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Vertex Covering with Monochromatic Pieces of few Colours

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Abstract

In 1995, Erdős and Gyárfás proved that in every 2-colouring of the edges of K_n, there is a vertex covering by $2\sqrt{n}$ monochromatic paths of the same colour, which is optimal up to a constant factor. The main goal of this paper is to study the natural multi-colour generalization of this problem: given two positive integers r, s, what is the smallest number $\mathrm{pc}_{r,s}(K_n)$ such that in every colouring of the edges of K_n with r colours, there exists a vertex covering of K_n by $\mathrm{pc}_{r,s}(K_n)$ monochromatic paths using altogether at most s different colours?

For fixed integers $r > s$ and as $n \to \infty$, we prove that $\mathrm{pc}_{r,s}(K_n) = \Theta(n^{1/\chi})$, where $\chi = \max\{1, 2 + 2s - r\}$ is the chromatic number of the Kneser graph $\mathcal{KG}(r, r - s)$. More generally, if one replaces K_n by an arbitrary n-vertex graph with fixed independence number α, then we have $\mathrm{pc}_{r,s}(G) = O(n^{1/\chi})$, where this time around χ is the chromatic number of the Kneser hypergraph $\mathcal{KG}^{(\alpha+1)}(r, r - s)$. This result is tight in the sense that there exist graphs with independence number α for which $\mathrm{pc}_{r,s}(G) = \Omega(n^{1/\chi})$. This is in sharp contrast to the case $r = s$, where it follows from a result of Sárközy (2012) that $\mathrm{pc}_{r,r}(G)$ depends only on r and α, but not on the number of vertices.

We obtain similar results for the situation where instead of using paths, one wants to cover a graph with bounded independence number by monochromatic cycles, or a complete graph by monochromatic d-regular graphs.

Mathematics Subject Classifications: 05C38, 05C55

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1 Introduction

Call a subgraph of an edge-coloured graph *monochromatic* if all its edges have the same colour. This paper is concerned with the general problem of covering all the vertices of an edge-coloured graph by monochromatic pieces. To be more precise, suppose that $\mathcal{F}$ is a fixed family of graphs, containing the 'pieces' that we can use for the covering. A *monochromatic $\mathcal{F}$-covering* of an edge-coloured graph G is then a collection of monochromatic subgraphs of G covering all the vertices, such that every subgraph in the collection is isomorphic to one of the graphs in $\mathcal{F}$. Typical choices for $\mathcal{F}$ include the the collection $\mathcal{F}_p$ of all paths or the collection $\mathcal{F}_c$ of all cycles, where it is customary to consider single vertices and edges as degenerate cycles. Given a graph G, we are interested in finding monochromatic $\mathcal{F}$-coverings that are as small as possible; for example, we might want to cover G using as few monochromatic paths or cycles as possible.

This type of problem goes back to a footnote in a 1967 paper of Gerencsér and Gyárfás [11] in which it is shown that in every colouring of the edges of the complete graph K_n with two colours, one can find two monochromatic paths that form a partition of (and, in particular, a covering) all the vertices. Over the last fifty years, such problems have been studied in many variations, including for more than two colours [10, 14, 24], for various other choices of $\mathcal{F}$ (most notably for the family of cycles [2, 6, 13, 16, 23], but also for regular graphs [25], bounded-degree graphs [12], trees [1, 7]), and for other choices of G (complete bipartite and multipartite graphs [7, 14, 17, 26], graphs satisfying a minimum degree condition [5, 8, 21], random graphs [4, 18, 20], graphs with bounded independence number [5, 27], . . .). We note that like the Gerencsér-Gyárfás result mentioned above, most (but not all) of these results apply to the stronger situation where one wants to *partition* the vertices of the graph into disjoint monochromatic pieces (as opposed to just covering the vertices). For more details we refer to the recent survey of Gyárfás [15].

The specific focus of this paper is on monochromatic $\mathcal{F}$-coverings that altogether *do not use too many different colours*. For a collection $\mathbf{S}$ of monochromatic edge-coloured graphs, we denote by $\operatorname{col}(\mathbf{S})$ the total number of different colours used by the graphs in $\mathbf{S}$. Then, given a graph G, a family $\mathcal{F}$, and positive integers r and s, we will write $c_{r,s}(G, \mathcal{F})$ for the smallest number with the property that every r-colouring of the edges of G admits a monochromatic $\mathcal{F}$-covering $\mathbf{S}$ such that $|\mathbf{S}| \leqslant c_{r,s}(G, \mathcal{F})$ and $\operatorname{col}(\mathbf{S}) \leqslant s$.

For the simplest case where $\mathcal{F} = \mathcal{F}_p$ is the collection of paths, where there are only two colours, and where G is the complete graph, Erdős and Gyárfás [9] proved that

$$\sqrt{n} \leqslant c_{2,1}(K_n, \mathcal{F}_p) \leqslant 2\sqrt{n}.^{1} \tag{1}$$

It is open which of the two bounds (if any) is correct; Erdős and Gyárfás conjectured that the true value is $\sqrt{n}$. In any case, we observe that this result is in stark contrast to the above-mentioned result of Gerencsér and Gyárfás [11], which implies that

$$c_{2,2}(K_n, \mathcal{F}_p) = 2,$$

[1] The quantity $c_{r,s}(G, \mathcal{F}_p)$ was denoted $\mathrm{pc}_{r,s}(G)$ in the abstract. Henceforth, we will only use the more flexible notation $c_{r,s}(G, \mathcal{F}_p)$.

which is a constant independent of n. One goal of this project was to see how the result (1) generalizes to to other values of r and s.

