Operator Preconditioning for Galerkin Boundary Element Methods on Screens

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Abstract

In this thesis we focus on acoustic and electromagnetic scattering at infinitely thin bounded objects, so-called screens in the literature. This kind of geometries arises in a broad variety of applications, from patch antenna design involved in wireless communications to the analysis of thin sound barriers for traffic noise isolation.

We solve the boundary value problems (BVPs) related to these scattering problems by means of boundary integral equations (BIEs). These BIEs give rise to boundary integral operators (BIOs), which we discretize using Galerkin boundary element methods (BEM), also known as method of moments.

Admittedly, one computational disadvantage of BEM is that the resulting Galerkin matrices are fully populated. For this reason, one resorts to compression techniques, such as hierarchical matrices. This permits, on the one hand, to build and store these matrices with reduced computational work and memory consumption, and, on the other hand, to be able to actually solve the obtained system with a reasonable amount of computational work. Nevertheless, by using these techniques, we also have to use iterative solvers and, given the ill-conditioned nature of the resulting Galerkin matrices for first-kind BIEs, these methods often converge prohibitively slowly. Thus, preconditioning becomes mandatory.

One popular approach to preconditioning is dubbed Calderón preconditioning, which is a particular case of operator preconditioning that uses Calderón identities that hold on closed surfaces and lead to well-conditioned systems that have condition numbers that are bounded independent of the mesh refinement [85, 26, 25, 4, 70]. Unfortunately, the fact that they are not valid on screens, makes that in that case these preconditioners achieve condition numbers that still grow like $\log(h)$, where $h$ is meshwidth [66].

The exterior Laplace BVP gives rise to BIEs involving the weakly singular operator in the case of Neumann boundary conditions and the hypersingular operator for Dirichlet data. This thesis presents for the first time exact inverses of these BIOs on disks. Moreover, we provide closed-form formulas that are amenable to Galerkin discretization and characterize the inverse operators as modifications of the standard BIOs on the disk. As expected, the correction in their kernels incorporates the distance to the boundary, in an analogous way to the 2D case [59, eq. (3.3)].

Additionally, we present Calderón-type identities and construct preconditioners using dual mesh operator preconditioning [46], which yield mesh-independent condition numbers for the BEM discretized BIEs for both the acoustic and the electromagnetic problems and for any screen that can be parametrized over the unit disk by a bi-Lipschitz diffeomorphism. Moreover, we can apply this preconditioning strategy to locally refined meshes. This is essential since solutions of boundary integral equations on screens have a square-root type singularity at the boundary of the screen. We provide numerical experiments to validate these theoretical findings.
Zusammenfassung


Wir verwenden Randintegralgleichungen, um die Randwertprobleme, die die Streuprobleme modellieren, zu lösen. Diese Randintegralgleichungen führen auf Randintegraloperatoren, die wir mit der Randelementmethode und dem Galerkinverfahren diskretisieren, auch bekannt unter dem Namen Momentenmethode.

Zugegeben,ein praktischer Nachteil der Randelementemethode ist, dass die entstehenden Galerkinmatrizen voll besetzt sind. Aus diesem Grund verwendet man Matrixkompresionstechniken, wie zum Beispiel hierarchische Matrizen. Dies erlaubt es uns, die Matrizen mit weniger Aufwand zu erstellen und abszuspeichern. Allerdings, wenn man diese Techniken benutzt, ist man gezwungen, iterative Löser einzusetzen und wegen der schlechten Kondition der Galerkinmatrizen für erster Art Randintegralgleichungen konvergieren diese Verfahren oft nur sehr langsam, was uns zur Vorkonditionierung zwingt.

Eine weitverbreitete Methode zur Vorkonditionierung ist bekannt unter dem Namen Calderón-Vorkonditionierung und ist eine spezielle Variante der Operatorvorkonditionierung die sich auf Calderón-Identitäten stützt. Sie führt auf gut konditionierte lineare Gleichungssysteme mit Konditionszahlen unabhängig von der Maschenweite \( h \) [85, 26, 25, 4, 70]. Leider sind diese Calderón-Identitäten auf Schirmen nicht verfügbar, so dass Calderón-Vorkonditionierung, wenn man sie trotzdem benutzt, nur zu Konditionszahlen führt, die immer noch logarithmisch in der Maschenweite zunehmen [66].

Äussere Randwertprobleme für den Laplace-Operator (führen auf) schwach singuläre und hypersinguläre Randintegraloperatoren beim Dirichlet- bzw. Neumannproblem. In der vorliegenden Arbeit gelang es zum ersten Mal, exakte Inverse dieser Randintegraloperatoren auf Kreisscheiben herzuleiten, gegeben durch analytische Ausdrücke, die sich als Grundlage für Galerkin-Diskretisierung eignen und die inversen Operatoren als abgewandelte Standard-Randintegraloperatoren formulieren. Wie erwartet, kommt die Entfernung zum Rand in den modifizierten Integralkernen vor, ganz aehnlich wie auch in zwei Dimensionen [59, eq. (3.3)].

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Introduction

At the time of writing this thesis, it has been 158 years since von Helmholtz first discussed how to solve the reduced wave equation named after him, and 153 years since James Clerk Maxwell published “A Dynamical Theory of the Electromagnetic Field” with the complete set of equations describing electromagnetism [64]. In the meantime we have gained some significant understanding of wave phenomena and yet, for every question physicists, engineers and mathematicians manage to answer, there are even more questions that we are able to ask and scattering theory continues to fascinate us and to provide a rich niche of research.

This thesis focuses in the acoustic and electromagnetic scattering of time-harmonic waves at infinitely thin bounded objects, so-called screens in the literature. This kind of geometries arises in a broad variety of applications, from patch antenna design involved in wireless communications to the analysis of thin sound barriers for traffic noise isolation. In practice, the thickness of these objects is negligible with respect to their other dimensions. For this reason, screens are modeled as “open surfaces” in 3D, which one describes mathematically as two-dimensional manifolds with boundary. This approximation naturally captures the extreme proximity of the two sides of the scatterer and thus the physical behavior of the scattered field.

The question we seek to answer in this dissertation is how to construct preconditioners such that the solution via iterative solvers to the linear systems arising from these screen problems are found faster. We dedicate the next paragraphs to make this statement clearer and to explain why this question is worth answering.

Broadly speaking, there are two types of numerical methods for the resolution of scattering problems on unbounded domains: volumetric PDE methods and boundary integral methods. The former covers discretization methods such as finite element methods and finite differences, where one needs to artificially impose and approximate absorbing boundary conditions to truncate the computational domain. The latter recasts the related scattering problem as a boundary integral formulation. In this way, finding the scattered field boils down to solving boundary integral equation (BIE). The formulation via BIEs deals with the unboundedness of the domain and respects decay conditions at infinity by construction. This offers an important advantage over volume-based discretization methods. Additionally, BIE methods reduce the problem dimensionality when discretizing and do not suffer from dispersion errors, which is an important property when dealing with wave propagation.

In this thesis, we pursue BIE methods using Galerkin boundary element methods (BEM), also known as method of moments. This choice of numerical treatment for the related BIE is fairly mature and builds on the analysis of the arising boundary integral operators (BIOs) in the framework of Sobolev spaces, which has been studied thoroughly also in the case of screens. We refer to [87, 24] for time-harmonic acoustic scattering and to [1, 16] for the time-harmonic electromagnetic problem. As we will see later in this thesis, the relevant BIOs for these scattering problems are the weakly singular and hypersingular operator for the acoustic case, and the EFIE operator for electromagnetic scattering at screens.

Admittedly, one computational disadvantage of BEM is that the resulting Galerkin matrices are fully populated. For this reason, one resorts to compression techniques, such as hierarchical matrices. This permits, on the one hand, to build and store these matrices with reduced computational work and memory consumption, and, on the other hand, to be able to actually solve the obtained system with a reasonable amount of computational work. Nevertheless, by using these techniques, we also have to use iterative solvers and, given the ill-conditioned nature of the resulting Galerkin matrices, these methods will converge prohibitively slowly, if they converge at all in the presence of round-off error. Moreover, it is also important to take into account that due to numerical approximations, we are actually solving a perturbed matrix system and the accuracy of the obtained solution degrades is presence of
ill-conditioning. These two facts motivate the use of preconditioners in order to achieve well-conditioned matrices such that these iterative solvers converge in a small number of iterations and no additional errors are introduced.

When aiming to build suitable preconditioners for screens problems, it is important to take into account some of their particular features that make the corresponding numerical treatment more difficult than when dealing with closed surfaces: we have that the domain is not Lipschitz; the mapping properties of our BIOs change; and we face edge singularities that behave like square root of the distance to the boundary of the screen. This last aspect causes slow convergence of the approximation error with respect to the mesh refinement. Since one can improve the approximation error convergence rate by refining towards the boundary of the screen [32], in this thesis we seek to build preconditioners that also work on non-uniform meshes.

There are many ways to construct preconditioners. One could choose for instance subspace correction methods, which build on multilevel decompositions of the boundary element spaces and are effective also on very fine meshes. These preconditioning methods have been developed for screens in [89, 3, 36] for the acoustic case and in [43, 54] for electromagnetism, sometimes even covering the case of locally refined meshes. Alternatively, one can opt for operator preconditioning that exploits the properties of the BIOs on the continuous level to establish endomorphisms that give rise to well-conditioned matrix products and thus provides suitable preconditioners.

One popular approach to preconditioning is dubbed Calderón preconditioning, which is a particular case of operator preconditioning that leverages the fact that specific compositions of BIOs, namely the Calderón identities, evaluate to the identity plus a compact operator [31]. Case of operator preconditioning that leverages the fact that specific compositions of BIOs, namely the Calderón identities, evaluate to the identity plus a compact operator [85, 26, 25, 4, 70]. These operator properties hold on closed surfaces and lead to mesh independent condition numbers. Unfortunately, they do not hold on screens. Indeed, when using them in the screen setting, they achieve condition numbers that still grow like \( \log(h) \), where \( h \) is the meshwidth [66].

Further improvement was achieved by Bruno and Lintner [14] who introduced weights in the kernels of the BIOs. Such weights depend on the distance to \( \partial \Gamma \) and, by incorporating information on the singular behavior of the solution, overall performance is improved, as in augmented schemes [86]. Bruno and coworkers extended these ideas to smooth screens in 3D: first for the acoustic case in [13], and afterwards for the electromagnetic problem in [75, Sect. 7.2.2], with some preliminary and very promising results. Still, no rigorous numerical analysis has been reported for either of the two extensions.

Concurrently to the above, exact variational inverses established in [59] for the Laplace problem on the line segment resulted in explicit Calderón-type preconditioners for open arcs [49], which are the two-dimensional counterpart to screens and sometimes even dubbed 2D screens. The inverses in [59] are introduced as modified weakly singular and hypersingular operators, and one observes that they also incorporate the distance from \( \partial \Gamma \) in their kernels. Still, a major tool is the use of spectral decompositions using Chebyshev polynomials. Indeed, it allows to show that the difference between the kernel of the modified BIOs and the standard ones also reflects the gap between the norms of the standard trace spaces and those particular of screen problems. This breakthrough in 2D was the starting point for this thesis and gave the key idea of finding exact inverses for the BIOs arising from the Laplace screen problem on the Disk and building Calderón-type identities from them.

In analogy to the Chebyshev spectral decompositions for the 2D case, Jerez-Hanckes and Nédélec started working on the spectral decompositions of the operators on 3D by means of spherical harmonics projected from the positive half sphere onto the unit disk. This collaboration was our starting point to begin looking for the inverse operators, although soon after finding their series expansion and deriving some basic properties, we noticed that unlike the two-dimensional case, the series provided no clear insight to find the closed-form formulas for these operators. Consequently, we changed our approach and in parallel Nédélec’s group continued working on the spectral approach which concluded in the thesis [77]. Although they could not find a closed-form formula for the inverse operators, they used the series and projections from the sphere instead to propose preconditioners for the acoustic problems on screens, which perform suboptimal in the sense that the resulting condition numbers grow with the mesh refinement.

Meanwhile, thanks to Fabrikant’s solutions to the BIEs for the Laplace equation on the unit disk [34] and their further discussion from the BIEs’ setting by Li and Rong [61], we were able to find the closed-
form formulas of the inverse BIOs for that case. Interestingly, these formulas characterize the inverse operators as modifications of the standard weakly singular and hypersingular operators over the disk. As expected, the correction in their kernels incorporates the distance to the boundary, in an analogous way to the function $M(x,y)$ in the 2D case [59, eq. (3.3)]. In this thesis we present these operators and propose new Calderón preconditioners using dual mesh operator preconditioning [46], which yield optimal condition numbers for the BEM discretized BIEs for both the acoustic and the electromagnetic problems.

**Outline of this thesis**

This thesis is structured in two parts: First, we present the results for the acoustic case, meaning we study the Helmholtz and Laplace equations for screen problems. Second, we tackle electromagnetic scattering at screens, which leads to the electric field integral equation (EFIE). These parts have 4 and 3 chapters, respectively. The outline of the chapters is the following:

**Chapter 1.** In view of the extensive yet scattered literature for this problem, we found it necessary to begin with a summary of the existing mathematical tools to solve the acoustic boundary value problems on screens using BEM.

**Chapter 2.** We develop one of the main results of this thesis: The closed-form formulas for the BIOs that are the exact inverse operators of the Laplacian over disks and that are actually amenable to Galerkin discretization. With these modified BIOs, we are able to show Calderón-type identities that will allow us to build preconditioners by means of the operator preconditioning technique as in [46].

**Chapter 3.** We begin by recapitulating operator preconditioner theory. Next we present the construction of all the required pieces and proofs to use the operators found in Chapter 2 to precondition the BIEs at hand. This includes the extension of the discrete stability of the $L^2$-duality pairing over non-uniform meshes under certain mesh assumptions from [84] to the Sobolev spaces involved in screen BIOs.

Combining all these pieces, we formulate preconditioners for Laplace and Helmholtz BIOs and display the obtained numerical results. We end Chapter 3 by extending our preconditioners to screens that can be parametrized over the unit disk by a bi-Lipschitz diffeomorphism and showing the corresponding numerical results for the Laplace BIOs.

**Chapter 4.** This last chapter of the acoustic part of the thesis is devoted to discuss some heuristic approximations of the inverses found in Chapter 2 on more general shapes of screens, and how to use them to build preconditioners. We numerically test the applicability of this construction and present the numerical results obtained for the Laplace equation.

**Chapter 5.** As for the acoustic case, we start the second part of this thesis by presenting the existing theory for the EFIE and BEM on screens.

**Chapter 6.** We derive an operator that is the inverse up to a compact operator of the EFIE on disks. For this we use the Hodge decomposition and the inverse operators found in Chapter 2. We prove that the resulting compact-equivalent inverse is continuous and satisfies its inf-sup condition. Furthermore, we take pains to formulate this operator such that is amenable to stable Galerkin discretization.

**Chapter 7.** We describe how to use the operator found in Chapter 6 to build a preconditioner employing the operator preconditioning theory. Again, we prove the discrete stability of our preconditioner on the same family of non-uniform meshes that was assumed in Chapter 3. Then we show the corresponding numerical results on the unit disk. Finally, as in Chapter 3, we finish our presentation with the extension of the EFIE preconditioner to parametrized screens and the exposition of the related numerical experiments.
Part I.

Acoustic Scattering at Screens
1. Boundary Element Methods on Screens

“Nanos gigantum humeris insidentes.”

– Attributed to Bernard of Chartres (12th century).

This Chapter introduces some of the notation we are going to use in this thesis and briefly summarizes the existing mathematical framework to solve the scattering of time-harmonic waves by screens using boundary integral equations and its numerical solution via boundary element methods (BEM). Most of this chapter is adapted from [58, 81, 87].

1.1. Geometric Considerations

Every story starts with the introduction of its main characters and a thesis is no exception. Although we have already mentioned that screens are mathematical representations of infinitely thin bounded objects in \( \mathbb{R}^3 \), we still have not given a specific description for it.

Screens are often portrayed as open surfaces on \( \mathbb{R}^3 \), where the term “open” refers to this surface having a non-empty boundary \( \partial \Gamma \), in contrast to “closed” surfaces that do not have a boundary, like the sphere. Rigorously speaking, this means that a screen \( \Gamma \subset \mathbb{R}^3 \) is a manifold of co-dimension equal to one and boundary \( \partial \Gamma \) of positive measure.

In this thesis, we consider two kinds of screens:

1. Planar Lipschitz or smoother screens. In other words, \( \Gamma \) is (at least) a Lipschitz screen such that \( \Gamma \subset \Gamma_\infty := \{x = (x_1, x_2, 0) \in \mathbb{R}^3 \} \).

   We point out that polygonal screens and smooth screens such as the unit disk \( D_1 = \{x \in \mathbb{R}^3 : x_3 = 0 \text{ and } ||x|| < 1 \} \) belong to this set.

   Here \( \Gamma_\infty \) splits \( \mathbb{R}^3 \) into an upper and lower halfspace. We denote the resulting two sides of the screen \( \Gamma \) by \( \Gamma_+ \) and \( \Gamma_- \), respectively.

2. \( \Gamma \) such that there are at least bi-Lipschitz mappings \( \phi : D_1 \rightarrow \Gamma \).

   As a consequence, \( \Gamma \) are orientable \( C^{0,1} \)-manifolds with boundary \( \partial \Gamma \). Moreover, they inherit the distinction of the two sides \( \Gamma_\pm \) from \( D_1 \).

From the point of view of the existing theory of BEM on screens, it is worth mentioning that the above definitions make that the screens under study in this thesis have the following properties:

- They are (globally) orientable Lipschitz screens.
- We can always find a fictitious Lipschitz domain \( \Omega_c \subset \mathbb{R}^3 \) such that \( \Gamma \subset \partial \Omega_c \).
- \( \Gamma \) has a tangent plane and an unit normal \( n(x) \) at almost every \( x \in \Gamma \).

We point out that the first difference between screen problems and scattering problems at closed surfaces is that, regardless of the smoothness of the boundary \( \Gamma \), our domain \( \Omega := \mathbb{R}^3 \setminus \Gamma \) fails to be Lipschitz, as \( \Omega \) is on both sides of its boundary. As we will see in the next sections, this fact will change some of the usual definitions and results but still allows us to construct the required pieces to use the BEM machinery.

---

1 See [28, Def. 2.1] or [16, Sect. 1.1] for the general definition of a Lipschitz screen (including the non-globally-orientable case).

2 This is a consequence of \( \Gamma \) being Lipschitz and Rademacher’s theorem [81, Theorem 2.7.1].
1.2. Acoustic Model Problem

We consider the reduced wave equation in the surrounding medium \( \Omega := \mathbb{R}^3 \setminus \Gamma \),

\[ -\Delta U - k^2 U = 0 \quad \text{in } \Omega, \]  

(1.1)

where \( k \geq 0 \) is the wave number. We point out that this corresponds to the Helmholtz equation when \( k > 0 \), while it yields the Laplace equation for \( k = 0 \).

We are interested in this partial differential equation in two situations: when our screen is a sound-soft obstacle, which is represented mathematically by a homogeneous Dirichlet boundary condition

\[ U = 0, \quad \text{on } \Gamma, \]  

(1.2)

and when our screen is a sound-hard obstacle, which corresponds to homogeneous Neumann boundary conditions

\[ \frac{\partial U}{\partial \mathbf{n}} = 0, \quad \text{on } \Gamma, \]  

(1.3)

where \( \mathbf{n} \) is the normal vector on \( \Gamma \) and \( \frac{\partial}{\partial \mathbf{n}} \) denotes the normal derivative.

Let us write \( U = U^s + U^i \), with \( U^i \) the incident wave and \( U^s \) the scattered wave. In order to ensure a unique solution, we will also impose the Sommerfeld radiation condition

\[ \lim_{r \to \infty} r \left( \frac{\partial U^s}{\partial r} - i k U^s \right) = 0, \quad r = \| \mathbf{x} \|, \]  

(1.4)

where \( \| \mathbf{x} \| \) designates the Euclidean norm of a point \( \mathbf{x} \) in \( \mathbb{R}^3 \).

We can now concretely formulate the exterior boundary value problems of interest.

**Problem 1.2.1 (EDP).** For \( k \geq 0 \), we define the exterior Dirichlet problem (EDP) as

\[
\begin{cases}
-\Delta U^s - k^2 U^s = 0 & \text{in } \Omega, \\
U^s = -U^i & \text{on } \Gamma,
\end{cases}
\]  

(1.5)

**Problem 1.2.2 (ENP).** For \( k \geq 0 \), we define the exterior Neumann problem (ENP) as

\[
\begin{cases}
-\Delta U^s - k^2 U^s = 0 & \text{in } \Omega, \\
\frac{\partial U^s}{\partial \mathbf{n}} = -\frac{\partial U^i}{\partial \mathbf{n}} & \text{on } \Gamma,
\end{cases}
\]  

(1.6)

**Remark 1.2.3.** Note that for the static case \( k = 0 \) (Laplace equation) the Sommerfeld radiation condition (1.4) is replaced by the decay condition \( U^s(\mathbf{x}) = O(\| \mathbf{x}^{-1} \|) \) as \( \| \mathbf{x} \| \to \infty \).

1.3. Spaces and Traces

1.3.1. Spaces

This section is mainly dedicated to present definitions and concepts that are relevant to the case of screens. We refer to [81, Chapter 2] for the standard definitions and begin by fixing some notation.

As usual, for all Banach spaces \( X \), we denote their norms by \( \| \cdot \|_X \) and their dual spaces by \( X' \). If the space \( X \) is also Hilbert, we put \( (\cdot, \cdot)_X \) for its inner product.

Let \( \mathcal{O} = \mathbb{R}^d \) or a domain \( \mathcal{O} \subset \mathbb{R}^d \) with \( d = 2, 3 \). We write:
1.3. Spaces and Traces

- \( C^k(O) \) with \( k \in \mathbb{N} \): Space of all \( k \) times continuously differentiable functions \( O \to \mathbb{C} \).
- \( C^{k, \lambda}(O) \) with \( k \in \mathbb{N} \) and \( 0 \leq \lambda \leq 1 \): \( C^k(O) \)-functions whose \( k \)-th derivative is Hölder continuous with exponent \( \lambda \).
- \( L^p(O) \) with \( 1 \leq p \leq \infty \): \( p \)-integrable measurable functions.
- \( \langle \cdot, \cdot \rangle_O \) for the \( L^2(O) \)-bilinear form \( \langle u, v \rangle_O := \int_O u(x)v(x)dx \), \( u, v \in L^2(O) \) and induced duality pairings.
- \( \mathcal{D}(O) = C_{\text{comp}}^\infty(O) \): Smooth compactly supported functions \( O \to \mathbb{C} \).
- \( H^s(O), H^s_0(O) \) with \( s \in \mathbb{R} \): Sobolev spaces \([81, \text{Sect. 2.3}]\).
- \( H^s_{\text{loc}}(O) := \{ v \in (C_{\text{comp}}^\infty(O))' : \varphi v \in H^s(O) \forall \varphi \in C_{\text{comp}}^\infty(O) \} \), for \( s \geq 0 \).

We also use this notation for spaces on boundaries \( \partial O \) of Lipschitz domains \( O \subset \mathbb{R}^3 \), although we remark that the associated definitions change in the following two situations where spaces rise from traces: \( \mathcal{D}(\partial O) := \mathcal{D}(\mathbb{R}^3)_{|\partial O} \) and \( H^s(\partial O), H^s_0(\partial O) \) with \( s \in \mathbb{R} \) (see \([81, \text{Sect. 2.4.1}]\)).

Unlike the case of closed surfaces, when dealing with screens \( \Gamma \), we work with the tilde spaces \( \tilde{H}^{1/2}(\Gamma) \) in addition to \( H^{1/2}(\Gamma) \), which are often called (single) trace spaces. Moreover, the definitions and duality relations of these four spaces also differ from the closed surface case.

There are several equivalent definitions for these spaces on screens, each of them providing valuable insight about the properties of solutions to BIEs at screen. For this reason, we will supply a brief survey on their definitions and properties.

In most of these definitions, the spaces of negative index \( H^{-1/2}(\Gamma) \) and \( \tilde{H}^{-1/2}(\Gamma) \) are introduced as the completion of \( L^2(\Gamma) \) with respect to the dual norm. In other words, they are defined as dual spaces with respect to the \( L^2(\Gamma) \)-bilinear pairing

\[
H^{-1/2}(\Gamma) = \left( \tilde{H}^{1/2}(\Gamma) \right)', \quad \tilde{H}^{-1/2}(\Gamma) = \left( H^{1/2}(\Gamma) \right)',
\]

and equipped with the usual dual norm.

a) Definition by extension to a closed surface

This is the more standard definition and it relies on the assumption that there is a Lipschitz domain \( \Omega_c \subset \mathbb{R}^3 \) such that \( \Gamma \subset \partial \Omega_c \) (see for example \([58, \text{Sect. 4.3}]\)[87][83, \text{Sect. 2.5}] [81, \text{Sect. 2.4.2}]\)).

We introduce the spaces \( H^s(\Gamma) \), for \( s \geq 0 \) by

\[
H^s(\Gamma) = \{ v = \tilde{v}\Gamma : \tilde{v} \in H^s(\partial \Omega_c) \},
\]

and furnish it with the norm

\[
\|v\|_{H^s(\Gamma)} = \inf_{\tilde{v} \in H^s(\partial \Omega_c): \tilde{v}_{|\Gamma} = v} \|\tilde{v}\|_{H^s(\partial \Omega_c)}.
\]

Let \( \mathcal{D}(\Gamma) := \{ v \in \mathcal{D}(\partial \Omega_c) : \supp v \subseteq \Gamma \} \). Then we can define \([65, \text{p. 99}]\]

\[
\tilde{H}^s(\Gamma) = \text{ closure of } \mathcal{D}(\Gamma) \text{ in } H^s(\partial \Omega_c).
\]

Alternatively, we can consider a function \( v \) defined on \( \tilde{\Gamma} \) and \( Zv \) its extension by zero from \( \tilde{\Gamma} \) to \( \partial \Omega_c \) given by

\[
Zv(x) = \begin{cases} 
  v(x), & \text{for } x \in \Gamma, \\
  0, & \text{for } x \in \partial \Omega_c \setminus \Gamma.
\end{cases}
\]

\(^3\)The spaces \( \tilde{H}^s(\Gamma) \) are also denoted \( H^s_{\text{loc}}(\Gamma) \) and are many times referred as Lions-Magenes spaces.
Then for $|s| \leq 1$ we also have the following characterization [58, Sect. 4.3]
\begin{equation}
\tilde{H}^s(\Gamma) := \{ v \in H^s(\partial\Omega_c) : v|_{\partial\Omega_c \setminus \bar{\Gamma}} = \{ v \in H^s(\partial\Omega_c) : \text{supp } v \subset \bar{\Gamma} \}
\end{equation}
as a subspace of $H^s(\partial\Omega_c)$ equipped with the corresponding norm
\begin{equation}
\|v\|_{\tilde{H}^s(\Gamma)} = \|Zv\|_{H^s(\partial\Omega_c)}.
\end{equation}
Finally, we introduce the space $H^s_0(\Gamma)$ as the closure of $D(\Gamma)$ in $H^s(\Gamma)$ [65, p. 99]. It is worth noticing that
\begin{equation}
\tilde{H}^s(\Gamma) = H^s_0(\Gamma)
\end{equation}
however, $\tilde{H}^{m+1/2}(\Gamma) \subsetneq H^{m+1/2}_0(\Gamma)$ for $m \in \mathbb{N}_0$.

b) Definition via interpolation spaces

Here, we borrow the notation and concepts introduced in [43, Chapter 2]. Let $A_0$ and $A_1$ be two Banach spaces with a continuous embedding $A_1 \hookrightarrow A_0$, then the interpolation space
\begin{equation}
A_s = [A_0, A_1]_s, \quad s \in (0, 1)
\end{equation}
is furnished with the norm
\begin{equation}
\|v\|_{A_s} := \left( \int_0^\infty (t^{-s} \inf_{v_1 \in A_1} \{ \|v - v_1\|_{A_0} + t \|v_1\|_{A_1} \})^2 \frac{dt}{t} \right)^{1/2}.
\end{equation}
We can use this technique and for $s \in (0, 1)$ define our spaces as
\begin{equation}
H^s(\Gamma) = [L^2(\Gamma), H^1(\Gamma)]_s \quad \tilde{H}^s(\Gamma) = [L^2(\Gamma), H^1_0(\Gamma)]_s.
\end{equation}

c) Definition via norm

Finally, we also have the following interesting expressions for the tilde spaces:
\begin{equation}
\tilde{H}^{1/2}(\Gamma) = \{ v \in H^{1/2}(\Gamma) : d(x, \partial\Gamma)^{-1/2}v \in L^2(\Gamma) \},
\end{equation}
where $d(x, \partial\Gamma)$ is the distance of $x$ to the boundary $\partial\Gamma$ [88, Chapter 33].
This definition can also be extended to $\tilde{H}^s(\Gamma)$ with $s = m + \frac{1}{2}$, $m \in \mathbb{N}_0$ as
\begin{equation}
\tilde{H}^{m+1/2}(\Gamma) = \{ v : v \in H^{m+1/2}_0(\Gamma), d(x, \partial\Gamma)^{-1/2} D^\alpha v \in L^2(\Gamma) \text{ for } |\alpha| = m \},
\end{equation}
where $D^\alpha$ stands for the covariant derivatives in a closed surface $\Gamma_c$ such that $\Gamma \subset \Gamma_c$ [58, Sect. 4.3].

Remark 1.3.1. One can also define these spaces via the trace and jump operators, as we will discuss in the next subsection.

4This property is not actually required but allows us to give a short introduction to the $K$-method of interpolation, also dubbed Real Interpolation Method. This property always holds in the cases treated in this thesis. We refer to [65, App. B] [81, Sect. 2.1.7] for further details.
Properties

There are some important features of the tilde and trace spaces on screens that we summarize in this subsection. We remark that all of them will hold for the screens considered in this thesis.

Let us begin by pointing out that our first two definitions (1.12) and (1.41) highlight that \( \tilde{H}^{1/2}(\Gamma) \) and also \( \tilde{H}^{-1/2}(\Gamma) \) contain functions that are zero on \( \partial \Gamma \). Our third definition also enforces this property for \( \tilde{H}^{1/2}(\Gamma) \) and, in fact, clearly portrays the square-root edge singularity one expects from those functions, as it will become clear later in this Chapter.

Along these lines we also have the following property:

**Property 1.3.2** ([62, Eq. (12.5)]). \( \nu \in H^{-1/2}(\Gamma) \Leftrightarrow \nu = \nu_0 + \nu_1, \nu_0 \in \tilde{H}^{-1/2}(\Gamma), \nu_1 \sqrt{d(x, \partial \Gamma)} \in L^2(\Gamma) \).

Now, it is manifest that the different definitions of \( \tilde{H}^{1/2}(\Gamma) \) imply norm equivalences, which are written explicitly in the next Corollary.

**Corollary 1.3.3.** If \( \Gamma \) is a globally orientable screen. The following norms for \( \nu \in \tilde{H}^{1/2}(\Gamma) \) are equivalent

\[
\begin{align*}
&i \left\{ \|v\|_{H^{1/2}(\Gamma)}^2 + \|d(x, \partial \Gamma)^{-1/2} v\|_{L^2(\Gamma)}^2 \right\}^{1/2}, \\
&ii \left( \int_0^\infty (t^{-1/2} \inf_{\nu_1 \in H^{1/2}(\Gamma)} \{\|v - \nu_1\|_{L^2(\Gamma)} + t \|\nu_1\|_{H^{1/2}(\Gamma)}\})^{2/3} \frac{dt}{2} \right)^{1/2}, \\
&iii \|Zv\|_{H^{1/2}(\partial \Omega_c)}, \text{ with } Z \text{ and } \partial \Omega_c \text{ defined as in (1.11) and subsection 1.3.2a), respectively.}
\end{align*}
\]

Furthermore, the extension by zero operator \( Z \) marks a difference between screens and closed surfaces.

**Proposition 1.3.4** ([47]). Let \( \Omega_c \subset \mathbb{R}^3 \) be a bounded Lipschitz domain with trivial topology such that \( \Gamma \subset \partial \Omega_c \).

Then the operator \( Z \) defined in (1.11) is continuous as a mapping \( L^2(\Gamma) \to L^2(\partial \Omega_c) \) and \( H^{1/2}_0(\Gamma) \to H^1(\partial \Omega_c) \), but unbounded as a mapping \( H^1(\Gamma) \to H^1(\partial \Omega_c) \) and \( H^{1/2}(\Gamma) \to H^{1/2}(\partial \Omega_c) \).

On the other hand, the properties of Sobolev spaces on Lipschitz domains in \( \mathbb{R}^3 \) carry over to Sobolev spaces on \( \Gamma \), subject to the condition that \( |s| \leq 1 \) [65, p. 99]. This gives us some important results.

**Property 1.3.5** ([58, Sect. 4.3]). We have the inclusions for \( s > 0 \):

\[
\tilde{H}^s(\Gamma) \subset H^s(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-s}(\Gamma) \subset H^{-s}(\Gamma).
\]

These inclusions are strict for \( s \geq \frac{1}{2} \), while

\[
\tilde{H}^s(\Gamma) = H^s(\Gamma) \quad \text{for} \quad |s| < \frac{1}{2}.
\]

In addition, the duality described in (1.7) holds for other Sobolev spaces for sufficiently smooth screens.

**Lemma 1.3.6** ([58, Lemma 4.3.1]). Let \( \Gamma_c \) be a closed surface such that \( \Gamma \subset \Gamma_c \). Moreover, let \( \Gamma_c \in C^{k,1}, k \in \mathbb{N}_0 \). For \( |s| \leq k + 1 \) we have

\[
H^{-s}(\Gamma) = \left( \tilde{H}^s(\Gamma) \right)', \quad \tilde{H}^{-s}(\Gamma) = (H^s(\Gamma))' \quad \text{.}
\]

We also have the following generalized Cauchy-Schwarz inequality.

---

5In case of \( \Gamma \subset \Gamma_c \) with \( \Gamma_c \) closed and \( C^{k,1}, k \in \mathbb{N} \), then the condition is \( |s| \leq k + 1 \). This is also the case for (1.12) and lemma 1.4.7 later on.
Lemma 1.3.7 ([58, Lemma 4.1.2]). The $H^s(\Gamma)$-scalar product extends to a continuous bilinear form on $H^{s+t}(\Gamma) \times H^{s-t}(\Gamma)$. Moreover, we have
\[
|(u, v)_{H^s(\Gamma)}| \leq \|u\|_{H^{s+t}(\Gamma)} \|v\|_{H^{s-t}(\Gamma)}
\] (1.23)
for all $(u, v) \in H^{s+t}(\Gamma) \times H^{s-t}(\Gamma)$ and for all $s, t \in \mathbb{R}$.

To conclude this subsection, we state the next Rellich’s Lemma.

Theorem 1.3.8 ([58, Theorem 4.1.6]). Let $\Gamma$ be bounded and Lipschitz. Then the following embedding are continuous and compact:
\begin{enumerate}
  \item $\tilde{H}^s(\Gamma) \hookrightarrow H^t(\Gamma)$ for $-\infty < t < s < \infty$,
  \item $H^s(\Gamma) \hookrightarrow H^t(\Gamma)$ for $-\infty < t < s < \infty$,
  \item $H^s(\Gamma) \hookrightarrow C^{m,\alpha}(\Gamma)$ for $m \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $m + \alpha < s - n/2$.
\end{enumerate}
For $s = m + \alpha + n/2$ and $0 < \alpha < 1$ the embedding $H^s(\Gamma) \hookrightarrow C^{m,\alpha}(\Gamma)$ is continuous.

1.3.2. Trace and Jump Operators

We will discuss two approaches to define trace and jump operators on screens. First, we present the standard and simplest procedure, which is to consider a Lipschitz domain $\Omega_c$ such that $\Gamma \subset \partial \Omega_c$ and use the standard definitions of trace operators on $\partial \Omega_c$ combined with the restriction operator from $\partial \Omega_c$ to $\Gamma$ (see for example [87, Sect. 2] and [24, Sect. 2.2]). Secondly, we introduce the approach developed by Buffa and Christiansen in [16] to tackle non-orientable Lipschitz screens, which in spite of its generality gives valuable insight and applies to our setting.

Standard definitions (via extension to a closed surface)

Let $\Omega_c$ be a Lipschitz domain such that $\Gamma \subset \partial \Omega_c$. For $V \in \mathcal{D}(\mathbb{R}^3)$, we define
\[
\tilde{\gamma}_0 V := V|_{\partial \Omega_c}
\] (1.24)
and
\[
\tilde{\gamma}_0^+ : \mathcal{D}(\Omega_c) \rightarrow \mathcal{D}(\partial \Omega_c) \quad \tilde{\gamma}_0^- : \mathcal{D}(\mathbb{R}^3 \setminus \Omega_c) \rightarrow \mathcal{D}(\partial \Omega_c)
\]
\[
\tilde{\gamma}_0^+ V := V|_{\partial \Omega_c} \quad \tilde{\gamma}_0^- V := V|_{\partial \Omega_c}.
\] (1.25)

There exist unique extensions to continuous linear trace operators
\[
\tilde{\gamma}_0 : H^1_{\text{loc}}(\mathbb{R}^3) \rightarrow H^{1/2}(\partial \Omega_c)
\]
\[
\tilde{\gamma}_0^+ : H^1_{\text{loc}}(\Omega_c) \rightarrow H^{1/2}(\partial \Omega_c)
\]
\[
\tilde{\gamma}_0^- : H^1_{\text{loc}}(\mathbb{R}^3 \setminus \Omega_c) \rightarrow H^{1/2}(\partial \Omega_c)
\]
and
\[
\tilde{\gamma}_0 V = \tilde{\gamma}_0^+ V = \tilde{\gamma}_0^- V, \quad \forall V \in H^1_{\text{loc}}(\mathbb{R}^3)
\] (1.26)
almost everywhere [81, Theorem 2.6.8].

Similarly, we introduce for $V \in \mathcal{D}(\mathbb{R}^3)$
\[
\tilde{\gamma}_1 V = \mathbf{n} \cdot \tilde{\gamma}_0 \nabla V
\] (1.27)
and
\[
\tilde{\gamma}_1^+ : \mathcal{D}(\Omega_c) \rightarrow \mathcal{D}(\partial \Omega_c) \quad \tilde{\gamma}_1^- : \mathcal{D}(\mathbb{R}^3 \setminus \Omega_c) \rightarrow \mathcal{D}(\partial \Omega_c)
\] (1.28)
where the symbol "\( \gamma_0 \)" denotes the unit normal vector pointing into \( \Omega_c \).

The operators \( \hat{\gamma}^\pm \) extend to continuous linear operators

\[
\hat{\gamma}^+: H^1(\Delta, \Omega_c) \rightarrow H^{-1/2}(\partial\Omega_c) \quad \text{and} \quad \hat{\gamma}^-: H^1(\Delta, \mathbb{R}^3 \setminus \Omega_c) \rightarrow H^{-1/2}(\partial\Omega_c)
\]

with \( H^1(\Delta, \mathcal{O}) := \{ v \in H^1(\mathcal{O}) : \Delta v \in L^2(\mathcal{O}) \} \).

We are now prepared to define the point trace or Dirichlet trace operator \( \gamma_0 : H^1_{\text{loc}}(\mathbb{R}^3) \rightarrow H^{1/2}(\Gamma) \) as

\[
\gamma_0 V := (\hat{\gamma}_0^+ V)_{|\Gamma}, \tag{1.29}
\]

and the conormal trace or Neumann trace operator

\[
\gamma_1 V = (n \cdot \hat{\gamma}_0^+ \nabla V)_{|\Gamma} \in H^{-1/2}(\Gamma), \tag{1.30}
\]

where the symbol "\( |_{\Gamma} \)" in (1.30) should be understood as the restriction operator in the sense of distributions on \( \partial\Omega_c \).

Let

\[
H^1_{\text{loc}}(\Delta, \Omega) := \{ v \in L^2_{\text{loc}}(\mathbb{R}^3) : \varphi v \in H^1(\Delta, \Omega) \forall \varphi \in \mathcal{D}(\Omega) \}. \tag{1.31}
\]

The traces \( \gamma_0 : H^1_{\text{loc}}(\Delta, \Omega) \rightarrow H^{1/2}(\Gamma) \) and \( \gamma_1 : H^1_{\text{loc}}(\Delta, \Omega) \rightarrow H^{-1/2}(\Gamma) \) are continuous mappings [28, Sect. 7.1].

In order to define the jump operator, we must first point out that by construction of the artificial boundary \( \partial\Omega_c \), solutions to (1.5) and (1.6) satisfy

\[
\hat{\gamma}_0^+ V|_{\partial\Omega_c \setminus \Gamma} - \hat{\gamma}_0^- V|_{\partial\Omega_c \setminus \Gamma} = 0, \quad \text{and} \quad \hat{\gamma}_1^+ V|_{\partial\Omega_c \setminus \Gamma} - \hat{\gamma}_1^- V|_{\partial\Omega_c \setminus \Gamma} = 0, \tag{1.32}
\]

respectively.

We then introduce

\[
[\gamma_0 V] := \hat{\gamma}_0^+ V - \hat{\gamma}_0^- V \in \bar{H}^{1/2}(\Gamma) \tag{1.33}
\]

\[
[\gamma_1 V] := \hat{\gamma}_1^+ V - \hat{\gamma}_1^- V \in \bar{H}^{-1/2}(\Gamma). \tag{1.34}
\]

From these definitions it becomes clear that, contrary to the situation with closed surfaces, the objects \( \gamma_0 V \) and \( \gamma_1 V \) do not belong to the same spaces as \( [\gamma_0 V] \) and \( [\gamma_1 V] \), respectively. As a matter of fact, this distinction motivates that in the context of screens, one often refers to \( H^{\pm 1/2}(\Gamma) \) as trace spaces and to \( \bar{H}^{\pm 1/2}(\Gamma) \) by jump spaces.

General definitions

For \( V \in \mathcal{D}(\mathbb{R}^3) \), we define the point trace or Dirichlet trace operator by

\[
\gamma_0 V = V_{|\Gamma}. \tag{1.35}
\]

This determines a continuous linear map from \( \mathcal{D}(\mathbb{R}^3) \) into \( L^\infty(\Gamma) \). Moreover, the following properties are known:

**Proposition 1.3.9** ([16, Sect. 2]).

i For any \( s \in (0, 1) \), \( \gamma_0 \) has a unique extension to a continuous and surjective operator

\[
\gamma_0 : H^{s+1/2}(\mathbb{R}^3) \rightarrow H^s(\Gamma). \tag{1.36}
\]

ii For any \( s > 0 \), \( \gamma_0 \) has a unique extension to a bounded linear map \( H^{s+3/2}(\mathbb{R}^3) \rightarrow H^1(\Gamma) \).
iii $\gamma_0$ has a unique extension to a continuous operator $\gamma_0 : H^1(\mathbb{R}^3) \to L^2(\Gamma)$. 

We also define the conormal trace or Neumann trace operator by

$$\gamma_1 V = n \cdot \gamma_0 \nabla V.$$  \hspace{1cm} (1.37)

We conclude this subsection by defining the jump of traces across $\Gamma$. In the case of globally orientable screens, one can separate the two sides of the screen and arbitrarily label them as $\Gamma_+$ and $\Gamma_-$. Then, one uses the same notation for the traces from each side and defines the jump operator as

$$[\gamma_i] := \gamma_i^+ - \gamma_i^-, \quad i = 0, 1.$$  \hspace{1cm} (1.38)

**Remark 1.3.10.** In the case of non-globally-orientable screens, one follows the cue from [16, Sect. 1.4] and considers local orientation. The idea is the following: For any $x \in \Gamma$, there is $r > 0$ such that $B(x, r) \setminus \Gamma$ has exactly two connected components $B^+$ and $B^-$. This allows us to define our trace operators from each side of the screen as

$$\gamma_0^+ V = \lim_{y \to x, y \in B^+} V(y), \quad \gamma_0^- V = \lim_{y \to x, y \in B^-} V(y).$$  \hspace{1cm} (1.39)

Moreover, one can define the normal vectors $n^+(x)$ and $n^-(x)$ as the outward-pointing normal on $B(x, r) \cap \Gamma$ relatively to $B^+$ and $B^-$, respectively. Then, one writes

$$\gamma_1^+ V = n^+ \cdot \gamma_0^+ \nabla V, \quad \gamma_1^- V = n^- \cdot \gamma_0^- \nabla V,$$  \hspace{1cm} (1.40)

and use the jump definition in (1.38).

**Remark 1.3.11.** As already mentioned, the spaces $H^{\pm 1/2}(\Gamma)$ and $\tilde{H}^{\pm 1/2}(\Gamma)$ can also be defined in terms of the trace and jump operators [28]:

$$H^{1/2}(\Gamma) = H^1(\mathbb{R}^3)/\ker \gamma_0,$$  \hspace{1cm} (1.41)

$$H^{-1/2}(\Gamma) = H(\text{div}, \mathbb{R}^3)/\ker \gamma_n,$$  \hspace{1cm} (1.42)

$$\tilde{H}^{1/2}(\Gamma) = [\gamma_0 (H^1_{\text{loc}}(\Delta, \mathbb{R}^3))],$$  \hspace{1cm} (1.43)

$$\tilde{H}^{-1/2}(\Gamma) = [\gamma_1 (H^1_{\text{loc}}(\Delta, \mathbb{R}^3))],$$  \hspace{1cm} (1.44)

where $\gamma_n : U \mapsto n \cdot U|_{\Gamma}$ is the normal component trace defined in [28, Sect. 3.2], and $H(\text{div}, \mathbb{R}^3)$ will be introduced in Section 5.2.

### 1.4. Boundary Integral Equations

We introduce the fundamental solution $G_k$ of the time-harmonic acoustic problem at hand

$$G_k(z) := \begin{cases} 
\frac{1}{4\pi ||z||}, & \text{if } k = 0 \text{ (Laplace)}, \\
\frac{1}{4\pi} e^{ik||z||}, & \text{if } k > 0 \text{ (Helmholtz)}, 
\end{cases}$$  \hspace{1cm} (1.45)

and the single layer and double layer potentials

$$\Psi_{SL}^k v(x) = \int_\Gamma v(y) G_k(x - y) d\Gamma(y), \quad x \in \Omega,$$  \hspace{1cm} (1.46)

$$\Psi_{DL}^k v(x) = \int_\Gamma v(y) \frac{\partial}{\partial n(y)} G_k(x - y) d\Gamma(y), \quad x \in \Omega,$$  \hspace{1cm} (1.47)
where \( \Omega := \mathbb{R}^3 \setminus \Gamma \), \( d\Gamma(y) \) designates the surface measure at \( y \in \Gamma \) and \( v \in L^1(\Gamma) \).

In the case of screens, the layer potentials extend to continuous mappings from the tilde spaces [16, Sect. 3][28, Sect. 8.1]

\[
\Psi^k_{\text{SL}} : \tilde{H}^{-1/2}(\Gamma) \to H^1_{\text{loc}}(\Omega), \quad (1.48)
\]

\[
\Psi^k_{\text{DL}} : \tilde{H}^{1/2}(\Gamma) \to H^1_{\text{loc}}(\Omega), \quad (1.49)
\]

and, as usual, satisfy the PDE (1.1) and Sommerfeld radiation conditions (1.4) [87, pp.243–245]. Thus, we can describe the solutions to Problems 1.2.1 and 1.2.2 using these layer potentials and pursue an indirect approach, in the same fashion as for closed surfaces.

With this purpose in mind, let us define the weakly singular and hypersingular BIEs on screens

\[
(V_k \sigma)(x) := \int_{\Gamma} \sigma(y)G_k(x - y)d\Gamma(y) \quad x \in \Gamma, \quad (1.50)
\]

\[
(W_k u)(x) := \frac{1}{4\pi} \int_{\Gamma} u(y) \frac{\partial^2}{\partial n(x) \partial n(y)} G_k(x - y) d\Gamma(y), \quad x \in \Gamma, \quad (1.51)
\]

for \( \sigma \) and \( u \) functions defined on \( \Gamma \) and such that \( u = 0 \) on \( \partial\Gamma \). Here, \( \int \) indicates finite part integrals with distributional meaning as in [65, Chapter 5].

We now formulate their associated BIEs:

**Problem 1.4.1 (Weakly singular IE).** For a given \( g \in H^{1/2}(\Gamma) \) and \( k \geq 0 \), seek \( \sigma \in \tilde{H}^{-1/2}(\Gamma) \) such that

\[
(V_k \sigma)(x) = g(x), \quad x \in \Gamma. \quad (1.52)
\]

**Problem 1.4.2 (Hypersingular IE).** For a given \( \mu \in H^{-1/2}(\Gamma) \) and \( k \geq 0 \), seek \( u \in \tilde{H}^{1/2}(\Gamma) \) such that

\[
(W_k u)(x) = \mu(x), \quad x \in \Gamma. \quad (1.53)
\]

**Theorem 1.4.3 ([87, Theorem 2.8]).** Let \( \exists(k) \geq 0 \), then Problem 1.4.1 and Problem 1.4.2 have exactly one solution \( \sigma \) and \( u \), respectively.

