# Fractional Harmonic Maps

Francesca Da Lio
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**Abstract:** The theory of $\alpha$-harmonic maps has been initiated some years ago by the author and Tristan Rivière in [8]. These maps are critical points of the following nonlocal energy

$$\mathcal{L}^{\alpha}(u) = \int_{\mathbb{R}^k} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 dx^k\,, \tag{2.0.1}$$

where $u \in \dot{H}^{\alpha}(\mathbb{R}^k, \mathcal{N})$, $\mathcal{N} \subset \mathbb{R}^m$ is an at least $C^2$ closed (compact without boundary) $n$-dimensional smooth manifold. In a recent paper [10] we also introduce the notion of *horizontal $\alpha$-harmonic maps*. Precisely, given a $C^1$ plane distribution $P_T$ on all $\mathbb{R}^m$, these are maps $u \in \dot{H}^{\alpha}(\mathbb{R}^k, \mathbb{R}^m)$, $\alpha \geqslant 1/2$, satisfying

$$\begin{cases} P_T(u)\nabla u = \nabla u & \text{in } \mathcal{D}'(\mathbb{R}^k) \\ P_T(u)(-\Delta)^{\alpha} u = 0 & \text{in } \mathcal{D}'(\mathbb{R}^k). \end{cases}$$

If the distribution of planes is integrable, then we recover the case of $\alpha$-harmonic maps with values into a manifold. We will concentrate here to the case $\alpha = 1/2$ and $k = 1$ which corresponds to a critical situation. Such maps arise from several geometric problems such as for instance in the study of free boundary manifolds. After giving an overview of the recent results on the regularity and the compactness of horizontal $1/2$-harmonic maps, we will describe the techniques that have been introduced in [8, 9] to investigate the regularity of such maps and mention some relevant applications to geometric problems.

## 2.1 Overview

Since the early 50's the analysis of critical points to conformal invariant Lagrangians has raised a special interest, due to the important role they play in physics and geometry.

The most elementary example of a 2-dimensional conformal invariant Lagrangian is the Dirichlet Energy

$$\mathcal{L}(u) = \int_{D} |\nabla u(x,y)|^2 dxdy\,, \tag{2.1.1}$$

where $D \subseteq R^2$ is an open set, $u\colon D \to \mathbb{R}^m$ and $\nabla u$ is the gradient of $u$.

We can define the Lagrangian (2.1.1) in the set of maps taking values in an at least $C^2$ closed $n$- dimensional submanifold $\mathcal{N} \subseteq \mathbb{R}^m$. In this case critical points $u \in W^{1,2}(D, \mathcal{N})$ of $\mathcal{L}$ satisfy in a weak sense the equation

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$$-\Delta u \perp T_u \mathcal{N}\,, \tag{2.1.2}$$

where $T_\xi \mathcal{N}$ is the tangent plane a $\mathcal{N}$ at the point $\xi \in \mathcal{N}$, or in a equivalent way

$$-\Delta u = A(u)(\nabla u, \nabla u) := A(u)(\partial_x u, \partial_x u) + A(u)(\partial_y u, \partial_y u), \tag{2.1.3}$$

where $A(\xi)$ is the second fundamental form at the point $\xi \in \mathcal{N}$ (see for instance [17]). The equation (2.1.3) is called the *harmonic map equation* into $\mathcal{N}$ .

In the case when $\mathcal{N}$ is an oriented hypersurface of $\mathbb{R}^m$ the harmonic map equation reads as

$$-\Delta u = \nu(u)\langle \nabla \nu(u), \nabla u\rangle\,, \tag{2.1.4}$$

where $\nu$ is the unit normal vector field to $\mathcal{N}$ .

The key point to get the regularity of the harmonic maps with values into the sphere $S^{m-1}$ was to rewrite the r.h.s of the equations as a sum of a Jacobians. More precisely Hélein in [17] wrote the equation (2.1.4) in the form

$$-\Delta u = \nabla^\perp B \cdot \nabla u, \tag{2.1.5}$$

where $\nabla^\perp B = (\nabla^\perp B_{ij})$ with $\nabla^\perp B_{ij} = u_i \nabla u_j - u_j \nabla u_i$, (for every vector field $v\colon \mathbb{R}^2 \to \mathbb{R}^m$, $\nabla^\perp v$ denotes the $\pi/2$ rotation of the gradient $\nabla v$, namely $\nabla^\perp v = (-\partial_y v, \partial_x v))$ .

The r.h.s of (2.1.5) can be written actually as a sum of Jacobians:

$$\nabla^\perp B_{ij} \nabla u_j = \partial_x u_j \partial_y B_{ij} - \partial_y u_j \partial_x B_{ij}\,.$$

This particular structure permitted to apply to the equation (2.1.5) the following result

**Theorem 2.1.** *[28] Let $D$ be a smooth bounded domain of $\mathbb{R}^2$. Let $a$ and $b$ be two measurable functions in $D$ whose gradients are in $L^2(D)$. Then there exists a unique solution $\varphi \in W^{1,2}(D)$ to*

$$\begin{cases} -\Delta\varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x}, & \text{in } D \\ \varphi = 0 & \text{on } \partial D\,. \end{cases} \tag{2.1.6}$$

*Moreover there exists a constant $C > 0$ independent of $a$ and $b$ such that*

$$\|\varphi\|_\infty + \|\nabla\varphi\|_{L^2} \leqslant C\|\nabla a\|_{L^2}\|\nabla b\|_{L^2}\,.$$

*In particular $\varphi$ is a continuous in $D$ .*

In the case of an oriented hypersurface $\mathcal{N}$ of $\mathbb{R}^m$ by using the fact that $\nabla u$ is orthogonal to $\nu(u)$ the equation (2.1.4) can be reformulated as follows

$$-\Delta u^i = \sum_{j=1}^m \left(\nu(u)^i\, \nabla(\nu(u))_j - \nu(u)_j\, \nabla(\nu(u))^i\right) \cdot \nabla u^j \quad . \tag{2.1.7}$$

Unlike the sphere case there is no reason for which the vector field

$$\nu(u)^i\,\nabla(\nu(u))_j-\nu(u)_j\,\nabla(\nu(u))^i$$

is divergence-free. What remains true is the anti-symmetry of the matrix

$$\Omega:=\left(\nu(u)^i\,\nabla(\nu(u))_j-\nu(u)_j\,\nabla(\nu(u))^i\right)_{i,j=1\cdots m}\quad. \tag{2.1.8}$$

Actually Rivière in [20] identified the anti-symmetry of the 1-form in (2.1.8) as the essential structure of equation (2.1.4) and he succeeded in writing the harmonic map system in the form of a conservation law whose constituents satisfy elliptic equations with a Jacobian structure to which Wente's regularity result (Theorem 2.1) could be applied.