1.1 Our results

In this paper, we restrict ourselves to graphs G with independence number at most $\alpha > 0$. We suppose that r, s, α are constants and that the size of G tends to infinity. Given r, s, α, we write

$$c_{r,s,\alpha}(n, \mathcal{F}) = \max_{\substack{|V(G)|=n \\ \alpha(G)\leqslant\alpha}} c_{r,s}(G, \mathcal{F}).$$

Thus $c_{r,s,\alpha}(n, \mathcal{F})$ is the minimum integer k such that in every graph G with independence number at most α and every r-colouring of the edges of G, there exists a monochromatic $\mathcal{F}$-covering $\mathbf{S}$ of G of size at most k that satisfies $\mathrm{col}(\mathbf{S}) \leqslant s$.

To state our results, we must first recall the notion of a Kneser hypergraph. The Kneser hypergraph $\mathcal{KG}^{(\alpha+1)}(r, r-s)$ is the $(\alpha+1)$-uniform hypergraph on the vertex set $\binom{[r]}{r-s} = \{X \subseteq [r] \colon |X| = r-s\}$ where the vertices $X_1, \dots, X_{\alpha+1} \in \binom{[r]}{r-s}$ form a hyperedge if and only if they are pairwise disjoint as subsets of $[r]$. A result of Alon, Frankl, and Lovász [3] states that the chromatic number of this hypergraph is

$$\chi(\mathcal{KG}^{(\alpha+1)}(r, r-s)) = \begin{cases} 1 & \text{if } 1 \leqslant s < \alpha r/(\alpha+1) \\ 1 + s - r + \lceil (s+1)\alpha^{-1} \rceil & \text{if } \alpha r/(\alpha+1) \leqslant s < r. \end{cases} \tag{2}$$

Note that the range $1 \leqslant s < \alpha r/(\alpha+1)$ corresponds precisely to the case where $\mathcal{KG}^{(\alpha+1)}(r, r-s)$ has no edges. The case $\alpha = 1$ (which corresponds here to the case where $G = K_n$) was conjectured by Kneser in 1955 and famously established by Lovász [22] in 1978 using topological methods.

Our first result gives a lower bound on $c_{r,s,\alpha}(n, \mathcal{F})$. Note that there are certain trivial cases where $c_{r,s,\alpha}(n, \mathcal{F})$ is very small simply because the graphs in $\mathcal{F}$ have many isolated vertices. To give an extreme example, if $\mathcal{F}$ contains for every $n \geqslant 0$ the graph with n vertices and no edges, then trivially $c_{r,s,\alpha}(n, \mathcal{F}) = 1$. The easiest way to avoid such issues is to insist that each graph in $\mathcal{F}$ has at most a bounded number of isolated vertices. In addition to this, we will assume that $\mathcal{F}$ is Δ-*bounded*, that is, that every graph in $\mathcal{F}$ has maximum degree at most Δ. Then we prove the following lower bound:

Theorem 1 (Lower bound)**.** *Given any positive integers r, s, α, Δ, K such that $r > s$, there exists $c > 0$ such that the following holds. Let $\mathcal{F}$ be a Δ-bounded family of graphs with at most K isolated vertices each. Then for every $n \in \mathbf{N}$, we have*

$$c_{r,s,\alpha}(n, \mathcal{F}) \geqslant cn^{1/\chi},$$

where $\chi = \chi(\mathcal{KG}^{(\alpha+1)}(r, r-s))$.

We remark that the conclusion of Theorem 1 fails when $r = s$; indeed, there are many situations where $c_{r,r,\alpha}(n, \mathcal{F})$ is known to be constant. For example, Gyárfás, Ruszinkó, Sárközy, and Szemerédi [16] proved that $c_{r,r,1}(n, \mathcal{F}_c) \leqslant 100r \log r$. Sárközy [27] proved that $c_{r,r,\alpha}(n, \mathcal{F}_c) \leqslant 25(\alpha r)^2 \log(\alpha r)$. Sárközy, Selkow, and Song [25] proved that if $\mathcal{F}$ contains the graph on a single vertex and all connected d-regular graphs, then $c_{r,r,1}(n, \mathcal{F}) \leqslant 100r \log r + 2rd$. For more general families, Grinshpun and Sárközy [12] showed that if $\mathcal{F}$ is Δ-bounded and contains at least one graph on i vertices for every $i \geqslant 1$, then $c_{2,2,1}(n, \mathcal{F}) \leqslant 2^{O(\Delta \log \Delta)}$.

We also prove an upper bound that matches the lower bound given by Theorem 1 in many cases. Note again that it is possible to choose $\mathcal{F}$ so that $c_{r,s,\alpha}(n, \mathcal{F})$ is trivially very large; for example, if $\mathcal{F}$ only contains a single fixed graph then it is obvious that $c_{r,s,\alpha}(n, \mathcal{F}) = \Omega(n)$. Our way to avoid this kind of problem will be to assume that there is some $\varepsilon > 0$ such that for every $i \geqslant 1$, $\mathcal{F}$ contains at least one graph F with $|V(F)| \in [\varepsilon i, i]$. In fact, our proof (but perhaps not the result) requires the stronger assumption that at least one such graph is *bipartite*. We prove:

Theorem 2 (Upper bound)**.** *Given any positive integers r, s, α, Δ such that $r > s$, and any $\varepsilon > 0$, there exists $C > 0$ such that the following holds. Let $\mathcal{F}$ be a Δ-bounded family $\mathcal{F}$ of graphs such that for every $i \geqslant 1$, there is a bipartite $F \in \mathcal{F}$ with $\varepsilon i \leqslant |V(F)| \leqslant i$. Then for every $n \in \mathbf{N}$, we have*

$$c_{r,s,\alpha}(n, \mathcal{F}) \leqslant Cn^{1/\chi} + c_{r,r,\alpha}(n, \mathcal{F}),$$

where $\chi = \chi(\mathcal{KG}^{(\alpha+1)}(r, r-s))$.