We remark that by using the potential ansatz

\[
U^s(x) = (\Psi^k_{\text{SL}} [\gamma_0 U^s])(x) \quad \text{for Problem 1.2.1},
\]

\[
U^s(x) = (\Psi^k_{\text{DL}} [\gamma_1 U^s])(x) \quad \text{for Problem 1.2.2}, \quad x \in \Omega = \mathbb{R}^3 \setminus \Gamma,
\]

and taking the corresponding traces, our BVPs (1.5) and (1.6) are converted into the BIEs (1.52) and (1.53), respectively (see [87, Sect. 2] for details). Moreover, after solving our BIEs, one recovers the solution to the BVPs using the corresponding potential ansatz. Indeed, due to this connection, the problems fulfill:

**Theorem 1.4.4 ([87, Theorem 2.5–2.6]).**

\[
i \quad U^s \in H^1_{\text{loc}}(\Omega) \text{ solves Problem 1.2.1 (EDP) if and only if } \sigma = [\gamma_1 U^s] \in \tilde{H}^{-1/2}(\Gamma) \text{ is the solution of Problem 1.4.1 with } g = \gamma_0(\neg U^i) \in H^{1/2}(\Gamma).
\]

\[
ii \quad U^s \in H^1_{\text{loc}}(\Omega) \text{ solves Problem 1.2.2 (ENP) if and only if } u = [\gamma_0 U^s] \in \tilde{H}^{1/2}(\Gamma) \text{ is the solution of Problem 1.4.2 with } \mu = \gamma_1(\neg U^i) \in H^{-1/2}(\Gamma).
\]

Now, let us move towards the solution of the BIEs via their variational formulations. Let us define the bilinear form

\[
a^k_{V,G}(\sigma, \sigma') := \langle V_k \sigma, \sigma' \rangle_G = \int_{\Gamma} \int_{\Gamma} \sigma(y) \sigma'(x) G_k(x - y) d\Gamma(y) d\Gamma(x), \quad (1.54)
\]

for all \( \sigma, \sigma' \in \tilde{H}^{-1/2}(\Gamma) \) and \( k \geq 0 \). Then, the symmetric variational formulation for (1.52) is:
Chapter 1. BEM on Screens

**Problem 1.4.5** (Variational weakly singular IE). Seek \( \sigma \in \tilde{H}^{-1/2}(\Gamma) \) such that for \( g \in H^{1/2}(\Gamma) \) and \( k \geq 0 \)

\[
a^k_{V, \Gamma}(\sigma, \sigma') = \langle g, \sigma' \rangle_{\Gamma}, \quad \forall \sigma' \in \tilde{H}^{-1/2}(\Gamma). \tag{1.55}
\]

In addition, we define the bilinear form

\[
a^k_{W, \Gamma}(u, v) := \langle W_k u, v \rangle_{\Gamma}, \quad \forall u, v \in \tilde{H}^{1/2}(\Gamma), \ k \geq 0. \tag{1.56}
\]

For the case \( k = 0 \) (Laplace operator), its explicit formula is given by

\[
a^0_{W, \Gamma}(u, v) = \frac{1}{4\pi} \int_{\Gamma} \frac{\text{curl}_{\Gamma} y u(y) \cdot \text{curl}_{\Gamma} x v(x)}{\|x - y\|} \, d\Gamma(y), \tag{1.57}
\]

where \( \text{curl}_{\Gamma} \) denotes the *vectorial* surface curl operator [83, p.133].

For \( k > 0 \) (Helmholtz operator), we have [44, Eq. (6)]

\[
a^k_{W, \Gamma}(u, v) = \frac{1}{4\pi} \int_{\Gamma} e^{ik\|x - y\|} \{ \text{curl}_{\Gamma} y u(y) \cdot \text{curl}_{\Gamma} x \tilde{v}(x) - k^2 n(x) \cdot n(y) u(y) \tilde{v}(x) \} \, d\Gamma(x) d\Gamma(y). \tag{1.58}
\]

**Problem 1.4.6** (Variational hypersingular IE). Seek \( u \in \tilde{H}^{1/2}(\Gamma) \) such that for \( \mu \in H^{-1/2}(\Gamma) \) and \( k \geq 0 \)

\[
a^k_{W, \Gamma}(u, v) = \langle \mu, v \rangle_{\Gamma}, \quad \forall v \in \tilde{H}^{1/2}(\Gamma). \tag{1.59}
\]

We point out that \( V_k \) and \( W_k \) enjoy the following properties that are relevant to solve these variational problems.

**Lemma 1.4.7** (Continuity and Gårding inequalities ([87, Lemma 2.8])).

i For any \( |s| < 1 \), the mappings

\[
V_k : \tilde{H}^s(\Gamma) \rightarrow H^{s+1}(\Gamma), \tag{1.60}
\]

\[
W_k : \tilde{H}^s(\Gamma) \rightarrow H^{s-1}(\Gamma) \tag{1.61}
\]

are continuous.

ii There exist operators

\[
C_1 : \tilde{H}^{-1/2}(\Gamma) \rightarrow H^{1/2+\epsilon}(\Gamma), \tag{1.62}
\]

\[
C_2 : \tilde{H}^{1/2}(\Gamma) \rightarrow H^{-1/2+\epsilon}(\Gamma), \tag{1.63}
\]

for some \( \epsilon > 0 \), such that for all \( \psi \in \tilde{H}^{-1/2}(\Gamma) \) and \( v \in \tilde{H}^{1/2}(\Gamma) \)

\[
\langle (V_k + C_1) \psi, \psi \rangle \geq c_1 \|\psi\|^2_{\tilde{H}^{-1/2}(\Gamma)}, \tag{1.64}
\]

\[
\langle (W_k + C_2) v, v \rangle \geq c_2 \|v\|^2_{\tilde{H}^{1/2}(\Gamma)}, \tag{1.65}
\]

with \( c_1, c_2 > 0 \).

**Remark 1.4.8.** By Rellich’s Lemma, the operators \( C_1 : \tilde{H}^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma) \) and \( C_2 : \tilde{H}^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma) \) are compact.

**Remark 1.4.9.** For the static case, the operators \( V_0 \) and \( W_0 \) are elliptic in their energy spaces [81, Theorem 3.5.9].

**Remark 1.4.10.** For \( k > 0 \), the operators \( V_k \) and \( W_k \) are also continuous and elliptic with respect to a certain kind of \( k \)-dependent norms \( \| \cdot \|_{\tilde{H}^k(\Gamma)} \) on planar screens \( \Gamma \) (i.e. any relatively open subset \( \Gamma \subset \{ x \in \mathbb{R}^3 : x_3 = 0 \} \)). We refer to [24] for the related proofs and definitions.
1.5. Regularity of Solutions

A particular feature of screen problems is that for sufficiently smooth $\Gamma$, when approaching the edges $\partial \Gamma$, solutions of (1.53) decay like the square-root of the distance to $\partial \Gamma$, whereas those of (1.52) blow-up like the reciprocal square-root.

This has important consequences in the performance of the numerical methods and also motivates the need for preconditioners that are suitable for meshes refined toward the boundary of the screen. For this reason, we dedicate this section to enunciate the existing regularity results for screens that are either smooth or polygonal.

**Theorem 1.5.1 ([87, Theorem 2.9]).** Let $\Gamma$ be bounded, simply connected and orientable, and smooth with a smooth boundary curve $\gamma$ which does not intersect itself. We identify $\Gamma$ with $[0,1] \times \gamma$.

Let $s$ denote the parameter of arclength of $\gamma$, $\rho$ the Euclidean distance to $\gamma$, and $\chi$ a $C^\infty$ cut-off function

$$\chi \equiv \begin{cases} 1 & \text{for } |\rho| < 1/2 \\ 0 & \text{for } |\rho| > 1 \end{cases}.$$  \hspace{1cm} (1.66)

Then, the following holds

1. Let $g \in H^{3/2+\alpha}(\Gamma)$ be given. Then the solution of the weakly singular BIE (1.52) has the form

$$\sigma = f_1(s)\rho^{-1/2}\chi(\rho) + v_R \text{ on } \Gamma,$$  \hspace{1cm} (1.67)

with $f_1 \in H^{1/2+\alpha}(\gamma)$, $v_R \in \tilde{H}^{1/2+\alpha'}(\Gamma)$, for $0 < \alpha' < \alpha < 1/2$.

2. Let $h \in H^{1/2+\alpha}(\Gamma)$ be given. Then the solution of the hypersingular BIE (1.53) has the form

$$u = f_2(s)\rho^{1/2}\chi(\rho) + v_r \text{ on } \Gamma,$$  \hspace{1cm} (1.68)

with $f_2 \in H^{1/2+\alpha}(\gamma)$, $v_r \in L^2(0,1;H^{1/2+\alpha}(\gamma)) \cap \tilde{H}^{3/2+\alpha'}(0,1;L^2(\gamma))$, for $0 < \alpha' < \alpha < 1/2$. \hspace{1cm} 6

**Theorem 1.5.2 ([33, Theorem 1.2] and [32, Theorem 1.3]).** Let the screen be the square plate $\Gamma = \{(x_0, x_1, 0) \in \mathbb{R}^3 : |x_0|, |x_1| < 1\}$ and denote the distance of $x \in \Gamma$ to the nearest edge by $\rho$. Let us consider plane polar coordinates $(r_j, \theta_j)$ for each vertex $v_j$, $j = 0, \ldots, 3$, of $\Gamma$ and define the $C^\infty$ cut-off functions

$$\chi(r_j) \equiv \begin{cases} 1 & \text{for } |r_j| < 1/4 \\ 0 & \text{for } |r_j| > 1 \end{cases}, \quad \tilde{\chi}(\theta_j) \equiv \begin{cases} 1 & \text{for } |\theta_j| < \pi/4 \\ 0 & \text{for } |\theta_j| > 1 \end{cases}, \quad j = 0, 1, 2, 3.$$  \hspace{1cm} (1.69)

1. If $g \in H^2(\Gamma)$, then the solution of (1.52) for $k = 0$ has near each vertex $v_j$ a decomposition into edge and corner singularities of the form

$$\sigma = \sigma_0 + \chi(r_j)a_1\rho^{-1}w_1(\theta_j) + \tilde{\chi}(\theta_j)e_1(r_j)\rho^{-1/2} + \tilde{\chi}(\theta_j)e_2(r_j)\rho^{1/2},$$  \hspace{1cm} (1.70)

with $\sigma_0 \in H^{1-\epsilon}, a_1 \in \mathbb{R}, w_1 \in H^{1-\epsilon}([0, \pi/2]), e_1 \sim b_1 r_j^{\beta-1}, b_1 \in \mathbb{R}, i = 1, 2$.

2. If $\mu \in H^2(\Gamma)$, then the solution of (1.53) for $k = 0$ has near each vertex $v_j$ the form

$$u = u_0 + \chi(r_j)a_2\rho\lambda^{-1}w_2(\theta_j) + \tilde{\chi}(\theta_j)e_3(r_j)\rho^{1/2} + \tilde{\chi}(\theta_j)e_3(r_j)\rho^{1/2},$$  \hspace{1cm} (1.71)

with $u_0 \in H^{3/2-\epsilon}, a_2 \in \mathbb{R}, w_2 \in H^{3/2-\epsilon}([0, \pi/2]), e_3 \sim r_j^{\beta-1/2}$.

Here $\lambda = 0.9266$ for any $\epsilon > 0$.

---

6Let $X$ be a Hilbert space and $a < b$ real numbers. $L^2(a,b;X)$ denotes the space of functions $v$ that are strongly measurable on $[a,b]$ with range in $X$ and such that $\|v\|_{L^2(a,b;X)} := \left(\int_a^b \|v(s)\|^2_X ds\right)^{1/2} < \infty$ [62, Sect. 1.1.3]. One defines $H^s(a,b;X)$ for $s > 0$ analogously to usual Sobolev spaces but using $L^2(a,b;X)$ [62, Sect. 4.2.1].
Remark 1.5.3. For $k \neq 0$ the first singularities are the same as for $k = 0$, although the resulting formulas are slightly more involved. For the sake of clarity, we only state the regularity for the static case in this thesis and refer to [57, Prop. 2] for the formulas for $k \neq 0$.

The first part of Theorem 1.5.2 is a particular case of [76, Theorem 8]. We enunciate a version of Corollary 5 in [76] that has been simplified to our setting.

**Theorem 1.5.4.** Let the screen $\Gamma$ be a polygon whose boundary $\partial \Gamma$ consists of straight sides.

Let us label each vertex of $\Gamma$ as $t_j$, $1 \leq j \leq J$ and denote its interior angle by $\beta_j$. We introduce $J(j)$ as the set of indices of the two edges neighbouring the vertex $t_j$. For simplicity, we additionally require that $\min \, \text{dist}(t_i, t_j) \geq 1$, $\forall i, j$.

Let us choose one vertex $t_j$ and introduce its local spherical polar coordinates $(r, \phi, \theta)$. Let $\lambda_k$ be the corner singularity orders determined by the Dirichlet Laplace-Beltrami eigenvalue problem on $S_0 := S_1 \cap \Omega$ as in [76, (3.1)]. We further assume that $\lambda_k \notin \mathbb{N}$.

If $g \in H^2(\Gamma)$, then the solution of (1.5) for $k = 0$ admits the following decomposition near each vertex

$$\gamma_1 U = \psi^0 + \chi(r) \sum_{k:|\lambda_k + 1/2 < 3/2} a_k r^{\lambda_k - 1} \frac{\partial}{\partial n} w_k |_{\partial S_0} + \sum_{l \in J(j)} \chi_l(\theta_l) r^{-1} e^{l}(r) \frac{\partial}{\partial n} S(\theta_l, \phi_l) |_{\partial S_0}, \quad (1.72)$$

where $\chi$ and $\chi_l$ are a $C^\infty$ cut-off functions concentrated near the chosen vertex and its neighbouring edges, respectively.

Here $\psi^0 \in H^{1-\epsilon}$, $\epsilon > 0$, $a_k \in \mathbb{R}$, $w_k \in H^{5/2-\epsilon}$, $e^{l}(1, 0) = b^{l}(1, 0) + \chi(r) \sum_{k:|\lambda_k + 1/2 < 3/2} a^{l}_{1, 0, k} r^{\lambda_k}$, with $b^{l}(1, 0) \in H^{1/2}(\mathbb{R}_+)$ and $a^{l}_{1, 0, k} \in \mathbb{R}$, and $S(\theta_l, \phi_l) = \theta_l^{1/2} \sin(\frac{\phi_l}{2})$.

This is a consequence of Theorem 8 in [76] with $\omega_j = 2\pi$, $s_0 = \frac{3}{2}$, $s_1 = \frac{1}{2}$ and $s = \frac{3}{2}$.

**Remark 1.5.5.** In the case that $\lambda_k \in \mathbb{N}$, one needs to modify (1.72) adding some logarithmic terms depending on $r$.

The values of the smallest eigenvalue $\lambda_1(\beta)$ have been computed analytically for some particular angles, but there is no general formula available. Nevertheless, they have also been studied numerically for different values of corner angles [90, 68]. For the sake of completeness, we include some of the values obtained in [68] and remark that these will define the leading corner singularity in formula (1.72).

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\lambda_1(\beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/6$</td>
<td>0.1811</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>0.2115</td>
</tr>
<tr>
<td>$3\pi/8$</td>
<td>0.2542</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>0.2966</td>
</tr>
</tbody>
</table>

**Remark 1.5.6.** We see that in all considered cases, the edge singularity still displays the square-root of the distance to $\partial \Gamma$ behaviour.

### 1.6. Galerkin BEM Numerical Approximation

Finally, one solves Problems 1.6.1 and 1.6.4 numerically. In particular, this thesis concerns the case when one pursues a Galerkin BEM numerical approximation and the ultimate goal is to construct an optimal preconditioner for the arising linear systems. We cannot properly achieve this without understanding its numerical context, as this shapes the tools and requirements we need to take into account.
This section gives a small review of the existing convergence theory and other discrete estimates. As we will see in Chapter 3, we specifically construct our preconditioners for lowest order Galerkin BEM. We restrict this presentation to said case. However, it is worth mentioning that the preconditioning strategy we propose in Chapter 3 holds independently of the discretization and thus can be applied to higher order piecewise polynomial boundary element discretizations. We refer the reader to [43, 57, 33] for the corresponding results in that case.

1.6.1. Weakly Singular BIE

We start by discretizing Problem 1.4.5 with the standard Galerkin scheme. For this, we consider conforming finite dimensional spaces $S_h^0 \subset \tilde{H}^{-1/2}(\Gamma)$ such that

$$\forall \psi \in \tilde{H}^{-1/2}(\Gamma) : \inf_{\psi_h \in S_h^0} \|\psi - \psi_h\|_{\tilde{H}^{-1/2}(\Gamma)} \to 0 \text{ for } h \to 0.$$ 

Furthermore, we assume that the family of finite element subspaces $S_h^0$ are piecewise polynomials of order zero, i.e. piecewise constants.

**Problem 1.6.1 (Weakly singular Galerkin Problem).** Find $\sigma_h \in S_h^0$ such that for $g \in H^{1/2}(\Gamma)$ and $k \geq 0$

$$a_{W,\Gamma}(\sigma_h, \psi_h) = \langle g, \psi_h \rangle_\Gamma, \quad \forall \psi_h \in S_h^0. \quad (1.73)$$

Hereunder, unless otherwise indicated, we write $\sigma$ for the unique solution of Problem 1.4.1 and $\sigma_h$ for the solution of Problem 1.6.1.

**Lemma 1.6.2 ([87, Theorem 3.1]).** There exists $h_0 > 0$ such that for any $h \leq h_0$ (1.73) are uniquely solvable. Moreover, there holds the quasi-optimal error estimates

$$\|\sigma - \sigma_h\|_{\tilde{H}^{-1/2}(\Gamma)} \leq c \inf_{\psi_h \in S_h^0} \|\sigma - \psi_h\|_{\tilde{H}^{-1/2}(\Gamma)}, \quad (1.74)$$

for a constant $c$ which is independent of $\sigma, \sigma_h$ and $h$.

Let us now consider $\Gamma$ piecewise smooth with partitioning $(\Gamma_i)_{i=1}^s$ and define $H_{pw}^s := \{ v \in H^1(\Gamma) : \forall \Gamma_i v|_{\Gamma_i} \in H^s(\Gamma_i) \}$ for $s > 1$ and $H_{pw}^1 = H^s(\Gamma)$ for $0 \leq s \leq 1$.

**Theorem 1.6.3 ([81, Theorem 4.1.54]).** If the exact solution $\sigma$ of the Problem 1.4.5 is contained in $H_{pw}^s(\Gamma)$ for $s \geq 0$ we have

$$\|\sigma - \sigma_h\|_{\tilde{H}^{-1/2}(\Gamma)} \leq C_1 h^{\min(s,1)+1/2} \|\sigma\|_{H_{pw}^s(\Gamma)}, \quad (1.75)$$

with constant $C_1$ depending only on $c$ from (1.74) and the shape regularity of the mesh.

1.6.2. Hypersingular BIE

In analogy to the previous subsection, we take conforming finite dimensional spaces $S_h^1 \subset \tilde{H}^{1/2}(\Gamma)$ such that

$$\forall v \in \tilde{H}^{1/2}(\Gamma) : \inf_{v_h \in S_h^1} \|v - v_h\|_{\tilde{H}^{1/2}(\Gamma)} \to 0 \text{ for } h \to 0.$$ 

Additionally, we assume that the family of finite element subspaces $S_h^1$ are piecewise polynomials of degree 1. Having these considerations in mind, we proceed to discretize Problem 1.6.4.

**Problem 1.6.4 (Hypersingular Galerkin problem).** Find $u_h \in S_h^1$ such that for $\mu \in H^{-1/2}(\Gamma)$ and $k \geq 0$

$$a_{W,\Gamma}(u_h, v_h) = \langle \mu, v_h \rangle_\Gamma, \quad \forall v_h \in S_h^1. \quad (1.76)$$
Lemma 1.6.5 ([87, Theorem 3.1]). There exists $h_0 > 0$ such that for any $h \leq h_0$ systems (1.76) are uniquely solvable. Moreover, there holds the quasi-optimal error estimates

\[
\|u - u_h\|_{\tilde{H}^{1/2}(\Gamma)} \leq c \inf_{v_h \in S^1_h} \|u - v_h\|_{\tilde{H}^{1/2}(\Gamma)},
\]

for a constant $c$ which is independent of $u, u_h$ and $h$.

Theorem 1.6.6 ([81, Theorem 4.1.54]). If the exact solution of Problem 1.4.6 is contained in $H^{s}_{pw}(\Gamma)$ for $s > 1/2$ we have

\[
\|u - u_h\|_{\tilde{H}^{1/2}(\Gamma)} \leq C_2 h^{\min\{s, 2\} - 1/2} \|\sigma\|_{H^{s}_{pw}(\Gamma)},
\]

with constant $C_2$ depending only on $c$ from (1.77), the shape regularity of the mesh and the polynomial degree $p_1$ of $S^1_h$.

Remark 1.6.7. As discussed in Section 1.5, the exact solutions of problems 1.4.1 and 1.4.2 do not have high order regularity $s$ due to their edge singularities (and even corner singularities, depending on the shape of $\Gamma$). Therefore, even for high order discretizations, the convergence rates in terms of meshwidth $h \to 0$ of Galerkin solutions is at most $1/2 - \epsilon$, $\epsilon > 0$.

Convergence rates can be improved via augmented finite element spaces [87]. However, implementing the related singular basis functions requires more work than simply refining the mesh towards the edge $\partial \Gamma$, which also improves the convergence rates [32].

Finally, we conclude this Chapter with an important connection between our fractional spaces in the discrete setting.

Proposition 1.6.8 ([66], [43, Lemma 2.8] or [47]). The following inverse inequalities hold:

\[
\|u_h\|_{\tilde{H}^{1/2}(\Gamma)} \leq c_1 (1 + |\log h|) \|u_h\|_{H^{1/2}(\Gamma)},
\]

\[
\|\varphi_h\|_{\tilde{H}^{-1/2}(\Gamma)} \leq c_2 (1 + |\log h|) \|\varphi_h\|_{H^{-1/2}(\Gamma)},
\]

for all $u_h \in S^1_h \subset \tilde{H}^{1/2}(\Gamma), \varphi_h \in S^0_h \subset \tilde{H}^{-1/2}(\Gamma)$, meshwidth $h \leq 1$ and with constants $c_1, c_2 > 0$ independent of $h$. 

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2. New Calderón-Type Identities

Q: “What does the $S$ stand for? Suspicious or Surprising?”
A: “It is not an $S$, but rather the mathematical symbol for hope.”

– Adapted from Man of Steel (2013).

On Lipschitz closed surfaces $\Gamma_c$, the weakly singular and hypersingular operators have the following mapping properties:

\[ V_k : H^{-1/2}(\Gamma_c) \rightarrow H^{1/2}(\Gamma_c), \]
\[ W_k : H^{1/2}(\Gamma_c) \rightarrow H^{-1/2}(\Gamma_c). \]

Moreover, they satisfy

\[ V_k W_k = \text{Id}/4 + K_1, \quad W_k V_k = \text{Id}/4 + K_2, \]

with $K_1, K_2$ compact operators that we do not define for the sake of simplicity.

These formulas are known as Calderón identities and play a key role in operator preconditioning. Indeed, the combination of the two is often dubbed as Calderón preconditioning and has been exhaustively studied and applied in scattering problems using BEM, for example in [85, 26, 25, 4, 70].

Unfortunately, these identities do not hold on screens, where the mapping properties of $V_k$ and $W_k$ also involve the tilde spaces (see Lemma 1.4.7). Furthermore, as already mentioned in the introduction, no analogous identities were available for the case of screens and the question remained open until very recently.

This Chapter presents the final answer for this question. We accomplished this result by first finding the exact inverse BIOs for $V_0$ and $W_0$ on the unit disk and then extending the results to the case $k > 0$.

The story of this mathematical quest would not be complete without mentioning that inverses for $V_0$ and $W_0$ on the unit disk have been extensively investigated and several characterizations for the solutions of the weakly singular and hypersingular BIEs have been found, even for $k \neq 0$! There is a very nice survey paper for $k = 0$ by Martin [63]. In addition to the works mentioned there, we would like to highlight other representations via Erdélyi-Kober Operators [72, 73, 71], integral transforms [37, 10], Wiener-Hopf techniques [74], orthogonal polynomials [35], and series expansions [77]. Unfortunately, although all these results have been important contributions, to the author’s knowledge, none of these solutions have been analyzed as inverse BIOs nor written as formulas that are amenable to Galerkin discretization. Consequently, they have been practically ignored by the BEM community until now.

Most of this chapter has been published in [48].

2.1. Inverse BIOs for Laplace on Disks

Throughout this section we consider our screen to be a flat circular disk $D_a$ with radius $a > 0$, defined as $D_a := \{ \mathbf{x} \in \mathbb{R}^3 : x_3 = 0 \text{ and } ||\mathbf{x}|| < a \}$. 
2.1.1. Modified Weakly Singular Integral Operator

We define the modified weakly singular operator through the improper integral

\[
(\mathcal{W}v)(x) := \frac{2}{\pi^2} \int_{\mathbb{D}_a} v(y) \frac{S_a(x, y)}{\|x - y\|} d\mathbb{D}_a(y), \quad x \in \mathbb{D}_a,
\]

(2.1)

with the bounded function on \( \mathbb{D}_a \times \mathbb{D}_a \)

\[
S_a(x, y) := \tan^{-1} \left( \frac{\sqrt{a^2 - \|x\|^2} \sqrt{a^2 - \|y\|^2}}{a \|x - y\|} \right), \quad x \neq y.
\]

(2.2)

**Remark 2.1.1.** The standard weakly singular operator \( V_0 \) (as defined in (1.50)) is given by (2.1) without \( S_a(x, y) \) and scaled by \( \frac{\pi}{8} \). Furthermore, since

\[
\lim_{x \to y} S_a(x, y) = \frac{\pi}{2}
\]

when \( x, y \in \mathbb{D}_a \),

(2.3)

the kernels of \( \mathcal{W} \) and \( V_0 \) have the same weakly singular behavior in (the interior of) \( \mathbb{D}_a \). Also note that \( S_a(x, y) = 0 \) if \( \|x\| = a \) or \( \|y\| = a \). As a consequence, \( S_a(x, y) \), though bounded, will be discontinuous on \( \partial \mathbb{D}_a \times \partial \mathbb{D}_a \). This is illustrated in Figure 2.1, where we plot \( y \mapsto S_1(x, y) \) for fixed \( x \in \mathbb{D}_1 \) in the interior (Figure 2.2a), and also for fixed \( x \in \mathbb{D}_1 \) near the boundary (Figure 2.2b). We can see that in all situations, \( S_1(x, y) \) is bounded.

![Figure 2.1: Plots of \( y \mapsto S_1(x, y) \) for fixed \( x \in \mathbb{D}_1 \).](image)

(a) \( x = (r_x = 0.5, \theta_x = \pi/4) \)
(b) \( x = (r_x = 0.99, \theta_x = \pi/4) \)

2.1.2. Modified Hypersingular Integral Operator

We define the modified hypersingular operator \( \mathcal{W} \) through the finite part integral

\[
(\mathcal{W}g)(x) := -\frac{2}{\pi^2} \int_{\mathbb{D}_a} g(y) K_{\mathcal{W}}(x, y) d\mathbb{D}_a(y), \quad x \in \mathbb{D}_a,
\]

(2.4)

---

\(^1\)Plot generated using Maple 2017.2. Maple is a trademark of Waterloo Maple Inc.
Due to \((2.1.3)\), we need to introduce and show some auxiliary results. Next, writing \(\theta \in [0, \pi]\) (cf. \cite{34}, Chap. 1.1). This function allows us to rewrite the kernels of \(V_0\) and \(W_0\) as will be shown in the following Lemma.

**Lemma 2.1.3 (Lemma 1 \cite{61}).** Let us consider points \(x, y\) on the disk \(\mathbb{D}_a\), satisfying \(x \neq y\). Then for a parameter \(\alpha \in (0, 4)\), such that \(\alpha \neq 2\), it holds

\[
\frac{1}{4\pi \|x - y\|^2} = \frac{1}{\pi} \sin \frac{\alpha \pi}{2} \int_0^{\min(r_x, r_y)} \frac{s^{\alpha-1}}{(r_x^2 - s^2)^{\alpha/2}(r_y^2 - s^2)^{\alpha/2}} \rho \left(\frac{s^2}{r_x r_y}, \theta_x - \theta_y\right) ds
\]

\[
= \frac{1}{\pi} \sin \frac{\alpha \pi}{2} \int_{\max(r_x, r_y)}^{\infty} \frac{s^{\alpha-1}}{(s^2 - r_x^2)^{\alpha/2}(s^2 - r_y^2)^{\alpha/2}} \rho \left(\frac{r_x r_y}{s^2}, \theta_x - \theta_y\right) ds.
\]

Here the integrals above are understood in the sense of finite-part integrals if \(\alpha > 2\).

**Sketch of Proof.** We outline the proof when \(\alpha \in (0, 2)\). First, one uses Sonine’s first integral \cite{42, Eq. 6.683.6} and \cite{42, Eq. 6.575.1} to derive

\[
\int_0^\infty s^{2\alpha - 1} J_n(s x) J_n(s y) ds = \frac{2^\alpha}{\Gamma\left(1 - \alpha/2\right)^2} \int_0^{\min(r_x, r_y)} \frac{s^{\alpha-1}}{(r_x^2 - s^2)^{\alpha/2}(r_y^2 - s^2)^{\alpha/2}} ds,
\]

with \(J_n\) is the \(n\)th Bessel function of the first kind \cite{30, § 10.2].

Next, writing \(R = \|x - y\|\), one can express \cite{42, Eq. 6.562.14} as

\[
\frac{1}{R^\alpha} = \frac{\Gamma\left(1 - \alpha/2\right)}{2^\alpha \Gamma\left(\alpha/2\right)} \int_0^\infty s^{\alpha-1} J_0(s R) ds, \quad 0 < \alpha < 2, x \neq y.
\]

2.1.3. Main Results

**Theorem 2.1.2.** The integral operators \(\nabla\) and \(\overline{\nabla}\) can be extended to continuous operators

\[
\nabla : H^{-1/2}(\mathbb{D}_a) \to \overline{H}^{1/2}(\mathbb{D}_a) \quad \text{and} \quad \overline{\nabla} : H^{1/2}(\mathbb{D}_a) \to \overline{H}^{-1/2}(\mathbb{D}_a).
\]

The extensions provide inverses of \(V_0\) and \(W_0\) on disks \(\mathbb{D}_a\):

\[
\overline{V}_0 = \text{Id}_{\overline{H}^{-1/2}(\mathbb{D}_a)} \quad \forall \overline{V} \overline{W}_0 = \text{Id}_{\overline{H}^{1/2}(\mathbb{D}_a)}.
\]

Before we proceed to the proof of Theorem 2.1.2 we need to introduce and show some auxiliary results. For simplicity, we will use the following polar coordinates notation: For \(x \in \mathbb{D}_a\) we write \(x = (r_x, \theta_x)\).

We begin by defining the function \(p(r, \theta)\) as

\[
p(r, \theta) = \frac{1 + r^2}{2\pi} \sum_{n=-\infty}^{\infty} r^n e^{in\theta} = \frac{1 + r^2}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}, \quad \forall |r| < 1,
\]

with \(\theta \in [0, 2\pi]\) (cf. \cite{34}, Chap. 1.1]). This function allows us to rewrite the kernels of \(V_0\) and \(W_0\) as will be shown in the following Lemma.
In addition, we have the following property [61, Eq. (13)]

$$J_0(tR) = J_0(tr_x)J_0(tr_y) + 2 \sum_{n=1}^{\infty} J_n(tr_x)J_n(tr_y) \cos[n(\theta_x - \theta_y)].$$  \hspace{1cm} (2.14)

Then, combining all the above we have

$$\frac{1}{R^\alpha} = \frac{\Gamma(1 - \alpha/2)}{2 \Gamma(\alpha/2)} \int_0^\infty t^{\alpha-1} \left( J_0(tr_x)J_0(tr_y) + 2 \sum_{n=1}^{\infty} J_n(tr_x)J_n(tr_y) \cos[n(\theta_x - \theta_y)] \right) dt$$

$$= \frac{2}{\Gamma(1 - \alpha/2)\Gamma(\alpha/2)} \int_0^{\min(r_x,r_y)} r^{\alpha-1} \frac{\Gamma(1 - \alpha/2)}{\Gamma(\alpha/2)} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{s^{2n}}{(s^2)^{\alpha/2}} \cos[n(\theta_x - \theta_y)] \right) ds$$

$$= \frac{2}{\Gamma(1 - \alpha/2)\Gamma(\alpha/2)} \int_0^{\min(r_x,r_y)} r^{\alpha-1} \frac{\Gamma(1 - \alpha/2)}{\Gamma(\alpha/2)} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{s^{2n}}{(s^2)^{\alpha/2}} \cos[n(\theta_x - \theta_y)] \right) ds$$

Finally, due to $\Gamma(1 - \alpha/2)\Gamma(\alpha/2) = \frac{\pi}{\sin \frac{\alpha\pi}{2}}$, one gets

$$\frac{1}{4\pi \|x - y\|^\alpha} = \frac{1}{\pi} \sin \frac{\alpha\pi}{2} \int_0^{\min(r_x,r_y)} \frac{r^{\alpha-1}}{(r_x^2 - s^2)^{\alpha/2}(r_y^2 - s^2)^{\alpha/2}} \left( \frac{s^2}{r_xr_y} \right) \cos(\theta_x - \theta_y) ds.$$

For $2 < \alpha < 4$ one uses the previous ingredients together with $R^{-\alpha/2} = \Delta R^{-\alpha}$ and the differential equation satisfied by the Bessel functions $J_\alpha$.

One obtains (2.11) from (2.10) by the change of variable $s' = \frac{r_xr_y}{s}$. \hspace{1cm} \Box

**Remark 2.1.4.** Note that Lemma 2.1.3 with $\alpha = 3$ gives us the kernel of $W_0$ for the case of the disk $D_\alpha$ with a minus sign (given the definition in (1.51)). Therefore, our results differ from [61] by a minus sign.

We can now follow Fabrikant’s work and introduce

$$L(\rho)v(r_x, \theta_x) := \int_0^{2\pi} p(\rho, \theta_x - \theta_0)v(r_x, \theta_0) d\theta_0$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \rho^n e^{in\theta} \int_0^{2\pi} e^{-in\theta_0} v(r_x, \theta_0) d\theta_0,$$

with $x = (r_x, \theta_x) \in \mathbb{D}_\alpha$. This integral operator is sometimes called Poisson integral over the disk [34, Chap. 1.1].

Combining the properties of $L(\rho)v(r_x, \theta_x)$ with the formulas from Lemma 2.1.3, one can recast our BIEs for $k = 0$ as

$$L^{-1}(\rho) = L(\rho^{-1})$$

$$\begin{align*}
(1.52) & \quad \Leftrightarrow \quad \frac{1}{\pi} L \left( \frac{1}{r_x} \right) \int_0^{r_x} \frac{1}{(r_x^2 - s^2)^{1/2}} \int_s^{r_y} \frac{1}{(r_y^2 - s^2)^{1/2}} \sigma(r_y, \theta_y) dr_y ds = g(r_x, \theta_x), \\
(1.53) & \quad \Leftrightarrow \quad \frac{1}{\pi} L \left( \frac{1}{r_x} \right) \int_0^{r_x} \frac{s^2}{(r_x^2 - s^2)^{1/2}} \int_s^{r_y} \frac{s^2}{(r_y^2 - s^2)^{1/2}} \mu(r_y, \theta_y) dr_y ds = \mu(r_x, \theta_x),
\end{align*}$$

which achieves a complete separation of variables. This decomposition is crucial since the resulting expressions can be solved using iteratively the property $L^{-1}(\rho) = L(\rho^{-1})$ and the techniques from Abel integral equations, as long as the right hand side is continuously differentiable.

We first discuss the case of the hypersingular BIE, whose solution is given in the next Theorem.
Finally, one simplifies
\[ r - \] where we used (\[2.21\]) after changing order of integration.

\[ \text{Sketch of Proof.} \] Let us recall the two following general Abel IEs and their solutions \([78, \text{Eq. (35)--(36)}]\) [41, Eq. (1.B.5i)]:

\[ \int_0^t \frac{F(t)}{(t^2 - s^2)^{(1+\nu)/2}} \, dt = f(s) \Leftrightarrow F(t) = \frac{\cos(\pi\nu/2)}{\pi} \frac{d}{dt} \int_t^a \frac{2sf(s)ds}{(s^2 - t^2)^{(1-\nu)/2}}, \]  
\[ \int_0^t \frac{F(s)}{\sqrt{t^2 - s^2}} \, ds = f(x) \Leftrightarrow F(s) = \frac{1}{\pi ds} \int_s^x \frac{2tf(t)dt}{\sqrt{s^2 - t^2}}. \tag{2.21} \]

for \(-1 < \nu < 1\) and \(f\) continuously differentiable.

We begin from (2.17) which is an Abel IE of the first type. We multiply (2.17) by \(\pi L(r_x)r_x\) and integrate over \(s\) from 0 to \(t\):

\[ \int_0^t \int_0^{r_x} \frac{r_y^2}{(r_y^2 - s^2)^{3/2}} L(s^2) \int_s^a \left( \frac{1}{r_y} \right) u(r_y, \theta_y) dr_y ds dr_x = \pi \int_0^t L(r_x) r_x \mu(r_x, \theta_x) dr_x. \tag{2.22} \]

Then, let us introduce the following property [61, Eq. (34)]

\[ \frac{d}{dr_x} \int_0^{r_x} \frac{sF(s)}{\sqrt{r_x^2 - s^2}} ds = -r_x \int_0^{r_x} \frac{sF(s)}{(r_x^2 - s^2)^{3/2}} ds \]  
\[ = F(s = 0)r_x^{-1/2} + r_x \int_0^s \frac{1}{\sqrt{r_x^2 - s^2}} ds \frac{dF(s)}{ds} ds. \tag{2.23} \]

We use (2.23) in (2.22) to get

\[ - \int_0^t \frac{\sqrt{r_x^2 - s^2}}{s^2} L(s^2) \int_s^a \left( \frac{1}{r_y} \right) u(r_y, \theta_y) dr_y ds = \pi \int_0^t L(r_x) r_x \mu(r_x, \theta_x) dr_x, \]

which is an Abel IE of the first kind. We solve this using (2.21) and obtain

\[ -s^2 L(s^2) \int_s^a \left( \frac{1}{r_y} \right) u(r_y, \theta_y) dr_y ds = \frac{2s}{ds} \int_0^s \frac{t}{\sqrt{s^2 - t^2}} \int_0^t L(r_x) r_x \mu(r_x, \theta_x) dr_x dt \]
\[ = 2s \int_0^s \frac{r_x}{\sqrt{s^2 - r_x^2}} L(r_x) \mu(r_x, \theta_x) dr_x, \]

where we used (2.24) in the last equality. Analogously, we multiply by \(s^{-1}L(s^{-2})\), integrate over \(s\) from \(t\) to \(a\), arrive to an Abel IE, employ (2.20) and get

\[ r_y L \left( \frac{1}{r_y} \right) u(r_y, \theta_y) = \frac{4}{\pi} r_y \int_{r_y}^a \frac{1}{\sqrt{s^2 - r_y^2}} L(s^{-2}) \int_0^s \frac{r_x}{\sqrt{s^2 - r_x^2}} L(r_x) \mu(r_x, \theta_x) dr_x ds. \]

Finally, one simplifies \(r_y\), multiply by \(L \left( \frac{1}{r_y} \right)\) and arrives to

\[ u(r_y, \theta_y) = \frac{4}{\pi} L(r_y) \int_{r_y}^a \frac{1}{\sqrt{s^2 - r_y^2}} L(s^{-2}) \int_0^s \frac{r_x}{\sqrt{s^2 - r_x^2}} L(r_x) \mu(r_x, \theta_x) dr_x ds, \]

which gives (2.19) after changing order of integration.
Remark 2.1.6. From (2.11) with \( \alpha = 1 \), we notice that \( R_{\Omega}(x, y) \) is a scaled restriction of \( \|x - y\| \) from \( \mathbb{R}^3 \) to \( \mathbb{D}_a \). Moreover, for \( a = \infty \), Theorem 2.1.5 implies \( R_{\Omega}(x, y) = \frac{1}{\|x - y\|} \).

Lemma 2.1.7. When \( \mu \) is continuously differentiable, the solution of the hypersingular BIE (1.53) can be written as \( u(x) = (\nabla \mu)(x) \), for all \( x \in \mathbb{D}_a \).

Proof. From Theorem 2.1.5, we know that if \( \mu \) is continuously differentiable, then the solution to (1.53) can be written as

\[
    u(x) = \frac{4}{\pi} \int_{\mathbb{D}_a} \mu(y) \int_{r_{\max}(r_x, r_y)}^{a} \frac{p\left(\frac{r_x r_y}{s}, \theta_x - \theta_y\right)}{(s^2 - r_x^2)^{1/2}(s^2 - r_y^2)^{1/2}} ds d\mathbb{D}_a(y). \tag{2.25}
\]

For \( a < \infty \), we apply [34, Eq. 1.2.18] and get

\[
    u(x) = \frac{2}{\pi^2} \int_{\mathbb{D}_a} \frac{\mu(y)}{\|x - y\|} \tan^{-1} \left( \frac{\sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2}}{s \|x - y\|} \right)_{r_{\max}(r_x, r_y)}^{a} d\mathbb{D}_a(y)
\]

\[
    = \frac{2}{\pi^2} \int_{\mathbb{D}_a} \frac{\mu(y)}{\|x - y\|} \left\{ \tan^{-1} \left( \frac{\sqrt{a^2 - r_x^2} \sqrt{a^2 - r_y^2}}{a \|x - y\|} \right) - \tan^{-1}(0) \right\} d\mathbb{D}_a(y)
\]

\[
    = \frac{2}{\pi^2} \int_{\mathbb{D}_a} \mu(y) S_a(x, y) d\mathbb{D}_a(y),
\]

with \( S_a(x, y) \) from (2.2).

Now we proceed with the weakly singular BIE.

Theorem 2.1.8 (Thm.1 [61]). Let \( g \in C^1(\mathbb{D}_a) \). Then, the solution \( \sigma(x) \) of (1.52) can be expressed in terms of a two-dimensional hypersingular integral as follows

\[
    \sigma(x) = -\frac{1}{\pi} \int_{\mathbb{D}_a} \frac{g(y)}{R_{\Omega}^3(x, y)} d\mathbb{D}_a(y), \quad x \in \mathbb{D}_a, \tag{2.26}
\]

where

\[
    \frac{1}{R_{\Omega}^3(x, y)} := -\frac{4}{\pi} \int_{r_{\max}(r_x, r_y)}^{a} \frac{s^2}{(s^2 - r_x^2)^{3/2}(s^2 - r_y^2)^{3/2}} p\left(\frac{r_x r_y}{s^2}, \theta_x - \theta_y\right) ds. \tag{2.27}
\]

We remind the reader that this result is analogous to that of Theorem 2.1.5 and that it is proven by rewriting (1.52) as (2.16) and then using the standard Abel integral equation solution techniques and the properties of the operator \( L \) defined in (2.15).

In analogy to what we did for the hypersingular BIE, converting (2.26) into (2.5) requires the following result.

Lemma 2.1.9. Let \( x, y \in \mathbb{D}_a \). If \( a \geq s \geq \max(r_x, r_y) \), we find the following primitive

\[
    \int \frac{s^2 p\left(\frac{r_x r_y}{s^2}, \theta_x - \theta_y\right)}{(s^2 - r_x^2)^{3/2}(s^2 - r_y^2)^{3/2}} ds =
\]

\[
    -\frac{1}{2\pi} \left( \frac{s}{\|x - y\|^2} \sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2} \right) + \tan^{-1} \left( \frac{\sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2}}{s \|x - y\|^3} \right) \tag{2.28}
\]
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\textbf{Proof.} This can be shown by direct calculation using the following change of variable \footnote{One uses the same change of variable to prove \cite[Eq. 1.2.18]{34}. To the author’s knowledge, however, it has never been used for this expression before.}:

\[ \eta := \frac{\sqrt{s^2 - r_x^2} \sqrt{s^2 - r_y^2}}{r}, \quad \frac{d\eta}{ds} = \frac{s^4 - r_x^2 r_y^2}{\eta s^3}, \]

which leads to

\[ \int \frac{s^2}{(s^2 - r_x^2)^{3/2}(s^2 - r_y^2)^{3/2}} \frac{1 - r_x^2 r_y^2}{1 + r_x^2 r_y^2 - 2 r_x r_y \cos(\theta_x - \theta_y)} ds = \int \frac{\eta^{-2}}{\|x - y\|^2 + \eta^2} d\eta, \]

where

\[ \int \frac{\eta^{-2}}{\|x - y\|^2 + \eta^2} d\eta = -\frac{1}{\eta \|x - y\|^2} \tan^{-1} \left( \frac{\eta}{\|x - y\|^2} \right). \] (2.29)

By definition of $\eta$ the result follows. \hfill \Box

\textbf{Lemma 2.1.10.} If $g$ is continuously differentiable, then the solution of the weakly singular BIE (1.52) can be written as $\sigma(x) = \langle Wg \rangle(x)$, for all $x \in D_a$.

\textbf{Proof.} From Theorem 2.1.8, we know that if $g$ is continuously differentiable, then the solution to (1.52) can be written as

\[ \sigma(x) = \frac{4}{\pi} \int_{\partial D_a} g(y) \int_{\max(r_x, r_y)}^a \frac{s^2 p \left( \frac{r_x r_y}{s}, \theta_x - \theta_y \right)}{(s^2 - r_x^2)^{3/2}(s^2 - r_y^2)^{3/2}} ds \, d\Omega_a(y). \] (2.30)

Now, let $\bar{r} := \max(r_x, r_y)$. For $a < \infty$, we use Lemma 2.1.9 to deal with the inner integral and get

\[ \int_{\bar{r}}^a \frac{s^2 p \left( \frac{r_x r_y}{s}, \theta_x - \theta_y \right)}{(s^2 - r_x^2)^{3/2}(s^2 - r_y^2)^{3/2}} ds = -\frac{1}{2\pi} \int_{\bar{r}}^a p \left( \frac{s}{\|x - y\|^2 \sqrt{s^2 - r_x^2 \sqrt{s^2 - r_y^2}}} \tan^{-1} \left( \frac{\sqrt{s^2 - r_x^2 \sqrt{s^2 - r_y^2}}}{s \|x - y\|^2} \right) \right) ds \bigg|_{s = \bar{r}} \]

Here the finite part ($fp$) of the last expression needs to be considered. This means that we keep only the first two terms because the third one tends to $\infty$ for $\epsilon \to 0$ and the fourth term vanishes in the limit. Hence, we arrive at

\[ \sigma(x) = \frac{2}{\pi^2} \int_{\partial D_a} g(y) \left( \frac{a}{\|x - y\|^2 \sqrt{a^2 - r_x^2 \sqrt{a^2 - r_y^2}}} + \frac{S_a(x, y)}{\|x - y\|^3} \right) d\Omega_a(y) \]

with $K_{\overline{W}}$ from (2.5) as stated. \hfill \Box
Proof of Theorem 2.1.2. Both identities in (2.7) follow from Lemma 2.1.10 combined with the density of \( C^\infty(\overline{D}_a) \) in \( H^{-1/2}(D_a) \). Analogously, the relations in (2.8) are consequences of Lemma (2.1.7) and the density of \( C^\infty(\overline{D}_a) \) in \( H^{1/2}(D_a) \). Thus, the mapping properties of \( \overline{W} \) and \( \overline{V} \) are induced by those of \( V_0 \) and \( W_0 \).

We mention a simple consequence of continuity and ellipticity of the standard BIOs \( W_0 \) and \( V_0 \) combined with Theorem 2.1.2.

**Corollary 2.1.11.** The bilinear forms:

\[
(\vartheta, \mu) \mapsto \langle \overline{V} \vartheta, \mu \rangle_{D_a}, \quad \vartheta, \mu \in H^{-1/2}(D_a),
\]

\[
(u, g) \mapsto \langle \overline{W} u, g \rangle_{D_a}, \quad u, g \in H^{1/2}(D_a),
\]

are elliptic and continuous in \( H^{-1/2}(D_a) \) and \( H^{1/2}(D_a) \), respectively.

Next, we look for a formula for the bilinear form (2.32) that can be used directly when implementing a Galerkin BE discretization. For the hypersingular operator \( W_0 \) this was achieved by (1.57), which connects \( W_0 \) with \( V_0 \). We aim to establish an analogous relation between \( \overline{W} \) and \( \overline{V} \). Let us denote \( \omega(x) := \sqrt{1 - r_x^2}, \ x \in D_1 \).

**Lemma 2.1.12.** The bilinear form associated to the modified hypersingular operator \( \overline{W} \) over \( D_1 \) satisfies

\[
\langle \overline{W} u, v \rangle_{D_1} = \frac{2}{\pi^2} \int_{D_1} \int_{D_1} \frac{S_1(x, y)}{||x-y||} \text{curl}_{D_1} u(x) \cdot \text{curl}_{D_1} v(y) dD_1(x)dD_1(y),
\]

for all \( u, v \in H^{1/2}(D_1) := \{ v \in H^{1/2}(D_1) : \langle v, \omega^{-1} \rangle_{D_1} = 0 \} \).