Let us now introduce $P_T(z)$, $P_N(z)$ the orthogonal projections respectively to the tangent space $T_z\mathcal{N}$ and to the normal space $(T_z\mathcal{N})^\perp$. Then the equation (2.1.2) can be re-formulated as follows

$$P_T(u)\Delta u=0,\ \text{ in }\mathcal{D}'(D). \tag{2.1.9}$$

We are going to release the assumption that the field of orthogonal projections is associated to a sub-manifold $\mathcal{N}$ and to consider the equation (2.1.9) for a general field of orthogonal projections $P_T$ and for horizontal maps $u$ satisfying

$$P_N(u)\nabla u=0,\ \text{ in }\mathcal{D}'(D). \tag{2.1.10}$$

We will assume that $P_T\in C^1(\mathbb{R}^m,\mathcal{M}_m(\mathbb{R}))$ and $P_N\in C^1(\mathbb{R}^m,\mathcal{M}_m(\mathbb{R}))$ satisfy

$$\begin{cases} P_T\circ P_T=P_T\quad P_N\circ P_N=P_N\\[2mm] P_T+P_N=I_m\\[2mm] \forall\, z\in\mathbb{R}^m\quad\forall\, U,V\in T_z\mathbb{R}^m\qquad <P_T(z)U,P_N(z)V>=0\\[2mm] \|\partial_z P_T\|_{L^\infty(\mathbb{R}^m)}<+\infty\end{cases} \tag{2.1.11}$$

where $<\cdot,\cdot>$ denotes the standard scalar product in $\mathbb{R}^m$. In other words $P_T$ is a $C^1$ map into the orthogonal projections of $\mathbb{R}^m$. For such a distribution of projections $P_T$ we denote by

$$n:=\operatorname{rank}(P_T).$$

Such a distribution identifies naturally with the distribution of $n$-planes given by the images of $P_T$ (or the Kernel of $P_T$) and conversely, any $C^1$ distribution of $n$-dimensional planes defines uniquely $P_T$ satisfying (2.1.11).

For any $\alpha\geqslant 1/2$ and for $k\geqslant 1$ we define the space of *$H^\alpha$-Sobolev horizontal maps*

$$\mathfrak{H}^\alpha(\mathbb{R}^k):=\left\{u\in H^\alpha(\mathbb{R}^k,\mathbb{R}^m);\quad P_N(u)\nabla u=0\quad \text{in }\mathcal{D}'(\mathbb{R}^k)\right\}.$$

Observe that this definition makes sense since we have respectively $P_N\circ u\in H^\alpha(\mathbb{R}^k,\mathcal{M}_m(\mathbb{R}))$ and $\nabla u\in H^{\alpha-1}(\mathbb{R}^k,\mathbb{R}^m)$. Next we give the following definition.

**Definition 2.2.** *Given a $C^1$ plane distribution $P_T$ in $\mathbb{R}^m$ satisfying (2.1.11), a map $u$ in the space $\mathfrak{H}^\alpha(\mathbb{R}^k)$ is called* **horizontal $\alpha$-harmonic** *with respect to $P_T$ if*

$$\forall\, i = 1 \cdots m \qquad \sum_{j=1}^{m} P_T^{ij}(u)(-\Delta)^\alpha u_j = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^k) \tag{2.1.12}$$

*and we shall use the following notation*

$$P_T(u)\,(-\Delta)^\alpha u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^k).$$

When the plane distribution $P_T$ is *integrable* that is to say when

$$\forall\, X, Y \in C^1(\mathbb{R}^m, \mathbb{R}^m) \quad P_N[P_T\, X, P_T\, Y] \equiv 0, \tag{2.1.13}$$

where $[\cdot, \cdot]$ denotes the Lie Bracket of vector-fields, using Frobenius theorem the plane distribution corresponds to the tangent plane distribution of a $n-$dimensional *foliation* $\mathcal{F}$. A smooth map $u$ in $\mathfrak{H}^\alpha(\mathbb{R}^m)$ takes values everywhere into a *leaf* of $\mathcal{F}$ that we denote $\mathcal{N}^n$ and we are back to the classical theory of harmonic maps into manifolds. Observe that our definition includes the case of $\alpha$-harmonic maps with values into a sub-manifold of the euclidean space and horizontal with respect to a plane distribution in this sub-manifold. Indeed it is sufficient to add to such a distribution the projection to the sub-manifold and extend the all to a tubular neighborhood of the sub-manifold.

In [10] we have proved the following result

**Theorem 2.3** (Theorem 2.1, [10]). *Let $P_T$ be a $C^1$ distribution of planes (or projections) satisfying (2.1.11). Any map $u \in \mathfrak{H}^1(D)$*

$$P_T(u)\,\Delta u = 0 \quad \text{in } \mathcal{D}'(D) \tag{2.1.14}$$

*is in $\cap_{\delta<1} C^{0,\delta}_{loc}(D)$.*

The main idea to prove Theorem 2.3 is to show that $u$ satisfies an elliptic Schrödinger type system with an antisymmetric potential $\Omega \in L^2(\mathbb{R}^k, \mathbb{R}^k \otimes so(m))$ (whose construction depends on $P_T$) of the form

$$-\Delta u = \Omega \cdot \nabla u. \tag{2.1.15}$$

Hence, following the analysis in [20] the authors deduced in dimension 2 the local existence on a disk $D$ of $A \in L^\infty \cap W^{1,2}(D, Gl_m(\mathbb{R}))$ and $B \in W^{1,2}(D, \mathcal{M}_m(\mathbb{R}))$, depending both on $P_T(u)$, such that

$$\operatorname{div}(A\, \nabla u) = \nabla^\perp B \cdot \nabla u \tag{2.1.16}$$

from which the regularity of $u$ can be deduced using Wente's Theorem 2.1.[2.1]

**2.1** We denote by $so(m)$ the space of antisymmetric matrices of order $m$ and by $GL_m$ the space of invertible matrices of order $m$.

