5) we have thus seen that

$$\begin{aligned} I_k &\leq C \cdot (I_{k-1} + \kappa(w))^{\frac{k-1}{2}} \cdot (I_{k+1})^{\frac{k+1}{2}} + C(I_{k-1} + \kappa(w)) \\ &\quad + C(I_{k-1} + \kappa(w))^{\frac{k-1}{k} \cdot \frac{2m-k}{2m-k+1}} \cdot (I_{2m})^{\frac{2m}{k} \cdot \frac{1}{2m-k+1}} \end{aligned}$$

for each $2 \leq k \leq 2m-1$.

This estimate may again be extended by induction leading to

$$\begin{aligned} I_k &\leq C \cdot (I_{k-h} + \kappa(w))^{\frac{k-h}{k} \cdot \frac{j}{j+h}} \cdot (I_{k+j})^{\frac{k+j}{k} \cdot \frac{h}{j+h}} + C \cdot (I_{k-h} + \kappa(w)) \\ &\quad + C \cdot (I_{k-h} + \kappa(w))^{\frac{k-h}{k} \cdot \frac{2m-k}{2m-k+h}} \cdot (I_{2m})^{\frac{2m}{k} \cdot \frac{h}{2m-k+h}} \end{aligned}$$

for every $2 \leq k \leq 2m-1$ and each $1 \leq h \leq k-1$ and $1 \leq j \leq 2m-k$.

Choosing $j = 2m-k$ and $h = k-1$ this is nothing else than the claim of proposition 3.31. \square

If the dimension is $d = 2m$, we may now prove the stronger interpolation statement of proposition 2.11, which we would like to recall was a key tool for the proof of the H^{2m} -estimates for almost harmonic maps of proposition 2.5.

PROOF OF PROPOSITION 2.11. Let $B_{2R}(x_0)$ be a ball in \mathbb{R}^{2m} and let $0 < \varepsilon_0 \leq 1$ be a constant that will be determined later on.

We consider functions $w \in H^{2m}(B_{2R}(x_0))$ with small total energy in the sense that

$$E(w, B_{2R}(x_0)) = \frac{1}{2} R^{2m} \kappa(w) \leq \varepsilon$$

for some $\varepsilon \leq \varepsilon_0$.

On the one hand, Sobolev's embedding theorem implies

$$\begin{aligned} I_1 &= \int (\varphi^2 |\nabla w|^2)^{2m} dx \leq C \cdot \left(\int |\nabla(\varphi^2 |\nabla w|^2)|^m dx \right)^2 \\ &\leq C \cdot \left[R^{-m} \int \varphi^m |\nabla w|^{2m} dx + \int \varphi^{2m} |\nabla^2 w|^m \cdot |\nabla w|^m dx \right]^2 \\ (A.6) \quad &\leq C \cdot E(w, B_{2R}(x_0)) \cdot (I_2 + \kappa(w)) \leq C\varepsilon(I_2 + \kappa(w)). \end{aligned}$$

On the other hand, according to proposition 3.31 the integral I_2 is bounded by

$$I_2 \leq C(I_1 + \kappa(w))^{\frac{m-1}{2m-1}} (I_{2m})^{\frac{m}{2m-1}} + CI_1 + C\kappa(w).$$

Combined with (A.6) and Young's inequality this implies

$$I_2 \leq \left(\frac{1}{2} + C\varepsilon\right)I_2 + C\varepsilon^{\frac{m-1}{m}}I_{2m} + C\varepsilon^{-1}\kappa(w).$$

If $\varepsilon_0 > 0$ is chosen small enough we recover the desired estimate for I_2 . By (A.6) also

$$I_1 \leq C \cdot \varepsilon(\varepsilon^{\frac{m-1}{m}}I_{2m} + \varepsilon^{-1}\kappa(w)) = C\varepsilon^{\frac{2m-1}{m}}I_{2m} + C\kappa(w)$$

as claimed. Together with the interpolation inequality of proposition 3.31 this finally implies the claim of proposition 2.11 for general values of $1 \leq k \leq 2m - 1$. \square

B. Collection of some additional proofs

B.1. Existence and uniqueness of solutions to (3.16). The proof of lemma 3.13 is based on well known methods in the theory of differential equations which were employed in a similar situation in [20]. We want to stress that condition (C2) is necessary only for the proof of the uniqueness statement.