This result was first reported as a definition by Nedélec and Ramacotti in their spectral study of the BIOs over \( D_1 \) and their variational inverses [77, Def. 2.7.12]. We provide a proof of this proposition using eigenfunctions of the BIOs at the end of this subsection. A proof by means of formal integration by parts remains elusive, as it encounters difficulties due to the finite part integrals involved in the definition of \( \overline{W} \) and its kernel introduced in (2.5).

Unfortunately, the expression in (2.33) vanishes for constant \( u \) or \( v \) and, thus, has a non-trivial kernel, if considered in the whole \( H^{1/2}(D_1) \) space. In other words, the bilinear form in (2.33) is \( H^{1/2}(D_1) \)-elliptic but does not have this property on \( H^{1/2}(D_1) \). For this reason, its extension to \( H^{1/2}(D_1) \) does not actually match the bilinear form of \( \overline{W} \) there, which is \( H^{1/2}(D_1) \)-elliptic. The correct extension involves an extra term and is presented in the next theorem.

**Theorem 2.1.13.** The symmetric bilinear form associated to the modified hypersingular operator \( \overline{W} : H^{1/2}(D_1) \rightarrow \overline{H}^{-1/2}(D_1) \) can be written as

\[
\langle \overline{W} u, v \rangle_{D_1} = \frac{2}{\pi^2} \int_{D_1} \int_{D_1} \frac{S_1(x, y)}{||x-y||} \text{curl}_{D_1} u(x) \cdot \text{curl}_{D_1} v(y) dD_1(x)dD_1(y)
\]

\[
+ \frac{2}{\pi^2} \int_{D_1} \int_{D_1} \frac{u(x)v(y)}{\omega(x)\omega(y)} dD_1(x)dD_1(y),
\]

\[\forall u, v \in H^{1/2}(D_1).\]

Proof. We start from (2.33) and add an appropriate regularizing term coming from the definition of \( H^{1/2}_*(D_1) \) and the known result [63, Eq. (12)]

\[
(\overline{W}1)(y) = \frac{4}{\pi} \omega^{-1}(y), \quad y \in D_1.
\]

From this, we see that for \( u_c \) constant, \( (\overline{W} u_c)(y) \) evaluates to

\[
(\overline{W} u_c)(y) = \frac{2}{\pi^2} \int_{D_1} u_c \omega^{-1}(x)\omega^{-1}(y) dD_1(x), \quad y \in D_1,
\]

\[
\text{(2.36)}
\]
since $\langle 1, \omega^{-1} \rangle_{\mathcal{D}_1} = 2\pi$. It is clear that the variational form of (2.36) corresponds to the additional term in (2.34).

This construction guarantees that our bilinear form (2.34) is equivalent to the bilinear form arising from our modified hypersingular operator $\mathcal{W}$ (2.4) on $H^{1/2}(\mathcal{D}_1)$. Thus, it is $H^{1/2}(\mathcal{D}_1)$-continuous and elliptic.

Theorem 2.1.13 gives us a variational form for $\mathcal{W}$ that can be easily implemented. Observe that the chosen regularization to extend the bilinear form (2.33) from $H^{1/2}(\mathcal{D}_1)$ to $H^{1/2}(\mathcal{D}_1)$ is analogous to the one needed for the modified hypersingular operator on a segment [49, Eq. (2.11)].

In addition, by linearity of the scaling map $\Psi_a : \mathcal{D}_1 \rightarrow \mathcal{D}_a$, we get the corresponding relationship on $\mathcal{D}_a$:

**Corollary 2.1.14.** For $a > 0$, the symmetric bilinear form associated to the modified hypersingular operator $\mathcal{W} : H^{1/2}(\mathcal{D}_a) \rightarrow \mathcal{H}^{-1/2}(\mathcal{D}_a)$ can be written as

$$
\langle \mathcal{W} u, v \rangle_{\mathcal{D}_a} = \frac{2}{\pi^2} \int_{\mathcal{D}_a} \int_{\mathcal{D}_a} \frac{S_a(x, y) \text{curl}_{\mathcal{D}_a, x} u(x) \cdot \text{curl}_{\mathcal{D}_a, y} v(y) \, d\mathcal{D}_a(x) \, d\mathcal{D}_a(y)}{|x - y|^3} + \frac{2}{a \pi^2} \int_{\mathcal{D}_a} \int_{\mathcal{D}_a} \frac{u(x) v(y) \, d\mathcal{D}_a(x) \, d\mathcal{D}_a(y)}{|x - y|^2},
$$

(2.37)

where $\omega_a(x) := \sqrt{a^2 - r_x^2}, x \in \mathcal{D}_a$.

**Remark 2.1.15.** The fact that

$$
\tan^{-1}(\frac{1}{x}) - \frac{\pi}{2} = -\tan^{-1}(x), \quad x > 0
$$

(2.38)

allows the following singularity subtraction in $\mathcal{B}_\mathcal{W}$:

$$
\frac{S_a(x, y) - \pi/2}{|x - y|^3} = -\frac{1}{|x - y|^3} \tan^{-1}\left(\frac{|x - y|}{\sqrt{a^2 - |x|^2} \sqrt{a^2 - |y|^2}}\right) = O(|x - y|^{-2}),
$$

(2.39)

for $x \rightarrow y, x, y \notin \partial \mathcal{D}_a$. This may pave the way to show (2.33) by means of the same steps as for $\mathcal{W}$.

**Eigenfunctions on the Disk and Proof of Lemma 2.1.12**

Our proof relies on “diagonalization”, the explicit characterization of the eigenfunctions of the BIONs. We begin by introducing key definitions and results on the unit disk $\mathcal{D}_1$. When needed, we provide a short proof for the results reported in [77, Chap. 2].

**Definition 2.1.16 (Projected Spherical Harmonics (PSHs) on $\mathcal{D}_1$ [77, Def. 2.7.4]).** For $l, m \in \mathbb{N}_0$ such that $|m| \leq l$ and $x = (r_x, \theta_x)$, we introduce the functions:

$$
y_l^m(x) := \gamma_l^m e^{im\theta_x} \mathbb{P}_l^m(\sqrt{1 - r_x^2}), \quad \gamma_l^m := (-1)^m \sqrt{\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}},
$$

(2.40)

where $\mathbb{P}_l^m$ denotes the associated Legendre functions of the first kind [30, § 14.3].

If $l_1 + m_1$ and $l_2 + m_2$ have the same parity, the following orthogonality holds [77, Prop. 2.7.4]

$$
\int_{\mathcal{D}_1} \frac{y_{l_1}^{m_1}(y) y_{l_2}^{m_2}(y)}{\omega(y)} \, d\mathcal{D}_1(y) = \frac{1}{2} \delta_{l_1}^{l_2} \delta_{m_1}^{m_2},
$$

(2.41)

where $\delta_{l_1}^{l_2}$ is the Kronecker symbol and again $\omega(y) = \sqrt{1 - r_y^2}$.
Lemma 2.1.17 ([91]). The PSHs solve a generalized eigenvalue problem for $V_0$ over $\mathbb{D}_1$ in the sense that
\[
\left( V_0 \frac{y^m}{\omega} \right)(x) = \frac{1}{4} \lambda_m^n y^m(x), \quad l + m \text{ even},
\]  
with $\lambda_m^n := \frac{\Gamma(\frac{l+m+1}{2}) \Gamma(\frac{l-m+1}{2})}{\Gamma(\frac{l+m+2}{2}) \Gamma(\frac{l-m+2}{2})}$, and $\Gamma$ being the Gamma function.

Sketch of Proof. Rewrite the weakly singular operator (1.50) on $\mathbb{D}_1$ using polar coordinates with $re^{i\theta} := x - y = r_x e^{i\theta_x} - r_y e^{i\theta_y}$:
\[
(V_0 \sigma)(x) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 \frac{\sigma(r_y, \theta_y)}{r_y} r_y dr_y d\theta_y, \quad x \in \mathbb{D}_1.
\]
Use the Ansatz
\[
\sigma(r_y, \theta_y) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{y^m(r_y)}{\omega(r_y)} \frac{1}{\omega(y)} e^{im\theta_y} d\theta_y,
\]
This gives
\[
(V_0 \sigma)(x) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{y^m(r_x)}{\omega(r_x)} \frac{1}{\omega(x)} e^{im\theta_x} d\theta_x.
\]
We notice that $I_{l,m}(r_x, \theta_x)$ is the convolution of $F(r_x, \theta_x) := \frac{1}{r_x}$ and
\[
G(r_x, \theta_x) := \begin{cases} P_m(\omega(x)) e^{im\theta_x} / \omega(x), & r_x < 1, \\ 0, & r_x > 1. \end{cases}
\]
So we can compute $I_{l,m}(r_x, \theta_x)$ using the product of their Fourier transforms $\mathcal{F}(F)$ and $\mathcal{F}(G)$. This leads to
\[
\mathcal{F}(I_{l,m}) = \mathcal{F}(F) \mathcal{F}(G) = (-1)^m \frac{\Gamma(\frac{l+m+1}{2})}{\Gamma(\frac{l-m+2}{2})} e^{im\pi/2} 2^{m+3/2} e^{im\theta} J_{l+1/2}(r_x),
\]
and
\[
I_{l,m} = \mathcal{F}^{-1}(\mathcal{F}(I_{l,m})) = \pi \frac{\Gamma(\frac{l+m+1}{2}) \Gamma(\frac{l-m+1}{2})}{\Gamma(\frac{l+m+2}{2}) \Gamma(\frac{l-m+2}{2})} J_{l+1/2}(r_x) e^{im\theta}.
\]
with $J_m$ is the $m$-th Bessel function of the first kind.\(^3\) Finally, the result follows plugging this expression in (2.43).

Lemma 2.1.18 ([60]). The PSHs solve a generalized eigenvalue problem for $W_0$ over $\mathbb{D}_1$ in the sense that
\[
(W_0 y^m)(x) = \frac{1}{\lambda_m^n} \frac{y^m(x)}{\omega(x)}, \quad l + m \text{ odd},
\]  
with $\lambda_m^n$ as in Proposition 2.1.17.

Remark 2.1.19. Propositions 2.1.17 and 2.1.18 nicely show how, in the case of a disk, the usual $V_0$ and $W_0$ have reciprocal symbols but $W_0 V_0 \neq \frac{1}{4} \text{Id}$ due to their mapping properties. This is evident from the parity of the PSHs involved.

\(^3\)Note that this value for $\mathcal{F}(I_{l,m})$ corresponds to the (non-standard) Fourier transform used in Wolfe’s paper and that constants would change with the normalized definition. Nevertheless, the resulting $I_{l,m}$ is independent of the chosen scaling.
Proposition 2.1.20. The sets of functions

\[
\begin{align*}
\{ y_l^m : l + m \text{ odd} \} \\
\{ y_l^m : l + m \text{ even} \} \\
\{ y_l^m \omega^{-1} : l + m \text{ odd} \} \\
\{ y_l^m \omega^{-1} : l + m \text{ even} \}
\end{align*}
\]
span a dense subspace of \( \tilde{H}^{1/2}(\mathbb{D}_1) \) and \( H^{1/2}(\mathbb{D}_1) \).

Proof. Since the bilinear form associated to \( \mathcal{W}_0 \) is symmetric, elliptic and continuous, it induces an energy inner product on \( \tilde{H}^{1/2}(\mathbb{D}_1) \). Then, the proof of the first assertion boils down to showing that for any \( u \in \tilde{H}^{1/2}(\mathbb{D}_1) \):

\[
(u, y_l^m)_{\tilde{H}^{1/2}(\mathbb{D}_1)} \approx (\mathcal{W}_0 u, y_l^m)_{\mathbb{D}_1} = 0, \quad \forall \, l + m \text{ odd} \quad \Leftrightarrow \quad u \equiv 0.
\]

Using the symmetry of the bilinear form and (2.44), we derive

\[
\langle \mathcal{W}_0 u, y_l^m \rangle_{\mathbb{D}_1} = \frac{1}{\lambda_l^m} \langle u, y_l^m \rangle_{\mathbb{D}_1} = \frac{1}{\lambda_l^m} (u, y_l^m)_{1/\omega},
\]

where

\[
(u, v)_{1/\omega} := \int_{\mathbb{D}_1} u(x)\overline{v(x)}\omega(x)^{-1}d\mathbb{D}_1(x),
\]
is the inner product corresponding to the weighted \( L^2 \)-space \( L^2_{1/\omega}(\mathbb{D}_1) \) [77, Def. 2.7.9].

From [77, Prop. 2.7.9] we know that \( \{ y_l^m : l + m \text{ odd} \} \) is an orthogonal basis for \( L^2_{1/\omega}(\mathbb{D}_1) \). This implies that \( (u, y_l^m)_{1/\omega} = 0 \) if and only if \( u \equiv 0 \), giving the desired density result.

The second assertion follows analogously while the third and fourth can be inferred by the same steps but using the inner product of \( L^2_{\omega}(\mathbb{D}_1) \) [77, Def. 2.7.9] and its corresponding orthogonal bases [77, Prop. 2.7.8]. \( \square \)

Remark 2.1.21. One can prove that odd PSHs can be factored as

\[
y_l^m(x) = e^{im\theta}\omega(x)\Psi(x), \quad l + m \text{ odd},
\]

where \( \Psi \) is a polynomial function—combine [30, Eq. 14.3.21], [6, Eq. 3.2(7)], and [6, Eq. 10.9(22)]. However, this is not true when \( l + m \) is even. In that case, the radial part of \( y_l^m \) is already a polynomial (since [6, 10.9(21)] holds instead of [6, 10.9(22)]). This property confirms that the basis functions of our four fractional Sobolev spaces have the correct behaviour at \( \partial\mathbb{D}_1 \). Namely, when \( l + m \) is odd, \( y_l^m \sim \omega \) near the boundary and belongs to \( \tilde{H}^{1/2}(\mathbb{D}_1) \); when \( l + m \) is even, \( y_l^m \omega^{-1} \sim \omega^{-1} \) near \( \partial\mathbb{D}_1 \) and lies in \( \tilde{H}^{-1/2}(\mathbb{D}_1) \); while the basis functions of \( H^{1/2}(\mathbb{D}_1) \) and \( H^{-1/2}(\mathbb{D}_1) \) have no singular behaviour.

We consider the differential operators \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) on \( \mathbb{D}_1 \) [77, Eq. (2.157)] defined as

\[
\mathcal{L}_\pm u := e^{\pm\theta}\left( \pm \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \right), \quad u \in C^\infty(\mathbb{D}_1).
\]

Lemma 2.1.22 ([77, Prop. 2.7.7, Cor. 2.7.2, Lemma 2.7.3]). We have the adjoint relationships \( \mathcal{L}_+^* = \mathcal{L}_- \), and \( \mathcal{L}_-^* = \mathcal{L}_+ \). Moreover, for \( u, v \in C^\infty(\mathbb{D}_1) \) and \( x, y \in \mathbb{D}_1 \) holds

\[
\text{curl}_{\mathbb{D}_1, x} u(x) \cdot \text{curl}_{\mathbb{D}_1} v(y) = -\frac{1}{2} \left( \mathcal{L}_+ u(x) \mathcal{L}_- v(y) + \mathcal{L}_- u(x) \mathcal{L}_+ v(y) \right),
\]

When applied to PSHs the differential operators \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) yield

\[
\mathcal{L}_+ y_l^{m+1} = \sqrt{(l+m)(l+m+1)} y_l^m / \omega,
\]

\[
\mathcal{L}_- y_l^{m-1} = \sqrt{(l-m)(l-m+1)} y_l^m / \omega,
\]

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Proof of Lemma 2.1.12. We begin our proof by inferring from $\nabla = \mathcal{V}_0^{-1}$, and (2.42) that

$$\nabla y_l^m = \frac{4}{\lambda_l^m} y_l^m, \quad \text{for } l + m \text{ even},$$

(2.49)

Next, we rely on the following recursion formula

$$\frac{4}{\lambda_l^m} = \frac{1}{2} \left[ (l + m)(l - m + 1)\lambda_l^{m-1} + (l - m)(l + m + 1)\lambda_l^{m+1} \right],$$

(2.50)

which can be verified by direct computations using the multiplicative property of the Gamma function, i.e., $\Gamma(z + 1) = z\Gamma(z)$. Plugging this recursion formula into (2.49) gives

$$\nabla y_l^m = \frac{4}{\lambda_l^m} y_l^m = \frac{1}{2} \left[ (l + m)(l - m + 1)\lambda_l^{m-1} + (l - m)(l + m + 1)\lambda_l^{m+1} \right] \frac{y_l^m}{\omega}.$$  

(2.51)

Then we plug in (2.47) and (2.48). Next, from the fact that $\mathcal{W}_0^{-1} = \nabla$, cf. Theorem 2.1.2, and (2.44), we conclude that

$$\nabla y_l^{m \pm 1} = \lambda_l^{m \pm 1} y_l^{m \pm 1}, \quad l + m \pm 1 \text{ odd}.$$  

(2.52)

Then, combining all these ingredients, it is clear that, for $(l, m) \neq (0, 0)$, $l + m$ even\(^4\), our expression is equivalent to

$$\nabla y_l^m = \frac{1}{2} \left( \mathcal{L}_+ \nabla \mathcal{L}_- y_l^m + \mathcal{L}_- \nabla \mathcal{L}_+ y_l^m \right).$$  

(2.53)

It follows that the associated bilinear form is

$$\langle \nabla y_{l_1}^{m_1}, y_{l_2}^{m_2} \rangle_{\mathcal{D}_1} = \frac{1}{2} \left( \langle \mathcal{L}_+ \nabla \mathcal{L}_- y_{l_1}^{m_1}, y_{l_2}^{m_2} \rangle_{\mathcal{D}_1} + \langle \mathcal{L}_- \nabla \mathcal{L}_+ y_{l_1}^{m_1}, y_{l_2}^{m_2} \rangle_{\mathcal{D}_1} \right)$$

$$= \frac{1}{2} \left( \langle \nabla \mathcal{L}_- y_{l_1}^{m_1}, \mathcal{L}_- y_{l_2}^{m_2} \rangle_{\mathcal{D}_1} + \langle \nabla \mathcal{L}_+ y_{l_1}^{m_1}, \mathcal{L}_+ y_{l_2}^{m_2} \rangle_{\mathcal{D}_1} \right),$$

(2.54)

for $(l_1, m_1) \neq (0, 0)$ and $(l_2, m_2) \neq (0, 0)$. Finally, (2.46) and $\mathcal{Z}_k = -\mathcal{L}_\omega$ imply that (2.54) can be rewritten as the desired formula.

It is worth noticing that the condition $(l, m) \neq (0, 0)$ only excludes the constants represented by $y_0^0$. Due to the orthogonality (2.41), this space is defined by

$$H_+^{1/2}(\mathbb{D}_1) = \{ v \in H^{1/2}(\mathbb{D}_1) : \langle v, \omega^{-1} \rangle_{\mathcal{D}_1} = 0 \},$$

as introduced in Proposition 2.1.12.

We conclude this proof by extending the obtained formula from $\{ y_l^m : (l, m) \neq (0, 0), l + m \text{ even} \}$ to $H_+^{1/2}(\mathbb{D}_1)$ using linearity and the density results from Lemma 2.1.20. \(\square\)

2.2. Calderón-Type Identities for Helmholtz BIOs

**Lemma 2.2.1** ([81, Lemma 3.9.8]). Let $\Gamma$ be a Lipschitz boundary in $\mathbb{R}^3$ and let $k \in \mathbb{R}$. Then the following operators are compact:

$$\mathcal{V}_k - \mathcal{V}_0 : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

$$\mathcal{W}_k - \mathcal{W}_0 : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

\(^4\)In the case $(l, m) = (0, 0)$, (2.50) reads $\frac{4}{\lambda_0^0} = \frac{1}{2} \left( (0)(1)\lambda_0^0 + (0)(1)\lambda_0^1 \right) = \frac{4}{\pi} \lim_{s \to 0} \Gamma(s)s = \frac{4}{\pi}$.
We note that although this theorem was presented in [81, Lemma 3.9.8] for closed surfaces, the proof also applies to orientable screens $\Gamma$ by means of the artificial boundary $\partial \Omega_0$ such that $\Gamma \subset \partial \Omega_0$ discussed in Chapter 1.

**Corollary 2.2.2 (Calderón-type Identities for Helmholtz).** The following identities hold for $k > 0$:

\begin{align*}
\mathcal{W} V_k &= \text{id}_{\tilde{H}^{-1/2}(\Gamma)} + K_1(k) & (2.55) \\
\nabla W_k &= \text{id}_{\tilde{H}^{1/2}(\Gamma)} + K_2(k), & (2.56)
\end{align*}

with $K_1(k), K_2(k)$ are compact operators that depend on $k$. 


3. Operator Preconditioning for Acoustic Scattering at Screens

“It is a truth numerically acknowledged, that an ill-conditioned matrix system solved with a Krylov subspace method, must be in want of a preconditioner.”

– Adapted from Pride and Prejudice by Jane Austen (1813).

The goal of this Chapter is to construct optimal preconditioners for screens using the Calderón-type identities presented in Chapter 2 and operator preconditioning, which is a general and powerful policy for devising mesh-robust preconditioners for Galerkin matrices.

Sections 3.1–3.3 are devoted to introduce the abstract theory and building blocks we need to pursue operator preconditioning for the variational problems 1.4.5 and 1.4.6, which we label as cases (EDP) and (ENP), respectively.

Finally, we put the pieces together and present preconditioning strategies and obtained numerical results for the unit disk in Section 3.4 and for mapped screens in Section 3.5. We confine ourselves to the case of lowest order Galerkin BEM discretization but remark that the strategy presented in this Chapter can also be applied to other choices of Galerkin discretizations, as the abstract theory does not depend on it.\(^1\)

The contents of this Chapter are adapted from [56], [51] and [52].

3.1. Abstract Theory

Operator preconditioning is best introduced through the abstract framework of [46], which we summarize in the next Theorem:

**Theorem 3.1.1.** Let \(X, Y\) be reflexive Banach spaces and \(X_h \subset X\) and \(Y_h \subset Y\) finite dimensional subspaces. Let \(a \in L(X \times X, \mathbb{C})\) and \(b \in L(Y \times Y, \mathbb{C})\) be continuous sesquilinear forms (with norms \(\|a\|\) and \(\|b\|\), resp.), each satisfying discrete inf-sup conditions with constants \(c_A, c_B > 0\) on \(X_h\) and \(Y_h\), respectively.

If there is a continuous sesquilinear form \(t \in L(X \times Y, \mathbb{C})\) (with norm \(\|t\|\)) that also satisfies discrete inf-sup conditions on \(X_h \times Y_h\) with constant \(c_T > 0\), then, for any bases \(\{\varphi_i\}_{i=0}^N\) of \(X_h\) and \(\{\phi_j\}_{j=0}^M\) of \(Y_h\) such that \(\dim X_h = M = \dim Y_h\), we have that the associated Galerkin matrices

\[
A_h := (a(\varphi_i, \varphi_j))_{i,j=1}^N, \quad B_h := (b(\phi_i, \phi_j))_{i,j=1}^M, \quad T_h := (t(\varphi_i, \phi_j))_{i,j=1}^{N,M},
\]

satisfy:

\[
\kappa(T_h^{-1}B_hT_h^{-H}A_h) \leq \frac{\|a\|\|b\|\|t\|^2}{c_Ac_Bc_T^2}, \tag{3.2}
\]

where \(\kappa\) designates the spectral condition number and the symbol \(\|\cdot\|\) must be understood here as operator norms for the induced operators.

**Remark 3.1.2.** The sesquilinearity of \(a, b\) and \(t\) can be replaced in Theorem 3.1.1 by bilinearity and (3.2) would still hold. We make use of this fact since we deal with bilinear forms induced by the \(L^2\)-bilinear pairing \(\langle \cdot, \cdot \rangle_T\) (instead of the usual sesquilinear forms arising from the \(L^2\)-inner product).

\(^1\)However, one would need to consider a construction of the dual BE spaces and to prove the \(L^2\)-stability developed in Sections 3.2–3.3 accordingly.
Chapter 3. Preconditioning for Acoustic Scattering

**Remark 3.1.3.** Spectral condition numbers are relevant for Hermitian $a$ and $b$ because they provide a direct prediction of the convergence rates for conjugated gradient method (CG) and minimal residual method (MINRES).

In the non-Hermitian case, one resorts to generalized minimal residual method (GMRES) to solve the resulting matrix system. Although in practice one observes that smaller spectral condition numbers lead to faster GMRES convergence, the theory needs additional information to predict its convergence rates. We will come back to this topic and discuss some details of GMRES convergence in the second part of this thesis (Chapter 7). For now we just warn the reader that according to the current GMRES convergence results, $h$-independence of the spectral condition number will not necessarily translate to $h$-independent numbers of iterations.

Of course, in this Chapter the bilinear forms $a_{V,1}^h$ and $a_{W,1}^h$ will play the role of $a$ with $X = \tilde{H}^{-1/2}(\Gamma)$ and $X = \tilde{H}^{1/2}(\Gamma)$, respectively. The corresponding finite dimensional spaces $X_h$ and their bases will be specified in the next Section.

The presence of the inverses $T_h^{-1}$ and $T_h^H$ in (3.2) hints that a practical preconditioner $T_h^{-1}B_hT_h^H$ can be obtained only if $T_h$ is sparse. This will be guaranteed by using localized basis functions if we stipulate that

$$t \in L(X_h \times Y_h, \mathbb{R})$$

is the $L^2(\Gamma)$-duality pairing: $t(\varphi, \psi) := (\varphi, \psi)_\Gamma$.

Accepting this and demanding that all constants in (3.2) be independent of $h \in \mathbb{N}$ for the given family $\{\Gamma_h\}_{h \in \mathbb{N}}$ of finer and finer meshes, we conclude that $X$ and $Y$ must be in $L^2(\Gamma)$-duality. Therefore, $Y = H^{1/2}(\Gamma)$ and $Y = H^{-1/2}(\Gamma)$ for $a_{V,1}^h$ and $a_{W,1}^h$, respectively.

The building blocks of operator preconditioning as they will be assembled are summarized in Table 3.1. We dedicate the remainder of this Chapter to construct the missing pieces.

| Case (EDP): | $a_{V,1}^h$, cf. (1.54) | $H^{-1/2}(\Gamma)$ | $H^{1/2}(\Gamma)$ |
| Case (ENP): | $a_{W,1}^h$, cf. (1.59) | $\tilde{H}^{1/2}(\Gamma)$ | $H^{-1/2}(\Gamma)$ |

**Remark 3.1.4.** It was proven that classical “opposite-order” preconditioning leads to a condition number that features a logarithmic growth on 2D screens [66]. We would like to stress the fact that their proof [66, Theorem 4.1] comes from the mismatch of norms between the standard fractional spaces $H^{\pm 1/2}(\Gamma)$ and the tilde ones $\tilde{H}^{\pm 1/2}(\Gamma)$, with $\Gamma$ being the screen. Therefore, it is also valid on 3D screens and applies, not only to preconditioning $W_k$ by $V_k$, but to precondition $V_k$ with $W_k$ too.

In order to see this, recall the mapping properties of these BIOs on a screen $\Gamma$:

$$V_k : \tilde{H}^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma), \quad W_k : \tilde{H}^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma),$$

and that, as shown in Proposition 1.6.8, there are inverse inequalities bounding the norms of $\tilde{H}^{\pm 1/2}(\Gamma)$ by those of $\tilde{H}^{\pm 1/2}(\Gamma)$ in the discrete setting and these contain a term $\log h$ with $h$ being the meshwidth.

In the scope of our operator preconditioning theory, these two ingredients together with (3.2) justify our claim.

**Remark 3.1.5.** If $b$ happens to be the exact inverse of $a$, the obtained condition number (3.2) will be minimal. However, this is not really required in order to have an optimal preconditioner, and there are several suitable candidates for the bilinear forms $b$. Indeed, if $a$ and $b$ are symmetric positive definite, it is sufficient that $b$ is spectrally equivalent to the exact inverse of $a$. 

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3.2. Discretization

In the spirit of [81, Definition 4.1.4] we equip the screen $\Gamma$ with a regular triangular surface mesh $\Gamma_h = \{\tau\}$, whose elements $\tau$ are supposed to be diffeomorphic images of a flat reference triangle.

In this thesis we adopt an asymptotic perspective, studying the behavior of algorithms for “meshwidth $\rightarrow$ zero”. To that end, we rely on a family of meshes $\{\Gamma_h\}_{h \in \mathbb{R}^+}$ of $\Gamma$, indexed by $h$ from an index set $\mathbb{R}^+ \subset \mathbb{R}$ with accumulation point zero; $h$ should be read as the meshwidth of $\Gamma_h$. Appealing to [81, Definition 4.1.12] we assume limited distortion of element shapes:

**Assumption 3.2.1.** The family $\{\Gamma_h\}_{h \in \mathbb{R}^+}$ of meshes is assumed to be uniformly shape-regular, i.e. the shape-regularity constants of the $\Gamma_h$ are $h$-uniformly bounded.

**Remark 3.2.2.** We must not assume quasi-uniformity of $\{\Gamma_h\}_{h \in \mathbb{R}^+}$ according to [81, Eq. (4.1.9)] as it rules out locally refined meshes, because the singular behavior of generic solutions of (1.55) and (1.59) at $\partial \Gamma$ badly calls for mesh refinement towards the boundary of the screen.

3.2.1. BE Spaces on Primal Meshes

For Galerkin discretization of the variational equations (1.55) and (1.59) we rely on standard $\Gamma_h$-piecewise (mapped) polynomial boundary element (BE) spaces as introduced in [81, Sections 4.1.3 & 4.1.7]:

- **(EDP), (1.55):** We use the space

  $X_h = \mathcal{S}^{-1.0}(\Gamma_h) := \text{“space of piecewise constant functions on } \Gamma_h \text{”} \subset \tilde{H}^{-1/2}(\Gamma)$.

  Its dimension agrees with the number of elements of $\Gamma_h$ and the characteristic functions of the mesh elements provide the customary basis $\{\phi_\tau\}_{\tau \in \Gamma_h}$.

- **(ENP), (1.59):** We employ the BE space

  $X_h = \mathcal{S}_0^{0,1}(\Gamma_h) := \text{“space of continuous piecewise linear functions on } \Gamma_h$, satisfying zero boundary conditions on } \partial \Gamma \text{”} \subset \tilde{H}^{1/2}(\Gamma)$.

  Its dimension agrees with the number of interior nodes of $\Gamma_h$ and we use the standard “nodal” basis comprised of (mapped) “tent functions” $\varphi_v$, with $v$ from the set of vertices of $\Gamma_h$.

In both cases, we arrive at dense linear systems of equations featuring symmetric coefficient matrices. Naturally, since we deal with first-kind boundary integral operators of order $\pm 1$ and use merely $L^2(\Gamma)$-stable bases, the spectral conditions numbers of the Galerkin matrices can be expected to increase like $O(h_{\text{min}}^{-1})$ for shrinking minimal element size $h_{\text{min}} \rightarrow 0$ of the mesh (cf. [81, Section 4.5] for a detailed exposition).

3.2.2. BE Spaces on Dual Meshes

We build the dual meshes based on a barycentric refinement following Steinbach [84, Chap. 2.2] and Buffa and Christiansen [15]. Starting from a triangular mesh $\Gamma_h$ of the screen\(^2\), dubbed *primal mesh*, we introduce its barycentric refinement $\hat{\Gamma}_h$ as follows:

**Definition 3.2.3.** For each element $\tau \in \Gamma_h$:

1. Locate its center of mass.

2. Compute the centers of the edges of $\tau$. From now on referred as *mid-edge vertices*.

\(^2\)as introduced at the beginning of this subsection
iii Create the child elements of $\tau$ by connecting the original vertices of $\tau$ and those computed in (i) and (ii) (see Figure 3.1).

We define the barycentric refinement $\hat{\Gamma}_h$ of $\Gamma_h$ as the union of all these children elements.

**Figure 3.1.: Barycentric refinement.** We illustrate the 6 children elements obtained for a triangular element. We show the original nodes using red dots, a green diamond for the center of mass and blue x’s for the mid-edges nodes.

---

**Definition 3.2.4.** The union of all those cells of $\hat{\Gamma}_h$ abutting a single node of the primal mesh, create the so called dual cells. These are (curved) polygons as shown in Figure 3.2b.

We define our dual mesh $\hat{\Gamma}_h$ to be the union of all the resulting dual cells. The nodes of $\hat{\Gamma}_h$ coincide with the barycenters of cells of $\Gamma_h$.

As mentioned at the beginning of this section, in this thesis we restrict to the case of triangular meshes. Specimens of primal and dual meshes are displayed in Figures 3.2 and 3.4.

BE spaces on $\hat{\Gamma}_h$, which will eventually serve as the spaces $Y_h$ for operator preconditioning, are spanned by linear combinations of basis functions of BE spaces $\hat{Y}_h$ built on $\hat{\Gamma}_h$. In order to outline this construction of $Y_h$ for our two main cases, we introduce the concept of linking matrices.

**Definition 3.2.5.** Let $\hat{X}_h \subset X$ and $\hat{Y}_h \subset Y$ the BE spaces spanned by the basis functions over the barycentric refinement $\hat{\Gamma}_h$. Let us introduce the two linking matrices:

- **Averaging matrix** $C_d : \hat{Y}_h \rightarrow Y_h$
  
  Tells us how to combine the barycentric basis functions of $\hat{Y}_h$ to get the basis functions of $Y_h$ over the dual mesh. Then, in order to establish $\dim Y_h = M$, it must hold that $C_d \in \mathbb{R}^{M,\hat{M}_Y}$, where $\hat{M}_Y := \dim \hat{Y}_h$.

- **Coupling matrix** $C_p : \hat{X}_h \rightarrow X_h$
  
  Basis representation of the embedding identity $X_h \rightarrow \hat{X}_h$, since $X_h \subset \hat{X}_h$. Therefore, if $\hat{M}_X := \dim \hat{X}_h$, and given that $M = \dim X_h$, we have that $C_p \in \mathbb{R}^{M,\hat{M}_X}$.

Here we have slightly abused notation and described these linking matrices as operators. On the one hand, we hope this makes their connection to our preconditioning policy from Theorem 3.1.1 clearer. On the other hand, we expect it also conveys their functionality regardless of the specific choice of basis functions.

The averaging matrix allows us to compute the Galerkin matrix $B_h$ associated to $b$ over the dual mesh $\hat{\Gamma}_h$: Let $B_h : \hat{Y}_h \rightarrow \hat{Y}_h$ denote the $\hat{M}_Y \times \hat{M}_Y$ Galerkin matrix of $b$ built over the barycentric refinement $\hat{\Gamma}_h$, then $B_h = C_d^T B_b C_d$, and thus $B_h \in \mathbb{R}^{M,M}$.

Analogously, the coupling matrix helps us to build the Galerkin matrix $T_h$ associated to the dual coupling $t$: Let $M_h : \hat{X}_h \rightarrow \hat{X}_h$ be the $\hat{M}_X \times \hat{M}_X$ mass matrix computed over the barycentric refinement $\hat{\Gamma}_h$. By using the linking matrices, it is clear that $T_h = C_p^T M_h^T C_d^T : Y_h \rightarrow X_h$, and that it is in $\mathbb{R}^{M,M}$.

---

3for details on quadrilateral meshes, we refer to [56].

4We borrow their name from [82]. However, we change the notation for the sake of clarity. $C_p$ ($C_d$ ) stands for primal (dual), to honor the space to which the matrix connects the associated barycentric BE space.
We can now discuss the construction of our linking matrices for our two main cases.

**Case (EDP), (1.55):**

Recall from Subsection 3.2.1 that in this case we use $X_h = S^{-1,0}(\Gamma_h) \subset \tilde{H}^{-1/2}(\Gamma)$ and have to satisfy both $Y_h \subset H^{1/2}(\Gamma)$ and $\text{dim } Y_h = \text{dim } X_h$, which is equal to the number of cells of $\Gamma_h$.

**Figure 3.2:** Primal and dual meshes for (EDP). Black lines represent primal edges, dashed gray lines barycentric ones and blue lines are used to highlight dual edges. Orange dots mark the locations of degrees of freedom (DOFs) in $X_h = S^{-1,0}(\Gamma_h)$ and $Y_h = S^{0,1}(\Gamma_h)$.

- **Averaging matrix** $C_d : \breve{Y}_h \rightarrow Y_h$
  
  We choose $Y_h$ as a subspace of the standard BE space on the barycentric refinement:
  
  $$\breve{Y}_h = S^{0,1}(\breve{\Gamma}_h) := \text{"space of continuous piecewise linear functions on } \breve{\Gamma}_h."$$
  
  The basis functions $\phi_{\tau}$, $\tau \in \Gamma_h$, of $Y_h$ are associated with the barycenters of primal cells and can be specified as local linear combinations of nodal basis functions of $S^{0,1}(\breve{\Gamma}_h)$. In particular, the basis function $\breve{\phi}_{\tau}$, $\tau \in \Gamma_h$, is a linear combination of those nodal basis functions $\breve{\phi}_v$ of $S^{0,1}(\breve{\Gamma}_h)$ that belong to vertices $v$ of $\Gamma_h$ that are contained in $\tau$.
  
  $$\breve{\phi}_{\tau} := \sum_{v \in \tau} c_v \breve{\phi}_v , \quad \tau \in \Gamma_h . \quad (3.4)$$
  
  The coefficient $c_v$ associated to a barycentric vertex $v$ is given by $c_v = \frac{1}{N_v}$, where $N_v$ equals the number of primal elements that are adjacent to $v$, which is also the number of dual nodal basis functions to which $\breve{\phi}_v$ will contribute. For the nodes $v$ at barycenters, we know that $c_v$ will always be equal to one. Analogously, the weight is always $\frac{1}{2}$ for the internal mid-edge nodes and 1 for those lying on $\partial \Gamma$. These weights are illustrated in Figure 3.3. They ensure that constant functions belong to
  
  $$Y_h = S^{0,1}(\Gamma_h) := \text{span}\{\breve{\phi}_{\tau} : \tau \in \Gamma_h \} \subset H^{1/2}(\Gamma) . \quad (3.5)$$

- **Coupling matrix** $C_p : \breve{X}_h \rightarrow X_h$
  
  Let us consider the standard BE space on the barycentric refinement:
  
  $$\breve{X}_h = S^{-1,0}(\breve{\Gamma}_h) := \text{"space of piecewise constant functions on } \breve{\Gamma}_h."$$
  
  We aim to connect primal piecewise (pw.) constant basis functions in $X_h$ with barycentric pw. constants in $\breve{X}_h$. Since in both cases the degrees of freedom (dofs) are the elements in which the functions are supported, this operator just connects primal elements to barycentric elements.
Figure 3.3.: Illustration of the linear combination coefficients to build $Y_h = S^{0,1}(\tilde{\Gamma}_h)$, case (EDP), see also [15, Figure 4]. On the left, we present the case when the primal cell has an edge on the boundary $\partial \Gamma$. On the right, we show the case when the primal cell has a single node located on $\partial \Gamma$. $N_v$ (resp. $N_w$) stands for the number of triangles sharing node $v$ (resp. $w$). In both situations, $\partial \Gamma$ is indicated by the thick gray line. We use black lines to draw primal edges, green lines for dual edges. Red crosses designate barycentric nodes.

Concretely, let $N$ and $\hat{N}$ be the number of elements in $\Gamma_h$ and $\hat{\Gamma}_h$, respectively. Additionally, consider $\Gamma_h = \bigcup_{i=1}^{N} \tau_i$ and $\hat{\Gamma}_h = \bigcup_{j=1}^{\hat{N}} \hat{\tau}_j$, then

$$C_p[i,j] = \begin{cases} 1 & \text{if } \hat{\tau}_j \text{ is a child of } \tau_i. \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

From where it is clear that each row (primal dof) will have 6 non-zero entries filled with 1 (related barycentric/children element).

**Case (ENP), (1.59):**
In this case $X_h = S^{0,1}(\Gamma_h) \subset \tilde{H}^{1/2}(\Gamma)$ and $\dim X_h$ agrees with the number of interior vertices of $\Gamma_h$.

Figure 3.4.: Primal and dual meshes for case (ENP). Black lines represent primal edges, dashed gray lines barycentric ones, and blue lines are used to highlight dual edges. Orange dots mark the DOFs in $X_h = S^{0,1}(\Gamma_h)$ and $Y_h = S^{-1,0}(\tilde{\Gamma}_h)$.

- **Averaging matrix $C_d : \hat{Y}_h \to Y_h$**

  We introduce $Y_h$ as subspace of

  $$\hat{Y}_h = S^{-1,0}(\tilde{\Gamma}_h) := \text{“space of piecewise constant functions on } \tilde{\Gamma}_h.$$
3.2. Discretization

and characterize it as the span of suitable basis functions. For each interior vertex \( p \) of the primal mesh we introduce the dual cell \( \hat{\tau}_p \) containing \( p \) and

\[
\hat{\phi}_p(x) := \begin{cases} 
  c_p, & \text{if } x \in \hat{\tau}_p, \\
  0, & \text{elsewhere in } \Gamma,
\end{cases}
\]

with \( c_p \) such that \( \int_{\hat{\tau}_p} c_p \, dx = 1 \). Then we set

\[
Y_h = S_{-1,0}^1(\hat{\Gamma}_h) := \text{span}\{\hat{\phi}_p : p \text{ vertex of } \Gamma_h, p \not\in \partial\Gamma\}.
\]

Hence, \( Y_h \) is spanned by the characteristic functions of only the interior cells of \( \hat{\Gamma}_h \), see Figure 3.4.

Matching dimensions \( \dim Y_h = \dim X_h \) is immediate.

Concretely, if \( \hat{N}_p \) denotes the number of barycentric triangles neighbouring the vertex \( p \) of \( \Gamma_h \), then we can express \( \hat{\phi}_p \in Y_h \) by the following linear combination of basis functions (normalized characteristic functions \( \hat{\phi}_{\tau} \) of cells of \( \hat{\Gamma}_h \)) in \( \hat{Y}_h \):

\[
\hat{\phi}_p := \frac{1}{\hat{N}_p} \sum_{\hat{\phi}_\tau \in S_{-1,0}^1(\hat{\Gamma}_h)} \hat{\phi}_\tau.
\]

Figure 3.5.: Illustration of the linear combination coefficients to construct \( Y_h = S_{-1,0}^1(\hat{\Gamma}_h) \), case (ENP). We use black lines to indicate primal triangles, green lines for dual cells, green dots for the dofs of the primal and dual meshes (nodes), and a shaded gray line for the boundary \( \partial\Gamma \). The green diamonds show the neglected dual mesh DOFs at \( \partial\Gamma \). Consider node \( p \) as the center of the dual mesh filled with gray, \( \hat{N}_p \) stands for the number of barycentric triangles neighboring this node (highlighted by a red ‘x’).

- **Coupling matrix** \( C_p : \hat{X}_h \to X_h \)

Here, the dofs corresponding to \( X_h \) are the internal vertices. Moreover, the associated barycentric space is given by

\[
\hat{X}_h = S_{0,1}^0(\hat{\Gamma}_h) := \text{“space of continuous piecewise linear functions on } \hat{\Gamma}_h, \text{ satisfying zero boundary conditions on } \partial\Gamma.\n\]

As shown in Figure 3.6b, the barycentric dofs associated with each primal dof lie in the vertices of the neighbouring barycentric elements.

As before, the coefficient \( c_v \) of each barycentric dof \( v \) is such that its total contribution to the primal BE space is one. Therefore, as it is summarized in Figure 3.6a, we have three possibilities: The barycentric node \( v \) is
located in the center of the parent triangle: As the barycentric edges connect this node with
3 primal mesh nodes, its coefficient is \( c_v = 1/3 \).

– in the middle of an edge: This node is connected with 2 primal mesh nodes. Hence, \( c_v = 1/2 \).

– in a parent vertex: \( c_v = 1 \), since it will contribute only to itself in the parent mesh.

**Figure 3.6.: Illustration of coefficients of \( C_p \) under Case (ENP) using triangular elements.** We use black lines to indicate primal triangles, green lines for dual cells. Green dots designate primal nodes, and red 'x's barycentric nodes.

Let us illustrate this with the case shown in Figure 3.6b. There we have

\[
\varphi_{nc} = \bar{\varphi}_{nc} + \frac{1}{3} \sum_{i=1}^{7} \bar{\varphi}_{n_{A_i}} + \frac{1}{2} \sum_{i=1}^{7} \bar{\varphi}_{n_{B_i}},
\]

where \( \varphi_{n(\cdot)} \) represent pw.linear basis functions over the primal mesh \( \Gamma_h \) and \( \bar{\varphi}_{n(\cdot)} \) pw.linear basis functions over the barycentric refinement \( \bar{\Gamma}_h \).

### 3.3. Stability of Discrete Duality Pairings on Non-Uniform Meshes

As mentioned before, solutions ofscreen problems have a singular behavior near the boundary (see Section 1.5) that can be resolved by refining the mesh towards it. This motivates our interest in applying the operator preconditioning strategy to non-uniform meshes. A key assumption of Theorem 3.1.1 is a discrete inf-sup condition for the duality pairing \( \langle \cdot, \cdot \rangle_{\Gamma} \) and the spaces \( X_h \) and \( Y_h \) discussed in subsection 3.2.2 for (EDP) and (ENP).

We extend the theory developed by Steinbach in [84] to \( \tilde{H}^{1/2}(\Gamma) \) and their corresponding dual spaces. We begin by presenting the mesh assumptions under which we will assert the desired stability results.

**Assumption 3.3.1.** The family of meshes \( \{ \Gamma_h \}_{h \in \mathbb{N}}, h > 0 \) of \( \Gamma \) is locally quasi-uniform.

Let us consider a given primal mesh \( \Gamma_h \), and denote the size of an arbitrary cell \( \tau \in \Gamma_h \) by \( h_{\tau} \). Let \( V_h \) be a BE space equipped with the standard nodal basis and \( W_h \) its dual BE space furnished with the usual indicator functions. Concretely,
3.3. Stability of Discrete Duality Pairings on Non-Uniform Meshes

for (EDP): \( V_h := \mathcal{S}^{0,1}(\Gamma_h) \subset H^{1/2}(\Gamma), \) \( W_h := \mathcal{S}^{-1,0}(\Gamma_h) \subset \tilde{H}^{-1/2}(\Gamma), \)
for (ENP): \( V_h := \mathcal{S}^{0,1}(\Gamma_h) \subset H^{1/2}(\Gamma), \) \( W_h := \mathcal{S}^{-1,0}(\Gamma_h) \subset H^{-1/2}(\Gamma). \)

Moreover, defining \( M := \dim V_h, \) we write

\[
V_h = \text{span}\{\varphi_k\}_{k=1}^M, \quad W_h = \text{span}\{\phi_k\}_{k=1}^M.
\]

We also introduce the local trial spaces

\[
V_h(\tau) := \{\varphi' : \exists \varphi_k \in V_h : \varphi'(x) = \varphi_k(x) \text{ for } x \in \tau\} = V_h|_{\tau}, \quad (3.9)
\]
\[
W_h(\tau) := \{\phi' : \exists \phi_k \in W_h : \phi'(x) = \phi_k(x) \text{ for } x \in \tau\} = W_h|_{\tau}, \quad (3.10)
\]

that allow us to define the local “mixed” Gram matrix as

\[
\tilde{G}_l[j, i] = \langle \varphi_j', \phi_i' \rangle_{L^2(\tau)}, \quad (3.11)
\]

with \( M_l = \dim X_h(\tau) = \dim Y_h(\tau). \)

On the other hand, as a consequence of local quasi-uniformity, we can introduce for each nodal basis function \( \varphi_k \in V_h \) an associated mesh size \( \hat{h}_k \) satisfying

\[
\frac{1}{c_Q} \leq \frac{\hat{h}_k}{h} \leq c_Q \quad \text{for all } \tau \text{ such that } \tau \cap \text{supp}\{\varphi_k\} \neq \emptyset, \quad k = 1, \ldots, M, \quad (3.12)
\]

with a global constant \( c_Q \geq 1. \)

Now we can state the following mesh condition.

**Assumption 3.3.2** ([Assumption 2.1 in [84]]). Let \( H_l = \text{diag}(\hat{h}_k)_{k=1}^{M_l}, \) and \( D_l := \text{diag}(G_l). \) We can find a constant \( c_0 > 0 \) such that

\[
(H_l \tilde{G}_l^T H_l^{-1} x_l, x_l) \geq c_0 \cdot (D_l x_l, x_l), \quad \forall x_l \in \mathbb{R}^{M_l}
\]

for all \( l \) and \( h. \)

Let \( \omega_k := \text{supp}\{\varphi_k\} \) and, for an arbitrary (open) cell \( \tau \in \Gamma_h, \) define the index set of "adjacent basis functions" \( J(l) := \{k \in \{1, \ldots, M\} : \omega_k \cap \tau \neq \emptyset\}. \)

We are also interested in the next mesh assumption.

**Assumption 3.3.3** ([84, Eq. (2.30)]). We assume that the primal mesh \( \Gamma_h \) satisfies the following local mesh condition:

\[
\frac{51}{7} - \sqrt{\sum_{k_1 \in J(l)} \hat{h}_{k_1} \cdot \sum_{k_2 \in J(l)} \hat{h}_{k_2}^{-1} \geq c_0 > 0} \quad \forall \tau \in \Gamma_h, \quad (3.15)
\]

with a global positive constant \( c_0. \)

We point out that the stability Assumption 3.3.2 is satisfied if the local mesh condition (3.15) holds. The proof is given by Steinbach in [84, Section 2.2].