Now we turn our attention to an analogous *fractional* problem in dimension 1. We consider the following Lagrangian that we will call $L-energy$ ($L$ stands for “Line”)

$$\mathcal{L}^{1/2}(u) := \int_{\mathbb{R}} |(-\Delta)^{1/4}u|^2 \, dx \tag{2.1.17}$$

within

$$\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) := \left\{ u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \; ; \; u(x) \in \mathcal{N} \text{ for a. e. } x \in \mathbb{R} \right\} .$$

The operator $(-\Delta)^{\alpha}$ on $\mathbb{R}$ is defined by means of the Fourier transform as follows

$$\widehat{(-\Delta)^{\alpha} u} = |\xi|^{2\alpha} \hat{u} \, ,$$

(given a function $f$, both $\hat{f}$ and $\mathcal{F}[f]$ denote the Fourier transform of $f$).

The Lagrangian (2.1.17) is invariant with respect to the Möbius group and it satisfies the following identity

$$\int_{\mathbb{R}} |(-\Delta)^{1/4}u(x)|^2 dx = \inf \left\{ \int_{\mathbb{R}^2_+} |\nabla \tilde{u}|^2 dx \, : \; \tilde{u} \in W^{1,2}(\mathbb{R}^2_+, \mathbb{R}^m), \;\; \text{trace } \tilde{u} = u \right\} .$$

In [8] we introduced the following Definition:

**Definition 2.4.** *A map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is called a weak $1/2$-harmonic map into $\mathcal{N}$ if for any $\varphi \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^{\infty}(\mathbb{R}, \mathbb{R}^m)$ there holds*

$$\frac{d}{dt} \mathcal{L}^{1/2}(\pi_{\mathcal{N}}(u + t\varphi))_{|t=0} = 0,$$

*where $\Pi_{\mathcal{N}}$ is the orthogonal projection on $\mathcal{N}$ .*

In short we say that *a weak $1/2$-harmonic map is a critical point of $\mathcal{L}^{1/2}$ in $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ for perturbations in the target*.

Weak $1/2$-harmonic maps satisfy the Euler-Lagrange equation

$$\nu(u) \wedge (-\Delta)^{1/2}u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{2.1.18}$$

Let $\Pi_{-i} \colon S^1 \backslash \{-i\} \to \mathbb{R}$, $\Pi_{-i}(\xi + i\eta) = \frac{\xi}{1+\eta}$ be the stereographic projection from the south pole, then the following relation between the $1/2$ Laplacian in $\mathbb{R}$ and in $S^1$ holds:

**Proposition 2.5** (Proposition 4.1, [7])**.** *Given $u : \mathbb{R} \to \mathbb{R}^m$, we set $v := u \circ \Pi_{-i} : S^1 \to \mathbb{R}^m$. Then $u \in L_{\frac{1}{2}}(\mathbb{R})$*[2.2] *if and only if $v \in L^1(S^1)$. In this case*

$$(-\Delta)^{\frac{1}{2}}_{S^1} v(e^{i\vartheta}) = \frac{((-\Delta)^{\frac{1}{2}}_{\mathbb{R}} u)(\Pi_{-i}(e^{i\vartheta}))}{1 + \sin \vartheta} \;\; \textit{in } \mathcal{D}'(S^1 \backslash \{-i\}), \tag{2.1.19}$$

**2.2** We recall that $L_{\frac{1}{2}}(\mathbb{R}) := \left\{ u \in L^1_{loc}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1+x^2} dx < \infty \right\}$

*Observe that* $(1+\sin(\vartheta))^{-1} = |\Pi'_{-i}(\vartheta)|$, *and hence we have*

$$\int_{S^1} (-\Delta)^{\frac{1}{2}} v(e^{i\vartheta})\,\varphi(e^{i\vartheta})\,d\vartheta = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u(x)\,\varphi(\Pi_{-i}^{-1}(x))\,dx \quad \textit{for every } \varphi \in C_0^\infty(S^1\backslash\{-i\}).$$

From (2.1.19) and the invariance of the Lagrangian (2.1.17) with respect to the trace of conformal maps in $\mathbb{C}$ it follows that a map $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is weak $1/2$-harmonic in $\mathbb{R}$ if and only if $v = u \circ \Pi_{-i} \in \dot{H}^{1/2}(S^1, \mathcal{N})$ is weak $1/2$-harmonic in $S^1$.

Indeed $v \in \dot{H}^{1/2}(S^1, \mathcal{N})$ satisfies

$$\nu(v) \wedge (-\Delta)^{1/2} v = 0 \qquad \text{in } \mathcal{D}'(S^1\backslash\{-i\}). \tag{2.1.20}$$

Consider now the stereographic projection from the north pole $\Pi_i \colon S^1\backslash\{i\} \to \mathbb{R}$, $\Pi_i(\xi + i\eta) = \frac{\xi}{1-\eta}$ and $\tilde{u} = v \circ \Pi_i^{-1} = u \circ \dfrac{1}{z}$. Since $\dfrac{1}{z} \colon \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ is a conformal map, $\tilde{u} \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is weak $1/2$-harmonic in $\mathbb{R}\backslash\{0\}$. By applying Proposition 2.2 in [5] (a singular point removability type result on $\mathbb{R}$) we deduce that $\tilde{u}$ is weak $1/2$-harmonic in $\mathbb{R}$ and in particular continuous in $\mathbb{R}$. Therefore not only $v$ is weak $1/2$-harmonic in $S^1$ but we deduce that

$$\lim_{x \to +\infty} u(x) = \lim_{x \to +\infty} u(x) \ \text{ and } \ \lim_{\substack{z \to -i^+ \\ z \in S^1}} v(z) = \lim_{\substack{z \to -i^- \\ z \in S^1}} v(z).$$

Fractional harmonic maps appear in several geometric problems and we mention some of them below.

1. The first application is the connection between weak $1/2$-harmonic maps and free boundary minimal disks. The following characterization of weak $1/2$-harmonic maps of $S^1$ into sub-manifolds of $\mathbb{R}^n$ holds, (see [7] and [18]).