This upper bound coincides asymptotically with the lower bound given by Theorem 1 whenever we know that $c_{r,r,\alpha}(n, \mathcal{F}) = O(n^{1/\chi})$. As mentioned above, in many situations it is even known that $c_{r,r,\alpha}(n, \mathcal{F}) = O(1)$. We can thus obtain asymptotically tight results in several different cases. From the above-mentioned result of Sárközy [27] we immediately obtain:

Corollary 3 (Paths and cycles)**.** *Let r, s, α be fixed positive integers such that $r > s$. Let $\chi = \chi(\mathcal{KG}^{(\alpha+1)}(r, r-s))$. Let $\mathcal{F}_p$ be the family of all paths and $\mathcal{F}_c$ be the family of all cycles. Then*

$$\Omega(n^{1/\chi}) \leqslant c_{r,s,\alpha}(n, \mathcal{F}_p) \leqslant c_{r,s,\alpha}(n, \mathcal{F}_c) \leqslant O(n^{1/\chi}).$$

In particular, setting $\alpha = 1$ and using (2) gives

$$c_{r,s}(K_n, \mathcal{F}_p) = \Theta(n^{1/\max\{1,2+2s-r\}})$$

thus generalizing the Erdős-Gyárfás result (1) to more colours (and the same holds for $\mathcal{F}_c$ instead of $\mathcal{F}_p$).

Similarly, using the result of Sárközy, Selkow, and Song [25], we get the following result for covering complete graphs by regular graphs:

Corollary 4 (***d***-regular graphs)**.** *Let r, s, d be fixed positive integers such that $r > s$. Let $\chi = \chi(\mathcal{KG}^{(2)}(r, r-s)) = \max\{1, 2+2s-r\}$. Let $\mathcal{F}_d$ be the family containing all connected d-regular graphs and also the graph with a single vertex and no edges. Then*

$$c_{r,s}(K_n, \mathcal{F}_d) = \Theta(n^{1/\chi}).$$

Note that the bounds in Corollaries 3 and 4 are only tight up to a large multiplicative factor depending on r, s, and α (resp. d). It would be interesting to determine these factors more precisely. As mentioned earlier, even the case where $r = 2$ and $s = \alpha = 1$ is still open.

It is perhaps interesting to note that the proof of Theorem 2 does not actually use the Alon-Frankl-Lovász result (2), but rather works directly with the definition of χ as the chromatic number of $\mathcal{KG}^{(\alpha+1)}(r, r-s)$. On the other hand, our proof of Theorem 1 really uses the value of χ given by (2), or, more precisely, it uses the lower bound on χ implied by (2), which is by far the more difficult direction.

1.2 Notation

We write $[k] = \{1, \ldots, k\}$. We write $\binom{A}{\ell}$ for the set of all ℓ-element subsets of the set A. If G is a graph and V_i, V_j are disjoint subsets of the vertices of G, then we denote by $G[V_i, V_j]$ the bipartite subgraph induced by the two parts V_i and V_j, and we write $e_G(V_i, V_j)$ for the number of edges of $G[V_i, V_j]$.

Since we are aiming for asymptotic statements, we routinely omit rounding brackets whenever they are not essential.

2 Proof of Theorem 1

Suppose that we are given positive integers r, s, α, Δ, K such that $r > s$. Let χ denote the chromatic number of $\mathcal{KG}^{(\alpha+1)}(r, r-s)$. We need to show that there is a constant $c = c(r, s, \alpha, \Delta, K) > 0$ such that

$$c_{r,s,\alpha}(n, \mathcal{F}) \geqslant cn^{1/\chi}$$

for all $n \in \mathbf{N}$ and all Δ-bounded families $\mathcal{F}$ of graphs with at most K isolated vertices each. In other words, we need to construct an r-coloured graph G with independence number at most α such that every monochromatic $\mathcal{F}$-covering $\mathbf{S}$ of G with $\mathrm{col}(\mathbf{S}) \leqslant s$ has size at least $cn^{1/\chi}$.

The construction will use *Johnson graphs*. The Johnson graph $J(a, b)$ is the graph with the vertex set $\binom{[a]}{b}$ where two vertices X and Y are joined by an edge if they have a non-empty intersection (so it is the complement of the Kneser graph $\mathcal{KG}^{(2)}(a, b)$). Is is easy to see that the independence number of $J(a, b)$ is at most $\lfloor a/b \rfloor$: every collection of $\lfloor a/b \rfloor + 1$ sets in $\binom{[a]}{b}$ covers in total $(\lfloor a/b \rfloor + 1)b > a$ elements, counted with multiplicities, so that at least two of the sets must intersect.

To prove Theorem 1, we use different constructions depending on the parameters. We distinguish between three cases.

Case 1. Suppose first that $1 \leqslant s < \alpha r/(\alpha + 1)$, i.e., that $\chi = 1$ by (2). Let G be a blow-up of $J(r, r-s)$ where every vertex is replaced by a clique on $n/\binom{r}{r-s}$ vertices and where every edge is replaced by a complete bipartite graph between the corresponding cliques. For a vertex X of $J(r, r-s)$, we write V_X for the vertices of G in the clique corresponding to X.

Note that G has the same independence number as $J(r, r-s)$, which is at most $\lfloor r/(r-s) \rfloor$. The assumption $\alpha r/(\alpha+1) > s$ implies that

$$\frac{r}{r-s} < \frac{r}{r - \alpha r/(\alpha+1)} = \alpha + 1$$

and so the independence number of G is at most α, as required.