**Remark 3.3.4.** We note that Assumption 3.3.3 turns out to be a mild condition and will always hold on meshes for which the local change in meshwidth is moderate. This can be verified as follows: From (3.12) we have that for each \( k \) and \( l \) such that \( \tau \cap \text{supp}\{\varphi_k\} \neq \emptyset \) it holds that \( \hat{h}_k \leq c_Q h_l. \) Then

\[
\frac{51}{7} - \sqrt{\sum_{k_1 \in J(l)} \hat{h}_{k_1} \sum_{k_2 \in J(l)} \hat{h}_{k_2}^{-1} \geq \frac{51}{7} - \sqrt{(\# J(l)c_Q h_l) \sum_{k_2 \in J(l)} \hat{h}_{k_2}^{-1} \forall \tau \in \Gamma_h}. \]

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Use (3.12) again and conclude \( \frac{h_k}{\hat{h}_k} \leq c_Q \) for any \( k \) and \( l \) such that \( \tau \cap \text{supp}\{\varphi_k\} \neq \emptyset \). We apply this and get
\[
\sum_{k \in J(l)} \frac{1}{h_k} \sum_{k_2 \in J(l)} \frac{1}{\hat{h}_{k_2}} \geq \frac{51}{7} - \sqrt{\frac{\#J(l)c_Q}{2}} \quad \forall \tau_l \in \Gamma_h.
\]

Since \( \#J(l) \leq 3 \) for all \( \tau_l \in \Gamma_h \), Assumption 3.3.3 holds, if
\[
2.429 \approx \frac{51}{21} \geq c_Q \geq 1, \quad \forall \tau_l \in \Gamma_h.
\]

In addition to the above conditions, we need the following technical Lemma for Case (ENP) to be able to show the desired stability result for the duality pairing \( t = \langle \cdot , \cdot \rangle_{\Gamma} \).

**Lemma 3.3.5.** (Extension of Lemma 2.3 [84], [50, Lemma B.2]) Let condition (3.13) from Assumption 3.3.2 be satisfied and \( \phi_k \in S_x^{-1,0}(\Gamma_h) \), \( k = 1, \ldots, M \). Then
\[
\sum_{i=1}^{N} \frac{1}{h_i^2} \|v_h\|^2_{L^2(\tau_i)} \leq c_{p_2} \sum_{k=1}^{M} \left( \frac{\langle v_h, \phi_k \rangle_{L^2(\Gamma)}}{h_k \|\phi_k\|_{L^2(\Gamma)}} \right)^2 , \quad (3.16)
\]
for all \( v_h \in S_x^{0,1}(\Gamma_h) \) with a constant \( c_{p_2} > 0 \) independent of the meshwidth.

The proof follows from adapting Steinbach’s original proof (similarly to what was shown in [12]). For the sake of clarity, we postpone this proof to the end of this Section, as it involves some additional notation.

Finally, we are in a position to prove the main result of this Section.

**Theorem 3.3.6.** Let Assumptions 3.3.1, 3.3.1 and 3.3.3 be satisfied. Then, for the following combinations of discrete spaces
\[
\text{(EDP): } X_h = S^{-1,0}(\Gamma_h) \subset X = \tilde{H}^{-1/2}(\Gamma), \quad Y_h = S_x^{0,1}(\Gamma_h) \subset Y = H^{1/2}(\Gamma), \\
\text{(ENP): } X_h = S_x^{0,1}(\Gamma_h) \subset X = \tilde{H}^{1/2}(\Gamma), \quad Y_h = S_x^{-1,0}(\Gamma_h) \subset Y = H^{-1/2}(\Gamma),
\]
the discrete inf-sup condition
\[
\sup_{v_h \in Y_h} \frac{|\langle w_h, v_h \rangle|}{\|v_h\|_Y} \geq \frac{1}{c_s} \|w_h\|_X \quad \forall w_h \in X_h, \quad (3.17)
\]
holds with a positive constant \( c_s \) independent of \( h \).

**Proof.** We prove each case separately

**Case (EDP):**

Under Assumptions 3.3.1 and 3.3.3, Theorems 2.1 and 2.2 in [84] give the discrete inf-sup condition for
\[
\text{(EDP): } \quad X_h^* = S^{0,1}(\Gamma_h) \subset H^{1/2}(\Gamma), \quad Y_h^* = S^{-1,0}(\Gamma_h) \subset \tilde{H}^{-1/2}(\Gamma):
\]
\[
\sup_{v_h \in Y_h^*} \frac{|\langle w_h, v_h \rangle|}{\|v_h\|_{\tilde{H}^{-1/2}(\Gamma)}} \geq \frac{1}{c_s} \|w_h\|_{H^{1/2}(\Gamma)} \quad \forall w_h \in X_h^*, \quad (3.18)
\]

Next, we appeal to an analogue of [84, Lemma 1.7] to define \( \tilde{Q}_h^* : L^2(\Gamma) \rightarrow S^{0,1}(\Gamma_h) \) for a given \( u \in L^2(\Gamma) \) as solution of the variational problem
\[
\langle \tilde{Q}_h u, \psi_h \rangle_{\Gamma} = \langle u, \psi_h \rangle_{\Gamma} \quad \forall \psi_h \in S^{-1,0}(\Gamma_h). \quad (3.19)
\]

Moreover, following the steps from [84, Thm. 2.1], one can prove that
\[
\|\tilde{Q}_h u\|_{H^{1/2}(\Gamma)} \leq c \|u\|_{H^{1/2}(\Gamma)} \quad \forall u \in H^{1/2}(\Gamma), \quad h \in \mathbb{H}. \quad (3.20)
\]
Finally, combining these results, we have for all \( v_h \in S^{-1,0}(\Gamma_h) \) that

\[
\|v_h\|_{\tilde{H}^{-1/2}(\Gamma)} = \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{\langle v_h, w \rangle_{\Gamma}}{\|w\|_{H^{1/2}(\Gamma)}} \leq c \sup_{0 \neq w \in H^{1/2}(\Gamma)} \frac{\langle v_h, \tilde{Q}_h w \rangle_{\Gamma}}{\|\tilde{Q}_h w\|_{H^{1/2}(\Gamma)}} \leq c \sup_{0 \neq w_h \in S^{0,1}(\Gamma_h)} \frac{\|v_h - \tilde{Q}_h w_h\|_{H^{1/2}(\Gamma)}}{\|w_h\|_{H^{1/2}(\Gamma)}},
\]

as studied.

**Case (ENP):**

We adapt and extend the proof of [84, Theorem 2.1 and 2.2]. The required extensions are analogous to those developed in [49] for the case of the segment and are discussed here for the sake of completeness.

1. From [84, Lemma 1.7] we learn that Assumption 3.3.3 guarantees

\[
\sup_{\psi_h \in S^{-1,0}(\Gamma_h)} \frac{\langle \psi_h, w_h \rangle_{\Gamma}}{\|\psi_h\|_{L^2(\Gamma)}} \geq c_{st} \|w_h\|_{L^2(\Gamma)}, \quad \forall w_h \in S^{0,1}_0(\Gamma_h), \quad (3.21)
\]

with \( c_{st} > 0 \) independent of \( h \). Thus, the generalized Galerkin \( L^2 \)-Projection \( \tilde{Q}_h : L^2(\Gamma) \to S^{0,1}_0(\Gamma_h) \), defined for \( u \in L^2(\Gamma) \) as follows:

\[
\langle \tilde{Q}_h u, \phi_h \rangle_{\Gamma} = \langle u, \phi_h \rangle_{\Gamma}, \quad \forall \phi_h \in S^{-1,0}_e(\Gamma_h), \quad (3.22)
\]

is well-defined and bounded in \( L^2(\Gamma) \).

2. To show the \( H^1(\Gamma) \)-stability of \( \tilde{Q}_h \), we employ a quasi-interpolation or Clement operator, as in [84, Section 1.5]. Let us define the local trial space related to the local basis function \( \varphi_k \) as

\[
V_h(\omega_k) := \{ \varphi_j |_{\omega_k} : \varphi_j \in S^{0,1}_0(\Gamma_h) \}.
\]

Let \( Q_h^k \) denote the standard \( L^2(\Gamma) \)-orthogonal projection onto \( V_h(\omega_k) \), such that for \( u \in L^2(\omega_k) \)

\[
\langle Q_h^k u, v_h \rangle_{\omega_k} = \langle u, v_h \rangle_{\omega_k}, \quad \forall v_h \in V_h(\omega_k). \quad (3.24)
\]

Due to Assumption 3.3.1, \( Q_h^k \) satisfies the following stability and quasi-optimal error estimates: there is a constant \( c_{st}^{loc} > 0 \) independent of \( h \)

\[
\|Q_h^k u\|_{L^2(\omega_k)} \leq \|u\|_{L^2(\omega_k)}, \quad \forall u \in L^2(\omega_k), \quad (3.25)
\]

\[
\| (I_d - Q_h^k) u \|_{L^2(\omega_k)} \leq c_{st}^{loc} \hat{h}_k \|u\|_{H^1(\omega_k)}, \quad \forall u \in H^1(\omega_k). \quad (3.26)
\]

Moreover, local quasi-uniformity gives us the following stability estimate via inverse estimates

\[
\exists c_{st}^{loc} > 0 : \|Q_h^k u\|_{H^1(\omega_k)} \leq c_{st}^{loc} \hat{h}_k \|u\|_{H^1(\omega_k)}, \quad \forall u \in H^1(\omega_k). \quad (3.27)
\]

Then, it is possible to define a quasi-interpolation operator \( P_h : L^2(\Gamma) \to S^{0,1}_0(\Gamma_h) \) by

\[
(P_h u)(x) := \sum_{k=1}^{M} (Q_h^k u)(x_k) \cdot \varphi_k(x), \quad (3.28)
\]

which is also a projection onto \( X_h \). Furthermore, \( P_h \) enjoys the following important properties: extending Lemma 1.9 of [84] to \( H^1_0(\Gamma) \) confirms that there exists a constant \( c_{p1} > 0 \) independent of \( h \) such that

\[
\|(I_d - P_h) u\|_{L^2(\gamma_t)} \leq c_{p1} \sum_{k \in J(\Gamma)} \hat{h}_k \|u\|_{H^1(\omega_k)}, \quad \forall \gamma_t \in \Gamma_h, \forall u \in H^1_0(\Gamma), \quad (3.29)
\]

\[
\|P_h u\|_{H^1(\Gamma)} \leq c_{p1} \|u\|_{H^1(\Gamma)}, \quad \forall u \in H^1_0(\Gamma), \quad (3.30)
\]
\[
\sum_{k=1}^{M} \hat{h}_k^{-2} \| (\text{id} - P_h)u \|_{L^2(\omega_k)}^2 \leq c_{p1} \| u \|_{\mathcal{H}_1^1(\Gamma)}^2, \quad \forall u \in H_0^1(\Gamma).
\] (3.31)

Combining all these results with Lemma 3.3.5, we find that under Assumptions 3.3.1 and 3.3.3 the generalized \(L^2(\Gamma)\)-orthogonal projection \(\tilde{Q}_h : H_0^1(\Gamma) \rightarrow X_h = S_0^{1,1}(\Gamma_h)\) as defined in (3.22) satisfies

\[
\| \tilde{Q}_h u \|_{H^1(\Gamma)} \leq \tilde{c}_{st} \| u \|_{H^1(\Gamma)}, \quad \forall u \in H_0^1(\Gamma),
\] (3.32)

with \(\tilde{c}_{st} > 0\) a constant independent of the meshwidth. This emerges from the following chain of estimates

\[
\| \tilde{Q}_h u \|_{H^1(\Gamma)}^2 \leq 2 \left\{ c_{p1} \| u \|_{H^1(\Gamma)}^2 + \| (\tilde{Q}_h - P_h)u \|_{H^1(\Gamma)}^2 \right\}
\] (3.30)

\[
\leq 2 \left\{ c_{p1} \| u \|_{H^1(\Gamma)}^2 + \sum_{\tau \in \Gamma_k} \hat{h}_\tau^{-2} \| (\tilde{Q}_h - P_h)u \|_{L^2(\tau)}^2 \right\}
\] (Inverse estimate)

\[
\leq 2 \left\{ c_{p1} \| u \|_{H^1(\Gamma)}^2 + c_{p2} \sum_{k=1}^{M} \frac{\langle (\tilde{Q}_h - P_h)u, \phi_k \rangle_\Gamma}{\hat{h}_k \| \phi_k \|_{L^2(\Gamma)}} \right\}
\]

\[
\leq 2 \left\{ c_{p1} \| u \|_{H^1(\Gamma)}^2 + c_{p2} \sum_{k=1}^{M} \hat{h}_k^{-2} \| (\text{id} - P_h)u \|_{L^2(\omega_k)}^2 \right\}
\] (3.31)

\[
\leq \tilde{c}_{st} \| u \|_{H^1(\Gamma)}^2, \quad \forall u \in H_0^1(\Gamma).
\] (3.33)

As discussed in subsection 1.3.1, the space \(\tilde{H}^{1/2}(\Gamma)\) can be defined by interpolating between \(L^2(\Gamma)\) and \(H_0^1(\Gamma)\), see (1.17). Thus, by interpolation of bounded linear operators we obtain from \(L^2\)-stability and (3.32) that

\[
\| \tilde{Q}_h u \|_{H^{1/2}(\Gamma)} \leq c_B \| u \|_{H^{1/2}(\Gamma)}, \quad \forall u \in \tilde{H}^{1/2}(\Gamma).
\] (3.34)

Introduce the projection operators \(\Pi_h : \tilde{H}^{1/2}(\Gamma) \rightarrow S_0^{-1.0}(\Gamma_h) \subset H^{-1/2}(\Gamma)\), satisfying

\[
\langle \Pi_h u, w_h \rangle_\Gamma = \langle u, w_h \rangle_{\tilde{H}^{1/2}(\Gamma)}, \quad \forall w_h \in S_0^{0,1}(\Gamma_h),
\] (3.35)

where \(\langle \cdot, \cdot \rangle_{\tilde{H}^{1/2}(\Gamma)}\) denotes the \(\tilde{H}^{1/2}(\Gamma)\)-inner product. Notice that (3.21) guarantees that \(\Pi_h\) is well-defined.

By the definition of the dual norm and continuity of \(\tilde{Q}_h,\) one can derive from (3.34) that

\[
\| \Pi_h u \|_{H^{-1/2}(\Gamma)} = \sup_{0 \neq w \in \tilde{H}^{1/2}(\Gamma)} \frac{\langle \Pi_h u, w \rangle_\Gamma}{\| w \|_{\tilde{H}^{1/2}(\Gamma)}} = \sup_{0 \neq w \in \tilde{H}^{1/2}(\Gamma)} \frac{\langle \Pi_h u, \tilde{Q}_h w \rangle_\Gamma}{\| \tilde{Q}_h w \|_{H^{1/2}(\Gamma)}} \leq c_B \| \tilde{Q}_h w \|_{H^{1/2}(\Gamma)}
\]

\[
\leq c_B \| u \|_{\tilde{H}^{1/2}(\Gamma)},
\]

for all \(u \in \tilde{H}^{1/2}(\Gamma)\).
Finally, by the above inequality, for any \( w_h \in S^0_1(\Gamma_h) \) we obtain the assertion

\[
\|w_h\|_{H^{1/2}(\Gamma)} = \frac{|\langle w_h, w_h \rangle_{H^{1/2}(\Gamma)}|}{\|w_h\|_{H^{1/2}(\Gamma)}} = \frac{|\langle w_h, \Pi_h w_h \rangle_{\Gamma}|}{\|w_h\|_{H^{1/2}(\Gamma)}} \\
\leq c_B \frac{|\langle w_h, \Pi_h w_h \rangle_{\Gamma}|}{\|\Pi_h w_h\|_{H^{-1/2}(\Gamma)}} \leq c_B \sup_{0 \neq \nu_h \in S^{-1,0}_\ast(\Gamma_h)} \frac{|\langle w_h, \nu_h \rangle_{\Gamma}|}{\|w_h\|_{H^{1/2}(\Gamma)}}
\]

**Remark 3.3.7.** The stability results discussed in this section are independent of the choice of basis functions. As a consequence, the considered mesh assumptions allow us to lift Buffa and Christiansen’s dual discrete space construction on quasi-uniform triangular meshes to non-uniform meshes.

It might be worth pointing out that in [15, Prop. 3.13], the discrete inf-sup condition (3.17) for Case (ENP) requires global quasi-uniformity together with the following local non-degeneracy condition [15, Prop 3.11]:

Let \( \Gamma_h^0 \) and \( \Gamma_h^1 \) denote the sets of vertices and edges on a given primal mesh \( \Gamma_h \), respectively. Let \( N_t \) denote the number of vertices \( v \in \Gamma_h^0 \) connected to the vertex \( t \in \Gamma_h^0 \). The family of meshes \( \{\Gamma_h\}_{h \in \mathbb{H}} \) is such that for some \( \delta' < 1 \) we have for each \( h \in \mathbb{H} \) and \( s \in \Gamma_h^0 \):

\[
\sum_{t \in \Gamma_h^0, (s,t) \in \Gamma_h^1} 1/N_t \leq \delta' \frac{1}{N_s}.
\]  \hfill (3.36)

**Proposition 3.3.8.** For Case (ENP), Assumption 3.3.1 implies the local non-degeneracy condition (3.36).

**Proof.** We define \( N_t \) as the number of vertices \( s \) such that \( (s,t) \in \Gamma_h^1 \), i.e. the number of vertices connected to \( t \) by a dual edge.

Define the dual mesh locally via a reference element (see Figure 3.7). Then, when computing the local “mixed” Gram matrix using Buffa and Christiansen’s approach, we get:

\[
\tilde{G}_l = \frac{1}{36} \begin{pmatrix}
22 & 7 & 7 \\
7 & 22 & 7 \\
7 & 7 & 22
\end{pmatrix}, \quad D_l = \begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]  \hfill (3.37)

Here notice that as the basis functions in [15] are normalized, we drop the coefficient \( \Delta_l \), with \( \Delta_l \) denoting the area of the element \( \tilde{\tau}_l \).

Then for any \( x_l = (x_0, x_1, x_2) \in \mathbb{R}^3 \)

\[
2C_1^2 \sum_{i=0}^2 x_i^2 \leq \sum_{j=0}^2 \left( \sum_{j \neq i} x_j x_i \frac{7}{N_{a_i}} + x_i^2 \frac{22}{N_{a_i}} \right) \leq 2C_2^2 \sum_{i=0}^2 x_i^2,
\]

which holds particularly when

\[
\sum_{j=0,2 : j \neq i} 1/N_j \leq \delta' \frac{22}{N_{a_i}},
\]  \hfill (3.38)

i.e. the local version of condition (3.36). Finally, if one sums this quantity over the \( \frac{N_{a_i}}{2} \) primal triangles containing the vertex \( a_i \), one gets the desired non-degeneracy condition. \( \square \)
Proof of Lemma 3.3.5

Let us recall the primal and dual spaces and the notations for Case (ENP):
\[ V_h = X_h = S_0^{0,1}(\Gamma_h) = \text{span}\{\varphi_k\}_{k=1}^M, \]
\[ W_h = Y_h = S^{-1,0}(\Gamma_h) = \text{span}\{\phi_k\}_{k=1}^M. \]

Using the definition of the local trial spaces \( V_h(\tau_l) \) and \( W_h(\tau_l) \) from (3.9), we introduce the following local Gram matrices
\[ G_l[j,i] = \langle \varphi_l^j, \varphi_l^i \rangle_{L^2(\tau_l)}, \quad \text{for } i,j = 1,\ldots,M_l, \quad (3.39) \]
\[ \hat{G}_l[j,i] = \langle \phi_l^j, \phi_l^i \rangle_{L^2(\tau_l)}, \quad \text{for } i,j = 1,\ldots,M_l. \quad (3.40) \]

In Steinbach's original work, the required mesh assumptions where defined as follows:

Assumption 3.3.9 (Assumptions 1.1 and 1.2 in [84]). Let \( D_l := \text{diag}(G_l) \). We assume that
\[ c_1 G_l x_l \leq \langle G_l x_l, x_l \rangle \leq c_2 G_l x_l, \quad (3.41) \]
\[ \tilde{c}_1 \hat{G}_l x_l \leq \langle \hat{G}_l x_l, x_l \rangle \leq \tilde{c}_2 \hat{G}_l x_l, \quad (3.42) \]
\[ \hat{c}_1 \hat{G}_l x_l \leq \langle \hat{G}_l x_l, x_l \rangle \leq \hat{c}_2 \hat{G}_l x_l, \quad (3.43) \]
hold uniformly for all \( x_l \in \mathbb{R}^M_l, (l = 1,\ldots,N) \) with positive constants.

However, we were able to only discuss Assumption 3.3.1 to enunciate and prove Theorem 3.3.6 due to the following result.

Proposition 3.3.10. Assumption 3.3.1 implies Assumption 3.3.9

Proof. This is done by local computations via the reference element (see Figure 3.7). In order to prove (3.41) we observe
\[ G_l = \frac{\Delta_l}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \]
where \( \Delta_l \) is the area of the element \( \tau_l \). Then we get the desired condition
\[ \frac{1}{2} (D_l x_l, x_l) \leq \langle G_l x_l, x_l \rangle \leq 4 (D_l x_l, x_l). \]
(3.45)

For (3.42) and (3.43) we refer to [84, Section 2.2].

Nonetheless, we presented Assumption 3.3.9 now because we explicitly use it to prove Lemma 3.3.5.

Proof of Lemma 3.3.5. We extend Lemma 2.3 [84] following the same recipe as for the 2D case. For the sake of clarity, we highlight the main differences from the steps given in [50, Lemma B.2] using bold letters (see local computations almost at the end).

Let \( N \) be the number of elements over the primal mesh \( \Gamma_h \). In addition, for each basis function \( \varphi_k \in V_h \) we define the set
\[ I(k) := \{ l \in \{1,\ldots,N\} : \tau_l \cap \text{supp } \{ \varphi_k \} \neq \emptyset \}, \quad (3.46) \]
and for each \( \tau_l \in \Gamma_h \), we recall the set \( J(l) \) defined in (3.14).
3.3. Stability of Discrete Duality Pairings on Non-Uniform Meshes

Seeing that \( v_h = \sum_{k=1}^{M} v_k \varphi_k \in S_0^{0,1}(\Gamma_h) \), we can write

\[
\sum_{l=1}^{N} h_l^{-2} \| v_h \|_{L^2(\tau_l)} \leq c_p \sum_{l=1}^{N} h_l^{-2} \sum_{k \in J(l)} v_k^2 \| \varphi_k \|_{L^2(\tau_l)}^2 \\
\leq c_p \sum_{k=1}^{M} v_k^2 \sum_{l \in I(k)} h_l^{-2} \| \varphi_k \|_{L^2(\tau_l)}^2 = c_p \sum_{k=1}^{M} v_k^2 \gamma_k^2,
\]

where \( \gamma_k := \sqrt{\sum_{l \in I(k)} h_l^{-2} \| \varphi_k \|_{L^2(\tau_l)}^2} \). Setting \( x_k := v_k \gamma_k \) this gives

\[
\sum_{l=1}^{N} h_l^{-2} \| v_h \|_{L^2(\tau_l)}^2 \leq c_p \| x \|_2^2.
\]

On the other hand,

\[
\sum_{k=1}^{M} \left( \frac{\langle v_k, \phi_k \rangle_{L^2(\Gamma)}}{h_k \| \phi_k \|_{L^2(\Gamma)}} \right)^2 = \sum_{k=1}^{M} \left( \frac{\langle \varphi_j, \phi_k \rangle_{L^2(\Gamma)}}{h_k \| \phi_k \|_{L^2(\Gamma)}} \right)^2 = \sum_{k=1}^{M} \left( \frac{\langle \varphi_j, \phi_k \rangle_{L^2(\Gamma)}}{\gamma_k h_k \| \phi_k \|_{L^2(\Gamma)}} \right)^2 = \| A x \|_2^2,
\]

where \( A \) is a matrix given by

\[
A := D_q^{-1} \bar{G}_h D_{\gamma}^{-1}, \quad D_q := \text{diag}(\hat{h}_k \| \phi_k \|_{L^2(\Gamma)}), \quad D_{\gamma} := \text{diag}(\gamma_k).
\]

Let \( \bar{G}_h = H^{-1} \bar{G}_h H \). Define for any \( y \in \mathbb{R}^M \)

\[
b_h := \sum_{k=1}^{M} h_k y_k \varphi_k \in S_0^{0,1}(\Gamma_h), \quad q_h := \sum_{k=1}^{M} h_k^{-1} y_k \phi_k \in S^{-1,0}(\Gamma_h).
\]

Then, using

\[
(H_l^{-1} \bar{G}_l H \mathbf{x}_l, \mathbf{x}_l) \geq c_0 (D_l \mathbf{x}_l, \mathbf{x}_l) \quad \text{for all } \mathbf{x}_l \in \mathbb{R}^M, \quad l = 1 \ldots N - 1,
\]

which is transposed to (3.13), we derive the following bound:

\[
(\bar{G}_h y, y) = (H^{-1} \bar{G}_h H y, y) = (\bar{G}_h H y, H^{-1} y) = \sum_{l=1}^{N} (b_h, q_h)_{L^2(\tau_l)} = \sum_{l=1}^{N} (b_h, q_h)_{L^2(\tau_l)}
\]

\[
\leq \sum_{l=1}^{N} (H_l^{-1} \bar{G}_l H \mathbf{y}_l, \mathbf{y}_l) \geq c_0 \sum_{l=1}^{N} (D_l \mathbf{y}_l, \mathbf{y}_l) = c_0 (D \mathbf{y}, \mathbf{y}).
\]

Now, set \( D_h^{1/2} := \text{diag}(\| \varphi_k \|_{L^2(\Gamma)}). \) From

\[
c_0 \| D_h^{1/2} y \|_2^2 = c_0 (D \mathbf{y}, \mathbf{y}) \leq (\bar{G}_h y, y) = (D_h^{-1/2} \bar{G}_h y, D_h^{1/2} y)
\]

\[
\leq \| D_h^{-1/2} \bar{G}_h y \|_2 \| D_h^{1/2} y \|_2,
\]

we conclude that

\[
c_0 \| D_h^{1/2} y \|_2 \leq \| D_h^{-1/2} \bar{G}_h y \|_2.
\]

Taking \( z := D_\gamma y \), this is equivalent to

\[
c_0 \| D_h^{1/2} D_\gamma^{-1} z \|_2 \leq \| D_h^{-1/2} D_q^{-1} \bar{G}_h D_\gamma^{-1} z \|_2.
\]
From the local quasi-uniformity, the ratio of the diagonal entries satisfies
\[
\frac{D_h^{1/2}[k,k]}{D_0[k,k]} = \frac{\| \varphi_k \|_{L^2(\Gamma)}}{\sqrt{\sum_{l \in I(k)} h_l^{-2} \| \varphi_k \|_{L^2(\tau_l)}^2}} \geq c h_k,
\]
due to
\[
\frac{D_h^{1/2}[k,k]}{D_0[k,k]} = \frac{\sqrt{\sum_{l \in I(k)} h_l^{-2} \| \varphi_k \|_{L^2(\tau_l)}^2}}{\sqrt{\sum_{l \in I(k)} h_l^{-2} \| \varphi_k \|_{L^2(\tau_l)}^2}} \geq \frac{1}{\sqrt{\max_{l \in I(k)} h_l^{-2}}} \geq c h_k,
\]
and
\[
\frac{D_0[k,k]}{D_h^{1/2}[k,k]} = \frac{h_k \| \varphi_k \|_{L^2(\Gamma)}}{\| \varphi_k \|_{L^2(\Gamma)}} \leq c h_k.
\]
We derive this last result from the fact that \(\| \varphi_k \|_{L^2(\Gamma)} = \sum_{l \in I(k)} c_l h_l \leq C Q h_k\) and \(h_k = \| \varphi_k \|_{L^2(\Gamma)}\).
Thus, by taking \(x = H z\)
\[
ce_p \| x \|_2 = c_p \| H z \|_2 \leq \| H D_q^{-1} \tilde{G}_h D^{-1}_\gamma z \|_2 = \| H D_q^{-1} H^{-1} \tilde{G}_h H D^{-1}_\gamma H^{-1} x \|_2 = \| A x \|_2.
\]

3.4. Preconditioning on Disks

The estimate of Theorem 3.3.6 ultimately justifies the choice of the spaces \(Y_h\) in subsection 3.2.2 and confirms the crucial stability requirement for the operator preconditioning approach summarized now in Table 3.2. There we use the notation

\[
\begin{align*}
b_{\nabla a}(\vartheta, \mu) &:= (\nabla \vartheta, \mu)_{D_a}, & \vartheta, \mu &\in H^{-1/2}(D_a), \quad (3.48a) \\
b_{\nabla W_a}(u, g) &:= (\nabla u, g)_{D_a}, & u, g &\in H^{1/2}(D_a). \quad (3.48b)
\end{align*}
\]

Table 3.2.: Completion of Table 3.2: Components of the \(L^2(\Gamma)\)-based dual mesh operator preconditioning approach for variational BIEs on \(\Gamma = D_a\).

<table>
<thead>
<tr>
<th>BVP</th>
<th>a</th>
<th>b</th>
<th>X</th>
<th>Y</th>
<th>(X_h)</th>
<th>(Y_h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EDP)</td>
<td>(a_{\nabla V D_a})</td>
<td>(b_{\nabla W_a})</td>
<td>(H^{-1/2}(D_a))</td>
<td>(H^{1/2}(D_a))</td>
<td>(S^{-1,0}(\Gamma_h))</td>
<td>(S_0^{-1,0}(\Gamma_h))</td>
</tr>
<tr>
<td>(ENP)</td>
<td>(a_{\nabla \nabla V D_a})</td>
<td>(b_{\nabla \nabla W_a})</td>
<td>(H^{1/2}(D_a))</td>
<td>(H^{-1/2}(D_a))</td>
<td>(S_0^{0,1}(\Gamma_h))</td>
<td>(S_1^{-1,0}(\Gamma_h))</td>
</tr>
</tbody>
</table>

Corollary 3.4.1. Borrowing notations from Theorem 3.1.1, consider the Galerkin BE discretization and the \(L^2(\Gamma)\)-based operator preconditioning approach for (1.55) \((EDP)\) and (1.59) \((ENP)\) as detailed in Table 3.2. Under Assumptions 3.3.1 and 3.3.3, the spectral condition number of the preconditioned Galerkin matrix \(T_h^{-1} B_h T_h^{-B} A_h\) is bounded independently of \(h \in \mathbb{H}\).

Remark 3.4.2. Although our preconditioners achieve \(h\)-independence, it is clear from Corollary 2.2.2, that the spectral condition number does depend on the wave number \(k\).
3.4.1. Numerical Results for Laplace BIOs on the Unit Disk

Numerical experiments were implemented employing BETL2 [53] and the required meshes were generated with Gmsh [38] using polygonal approximation of the boundaries. All required BEM operators were constructed with 12 quadrature points and regularizing transformations [81, Chap. 5]. For fine meshes, local low-rank compression of the BE matrices had to be employed. Specifically, BETL2 uses AHMED for its ACA implementation [7]. The ACA parameters used for these experiments were a tolerance of $10^{-5}$ and admissibility $\eta = 0.9$.

The measured condition numbers were computed via the ratio of the maximum and minimum eigenvalues using Preconditioned Conjugate Gradient (PCG) with the Lanczos algorithm [40, Ch.9–10]. We use PCG with a tolerance of $10^{-5}$ for the relative residual norm, initial guess equal to zero and, as right hand side, we considered a vector that had entries $+1$ in its upper half, $-1$ for the remaining components. Since a sufficiently precise computation of the eigenvalues usually requires a larger Krylov subspace and hence more PCG iterations, the algorithm continues iterating until the difference between the newly computed condition number and the old one is less or equal to $10^{-4}$ for two times consecutively. We measure the performance of a preconditioner by considering the obtained condition numbers $\kappa$ and the number of PCG Iterations, denoted by $\text{It}$, required to solve the linear system with the parameters and right hand side described above on a mesh with $N$ elements.

For each experiment presented, we provide two tables. The ones on the left, entitled (EDP), contain the preconditioning results for $V_0$, while the right-hand tables display the results for $W_0$ and is labeled as (ENP). In both situations we compare the performance of our preconditioner $\mathbf{P}_h$ constructed with our modified BIOs with diagonal preconditioning –denoted by $D_h^{-1}$–, and opposite-order operator preconditioning $\mathbf{P}_h$ arising from using the standard BIOs. We use the same notations for both cases. For the sake of clarity, they are detailed in the next paragraphs.

Since we denote the Galerkin matrices of $V_0, W_0, \nabla$ and $\overline{\nabla}$ by $V_h, W_h, \nabla_h$ and $\overline{\nabla}_h$, respectively, we have:

- (EDP): Here our preconditioner is $\mathbf{P}_h := T_{D,h}^{-1} \nabla_h T_{D,h}^{-T}$ and the opposite-order preconditioner is $\mathbf{P}_h := T_{D,h}^{-1} \nabla_h T_{D,h}^{-T}$, with $T_{D,h}$ the Galerkin matrix of the $L^2$-duality product between $S^{-1,0}(\Gamma_h)$ and $S^{0,1}(\Gamma_h)$.

- (ENP): In this case, our preconditioner is $\mathbf{P}_h := T_{N,h}^{-1} \overline{\nabla}_h T_{N,h}^{-T}$ and $\mathbf{P}_h := T_{N,h}^{-1} W_h T_{N,h}^{-T}$, with $T_{N,h}$ the Galerkin matrix of the $L^2$-duality product between $S^{0,1}(\Gamma_h)$ and $S^{-1,0}(\Gamma_h)$.

Notice that the second term in (2.32) can be regarded as a regularizing term. Without it, the expression would map constants to zero and lose the $H^{1/2}(\Omega_1)$-ellipticity of the bilinear form $b_{\overline{\nabla}}^{1/2}$, which is crucial to construct a suitable preconditioner for $V_0$, as discussed in Section 3.1. From an implementation point of view, this regularizing term $\mathbf{P}_h$ has been set to

$$a_{\overline{\nabla}} \langle u, \omega^{-1} \rangle_{\Omega_1} \langle v, \omega^{-1} \rangle_{\Omega_1}, \quad u, v \in H^{1/2}(\Omega_1), \quad (3.49)$$

with $a_{\overline{\nabla}} = \frac{1}{v} = 1/\langle 1, \omega^{-1} \rangle_{\Omega_1}$.

The bilinear form arising from $W_0$ also needs this kind of regularization in order to construct $\mathbf{P}_h$ as it maps constants to zero. In the same way as for $\mathbf{P}_h$, we recover the $H^{1/2}(\Omega_1)$-ellipticity by adding a regularization term [49, Sect. 5.1] equal to (3.49).

Tables 3.3 and 3.7 show the preconditioning results over a disk with two different families of triangular meshes: quasi-uniform and non-quasi-uniform (see Figure 3.10a for the latter). In both tables, we observe that the condition numbers achieved by $\mathbf{P}_h$ hardly increase with respect to the number of elements and are asymptotically smaller than those of $\mathbf{P}_h$. Nevertheless, the gain in terms of number of PCG iterations is not significant.

\[\text{Except for the rank-one regularizations (3.49) and (3.59), where BETL2’s default number of quadrature points was used, i.e. 7 points for the mass-matrix and 25 for } \langle 1, \omega^{-1} \rangle_{\Omega_1}.\]
Table 3.3.: Results on the unit disk with quasi-uniform family of triangular meshes.

<table>
<thead>
<tr>
<th>N</th>
<th>$\mathbf{D^{-1}_h V_h}$</th>
<th>$\mathbf{P_h V_h}$</th>
<th>$\mathbf{\bar{P}_h V_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>It</td>
<td>$\kappa$</td>
</tr>
<tr>
<td>64</td>
<td>22.55</td>
<td>13</td>
<td>7.22</td>
</tr>
<tr>
<td>256</td>
<td>49.41</td>
<td>20</td>
<td>2.84</td>
</tr>
<tr>
<td>1024</td>
<td>102.28</td>
<td>27</td>
<td>3.22</td>
</tr>
<tr>
<td>4096</td>
<td>206.81</td>
<td>40</td>
<td>3.85</td>
</tr>
<tr>
<td>16384</td>
<td>413.67</td>
<td>57</td>
<td>4.69</td>
</tr>
</tbody>
</table>

Table 3.4.: Results on the unit disk with non-quasi-uniform family of triangular meshes (see Figure 3.10a).

<table>
<thead>
<tr>
<th>N</th>
<th>$\mathbf{D^{-1}_h W_h}$</th>
<th>$\mathbf{P_h W_h}$</th>
<th>$\mathbf{\bar{P}_h W_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>It</td>
<td>$\kappa$</td>
</tr>
<tr>
<td>64</td>
<td>2.27</td>
<td>6</td>
<td>2.09</td>
</tr>
<tr>
<td>256</td>
<td>4.62</td>
<td>9</td>
<td>2.22</td>
</tr>
<tr>
<td>1024</td>
<td>9.48</td>
<td>14</td>
<td>2.61</td>
</tr>
<tr>
<td>4096</td>
<td>19.45</td>
<td>20</td>
<td>3.03</td>
</tr>
<tr>
<td>16384</td>
<td>39.27</td>
<td>30</td>
<td>3.83</td>
</tr>
</tbody>
</table>

A third triangular mesh of the unit disk is also studied. This time, each mesh is generated with local refinement on the boundary such that the meshwidth on $\partial D$ is half the one of the previous mesh. This locally refined mesh is displayed in Figure 3.10b and the obtained results are reported in Table 3.5. The condition numbers are just as expected. Moreover, this time we can see that our preconditioner $\mathbf{\bar{P}_h}$ not only achieves an almost constant $\kappa$, but also its number of PCG iterations becomes significantly smaller than those for $\mathbf{P_h}$. This of course reflects the fact that opposite order preconditioning achieves a condition number with a logarithmic growth depending on $h_{\min}$. In other words, the growth of the condition number is very slow on meshes obtained by regular refinement, as $h_{\min}$ is only halved, and thus it performs reasonably well until very refined levels. However, as soon as it faces a locally refined mesh where $h_{\min}$ decreases fast, the effect of this logarithmic growth is clear and our $h$-independent preconditioner gains a considerable advantage.

Figures 3.8 and 3.9 show the distribution of eigenvalues for cases (EDP) and (ENP), respectively, over the three first refinement levels of the considered meshes for the unit disk. There we can see that, for coarse meshes, the clustering obtained with $\mathbf{P_h}$ does not differ much from the one achieved with $\mathbf{\bar{P}_h}$ on the quasi-uniform and non-quasi-uniform meshes, which may explain the similar number of PCG iterations they took. This, is also true for (EDP) on the depicted levels of the locally refined meshes. However, for (ENP) on these meshes, the difference in clustering and PCG performance is significant, as one can see in Table 3.5b and Figure 3.9c.

Table 3.5.: Results on the unit disk with locally refined family of triangular meshes (see Figure 3.10b).

<table>
<thead>
<tr>
<th>N</th>
<th>$h_{\min}$</th>
<th>$\mathbf{D^{-1}_h V_h}$</th>
<th>$\mathbf{P_h V_h}$</th>
<th>$\mathbf{\bar{P}_h V_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>It</td>
<td>$\kappa$</td>
<td>It</td>
</tr>
<tr>
<td>162</td>
<td>0.0568</td>
<td>33.09</td>
<td>10</td>
<td>4.06</td>
</tr>
<tr>
<td>506</td>
<td>0.0244</td>
<td>51.23</td>
<td>28</td>
<td>4.11</td>
</tr>
<tr>
<td>1052</td>
<td>0.0124</td>
<td>64.43</td>
<td>33</td>
<td>4.70</td>
</tr>
<tr>
<td>2150</td>
<td>0.0050</td>
<td>79.70</td>
<td>42</td>
<td>5.59</td>
</tr>
<tr>
<td>4260</td>
<td>0.0030</td>
<td>96.09</td>
<td>50</td>
<td>6.83</td>
</tr>
<tr>
<td>8398</td>
<td>0.0015</td>
<td>113.61</td>
<td>59</td>
<td>7.92</td>
</tr>
<tr>
<td>16546</td>
<td>0.0008</td>
<td>130.95</td>
<td>68</td>
<td>9.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N</th>
<th>$\mathbf{D^{-1}_h W_h}$</th>
<th>$\mathbf{P_h W_h}$</th>
<th>$\mathbf{\bar{P}_h W_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>It</td>
<td>$\kappa$</td>
</tr>
<tr>
<td>162</td>
<td>2.24</td>
<td>6</td>
<td>2.17</td>
</tr>
<tr>
<td>506</td>
<td>2.83</td>
<td>7</td>
<td>3.12</td>
</tr>
<tr>
<td>1052</td>
<td>3.16</td>
<td>8</td>
<td>3.32</td>
</tr>
<tr>
<td>2150</td>
<td>3.57</td>
<td>9</td>
<td>3.85</td>
</tr>
<tr>
<td>4260</td>
<td>3.78</td>
<td>10</td>
<td>4.49</td>
</tr>
<tr>
<td>8398</td>
<td>3.99</td>
<td>10</td>
<td>5.06</td>
</tr>
<tr>
<td>16546</td>
<td>4.23</td>
<td>10</td>
<td>5.69</td>
</tr>
</tbody>
</table>
Figure 3.8.: Spectrum of standard $V_h$ is shown in black, while the preconditioned by $P_h$ is in red, and the one corresponding to $\tilde{P}_h$ is in blue.

(a) Quasi-uniform mesh

(b) Non-quasi-uniform mesh

(c) Locally-refined mesh
Figure 3.9.: Spectrum of standard $W_h$ is shown in black, while the preconditioned by $P_h$ is in red, and the one corresponding to $P_h$ is in blue.

(a) Quasi-uniform mesh

(b) Non-quasi-uniform mesh

(c) Locally-refined mesh
3.4. Preconditioning on Disks

Figure 3.10: Non-uniform triangular meshes

(a) Non-quasi-uniform: The coarsest mesh was created using the functions Attraction and Threshold iteratively in Gmsh. This means the mesh size is a piecewise linear function of the distance to the disk’s boundary. The subsequent meshes were obtained by standard refinement in Gmsh.

(b) Locally-refined: Each mesh was constructed with the functions Attraction and Matheval in Gmsh, where the evaluated function was the continuous distance to the boundary of the disk plus a parameter $h_*>0$. The subsequent meshes were obtained by halving $h_*$ and thus the minimum meshwidth on the boundary.

3.4.2. Numerical Results for Helmholtz BIOs on the Unit Disk

Numerical experiments were implemented employing BETL2 [53] with the same parameters and meshes as in subsection 3.4.1. Moreover, here we present results obtained with dense matrices, meaning that no ACA approximation was used. We measure the performance of a preconditioner by considering the obtained number of GMRES iterations required to solve the linear system. We use GMRES with a tolerance of $10^{-5}$ for the relative residual norm, initial guess equal to zero, and right-hand side given by a vector of ones on a mesh with $N$ elements.

For each experiment presented, we provide two tables. The ones on the left, entitled (EDP), contain the preconditioning results for $V_k$, while the right-hand tables display the results for $W_k$ and is labeled as (ENP). In both situations we compare the performance of our preconditioner $P_h$ constructed with our modified BIOs with opposite-order operator preconditioning $P_k^h$ arising from using the standard BIOs.

Let us consider the notations introduced in subsection 3.4.1 and also define $V^h_k$ and $W^h_k$ as the Galerkin matrices of $V_k$ and $W_k$, respectively. Then, we construct the preconditioners as follows:

- (EDP): Here our preconditioner $P_h$ is exactly the same as for $V_0$ and the opposite-order preconditioner is $P^h_k := T^{-1}_{D,h} V^h_k T_{D,h}^{-T}$.

- (ENP): In this case, our preconditioner $P_h$ is the same as for $W_0$ and $P^h_k := T^{-1}_{N,h} W^h_k T_{N,h}^{-T}$.

Note that the bilinear form arising from $W_k$, $k > 0$ has a trivial kernel in $H^{1/2}(D_1)$ and thus no regularization term is required.

Table 3.6 displays the preconditioning results over a disk considering different wave numbers $k$ and with a quasi-uniform family of triangular meshes. There we see that our preconditioner does not obtain a significant advantage over the standard opposite-order preconditioning. Moreover, for (ENP) with $k = 4$, $P^h_k$ performs slightly better than $P_h$. 

Table 3.6

<table>
<thead>
<tr>
<th>Wave Number</th>
<th>Preconditioner</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 1$</td>
<td>$P_h$</td>
<td>100</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>$P_h$</td>
<td>110</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>$P_h$</td>
<td>120</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>$P^h_k$</td>
<td>90</td>
</tr>
<tr>
<td>$k = 5$</td>
<td>$P_h$</td>
<td>130</td>
</tr>
<tr>
<td>$k = 6$</td>
<td>$P^h_k$</td>
<td>80</td>
</tr>
</tbody>
</table>

51
### Table 3.6.: GMRES iterations for Helmholtz BIOs on the unit disk with quasi-uniform family of triangular meshes.

#### (a) (EDP)

<table>
<thead>
<tr>
<th>N</th>
<th>$h$</th>
<th>$V_h^k$</th>
<th>$P_h^k V_h^k$</th>
<th>$F_h V_h^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.160</td>
<td>19</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>256</td>
<td>0.085</td>
<td>31</td>
<td>18</td>
<td>14</td>
</tr>
<tr>
<td>1024</td>
<td>0.042</td>
<td>43</td>
<td>21</td>
<td>17</td>
</tr>
<tr>
<td>4096</td>
<td>0.021</td>
<td>60</td>
<td>25</td>
<td>18</td>
</tr>
</tbody>
</table>

#### (b) (ENP)

<table>
<thead>
<tr>
<th>N</th>
<th>$h$</th>
<th>$W_h^k$</th>
<th>$P_h^k W_h^k$</th>
<th>$F_h W_h^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0.160</td>
<td>19</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>256</td>
<td>0.085</td>
<td>31</td>
<td>18</td>
<td>14</td>
</tr>
<tr>
<td>1024</td>
<td>0.042</td>
<td>43</td>
<td>21</td>
<td>17</td>
</tr>
<tr>
<td>4096</td>
<td>0.021</td>
<td>60</td>
<td>25</td>
<td>18</td>
</tr>
</tbody>
</table>

### Table 3.7.: Results for Helmholtz on the unit disk with non-quasi-uniform family of triangular meshes (see Figure 3.10a).

#### (a) (EDP)

<table>
<thead>
<tr>
<th>N</th>
<th>$V_h^k$</th>
<th>$P_h^k V_h^k$</th>
<th>$F_h V_h^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>92</td>
<td>36</td>
<td>27</td>
<td>23</td>
</tr>
<tr>
<td>384</td>
<td>54</td>
<td>23</td>
<td>21</td>
</tr>
<tr>
<td>1536</td>
<td>87</td>
<td>27</td>
<td>23</td>
</tr>
<tr>
<td>6144</td>
<td>121</td>
<td>30</td>
<td>25</td>
</tr>
</tbody>
</table>

#### (b) (ENP)

<table>
<thead>
<tr>
<th>N</th>
<th>$W_h^k$</th>
<th>$P_h^k W_h^k$</th>
<th>$F_h W_h^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>92</td>
<td>5</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td>384</td>
<td>16</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>1536</td>
<td>23</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>6144</td>
<td>34</td>
<td>15</td>
<td>9</td>
</tr>
</tbody>
</table>
3.5. Preconditioning for Laplace BIOs on Mapped Screens

We now extend the operator preconditioning strategy to more general screens $\Gamma \subset \mathbb{R}^3$, for which there is a bi-Lipschitz continuous \cite[Def. 2.2.5]{...} piecewise $C^1$-mapping $\tilde{\phi} : \overline{D_1} \times [-1, 1] \to \mathbb{R}^3$ such that $\phi(D_1) = \Gamma$ for $\phi := \tilde{\phi}|_{\overline{D_1} \times \{0\}}$. This ensured that $\Gamma$ is an orientable piecewise two-dimensional $C^1$-manifold with boundary $\partial \Gamma = \phi(\partial D_1)$.

- **Example 1:** In cylindrical coordinates $(r, \theta, z)$ of $\mathbb{R}^3$ define

$$
\tilde{\phi} \begin{bmatrix} r \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} r/\max(|\cos \varphi|, |\sin \varphi|) \\ \theta \\ z \end{bmatrix},
$$

(3.50)

for $r \geq 0, 0 \leq \theta \leq 2\pi$, and $z \in \mathbb{R}$. Then $\tilde{\phi}$ satisfies the above assumptions and maps the unit disk to a square $\phi(D_1) = [-1, 1]^2$.

- **Example 2:** If $f \in C^1(\overline{D_1}, \mathbb{R})$ and

$$
\tilde{\phi} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + f(x_1, x_2) \end{bmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R},
$$

then $\Gamma := \phi(D_1)$ yields the graph of $f$ over $D_1$.