**Theorem 2.6.** *Let* $u \in \dot{H}^{1/2}(S^1, \mathcal{N})$, *where* $\mathcal{N}$ *is a n-dimensional closed smooth sub-manifold of* $\mathbb{R}^m$. *If u is a nontrivial weak* $1/2$*-harmonic map, then its harmonic extension* $\tilde{u} \in W^{1,2}(D, \mathbb{R}^m)$ *is conformal and*

$$\nu(u) \wedge \frac{\partial \tilde{u}}{\partial r} = 0 \quad \textit{in } \mathcal{D}'(S^1). \tag{2.1.21}$$

From Theorem 2.6 it follows that $\tilde{u}$ is a minimal disk whose boundary lies in $\mathcal{N}$ and meets $\mathcal{N}$ orthogonally, namely its outward normal vector $\frac{\partial \tilde{u}}{\partial r}$ is othogonal to $\mathcal{N}$ at each point of $\tilde{u}(S^1)$. Moreover we can deduce the following two characterizations of $1/2$-harmonic maps in the case where $\mathcal{N} = S^1$ and $\mathcal{N} = S^2$.

**Theorem 2.7.** *i) Weak* $1/2$*-harmonic maps* $u \colon S^1 \to S^1$ *with* $deg(u) = 1$ *coincide with the trace of Möbius transformations of the disk* $B^2(0,1) \subseteq \mathbb{R}^2$.

*ii) If* $u \colon S^1 \to S^2$ *is a weak* $1/2$*-harmonic map then* $u(S^1)$ *is an equatorial plane and it is the composition of weak* $1/2$*-harmonic map* $u \colon S^1 \to S^1$ *with an isometry* $\tau \colon S^2 \to S^2$.

2. Another geometrical application concerns the so-called Steklov eigenvalue problem that is the first eigenvalue $\sigma_1$ of the Dirichlet-to-Neumann map on some Riemannian surfaces $(M, g)$ with boundary $\partial M$. In [14] the authors show the following

**Theorem 2.8** ( Proposition 2.8, [14]). *If M is a surface with boundary, and $g_0$ is a metric on M with*

$$\sigma_1(g_0)L_{g_0}(\partial M) = \max_g \sigma_1(g)L_g(\partial M),$$

*where $L_g(\partial M)$ is the lenght of $\partial M$, the max is over all smooth metrics on M in the conformal class of $g_0$. Then there exist independent eigenfunctions $u_1, \dots, u_n$ corresponding to the eigenvalue $\sigma_1(g_0)$ which give a conformal minimal immersion $u = (u_1, \dots, u_n)$ of M into the unit ball $B^n$ and $u(M)$ is a free boundary solution. That is, $u\colon (M, \partial M) \to (B^n, \partial B^n)$ is a harmonic map such that $u(\partial M)$ meets $\partial B^n$ orthogonally. Hence $u|_{\partial M}$ is $1/2$-harmonic.*

3. $1/2$-harmonic maps appear in the asymptotics of fractional Ginzburg-Landau equation, (see [18]) and in connections with regularity of critical knots of Möbius energy (see [2]).

The theory of weak $1/2$ harmonic maps with values into a closed $n$-dimensional sub-manifold $\mathcal{N}$ has been initiated some years ago by the author and Tristan Rivière in [8]. Since then several extensions have been considered (see [4, 12, 9]). The main novelty in the regularity of $1/2$-harmonic was the re-formulation of the Euler-Lagrange equation in terms of special algebraic quantities called *3-terms commutators* which are roughly speaking bilinear pseudo-differential operators satisfying some *integrability by compensation* properties.

As in the local case we can consider a plane distribution $P_T$ satisfying (2.1.11) and solutions of

$$P_T(u)\,(-\Delta)^{1/2}u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}) \tag{2.1.22}$$

under the constraint $P_N(u)\,\nabla u = 0$ in $\mathcal{D}'(\mathbb{R})$. Maps $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ satisfying (2.1.22) are called *horizontal* $1/2$-*harmonic maps*. One of the main result in [10] is the following Theorem.

**Theorem 2.9.** *Let $P_T$ be a $C^1$ distribution of planes satisfying (2.1.11). Any map $u \in \mathfrak{H}^{1/2}(\mathbb{R})$*

$$P_T(u)\,(-\Delta)^{1/2}u = 0 \quad in\ \mathcal{D}'(\mathbb{R}) \tag{2.1.23}$$

*is in $\cap_{\delta<1} C^{0,\delta}_{loc}(\mathbb{R})$.*

In [10] conservation laws corresponding to *horizontal* $1/2$-*harmonic maps* have been discovered: locally, modulo some smoother terms coming from the application of non-local operators on cut-off functions, we construct out of $P_T(u)\ A \in L^\infty \cap$

$\dot{H}^{1/2}(\mathbb{R}, Gl_m(\mathbb{R}))$ and $B \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_m(\mathbb{R}))$ such that

$$(-\Delta)^{1/4}(A\, v) = \mathcal{J}(B, v) + \text{cut-off}, \tag{2.1.24}$$

where $v := (P_T\,(-\Delta)^{1/4}u, \mathcal{R}(P_N(-\Delta)^{1/4}u))^t$, $\mathcal{R}$ denotes the Riesz operator defined by $\widehat{\mathcal{R}f}(\xi) = i\frac{\xi}{|\xi|}\hat{f}$ and $\mathcal{J}$ is a bilinear pseudo-differential operator satisfying

$$\|\mathcal{J}(B, v)\|_{\dot{H}^{-1/2}(\mathbb{R})} \leqslant C\,\|(-\Delta)^{1/4}B\|_{L^2(\mathbb{R})}\,\|v\|_{L^2(\mathbb{R})}. \tag{2.1.25}$$

As we will see later, the conservation law (2.1.24) will be crucial in the quantization analysis of sequences of horizontal 1/2-harmonic maps.

By assuming that $P_T \in C^2(\mathbb{R}^m)$ and by bootstrapping arguments one gets that every horizontal 1/2-harmonic map $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ is $C^{1,\alpha}_{loc}(\mathbb{R})$, for every $\alpha < 1$ (see [11]).

We would like to mention that in the non-integrable case it seems not feasible to get the regularity of the horizontal 1/2-harmonic maps by the techniques in [23] or [18] which consist in transforming the a-priori non-local PDE (2.1.18) into a local one and in performing *ad-hoc* extensions and reflections.

Also in the nonintegrable case the following geometric characterization holds.