We now colour the edges of G with colours from $[r]$ as follows. Let uv be an edge of G. Then there exist vertices X and Y of $J(r, r-s)$ such that $u \in V_X$ and $v \in V_Y$. Moreover, we either have $X = Y$, or $\{X, Y\}$ is an edge in $J(r, r-s)$, and in both cases, $X \cap Y \neq \emptyset$. We then colour uv with any colour belonging to the set $X \cap Y \subseteq [r]$.

Finally, suppose that $\mathbf{S}$ is a monochromatic $\mathcal{F}$-covering of G such that $\mathrm{col}(\mathbf{S}) \leqslant s$. Then there is some $X \subseteq [r]$ of size $r-s$ that is disjoint from the set of colors used by the graphs in $\mathbf{S}$. By our choice of colouring, all edges touching V_X have a colour in X, so the vertices in V_X can only be covered using isolated vertices. Since every graph in $\mathbf{S}$ has at most K isolated vertices, this means that $|\mathbf{S}| \geqslant |V_X|/K \geqslant n/(K\binom{r}{r-s})$, completing the proof in this case (since $\chi = 1$).

Case 2. Suppose now that $s \geqslant \alpha r/(\alpha+1)$ and assume additionally that $s < \chi\alpha$. Then by (2), we have $\chi = 1 + s - r + \lceil (s+1)\alpha^{-1} \rceil \leqslant r$, where the last inequality follows from $s + 1 \leqslant r$. Since additionally $s + 1 \leqslant \chi\alpha$, we can fix integers $1 \leqslant k_1, \ldots, k_\chi \leqslant \alpha$ such that $k := k_1 + \cdots + k_\chi \in \{s+1, \ldots, r\}$.

We now construct an n-vertex graph G as follows. We start with a blow-up of the complete graph K_χ where the i-th vertex is replaced by a set V_i of $n^{i/\chi}$ vertices, except for the χ-th vertex, which is replaced by a set set V_χ of

$$|V_\chi| = n - n^{1/\chi} - n^{2/\chi} - \cdots - n^{(\chi-1)/\chi} \geqslant n - o(n)$$

vertices. Each edge ij of K_χ is replaced by a complete bipartite graph between the corresponding sets V_i and V_j. We further partition each set V_i equitably into k_i parts $V_{i,1}, \ldots, V_{i,k_i}$, and insert all edges where both endpoints are contained in the same set $V_{i,j}$. Thus for each i, the graph $G[V_i]$ is the disjoint union of k_i cliques of size $|V_i|/k_i$. This defines the graph G. It is easy to see that G has independence number

$$\max\{k_i : 1 \leqslant i \leqslant \chi\} \leqslant \alpha.$$

Next, we colour the edges of G as follows. First, we fix an arbitrary bijection

$$\phi\colon \{(i,j) : 1 \leqslant i \leqslant \chi \text{ and } 1 \leqslant j \leqslant k_i\} \to [k].$$

Such a bijection exists because $k_1 + \cdots + k_\chi = k$. Then we distinguish two cases. If uv is an edge of G with both endpoints in the same set $V_{i,j}$, then uv receives the colour $\phi(i,j)$. On the other hand, if uv goes between the sets $V_{i,j}$ and $V_{i',j'}$ where $i < i'$, then we uv

receives the colour $\phi(i,j)$. Note that by construction, there are no edges going between to sets $V_{i,j}$ and $V_{i,j'}$ for $j \neq j'$. Since $k \leqslant r$, this is a colouring with at most r colours.

Now suppose that $\mathbf{S}$ is a monochromatic $\mathcal{F}$-covering of G such that $\operatorname{col}(\mathbf{S}) \leqslant s$. Since $s < k$, there is then some pair (i,j) with $1 \leqslant i \leqslant \chi$ and $1 \leqslant j \leqslant k_i$ such that $\phi(i,j)$ is not the colour of any graph in $\mathbf{S}$. Now observe that the only edges incident to $V_{i,j}$ that do not use the colour $\phi(i,j)$ are those that have an endpoint in $V_1 \cup \cdots \cup V_{i-1}$. In particular, every graph in $\mathbf{S}$, having maximum degree at most Δ and at most K isolated vertices, can cover at most $\Delta(|V_1| + \cdots + |V_{i-1}|) + K$ vertices of $V_{i,j}$. Now $|V_{i,j}| \geqslant n^{i/\chi}/r$ implies

$$\begin{aligned}\Delta(|V_1| + \cdots + |V_{i-1}|) + K &= \Delta(n^{1/\chi} + \cdots + n^{(i-1)/\chi}) + K \\ &\leqslant (1+o(1)) \cdot (\Delta + K) \cdot n^{(i-1)/\chi} \\ &\leqslant (1+o(1)) \cdot r(\Delta + K) \cdot n^{-1/\chi}|V_{i,j}|,\end{aligned}$$

and so to cover $V_{i,j}$ completely, $\mathbf{S}$ must contain at least $(1-o(1))n^{1/\chi}/(r(\Delta+K))$ graphs, completing the proof in this case.

Case 3. Finally, assume $s \geqslant \alpha r/(\alpha+1)$ and $s \geqslant \chi\alpha$. The construction in this case is a combination of the constructions used in the two previous cases. We will construct a graph G on n vertices as follows. As in Case 2, we start with a blow-up of the complete graph K_χ where the i-th vertex is replaced by a set V_i of $|V_i| = n^{i/\chi}$ vertices, except for the last vertex, which is replaced by a set V_χ of

$$|V_\chi| = n - n^{1/\chi} - n^{2/\chi} - \cdots - n^{(\chi-1)/\chi} \geqslant n - o(n)$$

vertices. Each edge ij of K_χ is replaced by a complete bipartite graph between the corresponding sets V_i and V_j. This defines the edges going between different sets V_i and V_j.