We consider the variational problems (1.55) and (1.59) on $\Gamma$. Writing $\phi^* v : D_1 \to \mathbb{R}$ for the pullback of a function $v : \Gamma \to \mathbb{R}$ to $D_1$ under $\phi$:

$$
\hat{v}(\hat{x}) := (\phi^* v)(\hat{x}) = v(\phi(\hat{x})), \quad \hat{x} \in D_1.
$$

We recall from \cite[Thm 3.23]{...} that

$$
\phi^* : L^2(\Gamma) \to L^2(\overline{D_1}), \quad \phi^* : H^1(\Gamma) \to H^1(D_1), \quad \phi^* : H^1_0(\Gamma) \to H^1_0(D_1),
$$

(3.51)

are isomorphisms, since $\phi$ is bi-Lipschitz. By interpolation arguments in Sobolev scales, we immediately infer that

$$
\phi^* : H^{1/2}(\Gamma) \to H^{1/2}(\overline{D_1}),
$$

(3.52)

$$
\phi^* : H^{-1/2}(\Gamma) \to H^{-1/2}(\overline{D_1}),
$$

(3.53)

are isomorphisms too. By duality, this will also hold for

$$
\phi^* : H^{-1/2}(\Gamma) \to H^{-1/2}(\overline{D_1}),
$$

(3.54)

$$
\phi^* : H^{-1/2}(\Gamma) \to H^{-1/2}(\overline{D_1}).
$$

(3.55)

### 3.5.1. Unit Disk Based Preconditioner for Mapped Screens

The idea is to rely on isomorphisms (3.52)–(3.55) to obtain suitable bilinear forms $b(\cdot, \cdot)$ for operator preconditioning according to Theorem 3.1.1. Also note that all constructions and results of Section 3.2 had already been formulated and proved for general screens as introduced above.

For the sake of Galerkin BE discretization, we equip $\Gamma$ with a uniformly shape regular family $\{\Gamma_h\}_{h \in \mathbb{H}}$ of surface meshes, obtained as the image of a family of meshes $\{D_h\}_{h \in \mathbb{H}}$ of the unit disk $D_1$ under $\phi$, i.e. $\Gamma_h := \phi(D_h)$, which is supposed to comply with Assumption 3.2.1.
Case (EDP)

We write \( V_{0, \Gamma} \) to denote the weakly singular operator (1.52) on \( \Gamma = \phi(\mathbb{D}_1) \) and with \( k = 0 \). Using the parametrization \( \phi \), it can be pulled back to \( \mathbb{D}_1 \): \( V_{0, \Gamma} = (\phi^*)^* \circ V_{0, \Gamma} \circ \phi^* \), where \( (\phi^*)^* \) denotes the adjoint operator of \( \phi^* \). Hence, \( V_{0, \Gamma} \) induces the bilinear form for \( \hat{\sigma}, \hat{\mu} \in \tilde{H}^{-1/2}(\mathbb{D}_1) \)

\[
a_{V, \Gamma}^{*}(\hat{\sigma}, \hat{\mu}) := \frac{1}{4\pi} \int_{\mathbb{D}_1} \int_{\mathbb{D}_1} \frac{\hat{\sigma}(\mathbf{y})\hat{\mu}(\mathbf{x})}{\|\phi(\mathbf{x}) - \phi(\mathbf{y})\|} g(\mathbf{x})g(\mathbf{y})d\mathbb{D}_1(\mathbf{x})d\mathbb{D}_1(\mathbf{y}),
\]

with \( g(\mathbf{x}) := \sqrt{\det(\nabla \phi(\mathbf{x})^T \nabla \phi(\mathbf{x}))} \), and \( \nabla \phi(\mathbf{x}) \) the Jacobian of \( \phi \) in \( \mathbf{x} \in \mathbb{D}_1 \). Thanks to (3.54), \( a_{V, \Gamma}^{*} \) is \( \tilde{H}^{-1/2}(\mathbb{D}_1) \)-elliptic and continuous. Thus, we can employ \( X_h = \mathcal{S}^{-1,0}(\mathbb{D}_h) \) for Galerkin discretization and \( b_{\mathcal{M}_h} \) from (3.48b) for operator preconditioning.

Case (ENP)

Let us write \( W_{0, \Gamma} \) for the hypersingular operator (1.53) on the screen \( \Gamma = \phi(\mathbb{D}_1) \) and with \( k = 0 \):

\[
(W_{0, \Gamma} u)(\mathbf{x}) := \int_{\Gamma} k_{W,0}(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \Gamma, \ u \in \tilde{H}^{1/2}(\Gamma),
\]

where \( k_{W,0} \) is the associated kernel. As for (EDP), we can use the parametrization of \( \Gamma \) over \( \mathbb{D}_1 \) to define \( W_{0, \Gamma} := (\phi^*)^* \circ W_{0, \Gamma} \circ \phi^* \), which induces the bilinear form

\[
a_{W, \Gamma}^{*}(\hat{u}, \hat{v}) := \frac{1}{4\pi} \int_{\mathbb{D}_1} \int_{\mathbb{D}_1} \frac{\text{curl}_{\Gamma}(\phi(\mathbf{x})) \hat{u}(\mathbf{x}) \cdot \text{curl}_{\Gamma}(\phi(\mathbf{y})) \hat{v}(\mathbf{y})}{\|\phi(\mathbf{x}) - \phi(\mathbf{y})\|} g(\mathbf{x})g(\mathbf{y})d\mathbb{D}_1(\mathbf{x})d\mathbb{D}_1(\mathbf{y})
\]

for \( \hat{u}, \hat{v} \in \tilde{H}^{1/2}(\mathbb{D}_1) \), and \( g \) as before.

The norm equivalence implied by (3.53) gives \( \tilde{H}^{1/2}(\mathbb{D}_1) \)-ellipticity and continuity of \( a_{W, \Gamma}^{*} \). Therefore we can use \( X_h = \mathcal{S}^{0,1}(\mathbb{D}_h) \) for Galerkin discretization and \( b_{\mathcal{M}_h} \) from (3.48a) for operator preconditioning.

Table 3.8.: Components of \( L^2(\Gamma) \)-based dual mesh operator preconditioning approach for Laplace’s variational BIEs on \( \Gamma = \phi(\mathbb{D}_1) \) under Assumptions 3.3.1 and 3.3.3. Recall that \( \mathbb{D}_h \) denotes the primal mesh of \( \mathbb{D}_1 \) and here we write \( \mathbb{D}_h \) for its dual mesh.

<table>
<thead>
<tr>
<th>BVP</th>
<th>( a )</th>
<th>( b )</th>
<th>( X )</th>
<th>( Y )</th>
<th>( X_h )</th>
<th>( Y_h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EDP)</td>
<td>( a_{V, \Gamma}^{*} )</td>
<td>( b_{\mathcal{M}_h} )</td>
<td>( \tilde{H}^{-1/2}(\mathbb{D}_1) )</td>
<td>( H^{1/2}(\mathbb{D}_1) )</td>
<td>( \mathcal{S}^{-1,0}(\mathbb{D}_h) )</td>
<td>( \mathcal{S}^{0,1}(\mathbb{D}_h) )</td>
</tr>
<tr>
<td>(ENP)</td>
<td>( a_{W, \Gamma}^{*} )</td>
<td>( b_{\mathcal{M}_h} )</td>
<td>( \tilde{H}^{1/2}(\mathbb{D}_1) )</td>
<td>( H^{-1/2}(\mathbb{D}_1) )</td>
<td>( \mathcal{S}^{0,1}(\mathbb{D}_h) )</td>
<td>( \mathcal{S}^{-1,0}(\mathbb{D}_h) )</td>
</tr>
</tbody>
</table>

**Corollary 3.5.1.** Borrowing notations from Theorem 3.1.1, we consider the variational BIEs (EDP) and (ENP), and their Galerkin BE discretization according to Table 3.8. Then, the operator preconditioning strategy proposed in Table 3.8 yields \( h \)-uniformly bounded spectral condition number \( \kappa(T_h^{-1} B_h T_h^{-H} A_h) \).

**Remark 3.5.2.** The isomorphisms (3.52)–(3.55) imply equivalence of corresponding norms of a function \( v \) on \( \Gamma \) and its pullback \( \phi^* v \). The constants in these equivalences will have a direct impact on the final bounds for the spectral condition numbers of the preconditioned Galerkin matrices through \( \|a\| \) and \( c_A \), see Theorem 3.1.1 for notations.

This means that Corollary 3.5.1 guarantees that the condition number (3.2) will remain \( h \)-uniformly bounded for the preconditioned Galerkin matrices of \( V_{0, \Gamma} \) and \( W_{0, \Gamma} \), albeit affected by a constant depending on the distortion effected by \( \phi \). Crudely speaking, in case \( \phi \) causes large distortions, that is, if \( \|D\phi\|_\infty \) or \( |D\phi^{-1}|_\infty \) are large, we should expect poorer performance of the preconditioner. This will be reflected in the numerical experiments in subsection 3.5.2 through a pre-asymptotic phase in which the behavior of the preconditioner is not as good as expected.

**Remark 3.5.3.** In this section we constrained ourselves to the Laplace BIOs, however, the same preconditioning strategy can be applied to Helmholtz BIOs.
3.5.2. Numerical Results

In this subsection we study the preconditioning results achieved when applying the approach described in subsection 3.5.1 to precondition $V_0$ and $W_0$. This means that $V_h$ (resp. $W_h$) corresponds to the Galerkin matrix of the weakly singular (resp. hypersingular) operator mapped from the disk $D_1$ to the target screen $\Gamma$ via $\phi$, whereas $P_h$ is constructed on the disk, i.e. using (3.48b) (resp. (3.48a)), whereas $P_h$ arises from the standard hypersingular (weakly singular) operator mapped from $D_1$ to $\Gamma$. Indeed, for $P_h$, the dual-pairing matrix $T_h$ is transformed from $D_1$ to $\Gamma$, whereas for $P_h$, it is computed on $D_1$.

For this particular case, instead of using (3.49), we regularize $W$ by adding

$$\alpha_W(u,1)_{\Gamma} (v,1)_{\Gamma}, \quad u,v \in \widetilde{H}^{1/2}(\Gamma),$$

for general $\Gamma$, where $\alpha_W \in \mathbb{R}^+$ bounded. In this subsection we choose $\alpha_W = 0.3 \approx \frac{1}{\pi} = ((1,1)_{D_1})^{-1}$.  

**Remark 3.5.4.** Currently, we do not have a rule of thumb to choose the parameter $\alpha_W$ and $\alpha_{\Psi}$. In fact, we selected each one of them empirically in the following way: First, we set an initial guess for both parameters and computed the resulting full Galerkin matrices and their spectra. By changing parameter values, we then sought to locate eigenvalues related to the regularizing term inside the remaining spectra, thus preventing an artificial enlargement of condition numbers. Consequently, they do not alter the performance of the preconditioners and allow us to make fair comparisons between $P_h$ and $P_h$. As our eigenvalues cluster around one, we have found that $\alpha_{\Psi} = \frac{1}{\pi^2} = 1/(1, \omega^{-1})_{D_1}$ does a good job for the regularization in (3.49), while for (3.59), a good initial guess is given by the inverse of the area of the screen. Since in this subsection we study screens that are mapped from the unit disk, this motivates the choice $\alpha_W = 0.3 \approx \frac{1}{\pi} = ((1,1)_{D_1})^{-1}$.

We show numerical results for four different shapes: The first data are listed in Table 3.9 and correspond to the square screen introduced in Example 1 at the beginning of this Section (see formula in (3.50)). The other three shapes are inspired by Example 2 for different functions $f$ and the corresponding results are shown in Tables 3.10-3.12. For all studied mapped screens we compared the performance of the preconditioners on three families of triangular meshes: quasi-uniform, non-quasi-uniform (see Figure 3.10a), and locally refined (see Figure 3.10b).

**Table 3.9:** Results on mapped square (see formula in (3.50)).

<table>
<thead>
<tr>
<th>N</th>
<th>$D_1^{-1}V_h$</th>
<th>$P_hV_h$</th>
<th>$P_hV_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>$k$</td>
<td>$k$</td>
</tr>
<tr>
<td>Quasi-uniform</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>24.08</td>
<td>14</td>
<td>5.15</td>
</tr>
<tr>
<td>256</td>
<td>51.46</td>
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<td>6.03</td>
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<td>105.18</td>
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<tr>
<td>4096</td>
<td>217.04</td>
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</tr>
<tr>
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<td>441.72</td>
<td>57</td>
<td>9.90</td>
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<tr>
<td>Non-quasi-uniform</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>92</td>
<td>26.09</td>
<td>18</td>
<td>8.24</td>
</tr>
<tr>
<td>384</td>
<td>57.32</td>
<td>23</td>
<td>7.24</td>
</tr>
<tr>
<td>1536</td>
<td>131.34</td>
<td>33</td>
<td>9.22</td>
</tr>
<tr>
<td>6144</td>
<td>291.62</td>
<td>48</td>
<td>10.97</td>
</tr>
<tr>
<td>Locally-refined</td>
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<td></td>
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<tr>
<td>162</td>
<td>34.05</td>
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<td>8.56</td>
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<td>506</td>
<td>52.40</td>
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<td>9.19</td>
</tr>
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<td>1052</td>
<td>67.15</td>
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<td>82.39</td>
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<td>100.78</td>
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<td>132.73</td>
<td>69</td>
<td>18.68</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N</th>
<th>$D_1^{-1}W_h$</th>
<th>$P_hW_h$</th>
<th>$P_hW_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>$k$</td>
<td>$k$</td>
</tr>
<tr>
<td>Quasi-uniform</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>2.26</td>
<td>5</td>
<td>2.38</td>
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<td>4.85</td>
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<td>16384</td>
<td>44.03</td>
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<tr>
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<td>11</td>
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<tr>
<td>16546</td>
<td>4.78</td>
<td>11</td>
<td>6.12</td>
</tr>
</tbody>
</table>
For the square screen, we observe in Table 3.9 that $\kappa(\mathbf{P}_h\mathbf{V}_h)$ has a small grow with respect to $N$. Moreover, the change in the condition number decreases when increasing the number of elements in our meshes. We believe this reflects a pre-asymptotic phase together with numerical errors introduced by quadrature and the ACA approximation, which have a stronger influence on coarser meshes. Nevertheless, our preconditioner achieves lower condition numbers than the other two approaches, particularly on the locally refined mesh. In terms of PCG iterations, we only observe that $\mathbf{P}_h$ performs significantly better than $\mathbf{P}_h$ when preconditioning $\mathbf{V}_h$.

Tables 3.10, 3.11, and 3.12 reveal the preconditioning results for the other three different shapes mapped from the unit disk. In all of them, we see how the opposite-order preconditioner $\mathbf{P}_h$ displays the expected logarithmic growth while our proposal shows a small rise that seems to be less pronounced the larger the mesh. Naturally, this behaviour is also reflected in the number of PCG iterations. Whereas the number of PCG counts for $\mathbf{P}_h$ increases with respect to $N$, those of $\mathbf{P}_h$ remain constant in the last two levels for most considered mappings.

Furthermore, we can see that our proposed preconditioner performs considerably better for locally refined meshes when the distortion of the mesh is moderate, as in Tables 3.10 and 3.11 where $f(x_1, x_2) = x_1 + x_2$ and $f(x_1, x_2) = x_1x_2$, respectively. However, as predicted in Remark 3.5.2, this is not the case for $f(x_1, x_2) = x_1^2 + x_2^2$. In that case, $\mathbf{P}_h$ does not actually offer an advantage over $\mathbf{P}_h$ in terms of PCG counts, in spite of achieving an almost constant condition number.

Table 3.10.: Results on mapped screens $\phi(x) = (x_1, x_2, x_1 + x_2)^T$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\mathbf{D}_h^{-1}\mathbf{V}_h$</th>
<th>$\mathbf{P}_h\mathbf{V}_h$</th>
<th>$\mathbf{P}_h\mathbf{V}_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>$\kappa$</td>
<td>$\kappa$</td>
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<td>It</td>
<td>It</td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>64</td>
<td>24.15 14</td>
<td>3.17 7</td>
<td>3.02 7</td>
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<td>3.28 8</td>
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<td>1024</td>
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<td>3.45 8</td>
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<td>4096</td>
<td>223.37 43</td>
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<td>3.61 9</td>
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<td>6.70 12</td>
<td>3.72 9</td>
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<td>6.72 13</td>
</tr>
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<td>384</td>
<td>55.80 24</td>
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<td>3.87 10</td>
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<td>5.49 11</td>
<td>4.23 10</td>
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<td>4.46 11</td>
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<td>162</td>
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<td>16546</td>
<td>133.70 69</td>
<td>11.79 25</td>
<td>5.76 18</td>
</tr>
</tbody>
</table>

(a) (EDP)  
(b) (ENP)
### 3.5. Preconditioning for Laplace BIOs on Mapped Screens

#### Table 3.11: Results on mapped screens $\phi(x) = (x_1, x_2, x_1x_2)^T$.

<table>
<thead>
<tr>
<th>N</th>
<th>$D_h^{-1}V_h$</th>
<th>$P_hV_h$</th>
<th>$F_hV_h$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$\text{It}$</td>
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<td></td>
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<td></td>
<td></td>
</tr>
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<td>135.74</td>
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<td>12.64</td>
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</tbody>
</table>

#### Table 3.12: Results on mapped screens $\phi(x) = (x_1, x_2, x_1 + x_2)^T$.

<table>
<thead>
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<th>N</th>
<th>$D_h^{-1}W_h$</th>
<th>$P_hW_h$</th>
<th>$F_hW_h$</th>
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<td>175.91</td>
<td>73</td>
<td>7.71</td>
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</tbody>
</table>
4. Shape-aware Preconditioners

“Accursed creator! Why did you form a monster so hideous that even you turned from me in disgust?”
– Frankenstein by Mary W. Shelley (1818).

Motivated by the deterioration of the performance of our preconditioners on “distorted” screens (see subsection 3.5.2), we propose an alternative approach to build preconditioners on general screens. In contrast to what we did in subsection 3.5.1, the idea here is to compute all operators in the target meshes of $\Gamma$. Hence, we refer to this preconditioning strategy by the name of shape-aware preconditioners.

Shape-aware preconditioners follow an heuristic construction based on the operator preconditioning policy. This means that we employ the notation and building blocks introduced in Chapter 3 but use bilinear forms $b$ arising from heuristic extensions of $b_{W,0}^a$ and $b_{V,0}^a$ to more general screens. These extensions are meant as approximate inverses of $W_{0,1}^a$ and $V_{0,1}^a$ and thus follow the intuition behind operator preconditioning. However, their exact properties remain unknown and we do not show the required continuity and inf-sup conditions. Consequently, we cannot prove that shape-aware preconditioners comply with operator preconditioning theory nor that they achieve an $h$-uniformly bounded spectral condition number. Nevertheless, they do attain this optimality in practice for a large family of screen shapes, as we will see later in the numerical experiments. For this reason, although of little mathematical rigor, we have included this chapter to report and discuss this approach.

Table 4.1 summarizes the elements from operator preconditioning that we will use throughout this chapter. We dedicate the next Sections to introduce the two heuristic extensions we developed and study their applicability as preconditioners. This Chapter is based on [51, Sect. 3.6–3.7] and [52, Sect. 3.5].

### Table 4.1.: Components of the $L^2(\Gamma)$-based dual mesh operator preconditioning approach for variational boundary integral equations on $\Gamma$.

<table>
<thead>
<tr>
<th>BVP</th>
<th>$a$</th>
<th>$X$</th>
<th>$Y$</th>
<th>$X_h$</th>
<th>$Y_h$</th>
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<tbody>
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<td>(EDP)</td>
<td>$a_{W,1}^a$</td>
<td>$H^{-1/2}(\Gamma)$</td>
<td>$H^{1/2}(\Gamma)$</td>
<td>$S^{-1,0}(\Gamma_h)$</td>
<td>$S^{0,1}(\Gamma_h)$</td>
</tr>
<tr>
<td>(ENP)</td>
<td>$a_{W,1}^a$</td>
<td>$H^{1/2}(\Gamma)$</td>
<td>$H^{-1/2}(\Gamma)$</td>
<td>$S^{0,1}(\Gamma_h)$</td>
<td>$S^{-1,0}(\Gamma_h)$</td>
</tr>
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</table>

#### 4.1. Flat Screens

##### 4.1.1. Approximate Inverses

Let us consider again the disk $D_a$ of radius $a$ centered at the origin. The kernel $k_{\Gamma}$ of $\nabla$ over $D_a$ is given by

$$k_{\Gamma}(x,y) := \frac{2}{\pi^2} \frac{S_a(x,y)}{\|x - y\|^2},$$

with

$$S_a(x,y) = \tan^{-1}\left(\frac{\sqrt{a^2 - \|x\|^2} \sqrt{a^2 - \|y\|^2}}{a \|x - y\|}\right)$$

(4.1)

for $x \neq y$, $x, y \in D_a$. 

\[59\]
Chapter 4. Shape-aware Preconditioners

Note that the boundary of the disk $D_a$ is given in polar coordinates as $r = a(\theta)$. Then, the kernel of the modified weakly singular integral operator can be rewritten with

$$S_a(x, y) = \left( \frac{\sqrt{a(\theta_x)^2 - \|x\|^2} \sqrt{a(\theta_y)^2 - \|y\|^2}}{a(\theta_x)a(\theta_y)\|x - y\|} \right), \quad x \neq y, \ x, y \in D_a, \quad (4.2)$$

where $\theta_x$ and $\theta_y$ indicate the polar angle of $x$ and $y$, respectively.

Although $a(\theta) = a$ for $D_a$ –and the expression above is unduly complicated–, it can be used as a starting point to develop an approximation of $W^{-1}$ for general flat open surfaces that allow polar angle parametrization of their boundary, i.e. boundaries that can be described by a function $a(\theta), \theta \in [0, 2\pi]$.

In this case, we can use

$$S_\Gamma(x, y) = \tan^{-1}\left( \frac{\sqrt{a(\theta_x)^2 - \|x\|^2} \sqrt{a(\theta_y)^2 - \|y\|^2}}{\sqrt{a(\theta_x)a(\theta_y)}\|x - y\|} \right), \quad \text{for} \ x \neq y, \ x, y \in \Gamma, \quad (4.3)$$

to construct an approximation of $W^{-1}$ and $V^{-1}$.

4.1.2. Shape-aware Preconditioning

Case (EDP), (1.55):

The approximation of the bilinear form related to $W$ that we pursue is

$$b_W(u, v) := \frac{2}{\pi^2} \int_{\Gamma} \int_{\Gamma} S_\Gamma(x, y) \frac{\|x - y\|}{\|x\| \cdot \|y\|} \text{curl}_\Gamma \cdot u(x) \cdot \text{curl}_\Gamma \cdot v(y) d\Gamma(x) d\Gamma(y) + a_W(u, 1)_{\Gamma} \langle v, 1 \rangle_{\Gamma} \quad (4.4)$$

for $u, v \in H^{1/2}(\Gamma)$ and with $a_W \in \mathbb{R}^+$ bounded.

Here, we have additionally replaced the function $\omega^{-1}$ by the constant 1 in the correction term for implementation simplicity. We justify this choice on account of the fact that we will use PCG/GMRES to solve the arising system and said method will not perceive a significant difference among these two regularizations. Moreover, this alternative is computationally cheaper than the original choice in (3.48b).

Case (ENP), (1.59):

We build an approximation of the bilinear form associated to $V$ using (4.3). In other words

$$b_V(\vartheta, \mu) := \frac{2}{\pi^2} \int_{\Gamma} \int_{\Gamma} S_\Gamma(x, y) \frac{\|x - y\|}{\|x\| \cdot \|y\|} \vartheta(x) \mu(y) d\Gamma(x) d\Gamma(y) \quad (4.5)$$

for $\vartheta, \mu \in H^{-1/2}(\Gamma)$.

Remark 4.1.1. The flat screen need not be the result of a transformed unit disk via a bi-Lipschitz mapping as in the previous subsection. However, a piecewise Lipschitz transformation is still required.

Borrowing the notation from Theorem 3.1.1, we can now state the shape-aware preconditioning strategy for flat screens that allow polar angle parametrization of their boundary. This is outlined in Table 4.2. Note that all BE spaces are intrinsically defined on $\Gamma$ now.
Table 4.2.: Components of Shape-aware preconditioning for variational BIEs on flat screens $\Gamma$ as defined in subsection 4.1.1.

<table>
<thead>
<tr>
<th>BVP</th>
<th>$a_{\nu,\Gamma}$</th>
<th>$b_{\nu,\Gamma}$</th>
<th>$H^{-1/2}(\Gamma)$</th>
<th>$H^{1/2}(\Gamma)$</th>
<th>$S^{-1,0}(\Gamma_h)$</th>
<th>$S^{0,1}(\tilde{\Gamma}_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EDP)</td>
<td>$a_{\nu,\Gamma}$</td>
<td>$b_{\nu,\Gamma}$</td>
<td>$\bar{H}^{-1/2}(\Gamma)$</td>
<td>$\bar{H}^{1/2}(\Gamma)$</td>
<td>$\bar{S}^{-1,0}(\Gamma_h)$</td>
<td>$\bar{S}^{0,1}(\tilde{\Gamma}_h)$</td>
</tr>
<tr>
<td>(ENP)</td>
<td>$a_{\nu,\Gamma}$</td>
<td>$b_{\nu,\Gamma}$</td>
<td>$\bar{H}^{1/2}(\Gamma)$</td>
<td>$\bar{H}^{-1/2}(\Gamma)$</td>
<td>$\bar{S}^{0,1}(\Gamma_h)$</td>
<td>$\bar{S}^{-1,0}(\tilde{\Gamma}_h)$</td>
</tr>
</tbody>
</table>

4.1.3. Numerical Experiments for Laplace BIOs

We present numerical results to illustrate the approach proposed in Section 4.1.2 to precondition the standard BIOs for the Laplace equation. For this we considered 5 different polygonal shapes: unit square, three different triangles featuring distinct minimal angles and a set of rectangles with varying aspect ratios. This choice seeks to elucidate the performance of shape-aware preconditioners under different kind of “deformations” with respect to the unit disk.

Numerical experiments were implemented with BETL2 and run with the same parameters described in subsection 3.4.1. It is worth mentioning that this includes the ACA approximation and PCG Lanczos algorithm to compute the spectral condition numbers.

As usual, we provide two tables for each experiment. The ones on the left, labeled as (EDP) with the preconditioning results for $V_0$, an the ones on the right, report the results for $W_0$ and entitled (ENP). Let us briefly recall the notation used: $V_h$ and $W_h$ correspond to the matrices coming from $V_0$ and $W_0$ on the flat screen $\Gamma$, $D_h$ denotes diagonal preconditioning, $P_h$ comes from the standard BIOs discretized on $\tilde{\Gamma}_h$, and $P_{\nu,h}$ is constructed using the Galerkin matrices of the approximate modified BIOs introduced in subsection 4.1.1.

Unit square

The first example is a unit square. Here we considered the following radius function:

$$a(\theta) := \begin{cases} 
1/ \cos \theta, & -\pi/4 < \theta < \pi/4 \\
1/ \sin \theta, & \pi/4 < \theta < 3\pi/4 \\
-1/ \cos \theta, & 3\pi/4 < \theta < 5\pi/4 \\
-1/ \sin \theta, & 5\pi/4 < \theta < 7\pi/4 
\end{cases}, \quad (4.6)
$$

to be used in (4.3) to build our preconditioner $\bar{P}_h$.

Figure 4.1.: $a(\theta)$ on square

We ran our experiments on two families of triangular meshes; quasi-uniform and non-uniform (see Figure 4.2 for a specimen of each type). Tables 4.3 and 4.4 give the numerical results for preconditioning on a unit square screen over these meshes and specify the parameters $a_{\nu,\Gamma}$ and $a_{\nu,\Gamma}$ considered in Case (EDP) for $P_h$ and $\bar{P}_h$, respectively. We observe that the shape-aware preconditioners behave qualitatively in the same fashion as the unit disk based preconditioner for mapped screens presented in section 3.5, taking about two iterations less to reach the desired tolerance in most cases.

For the quasi-uniform mesh, there is a slight increase in the condition number that is probably coming from numerical error. In the non-uniform case, the slope decreases but it is not as small. As before, this is also reflected in the number of PCG iterations obtained with $\bar{P}_h$, which, although asymptotically constant, are not significantly smaller than those achieved by $P_h$. 
Chapter 4. Shape-aware Preconditioners

Table 4.3.: Results over a square screen with quasi-uniform triangular meshes. $P_h$ built using shape-aware preconditioner for flat screens.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$D_h^{-1}V_h$</th>
<th>$P_hV_h$</th>
<th>$P_hV_h$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$\kappa$ $\text{It}$</td>
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<td>25.63 11 3.38 4 3.35 4</td>
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<tr>
<td>256</td>
<td>52.00 18 4.30 6 3.71 6</td>
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<tr>
<td>1024</td>
<td>104.29 25 5.31 7 3.94 6</td>
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<tr>
<td>4096</td>
<td>208.65 37 6.41 9 4.10 6</td>
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<tr>
<td>16384</td>
<td>417.32 51 7.64 8 4.21 6</td>
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</table>

(b) (ENP)

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<th>$P_hW_h$</th>
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<td>1024</td>
<td>9.74 14 2.87 6 1.45 4</td>
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<td>4096</td>
<td>19.43 21 4.19 8 1.83 6</td>
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<td>16384</td>
<td>41.79 33 7.69 12 3.41 9</td>
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</table>

Table 4.4.: Results over a square screen with locally refined meshes. $P_h$ built using shape-aware preconditioner for flat screens.

(a) (EDP). $\alpha_W = \alpha_{\text{qu}} = 0.3$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$D_h^{-1}V_h$</th>
<th>$P_hV_h$</th>
<th>$P_hV_h$</th>
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<td>$\kappa$ $\text{It}$</td>
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<tr>
<td>108</td>
<td>29.66 16 3.81 8 4.26 9</td>
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<tr>
<td>432</td>
<td>59.84 24 4.73 10 4.98 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1728</td>
<td>120.02 33 5.78 11 5.52 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6912</td>
<td>241.51 46 6.94 12 5.59 12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27648</td>
<td>484.43 63 8.19 13 5.92 12</td>
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</tbody>
</table>

(b) (ENP)

<table>
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<th>$P_hW_h$</th>
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<td>$\kappa$ $\text{It}$</td>
<td>$\kappa$ $\text{It}$</td>
<td>$\kappa$ $\text{It}$</td>
</tr>
<tr>
<td>108</td>
<td>2.04 5 2.29 5 1.46 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>432</td>
<td>4.40 9 2.60 5 1.46 4</td>
<td></td>
<td></td>
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<tr>
<td>1728</td>
<td>9.29 14 3.08 7 1.50 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6912</td>
<td>19.18 21 3.69 7 1.48 5</td>
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<tr>
<td>27648</td>
<td>39.06 30 4.70 9 1.68 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.2.: Triangular meshes for unit square

(a) Quasi-uniform

(b) Non-uniform

Equilateral triangle

In our second example, we consider a screen given by an equilateral triangle $T$ with vertices $(0, 4.6, 0), (-4, -2.3, 0)$ and $(4, -2.3, 0)$. Here we considered the following Lipschitz continuous radius function

$$a(\theta_x) := \min_{i=0,1,2} d(x, e_i),$$

(4.7)

where $d(x, e_i)$ is the distance of the point $x$ to the $i$th-edge, computed using barycentric coordinates $\Lambda_0, \Lambda_1, \Lambda_2$:

$$d(x, e_i) := \frac{2\Lambda_i |T|}{|e_i|}, \quad i = 0, 1, 2,$$

(4.8)

with $|T|$ being the area of the equilateral triangle $T$ and $|e_i|$ the length of the $i$th-edge.
Table 4.5 provides the condition numbers for both cases over the equilateral triangle under consideration. We see that the standard choice $\alpha_W = \alpha_W = 0.3$ leads to unsatisfactory results for both preconditioners for (EDP). Since the resulting spectra showed that this could be improved by using other values of $\alpha_W$ and $\alpha_W$, we searched for better ones using the procedure described in Remark 3.5.4. The Table on the right reports the results corresponding to each operator using no approximation and the adjusted values of $\alpha_W$ and $\alpha_W$. However, even after tuning the parameters, it is clear that both preconditioners achieve similar condition numbers for (EDP), while $P_h$ performs well on the case (ENP) (even comparable to how it did on the unit disk, see Table 3.3).

**Figure 4.3.** Eigenvalues distribution for equilateral triangle. Spectrum of standard BI0s is shown in black, while the preconditioned by $P_h$ is in red, and the one corresponding to $\tilde{P}_h$ is in blue. $P_h$ built using shape-aware preconditioner for flat screens.

(a) (EDP). $\alpha_W = 0.05$, $\alpha_W = 0.01$
Chapter 4. Shape-aware Preconditioners

Table 4.5.: Results over equilateral triangle screen with quasi-uniform triangular meshes. $\tilde{F}_h$ built using shape-aware preconditioner for flat screens.

<table>
<thead>
<tr>
<th></th>
<th>$\kappa(D_h^{-1}V_h)$</th>
<th>$\kappa(P_h V_h)$</th>
<th>$\kappa(\tilde{F}_h V_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>74</td>
<td>25.34</td>
<td>28.36</td>
<td>27.55</td>
</tr>
<tr>
<td>296</td>
<td>53.11</td>
<td>30.19</td>
<td>30.45</td>
</tr>
<tr>
<td>1184</td>
<td>107.96</td>
<td>31.73</td>
<td>32.76</td>
</tr>
</tbody>
</table>

Table 4.6.: ACA with Lanczos results over equilateral triangle screen with quasi-uniform triangular meshes. $\tilde{F}_h$ built using shape-aware preconditioner.

<table>
<thead>
<tr>
<th></th>
<th>$\kappa(D_h^{-1}W_h)$</th>
<th>$\kappa(P_h W_h)$</th>
<th>$\kappa(\tilde{F}_h W_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>74</td>
<td>2.30</td>
<td>2.03</td>
<td>1.47</td>
</tr>
<tr>
<td>296</td>
<td>4.63</td>
<td>2.38</td>
<td>1.40</td>
</tr>
<tr>
<td>1184</td>
<td>9.61</td>
<td>2.87</td>
<td>1.40</td>
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</table>

Isosceles triangle

We now consider a right Isosceles triangle with vertices $(-1, -2, 0), (2, -2, 0)$ and $(-1, 1, 0)$. We describe its boundary also by using (4.7). Table 4.7 exhibits the resulting condition number for both cases. As we observed for the equilateral triangle, the standard choice of parameters $\alpha_W = \alpha_{\tilde{\omega}} = 0.3$ gives results that are not as good as expected for the case (EDP). However, for this Isosceles triangle, different parameter choices did not provide a significant improvement on the condition numbers.

Table 4.7.: Results over right isosceles triangle screen with quasi-uniform triangular meshes. $\tilde{F}_h$ built using shape-aware preconditioner for flat screens.

<table>
<thead>
<tr>
<th></th>
<th>$\kappa(D_h^{-1}V_h)$</th>
<th>$\kappa(P_h V_h)$</th>
<th>$\kappa(\tilde{F}_h V_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>22.04</td>
<td>6.66</td>
<td>8.89</td>
</tr>
<tr>
<td>240</td>
<td>46.50</td>
<td>8.90</td>
<td>10.63</td>
</tr>
<tr>
<td>960</td>
<td>94.53</td>
<td>11.36</td>
<td>11.96</td>
</tr>
</tbody>
</table>

(c) (ENP)

<table>
<thead>
<tr>
<th></th>
<th>$\kappa(D_h^{-1}W_h)$</th>
<th>$\kappa(P_h W_h)$</th>
<th>$\kappa(\tilde{F}_h W_h)$</th>
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</thead>
<tbody>
<tr>
<td>60</td>
<td>2.07</td>
<td>1.80</td>
<td>1.25</td>
</tr>
<tr>
<td>240</td>
<td>3.98</td>
<td>2.31</td>
<td>1.33</td>
</tr>
<tr>
<td>960</td>
<td>8.05</td>
<td>2.79</td>
<td>1.36</td>
</tr>
</tbody>
</table>
Figure 4.4.: Eigenvalues distribution for isosceles triangle. Spectrum of standard BIOs is shown in black, while the preconditioned by $P_h$ is in red, and the one corresponding to $P_h$ is in blue. $P_h$ built using shape-aware preconditioner for flat screens.

(a) (EDP). $\alpha_W = 0.3$, $\alpha_{\overline{W}} = 0.1$

Table 4.8.: ACA with Lanczos results over isosceles triangle screen with quasi-uniform triangular meshes. $P_h$ built using shape-aware preconditioner.

(a) (EDP). $\alpha_W = 0.3$, $\alpha_{\overline{W}} = 0.1$

<table>
<thead>
<tr>
<th>N</th>
<th>$D_h^{-1}V_h$</th>
<th>$P_hV_h$</th>
<th>$P_h^2V_h$</th>
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<td>22.04</td>
<td>14</td>
<td>6.51</td>
</tr>
<tr>
<td>240</td>
<td>46.50</td>
<td>21</td>
<td>8.90</td>
</tr>
<tr>
<td>960</td>
<td>94.47</td>
<td>29</td>
<td>11.36</td>
</tr>
<tr>
<td>3840</td>
<td>190.05</td>
<td>41</td>
<td>13.97</td>
</tr>
<tr>
<td>15360</td>
<td>381.03</td>
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<td>16.76</td>
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</table>

(b) (ENP)

<table>
<thead>
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<th>$D_h^{-1}W_h$</th>
<th>$P_hW_h$</th>
<th>$P_h^2W_h$</th>
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</thead>
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<td>$\text{lt}$</td>
<td>$\kappa$</td>
</tr>
<tr>
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<td>5</td>
<td>1.80</td>
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<tr>
<td>15360</td>
<td>33.76</td>
<td>30</td>
<td>5.66</td>
</tr>
</tbody>
</table>


**General triangle**

For our last triangular screen, we deal with a general triangle with vertices \((-1, -2, 0), (2, -2, 0)\) and \((-1, 4)\), such that its angles are \(\pi/2, \pi/3\) and \(\pi/6\). As before, we describe its boundary using the formula (4.7). Table 4.9 shows the condition numbers we obtained. We observe that although the condition numbers are larger, the preconditioners behave similarly to the isosceles triangle and there is no advantage of \(P_h\) over \(P_h^o\) for the case (EDP). Moreover, as illustrated in Figure 4.6, the performance of \(P_h\) does not improve when changing the parameter \(\alpha_W\). Here we again see how the performance of the preconditioner deteriorates when compared to the previous triangular screens, presumably due to larger deformations with respect to the unit disk.

Table 4.9.: Results over general triangle screen with quasi-uniform triangular meshes. \(P_h\) built using shape-aware preconditioner for flat screens.

<table>
<thead>
<tr>
<th></th>
<th>(\kappa(D_h^{-1}V_h))</th>
<th>(\kappa(P_hV_h))</th>
<th>(\kappa(P_hV_h))</th>
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<tr>
<td>32</td>
<td>14.49</td>
<td>14.11</td>
<td>18.39</td>
</tr>
<tr>
<td>128</td>
<td>31.82</td>
<td>20.15</td>
<td>22.18</td>
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<tr>
<td>512</td>
<td>70.41</td>
<td>27.10</td>
<td>28.73</td>
</tr>
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</table>

(a) (EDP). \(\alpha_W = 0.3\)

<table>
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<th>(\kappa(P_hV_h))</th>
<th>(\kappa(P_hV_h))</th>
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<td>18.39</td>
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<tr>
<td>512</td>
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<td>27.08</td>
<td>28.73</td>
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(b) (EDP). \(\alpha_W = 0.4, \alpha_W = 0.3\)

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<td>1.29</td>
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<tr>
<td>128</td>
<td>2.59</td>
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<tr>
<td>512</td>
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<td>2.02</td>
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Table 4.10.: ACA with Lanczos results over general triangle screen with quasi-uniform triangular meshes. \(P_h\) built using shape-aware preconditioner for flat screens.

<table>
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<tr>
<th></th>
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<tr>
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<td>27.05</td>
<td>19</td>
</tr>
<tr>
<td>2048</td>
<td>147.34</td>
<td>34.23</td>
<td>21</td>
</tr>
<tr>
<td>8192</td>
<td>300.20</td>
<td>41.42</td>
<td>24</td>
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</table>

(a) (EDP). \(\alpha_W = 0.4, \alpha_W = 0.3\)

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<td>16</td>
</tr>
<tr>
<td>512</td>
<td>70.93</td>
<td>27.05</td>
<td>19</td>
</tr>
<tr>
<td>2048</td>
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<td>34.23</td>
<td>21</td>
</tr>
<tr>
<td>8192</td>
<td>300.20</td>
<td>41.42</td>
<td>24</td>
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(b) (ENP)

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<th>(\kappa(P_hV_h))</th>
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</thead>
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<tr>
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<tr>
<td>8192</td>
<td>300.20</td>
<td>41.42</td>
<td>24</td>
</tr>
</tbody>
</table>

Figure 4.5.: Spectrum for \(W_h\) over general triangle.
Figure 4.6: Eigenvalue distribution for $V_h$ over general triangle. The spectrum of matrix $D_h^{-1}V_h$ is shown in black, while that preconditioned by $W_h$ is depicted in red. The one corresponding to $W_h$ is in blue.

$\alpha_W = 0.4, \alpha_W = 0.2$

$\alpha_W = 0.4, \alpha_W = 0.4$

$\alpha_W = 0.4, \alpha_W = 0.3$
Rectangles with different aspect ratio

As last example, we study three rectangles \([−n, n] \times [−1, 1]\), with \(n = 2, 5, 10\) using the radius function

\[
a(\theta) := \begin{cases} 
\frac{n}{\cos \theta}, & -t_1 \leq \theta < t_1 \\
\frac{1}{\sin \theta}, & t_1 \leq \theta < t_2 \\
\frac{n}{\cos (\theta - \pi)}, & t_2 \leq \theta \leq \pi \\
\frac{1}{\cos (\theta + \pi/2)}, & -t_2 \leq \theta < -t_1 \\
\frac{n}{\cos (\theta + \pi)}, & -\pi \leq \theta < -t_2
\end{cases} ,
\]

(4.9)

where \(t_1 = \tan^{-1}(1/n), \ t_2 = \pi - t_1\).

Tables 4.11–4.13 display the obtained results using BETL2 with dense matrices. As expected, the performance of both preconditioners worsens when the aspect ratio increases. Indeed, our preconditioner although \(h\)-independent, fails for \(V_h\) over the rectangle \([−10, 10] \times [−1, 1]\).

Table 4.11.: Results over rectangle 2:1 screen with quasi-uniform triangular meshes. \(P_h\) built using shape-aware preconditioner for flat screens.

<table>
<thead>
<tr>
<th>(N)</th>
<th>(D_h^{-1}V_h)</th>
<th>(P_hV_h)</th>
<th>(P_hV_h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\kappa)</td>
<td>(\text{It})</td>
<td>(\kappa)</td>
<td>(\text{It})</td>
</tr>
<tr>
<td>86</td>
<td>44.92</td>
<td>16</td>
<td>4.73</td>
</tr>
<tr>
<td>344</td>
<td>111.28</td>
<td>23</td>
<td>5.73</td>
</tr>
<tr>
<td>1376</td>
<td>251.52</td>
<td>32</td>
<td>6.89</td>
</tr>
</tbody>
</table>

Table 4.12.: Results over rectangle 5:1 screen with quasi-uniform triangular meshes. \(P_h\) built using shape-aware preconditioner for flat screens.

<table>
<thead>
<tr>
<th>(N)</th>
<th>(D_h^{-1}V_h)</th>
<th>(P_hV_h)</th>
<th>(P_hV_h)</th>
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</thead>
<tbody>
<tr>
<td>(\kappa)</td>
<td>(\text{It})</td>
<td>(\kappa)</td>
<td>(\text{It})</td>
</tr>
<tr>
<td>206</td>
<td>63.71</td>
<td>21</td>
<td>35.57</td>
</tr>
<tr>
<td>824</td>
<td>165.36</td>
<td>29</td>
<td>36.55</td>
</tr>
<tr>
<td>3296</td>
<td>378.46</td>
<td>41</td>
<td>36.98</td>
</tr>
</tbody>
</table>

Table 4.13.: Results over rectangle 10:1 screen with quasi-uniform triangular meshes. \(P_h\) built using shape-aware preconditioner for flat screens.

<table>
<thead>
<tr>
<th>(N)</th>
<th>(D_h^{-1}V_h)</th>
<th>(P_hV_h)</th>
<th>(P_hV_h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\kappa)</td>
<td>(\text{It})</td>
<td>(\kappa)</td>
<td>(\text{It})</td>
</tr>
<tr>
<td>426</td>
<td>73.54</td>
<td>25</td>
<td>201.35</td>
</tr>
<tr>
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<td>193.93</td>
<td>35</td>
<td>203.93</td>
</tr>
<tr>
<td>6816</td>
<td>443.79</td>
<td>48</td>
<td>204.68</td>
</tr>
</tbody>
</table>
4.2. Parametrized Screens

In this Section, we consider open surfaces $\Gamma$ defined by a $C^1$-diffeomorphism $\phi : \overline{D}_1 \to \Gamma$.

4.2.1. Approximate Inverses

We again start our derivation from the situation on disks. Recall from (4.1) the kernel $k_V$ of $V$ over the disk $D_a$. Since $\phi : \{D_1 \to D_a \hat{x} \mapsto \hat{a} \hat{x}$, $\phi^{-1} : \{D_a \to D_1 \hat{x} \mapsto \hat{x}/a$,

$S_a(x, y)$ can be rephrased over $D_1$ as

$$S_\phi(\hat{x}, \hat{y}) := \tan^{-1} \left( \frac{\sqrt{\phi(\frac{2}{\pi a})}^2 - \|\phi(\hat{x})\|^2}{\sqrt{g_{\phi}(\hat{x})g_{\phi}(\hat{y})} \|\phi(\hat{x}) - \phi(\hat{y})\|^2} \right), \quad (4.10)$$

for $\hat{x} \neq \hat{y}$, and where $g_{\phi}(\hat{x})$ is the Gram determinant of $\phi$ on $\hat{x}$. This expression is somehow analogous to the approach developed in subsection 4.1.1 with $a(\theta_x) = \phi(\frac{\hat{x}}{\|\hat{x}\|})$.

On the other hand, we could map from $D_a$ to $D_1$ instead, since $S_1(x, y)$ can be rewritten as

$$S_1(x, y) = \tan^{-1} \left( \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \frac{a}{\|x - y\|} \right), \quad x \neq y, \quad (4.11)$$

and recast as

$$S_\phi(x, y) := \tan^{-1} \left( \sqrt{1 - \|\phi^{-1}(x)\|^2} \sqrt{1 - \|\phi^{-1}(y)\|^2} \frac{a}{\|x - y\|} \right), \quad (4.12)$$

for $x \neq y$, and where $g_{\phi^{-1}}(x)$ is the Gram determinant of $\phi^{-1}$ on $x$.

Then, we build approximate $\nabla$ and $\nabla$, for which $S_1(x, y)$ is replaced by $S_\phi^{-1}(x, y)$.

4.2.2. Shape-aware Preconditioning

Case (EDP), (1.55):

Again, we base our preconditioner on an approximate bilinear form for the modified hypersingular operator $\nabla$, which is given by

$$b_{\nabla, \phi}(u, v) := \frac{2}{\pi^2} \int_{\Gamma} \int_{\Gamma} \frac{S_{\phi}(x, y)}{\|x - y\|^2} \mathbf{curl}_\Gamma u(x) \cdot \mathbf{curl}_\Gamma v(y) \, d\Gamma(x) \, d\Gamma(y)$$

$$+ a_{\nabla}(u, 1)_\Gamma (v, 1)_\Gamma \quad (4.13)$$

for $u, v \in H^{1/2}(\Gamma)$ and with $a_{\nabla} \in \mathbb{R}^+$ bounded.
Case (ENP), (1.59):
We construct our preconditioner for $W$ over $\Gamma$ via an approximate modified weakly singular operator $V$, for which $S_1(x, y)$ is replaced by $S_\phi(x, y)$ in the kernel of (2.1). Concretely, we use
\[ b_{\phi, \Gamma}(\theta, \mu) := \frac{2}{\pi^2} \int_{\Gamma} \int_{\Gamma} S_{\phi}(x, y) \frac{\partial(x) \mu(y)}{\|x - y\|} d\Gamma(x) d\Gamma(y) \] (4.14)
for $\theta, \mu \in H^{-1/2}(\Gamma)$.

Remark 4.2.1. This approach has the advantage of enforcing axisymmetry and being directly implementable on the given mesh $\Gamma_h$.

Table 4.14 provides the summary of the shape-aware preconditioning strategy for parametrized screens (see Theorem 3.1.1 for notation).