**Proposition 2.10.** *An element in $\mathfrak{H}^{1/2}$ satisfying (2.1.22) has a harmonic extension $\tilde{u}$ in $B^2(0,1)$ which is conformal and hence it is the boundary of a minimal disk whose exterior normal derivative $\partial_r\tilde{u}$ is orthogonal to the plane distribution given by $P_T$.*

**Example** : We consider the following field of non-integrable projections in $\mathbb{C}^2\setminus\{0\}$.

$$P_T(z)\,Z := Z - |z|^{-2}\;[Z\cdot(z_1, z_2)\,(z_1, z_2) + Z\cdot(iz_1, iz_2)\,(iz_1, iz_2)]\,. \tag{2.1.26}$$

An example of $u$ satisfying (2.1.23) is given by solutions to the system

$$\left\{\begin{array}{ll} \dfrac{\partial\tilde{u}}{\partial r}\wedge u\wedge iu = 0 \quad \text{in } \mathcal{D}'(S^1) & \\[2ex] u\cdot\dfrac{\partial u}{\partial\vartheta} = 0 \quad \text{in } \mathcal{D}'(S^1) & \qquad \text{an at least} \\[2ex] i\,u\cdot\dfrac{\partial u}{\partial\vartheta} = 0 \quad \text{in } \mathcal{D}'(S^1) & \end{array}\right. \tag{2.1.27}$$

where $\tilde{u}$ denotes the harmonic extension of $u$ which happens to be conformal due to Proposition 2.10 and define a minimal disk. An example of such maps is given by

$$u(\vartheta) := \frac{1}{\sqrt{2}}(e^{i\vartheta}, e^{-i\vartheta}) \quad \text{where} \quad \tilde{u}(z,\overline{z}) = \frac{1}{\sqrt{2}}(z,\overline{z}). \tag{2.1.28}$$

Observe that the solution in (2.1.28) is also a 1/2-harmonic map into $S^3$ and it would be interesting to investigate whether this is the unique solution.

From a geometrical point of view to find a solution to (2.1.23) means to find a minimal disk whose boundary is horizontal and the normal direction is vertical.

One natural question is to see if this problem is variational. A priori if $\tilde{u}$ is a critical point of the Dirichlet energy whose boundary is horizontal, then its exterior normal derivative $\partial_r \tilde{u}$ does not belong necessarily to $\mathrm{Im}(P_N)$. Despite the geometric relevance of equations (2.1.12) in the non-integrable case, it is however a-priori not the *Euler-Lagrange* equation of the variational problem consisting in finding the critical points of $\|(-\Delta)^{\alpha/2}u\|^2_{L^2}$ within $\mathfrak{H}^\alpha$ when $P_T$ is not satisfying (2.1.13). This can be seen in the particular case where $\alpha = 1$ where the critical points to the *Dirichlet Energy* have been extensively studied in relation with the computation of *normal geodesics* in sub-riemannian geometry. We then introduce the following definition:

**Definition 2.11.** *A map $u$ in $\mathfrak{H}^\alpha$ is called* **variational** $\alpha-$**harmonic** *into the plane distribution $P_T$ if it is a critical point of the $\|(-\Delta)^{\alpha/2}u\|^2_{L^2}$ within variations in $\mathfrak{H}^\alpha$ i.e. for any $u_t \in C^1((-1,1),\mathfrak{H}^\alpha)$ we have*

$$\frac{d}{dt}\|(-\Delta)^{\alpha/2}u_t\|^2_{L^2}\Big|_{t=0} = 0. \tag{2.1.29}$$

Example of variational harmonic maps from $S^1$ into a plane distribution is given by the sub-riemannian geodesics.

A priori the equation (2.1.22) is not the Euler-Lagrange equation associated to (2.1.29). The main difficulty is that we have not a pointwise constraint but a constraint on the gradient. In order to study critical points of (2.1.29) we use a convexification of the above variational problem following the spirit of the approach introduced by Strichartz in [27] for *normal geodesics* in sub-riemannian geometry. We prove in particular for the case $\alpha = 1/2$ that the smooth critical points of

$$\begin{aligned}\mathcal{L}^{1/2}(u,\xi) := &\int_{S^1} \frac{|(-\Delta)_0^{-1/4}(P_T(u)\xi)|^2}{2}\, d\vartheta \\ &- \int_{S^1}\left\langle (-\Delta)_0^{-1/4}(P_T(u)\xi), (-\Delta)_0^{-1/4}\left(P_T(u)\frac{du}{d\vartheta}\right)\right\rangle d\vartheta \\ &- \int_{S^1}\left\langle (-\Delta)_0^{-1/4}(P_N(u)\xi), (-\Delta)_0^{-1/4}\left(P_N(u)\frac{du}{d\vartheta}\right)\right\rangle d\vartheta\end{aligned} \tag{2.1.30}$$

in the co-dimension $m$ <u>Hilbert subspace</u> of $\dot{H}^{1/2}(S^1,\mathbb{R}^m)\times \dot{H}^{-1/2}(S^1,\mathbb{R}^m)$ given by[2.3]

$$\mathfrak{E} := \left\{\begin{array}{c} (u,\xi)\in \dot{H}^{1/2}(S^1,\mathbb{R}^m)\times H^{-1/2}(S^1,\mathbb{R}^m) \quad \text{s. t.} \\[2ex] \left(P_N(u),\dfrac{du}{d\vartheta}\right)_{\dot{H}^{1/2},\dot{H}^{-1/2}} = 0 \\[2ex] (-\Delta)_0^{-1/4}(P_T(u)\xi)\in L^2(S^1) \quad \text{and} \quad (-\Delta)_0^{-1/4}\left(P_T(u)\dfrac{du}{d\vartheta}\right)\in L^2(S^1)\end{array}\right\}$$

**2.3** Given $f\in \dot{H}^{1/2}$, $g\in \dot{H}^{-1/2}$ we denote by $(f,g)_{\dot{H}^{1/2},\dot{H}^{-1/2}}$ the duality between $f$ and $g$.

at the point where the constraint $\left(P_N(u), \frac{du}{d\vartheta}\right)_{\dot H^{1/2},\dot H^{-1/2}} = 0$ is **non-degenerate** are "variational 1/2-harmonic" into the plane distribution $P_T$ in the sense of definition 2.11. It remains open the regularity of critical points of (2.1.30) or even of the 1/2 energy (2.1.17) in $\mathfrak{H}^{1/2}$ in the case when the constraint $\left(P_N(u), \frac{du}{d\vartheta}\right)_{\dot H^{1/2},H^{-1/2}} = 0$ is degenerate.