Next, we specify what each graph $G[V_i]$ looks like. For $G[V_1]$, we use a similar construction as in Case 1. Let $t := r - \alpha(\chi-1)$ and note that since $s \geqslant \chi\alpha > \alpha(\chi-1)$, we have $t > r-s$. We let $G[V_1]$ be a blow-up of the Johnson graph $J(t, r-s)$ where every vertex is replaced by a clique on $|V_1|/\binom{t}{r-s}$ vertices, and where every edge is replaced by a complete bipartite graph between the corresponding cliques. For later reference, we define $V_{1,X} \subseteq V_1$ to be the vertex set of the clique corresponding to the vertex X of $J(t, r-s)$. For $1 < i \leqslant \chi$, we let $G[V_i]$ be the union of α vertex-disjoint cliques of size $|V_i|/\alpha$, somewhat similarly as in Case 2. We will write $V_{i,1}, \ldots, V_{i,\alpha} \subseteq V_i$ for the vertex sets of these cliques. This completes the definition of G.

We first check that G really has independence number at most α. It is immediate from the construction that $\alpha(G) = \max\{\alpha(G[V_i]) : 1 \leqslant i \leqslant \chi\}$. Moreover, it is easy to see that for $i > 1$, we have $\alpha(G[V_i]) = \alpha$. So it remains only to consider $i = 1$. Observe that $G[V_1]$ has the same independence number as $J(t, r-s)$, which is at most $\lfloor t/(r-s) \rfloor$. It is thus sufficient to prove that $t/(r-s) < \alpha + 1$, which is easily seen to be true using the definition of χ. Indeed, since $t = r - \alpha(\chi-1)$, the inequality $t < (\alpha+1)(r-s)$ is equivalent to

$$s < \alpha(r - s + \chi - 1),$$

which is true because $r - s + \chi - 1 = \lceil (s+1)/\alpha \rceil$ using (2) and the assumption $s \geqslant \alpha r/(\alpha+1)$. Hence we have $\alpha(G) \leqslant \alpha$, as required.

We now define a colouring of the edges of G with r colours, where we distinguish several cases. First, suppose that uv is an edge with $u, v \in V_1$. Then there exist vertices X, Y of $J(t, r-s)$ such that $u \in V_{1,X}$ and $v \in V_{1,Y}$; moreover, for these X, Y it holds that $X \cap Y \neq \emptyset$ (they are either identical or represent an edge in $J(t, r-s)$). We then colour uv with any colour in $X \cap Y$. Second, assume that uv has exactly one endpoint (say, u) in V_1 and the other in V_i for some $i > 1$. Then there is some vertex X of $J(t, r-s)$ such that $u \in V_{1,X}$, and we colour uv with any colour in X. Lastly, to colour the remaining edges, fix any bijection

$$\phi\colon \{(i,j) : 1 < i \leqslant \chi \text{ and } 1 \leqslant j \leqslant \alpha\} \to [r] \setminus [t].$$

Such a bijection exists because $r - t = \alpha(\chi - 1)$. If uv is an edge with both endpoints in the same set V_i for $i > 1$, say $u, v \in V_{i,j}$, then we colour uv with the colour $\phi(i,j)$ (note that there are no edges between $V_{i,j}$ and $V_{i,j'}$ for $j \neq j'$). If uv is an edge going between $u \in V_{i,j}$ and $v \in V_{i',j'}$ where $i < i'$, then we colour uv with the colour $\phi(i,j)$. Thus we have coloured all the edges.

We make two observations at this point:

(i) Every edge incident to $V_{1,X}$ is coloured with a colour from X, for every vertex X of $J(t, r-s)$;

(ii) For every $1 < i \leqslant \chi$ and $1 \leqslant j \leqslant \alpha$, the only edges incident to $V_{i,j}$ that do not use the colour $\phi(i,j)$ are those that are incident to a set $V_{i'}$ where $i' < i$. In particular, every monochromatic copy of a graph $F \in \mathcal{F}$ that uses a colour different from $\phi(i,j)$ can cover at most

$$\begin{aligned} \Delta(|V_1| + \cdots + |V_{i-1}|) + K &\leqslant \Delta(n^{1/\chi} + \cdots + n^{(i-1)/\chi}) + K \\ &\leqslant (1+o(1)) \cdot (\Delta + K) \cdot n^{(i-1)/\chi} \\ &\leqslant (1+o(1)) \cdot \alpha(\Delta + K) \cdot n^{-1/\chi} |V_{i,j}| \end{aligned}$$

vertices of $V_{i,j}$, where we use that F has maximum degree at most Δ and at most K isolated vertices.

To complete the proof, suppose that $\mathbf{S}$ is a monochromatic $\mathcal{F}$-covering of G such that $\mathrm{col}(\mathbf{S}) \leqslant s$. Denoting by $\mathrm{Col}(\mathbf{S})$ the set of all colours used by graphs in $\mathbf{S}$, we distinguish two possible cases.

The first case is when $\mathrm{Col}(\mathbf{S})$ contains at most $t - (r - s)$ colours from $[t]$. In this case, there is some set X of $r - s$ colours in $[t]$ that do not belong to $\mathrm{Col}(\mathbf{S})$. But then, as all edges incident to $V_{1,X}$ use a colour from X (see (i)), the only way in which $\mathbf{S}$ can cover the vertices in $V_{1,X}$ is by using isolated vertices. Since each graph in $\mathbf{S}$ has at most K isolated vertices, this implies $|\mathbf{S}| \geqslant |V_{1,X}|/K \geqslant n^{1/\chi}/K$, completing the proof in this case.