PROOF OF THE EXISTENCE STATEMENT OF LEMMA 3.13. Let s_0 be any local minimum of g^2 , let $\gamma > 0$ be as in lemma 3.13 and let T_0 be the operator defined in (3.19) corresponding to equation (3.16). Since T_0 is regular away from $r = 0$, it is enough to consider equation (3.16) on a small interval $[0, \delta]$, $\delta > 0$. We show

Claim. There exist numbers $\delta > 0$, $M > 0$ such that

$$f_{\pm}(r) = s_0 + r^{\gamma} \pm M \cdot r^{\min(\gamma+1, 2\gamma)}$$

is a supersolution (respectively subsolution) of (3.16) on $(0, \delta]$, i.e. $T_0 f_+ \geq 0$ and $T_0 f_- \leq 0$ on $(0, \delta]$.

PROOF. It is important to remark that γ is chosen such that $\gamma^2 + (d-2)\gamma - kG'(s_0) = 0$. If $\gamma \geq 1$, we find by Taylor expansion

$$\begin{aligned} T_0 f_{\pm} &= r^{\gamma-2}[\gamma(\gamma-1) + (d-1)\gamma - kG'(s_0)] \\ &\quad \pm Mr^{\gamma-1}[\gamma(\gamma+1) + (d-1)(\gamma+1) - kG'(s_0) \mp k\frac{G''(s_0)}{2M}r^{\gamma-1} + O(r^{\gamma})] \\ &= \pm M \cdot r^{\gamma-1}\left[2\gamma + d - 1 \mp \frac{kG''(s_0)}{2M}r^{\gamma-1} + O(r^{\gamma})\right]. \end{aligned}$$

The claim follows for r small and M large enough. The second case $\gamma \in (0, 1)$ can be treated by a similar computation. \square

We choose the numbers $\delta > 0$ and M in the above claim such that $f_+ \leq b$ on $[0, \delta]$ for a number $b > s_0$ with $G'|_{[s_0, b]} > 0$.

We then define a sequence of solutions (f_n) of (3.16) by the initial conditions

$$\begin{aligned} f_n\left(\frac{1}{n}\right) &= \frac{1}{2}\left[f_+\left(\frac{1}{n}\right) + f_-\left(\frac{1}{n}\right)\right] = s_0 + \left(\frac{1}{n}\right)^{\gamma}, \\ f_n'\left(\frac{1}{n}\right) &= \frac{1}{2}\left[f_+' \left(\frac{1}{n}\right) + f_-' \left(\frac{1}{n}\right)\right] = \gamma\left(\frac{1}{n}\right)^{\gamma-1}. \end{aligned}$$

The second statement of lemma 3.18 implies that for every $n \in \mathbb{N}$

$$f_- < f_n < f_+ \text{ and } f'_- < f'_n < f'_+$$

on the interval $[\frac{1}{n}, \delta]$. Since the operator T_0 is regular away from $r = 0$, the family $\{f_n\}$ of solutions to (3.16) is thus uniformly bounded in $C^k([\varepsilon, \delta])$ for each $k \in \mathbb{N}$ and each $\varepsilon > 0$. By the theorem of Arzela-Ascoli we may extract a subsequence which converges in $C^2_{loc}((0, \delta])$ to a solution \bar{h} of (3.16). Because the initial conditions and the additional claim about the derivative of \bar{h} are satisfied by both f_- and f_+ , the same holds true also for \bar{h} . \square

PROOF OF THE UNIQUENESS STATEMENT OF LEMMA 3.13. Let s_0 be a local minimum of g^2 for which condition (C2) is satisfied. Suppose there exist two different solutions $h_1 \neq h_2$ of (3.16) with $h_1(0) = h_2(0) = s_0$ and $\lim_{r \searrow 0} r^{-\gamma}(h_1 - h_2)(r) = 0$. We may apply lemma 3.18 on a small interval $(0, \delta)$ and conclude that on this interval one of the solutions is strictly larger than the other, say $h_1 > h_2$.

The rescaled difference

$$f(r) = r^{1-\gamma}(h_1(r) - h_2(r)) \in C^1([0, \delta])$$

thus achieves a local minimum at $f(0) = 0$. It is a solution of the equation

$$(B.1) \quad f'' + a_1(r)f' + a_2(r)f = 0$$

for $a_1(r) = \frac{2\gamma+d-3}{r}$ and $a_2(r) = \frac{-2\gamma-d+3+k(G'(s_0)-G'(\xi(r)))}{r^2}$. Here $\xi = \xi(h_1, h_2)$ denotes an appropriate function with values $h_2 \leq \xi \leq h_1$ which is obtained by Taylor expansion. We make use of the following boundary point lemma (see for example [43], Theorem I.4).

PROPOSITION 3.33. *Consider a differential equation of the form (B.1) with coefficients $a_1, a_2 \in C((0, R])$ for some $R > 0$. Assume that $a_2 \leq 0$ and that $a_1(r) + ra_2(r)$ is bounded from below on $(0, R]$.*

Then for each non-constant solution f of (B.1) which achieves a non-positive local minimum at $r = 0$ and for which the one-sided derivative $f'(0^+)$ exists, we have

$$f'(0^+) > 0.$$

Turning back to the proof of lemma 3.13, we observe first of all that $s_0 - \xi(r) = O(r^\gamma)$ and thus also $G'(s_0) - G'(\xi(r)) = O(r^\gamma)$. Since $\gamma > 0$, the coefficient a_2 from above is thus negative for r small enough.