In a joint paper with P. Laurain and T. Rivière we investigate compactness and quantization properties of sequences of horizontal 1/2 harmonic maps $u_k \in \mathfrak{H}^{1/2}(\mathbb{R})$ by extending the results obtained by the author in [5] in the case of 1/2-harmonic maps with values into a sphere. Our first main result is the following:

**Theorem 2.12. [Theorem 1.2 in [6]]** *Let $u_k \in \mathfrak{H}^{1/2}(\mathbb{R})$ be a sequence of horizontal 1/2-harmonic maps such that*

$$\|u_k\|_{\dot H^{1/2}} \leqslant C, \quad \|(-\Delta)^{1/2}u_k\|_{L^1} \leqslant C\,. \tag{2.1.31}$$

*Then it holds:*

1. *There exist $u_\infty \in \mathfrak{H}^{1/2}(\mathbb{R})$ and a possibly empty set $\{a_1,\dots,a_\ell\}$, $\ell \geqslant 1$, such that up to subsequence*
$$u_k \to u_\infty \quad \text{in } \dot W^{1/2,p}_{loc}(\mathbb{R}\backslash\{a_1,\dots,a_\ell\}),\ \ p \geqslant 2 \text{ as } k \to +\infty \tag{2.1.32}$$
*and*
$$P_T(u_\infty)(-\Delta)^{1/2}u_\infty = 0, \quad \text{in } \mathcal{D}'(\mathbb{R})\,. \tag{2.1.33}$$
2. *There is a family $\tilde u^{i,j}_\infty \in \dot{\mathfrak{H}}^{1/2}(\mathbb{R})$ of horizontal 1/2-harmonic maps ($i \in \{1,\dots,\ell\}$, $j \in \{1,\dots,N_i\}$), such that up to subsequence*
$$\left\|(-\Delta)^{1/4}\left(u_k - u_\infty - \sum_{i,j}\tilde u^{i,j}_\infty((x - x^k_{i,j})/r^k_{i,j})\right)\right\|_{L^2_{loc}(\mathbb{R})} \to 0, \ \ \text{as } k \to +\infty\,. \tag{2.1.34}$$
*for some sequences $r^k_{i,j} \to 0$ and $x^k_{i,j} \in \mathbb{R}$.*

As we have already remarked in [6] the condition $\|(-\Delta)^{1/2}u_k\|_{L^1} \leqslant C$ is always satisfied in the case the maps $u_k$ take values into a closed manifold of $\mathbb{R}^m$ (case of sequences of 1/2 harmonic maps) as soon as $\|u_k\|_{\dot H^{1/2}} \leqslant C$. This follows from the fact that if $u$ is a 1/2-harmonic maps with values into a closed manifold of $\mathcal{N}$ of $\mathbb{R}^m$ then the following inequality holds (see Proposition 5.1 in [6])

$$\|(-\Delta)^{1/2}u\|_{L^1(\mathbb{R})} \leqslant C\|(-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R})}. \tag{2.1.35}$$

Hence in the case of 1/2-harmonic maps defined in $S^1$ we have the following corollary.

**Corollary 2.13. [Corollary 1.1 in [6]]** *Let $\mathcal{N}$ be a closed $C^2$ submanifold of $\mathbb{R}^m$ and let $u_k \in H^{1/2}(S^1,\mathcal{N})$ be a sequence of 1/2-harmonic maps such that*

$$\|u_k\|_{\dot H^{1/2}(S^1)} \leqslant C \tag{2.1.36}$$

*then the conclusions of Ttheseheorem 2.12 hold. In particular up to subsequence we have the following energy identity*

$$\lim_{k\to+\infty}\int_{S^1}|(-\Delta)^{1/4}u_k|^2\,d\vartheta=\int_{S^1}|(-\Delta)^{1/4}u_\infty|^2\,d\vartheta+\sum_{i,j}\int_{S^1}|(-\Delta)^{1/4}\tilde{u}_\infty^{i,j}|^2\,d\vartheta \quad (2.1.37)$$

*where $\tilde{u}_\infty^{i,j}$ are the* **bubbles** *associated to the weak convergence.*

For the moment it remains open whether the bound (2.1.35) holds or not in the general case of horizontal $1/2$-harmonic maps.

The compactness issue (first part of Theorem 2.12) is quite standard. The most delicate part is the quantization analysis consisting in verifying that there is no dissipation of the energy in the region between $u_\infty$ and the *bubbles* $\tilde{u}_\infty^{i,j}$ and between the bubbles themselves (the so-called *neck-regions*). Such an analysis has been achieved in [6] by performing a precise asymptotic development of horizontal $1/2$-harmonic maps in these neck-regions, that was possible thanks to the conservation law (2.1.24) and an application of new Pohozaev-type identities in 1-D discovered in [6]. We refer the reader to [6] for a complete description of compactness and quantization issues of horizontal $1/2$-harmonic maps.

We conclude this section by mentioning that the partial regularity of $1/2$-harmonic map in dimension $k\geqslant 2$ with values into a sphere has been been deduced in [18] from existing regularity results of harmonic maps with free boundary. Schikorra [25] has also studied the partial regularity of weak solutions to nonlocal linear systems with an antisymmetric potential in the supercritical case under a crucial monotonicity assumption on the solutions which allows us to reduce it to the critical case.

It still remains open a direct proof of the partial regularity without an ad-hoc monotonicity assumption.

## 2.2 3-Commutators Estimates

As we have already mentioned in the previous section, when the notion of $1/2$-harmonic map was introduced in [8], one of the main novelty was the re-formulation of the Euler-Lagrange equation in terms of *three-terms-commutators* which have played a key role in all the results that have been obtained later.

In this section we will introduce such commutators and recall some important estimates and properties. Such properties will be crucial to get regularity results of $1/2$-harmonic maps and to re-write the system (satisfied by a horizontal $1/2$-harmonic map)

$$\begin{cases} P_T(u)(-\Delta)^{1/2}u=0 \\ \\ P_N(u)\nabla u=0 \end{cases} \quad (2.2.1)$$

in term of a conservation law.

We first introduce some functional spaces.

$\mathcal{H}^1(\mathbb{R}^n)$ denotes the Hardy space which is the space of $L^1$ functions $f$ on $\mathbb{R}^n$satisfying

$$\int_{\mathbb{R}^n} \sup_{t\in\mathbb{R}} |\varphi_t * f|(x)\, dx < +\infty \quad ,$$

where $\varphi_t(x) := t^{-n}\ \varphi(t^{-1}x)$ and where $\varphi$ is some function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. For more properties on the Hardy space $\mathcal{H}^1$ we refer to [15, 16, 26].