In the other case, $\mathrm{Col}(\mathbf{S})$ contains at least $t-(r-s)+1$ colours from $[t]$. Since $\mathrm{col}(\mathbf{S}) \leqslant s$, this means that at most $s-t+(r-s)-1 = r-t-1$ colours from $\mathrm{Col}(\mathbf{S})$ can be contained in $[r] \setminus [t]$. In particular, there is a colour $a \in [r] \setminus [t]$ that is not used by any of the graphs in $\mathbf{S}$. Let $(i,j) = \phi^{-1}(a)$ and consider the set $V_{i,j}$. Then by (ii), every graph in $\mathbf{S}$ can cover at most $(1+o(1)) \cdot \alpha(\Delta+K) \cdot n^{-1/\chi}|V_{i,j}|$ vertices of $V_{i,j}$, so $|\mathbf{S}| \geqslant (1-o(1)) \cdot n^{1/\chi}/(\alpha(\Delta+K))$. This completes the proof of Theorem 1.

3 Proof of Theorem 2

Let r, s, α be positive integers with $r > s$. Let $\mathcal{K} := \mathcal{KG}^{(\alpha+1)}(r, r-s)$ and $\chi := \chi(\mathcal{K})$. Let G be a graph on n vertices with independence number at most α, and suppose that the edges of G are coloured with r colours, which we assume to come from the set $[r] = \{1, \ldots, r\}$. Then the vertices of $\mathcal{K}$ correspond naturally to sets of $r-s$ colours. Let $\Delta, \varepsilon > 0$ and let $\mathcal{F}$ be a Δ-bounded family of graphs with such that for every $i \geqslant 1$, $\mathcal{F}$ contains at least one bipartite graph with at least εi and at most i vertices. In particular, $\mathcal{F}$ contains the graph on a single vertex and with no edges. We will show that there is a monochromatic $\mathcal{F}$-covering $\mathbf{S}$ of G such that

$$|\mathbf{S}| \leqslant Cn^{1/\chi} + c_{r,r}(G, \mathcal{F}) \text{ and } \mathrm{col}(\mathbf{S}) \leqslant s,$$

where $C = C(r, s, \alpha, \varepsilon) > 0$ is a suitable constant.

We first note that if $s < \alpha r/(\alpha+1)$, then by (2), we have $\chi = 1$. In this case, we can simply cover G by n single vertices, and we are done. Therefore, we will assume from now on that $s \geqslant \alpha r/(\alpha+1)$.

We start by introducing some notation. If $\mathbf{S}$ is a monochromatic $\mathcal{F}$-covering of G and $X \in V(\mathcal{K})$ is a set of $r-s$ colours, then we write $V_{\mathbf{S},X} \subseteq V(G)$ for the set of all vertices of G that are covered in $\mathbf{S}$ exclusively by graphs having a colour in X, that is,

$$V_{\mathbf{S},X} := \{v \in V(G) \colon \text{every } H \in \mathbf{S} \text{ such that } v \in V(H) \text{ has a colour in } X\}.$$

Note that $\mathbf{S} \subseteq \mathbf{S}'$ implies $V_{\mathbf{S}',X} \subseteq V_{\mathbf{S},X}$ for all $X \in V(\mathcal{K})$: adding more graphs to $\mathbf{S}$ can never increase one of the sets $V_{\mathbf{S},X}$. Our goal will be to construct a small monochromatic $\mathcal{F}$-covering $\mathbf{S}$ such that $V_{\mathbf{S},X} = \emptyset$ for some $X \in V(\mathcal{K})$. Note that in this case, G is completely covered by the graphs in $\mathbf{S}$ that have colours not in X, so by removing all graphs with a colour in X from $\mathbf{S}$, we can obtain a monochromatic $\mathcal{F}$-covering $\mathbf{S}' \subseteq \mathbf{S}$ with $\mathrm{col}(\mathbf{S}') \leqslant s$.

With this goal in mind, we define a quantity to track the sizes of the sets $|V_{\mathbf{S},X}|$:

$$\delta(\mathbf{S}) := \sum_{X \in V(\mathcal{K})} \log |V_{\mathbf{S},X}|,$$

where we can set $\delta(\mathbf{S}) = -\infty$ if $|V_{\mathbf{S},X}| = 0$ holds for some $X \in V(\mathcal{K})$. Note that since $|V_{\mathbf{S},X}| \leqslant n$, we always have the bound $\delta(\mathbf{S}) \leqslant \binom{r}{r-s} \log n$. Our central claim is:

Claim 5. *There is a constant $\beta > 0$ such that the following holds. If $\mathbf{S}$ is a monochromatic $\mathcal{F}$-covering of G such that $|V_{\mathbf{S},X}| > n^{1/\chi}$ for all $X \in V(\mathcal{K})$, then G contains a (nonempty) collection $\mathcal{H} = \{H_1, \ldots, H_t\}$ of monochromatic copies of graphs in $\mathcal{F}$ such that*

$$\delta(\mathbf{S}) - \delta(\mathbf{S} \cup \mathcal{H}) \geqslant \beta t n^{-1/\chi} \log n. \tag{3}$$