<table>
<thead>
<tr>
<th>BVP</th>
<th>$a_b^{W, \Gamma}$</th>
<th>$b_{\phi, \Gamma}$</th>
<th>$X$</th>
<th>$Y$</th>
<th>$X_h$</th>
<th>$Y_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(EDP)</td>
<td>$b_{\phi, \Gamma}$</td>
<td>$H^{-1/2}(\Gamma)$</td>
<td>$H^{1/2}(\Gamma)$</td>
<td>$S^{-1,0}(\Gamma_h)$</td>
<td>$S^{0,1}(\Gamma_h)$</td>
<td></td>
</tr>
<tr>
<td>(ENP)</td>
<td>$b_{\phi, \Gamma}$</td>
<td>$H^{1/2}(\Gamma)$</td>
<td>$H^{-1/2}(\Gamma)$</td>
<td>$S^{0,1}(\Gamma_h)$</td>
<td>$S^{-1,0}(\Gamma_h)$</td>
<td></td>
</tr>
</tbody>
</table>

4.2.3. Numerical results for Laplace BIEs

We finally consider the same setting as in subsection 4.1.3 but now $P_h$ follows the cue from subsection 4.2.2 with parameters $\alpha_W = \alpha_{\phi} = 0.3$ in the case (EDP) for the sake of simplicity.

For comparison purposes, Tables 4.15, 4.16, and 4.18 illustrate the results obtained for the same mappings studied in subsection 3.5.2. From this, we validate our results for the two first columns, and, additionally, remark that the results obtained by $P_h$ are fairly similar to those achieved with the unit disk based preconditioner in subsection 3.5.2. Indeed, the data in Tables 4.15–4.19 reveals that our shape-aware preconditioners $P_h$ achieve condition numbers $\kappa$ that are practically $h$-independent. Still, this optimality in terms of condition numbers does not meaningfully reduce the number of PCG iterations when compared to the opposite-order preconditioner $P_h$.

Nevertheless, the approximation proposed here is sometimes successful in lowering the bound for the condition number, but in other situations it is not much better than the unit disk based preconditioner.

Table 4.15.: Results over parametrized screens with quasi-uniform triangular meshes. $P_h$ built using shape-aware preconditioner. $\phi(x) = (x_1, x_2, x_1 + x_2)^T$.

| | (a) (EDP) | \begin{tabular}{cc|cc|cc}
<table>
<thead>
<tr>
<th>N</th>
<th>$D_h^{-1}V_h$</th>
<th>$P_hV_h$</th>
<th>$P_hW_h$</th>
<th>$P_hW_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa$</td>
<td>$\text{It}$</td>
<td>$\kappa$</td>
<td>$\text{It}$</td>
</tr>
<tr>
<td>64</td>
<td>24.15</td>
<td>14</td>
<td>3.17</td>
<td>7</td>
</tr>
<tr>
<td>256</td>
<td>51.67</td>
<td>22</td>
<td>4.17</td>
<td>8</td>
</tr>
<tr>
<td>1024</td>
<td>108.94</td>
<td>30</td>
<td>4.92</td>
<td>9</td>
</tr>
<tr>
<td>4096</td>
<td>223.40</td>
<td>43</td>
<td>5.76</td>
<td>10</td>
</tr>
<tr>
<td>16384</td>
<td>435.05</td>
<td>58</td>
<td>6.70</td>
<td>12</td>
</tr>
</tbody>
</table>

| | (b) (ENP) | \begin{tabular}{cc|cc|cc}
<table>
<thead>
<tr>
<th>N</th>
<th>$D_h^{-1}W_h$</th>
<th>$P_hW_h$</th>
<th>$P_hW_h$</th>
<th>$P_hW_h$</th>
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<td></td>
<td>$\kappa$</td>
<td>$\text{It}$</td>
<td>$\kappa$</td>
<td>$\text{It}$</td>
</tr>
<tr>
<td>64</td>
<td>2.69</td>
<td>7</td>
<td>2.16</td>
<td>5</td>
</tr>
<tr>
<td>256</td>
<td>5.39</td>
<td>10</td>
<td>2.30</td>
<td>5</td>
</tr>
<tr>
<td>1024</td>
<td>11.03</td>
<td>15</td>
<td>2.72</td>
<td>6</td>
</tr>
<tr>
<td>4096</td>
<td>22.64</td>
<td>23</td>
<td>3.15</td>
<td>7</td>
</tr>
<tr>
<td>16384</td>
<td>43.61</td>
<td>33</td>
<td>3.64</td>
<td>8</td>
</tr>
</tbody>
</table>
Table 4.16.: Results over parametrized screens with quasi-uniform triangular meshes. $\mathbf{P}_h$ built using shape-aware preconditioner. $\phi(x) = (x_1, x_2, x_1^2 + x_2^2)^T$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
<th>$\mathbf{P}_h V_h$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
<th>$\mathbf{P}_h V_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>24.33 13</td>
<td>1.89 6</td>
<td>2.24 6</td>
<td>64</td>
<td>3.92 7</td>
<td>2.18 5</td>
</tr>
<tr>
<td>256</td>
<td>53.62 21</td>
<td>2.19 6</td>
<td>2.72 7</td>
<td>256</td>
<td>8.40 11</td>
<td>2.08 5</td>
</tr>
<tr>
<td>1024</td>
<td>111.02 30</td>
<td>2.57 7</td>
<td>2.92 8</td>
<td>1024</td>
<td>17.48 16</td>
<td>2.46 6</td>
</tr>
<tr>
<td>4096</td>
<td>228.16 41</td>
<td>3.06 8</td>
<td>3.05 8</td>
<td>4096</td>
<td>36.32 24</td>
<td>2.89 6</td>
</tr>
<tr>
<td>16384</td>
<td>462.61 60</td>
<td>3.70 8</td>
<td>3.16 8</td>
<td>16384</td>
<td>73.67 35</td>
<td>3.38 7</td>
</tr>
</tbody>
</table>

Table 4.17.: Results over parametrized screens with quasi-uniform triangular meshes. $\mathbf{P}_h$ built using shape-aware preconditioner. $\phi(x) = (x_1, x_2, \frac{x_1^2 + x_2^2}{x_1})^T$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
<th>$\mathbf{P}_h V_h$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
<th>$\mathbf{P}_h V_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>23.20 13</td>
<td>2.92 7</td>
<td>2.46 7</td>
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<td>2.67 6</td>
<td>2.03 5</td>
</tr>
<tr>
<td>256</td>
<td>50.93 21</td>
<td>3.41 8</td>
<td>2.74 7</td>
<td>256</td>
<td>5.57 9</td>
<td>2.08 5</td>
</tr>
<tr>
<td>1024</td>
<td>105.40 27</td>
<td>4.01 8</td>
<td>2.87 8</td>
<td>1024</td>
<td>11.51 14</td>
<td>2.42 5</td>
</tr>
<tr>
<td>4096</td>
<td>213.11 40</td>
<td>4.74 9</td>
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<td>23.85 21</td>
<td>2.81 6</td>
</tr>
<tr>
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<td>427.56 59</td>
<td>5.66 10</td>
<td>3.12 8</td>
<td>16384</td>
<td>48.41 31</td>
<td>3.23 7</td>
</tr>
</tbody>
</table>

Table 4.18.: Results over parametrized screens with quasi-uniform triangular meshes. $\mathbf{P}_h$ built using shape-aware preconditioner. $\phi(x) = (x_1, x_2, x_1 x_2)^T$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
<th>$\mathbf{P}_h V_h$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
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<tbody>
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<td>64</td>
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<td>2.30 5</td>
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<tr>
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<td>104.54 29</td>
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</tr>
<tr>
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<td>211.33 42</td>
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<td>3.26 8</td>
<td>4096</td>
<td>19.98 21</td>
<td>3.14 6</td>
</tr>
<tr>
<td>16384</td>
<td>423.26 57</td>
<td>6.18 11</td>
<td>3.43 9</td>
<td>16384</td>
<td>40.30 31</td>
<td>3.63 7</td>
</tr>
</tbody>
</table>

Table 4.19.: Results over parametrized screens with quasi-uniform triangular meshes. $\mathbf{P}_h$ built using shape-aware preconditioner. $\phi(x) = (x_1, x_2, \exp(\frac{x_1^2 + x_2^2}{x_1}))^T$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
<th>$\mathbf{P}_h V_h$</th>
<th>$D^{-1}_h V_h$</th>
<th>$P_h V_h$</th>
<th>$\mathbf{P}_h V_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>24.02 14</td>
<td>3.47 8</td>
<td>2.66 8</td>
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</tr>
<tr>
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<td>2.63 6</td>
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<tr>
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<td>3.31 8</td>
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<td>3.05 7</td>
</tr>
<tr>
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<td>6.64 12</td>
<td>3.47 9</td>
<td>16384</td>
<td>41.63 32</td>
<td>3.53 7</td>
</tr>
</tbody>
</table>
Part II.

Electromagnetic Scattering at Screens


5. Preliminaries

"Science is made up of so many things that appear obvious after they are explained."


We now turn our attention to the scattering of time-harmonic electromagnetic waves by screens. As in our study of operator preconditioners for acoustic scattering, we start by summarizing the existing mathematical framework to solve this kind of problems by means of boundary element methods.

Naturally, we keep the notation that was already adopted in the first part of this thesis and subscribe to the same geometrical considerations described in Section 1.1.

5.1. Model Problem

For a given wave number \( k > 0 \) and a screen \( \Gamma \), we consider homogeneous time-harmonic Maxwell equations

\[
\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = 0, \quad \text{in } \Omega = \mathbb{R}^3 \setminus \Gamma
\]

subject to perfect conductor boundary conditions

\[
\n \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = 0 \quad \text{on } \Gamma,
\]

and Silver-Müller radiation conditions at infinity.

\[
\lim_{|x| \to \infty} |x| \left( \nabla \times \mathbf{E}(x) \times \frac{x}{|x|} - ik \mathbf{E}(x) \right) = 0
\]

Since the total field \( \mathbf{E} \) can be decomposed into the scattered and incident fields \( \mathbf{E} = \mathbf{E}^{sc} + \mathbf{E}^{in} \), we actually formulate our problem as:

**Problem 5.1.1.** Find \( \mathbf{E}^{sc} \) such that

\[
\begin{aligned}
\nabla \times \nabla \times \mathbf{E}^{sc} - k^2 \mathbf{E}^{sc} &= 0, \\
\mathbf{n} \times (\mathbf{E}^{sc} \times \mathbf{n}) &= -\mathbf{n} \times (\mathbf{E}^{in} \times \mathbf{n}) \quad \text{on } \Gamma, \\
\lim_{|x| \to \infty} |x| \left( \nabla \times \mathbf{E}^{sc}(x) \times \frac{x}{|x|} - ik \mathbf{E}^{sc}(x) \right) &= 0
\end{aligned}
\]

5.2. Spaces, Traces and Jump Operators

It is time to present the mathematical tools we will need to solve Problem 5.1.1 using BEM. We refer to subsection 1.3.1 for the introduction of the scalar Sobolev spaces used we are going to use and proceed to present the notation for the additional spaces we will need throughout this part of the thesis. We will use boldface for vector fields and related function spaces to distinguish them from the scalar ones.

Let \( \mathcal{O} \subset \mathbb{R}^d, d = 2, 3 \) be open and \( s \in \mathbb{R} \), we also consider vector Sobolev spaces \( \mathbf{H}^s(\mathcal{O}) := (H^s(\mathcal{O}))^d \) and \( \mathbf{H}^s(\partial \mathcal{O}) := (H^s(\mathcal{O}))^d \) in \( \mathbb{R}^d \), with \( d \) depending on \( \mathcal{O} \). They are Hilbert spaces and their inner product is given by the sum of the inner products of their components. In other words

\[
(u, v)_{\mathbf{H}^s(\mathcal{O})} := \sum_{i=1}^{d} (u_i, v_i)_{H^s(\mathcal{O})},
\]
Chapter 5. Preliminaries

where \( u_i (v_i) \) denote the \( i \)-th component of the vector valued function \( u (v) \).

In the particular case that \( s = 0 \) in the latter, we write \( L^2 (O) \) and use \( \langle \cdot , \cdot \rangle_O \) also to denote the \( L^2 \)-induced bilinear form and duality pairing. We admit this slight abuse of notation as the context will clearly distinguish it from the scalar case \( L^2 (O) \).

The following spaces are also relevant in the study of Maxwell’s equations:

\[
H(\text{curl}, O) := \{ u \in L^2 (O) : \text{curl} u \in L^2 (O) \}, \tag{5.6}
\]

\[
H(\text{div}, O) := \{ u \in L^2 (O) : \text{div} u \in L^2 (O) \}, \tag{5.7}
\]

\[
H_{\text{loc}}(\text{curl}, O) := \{ u \in \left( L^2_{\text{loc}} (O) \right)^3 : \text{curl} u \in \left( L^2_{\text{loc}} (O) \right)^3 \}. \tag{5.8}
\]

In the following we address the boundary spaces and operators that are relevant to treat Maxwell’s equations in the case of screens. We remind the reader that in this thesis we denote the screen by \( \Gamma \) and define our domain as \( \Omega = \mathbb{R}^3 \setminus \Gamma \).

Before introducing new spaces, we point out that in the case of Lipschitz screens and when \( s = 1/2 \), the vector Sobolev spaces defined above satisfy duality relations analogous to the scalar case, i.e.

\[
\tilde{H}^{-1/2} (\Gamma) \equiv \left( H^{1/2} (\Gamma) \right)' \quad \text{and} \quad H^{-1/2} (\Gamma) \equiv \left( \tilde{H}^{1/2} (\Gamma) \right)', \tag{5.9}
\]

now with \( L^2 (\Gamma) \) as pivot space.

That being said, let us begin by mentioning that an element of \( L^p (\Gamma) := \left( L^p (\Gamma) \right)^3 \) for \( 1 \leq p \leq \infty \) can be split into the sum of a tangential and a normal vector-field. To that end, we introduce the operator

\[
\pi_t : \begin{cases} L^p (\Gamma) & \rightarrow L^p (\Gamma), \\
\quad u(x) & \mapsto u(x) - (u(x) \cdot n(x)) n(x)
\end{cases} \tag{5.10}
\]

which is a continuous projector with norm 1. Moreover, we use this projector to define the subspaces

\[
L^p_t (\Gamma) := \pi_t (L^p (\Gamma)), \quad L^p_n (\Gamma) := \ker (\pi_t), \tag{5.11, 5.12}
\]

that satisfy the following direct sum

\[
L^p (\Gamma) = L^p_t (\Gamma) \oplus L^p_n (\Gamma). \tag{5.13}
\]

We will mostly work with the particular case \( p = 2 \), where \( \pi_t \) is an orthogonal projector and (5.13) is an orthogonal sum. Additionally, we can characterize the space of tangential square-integrable vector fields as

\[
L^2_t (\Gamma) := \{ u \in L^2 (\Gamma) \mid u \cdot n = 0 \, \text{a.e. on} \, \Gamma \}, \tag{5.14}
\]

endowed with the \( L^2 \)-inner product.

We continue our journey to define traces and jump operators on \( \Gamma \). For this, we proceed in a similar manner as we did in the acoustic case and discuss two possible approaches to introduce these concepts, given that they both provide valuable insight regarding these objects on screens and how they differ from the usual closed surface case. The first and more standard idea is to consider a Lipschitz domain \( \Omega_c \) such that \( \Gamma \subset \partial \Omega_c \) and use the standard definitions of trace operators on \( \partial \Omega_c \) combined with the restriction operator from \( \partial \Omega_c \) to \( \Gamma \) (see for example [1, 21, 17, 18]), however, this procedure is only valid for globally orientable screens and for Maxwell equations it additionally requires that \( \Omega_c \) is at least polyhedral Lipschitz. On the other hand, the second approach we present is based on [16], and it has the advantage of applying to any Lipschitz screen, although it is slightly more involved.
I. Standard definitions (assuming \( \Gamma \) is a part of a closed surface)

As discussed previously, our geometrical assumptions allow us to find an artificial domain \( \Omega_c \) such that \( \Gamma \subset \partial \Omega_c \). The case when \( \Omega_c \) is a smooth domain is discussed in [1, 21]. Here, we will follow [17, 18] and consider \( \Omega_c \) to be a Lipschitz polyhedron.

Let us split the boundary \( \partial \Omega_c \) in \( N \) (open) faces \( (\partial \Omega^j_c)_{j=1}^N \), with \( \partial \Omega_c = \bigcup_{j=1}^N \partial \Omega^j_c \).

**Assumption 5.2.1.** In this particular subsection, we further assume that \( \hat{\Gamma} \) is a collection of closed faces of \( \partial \Omega \) such that \( \hat{\Gamma} \) is connected.

Let us set

\[
H_{1/2}^{1/2}(\partial \Omega_c) := \{ u \in L^2_t(\partial \Omega_c) : u|_{\partial \Omega_c^\ell} \in H^{1/2}(\partial \Omega_c^\ell) \},
\]

and define the interior and exterior tangential trace operators \(^1\)

\[
\begin{align*}
\hat{\gamma}^+_t : (\mathcal{D}(\Omega_c))^3 &\to H^{1/2}_t(\partial \Omega_c) &\hat{\gamma}^-_t : (\mathcal{D}(\Omega_c))^3 &\to H^{-1/2}_t(\partial \Omega_c) \\
\hat{\gamma}^+_x : (\mathcal{D}(\Omega_c))^3 &\to H^{1/2}_x(\partial \Omega_c) &\hat{\gamma}^-_x : (\mathcal{D}(\Omega_c))^3 &\to H^{-1/2}_x(\partial \Omega_c) \\
\end{align*}
\]

and the interior and exterior twisted tangential trace operators

\[
\begin{align*}
\hat{\gamma}^+_t : (\mathcal{D}(\Omega_c))^3 &\to H^{1/2}_t(\partial \Omega_c) &\hat{\gamma}^-_t : (\mathcal{D}(\Omega_c))^3 &\to H^{-1/2}_t(\partial \Omega_c) \\
\hat{\gamma}^+_x : (\mathcal{D}(\Omega_c))^3 &\to H^{1/2}_x(\partial \Omega_c) &\hat{\gamma}^-_x : (\mathcal{D}(\Omega_c))^3 &\to H^{-1/2}_x(\partial \Omega_c) \\
\end{align*}
\]

where \( \hat{\gamma}^\pm_0 \) are the Dirichlet trace introduced in (1.25) acting component-wise.

We write

\[
H^{1/2}_t(\partial \Omega_c) := \hat{\gamma}^+_t(H^1(\Omega_c)), \quad H^{1/2}_x(\partial \Omega_c) := \hat{\gamma}^+_x(H^1(\Omega_c)),
\]

and remark that in the case of non-smooth \( \partial \Omega_c \), these spaces are different from each other and do not coincide with the usual spaces of tangential surface vector fields. Indeed, sloppily speaking, \( H^{1/2}_t(\partial \Omega_c) \) contains tangential surface vector fields that satisfy a kind of “weak tangential continuity”, while the tangential surface vector fields in \( H^{1/2}_x(\partial \Omega_c) \) fulfill a corresponding ”weak normal continuity” [55, Sect. 2].

We consider the spaces

\[
\begin{align*}
H^{-1/2}(\text{div}_{\partial \Omega_c}, \partial \Omega_c) := \{ u \in H^{-1/2}(\partial \Omega_c) : \text{div}_{\partial \Omega_c} u \in H^{-1/2}(\partial \Omega_c) \}, \\
H^{-1/2}(\text{curl}_{\partial \Omega_c}, \partial \Omega_c) := \{ u \in H^{-1/2}(\partial \Omega_c) : \text{curl}_{\partial \Omega_c} u \in H^{-1/2}(\partial \Omega_c) \},
\end{align*}
\]

and note that

\[
\begin{align*}
\hat{\gamma}^+_t : H(\text{curl}, \Omega_c) &\to H^{-1/2}(\text{div}_{\partial \Omega_c}, \partial \Omega_c), &\hat{\gamma}^-_t : H(\text{curl}, \Omega_c) &\to H^{1/2}(\text{div}_{\partial \Omega_c}, \partial \Omega_c), \\
\hat{\gamma}^+_x : H(\text{curl}, \Omega_c) &\to H^{-1/2}(\text{curl}_{\partial \Omega_c}, \partial \Omega_c), &\hat{\gamma}^-_x : H(\text{curl}, \Omega_c) &\to H^{1/2}(\text{curl}_{\partial \Omega_c}, \partial \Omega_c),
\end{align*}
\]

are likewise linear, continuous [17, Theorem 3.9–3.10], [1, Theorem 2.2] and surjective [18, Theorem 5.4].

Now, let us use the above to actually introduce the pieces necessary to properly work with screens. First, we recall the extension operator \( \widetilde{Z} \) from (1.11) and define the following spaces on \( \Gamma \)

\[
\begin{align*}
H^{1/2}_{t}(\Gamma) := \{ u = v|_\Gamma : v \in H^{1/2}_t(\partial \Omega_c) \}, \\
H^{1/2}_{x}(\Gamma) := \{ u = v|_\Gamma : v \in H^{1/2}_x(\partial \Omega_c) \},
\end{align*}
\]

and define their corresponding tilde spaces

\[
\begin{align*}
\tilde{H}^{1/2}_{t}(\Gamma) := \{ u \in H^{1/2}_{t}(\Gamma) : Z u \in H^{1/2}_t(\partial \Omega_c) \}, \\
\tilde{H}^{1/2}_{x}(\Gamma) := \{ u \in H^{1/2}_{x}(\Gamma) : Z u \in H^{1/2}_x(\partial \Omega_c) \},
\end{align*}
\]

\(^1\)These are often referred as tangential components traces, see for example [17, 55].
\[ \tilde{H}^{1/2}(\Gamma) := \{ u \in H^{1/2}_x(\Gamma) : Z u \in H^{1/2}_x(\partial \Omega_c) \}, \quad (5.24) \]

and their dual spaces
\[ \tilde{H}^{-1/2}(\Gamma) = \left( \tilde{H}^{1/2}(\Gamma) \right)', \quad H^{-1/2}(\Gamma) = \left( H^{1/2}(\Gamma) \right)', \quad (5.25) \]
\[ \tilde{H}_x^{-1/2}(\Gamma) = \left( \tilde{H}_x^{1/2}(\Gamma) \right)', \quad H_x^{-1/2}(\Gamma) = \left( H_x^{1/2}(\Gamma) \right)', \quad (5.26) \]

with \( L^2(\Gamma) \) as pivot space.

**Remark 5.2.2.** We point out that one can also define the spaces \( H^s(\Gamma), \tilde{H}^s(\Gamma), \tilde{H}_x^s(\Gamma) \) and \( H_x^s(\Gamma) \) for \( 0 \leq s \leq 1 \), although they are all equivalent when \( 0 \leq s < 1/2 \). Moreover, in analogy to the scalar case, one can characterize these spaces for \( 0 < s < 1 \) by interpolation [8, Sect. 3].

Then we can introduce the tangential and twisted trace operators for \( \Gamma \) by
\[ \gamma_t^\pm U = \tilde{\gamma}_t^\pm U|_\Gamma, \quad \gamma_x^\pm U = \tilde{\gamma}_x^\pm U|_\Gamma, \quad (5.27) \]
which define continuous and surjective linear mappings [17, Theorem 3.15–3.16]
\[ \gamma_t^+: H(\text{curl}, \Omega_c) \to H^{-1/2}(\text{div}_\Gamma, \Gamma), \quad \gamma_x^+: H(\text{curl}, \Omega_c) \to H^{-1/2}(\text{curl}_\Gamma, \Gamma), \]
\[ \gamma_t^-: H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \Omega_c) \to H^{-1/2}(\text{div}_\Gamma, \Gamma), \quad \gamma_x^-: H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \Omega_c) \to H^{-1/2}(\text{curl}_\Gamma, \Gamma), \]

with
\[ H^{-1/2}(\text{div}_\Gamma, \Gamma) := \{ u \in H^{-1/2}_x(\Gamma) : \text{div}_\Gamma u \in H^{-1/2}(\Gamma) \}, \quad (5.28) \]
\[ H^{-1/2}(\text{curl}_\Gamma, \Gamma) := \{ u \in H^{-1/2}_x(\Gamma) : \text{curl}_\Gamma u \in H^{-1/2}(\Gamma) \}. \quad (5.29) \]

In addition, we present the associated dual spaces (with respect to \( L^2(\Gamma) \))
\[ \tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) = \left( \tilde{H}^{1/2}(\text{curl}_\Gamma, \Gamma) \right)', \quad (5.30) \]
\[ \tilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) := \left( \tilde{H}^{1/2}(\text{div}_\Gamma, \Gamma) \right)', \quad (5.31) \]

which can also be characterized as [21, Sect. 2]
\[ \tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) = \{ u \in H^{-1/2}(\text{div}_\Gamma, \Gamma) : Z u \in H^{-1/2}(\text{div}_{\partial \Omega_c}, \partial \Omega_c) \}, \quad (5.32) \]
and [27, p. 42]
\[ \tilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) = \{ u \in H^{-1/2}(\text{curl}_\Gamma, \partial \Omega_c) : u = 0 \text{ on } \partial \Omega_c \setminus \Gamma \}. \quad (5.33) \]

In order to define the *twisted jump* operator we remark that by construction of the artificial boundary \( \partial \Omega_c \), solutions \( E \) to Problem 5.1.1 satisfy
\[ \tilde{\gamma}_t^+ E|_{\partial \Omega_c \setminus \Gamma} - \tilde{\gamma}_x^+ E|_{\partial \Omega_c \setminus \Gamma} = 0. \quad (5.34) \]

Considering this, we define the twisted jump operator as
\[ [\gamma_x U] := \gamma_t^+ U - \gamma_x^+ U, \quad (5.35) \]
which is a linear continuous mapping from \( H(\text{curl}, \Omega) \) to \( \tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \), as a consequence of (5.34) and [17, Theorem 3.16].

**Remark 5.2.3.** Even though it will not be used in this thesis, we point out that one can analogously build the tangential jump operator \( [\gamma_t] : H(\text{curl}, \Omega) \to \tilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \), which is linear continuous and surjective [17, Rmk. 6.7].
5.2. Spaces, Traces and Jump Operators

Remark 5.2.4. As mentioned before, if $\partial \Omega_c$ were a smooth surface (as in [1]), the spaces $H_{1/2}^t(\partial \Omega_c)$ and $H_{-1/2}^t(\partial \Omega_c)$ would be equivalent [20, Sect. 1], [55, Sect. 2]. Consequently, it would also hold $H_{1/2}^t(\Gamma) \cong H_{-1/2}^t(\Gamma)$ (cf. [1, Sect. 2.3], [21, Sect. 2]). This will also be the case on planar Lipschitz screens such as the unit square.

On the other hand, Assumption 5.2.1 allows for example, for $\Gamma$ such as in Figure 5.1, which is a globally orientable Lipschitz screen and thus considered in this thesis. We note that in such geometries, where two noncoplanar faces meet, we lose continuity of the tangential trace across the junction edge and the distinction between $H_{1/2}^t(\Gamma)$ and $H_{-1/2}^t(\Gamma)$ is unavoidable.

Figure 5.1.: We illustrate a screen $\Gamma^*$ that is admitted by Assumption 5.2.1 and that requires the distinction between $H_{1/2}^t(\Gamma)$ and $H_{-1/2}^t(\Gamma)$. We use black lines to draw the boundary $\partial \Omega_c$ and depict in green the two faces that comprise $\Gamma^*$.

II. More General definitions of trace spaces

Now we present, as an alternative, the more general definitions from [16].

We denote by $\mathcal{D}_t(\mathbb{R}^3)$ the space of smooth compactly supported vector fields on $\mathbb{R}^3$. We are interested in this space because for $U \in \mathcal{D}_t(\mathbb{R}^3)$ one can also consider its Dirichlet trace component-wise, which we write as $\gamma_0 U \in L^\infty_t(\bar{\Gamma})$.

We are now in a position to introduce the **tangential trace** operator $\gamma_t$ as the operator that satisfies

$$
\gamma_t(U) := \left(\pi_t \circ \gamma_0\right)(U) = n \times (U|_\Gamma \times n), \quad \forall U \in (\mathcal{D}(\mathbb{R}^3))^3,
$$

(5.36)

which is a linear continuous map $\gamma_t : \mathcal{D}_t(\mathbb{R}^3) \to L^\infty_t(\bar{\Gamma})$.

Furthermore, these two traces can be extended to Sobolev spaces and for any $s > 0$ satisfy the following commuting diagram [16, Prop. 2.3]:

$$
\begin{array}{c}
H^{3/2+s}(\mathbb{R}^3) \xrightarrow{\text{grad}} H^{1/2+s}(\mathbb{R}^3) \\
\downarrow {\gamma_0} \quad \downarrow {\gamma_t} \\
H^1(\Gamma) \xrightarrow{\text{grad}_t} L^2_t(\Gamma)
\end{array}
$$

and it is natural to consider the **tangential trace space** as

$$
H^{1/2}_t(\Gamma) := \gamma_t(H^1(\mathbb{R}^3)),
$$

(5.37)

together with its $L^2_t(\Gamma)$-dual space $H^{-1/2}_t(\Gamma)$ (c.f. (5.25)).

It remains now to define the jump operator that is relevant in the current setting. With this purpose in mind, we consider $\mathcal{D}^{ slit}_t(\mathbb{R}^3)$ ($\mathcal{D}^{ slit}_t(\mathbb{R}^3)$) as the space of the functions $V \in C^\infty(\Omega)$ ($U \in (C^\infty(\Omega))^3$) with bounded support in $\mathbb{R}^3$ and such that:
Chapter 5. Preliminaries

- \( V(U) \) vanishes on a neighborhood of \( \partial \Gamma \).
- for any \( x \in \Gamma \) there is \( r > 0 \) such that \( B(x, r) \setminus \Gamma \) has exactly two connected components \( B^\pm \) and such that \( V_{|B^\pm} (U_{|B^\pm}) \) both admit compactly supported extensions to \( \mathbb{R}^3 \).

Alternatively, one can write [16, Sect. 1.4]

\[
\mathcal{D}^{slit}_t(\mathbb{R}^3) = \{ \text{Vector fields } U \in \Omega : l \circ U \in \mathcal{D}^{slit}(\mathbb{R}^3), \text{ for all linear form } l \in \mathbb{R}^3 \}. \tag{5.38}
\]

Finally, we write the jump operator for \( U \in \mathcal{D}^{slit}_t(\mathbb{R}^3) \) as

\[
[U]_x := \gamma_0^+ U \times n + \gamma_0^- U \times n, \tag{5.39}
\]

where \( \gamma_0^\pm \) are understood as the Dirichlet traces taken from each side \( \Gamma^\pm \) of the screen \( \Gamma \), and \( n \) is the unit normal pointing outwards from \( \Gamma^+ \).

The jump operator \( [\cdot]_x : \mathcal{D}^{slit}_t(\mathbb{R}^3) \rightarrow L^\infty_t(\Gamma) \) is continuous. Moreover, it can be extended to a continuous operator mapping between the spaces that we need for our boundary integral formulation. In order to state this, we characterize \( \widetilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \) as [16, Def. 1]

\[
\widetilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) := \left\{ u \in \widetilde{H}^{-1/2}(\Gamma) | \text{ div}_\Gamma u \in \widetilde{H}^{-1/2}(\Gamma) \right\}, \tag{5.40}
\]

and recall that its dual space (with respect to \( L^2_t(\Gamma) \)) is \( \widetilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \).

At last, we can present the desired continuous extensions of \( [\cdot]_x \) and \( \gamma_t \) that will give rise to our BIE.

**Theorem 5.2.5** ([16, Theorem 2.14]). The jump operator \( [\cdot]_x \) determines a linear continuous surjection \( H(\text{curl}, \Omega) \rightarrow \widetilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \).

**Corollary 5.2.6** ([16, Cor. 2.15]). The operator \( \gamma_t \) extends uniquely to a continuous operator \( H(\text{curl}, \mathbb{R}^3) \rightarrow \widetilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \). We have \( \forall U \in H(\text{curl}, \mathbb{R}^3) \), \( \forall V \in H(\text{curl}, \Omega) \)

\[
\int_{\mathbb{R}^3} \text{curl} U \cdot V dx + \int_{\mathbb{R}^3} U \cdot \text{curl} V dx = \langle \gamma_t U, [V]_x \rangle_{\Gamma}. \tag{5.41}
\]

We conclude this Section by introducing

\[
\widetilde{H}^{0,-1/2}(\text{div}_\Gamma, \Gamma) := \left\{ u \in L^2_t(\Gamma) | \text{ div}_\Gamma u \in \widetilde{H}^{-1/2}(\Gamma) \right\}, \tag{5.42}
\]

which is a subspace of \( \widetilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \).

**Remark 5.2.7.** Surface differential operators map continuously between ”tilde spaces”. For the sake of clarity, we introduce the notation:

- \( \widetilde{\text{grad}}_\Gamma : \widetilde{H}^{1/2}(\Gamma) \rightarrow \widetilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \)
- \( \text{div}_\Gamma = -(\text{grad}_\Gamma)^* : \widetilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow \widetilde{H}^{-1/2}(\Gamma) \)
- \( \text{grad}_\Gamma : \widetilde{H}^{1/2}(\Gamma) \rightarrow \widetilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \)
- \( \text{div}_\Gamma = -(\text{grad}_\Gamma)^* : \widetilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow \widetilde{H}^{-1/2}(\Gamma) \)
- \( \text{curl}_\Gamma : \widetilde{H}^{1/2}(\Gamma) \rightarrow \widetilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \)
- \( \text{curl}_\Gamma = (\text{curl}_\Gamma)^* : \widetilde{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \rightarrow \widetilde{H}^{-1/2}(\Gamma) \)

where * denotes the adjoint operators with respect to \( L^2_t(\Gamma) \).
5.3. Hodge Decompositions

Now that we have defined our spaces, we can properly assert that the Problem 5.1.1 has a unique solution $E \in H_{loc}(\text{curl}, \Omega)$ [16, Theorem 3.1].

We recall that we plan to find this solution using the BEM machinery in a similar way to what was done for acoustic scattering in the first part of this thesis. Indeed, we aim to solve the variational problem arising from the so-called Electric Field Integral Equation (EFIE), that will be properly defined in the next Section. For now, the important message is that, similarly to the situation of closed surfaces, in order to properly state and understand the EFIE and its properties, we are still missing one mathematical tool: The Hodge decomposition \(^2\) (see for example [20, Theorem 2] for the corresponding Hodge decomposition on closed surfaces).

As usual, these decompositions are slightly different in the case of screens. Moreover, as we will see later on, the EFIE is related to the spaces $\tilde{H}^{-1/2}(\text{div}, \Gamma)$ and $H^{-1/2}(\text{curl}, \Gamma)$ in the case of screens. Thus, we proceed to discuss the Hodge decompositions of these spaces.

Please note that in this thesis we assume $\Gamma$ to have trivial topology (e.g. disk).

5.3.1. $\tilde{H}^{-1/2}(\text{div}, \Gamma)$

We decompose the space $\tilde{H}^{-1/2}(\text{div}, \Gamma)$ as the direct sum of two closed subspaces [16, Sect. 2.4]:

$$\tilde{H}^{-1/2}(\text{div}, \Gamma) = X_0(\Gamma) \oplus X_\perp(\Gamma),$$

with

$$X_0(\Gamma) := \{ v \in \tilde{H}^{-1/2}(\text{div}, \Gamma) : \text{div}_\Gamma v = 0 \} = \text{curl}_\Gamma(\tilde{H}^{1/2}(\Gamma)).$$

and

$$X_\perp(\Gamma) = \text{grad}_\Gamma \mathcal{H}(\Gamma),$$

where

$$\mathcal{H}(\Gamma) := \{ v \in H^1(\Gamma) : \Delta_\Gamma v \in \tilde{H}^{-1/2}(\Gamma) \},$$

$$H^1_0(\Gamma) := \{ v \in H^1(\Gamma) : \langle v, 1 \rangle_\Gamma = 0 \}.$$

Therefore, we can rewrite (5.43) as [18, Theorem 6.4]

$$\tilde{H}^{-1/2}(\text{div}, \Gamma) = \text{curl}_\Gamma(\tilde{H}^{1/2}(\Gamma)) \bigoplus \text{grad}_\Gamma \mathcal{H}(\Gamma).$$

Parametrization of $X_\perp(\Gamma)$

Driven by some technical aspects that will be addressed in Chapter 6, we devote this additional subsection to develop a parametrization for $X_\perp(\Gamma)$ that we can describe explicitly.

Let us consider

$$\tilde{H}^{-1/2}_e(\Gamma) := \{ \varphi \in \tilde{H}^{-1/2}(\Gamma) : \langle \varphi, 1 \rangle_\Gamma = 0 \}. \quad (5.49)$$

We leverage the fact that $\text{div}_\Gamma : \tilde{H}^{-1/2}(\text{div}, \Gamma) \to \tilde{H}^{-1/2}_e(\Gamma)$ is surjective, to introduce the divergence lifting $L : \tilde{H}^{-1/2}_e(\Gamma) \to X_\perp(\Gamma)$ such that

$$\text{div}_\Gamma \circ L = \text{id}. \quad (5.50)$$

\(^2\)sometimes also called Helmholtz decomposition [16, Sect. 2.4].
Hence we can write
\[ X_{\perp}(\Gamma) = L \left( \tilde{H}_s^{-1/2}(\Gamma) \right), \]
where \( L \) can be defined directly as follows
\[ L = - \nabla_{\Gamma} \circ (-\Delta_N)^{-1}, \]
with \((\Delta_N)^{-1} : \tilde{H}^{-1/2}(\Gamma) \to \mathcal{H}(\Gamma)\) denoting the inverse of the Laplace Beltrami operator with Neumann boundary conditions, which is to be understood in the variational sense.

More concretely, one computes \( L v \), for \( v \in \tilde{H}_s^{-1/2}(\Gamma) \) first by solving the variational problem associated to \(-\Delta_N\): seek \( w \in H^2_s(\Gamma)\):
\[ \int_{\Gamma} \nabla_{\Gamma} w(y) \cdot \nabla_{\Gamma} w'(y) d\Gamma(y) = \int_{\Gamma} v(y) w'(y) d\Gamma(y), \quad \forall w' \in H^1_s(\Gamma), \]
and then by evaluating \(- \nabla_{\Gamma} w(= L v)\).

By means of the lifting operator \( L \), we find the following representation
\[ X_{\perp}(\Gamma) = L \left( \tilde{H}_s^{-1/2}(\Gamma) \right) = - \nabla_{\Gamma} \circ (-\Delta_N)^{-1} \tilde{H}_s^{-1/2}(\Gamma). \]

5.3.2. \( H^{-1/2}(\text{curl}_\Gamma, \Gamma) \)
Similarly, the following decomposition is valid \([18, \text{Eq. (53)}]\)
\[ H^{-1/2}(\text{curl}_\Gamma, \Gamma) := Y_0(\Gamma) \bigoplus Y_{\perp}(\Gamma), \]
with
\[ Y_0(\Gamma) = \nabla_{\Gamma} H^{1/2}(\Gamma), \]
\[ Y_{\perp}(\Gamma) = \text{curl}_\Gamma \mathcal{H}_{00}(\Gamma), \]
and
\[ \mathcal{H}_{00}(\Gamma) := \{ u \in H^1_0(\Gamma) : \Delta_\Gamma u \in H^{-1/2}(\Gamma) \}. \]

Since \( H^{-1/2}(\text{curl}_\Gamma, \Gamma) = (\tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma))' \), one can show that
\[ Y_0(\Gamma) = (X_{\perp}(\Gamma))', \]
\[ Y_{\perp}(\Gamma) = (X_0(\Gamma))', \]
with pivot space \( L^2_0(\Gamma) \).

Indeed, take \( y_0 \in Y_0(\Gamma) \) and \( x_\perp \in X_{\perp}(\Gamma) \), we see that \( \langle y_0, x_\perp \rangle_\Gamma \) is well-defined:
\[ \langle y_0, x_\perp \rangle_\Gamma = \langle \nabla_{\Gamma} u_0, L \psi_{\perp} \rangle_\Gamma, \quad u_0 \in H^{1/2}(\Gamma), \psi_{\perp} \in \tilde{H}_s^{-1/2}(\Gamma) \]
\[ = - \langle u_0, \text{div}_\Gamma L \psi_{\perp} \rangle_\Gamma \]
\[ = - \langle u_0, \psi_{\perp} \rangle_\Gamma. \]

Analogously, for any \( y_\perp \in Y_{\perp}(\Gamma) \) and \( x_0 \in X_0(\Gamma) \)
\[ \langle x_0, y_\perp \rangle_\Gamma = \langle \text{curl}_\Gamma v_0, \text{curl}_\Gamma \phi_{\perp} \rangle_\Gamma, \quad v_0 \in \tilde{H}^{1/2}(\Gamma), \phi_{\perp} \in \mathcal{H}_{00}(\Gamma) \]
\[ = - \langle v_0, \text{curl}_\Gamma \phi_{\perp} \rangle_\Gamma \]
\[ = - \langle v_0, \phi^* \rangle_\Gamma, \quad \text{for } v_0 \in \tilde{H}^{1/2}(\Gamma), \phi^* \in H^{-1/2}(\Gamma). \]
5.4. Electric Field Integral Equations on Screens

Recall the single layer potential $\Psi_{\text{SL}}^k$ for $k \geq 0$ introduced in (1.46), which extends to a linear and continuous map from $\tilde{H}^{-1/2}(\Gamma)$ to $H^1_{\text{loc}}(\text{curl}, \Omega)$ [16, Sect. 3.2].

Likewise, we put $\Psi_{\text{SL}}^k$ for the single layer potential acting on tangential fields, which is continuous from $\tilde{H}^{-1/2}(\Gamma) \rightarrow H^1_{\text{loc}}(\mathbb{R}^3)$

Now, combining these two potentials, we introduce the EFIE potential on screens as

$$A(k)\xi = \Psi_{\text{SL}}^k \xi + \frac{1}{k^2} \text{grad} \Psi_{\text{SL}}^k \text{div}_T \xi, \quad k > 0,$$

and note that $A(k)$ is linear and continuous from $\tilde{H}^{-1/2}(\text{div}_T, \Gamma) \rightarrow H^1_{\text{loc}}(\text{curl}, \mathbb{R}^3) \cap H(\text{curl}^2, \mathbb{R}^3 \setminus \Gamma)$.

As usual, we apply the tangential trace to this potential and define the EFIE boundary integral operator for $k > 0$ by

$$A(k)\xi = \gamma_t(A(k)\xi) = V_k\xi + \frac{1}{k^2} \text{grad}_T V_k \text{div}_T \xi,$$

where $V_k$ is the standard weakly singular operator for Helmholtz equation introduced in (1.50) and $V_k$ corresponds to its vectorial form.

It holds that $A(k)$ is a bounded and injective operator mapping from $\tilde{H}^{-1/2}(\text{div}_T, \Gamma)$ to $H^{-1/2}(\text{curl}_T, \Gamma)$. For simplicity, in this second part of this thesis, we will drop the subscript whenever $k = 0$ and just write $V, V$ and $A$.

**Problem 5.4.1 (EFIE).** For $g \in H^{-1/2}(\text{curl}_T, \Gamma)$ and $k > 0$, seek $\xi \in \tilde{H}^{-1/2}(\text{div}_T, \Gamma)$ such that

$$A(k)\xi = g.$$  

**Theorem 5.4.2 ([16, Theorem 3.2]).**

1. Let $E \in H^1_{\text{loc}}(\text{curl}, \Omega)$ solve Problem 5.1.1. If $g = \gamma_t(-E^{\text{int}})$, then $\xi$ solves Problem 5.4.1 if and only if $\xi = [\text{curl}E]_\Gamma$.

2. If $\xi \in \tilde{H}^{-1/2}(\text{div}_T, \Gamma)$ solves Problem 5.4.1 with $g = \gamma_t(-E^{\text{int}})$, then $E = A(k)\xi \in H^1_{\text{loc}}(\text{curl}, \Omega)$ solves Problem 5.1.1.

We now consider the associated bilinear form

$$a^k_{A,T}(\xi, \xi') := \langle A(k)\xi, \xi' \rangle_{\Gamma}$$

$$\quad = \langle V_k\xi, \xi' \rangle_{\Gamma} - \frac{1}{k^2} \langle \text{grad}_T V_k \text{div}_T \xi, \text{div}_T \xi' \rangle_{\Gamma}$$

for $\xi, \xi' \in \tilde{H}^{-1/2}(\text{div}_T, \Gamma)$ and $k > 0$. Then, the symmetric variational formulation of Problem 5.4.1 is:

**Problem 5.4.3 (EFIE Variational Problem).** For $g \in H^{-1/2}(\text{curl}_T, \Gamma)$ and $k \geq 0$, seek $\xi \in \tilde{H}^{-1/2}(\text{div}_T, \Gamma)$ such that

$$a^k_{A,T}(\xi, \xi') = \langle g, \xi' \rangle_{\Gamma},$$

for all $\xi' \in \tilde{H}^{-1/2}(\text{div}_T, \Gamma)$.

It is easy to see that $a^k_{A,T}$ inherits the continuity from $A(k)$ and that it is an indefinite bilinear form. Nevertheless, for the sake of solving via Galerkin BEM, this will not pose a problem as the next $T$-coercivity is verified.

**Theorem 5.4.4 ([16, Theorem 3.4]).** There is a compact bilinear form $b$ such that for all $(\xi_0, \xi_\perp) \in X_0(\Gamma) \times X_\perp(\Gamma)$ we have:

$$\left| (a^k_{A,T} + b)(\xi_0 + \xi_\perp, \xi_0 - \xi_\perp) \right| \geq (1/C) \| \xi_0 + \xi_\perp \|^2_{\tilde{H}^{-1/2}(\text{div}_T, \Gamma)}.$$  

As a consequence, the operator $A(k) : \tilde{H}^{-1/2}(\text{div}_T, \Gamma) \rightarrow \tilde{H}^{-1/2}(\text{curl}_T, \Gamma)$ is Fredholm of index 0, invertible and Problem 5.4.1 has a unique solution.
5.5. Galerkin BEM Numerical Approximation

In the same spirit as Section 1.6, the main focus of this Section is to review the ingredients required to solve Problem 5.4.3 numerically using a lowest order Galerkin BEM discretization.

Let \( \Gamma_h \) be a given mesh of \( \Gamma \) as in Section 3.2. We begin by mentioning the vector valued boundary element spaces we are going to use in our discretization. For \( \mathbf{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \) we consider the curl-conforming lowest order Nédélec edge elements \( \mathcal{N}_0(\Gamma_h) \) as introduced in [45, Sect. 7.2].

Similarly, for \( \tilde{\mathbf{H}}^{-1/2}(\text{div}_\Gamma, \Gamma) \) we take the div-conforming lowest order Raviart-Thomas edge elements \( \mathcal{R}^0(\Gamma_h) \) as defined in [80] with zero normal boundary conditions, which we denote by \( \mathcal{E}_0(\Gamma_h) \).

**Remark 5.5.1.** In analogy to the case of closed surfaces [20, Sect. 2], we can consider a vector field \( \mathbf{u} \) defined on \( \Gamma \) and introduce the rotation operator \( R_x \) defined as \( R_x \mathbf{u} := \mathbf{u} \times \mathbf{n} \).

For any \( \mathbf{u} \in \mathbf{L}^2(\Gamma) \) we have \( \gamma_r(\mathbf{u}) = R_x(\gamma_x(\mathbf{u})) \) and \( \gamma_x(\mathbf{u}) = - R_x(\gamma_r(\mathbf{u})) \), from where it follows that \( R_x^{-1} = - R_x \).

Moreover, \( R_x : \mathbf{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \rightarrow \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \) and \( R_x : \tilde{\mathbf{H}}^{-1/2}(\text{curl}_\Gamma, \Gamma) \rightarrow \tilde{\mathbf{H}}^{-1/2}(\text{div}_\Gamma, \Gamma) \) are isometric isomorphisms. Hence, Nédélec and Raviart-Thomas edge elements are connected by the rotation operator \( R_x \) and thus we can build \( \mathcal{R}^0(\Gamma_h) \) using rotated surface edge elements from \( \mathcal{N}_0(\Gamma_h) \).

Let us use the discrete space \( \mathcal{E}_0(\Gamma_h) \) and define the Galerkin problem for Problem 5.4.3.

**Problem 5.5.2 (EFIE Galerkin Problem).** Find \( \xi_h \in \mathcal{E}_0(\Gamma_h) \) such that for \( \mathbf{g} \in \mathbf{H}^{-1/2}(\text{curl}_\Gamma, \Gamma) \) and \( k \geq 0 \)

\[
\mathbf{a}_k(\xi_h, \xi'_h) = \langle \mathbf{g}, \xi'_h \rangle_{\Gamma}, \quad \forall \xi'_h \in \mathcal{E}_0(\Gamma_h). \tag{5.66}
\]

The family of finite dimensional spaces \( (\mathcal{E}_0(\Gamma_h)) \) approximate \( \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \) in the sense that

\[
\forall \mathbf{u} \in \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \quad \lim_{h \to 0} \sup_{\mathbf{u}' \in \mathcal{E}_0(\Gamma_h)} \inf_{\mathbf{u} \in \mathbf{E}(\Gamma)} \| \mathbf{u} - \mathbf{u}' \|_{\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)} = 0. \tag{5.67}
\]

Moreover, \( \mathcal{E}_0(\Gamma_h) \) can be split into direct sums

\[
\mathcal{E}_0(\Gamma_h) = X_{0,h} \bigoplus X_{\perp,h}
\]

of closed subspaces of \( \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \) that satisfy

\[
\sup_{\mathbf{v}_{0,h} \in X_{0,h}} \inf_{\mathbf{v}_0 \in \mathcal{E}(\Gamma)} \frac{\| \mathbf{v}_0 - \mathbf{v}_{0,h} \|_{\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)}}{\| \mathbf{v}_{0,h} \|_{\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)}} \to 0 \tag{5.68}
\]

\[
\sup_{\mathbf{v}_{\perp,h} \in X_{\perp,h}} \inf_{\mathbf{v}_\perp \in \mathcal{E}(\Gamma)} \frac{\| \mathbf{v}_\perp - \mathbf{v}_{\perp,h} \|_{\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)}}{\| \mathbf{v}_{\perp,h} \|_{\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)}} \to 0 \tag{5.69}
\]

when \( h \to 0 \), and are stable under conjugate conjugation [16, Sect. 4.2].