The $L^{2,\infty}(\mathbb{R})$ is the space of measurable functions $f$ such that

$$\sup_{\lambda>0} \lambda |\{x \in \mathbb{R}\ : |f(x)| \geqslant \lambda\}|^{1/2} < +\infty\,.$$

$L^{2,1}(\mathbb{R})$ is the Lorentz space of measurable functions satisfying

$$\int_0^{+\infty} |\{x \in \mathbb{R}\ : |f(x)| \geqslant \lambda\}|^{1/2} d\lambda < +\infty\ .$$

In [8] the following two *three-terms commutators* have been introduced:

$$T(Q,v) := (-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4}v + (-\Delta)^{1/4}Qv \tag{2.2.2}$$

and

$$S(Q,v) := (-\Delta)^{1/4}[Qv] - \overline{\mathcal{R}}(Q\mathcal{R}(-\Delta)^{1/4}v) + \overline{\mathcal{R}}((-\Delta)^{1/4}Q\mathcal{R}v), \tag{2.2.3}$$

where $\mathcal{R}$ is the Riesz operator.

In [8] the authors obtained the following estimates.

**Theorem 2.14.** *Let $v \in L^2(\mathbb{R})$, $Q \in \dot{H}^{1/2}(\mathbb{R})$ . Then $T(Q,v), S(Q,v) \in H^{-1/2}(\mathbb{R})$ and*

$$\|T(Q,v)\|_{H^{-1/2}(\mathbb{R})} \leqslant C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}(\mathbb{R})}\,; \tag{2.2.4}$$

$$\|S(Q,v)\|_{H^{-1/2}(\mathbb{R})} \leqslant C\ \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^{2,\infty}(\mathbb{R})}\,. \tag{2.2.5}$$

We observe that under our assumptions $u \in \dot{H}^{1/2}(\mathbb{R},\mathbb{R}^m)$ and $Q \in \dot{H}^{1/2}(\mathbb{R},\mathcal{M}_{\ell\times m}(\mathbb{R}))$ each term individually in $T$ and $S$ - like for instance $(-\Delta)^{1/4}(Q(-\Delta)^{1/4}u)$ or $Q(-\Delta)^{1/2}u$ ... - are not in $H^{-1/2}$ but the special linear combination of them constituting $T$ and $S$ are in $H^{-1/2}$. In a similar way, <u>in dimension 2</u>, $J(a,b) := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ satisfies, as a direct consequence of Wente's theorem 2.1

$$\|J(a,b)\|_{\dot{H}^{-1}} \leqslant C\ \|a\|_{\dot{H}^1}\ \|b\|_{\dot{H}^1} \tag{2.2.6}$$

whereas, individually, the terms $\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ are not in $H^{-1}$.

Actually in [5] we improve the estimates on the operators $T, S$.

**Theorem 2.15.** *Let $v \in L^2(\mathbb{R})$, $Q \in \dot{H}^{1/2}(\mathbb{R})$. Then $T(Q,v), S(Q,v) \in \mathcal{H}^1(\mathbb{R})$ and*

$$\|T(Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} . \tag{2.2.7}$$

$$\|S(Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|Q\|_{\dot{H}^{1/2}(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} . \tag{2.2.8}$$

We refer the reader to [8] and [5] for the proof of respectively Theorem 2.14 and Theorem 2.15. We just mention that the above estimates is based on a well-known tool in harmonic analysis, the Littlewood-Paley dyadic decomposition of unity that we briefly recall here. Such a decomposition can be obtained as follows. Let $\varphi(\xi)$ be a radial Schwartz function supported in $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}$, which is equal to 1 in $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\}$. Let $\psi(\xi)$ be the function given by

$$\psi(\xi) := \varphi(\xi) - \varphi(2\xi)$$

$\psi$ is then a "bump function" supported in the annulus $\{\xi \in \mathbb{R}^n : 1/2 \leqslant |\xi| \leqslant 2\}$.

Let $\psi_0 = \varphi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions $\psi_j$, for $j \in \mathbb{Z}$, are supported in $\{\xi \in \mathbb{R}^n : 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$ and they realize a dyadic decomposition of the unity:

$$\sum_{j\in\mathbb{Z}} \psi_j(x) = 1 .$$

We further denote

$$\varphi_j(\xi) := \sum_{k=-\infty}^{j} \psi_k(\xi) .$$

The function $\varphi_j$ is supported on $\{\xi,\ |\xi| \leqslant 2^{j+1}\}$.

For every $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R})$ we define the Littlewood-Paley projection operators $P_j$ and $P_{\leqslant j}$ by

$$\widehat{P_j f} = \psi_j \hat{f} \quad \widehat{P_{\leqslant j} f} = \varphi_j \hat{f} .$$

Informally $P_j$ is a frequency projection to the annulus $\{2^{j-1} \leqslant |\xi| \leqslant 2^j\}$, while $P_{\leqslant j}$ is a frequency projection to the ball $\{|\xi| \leqslant 2^j\}$. We will set $f_j = P_j f$ and $f^j = P_{\leqslant j} f$.

We observe that $f^j = \sum_{k=-\infty}^{j} f_k$ and $f = \sum_{k=-\infty}^{+\infty} f_k$ (where the convergence is in $\mathcal{S}'(\mathbb{R})$).

Given $f, g \in \mathcal{S}'(\mathbb{R})$ we can split the product in the following way

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{2.2.9}$$

where

$$\begin{aligned} \Pi_1(f,g) &= \sum_{-\infty}^{+\infty} f_j \sum_{k\leqslant j-4} g_k = \sum_{-\infty}^{+\infty} f_j g^{j-4} ; \\ \Pi_2(f,g) &= \sum_{-\infty}^{+\infty} f_j \sum_{k\geqslant j+4} g_k = \sum_{-\infty}^{+\infty} g_j f^{j-4} ; \end{aligned}$$

$$\Pi_3(f,g) = \sum_{-\infty}^{+\infty} f_j \sum_{|k-j|<4} g_k\,.$$

We observe that for every $j$ we have

$$\mathrm{supp}\mathcal{F}[f^{j-4}g_j] \subset \{2^{j-2} \leqslant |\xi| \leqslant 2^{j+2}\};$$

$$\mathrm{supp}\mathcal{F}[\textstyle\sum_{k=j-3}^{j+3} f_j g_k] \subset \{|\xi| \leqslant 2^{j+5}\}\,.$$

The three pieces of the decomposition (2.2.9) are examples of paraproducts. Informally the first paraproduct $\Pi_1$ is an operator which allows high frequences of $f$ ($\sim 2^j$) multiplied by low frequences of $g$ ($\ll 2^j$) to produce high frequences in the output. The second paraproduct $\Pi_2$ multiplies low fequences of $f$ with high frequences of $g$ to produce high fequences in the output. The third paraproduct $\Pi_3$ multiply high frequences of $f$ with high frequences of $g$ to produce comparable or lower frequences in the output. For a presentation of these paraproducts we refer to the reader for instance to the book [16] .