We claim that the second assumption of proposition 3.33 is also fulfilled for equation (B.1). Since $a_1(r) + ra_2(r) = r^{-1}(G'(s_0) - G'(\xi)) = O(r^{\gamma-1})$ this is obviously true if $\gamma \geq 1$, i.e. if $G'(s_0) \geq \frac{d-1}{k}$. Since $k \geq d-1$ and since condition (C2) is satisfied this inequality can only be violated if s_0 is a local maximum of G' . But then of course $a_1 + ra_2 \geq 0$ for r small and the assumptions of lemma 3.33 are once more satisfied.

Consequently we find that $f'(0^+) > 0$ for the function f defined as above. This contradicts the assumption that $f'(0^+) = \lim_{r \rightarrow 0} r^{-\gamma}(h_1(r) - h_2(r)) = 0$. \square

B.2. Proof of proposition 3.17. We present a sketch of the proof of proposition 3.17 based on the ideas of the original work of Jäger and Kaul [32].

For any given function $h \in C^2((0, \infty))$ we define $(q, p) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$ by

$$q(t) := h(e^t) \text{ and } p(t) = q'(t) = e^t h'(e^t).$$

Equation (3.16) can then be rewritten as an autonomous system of first order differential equations

$$(B.2) \quad \begin{pmatrix} q' \\ p' \end{pmatrix} = X(q, p) := \begin{pmatrix} p \\ -(d-2)p + kG(q) \end{pmatrix}.$$

As usual, we denote by $G = g' \cdot g$. The function $\bar{V} : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$\bar{V}(q, p) = p^2 - kg^2(q)$$

is a Lyapunov-function for the system (B.2); indeed,

$$\frac{d}{dt} \bar{V}(q(t), p(t)) = -2(d-2)p^2(t)$$

for every solution (q, p) of the system (B.2). We conclude that any bounded trajectory $t \mapsto (q(t), p(t)) \in \mathbb{R}^2$ of (B.2) converges to a critical point of \bar{V} as $t \rightarrow \infty$. These critical points are of the form $(\tilde{s}, 0)$, where \tilde{s} is a local extremum of g^2 .

To further analyse the asymptotic behaviour of trajectories of (B.2), we study the linearization

$$(B.3) \quad \begin{aligned} \begin{pmatrix} q' \\ p' \end{pmatrix} &= X(\tilde{s}, 0) + dX(\tilde{s}, 0) \cdot \begin{pmatrix} q - \tilde{s} \\ p \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ kG'(\tilde{s}) & -(d-2) \end{pmatrix} \cdot \begin{pmatrix} q - \tilde{s} \\ p \end{pmatrix} \end{aligned}$$

of the system (B.2) around these points $(\tilde{s}, 0)$.

If $G'(\tilde{s}) > 0$, i.e. if \tilde{s} is a local minimum of g^2 , the point $(\tilde{s}, 0)$ is a heteroclinic fixed point of X . A trajectory (q, p) of (B.2) converging to $(\tilde{s}, 0)$ is thus contained in the corresponding stable manifold of the system B.2. At the point $(\tilde{s}, 0)$ this one dimensional manifold is tangential to the eigenvector $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$ corresponding to the negative eigenvalue $\lambda_2 < 0$ of $dX(\tilde{s}, 0)$. Therefore, the sign of $p(t)$ is constant at least for large times t . We conclude that the convergence $h(r) \rightarrow \tilde{s}$ for $r \rightarrow \infty$ of the corresponding solution h of (3.6) is monotone.

If $C_{\tilde{s}}$ is an equator of N the fixed point $(\tilde{s}, 0)$ is asymptotically stable.

If in addition $-4kG'(\tilde{s}) > (d-2)^2$ then the eigenvalues $\lambda_1 = \bar{\lambda}_2$ are complex with $\text{Re}(\lambda_i) < 0$. Trajectories (q, p) of (B.2) which converge to $(\tilde{s}, 0)$ thus spiral into $(\tilde{s}, 0)$. The corresponding solution h of equation (3.16) therefore oscillates around the level $s = \tilde{s}$ infinitely many times.