With these properties one can prove the following stability result that gives existence and uniqueness of solutions to the EFIE Galerkin Problem 5.5.2.

**Proposition 5.5.3 ([16, Cor. 4.2]).** The bilinear form \( \mathbf{a}_k(\cdot, \cdot) \) satisfies the inf-sup condition

\[
\inf_{\xi_h \in \mathcal{E}_0(\Gamma_h)} \sup_{\xi'_h \in \mathcal{E}_0(\Gamma_h)} \left| \frac{\mathbf{a}_k(\xi_h, \xi'_h)}{\| \xi_h \|_{\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)} \| \xi'_h \|_{\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)}} \right| \geq 1/C, \tag{5.70}
\]

on \( (\mathcal{E}_0(\Gamma_h))_{h < h_0} \) for some \( h_0 > 0 \), and with \( C > 0 \) independent of \( h \).

---

\(^3\)Raviart-Thomas edge elements are also known in the electromagnetism literature as Rao-Wilton-Glisson (RWG) basis functions [79].
In addition, we have the next quasi-optimal convergence estimate.

**Theorem 5.5.4** ([16, Theorem 4.5]). Let \( \xi \in \tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \) solve Problem 5.4.3 and \( \xi_h \in \mathcal{E}_0(\Gamma_h) \) be the solution to Problem 5.5.2. Then we have

\[
\|\xi - \xi_h\|_{\tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma)} \leq C_d \inf_{\nu_h \in \mathcal{E}_0(\Gamma_h)} \|\xi - \nu_h\|_{\tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma)},
\]

(5.71)

with \( C_d \) independent of \( h \).

Finally, we wish to state the implied error estimate. Before doing this, however, we need to define the space

\[
X^s(\Gamma) := \left\{ u \in \tilde{H}^s(\Gamma) \mid \tilde{\text{div}}_\Gamma u \in \tilde{H}^s(\Gamma) \text{ and } (u', \text{grad}_\Gamma v')_\Gamma + \left\langle \text{div}_\Gamma u, v \right\rangle_\Gamma = 0 \forall v \in \mathcal{D}(\Gamma) \right\},
\]

(5.72)

for any \( s \in [-1/2, 0) \).

Then, combining Theorem 5.5.4 with the regularity of the solution \( \xi \) of Problem 5.4.3 [29], we can state the following convergence result.

**Theorem 5.5.5** ([16, Theorem 4.5]). Let \( \xi \in \tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \) solve Problem 5.4.3 and \( \xi_h \in \mathcal{E}_0(\Gamma_h) \) be the solution to Problem 5.5.2. Then there exists \( s \in (-1/2, 0) \) such that \( \xi \in X^s(\Gamma) \)

\[
\|\xi - \xi_h\|_{\tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma)} \leq Ch^{1/2+s} \|\xi\|_{X^s(\Gamma)}.
\]

(5.73)
6. Compact-Equivalent Inverse of the EFIE Operator on Disks

“As wir müssen wissen, wir werden wissen”
– David Hilbert (1930).

As discussed for the acoustic part, Calderón-type identities play an important role in operator preconditioning on closed surfaces. Since such relations do not exist for the EFIE on screens, we take a first step towards finding suitable substitutes.

The objective of this Chapter is to develop the ingredients to build an operator \( N(k) \) such that

\[
N(k)A(k) = \text{Id} + K(k), \quad \tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow H^{-1/2}(\text{curl}_\Gamma, \Gamma),
\]

(6.1)

with a compact operator \( K(k) \) that might also depend on the wave number \( k \). Furthermore, as we are interested in this operator for preconditioning purposes, we take pains in this Chapter to show that it fulfills all requirements on the continuous and discrete level to provide a suitable preconditioner.

Our quest to attain these goals will require several steps. On account of this, let us outline the path in front of us. We begin in Section 6.1 by discussing some simplifications to the variational EFIE Problem 5.4.3 that result in a reduced static EFIE operator, which is a compact perturbation of the original one. We then derive \( N \) as an inverse to said reduced EFIE operator on Disks. After a brief discussion in Section 6.2 of the properties of \( N \) in the continuous setting, we dedicate the last Section to derive a formulation of \( N \) that is amenable to conforming discretization on the unit disk.

**Remark 6.0.1.** \( \mathcal{D}_a \) has trivial topology.

6.1. Abstract Construction of \( N \) on \( \Gamma \)

Let us first consider the *positive definite static EFIE* operator \( S \) defined as

\[
S \xi := \mathcal{V} \xi - \text{grad}_\Gamma \mathcal{V} \tilde{\text{div}}_\Gamma \xi,
\]

(6.2)

and note that \( S : \tilde{H}^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) is an isomorphism and that the associated bilinear form supplies an equivalent inner product.

Now, let us plug in the Hodge decomposition \( \xi = \xi_0 + \xi_\perp \) with \( \xi_0 \in X_0(\Gamma) \), \( \xi_\perp \in X_\perp(\Gamma) \), and write

\[
S(\xi_0 + \xi_\perp) = \mathcal{V} \xi_0 + \mathcal{V} \xi_\perp - \text{grad}_\Gamma \mathcal{V} \tilde{\text{div}}_\Gamma \xi_\perp.
\]

(6.3)

We realize that the second term is a compact perturbation, due to the compact embedding \( X_\perp(\Gamma) \subset \tilde{H}^{-1/2}_r(\Gamma) \). Moreover, by neglecting this term, we define the *reduced static EFIE* operator \( \tilde{S} \):

\[
\tilde{S}(\xi_0 + \xi_\perp) = \mathcal{V} \xi_0 - \text{grad}_\Gamma \mathcal{V} \tilde{\text{div}}_\Gamma \xi_\perp,
\]

(6.4)

which is still an isomorphism from \( \tilde{H}^{-1/2}_r(\text{div}_\Gamma, \Gamma) \) to \( H^{-1/2}(\text{curl}_\Gamma, \Gamma) \).

We use \( \tilde{S} \) to introduce the following variational problem.
Problem 6.1.1 (Reduced Static EFIE Variational Problem). For \( g \in H^{-1/2}(\text{curl}_r, \Gamma) \), find \( \xi_0 \in X_0(\Gamma) \) and \( \xi_\perp \in X_\perp(\Gamma) \) such that

\[
\left\langle \tilde{S}_0 \xi_0, \xi_0' \right\rangle_{\Gamma} := \left\langle \mathbf{V} \xi_0, \xi_0' \right\rangle_{\Gamma} = \left\langle g, \xi_0' \right\rangle_{\Gamma}, \quad \forall \xi_0' \in X_0(\Gamma), \tag{6.5}
\]

\[
\left\langle \tilde{S}_\perp \xi_\perp, \xi_\perp' \right\rangle_{\Gamma} := \left\langle \mathbf{V} \text{divr} \xi_\perp, \text{divr} \xi_\perp' \right\rangle_{\Gamma} = \left\langle g, \xi_\perp' \right\rangle_{\Gamma}, \quad \forall \xi_\perp' \in X_\perp(\Gamma). \tag{6.6}
\]

We see that Problem 6.1.1 decouples the part acting in \( X_0(\Gamma) \) from the one in \( X_\perp(\Gamma) \). This allows us to write \( \tilde{S} = \tilde{S}_0 + \tilde{S}_\perp \) with

\[
\tilde{S}_0 : \begin{cases} X_0(\Gamma) & \rightarrow (X_0(\Gamma))', \\ \xi_0 & \mapsto \mathbf{V} \xi_0 \end{cases}, \quad \tilde{S}_\perp : \begin{cases} X_\perp(\Gamma) & \rightarrow (X_\perp(\Gamma))', \\ \xi_\perp & \mapsto -\text{grad}_r \text{divr} \xi_\perp, \end{cases}
\]

and define

\[
N = \tilde{S}^{-1} : H^{-1/2}(\text{curl}_r, \Gamma) \rightarrow \tilde{H}^{-1/2}(\text{divr}, \Gamma),
\]

by \( N = N_0 + N_\perp \), with \( N_0 = \tilde{S}_0^{-1} \) and \( N_\perp = \tilde{S}_\perp^{-1} \).

In other words, given \( g \in H^{-1/2}(\text{curl}_r, \Gamma) \), we can compute \( \xi = N g = N_0 g + N_\perp g \) as follows:

\[ \boxed{N_0 \text{ Find } \xi_0 \in X_0(\Gamma):} \]

\[
\langle \mathbf{V} \xi_0, \xi_0' \rangle_{\Gamma} = \left\langle g, \xi_0' \right\rangle_{\Gamma}, \quad \forall \xi_0' \in X_0(\Gamma). \tag{6.9}
\]

We use the scalar potential representation of \( X_0(\Gamma) \) and solve: \( \varphi \in \tilde{H}^{1/2}(\Gamma) \)

\[
\langle \mathbf{V} \text{curl}_r \varphi, \text{curl}_r \varphi' \rangle_{\Gamma} = \left\langle g, \text{curl}_r \varphi' \right\rangle_{\Gamma}, \quad \forall \varphi' \in \tilde{H}^{1/2}(\Gamma), \tag{6.10}
\]

which corresponds component-wise to the hypersingular boundary integral equation for the Laplacian over the disk. We can therefore use \( W^{-1} : \tilde{H}^{-1/2}(\Gamma) \rightarrow \tilde{H}^{1/2}(\Gamma) \) and express it as

\[
\varphi = W^{-1} \circ \text{curl}_r^* g.
\]

Finally, we conclude that

\[
N_0 = \text{curl}_r \circ W^{-1} \circ (\text{curl}_r^*)^* = \text{curl}_r \circ W^{-1} \circ \text{curl}_r.
\]

\[ \boxed{N_\perp \text{ Find } \xi_\perp \in X_\perp(\Gamma):} \]

\[
\langle \mathbf{V} \text{divr} \xi_\perp, \text{divr} \xi_\perp' \rangle_{\Gamma} = \left\langle g, \xi_\perp' \right\rangle_{\Gamma}, \quad \forall \xi_\perp' \in X_\perp(\Gamma). \tag{6.13}
\]

It is worth noticing that (5.53) allows us to write \( \xi_\perp = L \psi, \ \psi \in \tilde{H}_s^{-1/2}(\Gamma) \) and recast (6.13) as

\[
\langle \mathbf{V} \text{divr} L \psi, \text{divr} L \psi' \rangle_{\Gamma} = \left\langle g, L \psi' \right\rangle_{\Gamma}, \quad \forall \psi' \in \tilde{H}_s^{-1/2}(\Gamma), \tag{6.14}
\]

which reduces to

\[
\langle \mathbf{V} \psi, \psi' \rangle_{\Gamma} = \langle L^* g, \psi' \rangle_{\Gamma}, \quad \forall \psi' \in \tilde{H}_s^{-1/2}(\Gamma), \tag{6.15}
\]

when using \( \text{divr} \circ L = \text{id} \) and the adjoint operator \( L^* \) of \( L \).

Rewriting the above with \( V^{-1} : H^{1/2}(\Gamma) \rightarrow \tilde{H}^{-1/2}(\Gamma) \), we have

\[
N_\perp = \text{id} \circ V^{-1} \circ L^*.
\]

We remind the reader that we will postpone the concrete construction of \( N \) until Section 6.3. First we proceed to investigate its continuity and inf-sup condition, which are the two properties that \( N \) needs to fulfill on the continuous level to give rise to a suitable preconditioner.
6.2. Continuity and inf-sup condition of $N$

Let us begin by defining the projections

$$
\Pi_0 : \tilde{H}^{-1/2}(\text{div}, \Gamma) \to X_0(\Gamma) 
$$

(6.17)

$$
\Pi_\perp := (\text{Id} - \Pi_0) : \tilde{H}^{-1/2}(\text{div}, \Gamma) \to X_\perp(\Gamma) 
$$

(6.18)

induced by the Hodge decomposition of $\tilde{H}^{-1/2}(\text{div}, \Gamma)$. We note that they are continuous operators because the Hodge decomposition is stable.

**Proposition 6.2.1.** The operator $N_0$ is elliptic and continuous.

**Proof.** Let us recall that $N_0 = \tilde{S}_0^{-1} : H^{-1/2}(\text{curl}, \Gamma) \to X_0(\Gamma)$, with

$$
\tilde{S}_0 \xi_0 = \text{V} \xi_0 = \Pi_0^* \circ \text{V} \circ \Pi_0 \xi, 
$$

(6.19)

for all $\xi = \xi_0 + \xi_\perp \in \tilde{H}^{-1/2}(\text{div}, \Gamma)$ with $(\xi_0, \xi_\perp) \in X_0(\Gamma) \times X_\perp(\Gamma)$.

It follows from the ellipticity of $\text{V}$ and the fact that $\Pi_\perp u_0 = u_0, \forall u_0 \in X_0(\Gamma)$, that this operator is elliptic on $X_0(\Gamma)$. Analogously, we have that $\tilde{S}_0$ is continuous. These two properties imply that

$$
\|N_0 u\|_{\tilde{H}^{-1/2}(\text{curl}, \Gamma)} \leq C_0 \|u\|_{H^{-1/2}(\text{curl}, \Gamma)}, \quad \forall u \in H^{-1/2}(\text{curl}, \Gamma), 
$$

(6.20)

$$
\|u\|_{H^{-1/2}(\text{curl}, \Gamma)} \leq c_0 \|N_0 u\|_{\tilde{H}^{-1/2}(\text{curl}, \Gamma)}, \quad \forall u \in Y_\perp(\Gamma), 
$$

(6.21)

as desired. \qed

**Proposition 6.2.2.** The operator $N_\perp : H^{-1/2}(\text{curl}, \Gamma) \to X_\perp(\Gamma)$ satisfies

$$
\|N_\perp u\|_{\tilde{H}^{-1/2}(\text{curl}, \Gamma)} \leq C_\perp \|u\|_{H^{-1/2}(\text{curl}, \Gamma)}, \quad \forall u \in H^{-1/2}(\text{curl}, \Gamma) 
$$

(6.22)

$$
\|u\|_{H^{-1/2}(\text{curl}, \Gamma)} \leq c_\perp \|N_\perp u\|_{\tilde{H}^{-1/2}(\text{curl}, \Gamma)}, \quad \forall u \in Y_0(\Gamma). 
$$

(6.23)

**Proof.** Recall that $N_\perp = \tilde{S}_\perp^{-1}$, with $\tilde{S}_\perp$ inducing the bilinear form

$$
\left\langle \tilde{S}_\perp u_\perp, u'_\perp \right\rangle_F = \left\langle \text{V} \tilde{\text{div}} u_\perp, \tilde{\text{div}} u'_\perp \right\rangle_F, \quad u_\perp, u'_\perp \in X_\perp(\Gamma), 
$$

(6.24)

which is continuous and elliptic on $X_\perp(\Gamma)$. This is clear from the characterization $X_\perp(\Gamma) = L(\tilde{H}^{-1/2}(\Gamma))$ and the properties of the weakly singular operator $\text{V}$.

The remainder of the proof follows straightforwardly from the definitions of $N_\perp$. \qed

6.3. Discretization-oriented Formulation of $N$ on the Unit Disk

In light of (6.12) and (6.16), it becomes clear that the explicit computation of $N$ relies on the availability of closed-forms for $W^{-1}$ and $V^{-1}$. As we learnt in Chapter 2, and particularly from Theorem 2.1.2, we have explicit formulas for these inverse operators on disks $\mathbb{D}_a := \{x \in \mathbb{R}^3 : x_3 = 0 \text{ and } \|x\| < a\}$, where it holds that

$$
W^{-1} = \overline{\text{V}} : H^{-1/2}(\mathbb{D}_a) \to \tilde{H}^{1/2}(\mathbb{D}_a), 
$$

$$
V^{-1} = \overline{\text{W}} : H^{1/2}(\mathbb{D}_a) \to \tilde{H}^{-1/2}(\mathbb{D}_a), 
$$

with $\overline{\text{V}}$ and $\overline{\text{W}}$ defined in (2.1) and (2.4), respectively.

Hence, we can construct $N$ on disks $\mathbb{D}_a$ using

$$
N_0 = \text{curl}\Gamma \circ \overline{\text{V}} \circ \text{curl}\Gamma. 
$$

(6.25)
Chapter 6. Compact-Equivalent Inverse of the EFIE Operator on Disks

\[ N_{\perp} = L \circ \overline{W} \circ L^* \]. \quad (6.26)

For simplicity and without loss of generality, below we will only discuss \( N \) on the unit disk \( D_1 \).

This Section is devoted to derive a formulation of \( N \) that can be discretized using conforming Galerkin BEM. The issue arises specifically for \( N_{\perp} \) and we start by briefly showing why one cannot pursue its usual formulation.

6.3.1. A Flawed Formulation for \( N_{\perp} \)

From (6.26), one can naively deduce that for \( g \in H^{-1/2}(\text{curl} D_1, D_1) \), \( N_{\perp} \) is computed with the four following steps:

I. Seek \( L^* g = v \in H^1_*(D_1) \) such that

\[ \int_{D_1} \nabla_{D_1} v \cdot \nabla_{D_1} v' \, dS = \int_{D_1} g \cdot \nabla_{D_1} v' \, dS, \quad \forall v' \in H^1_*(D_1). \quad (6.27) \]

II. Take \( \mu = \overline{W} v \in \widetilde{H}^{-1/2}_*(D_1) \).

III. Find \( w \in H^1_*(D_1) \) such that

\[ \int_{D_1} \nabla_{D_1} w \cdot \nabla_{D_1} w' \, dS = \int_{D_1} \mu w' \, dS, \quad \forall w' \in H^1_*(D_1). \quad (6.28) \]

IV. Compute \( N_{\perp} g = - \nabla_{D_1} w \).

However, this is not suitable for discretization.

The first problem arises from the right hand side of the variational problem I. In order to be meaningful, one requires \( \nabla_{D_1} v' \in \tilde{H}^{-1/2}_*(\text{div} D_1, D_1) \), and thus \( v' \in H(\text{div} D_1) \), with \( H(\text{div} D_1) \) as defined in (5.46). The issue here is that no simple finite dimensional subspaces are known for \( H(\text{div} D_1) \).

The second issue comes from variational problem III. If \( w \) is approximated by piecewise linear basis functions, then it is possible that \( \nabla_{D_1} w' \notin \tilde{H}^{-1/2}_*(\text{div} D_1, D_1) \).

These difficulties motivate the use of a mixed formulation to compute \( N_{\perp} g \).

6.3.2. Mixed formulation for \( N_{\perp} \)

Recall the formula

\[ N_{\perp} g = L \circ \overline{W} \circ L^* g, \quad (6.29) \]

with \( L = - \nabla_{D_1} \circ (-\Delta_{D_1}^N)^{-1} \) and \( L^* = (-\Delta_{D_1}^N)^{-1} \circ \tilde{\text{div}}_{D_1} \).

Let us recall the definition of \( \tilde{H}^{0,-1/2}(\text{div} D_1, D_1) \) from (5.42):

\[ \tilde{H}^{0,-1/2}(\text{div} D_1, D_1) = \tilde{H}^{-1/2}(\text{div} D_1, D_1) \cap L^2(D_1), \]

and note that \( X_{\perp}(D_1) \subset \tilde{H}^{0,-1/2}(\text{div} D_1, D_1) \), since \( L \) maps into that space.

We also remind the reader about the following space introduced in Lemma 2.1.12

\[ H^{1/2}(D_1) = \{ g \in H^{1/2}(D_1) \, : \, \langle g, \omega^{-1} \rangle_{D_1} = 0 \}, \quad (6.30) \]

with \( \omega(x) = \sqrt{1 - r^2_x}, \, x \in D_1 \).
We now analyze each of these steps separately:

- \( u := L^* g \in H^{1/2}(D_1) \) solves
  \[
  - \Delta_{D_1}^N u = \overline{\text{div}}_{D_1} g,
  \]
  which holds if and only if \( \overline{\text{div}}_{D_1} (\text{grad}_{D_1} u + g) = 0 \). This leads to the next mixed formulation for \( q \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \):
  \[
  q = \text{grad}_{D_1} u + g
  \]
  \[
  \overline{\text{div}}_{D_1} q = 0,
  \]
  from where we deduce the following mixed variational problem: Find \( q \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \) and \( u \in H^{1/2}(D_1) \) such that
  \[
  \langle q, j' \rangle_{D_1} + \langle u, \overline{\text{div}}_{D_1} j' \rangle_{D_1} = \langle g, j' \rangle_{D_1}, \quad \forall j' \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1)
  \]
  \[
  \langle \overline{\text{div}}_{D_1} q, v' \rangle_{D_1} = 0, \quad \forall v' \in H^{1/2}(D_1)
  \]

- \( L \overline{W} u \) follows from solving
  \[
  - \Delta_{D_1}^N v = \overline{W} u
  \]
  and taking \( - \text{grad}_{D_1} v \).

Its mixed formulation is given by:

\[
\text{grad}_{D_1} v = j,
\]
\[
\overline{\text{div}}_{D_1} j = \overline{W} u
\]

From where we get the following mixed variational problem: Find \( j \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1), v \in H^{1/2}(D_1) \) such that

\[
\langle j, q' \rangle_{D_1} + \langle v, \overline{\text{div}}_{D_1} q' \rangle_{D_1} = 0, \quad \forall q' \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1)
\]
\[
\langle \overline{\text{div}}_{D_1} j, \varphi' \rangle_{D_1} = \langle \overline{W} u, \varphi' \rangle_{D_1}, \quad \forall \varphi' \in H^{1/2}(D_1)
\]
6.3.3. Evaluation of N

With the derivation in the preceding subsection, we obtain a formulation of \( N \) for which a conforming BE discretization is available, since we split the evaluation of \( N \) into the computation of its two components \( N_0 \) and \( N_\perp \) as follows:

1. \[ N_0 \, g = \overline{\text{curl}}_{D_1} \nabla \text{curl}_{D_1} \, g. \] (6.38)

2. \[ N_\perp \, g = L \circ \overline{W} \circ L^* \, g. \] (6.39)

As previously discussed, this boils down to the following two steps:

(i) Seek \( q \in \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \), \( u \in H^{1/2}_\ast(D_1) \) such that

\[
\langle q, j' \rangle_{D_1} + \left\langle u, \text{div}_{D_1} \, j' \right\rangle_{D_1} = \left\langle g, j' \right\rangle_{D_1} \quad \forall j' \in \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \] (6.40)

\[
\left\langle \text{div}_{D_1} \, q, v' \right\rangle_{D_1} = 0 \quad \forall v' \in H^{1/2}_\ast(D_1).
\]

(ii) Seek \( \xi_\perp \in \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \), \( v \in H^{1/2}_\ast(D_1) \) such that:

\[
\langle \xi_\perp, q' \rangle_{D_1} + \left\langle v, \overline{\text{div}}_{D_1} q' \right\rangle_{D_1} = 0 \quad \forall q' \in \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \] (6.41)

\[
\left\langle \overline{\text{div}}_{D_1} \, \xi_\perp, u' \right\rangle_{D_1} = \langle \overline{W} u, u' \rangle_{D_1} \quad \forall u' \in H^{1/2}_\ast(D_1).
\]

Result: \( \xi_\perp \in \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \subset \widetilde{H}^{1/2}(\text{div}_{D_1}, D_1) \).

From where it is clear that

\[ N_\perp \, g = \xi_\perp. \] (6.42)

**Remark 6.3.1.** Note that due to the elliptic lifting of the Laplace-Beltrami operator [19], we have

\[ X_\perp(D_1) = \text{grad} \, H^{1/2}_\ast(D_1) \subset L^2(D_1), \] (6.43)

and therefore it is correct that \( \xi_\perp \in \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \).

**Note on existence and uniqueness of solutions to saddle point problems**

Due to the fact that evaluating \( N_\perp \) requires solving two saddle point problems, we now turn our attention to the existence and uniqueness of solutions for this kind of problems. We take the cue from [11, p. III.4] and begin by noticing that problems (6.40) and (6.41) define linear mappings

\[ T_1 : \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \times H^{1/2}_\ast(D_1) \rightarrow \left( \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \right)' \times \widetilde{H}^{-1/2}_\ast(D_1) \]

\[ (q, u) \rightarrow (g, 0), \]

and

\[ T_2 : \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \times H^{1/2}_\ast(D_1) \rightarrow \left( \widetilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \right)' \times \widetilde{H}^{-1/2}_\ast(D_1) \]

\[ (\xi_\perp, v) \rightarrow (0, \overline{W} u). \]

In order to get the existence and uniqueness of solutions, we need to prove that they are isomorphisms. For this, we use the following result:
6.3. Discretization-oriented Formulation of $N$ on the Unit Disk

**Theorem 6.3.2** (Brezzi’s splitting theorem [11, Theorem 4.3]). Let $X$ and $M$ be Hilbert spaces. Let $\Upsilon$ be the linear mapping

$$\Upsilon : X \times M \to X' \times M'$$

$$\langle u, \alpha \rangle \to (f, g),$$

associated to the saddle point problem: Find $(u, \alpha) \in X \times M$ such that

$$a(u, v) + b(\alpha, v) = \langle f, v \rangle, \quad \forall v \in X$$

$$b(u, \beta) = \langle g, \beta \rangle, \quad \forall \beta \in M,$$

with $a : X \times X \to \mathbb{R}, b : X \times M \to \mathbb{R}$ continuous bilinear forms, $f \in X'$, $g \in M'$ and $\langle \cdot, \cdot \rangle$ denoting both the dual pairing of $X$ and $X'$ and that of $M$ and $M'$.

The mapping $\Upsilon : X \times M \to X' \times M'$ defines an isomorphism if an only if the following two conditions are satisfied:

i) The bilinear form $a(\cdot, \cdot)$ verifies

$$a(u, v) \geq C \|v\|_X^2$$

for all $v \in V := \{v \in X : b(v, \beta) = 0 \forall \beta \in M\}$ and with $C > 0$.

ii) The bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition

$$\inf_{\beta \in M} \sup_{v \in X} \frac{b(v, \beta)}{\|v\|_X \|\beta\|_M} \geq c_b > 0.$$

Given the structure of $\Upsilon_1$ and $\Upsilon_2$, it suffices to show that the following two assertions hold:

(c1) $\left|\left(q, q_{D_1}\right)\right|_V \geq C \|q\|_{H_0^{0,-1/2}(\text{div} \cdot, D_1)}^2$, $\forall q \in V$ with $C > 0$ and

$$V := \{j \in H_0^{0,-1/2}(\text{div} \cdot, D_1) : \left\langle \text{div}_{D_1} j, u \right\rangle_{D_1} = 0 \forall u \in H_{1/2}(D_1)\}.$$

(c2) $\sup_{j \in H_0^{0,-1/2}(\text{div} \cdot, D_1)} \frac{\left|\left\langle \text{div}_{D_1} j, u \right\rangle_{D_1}\right|}{\|j\|_{H_0^{0,-1/2}(\text{div} \cdot, D_1)}} \geq c_b \|u\|_{H_{1/2}(D_1)}$, $\forall u \in H_{1/2}(D_1)$.

On the one hand, (c1) follows from the definition of $V$ and the graph norm

$$\|j\|_{H_0^{0,-1/2}(\text{div} \cdot, D_1)}^2 = \|j\|_{L^2(D_1)}^2 + \|\text{div}_{D_1} j\|_{L^2(D_1)}^2.$$

On the other hand, (c2) is verified by the surjectivity of $\text{div}_{D_1} : X_\perp(D_1) \to H_{-1/2}(D_1)$ and

$$\sup_{j \in H_0^{0,-1/2}(\text{div} \cdot, D_1)} \frac{\left|\left\langle \text{div}_{D_1} j, u \right\rangle_{D_1}\right|}{\|j\|_{H_0^{0,-1/2}(\text{div} \cdot, D_1)}} \geq \sup_{\varphi \in H_{-1/2}(D_1)} \frac{\langle \varphi, u \rangle_{L^2(D_1)}}{\|\varphi\|_{H_{-1/2}(D_1)}} = \|u\|_{H_{1/2}(D_1)}$, $\forall u \in H_{1/2}(D_1)$.

(6.44)

### 6.3.4. Augmented saddle point problem for $N_\perp$

Since we cannot construct a discretization that fulfills the orthogonality in the definition of the space $H_{1/2}^2(D_1)$, we resort to reformulate the saddle point problems in (2) as augmented variational problems.

We remark that when using low order Galerkin BEM discretization, the formula $\langle g, \omega^{-1} \rangle_{D_1} = 0$ removes the constants from the discrete boundary element space. Therefore, from a numerical point of view, the
usual characterization of the vanishing mean serves the same purpose and is simpler to compute. For this reason, we use the condition $\langle g, 1 \rangle_{D_1} = 0$ when formulating the "augmented saddle point problem" for the discrete evaluation of $N_{\perp}$.

With all these considerations, the resulting evaluation of $N_{\perp}$ reads:

i) Seek $q \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1)$, $u \in H^{1/2}(D_1)$, and $\alpha \in \mathbb{R}$ such that

$$
\langle q, j' \rangle_{D_1} + \langle u, \text{div}_{D_1} j' \rangle_{D_1} = \langle g, j' \rangle_{D_1} \quad \forall j' \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \tag{6.45}
$$

$$
\langle \text{div}_{D_1} q, v' \rangle_{D_1} + \alpha \langle 1, v' \rangle_{D_1} = 0 \quad \forall v' \in H^{1/2}(D_1)
$$

$$
\langle u, 1 \rangle_{D_1} = 0.
$$

ii) Seek $\xi_{\perp} \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1)$, $v \in H^{1/2}(D_1)$, and $\beta \in \mathbb{R}$ such that:

$$
\langle \xi_{\perp}, q' \rangle_{D_1} + \langle v, \text{div}_{D_1} q' \rangle_{D_1} = 0 \quad \forall q' \in \tilde{H}^{0,-1/2}(\text{div}_{D_1}, D_1) \tag{6.46}
$$

$$
\langle \text{div}_{D_1} \xi_{\perp}, u' \rangle_{D_1} + \beta \langle 1, u' \rangle_{D_1} = \langle \nabla u, u' \rangle_{D_1} \quad \forall u' \in H^{1/2}(D_1)
$$

$$
\langle v, 1 \rangle_{D_1} = 0.
$$

Remark 6.3.3. In view of Theorem 6.3.2 it is clear that the augmented saddle point problems (6.45) and (6.46) also have unique solutions.
7. Operator Preconditioning for EFIE

"If you have built castles in the air, your work need not be lost; that is where they should be. Now put the foundations under them. “ – Walden by Henry David Thoreau (1854).

With the analysis of the preceding Chapters, we are now prepared to discuss how to use N to build a preconditioner for the EFIE on screens. Although operator preconditioning as we first discussed in Section 3.1 also applies to the current setting, we will take a different approach in this Chapter and will study operator preconditioning from the perspective of its underlying reasoning.

As explained in [46, Sect. 1], given function spaces X and Y, a continuous bijective linear operator $A : X \rightarrow Y$ and another isomorphism $B : Y \rightarrow X$, the composition $BA$ will provide an endomorphism of X that, under certain discretization choices, will give rise to well-conditioned matrices. Consequently, the challenge of operator preconditioning is establishing and fulfilling the required conditions leading to these well-conditioned matrices.

One general way of accomplishing this purpose was proposed in Theorem 3.1.1, but one can put forth this theory also for particular contexts. Indeed, we rephrase this theorem to specifically treat the approach we pursue to precondition the Helmholtz BIEs and EFIE on screens:

Let $X$ be a Hilbert space and $A, B : X \rightarrow X'$ linear bijective operators such that

$$B^{-1}A = \text{Id} + K : X \rightarrow X,$$

with K a compact operator.

Let $X_h \subset X$, $h \in \mathbb{H}$ be a family of finite-dimensional subspaces such that $\dim X_h \rightarrow \infty$ for $h \rightarrow 0$. Moreover, for each $X_h$ we choose a basis $\{b_i\}_{i=1}^N$ of $X_h$ and define $A_h = ((A b_i, b_j))_{i,j=1}^N$.

If A and B induce bilinear forms satisfying $h$-uniform inf-sup conditions on the family $X_h \subset X$, $h \in \mathbb{H}$, and there is a stable discretization $B_h$ of B on $X_h$. Then the abstract theory of operator preconditioning gives

$$\kappa((B_h^{-1}A_h)) \leq \frac{\theta_2}{\theta_1},$$

with $\theta_2/\theta_1$ does not depend on any discretization parameter.

From Riesz-Schauder theory of functional analysis, we know that the point spectrum of $\text{Id} + K$, denoted by $\sigma_p(\text{Id} + K)$, will cluster around one. The intuition exploiting this property to build a preconditioner is that if we consider again the family of finite-dimensional subspaces $X_h \subset X$, $h \in \mathbb{H}$ (such that $\dim X_h \rightarrow \infty$ for $h \rightarrow 0$), then 1 is the only accumulation point of

$$\bigcup_{h \in \mathbb{H}} \sigma_p(B_h^{-1}A_h).$$

As a consequence, the resulting preconditioned system matrix should achieve very fast convergence when using Krylov subspace methods. Moreover, given the asymptotic $h$-independence of the spectral condition number, one might even expect that this fast convergence is also independent of $h$ after certain mesh refinement level. Generally one can prove this type of statements for Hermitian matrices, for which the spectral condition number gives precise bounds for convergence of CG and of GMRES (that boils down to MINRES in that case). Unfortunately, this is false for non-Hermitian matrices, where $h$-uniform...
spectral condition numbers are not enough to guarantee convergence of this kind of methods. Nevertheless, the numerical evidence that this approach speeds-up the convergence of Krylov subspace methods even for (indefinite) non-Hermitian operators motivates that we still adopt this operator preconditioning strategy. We refer to [26, 25, 4] for a couple of examples.

Recall the operators $N_0$ and $N_\bot$ from (6.25) and (6.26). We define our preconditioning operator for $k > 0$ as

$$P(k) = N_0 - k^2 N_\bot : H^{1/2}(\text{curl}_{\mathcal{D}_1}, \mathcal{D}_1) \to H^{1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1).$$

(7.4)

By construction, we have the following result.

**Proposition 7.0.1.** For $k > 0$ it holds

$$P(k) A(k) = \text{Id} + K(k) : H^{1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1) \to H^{1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1),$$

(7.5)

with $K(k)$ a compact operator.

**Proof.** Let us recall from Section 6.1 the operators $S = V - \text{grad}_{\mathcal{D}_1} V \text{div}_{\mathcal{D}_1}$ and $\tilde{S} = \tilde{S}_0 + \tilde{S}_\bot$ with

$$\tilde{S}_0 \xi_0 = V \xi_0 = \Pi_o^* \circ V \circ \Pi_o \xi,$$

$$\tilde{S}_\bot \xi_\bot = - \text{grad}_{\mathcal{D}_1} V \text{div}_{\mathcal{D}_1} \xi_\bot = - \Pi_\bot^* \circ \text{grad}_{\mathcal{D}_1} V \text{div}_{\mathcal{D}_1} \circ \Pi_\bot \xi$$

for all $\xi_0 \in X_0(\mathcal{D}_1)$ and $\xi_\bot \in X_\bot(\mathcal{D}_1)$, and with $\Pi_0$ and $\Pi_\bot$ the projections induced by the Hodge decomposition.

Now, let us note that, by construction, the operator $S - \tilde{S} : H^{1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1) \to H^{1/2}(\text{curl}_{\mathcal{D}_1}, \mathcal{D}_1)$ is compact.

Next, abusing notation we write the static EFIE as

$$A(0) := V + \frac{1}{k^2} \text{grad}_{\mathcal{D}_1} \text{div}_{\mathcal{D}_1}.$$

We point out that the difference between $S$ and $A(0)$ is the sign and scaling on the second term. Ergo $A(0) - \tilde{S}_0 + \frac{1}{k^2} \tilde{S}_\bot$ is also compact. Indeed

$$(A(0)(\xi_0 + \xi_\bot) - \tilde{S}_0 \xi_0 + \frac{1}{k^2} \tilde{S}_\bot \xi_\bot = V \xi_0 + \xi_\bot + \frac{1}{k^2} \text{grad}_{\mathcal{D}_1} \text{div}_{\mathcal{D}_1} \xi_\bot = V \xi_\bot,$$

which is a compact perturbation due to the compact embedding $X_\bot(\Gamma) \subset H^{1/2}_r(\Gamma)$.

On the other hand, one can prove using Lemma 2.2.1 that $A(k) - A(0) = \mathcal{V}_k - V + \frac{1}{k^2} \text{grad}_{\mathcal{D}_1}(\mathcal{V}_k - V) \text{div}_{\mathcal{D}_1}$ is also a compact operator (c.f. [55, Lemma 3.2]). Thus, we get that $A(k) - \left(\tilde{S}_0 - \frac{1}{k^2} \tilde{S}_\bot\right)$ is compact.

Finally, since $N_0 = \tilde{S}_0^{-1}$ and $N_\bot = \tilde{S}_\bot^{-1}$, we have

$$(P(k) A(k) \xi = P(k) \left(A(k) - (\tilde{S}_0 - \frac{1}{k^2} \tilde{S}_\bot)\right) \xi = P(k)(\tilde{S}_0 - \frac{1}{k^2} \tilde{S}_\bot) \xi$$

$$= \text{Id} + P(k) V \xi_\bot - \frac{1}{k^2} N_0 \tilde{S}_\bot \xi_0 - k^2 N_\bot \tilde{S}_0 \xi_\bot,$$

where the second term is compact due and the third and fourth terms vanish due to the fact that by definition of $Y_0(\mathcal{D})$ and $Y_\bot(\mathcal{D}_1)$:

$$N_0 w_0 = 0 \quad \forall w_0 \in Y_0(\mathcal{D}_1) = (X_\bot(\mathcal{D}_1))^\prime,$$

$$N_\bot w_\bot = 0 \quad \forall w_\bot \in Y_\bot(\mathcal{D}_1) = (X_0(\mathcal{D}_1))^\prime.$$
The remainder of this Chapter is structured as follows: First we prove that \( P(k) \) fulfills all the requirements from operator preconditioning theory, including a stable discretization. Next, in Section 7.3, we will briefly review the existing GMRES convergence results in the hope that we can assess if they can predict our numerical findings. Then, we will test numerically our preconditioner on the unit disk and some GMRES bounds. Finally, we conclude by extending our preconditioner to mapped screens and presenting the corresponding numerical experiments.

### 7.1. Continuity and inf-sup condition of \( P(k) \)

**Theorem 7.1.1.** The operator \( P(k) = N_0 - k^2 N_\perp \) is continuous.

**Proof.** We use the continuity of \( N_0 \) and \( N_\perp \) to derive

\[
\|P(k)u\|_{H^{-1/2}(\text{div}, D_1)} \leq \|N_0 u\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)} + \| -k^2 N_\perp u\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)} \tag{7.6}
\]

\[
\leq C_0 \|u\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)} + C_\perp k^2 \|u\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)} \tag{7.7}
\]

\[
= (C_0 + C_\perp k^2) \|u\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)}. \tag{7.8}
\]

**Theorem 7.1.2 (T-coercivity of \( P(k) \)).** Consider \( w = w_0 + w_\perp \in H^{-1/2}(\text{curl} \perp D_1, D_1) \) with \( (w_0, w_\perp) \in Y_0(\mathbb{D}) \times Y_\perp(\mathbb{D}_1) \). The operator \( P(k) \) satisfies

\[
\langle P(k)(w_0 + w_\perp), w_0 - w_\perp \rangle_{D_1} \geq c_0 \|w_0 + w_\perp\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)}^2. \tag{7.9}
\]

**Proof.** We begin by writing

\[
\langle P(k)(w_0 + w_\perp), w_0 - w_\perp \rangle_{D_1} = \langle P(k)w_0, w_0 \rangle_{D_1} + \langle P(k)w_\perp, w_0 \rangle_{D_1} \tag{7.10}
\]

\[
- \langle P(k)w_0, w_\perp \rangle_{D_1} - \langle P(k)w_\perp, w_\perp \rangle_{D_1}.
\]

Note that \( P(k) = N_0 - k^2 N_\perp \) and that, by definition of \( Y_0(\mathbb{D}) \) and \( Y_\perp(\mathbb{D}_1) \)

\[
N_0 w_0 = 0 \quad \forall w_0 \in Y_0(\mathbb{D}_1) = (X_\perp(\mathbb{D}_1))^\perp, \\
N_\perp w_\perp = 0 \quad \forall w_\perp \in Y_\perp(\mathbb{D}_1) = (X_0(\mathbb{D}_1))^\perp.
\]

Combining this with the fact that these operators are self-adjoint, we get

\[
\langle P(k)(w_0 + w_\perp), w_0 - w_\perp \rangle_{D_1} = \langle N_0 w_\perp, w_\perp \rangle_{D_1} - \langle -k^2 N_\perp w_0, w_\perp \rangle_{D_1}. \tag{7.11}
\]

Finally, we use the ellipticity of \( N_0 \) and \( N_\perp \) shown in Propositions 6.2.1 and 6.2.2, and the definitions of \( N_0 \) and \( N_\perp \) to get

\[
\langle P(k)(w_0 + w_\perp), w_0 - w_\perp \rangle_{D_1} = \langle N_0 w_\perp, w_\perp \rangle_{D_1} + k^2 \langle N_\perp w_0, w_\perp \rangle_{D_1} \tag{7.12}
\]

\[
\geq c_0 \|w_\perp\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)}^2 + k^2 c_\perp \|w_0\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)}^2 \tag{7.13}
\]

\[
\geq \frac{\min(c_0, k^2 c_\perp)}{2} \|w_0 + w_\perp\|_{H^{-1/2}(\text{curl} \perp D_1, D_1)}^2. \tag{7.14}
\]

**Corollary 7.1.3.** The operator \( P(k) \) satisfies the inf-sup condition:

\[
\sup_{w \in H^{-1/2}(\text{curl} \perp D_1)} \frac{|\langle P(k)w, v \rangle_{D_1}|}{\|w\|_{H^{-1/2}(\text{curl} \perp D_1)}} \geq c_1 \|v\|_{H^{-1/2}(\text{curl} \perp D_1)} \quad \forall v \in H^{-1/2}(\text{curl} \perp D_1), \tag{7.15}
\]

with \( c_1 > 0 \).
7.2. Stable Discretization of $P(k)$

We again consider a primal mesh $\mathbb{D}_h$ and a dual mesh $\mathbb{\hat{D}}_h$ of $\mathbb{D}_1$ as defined in Section 3.2.2, and rely on a low-order Lagrangian BEM discretization.

We remind the reader of our notation introduced in Sections 3.2.2 and 5.5:

- $S^{-1,0}(\mathbb{D}_h) \subset H^{-1/2}(\mathbb{D}_1)$: p.w. constants on primal mesh,
- $S^{0,1}(\mathbb{D}_h) \subset \tilde{H}^{1/2}(\mathbb{D}_1)$: p.w. linear cont. with zero b.c. on primal mesh,
- $S_0(\mathbb{D}_h) \subset \tilde{H}^{-1/2}(\text{div}_{\mathbb{D}_1}, \mathbb{D}_1)$: rotated surface edge elements on primal mesh, with zero tangential b.c., \(^1\)
- $S^{-1,0}(\mathbb{\hat{D}}_h) \subset H^{-1/2}(\mathbb{D}_1)$: p.w. constants on dual mesh,
- $S^{0,1}(\mathbb{\hat{D}}_h) \subset H^{1/2}(\mathbb{D}_1)$: p.w. linear cont. on dual mesh.

We remark from the above that our discretization strategy is as follows: We discretize functions in "tilde"-spaces on the primal mesh $\mathbb{D}_h$, and functions in spaces "without boundary conditions" over the dual mesh $\mathbb{\hat{D}}_h$.

Let $(\mathcal{Y}(\mathbb{D}_h))_h$ be a family of finite dimensional subspaces approximating $H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)$ in the sense that

$$\forall u \in H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1) \quad \lim_{h \to 0} \inf_{u' \in \mathcal{Y}(\mathbb{D}_h)} \|u - u'\|_{H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)} = 0, \quad (7.16)$$

and such that it can be split into direct sums

$$\mathcal{Y}(\mathbb{D}_h) = Y_{0,h} \bigoplus Y_{\perp,h}$$

of closed subspaces of $H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)$ that satisfy

$$\sup_{v_{0,h} \in Y_{0,h}} \inf_{v_0 \in \mathcal{Y}(\mathbb{D}_h)} \frac{\|v_0 - v_{0,h}\|_{H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)}}{\|v_0\|_{H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)}} \to 0 \quad (7.17)$$

$$\sup_{v_{\perp,h} \in Y_{\perp,h}} \inf_{v_{\perp} \in \mathcal{Y}_{\perp}(\mathbb{D}_h)} \frac{\|v_{\perp} - v_{\perp,h}\|_{H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)}}{\|v_{\perp,h}\|_{H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)}} \to 0. \quad (7.18)$$

In this Section we aim to prove

**Proposition 7.2.1.** For some $h_0 > 0$, the operator $P(k)$ satisfies the following discrete inf-sup condition:

$$\sup_{w_h \in \mathcal{Y}(\mathbb{D}_h)} \frac{|(P(k)w_h, v_h)_{D_1}|}{\|w_h\|_{H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)}} \geq c_p \|v_h\|_{H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)} \quad \forall v_h \in \mathcal{Y}(\mathbb{D}_h), h < h_0, \quad (7.19)$$

with $c_p > 0$ independent of $h$ and $h_0$.

And for this we first need to prove the stability of the components of $P(k)$.

---

\(^1\)Note that (after rotation) this corresponds to lowest order Raviart-Thomas edge elements with zero normal b.c.
7.2.1. Stability of \( N_\perp \)

In order to prove the stability of \( N_\perp \), we need to show the stability of the discrete saddle point problems introduced in subsection 6.3.4 following the proposed discretization strategy.

Let us begin by the discrete version of the first saddle point problem (6.45): Let \( g_h \in \mathcal{V}(\mathcal{D}_h) \). Seek \( q_h \in \mathcal{E}_0(\mathcal{D}_h) \), \( \mu_h \in S^{0,1}(\mathcal{D}_h) \) such that

\[
\begin{align*}
\langle q_h, j_h' \rangle_{\mathcal{D}_1} + \langle \mu_h, \text{div}_{\mathcal{D}_1} j_h' \rangle_{\mathcal{D}_1} &= \langle g_h, j_h' \rangle_{\mathcal{D}_1} & \forall j_h' \in \mathcal{E}_0(\mathcal{D}_h) \\
\langle \text{div}_{\mathcal{D}_1} q_h, \psi_h' \rangle_{\mathcal{D}_1} + \alpha \langle 1, \psi_h' \rangle_{\mathcal{D}_1} &= 0 & \forall \psi_h' \in S^{0,1}(\mathcal{D}_h)
\end{align*}
\]

Following the discrete version of Brezzi’s splitting Theorem (see [11, p. III.4], c.f. Theorem 6.3.2), showing its stability for non-uniform meshes entails proving that:

(C1) \( \|v_h, v_h\|_{D_1} \geq \alpha \|v_h\|_{H^{0.5}}^2 \), \( \forall v_h \in V_h \)

with \( V_h := \{ v_h \in E_0(\mathcal{D}_h) : \langle \text{div}_{\mathcal{D}_1} v_h, \psi_h' \rangle_{\mathcal{D}_1} = 0, \forall \psi_h' \in S^{0,1}(\mathcal{D}_h) \} \) and \( \alpha > 0 \).

(C2)

\[
\sup_{v_h \in E_0(\mathcal{D}_h)} \frac{\langle \text{div}_{\mathcal{D}_1} v_h, \psi_h' \rangle_{\mathcal{D}_1}}{\|v_h\|_{H^{0.5}}(\mathcal{D}_1)} \geq \beta_1 \|\psi_h\|_{H^{-1/2}(\mathcal{D}_1)} \quad \forall \psi_h \in S^{0,1}(\mathcal{D}_h),
\]

with \( \beta_1 > 0 \) independent of \( h \).

(C3)

\[
\sup_{w_h \in S^{-1,0}(\mathcal{D}_h)} \frac{\langle w_h, 1 \rangle_{\mathcal{D}_1}}{\|w_h\|_{H^{1/2}(\mathcal{D}_1)}} \geq \beta_2 \|1\|_{H^{-1/2}(\mathcal{D}_1)},
\]

with \( 1 \in S^{-1,0}(\mathcal{D}_h) \) and \( \beta_2 > 0 \) independent of \( h \).

Condition (C1) clearly holds, while (C3) holds if and only if the \( L^2 \)-duality (bilinear) pairing fulfills the discrete inf-sup condition (7.21). In subsection 3.3, we learnt that this is the case as long as the mesh Assumptions 3.2.1, 3.3.1 and 3.3.3 are satisfied (see Theorem 3.3.6, case (EDP)).