We postpone the proof of this claim and first show how it can serve to imply the theorem. We construct a monochromatic $\mathcal{F}$-covering step by step, starting with some monochromatic $\mathcal{F}$-covering $\mathbf{S}_0$ of size $c_{r,r}(G, \mathcal{F})$ (which exists by definition). Then as long as $|V_{\mathbf{S}_i,X}| > n^{1/\chi}$ for all $X \in V(\mathcal{K})$, we construct $\mathbf{S}_{i+1}$ from $\mathbf{S}_i$ by setting $\mathbf{S}_{i+1} = \mathbf{S}_i \cup \mathcal{H}$ for a collection $\mathcal{H}$ as given by Claim 5. Note that since $\delta(\mathbf{S}_0) \leqslant \binom{r}{r-s} \log n$, and since $\delta(\mathbf{S}) \leqslant 0$ implies that $|V_{\mathbf{S},X}| \leqslant 1 \leqslant n^{1/\chi}$ for some $X \in V(\mathcal{K})$, it follows from (3) that this process must end after adding at most $\binom{r}{r-s} n^{1/\chi}/\beta$ graphs to $\mathbf{S}_0$. In other words, we end up with a monochromatic $\mathcal{F}$-covering $\mathbf{S}^*$ of size $|\mathbf{S}^*| \leqslant c_{r,r}(G, \mathcal{F}) + \binom{r}{r-s} n^{1/\chi}/\beta$ such that $|V_{\mathbf{S}^*,X}| \leqslant n^{1/\chi}$ holds for at least one $X \in V(\mathcal{K})$. From this we obtain another monochromatic $\mathcal{F}$-covering $\mathbf{S}$ by adding to $\mathbf{S}^*$ at most $n^{1/\chi}$ single-vertex graphs covering the vertices in $V_{\mathbf{S}^*,X}$. Note that then $V_{\mathbf{S},X} = \emptyset$ and $|\mathbf{S}| \leqslant c_{r,r}(G, \mathcal{F}) + \binom{r}{r-s} n^{1/\chi}/\beta + n^{1/\chi}$. As mentioned above, we can then find a monochromatic $\mathcal{F}$-covering $\mathbf{S}' \subseteq \mathbf{S}$ with $\mathrm{col}(\mathbf{S}') \leqslant s$, completing the proof of the theorem.

3.1 Proof of Claim 5

It remains to give the proof of Claim 5. The proof will use the following lemma, whose proof we omit (it is a standard application of Szemerédi's regularity lemma, see for example [19, Theorem 2.1]).

Lemma 6. *For every $\varepsilon > 0$ and $\Delta > 0$ there is a constant $\delta > 0$ such that the following holds for all sufficiently large n. If $G = (A, B, E)$ is a bipartite graph with $|A| = |B| = n$ and $|E| \geqslant \varepsilon n^2$, then it contains as a subgraph every bipartite graph with maximum degree at most Δ and at most δn vertices.*

In the following, let $\mathbf{S}$ be a monochromatic $\mathcal{F}$-covering of G such that $|V_{\mathbf{S},X}| > n^{1/\chi}$ for all $X \in V(\mathcal{K})$. We first show:

Claim 7. *There exists a hyperedge $\mathcal{E} = \{X_1, \ldots, X_{\alpha+1}\}$ of $\mathcal{K}$ such that*

$$n^{-1/\chi} \leqslant \frac{|V_{\mathbf{S},X_i}|}{|V_{\mathbf{S},X_j}|} \leqslant n^{1/\chi} \quad \text{for all } i, j \in [\alpha+1]. \tag{4}$$

Proof. Fix any $c > 1$ and let $b \in (n^{1/\chi}, cn^{1/\chi})$ be such that $b \leqslant |V_{\mathbf{S},X}|$ holds for all $X \in V(\mathcal{K})$. This is possible because we assume that $|V_{\mathbf{S},X}| > n^{1/\chi}$ for all $X \in V(\mathcal{K})$. Then, because $b \leqslant |V_{\mathbf{S},X}| \leqslant n$, the map $X \mapsto \lfloor \log_b |V_{\mathbf{S},X}| \rfloor$ assigns each vertex of $\mathcal{K}$ a number between 1 and $\lfloor \log_b n \rfloor \leqslant \chi - 1$. Hence, by definition of the chromatic number, there is a hyperedge $\mathcal{E} = \{X_1, \ldots, X_{\alpha+1}\}$ in which all vertices receive the same number. Then for all $i, j \in [\alpha+1]$, we have

$$-1 < \log_b |V_{\mathbf{S},X_i}| - \log_b |V_{\mathbf{S},X_j}| < 1,$$

so $n^{-1/\chi}/c < |V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_j}| < cn^{1/\chi}$. Since c can be arbitrarily close to 1, and as $\mathcal{K}$ is finite, this implies the claim. □

Let now $\mathcal{E} = \{X_1, \ldots, X_{\alpha+1}\}$ be a hyperedge of $\mathcal{K}$ satisfying (4). We will assume the elements of $\mathcal{E}$ are ordered so that

$$|V_{\mathbf{S},X_1}| \geqslant |V_{\mathbf{S},X_2}| \geqslant \cdots \geqslant |V_{\mathbf{S},X_{\alpha+1}}|.$$

Definition 8 (Removable set)**.** Let us say that a subset $W \subseteq V_{\mathbf{S},X_i}$ is *removable* if G contains a monochromatic copy H of some graph in $\mathcal{F}$ such that (i) the colour of H is in $[r] \setminus X_i$ and (ii) $W \subseteq V(H)$.

The idea behind this definition is that if $W \subseteq V_{\mathbf{S},X_i}$ is removable, then by adding the graph H to $\mathbf{S}$, we can decrease the size of $|V_{\mathbf{S},X_i}|$ by at least $|W|$: indeed, recalling the definition of $V_{\mathbf{S},X_i}$, we see that $V_{\mathbf{S}\cup\{H\},X_i} \subseteq V_{\mathbf{S},X_i} \setminus W$.

Claim 9. *There is a constant $C > 0$ and some $i \in [\alpha+1]$ such that the following holds: There exist $t \leqslant C|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|$ disjoint removable sets $W_1, \ldots, W_t \subseteq V_{\mathbf{S},X_i}$ covering all except for at most $|V_{\mathbf{S},X_{\alpha+1}}|/2$ vertices in $|V_{\mathbf{S},X_i}|$.*

Proof. Observe first that it is enough to show the following statement: for every choice of subsets $V_1, \ldots, V_{\alpha+1}$ where $V_i \subseteq V_{\mathbf{S},X_i}$ and where each V_i has size $|V_{\mathbf{S},X_{\alpha+1}}|/2$, there is some $i \in [\alpha+1]$ and a subset $W \subseteq V_i$ of size at least $|V_{\mathbf{S},X_{\alpha+1}}|/C$ that is removable. Indeed, we can then repeatedly apply this statement until we have covered all but $|V_{\mathbf{S},X_{\alpha+1}}|/2$ vertices in at least one set $V_{\mathbf{S},X_i}$, and it is clear that this requires at most $C|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|$ subsets of $V_{\mathbf{S},X_i}$. So we will now prove this other statement instead.