Finally, if $C_{\tilde{s}}$ is an equator with $-4kG'(\tilde{s}) \leq (d-2)^2$, the eigenvalues of $dX(\tilde{s}, 0)$ are real and non-positive $\lambda_2 < \lambda_1 \leq 0$. Let now s_0 be the local minimum of g^2 to the left of \tilde{s} . We consider the triangle

$$\Delta := \{(q, p) : s_0 < q < \tilde{s}, 0 < p < \lambda_2 \cdot (q - \tilde{s})\} \subset \mathbb{R}^2.$$

Since $G(q) > 0$ for $q \in (s_0, \tilde{s})$ the vector $X(q, p)$ points into the triangle on the parts of the boundary of Δ contained either in $\{p = 0\}$ or in $\{q = s_0\}$. In addition,

condition (C1) and the assumption that G' is not constant on any interval of positive length imply

$$G(s) = G(\tilde{s}) - \int_s^{\tilde{s}} G'(\xi) d\xi < -G'(\tilde{s})(\tilde{s} - s)$$

for $s_0 < s < \tilde{s}$. Therefore

$$X(q, p) = \begin{pmatrix} p \\ -(d-2)p + kG(q) \end{pmatrix} = dX(\tilde{s}, 0) \cdot \begin{pmatrix} q - \tilde{s} \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ -f(q) \end{pmatrix}$$

for a positive function f on (s_0, \tilde{s}) . Since the third edge of the triangle Δ is in direction of the eigenvector $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$ of $dX(\tilde{s}, 0)$ we find that $X(q, p)$ points into the triangle Δ on the whole boundary of Δ .

This means that if a trajectory $t \mapsto (q(t), p(t))$ of the system (B.2) is in the triangle Δ at some time $t_0 \in \mathbb{R}$, it will remain in Δ for all times $t \geq t_0$. In particular, $p(t) > 0$ for $t \geq t_0$ and thus q is increasing on $[t_0, \infty)$.

On the other hand, if a solution (q, p) of (B.2) converges to $(\tilde{s}, 0)$ without entering Δ at all, q cannot achieve a local minimum for large times t according to lemma 3.18. Therefore, the convergence of $q(t) \rightarrow \tilde{s}$ is monotone also in this case.

Let now \bar{h} be the solution of equation (3.16) given by lemma 3.13 for a local minimum s_0 of g^2 . According to remark 3.12 and since N is compact, the functions \bar{h} and $r \mapsto r\bar{h}'(r)$ are bounded on $(0, \infty)$. The corresponding solution $t \mapsto (\bar{h}(e^t), \bar{h}'(e^t)e^t)$ of (B.2) is therefore bounded in \mathbb{R}^2 . The second statement of proposition 3.17 immediately follows from the discussion above.

For the proof of the first statement we assume that $-4kG'(s^*) \leq (d-2)^2$ for the equator C_{s^*} to the right of s_0 . Let now Δ be the triangle defined above. We observe that $(q(t), p(t)) \in \Delta$ for $t \ll 0$ since

$$\lim_{t \rightarrow -\infty} (q(t), p(t)) = (\bar{h}(0), \lim_{r \rightarrow 0} r\bar{h}'(0)) = (s_0, 0) \text{ and } q(t) > s_0, p(t) > 0 \text{ for } t \ll 0.$$

Thus the trajectory $t \mapsto (q(t), p(t))$ is contained in Δ for all times $t \in \mathbb{R}$. Therefore q , and consequently also \bar{h} , is increasing on the whole domain. Since $(s^*, 0)$ is the only critical point of \bar{V} in $\bar{\Delta}$ different from $(s_0, 0)$ we find that $(q(t), p(t)) \rightarrow (s^*, 0)$ as $t \rightarrow \infty$. This establishes statement (i) and thus concludes the proof of proposition 3.17.

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Curriculum Vitae

PERSONAL DATA

Name: Melanie Rupflin
Date of birth: March 27, 1982
Place of birth: St. Gallen, Switzerland
Citizenship: Swiss
Civil status: single

EDUCATION

since July 2006 Doctoral studies in Mathematics under the supervision of Prof. Michael Struwe, ETH Zurich, Switzerland

Oct 2001 - Mar 2006 Diploma studies in Mathematics, ETH Zurich, Switzerland
Diploma with distinction, Willi-Studer price for best diploma in Mathematics and ETH Medal for diploma thesis
Existenz schnellschwingender periodischer Lösungen bei steifen Differentialgleichungen mit konstanter Verzögerung

Aug 1997 - June 2001 Gymnasium with focus on natural sciences,
Kantonsschule Romanshorn, Switzerland

Aug 1989 - July 1997 Primary and Secondary School, Arbon, Switzerland

PROFESSIONAL EXPERIENCE

Apr 2006 - May 2010 Teaching assistant
Department of Mathematics, ETH Zurich, Switzerland

Mar 2003 - Mar 2006 Tutor and junior assistant
Department of Mathematics, ETH Zurich, Switzerland