Thus, it remains to prove (C2). It is worth noticing that, from the proof of the corresponding (continuous) inf-sup condition in (6.44), we know that (C2) will rely on the stability of the \( L^2 \)-duality pairing of \( H^{1/2}(\mathcal{D}_1) \) and \( H^{-1/2}(\mathcal{D}_1) \), which holds also under the mesh Assumptions 3.2.1, 3.3.1 and 3.3.3 [84, Theorems 2.1–2.2].

With this purpose in mind, we prove the following:

**Proposition 7.2.2.** The discrete operator

\[
\text{div}_{\mathcal{D}_1} : E_0(\mathcal{D}_h) \subset H^{0,-1/2}(\mathcal{D}_1) \to S^{-1,0}(\mathcal{D}_h) \cap H^{-1/2}_*(\mathcal{D}_1) \subset \tilde{H}^{-1/2}(\mathcal{D}_1)
\]

is stable:

\[
\forall \varphi_h \in S^{-1,0}(\mathcal{D}_h) \cap H^{-1/2}_*(\mathcal{D}_1) : \exists v_h \in E_0(\mathcal{D}_h) : \text{div}_{\mathcal{D}_1} v_h = \varphi_h
\]

\[
\|v_h\|_{H^{0,-1/2}(\mathcal{D}_1)} \leq C \|\varphi_h\|_{\tilde{H}^{-1/2}(\mathcal{D}_1)},
\]

with \( C \) independent of \( h \).
Proof. Let us introduce the nodal interpolation operator \( \Pi_h \) onto \( \mathcal{E}_0(\mathbb{D}_h) \) and the \( L^2 \)-orthogonal projection \( Q_h \) onto \( \mathcal{S}^{-1,0}(\mathbb{D}_h) \) (see for example [55, Sect. 5] [9, Sect. 5]. Note that \( v := \text{grad}_{\mathcal{D}_1} \mathcal{H}(\mathbb{D}_1) \) and that \( v_h := \Pi_h v = \Pi_h L v \) is well-defined.

Next we have
\[
\text{div}_{\mathcal{D}_1} v_h = \text{div}_{\mathcal{D}_1} \Pi_h L v_h = Q_h \text{div}_{\mathcal{D}_1} L v_h = v_h,
\]
where we have used the commuting diagram properties of \( Q_h \) and \( \Pi_h \) (c.f.[9, Eq. (5.5)]).

In addition, the surjectivity of \( \text{div}_{\mathcal{D}_1} : \mathring{H}^{0,-1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1) \rightarrow \mathring{H}^{-1/2}(\mathcal{D}_1) \) gives
\[
\inf_{\text{div}_{\mathcal{D}_1} u = v} \| u \|_{\mathring{H}^{0,-1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1)} \leq \| v \|_{\mathring{H}^{-1/2}(\mathcal{D}_1)}, \quad \forall u \in \mathring{H}^{0,-1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1).
\]

Finally, we combine all these ingredients and obtain (7.23).

\[\square\]

**Corollary 7.2.3.** Let \( \mathbf{q}_h \in \mathcal{E}_0(\mathbb{D}_h) \) and \( \mathbf{\mu}_h \in \mathcal{S}^{0,1}(\mathring{\mathbb{D}}_h) \) be solutions of the discrete version of the saddle point problem (6.45).

If Assumptions 3.2.1, 3.3.1 and 3.3.3 are satisfied, then:
\[
\| \mathbf{q}_h \|_{\mathring{H}^{0,-1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1)} + \| \mathbf{\mu}_h \|_{H^{1/2}(\mathcal{D}_1)} \leq C \| \mathbf{g}_h \|_{H^{-1/2}(\text{curl}_{\mathcal{D}_1}, \mathcal{D}_1)}
\]
with \( C \) independent of \( h \).

Proof. As a consequence of the inf-sup condition, we have
\[
\| \mathbf{q}_h \|_{\mathring{H}^{0,-1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1)} + \| \mathbf{\mu}_h \|_{H^{1/2}(\mathcal{D}_1)} \leq C \sup_{J_h \in \mathcal{E}_0(\mathbb{D}_h)} \frac{\| \mathbf{g}_h \cdot J_h \|_{\mathcal{D}_1}}{\| J_h \|_{\mathring{H}^{0,-1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1)}} \leq C \sup_{J_h \in \mathcal{E}_0(\mathbb{D}_h)} \frac{\| \mathbf{g}_h \cdot J_h \|_{\mathcal{D}_1}}{\| J_h \|_{\mathring{H}^{-1/2}(\text{div}_{\mathcal{D}_1})}} \leq C \| \mathbf{g}_h \|_{H^{-1/2}(\text{curl}_{\mathcal{D}_1}, \mathcal{D}_1)}.
\]

\[\square\]

We now take care of the discrete version of the second saddle point problem (6.46): Seek \( \mathbf{\xi}_{\perp,h} \in \mathcal{E}_0(\mathbb{D}_h), \psi_h \in \mathcal{S}^{0,1}(\mathring{\mathbb{D}}_h) \) such that
\[
\langle \mathbf{\xi}_{\perp,h} , \mathbf{q}_h' \rangle_{\mathcal{D}_1} + \langle \psi_h , \text{div}_{\mathcal{D}_1} \mathbf{q}_h' \rangle_{\mathcal{D}_1} = 0 \quad \forall \mathbf{q}_h' \in \mathcal{E}_0(\mathbb{D}_h)
\]
\[
\langle \text{div}_{\mathcal{D}_1} \mathbf{\xi}_{\perp,h} , \mathbf{\mu}_h' \rangle_{\mathcal{D}_1} + \beta \langle 1 , \mathbf{\mu}_h' \rangle_{\mathcal{D}_1} = \langle \mathcal{W} \mathbf{\mu}_h , \mathbf{\mu}_h' \rangle_{\mathcal{D}_1} \quad \forall \mathbf{\mu}_h' \in \mathcal{S}^{0,1}(\mathring{\mathbb{D}}_h)
\]
\[
\langle \psi_h , 1 \rangle_{\mathcal{D}_1} = 0.
\]

**Corollary 7.2.4.** Let \( \mathbf{\xi}_{\perp,h} \in \mathcal{E}_0(\mathbb{D}_h) \) and \( \psi_h \in \mathcal{S}^{0,1}(\mathring{\mathbb{D}}_h) \) be solutions of the discrete version of the saddle point problem (6.46).

If Assumptions 3.2.1, 3.3.1 and 3.3.3 are satisfied, then:
\[
\| \mathbf{\xi}_{\perp,h} \|_{\mathring{H}^{0,-1/2}(\text{div}_{\mathcal{D}_1}, \mathcal{D}_1)} + \| \psi_h \|_{H^{1/2}(\mathcal{D}_1)} \leq \tilde{C} \| \mathbf{\mu}_h \|_{H^{1/2}(\mathcal{D}_1)}
\]
with \( \tilde{C} \) independent of \( h \).
7.2. Stable Discretization of $P(k)$

Proof. From the above stability analysis, we have

$$
\|\xi_{\perp,h}\|_{\bar{H}^{0,\perp}({\text{div}}_{2},D_1)} + \|\psi_h\|_{H^{1/2}(D_1)} \leq \hat{c} \sup_{\psi_h' \in S_h^1(D_h)} \frac{\|\mathbf{W} \psi_h', \psi_h'\|_{D_1}}{\|\psi_h'\|_{H^{1/2}(D_1)}},
$$

(7.31)

and the result follows from continuity of $\mathbf{W}$. \hfill \Box

Finally, we have the desired stability result for $N_{\perp}$.

Lemma 7.2.5. Under Assumptions 3.2.1, 3.3.1 and 3.3.3, the discretization $N_{\perp,h}$ of $N_{\perp}$ is stable and satisfies

$$
\|N_{\perp,h} g_h\|_{\bar{H}^{-1/2}({\text{div}}_{2},D_1)} \leq C_{\perp} \|g_h\|_{\bar{H}^{-1/2}({\text{curl}}_{2},D_1)}, \quad \forall g_h \in \mathcal{Y}(D_h),
$$

(7.32)

with $C_{\perp}$ independent of $h$.

Proof. Let $\xi_{\perp,h} := N_{\perp,h} g$. From the above results we have

$$
\|\xi_{\perp,h}\|_{\bar{H}^{-1/2}({\text{div}}_{2},D_1)} \leq c_1 \|\xi_{\perp,h}\|_{\bar{H}^{0,\perp}({\text{div}}_{2},D_1)} \\
\leq c_2 \|\mu_h\|_{H^{1/2}(D_1)} \\
\leq c_2 (\|\mu_h\|_{H^{1/2}(D_1)} + \|q_h\|_{\bar{H}^{0,\perp}({\text{div}}_{2},D_1)}) \\
\leq C_{\perp} \|g_h\|_{\bar{H}^{-1/2}({\text{curl}}_{2},D_1)}.
$$

(7.33)

\hfill \Box

7.2.2. Stability of $N_0$

Let us first note that the operators

$$
\text{curl}_{D_1} : \tilde{H}^{1/2}(D_1) \to X_0(D_1) \subset \bar{H}^{-1/2}({\text{div}}_{2},D_1) \\
\nabla : H^{-1/2}(D_1) \to \tilde{H}^{1/2}(D_1)
$$

are continuous and hence we have on the discrete setting

$$
\text{curl}_{D_1} : S^0_0(D_h) \subset \tilde{H}^{1/2}(D_1) \to \mathcal{E}_0(D_h) \subset \bar{H}^{-1/2}({\text{div}}_{2},D_1) \\
\nabla : S^{-1,0}_-(D_h) \subset H^{-1/2}(D_1) \to S^0_0(D_h) \subset \tilde{H}^{1/2}(D_1)
$$

are continuous. Consequently, it follows that:

Lemma 7.2.6. The discretization $N_{0,h}$ of $N_0$ is stable and satisfies

$$
\|N_{0,h} g_h\|_{\bar{H}^{-1/2}({\text{div}}_{2},D_1)} \leq C_0 \|g_h\|_{\bar{H}^{-1/2}({\text{curl}}_{2},D_1)}, \quad \forall g_h \in \mathcal{Y}(D_h),
$$

(7.33)

with $C_0$ independent of $h$.

Proposition 7.2.7. Under Assumptions 3.2.1, 3.3.1 and 3.3.3, the discretization of $P(k)$ is stable and satisfies

$$
\|\xi_k\|_{\bar{H}^{-1/2}({\text{div}}_{2},D_1)} \leq C(k) \|g_h\|_{\bar{H}^{-1/2}({\text{curl}}_{2},D_1)}, \quad \forall g_h \in \mathcal{Y}(D_h),
$$

(7.34)

with $C(k)$ a constant depending on $k$ but independent of $h$.

Finally, the inf-sup condition in Proposition 7.2.1 follows from plugging (7.34) on the left hand side of (7.19) and using the definition of dual norms.
7.3. GMRES convergence for Preconditioned Systems

In the first part of this thesis, when dealing with the Laplace operator, our preconditioning strategy was based on the fact that, for the method of Conjugate Gradients (CG), the relation between the spectral condition number of the system matrix and the number of iterations needed for the algorithm to converge has long been known for symmetric positive definite matrices (see for example [83, Sect. 13.1]).

Moreover, it is observed in practice that the condition number is also connected to the convergence of other Krylov subspace iterative solvers, like GMRES. Indeed, the performance of the algorithm is usually degraded when the system matrix has a large condition number and one would expect that it improves for smaller ones. However, unlike for CG, there is no proof available yet for this observation. There are, however, some GMRES convergence results relying on additional information of the system matrix [69, 31]. This subsection is intended to survey these estimates for GMRES convergence and discuss their applicability to our setting. Please note that we are interested in GMRES implemented with the Euclidean inner product.

Let us begin by describing what kind of matrices we will consider. Inspired by our BEM preconditioned system, we naturally focus on Galerkin matrices resulting from

$$P(k) A(k) = I_d + K(k),$$

defined in (7.5) and where $K(k)$ is a compact operator.

Hence we study the convergence results applying to complex matrices that are non-Hermitian, non-normal and diagonizable. Let us assume without loss of generality that for a given discretization, we have the following equation

$$Bu = f,$$  \hspace{1cm} (7.35)

where $B \in \mathbb{C}^{N \times N}$ for some $N \in \mathbb{N}$.

We write $u_n$ for the approximated solution of (7.35) computed by GMRES after $n$ iterations. We define the $n$-th GMRES residual as $r_n := f - Bu_n$. For the sake of simplicity, we assume the initial guess to be $u_0 = 0$ and thus $r_0 := f$.

The goal is to establish convergence bounds that guarantee that GMRES will solve (7.35) in a small number of iterations. We write this condition as $k_{end} \ll N$, where $k_{end} \in \mathbb{N}$ is the GMRES iteration where the algorithm reaches the required residual error and stops. Therefore, as expected, all GMRES convergence estimates boil down to sharply characterizing the norm $\|r_n\|_2$ at each step $n$.

The classical GMRES convergence results depend for example on the matrix spectrum, pseudospectra and field of values [69, 31, 39]. Unfortunately, these estimates have been shown to be exceedingly pessimistic for non-normal matrices [31]. Furthermore, they predict rates of linear convergence and thus cannot be used to prove the fast convergence one observes in our setting.

As a matter of fact, one often shows the fast convergence of GMRES by proving that it attains so-called superlinear convergence after some few iterations. In other words, one proves that the convergence rate of GMRES increases when the step $n$ increases.

The following result is often cited to justify the superlinear convergence of GMRES in presence of $I_d + K$ with $K$ compact.

**Theorem 7.3.1** ([67, Theorem 1]). Let $H$ be a complex separable Hilbert space with scalar product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$.

Let $B : H \rightarrow H$ be an invertible linear operator and consider the equation (7.35). For $\lambda \in \mathbb{R}$ such that $\lambda \neq 0$, let $K = B - \lambda I_d : H \rightarrow H$ be compact.

Then GMRES converges superlinearly and

$$\frac{\|r_n\|_H}{\|r_0\|_H} \leq \prod_{1 \leq j \leq n} \sigma_j(K) \sigma_j(B^{-1}),$$  \hspace{1cm} (7.36)

where $\sigma_j$ denotes the corresponding singular values (sorted in descending order).
We remark that this estimate applies when the norm in GMRES corresponds to the norm of the space $H$ where $K$ is compact, which is not exactly what we aim for here for the EFIE with $P(k) A(k) - \text{Id}$ compact in $H^{-1/2}(\text{div}_{D_1}, D_1)$.

Of course, since in finite dimensions every linear operator is "compact", Theorem 7.3.1 always applies in a matrix setting in $\mathbb{C}^{N \times N}$ and one is tempted to anyway use this result to explain the fast GMRES convergence that one observes in practice for compact-equivalent identities even with GMRES using the Euclidean norm. However, this is a flawed idea, as pointed out in [69, Sect. 1.8]: although in the finite dimensional case the superlinear behaviour is formally always there for GMRES, the finite termination property will take over if the singular values decay too slowly. And indeed that is what actually happens to GMRES when solving the system given by the (unpreconditioned) Galerkin matrix of $A(k)$, that also fulfills the requirements for the theorem above and for which we know that GMRES does not converge fast. Therefore, the take home message we want to convey is that we need to characterize the decay of the singular values to actually benefit from (7.36).

On the other hand, one would like to know if like for CG, one can show that in presence of $\text{Id} + K$ GMRES convergences after the same number of steps regardless of the mesh refinement. The estimate (7.36) hints that for this we would need to prove that the decay of singular values is $h$-independent. To the author’s knowledge, this has only been shown in some situations with very strong convergence assumptions [23].

In light of these open questions, we will provide an extensive study of the discrete eigenvalues and singular values when discussing our numerical results. With this we aim to numerically establishing $\text{Id}$-independence for the problem at hand. From these observations we will conclude that the bound (7.36) cannot be used to justify the fast convergence found in our numerical results.

Nonetheless, it is worth noticing that in some cases one can actually obtain the $h$-independence in (7.36) that we fail to observe in our setting. For example when solving the Helmholtz equation using FEM [5], where one wants to use the $H^1$-norm that is easy to approximate numerically, one implements GMRES using that norm instead of the usual Euclidean one, and achieves superlinear convergence bounds that are mesh independent. However, even if these results could be extended to other norms, implementing GMRES with the $H^{-1/2}(\text{div}_{D_1}, D_1)$-norm would most likely require a larger computational cost than its possible gain.

Finally, to complete the investigation of which existing GMRES convergence bounds actually describe the behaviour observed in our experiments with preconditioned systems, we discuss the following:

**Proposition 7.3.2 ([22, Prop. 5.1]).** Write $\gamma_j, 1 \leq j \leq J$ for the distinct eigenvalues of $B$ and denote the minimal polynomial of $B$ by

$$p_{\min}(z) := \prod_{1 \leq j \leq J} (z - \lambda_j)^{k_j},$$

with $k_j$ the smallest positive integer such that $\ker(\lambda_j I - B)^{k_j} = \ker(\lambda_j I - B)^{k_j + 1}$ for each $1 \leq j \leq J$.

Let $B$ have $M_1$ outlying eigenvalues and $P$ non-intersecting clusters of eigenvalues. Let $d := \sum_{j=1}^{M_1} k_j$ be the degree of the minimal polynomial associated with the outliers $\{\lambda_j\}_{j=1}^{M_1}$. The clusters are centered at distinct non-zero (complex) points $\{\gamma_p\}_{p=1}^{M+1}$.

Given $\rho > 0$, determine $0 \leq M_1 \leq M_2 \leq \cdots \leq M_{P+1} = J$ so that the non-intersecting sets

$$\{\lambda_j\}_{j=M_p-1}^{M_p} \subset \{z : |z - \gamma_p| < \rho |\gamma_p|\}, \quad 2 \leq p \leq P + 1,$$

are the clusters and

$$\{\lambda_j\}_{j=1}^{M_1} \subset \{z : |z - \gamma_p| > \rho |\gamma_p| \text{ for } 2 \leq p \leq P + 1\},$$

are the clusters and
are the outliers. Denote the corresponding spectral projectors by $Z_p$ with $1 \leq p \leq P + 1$. Define the distance of the outliers from the clusters as

$$
\delta := \max_{2 \leq p \leq P + 1} \max_{|z - \gamma_p| = \rho |\gamma_p|} \max_{1 \leq j \leq M_1} \frac{|\lambda_j - z|}{|\lambda_j|},
$$

(7.37)

and the maximal distance between clusters as

$$
\sigma := \max_{2 \leq p \leq P + 1} \max_{|z - \gamma_p| = \rho |\gamma_p|} \max_{q \neq p} \frac{|\gamma_q - z|}{|\gamma_q|}.
$$

(7.38)

Then for any $f$ and $u_0$

$$
\frac{\|r_{d+nP}\|_2}{\|r_0\|_2} \leq C (\sigma^{P-1} \rho)^n,
$$

(7.39)

where the constant $C := \delta^d P \rho \max_{2 \leq p \leq P + 1} \max_{|z - \gamma_p| = \rho |\gamma_p|} \|(z I - Z_p B)^{-1}\|$ is independent of $n$.

This result is often labeled as qualitative since it includes asymptotic constants that are hard to compute and could potentially be unbounded. However, it is due to the fact that it depends only on asymptotic constants that this bound is mesh independent for fine enough discretizations. Moreover, we will see in our numerical results in Section 7.4 that, unlike the previous bounds, we can actually verify these results by plotting our eigenvalues, marking the clusters and outliers and noting how the convergence deteriorates in the presence of larger clusters or more outliers.

On the other hand, another reason to consider the results in [22] as qualitative is that currently it is not clear how sensitive is GMRES to the distance of the eigenvalues and thus how it differentiates among clusters. This is important to take into account, as it prevents us from arbitrarily choosing the parameters such that the resulting eigenvalue clusters and outliers make the right hand side of (7.39) predict a number of GMRES iterations that matches that observed in the numerical experiments, reaching wrong conclusions about the accuracy of this estimate.
7.4. Numerical Results on Disks

Numerical experiments were implemented using BETL2 [53] with the same parameters and considerations as in subsection 3.4.1. Likewise, we employ the same meshes as in Chapter 3. In all experiments, we use either CG or GMRES with a tolerance of $10^{-5}$ for the relative residual norm, initial guess equal to zero and, as right hand side, we considered a vector that had entries +1 in its upper half, −1 for the remaining components. As in subsection 3.4.1, when using CG with ACA-compressed matrices, we will estimate the spectral condition number using CG with Lanczos. This motivates that we set MaxIter equal two times the number of columns of the system matrix for CG. On the other hand, for GMRES we put MaxIter to three times the number of columns of the system matrix.

Let $N_{0,h}$ and $N_{\perp,h}$ represent the discrete application of $N_0$ and $N_{\perp}$. We first present numerical results for the positive definite static EFIE introduced in (6.2), which can be solved with CG and has some nice properties that we would like to verify as a sanity check. Then we exhibit the numerical experiments corresponding to preconditioning a positive definite EFIE that will be defined in subsection 7.4.2. Finally, we show the numerical results for the standard EFIE and study the properties related to the GMRES convergence discussed in the previous Section.

7.4.1. Positive Definite Static EFIE

First we consider the variational problem arising from $S$: Given $g \in H^{-1/2}(\text{curl}_{D_1}, \mathbb{D}_1)$, seek $\xi \in \widetilde{H}^{-1/2}(\text{div}_{D_1}, \mathbb{D}_1)$ such that

$$\langle \nabla \xi, \xi \rangle_{D_1} + \langle \nabla \text{div}_{D_1} \xi, \text{div}_{D_1} \xi \rangle_{D_1} = \langle g, \xi \rangle_{D_1}, \quad \forall \xi' \in \widetilde{H}^{-1/2}(\text{div}_{D_1}, \mathbb{D}_1). \quad (7.40)$$

We denote the corresponding Galerkin matrix by $S_h$. Moreover, we consider the splitting $S = S_{0,h} + S_{\perp,h}$, with $S_{0,h}$ and $S_{\perp,h}$ being the Galerkin matrices of the first and second term on the left hand side of (7.40), respectively. Then, we precondition $S_h$ using $P_h = N_{0,h} + N_{\perp,h}$.

We begin our study of the obtained numerical results by a small consistency check. From the properties of $S$ and $N_0 + N_{\perp}$, we know that the matrices $S_h$ and $P_h$ should both be non-singular. On the other hand, from the Hodge decomposition we have

$$\ker(S_{\perp}) = X_0(\mathbb{D}_1) = \text{curl}_{D_1} \widetilde{H}^{1/2}(\mathbb{D}_1),$$
$$\ker(N_0) = (X_\perp(\mathbb{D}_1))' = Y_0(\mathbb{D}_1),$$
$$\ker(N_{\perp}) = (X_0(\mathbb{D}_1))' = Y_\perp(\mathbb{D}_1) = \text{curl}_{D_1} \mathcal{H}_{00}(\mathbb{D}_1).$$

Therefore, in the discrete setting, we should have that $\dim(\ker(S_{\perp})) = \dim(S_{0,1}^{0,1}(\mathbb{D}_h)) = \dim(\ker(N_{\perp,h}))$, and $\dim(\ker(N_{0,h})) + \dim(\ker(N_{\perp,h})) = \dim(\mathcal{E}_0(\mathbb{D}_h))$, and this is exactly what we want to verify. We study this on a family of quasi-uniform meshes of $\mathbb{D}_1$ whose data of interest is displayed in Table 7.1.

<table>
<thead>
<tr>
<th>$N$</th>
<th># vertices</th>
<th># edges</th>
<th># interior vertices</th>
<th># interior edges</th>
</tr>
</thead>
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<tr>
<td>64</td>
<td>42</td>
<td>104</td>
<td>25</td>
<td>88</td>
</tr>
<tr>
<td>256</td>
<td>146</td>
<td>400</td>
<td>113</td>
<td>368</td>
</tr>
<tr>
<td>1024</td>
<td>546</td>
<td>1568</td>
<td>481</td>
<td>1504</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\ker(S_h)$</th>
<th>$\ker(S_{0,h})$</th>
<th>$\ker(S_{\perp,h})$</th>
<th>$\ker(P_h)$</th>
<th>$\ker(P_{0,h})$</th>
<th>$\ker(P_{\perp,h})$</th>
<th>$\ker(P_{0,h}S_{0,h})$</th>
<th>$\ker(P_{\perp,h}S_{\perp,h})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>63</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>256</td>
<td>0</td>
<td>0</td>
<td>13</td>
<td>255</td>
<td>113</td>
<td>255</td>
<td>113</td>
<td>113</td>
</tr>
<tr>
<td>1024</td>
<td>0</td>
<td>0</td>
<td>481</td>
<td>1023</td>
<td>481</td>
<td>1023</td>
<td>481</td>
<td>481</td>
</tr>
</tbody>
</table>
Table 7.2 shows the dimension for the different matrix kernels. We see that
- $S_h$ and $P_h$ are non-singular matrices as expected.
- $\dim(\text{Ker}(S_{1,h})) = \dim(\text{Ker}(N_{1,h})) = \# \text{ interior vertices}$.
- $\dim(\text{Ker}(N_{0,h})) + \dim(\text{Ker}(N_{1,h})) = \# \text{ interior edges}$.
- $\dim(\text{Ker}(N_{0,h}S_{0,h})) + \dim(\text{Ker}(N_{1,h}S_{1,h})) = \# \text{ interior edges}$.

Hence the matrices have the expected kernel dimensions. This is coherent with Figure 7.1 that exhibits the spectra following the splitting $P_{0,h}N_{0,h}$ and $N_{1,h}S_{1,h}$. There we clearly see the cluster around 1, the kernel modes of these operators and the complementarity of their images and kernels. Additionally, in Figure 7.2 we plot the spectra following this splitting but suppressing the eigenvalues that are smaller than machine precision. By doing this we can better appreciate the clustering of the non-zero eigenvalues around 1.

**Figure 7.1.:** Spectra of split operators $N_{0,h}S_{0,h}$ and $N_{1,h}S_{1,h}$ arising from preconditioning the positive definite static EFIE $S$ on a quasi-uniform mesh with $N=1024$ elements.

**Figure 7.2.:** Non-zero spectra of split operators $N_{0,h}S_{0,h}$ and $N_{1,h}S_{1,h}$ arising from preconditioning the positive definite static EFIE on a quasi-uniform mesh with $N=1024$ elements.

Figure 7.3 shows the spectra of $S_h$ and $P_hS_h$, where we observe that the preconditioned matrix achieves a $h$-independent eigenvalues clustering as expected. In addition, we present the achieved CG results

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2The computation of these kernel’s dimensions and the condition numbers of dense matrices, together with all plots in the numerical experiments' sections in this thesis, were done using MATLAB R2017a, The MathWorks, Inc., Natick, Massachusetts, United States.

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using ACA compression with the same parameters as in subsection 3.4.1. Table 7.3 contains the results for the usual quasi-uniform mesh, together with those for the non-quasi-uniform and locally-refined meshes from Figures 3.10a and 3.10b, respectively. We observe that the spectral condition number of the preconditioned matrix that has a minor increase when refining the mesh and that CG converges after a small number of iterations as expected.

Figure 7.3.: Spectra for positive definite static EFIE on a sequence of three quasi-uniform meshes.

Table 7.3.: Preconditioning results for static positive definite EFIE. The condition numbers of $S_h$ over the last two levels of the locally refined meshes were too expensive to compute with the Lanczos algorithm, and thus left empty. The numbers with * show the iteration where the Lanczos algorithm reached the maxIter limit without achieving stagnation (see details in Section 3.4.1).

<table>
<thead>
<tr>
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<th>CG it</th>
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<th>PCG it</th>
<th>$\kappa(P_hS_h)$</th>
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<td>16546</td>
<td>7618</td>
<td>–</td>
<td>8</td>
<td>2.62</td>
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</table>
7.4.2. Positive Definite EFIE

We define the positive definite EFIE operator $S(k)$ for $k > 0$ as

$$S(k) := \nabla_k - \frac{1}{k^2} \text{grad}_{D_1} \nabla_k \text{div}_{D_1}.$$  (7.41)

Its associated variational problem is: Given $g \in H^{-1/2}(\text{curl}_{D_1}, D_1)$ and $k > 0$, seek $\xi \in \tilde{H}^{-1/2}(\text{div}_{D_1}, D_1)$ such that

$$\langle \nabla_k \xi, \xi' \rangle_{D_1} + \frac{1}{k^2} \langle \nabla_k \text{div}_{D_1} \xi, \text{div}_{D_1} \xi' \rangle_{D_1} = \langle g, \xi' \rangle_{D_1}, \quad \forall \xi' \in \tilde{H}^{-1/2}(\text{div}_{D_1}, D_1).$$  (7.42)

We denote the corresponding Galerkin matrix by $S_h^k$ and consider $P_h^k = N_{0,h} + k^2 N_{\perp,h}$ as a preconditioner for it.

Table 7.4 shows the achieved GMRES results on quasi-uniform and non-quasi-uniform (see Figure 3.10a) meshes for ACA-compressed matrices and considering six different wave numbers. As expected, for all of them we note that the number of iterations it takes GMRES to converge for $S_h^k$ increases like $h$, while for the preconditioned matrix, it remains almost constant. On the other hand, we also observe that GMRES convergence improves for $S_h^k$ when $k$ grows. On the contrary, although our preconditioner still reduces considerably the number of needed GMRES iterations, its performance deteriorates when $k$ increases. This is not surprising because the $P_h^k$ was constructed from the static case $k = 0$.

| Table 7.4.: GMRES iterations for positive definite EFIE and different wave numbers $k$. |
|---|---|---|---|---|---|---|
| $N$ | $k = 0.01$ | $k = 0.1$ | $k = 0.5$ | $k = 1$ | $k = 2$ | $k = 4$ |
| Quasi-uniform | | | | | | |
| 64 | 77 | 6 | 75 | 8 | 67 | 10 | 58 | 10 | 40 | 12 | 30 | 20 |
| 256 | 215 | 7 | 173 | 9 | 135 | 10 | 111 | 10 | 85 | 13 | 57 | 21 |
| 1024 | 414 | 8 | 329 | 9 | 259 | 10 | 224 | 11 | 168 | 14 | 112 | 22 |
| 4096 | 815 | 9 | 639 | 9 | 510 | 11 | 436 | 12 | 344 | 15 | 225 | 23 |
| 16384 | 1552 | 10 | 1241 | 10 | 972 | 12 | 849 | 13 | 690 | 16 | 467 | 24 |
| Non-quasi-uniform | | | | | | |
| 96 | 108 | 6 | 107 | 8 | 80 | 9 | 61 | 10 | 45 | 13 | 34 | 21 |
| 384 | 295 | 7 | 232 | 9 | 179 | 10 | 148 | 10 | 109 | 14 | 71 | 22 |
| 1536 | 660 | 8 | 508 | 10 | 397 | 11 | 335 | 12 | 259 | 15 | 170 | 24 |
| 6144 | 1241 | 10 | 984 | 10 | 784 | 12 | 678 | 13 | 534 | 16 | 359 | 24 |

7.4.3. Standard EFIE

We finally use $P_h^k = N_{0,h} - k^2 N_{\perp,h}$ to precondition the Galerkin matrix $A_h^k$, arising from the EFIE variational Problem 5.4.3. As before, we present the results obtained using ACA-compressed matrices on quasi-uniform meshes and also on non-quasi-uniform meshes (see Figure 3.10a). Moreover, since this is the operator we are interested the most in the second part of this thesis, we carry out the computation of the anticipated spectra and singular values analysis on quasi-uniform meshes in order to make the connection between these properties, the numerical results, and the GMRES bounds (7.36) and (7.36).

Tables 7.5 and 7.6 report the GMRES results using ACA-compressed matrices for six different wave numbers. We see that the number of iterations it takes GMRES to converge for $A_h^k$ increases like the meshwidth, while it remains almost constant for $P_h^k A_h^k$. Similarly to $S_h^k$, we also observe that GMRES convergence improves for $A_h^k$ for larger wave number $k$. Conversely, the performance of our preconditioner declines when $k$ increases, which again makes sense, as it was constructed for the static case $k = 0$. In addition, Figure 7.4 illustrates how the number of iterations change on a fixed mesh when considering an extended range of wave numbers $k$. This plot was generated with a $\Delta k = 0.1$ and $0.1 \leq k \leq 6$. There we see that GMRES needs considerably more iterations to solve $A(k)$ when $k$ is smaller. This reflects
the so-called low-frequency break-down, i.e. the condition number of the EFIE operator increases when $k$ decreases. Figure 7.4 also shows the remarkable robustness of our preconditioner in the "low-frequency limit" $k \to 0$, preventing the low-frequency break-down.

**Table 7.5.** GMRES iterations for (indefinite) EFIE and different wave numbers $k$ over quasi-uniform mesh.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$A_h^k D_h^{-1} A_h^k P_h^k A_h^k$</th>
<th>$A_h^k D_h^{-1} A_h^k P_h^k A_h^k$</th>
<th>$A_h^k D_h^{-1} A_h^k P_h^k A_h^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
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<td>85</td>
<td>6</td>
</tr>
<tr>
<td>1024</td>
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<td>168</td>
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</tr>
<tr>
<td>16384</td>
<td>1533</td>
<td>573</td>
<td>8</td>
</tr>
</tbody>
</table>

**Table 7.6.** GMRES iterations for (indefinite) EFIE and different wave numbers $k$ over non-uniform mesh.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$A_h^k D_h^{-1} A_h^k P_h^k A_h^k$</th>
<th>$A_h^k D_h^{-1} A_h^k P_h^k A_h^k$</th>
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<tr>
<td>6144</td>
<td>1241</td>
<td>520</td>
<td>8</td>
</tr>
</tbody>
</table>

**Figure 7.4.** Results for EFIE on a quasi-uniform mesh with $N=1024$ elements and different values of $k$. 

![Figure 7.4](image-url)
Figure 7.5.: GMRES convergence history for EFIE and different wave numbers $k$.

$k = 0.1$

$k = 1$

$k = 2$
7.4. Numerical Results on Disks

Figure 7.6.: Spectra of $A^k_h$ and $P^k_h A^k_h$ for different wave numbers $k$.

In addition, Figure 7.5 depicts how the residual norm approximated by GMRES decays at each iteration. There we can observe how when increasing $k$, GMRES takes longer to attain the superlinear phase. We would like to analyze if one could explain this behaviour making use of the two GMRES bounds we discussed earlier.

Let us begin considering Proposition 7.3.2. For this we present the eigenvalues of the EFIE operator and of its preconditioned version for three different levels of refinement and wave numbers in Figure 7.6. There we can clearly see how the cluster of eigenvalues of the original operator grows when increasing the
mesh refinement but shrinks for larger k, which is consistent with the GMRES results we get for $A(k)$. On the other hand, we see how for the preconditioned eigenvalues cluster around one almost perfectly for $k = 0.1$ and begin spreading when increasing the wave number $k$. This matches the observations made from Tables 7.5 and 7.6.

Figure 7.7 offers a closer look to the eigenvalues of $P(k)A(k)$ for $k = 0.1$ and $k = 2$ and the matrices computed on the same three levels of quasi-uniform meshes as before. Here we arbitrarily drew the most prominent eigenvalue clusters and remark that their radii are almost independent of the refinement level. Moreover, when comparing the situation of $k = 0.1$ with $k = 2$, we notice that in spite of the fact that both have three eigenvalues clusters, $\rho$ for $k = 0.1$ is slightly smaller than for $k = 2$. Additionally, the maximal distance between clusters and the number of outliers are both larger for $k = 2$. All these facts contribute to the fact that the right hand side of (7.39) is bigger for $k = 2$ than for $k = 0.1$, but as we cannot compute $C$, we cannot conclude whether this accounts for the almost 10 additional iterations that GMRES takes to converge on the former case.

Figure 7.7.: Zoomed-in spectra of $P(k)A(k)$ for $k = 0.1$ and $k = 2$.

Figure 7.8.: Singular values for $P(k)A(k)$ with different values of $k$ and level of meshes. We use black for $k = 0.1$, red for $k = 1$ and blue for $k = 2$.
Figure 7.9: Singular values of EFIE with different values of $k$ and three mesh levels. The black circles show the values corresponding to $A(k)$ while the red circles depict those of $P(k)A(k)$. 

For $k=0.1$:

For $k=1$:

For $k=2$:
Figure 7.10.: Loglog plot of the singular values for $\mathbf{I d} - \mathbf{P}(k)\mathbf{A}(k)$ and $(\mathbf{P}(k)\mathbf{A}(k))^{-1}$ with different values of $k$ over three levels of a family of quasi-uniform triangular meshes. We use black to show the results for $N=64$ elements, red for $N=256$ elements and blue for $N=1024$ elements.

(a) $\mathbf{I d} - \mathbf{P}(k)\mathbf{A}(k)$

(b) $(\mathbf{P}(k)\mathbf{A}(k))^{-1}$

Now, let us study (7.36). With this purpose in mind, we plot the singular values of $\mathbf{P}^h\mathbf{A}^h$ in Figures 7.8 and 7.9. The former highlights how the clustering of singular values changes for different wave numbers $k$, while the latter shows the contrast between the singular values of $\mathbf{A}^h$ and $\mathbf{P}^h\mathbf{A}^h$.

Additionally, Figure 7.10 exhibits the singular values of the compact operator $\mathbf{I d} - \mathbf{P}(k)\mathbf{A}(k)$, which
Figure 7.11.: Loglog plot of the distribution of singular values for $I_d - P(k)A(k)$ and $(P(k)A(k))^{-1}$ with different values of $k$ on a quasi-uniform triangular mesh with $N=1024$ elements.

(a) $I_d - P(k)A(k)$

(b) $(P(k)A(k))^{-1}$


corresponds to the matrix $K_h = M_h - P_h^kA_h^k$, and those of $(P_h^kA_h^k)^{-1}$. As expected, we see that $\sigma_j(K)$ go to zero, but we also note that $\sigma_{\min}(K)$ depends on $h$. Similarly, the singular values of $(P_h^kA_h^k)^{-1}$ also seem to depend on the mesh. Figure 7.11 further illustrates how the singular values are distributed when plotted in descending order for a given mesh. We find that the singular values of $K_h$ do decay exponentially but almost at the end of the set of indices.

We compute the right hand side of (7.36) and display it in Figure 7.12 for the same three mesh levels and wave numbers as in the previous figures. We also present the values of taking the $n$-th root for
(7.36) on each step $n$ in order to visualize a sort of convergence rate given by the bound. We see that the estimate predicts the superlinear convergence but way later than actually observed in our numerical experiments, so it is not sharp enough to explain the resulting GMRES fast convergence. Moreover, it is clear from the plots that the bound (7.36) is not $h$-independent for our setting.

**Figure 7.12.** Loglog plot of computed values of (7.36) for each step $n$ and over three levels of a family of quasi-uniform triangular meshes. The results for a mesh with $N=64$ elements are shown in black, those for $N=256$ elements are depicted in red, while the results for a mesh with $N=1024$ elements are drawn in blue.

(a) right hand side of (7.36) 

(b) $n$-th root of right hand side of (7.36)
7.5. Preconditioning on more general screens

We take the cue from Section 3.5 and extend our operator preconditioning strategy for the EFIE to more general screens $\Gamma \subset \mathbb{R}^3$, for which there is a bi-Lipschitz continuous piecewise $C^1$-mapping $\phi : \mathbb{D}_1 \times [-1,1] \to \mathbb{R}^3$ such that $\phi(\mathbb{D}_1) = \mathbb{T}$ for $\phi := \hat{\phi}|_{\mathbb{D}_1 \times \{0\}}$.

Recall from Section 3.5 that we write $\phi^* v : \mathbb{D}_1 \to \mathbb{R}$ for the pullback of a function $v : \Gamma \to \mathbb{R}$ to $\mathbb{D}_1$ under $\phi$:

$$\hat{\phi}(\hat{x}) := (\phi^* v)(\hat{x}) = v(\phi(\hat{x})), \quad \hat{x} \in \mathbb{D}_1.$$ 

We also remind the reader that

$$\phi^* : L^2(\Gamma) \to L^2(\mathbb{D}_1), \quad \phi^* : H^1(\Gamma) \to H^1(\mathbb{D}_1), \quad \phi^* : H^1_0(\Gamma) \to H^1_0(\mathbb{D}_1),$$

are isomorphisms, since $\phi$ is bi-Lipschitz [65, Thm 3.23]. Additionally, we proved that

$$\phi^* : H^{\pm 1/2}(\Gamma) \to H^{\pm 1/2}(\mathbb{D}_1), \quad \phi^* : \tilde{H}^{\pm 1/2}(\Gamma) \to \tilde{H}^{\pm 1/2}(\mathbb{D}_1),$$

were also isomorphisms.

Now, by definition of the vector Sobolev spaces, these isomorphisms carry over to $L^2(\Gamma), H^1(\Gamma)$ and $H^1_0(\Gamma)$ and by interpolation arguments to $H^{\pm 1/2}(\Gamma)$ and $\tilde{H}^{\pm 1/2}(\Gamma)$. Via the definition

$$\|u\|_{H^{\pm 1/2}(\Gamma)} = \inf_{\gamma \in H^{\pm 1/2}(\Gamma)} \{\|u\|_{H^{\pm 1/2}(\Gamma)} : \gamma(\mathbf{v})\}.$$ 

(7.43)

it is clear how we can likewise show that $\phi^* : H^{\pm 1/2}(\Gamma) \to H^{\pm 1/2}_x(\mathbb{D}_1)$ is an isomorphism. Then, again by interpolation and duality, we also get that property for $\phi^* : \tilde{H}^{\pm 1/2}(\Gamma) \to \tilde{H}^{\pm 1/2}_x(\mathbb{D}_1)$.

Finally, using the characterization of $H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1)$ as a graph space equipped with the corresponding graph norm, we immediately infer that

$$\phi^* : H^{-1/2}(\text{curl}_{\Gamma}, \Gamma) \to H^{-1/2}(\text{curl}_{\mathbb{D}_1}, \mathbb{D}_1),$$

(7.44)

is an isomorphism too and by duality, this will also hold for

$$\phi^* : \tilde{H}^{-1/2}(\text{div}_{\Gamma}, \Gamma) \to \tilde{H}^{-1/2}(\text{div}_{\mathbb{D}_1}, \mathbb{D}_1),$$

(7.45)

Similarly to what we did in Section 3.5, one uses isomorphisms (7.44) and (7.45) to show that $P(k)$ (defined on $\mathbb{D}_1$) is a suitable preconditioner for $A(k)$ on the screen $\Gamma$. Hence this strategy should again conserve the $h$-independence of the resulting spectral condition number, although we do lose the "identity plus compact" property from the unit disk.

7.5.1. Numerical Results for EFIE on Mapped Screens

We consider ACA-compressed matrices and the same implementation details and parameters as in subsection 3.5.2 but using GMRES instead of CG. For GMRES itself, we use the same parameters as in Section 7.4.

Here $A(k)$ is mapped from the unit disk $\mathbb{D}_1$ to the target screen $\Gamma := \phi(\mathbb{D}_1)$ and $P(k) = N_0 - k^2 N_\perp$ is computed on $\mathbb{D}_1$. We employ the same matrix notation as in subsection 7.4.3: $A_h^k$ stands for the Galerkin matrix of $A(k)$, $D_h^{-1}$ denotes diagonal preconditioning, and $P_h^k$ the discretization of $P(k)$.

We show numerical results for six different wave numbers and four different shapes: The first data are listed in Table 7.7 and correspond to the square screen introduced in Example 1 from Section 3.5 (see formula in (3.50)). The other three shapes follow Example 2 in Section 3.5 with diverse functions $f$ and the corresponding results are shown in Tables 7.8-7.10. For all studied mapped screens we considered the usual family of quasi-uniform triangular meshes of $\mathbb{D}_1$. 

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In all these cases we see that the preconditioner reduces significantly the number of GMRES iterations, although this time the GMRES iteration counts change more noticeably with the refinement level. Surprisingly, for the set of chosen shapes, the difference between the number of GMRES iterations on the finest and coarsest levels for \(P_h^kA_h^k\) is the largest for the square (\(\sim 10\)) and not for the parabola, which has the largest distortion in terms of the Gram matrix. Figure 7.13 shows the eigenvalues for the mapped square using \(k = 0.1\) and \(k = 2\). There we can see that unlike for the unit disk (c.f. Figure 7.6), the cluster of the eigenvalues does seem to expand when refining the mesh, thus explaining the increase on GMRES iterations, but contradicting what one would expect from the theory.

We also note that the unpreconditioned and diagonally preconditioned results are qualitatively similar to those achieved for the EFIE on the unit disk (see Table 7.5). However, the performance of the preconditioner \(P(k)\) seems more sensitive to the increase of wave number than its unit disk counterpart, and also more sensitive to the transformations than its scalar analogous from subsection 3.5.2.

### Table 7.7.: GMRES iterations for EFIE and different wave numbers \(k\) on mapped square (from (3.50)).

<table>
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<th>(k = 0.01)</th>
<th>(k = 0.1)</th>
<th>(k = 0.5)</th>
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<td>(P_h^kA_h^k)</td>
<td>(A_h^k)</td>
</tr>
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<td>64</td>
<td>77</td>
<td>44</td>
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</tr>
<tr>
<td>16384</td>
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<td>656</td>
<td>18</td>
</tr>
</tbody>
</table>

### Table 7.8.: GMRES iterations for EFIE and different wave numbers \(k\) on mapped screens \(\phi(x) = (x_1, x_2, x_1 + x_2)^T\).

<table>
<thead>
<tr>
<th>(N)</th>
<th>(k = 0.01)</th>
<th>(k = 0.1)</th>
<th>(k = 0.5)</th>
</tr>
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<td>(A_h^k)</td>
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<td>(P_h^kA_h^k)</td>
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<td>(P_h^kA_h^k)</td>
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<td>16384</td>
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<td>783</td>
<td>27</td>
</tr>
</tbody>
</table>
7.5. Preconditioning on more general screens

Figure 7.13.: Spectra of $A_k^h$ and $P_k^h A_k^h$ on mapped square for different wave numbers $k$.

Table 7.9.: GMRES iterations for EFIE and different wave numbers $k$ on mapped screens $\phi(x) = (x_1, x_2, x_1 x_2)^T$.
### Table 7.10.: GMRES iterations for EFIE and different wave numbers $k$ on mapped screens $\phi(x) = (x_1, x_2, x_1^2 + x_2^2)^T$.

<table>
<thead>
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Conclusion and Closing Remarks

"One never notices what has been done; one can only see what remains to be done."
– Marie Curie (1894).

With this thesis, we achieved for the first time operator identities that offer a natural replacement for Calderón identities in the case of screens. Furthermore, we accomplished operator preconditioners for screens that are furnished with a complete mathematical analysis, in most cases attain optimality in the sense of $h$-uniform spectral condition numbers, and are independent of the choice of mesh and discretization spaces. We point out that this includes a broader set of shapes of screens than previous works and, as indicated before, our proposal has the advantage of being able to tackle a wide range of non-uniform meshes. The only case where we did not observe mesh-independent spectra was for the EFIE on mapped screens. Further investigations are needed to understand whether those results are reflecting a pre-asymptotic phase or if there are additional aspects to be considered in its analysis.

Although we were not able to prove that this optimality translates into GMRES $h$-independent superlinear convergence, we provided numerical results that show the desired fast GMRES convergence. Additionally, we studied numerically the applicability of some current GMRES convergence estimates.

It is important to note that by construction, our preconditioners for the Helmholtz BIzos and for the EFIE are effective only for small wave numbers. The numerical results in subsections 3.4.2, 7.4.3 and 7.5.1 clearly support this notion and show that the preconditioner for the EFIE does an excellent job for wave numbers closer to zero, hence preventing the so-called low-frequency break down.

Our preconditioning results are robust and significantly reduce the number of iterations needed for CG or GMRES to converge. It is however important to point out that one should keep in mind also the computational cost of setting-up the preconditioners to assess whether they offer an efficient alternative or not. This is currently the main disadvantage of the operator preconditioning technique, as its stable construction uses dual meshes that are built by means of a barycentric refinement that increase the number of elements by six on triangular meshes, as we saw in subsection 3.2.2.

The implementation developed for the numerical experiments in this thesis was conceived as a proof of concept and not focused on efficiency. This is the reason why we could not give fair measurements for the computational times and did not include this aspect in our analysis. Thus, we cannot conclude that our preconditioner is efficient, and we even observe in our results that if not implemented adequately, the total computational time with preconditioning is larger than the solving time with no preconditioning. Nonetheless, in view of the computational complexity of the EFIE preconditioner, which is similar to that of the multiplicative preconditioner in [4], we have reasons to believe that it should be competitive also in terms of computational time when using an efficient implementation. On the other hand, our scalar preconditioners from Chapter 3 are computationally more expensive than the usual opposite order approach, due to the arctan function in the kernels of $\overline{V}$ and $\overline{W}$. For this reason, we predict from the results in subsection 3.4.1 that even with efficient implementations, both preconditioners will perform equally good in most situations, in spite of the suboptimality of the opposite order strategy on screens. Yet, recent work developed in the thesis of Adrian [2, Chapter 8] constructs operator preconditioning without the overhead of working on the barycentric refinements. This is a promising approach that could be adapted to notably reduced the set-up time also for our preconditioners.

Finally, we highlight that the Calderón-type identities for screens shown in Chapters 2 and 6 are indeed a powerful tool. Not only they can be applied to other computationally more efficient preconditioning strategies, but they can also be extended to other screen problems like linear elasticity and continue to shed some new light into these interesting geometries.
Bibliography