Fix sets $V_1, \ldots, V_{\alpha+1}$ as above. For brevity, write $\eta := |V_{\mathbf{S},X_{\alpha+1}}|/2 = |V_1| = \cdots = |V_{\alpha+1}|$. From the fact that G has independence number at most α it follows that there exist distinct $i, j \in [\alpha+1]$ such that $e_G(V_i, V_j) \geqslant \eta^2/(\alpha+1)^2$. This can be seen by simple double counting: for every choice of $\alpha+1$ vertices $v_i \in V_i$ for $i \in [\alpha+1]$, there must be two vertices that are connected by an edge. Going over all ways to choose such vertices, we thus obtain $\eta^{\alpha+1}$ edges, where every edge is obtained at most $\eta^{\alpha-1}$ times; so there must be η^2 edges going between the sets $V_1, \ldots, V_{\alpha+1}$. In particular, for some $i \neq j$, we have $e_G(V_i, V_j) \geqslant \eta^2/(\alpha+1)^2$.

Suppose now that $e_G(V_i, V_j) \geqslant \eta^2/(\alpha+1)^2$. Let $k \in [r]$ denote the majority colour of the edges in $G[V_i, V_j]$ and write $G_k[V_i, V_j]$ for the subgraph consisting only of the edges having colour k. Then it is clear that $G_k[V_i, V_j]$ has at least $\eta^2/(r(\alpha+1)^2)$ edges.

Recall that we assume that $\mathcal{F}$ is Δ-bounded and that there is some $\varepsilon > 0$ such that for every $n' \geqslant 1$, the family $\mathcal{F}$ contains at least one bipartite subgraph $F \in \mathcal{F}$ with $\varepsilon n' \leqslant |V(F)| \leqslant n'$.

Applying Lemma 6 to $G_k[V_i, V_j]$ (which is possible for large n since $|V_i| = |V_j| = \eta > n^{1/\chi}/2$), we obtain that $G_k[V_i, V_j]$ contains as a subgraph every Δ-bounded bipartite graph on at most $2(\Delta+1)\eta/(C\varepsilon)$ vertices, for some sufficiently large constant $C > 0$. In particular, $G_k[V_i, V_j]$ contains a copy of a graph $F \in \mathcal{F}$ with at least $2(\Delta+1)\eta/C$ vertices. In fact, since F has maximum degree at most Δ, it can be embedded in such a way that is uses at least $2\eta/C$ vertices of V_i and at least $2\eta/C$ vertices of V_j (for every Δ non-isolated vertices in V_i we must embed at least one vertex in V_j, whereas the isolated vertices can be embedded arbitrarily). Denote this copy by H and note that as a subgraph of $G_k[V_i, V_j]$ it is clearly monochromatic in colour k. Since the sets X_i and X_j are disjoint (they are part of a hyperedge in $\mathcal{K}$), they cannot both contain k, and so at least one of the sets $V(H) \cap V_i$ or $V(H) \cap V_j$ is removable, and both these sets have size $2\eta/C = |V_{\mathbf{S},X_{\alpha+1}}|/C$. □

Let $W_1, \ldots, W_t \subseteq V_{\mathbf{S},X_i}$ be disjoint removable sets as given by Claim 9 and let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be the corresponding collection of subgraphs, so that H_j is a monochromatic copy of a graph in $\mathcal{F}$ that covers W_j and uses a colour outside X_i. By Claim 9 and the definition of *removable*, we have $|V_{\mathbf{S}\cup\mathcal{H},X_i}| \leqslant |V_{\mathbf{S},X_{\alpha+1}}|/2 < |V_{\mathbf{S},X_i}|$. This implies immediately that the collection $\mathcal{H}$ is nonempty. It also implies that

$$\begin{aligned}
\delta(\mathbf{S} \cup \mathcal{H}) &= \sum_{j\in[\alpha+1]} \log |V_{\mathbf{S}\cup\mathcal{H},X_j}| \\
&\leqslant \sum_{j\in[\alpha+1]\setminus\{i\}} \log |V_{\mathbf{S},X_j}| + \log |V_{\mathbf{S}\cup\mathcal{H},X_i}| \\
&\leqslant \delta(\mathbf{S}) - \log |V_{\mathbf{S},X_i}| + \log(|V_{\mathbf{S},X_{\alpha+1}}|/2) \\
&= \delta(\mathbf{S}) - \log(2|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|),
\end{aligned}$$

and so

$$\delta(\mathbf{S}) - \delta(\mathbf{S} \cup \mathcal{H}) \geqslant \log(2|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|).$$

At the same time, using $1 \leqslant t \leqslant C|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|$ and $|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}| \leqslant n^{1/\chi}$, we get

$$\frac{\log(2|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|)}{t} \geqslant \frac{\log(2|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|)}{C|V_{\mathbf{S},X_i}|/|V_{\mathbf{S},X_{\alpha+1}}|} \geqslant \frac{\log(2n^{1/\chi})}{Cn^{1/\chi}} \geqslant \frac{n^{-1/\chi}\log n}{C\chi},$$

so

$$\delta(\mathbf{S}) - \delta(\mathbf{S} \cup \mathcal{H}) \geqslant \frac{tn^{-1/\chi}\log n}{C\chi},$$

completing the proof of Claim 5.

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