The compensations of the 3 different terms in $T(Q,v)$ will be clear just from the Littlewood-Paley decomposition of the different products. With this regards to get for instance the estimate (2.2.7) we shall need the following groupings

– i) For $\Pi_1(T(Q,v))$ we proceed to the following decomposition

$$\Pi_1(T(Q,v)) = \underbrace{\Pi_1((-\Delta)^{1/4}(Qv))} + \underbrace{\Pi_1 Q(-\Delta)^{1/4}v + (-\Delta)^{1/4}Qv}\,.$$

– ii) For $\Pi_2(R(Q,u))$ we decompose as follows

$$\Pi_2(T(Q,v)) = \underbrace{\Pi_2((-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4}v)} + \underbrace{\Pi_2((-\Delta)^{1/4}Qv)}\,.$$

– ii) Finally, for $\Pi_3(R(Q,u))$ we decompose as follows

$$\Pi_3(T(Q,v)) = \underbrace{\Pi_3((-\Delta)^{1/4}(Qv))} - \underbrace{\Pi_3(Q(-\Delta)^{1/4}v)} + \underbrace{\Pi_3((-\Delta)^{1/4}Qv)}\,.$$

The following *2-terms commutators* have also been used in [9, 10]:

$$F(Q,v) := \mathcal{R}[Q]\mathcal{R}[v] - Qv. \tag{2.2.10}$$

$$\Lambda(Q,v) := Qv + \mathcal{R}[Q\mathcal{R}[v]]. \tag{2.2.11}$$

**Theorem 2.16. [Theorem 3.6 in [10]]** *For $f, v \in L^2$ it holds*

$$\|F(f,v)\|_{H^{-1/2}(\mathbb{R})} \leqslant C\|f\|_{L^2(\mathbb{R})}\|v\|_{L^{2,\infty}(\mathbb{R})}, \tag{2.2.12}$$

*and*

$$\|F(f,v)\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|f\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\,.\,\square \tag{2.2.13}$$

**Theorem 2.17. [Theorem 3.7 in [10]]** *For $Q \in \dot{H}^{1/2}(\mathbb{R})$, $v \in L^2(\mathbb{R})$ it holds*

$$\|(-\Delta)^{1/4}(\Lambda(Q,v))\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|Q\|_{H^{1/2}(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\,. \tag{2.2.14}$$

Actually the estimate (2.2.13) is a consequence of the Coifman-Rochberg-Weiss estimate [3].

From Theorem 2.17 we deduce that under the same assumptions it holds $\Lambda(Q,v) \in L^{2,1}(\mathbb{R})$ with

$$\|\Lambda(Q,v)\|_{L^{2,1}(\mathbb{R})} \leqslant C\|Q\|_{H^{1/2}(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}.$$

We finally remark that we can simply write the operator $S$ as follows:

$$S(Q,v) = \overline{\mathcal{R}}T(Q,\mathcal{R}v) - \overline{\mathcal{R}}(-\Delta)^{1/4}[\Lambda(Q,\mathcal{R}v)]. \tag{2.2.15}$$

Therefore the estimate (2.2.8) for $S$ can be deduced from the estimate (2.2.8) for the operator $T$ and Theorem 2.17.

In [10] we have proved a sort of stability of of the operators $T, S$ with respect to the multiplication by a function $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Roughly speaking if we multiply $T(Q,v)$ or $S(Q,v)$ by a function $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we get a decomposition into the sum of a function in the Hardy Space and a term which is the product of function in $L^{2,1}$ by one in $L^2$.

**Theorem 2.18. [Multiplication of $T$ by $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$]** *Let $P, Q \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v \in L^2(\mathbb{R})$. Then*

$$PT(Q,v) = J_T(P,Q,v) + \mathcal{A}_T(P,Q)v, \tag{2.2.16}$$

*where*

$$\mathcal{A}_T(P,Q) = P(-\Delta)^{1/4}[Q] + (-\Delta)^{1/4}[P]Q - (-\Delta)^{1/4}[PQ] \in L^{2,1}$$

*with*

$$\|\mathcal{A}_T(P,Q)\|_{L^{2,1}} \leqslant C\|(-\Delta)^{1/4}[P]\|_{L^2}\|(-\Delta)^{1/4}[Q]\|_{L^2}, \tag{2.2.17}$$

*and*

$$J_T(P,Q,v) := T(PQ,v) - T(P,Qv) \in \mathcal{H}^1(\mathbb{R})$$

*with*

$$\|J_T(P,Q,v)\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C(\|P\|_{L^\infty} + \|Q\|_{L^\infty})\left(\|(-\Delta)^{1/4}[P]\|_{L^2} + \|(-\Delta)^{1/4}[Q]\|_{L^2}\right)\|v\|_{L^2}. \tag{2.2.18}$$

**Proof of Theorem 2.18.** We have

$$\begin{aligned} PT(Q,v) &= P(-\Delta)^{1/4}[Qv] - PQ(-\Delta)^{1/4}[v] + P(-\Delta)^{1/4}[Q]v \\ &= \{P(-\Delta)^{1/4}[Q] - (-\Delta)^{1/4}[PQ] + (-\Delta)^{1/4}[P]Q\}v \\ &+ (-\Delta)^{1/4}[PQv] - PQ(-\Delta)^{1/4}v + (-\Delta)^{1/4}[PQ]v \end{aligned}$$

$$
\begin{aligned}
& - \quad \left((-\Delta)^{1/4}[PQv] + P(-\Delta)^{1/4}(Qv) - (-\Delta)^{1/4}[P]Qv\right) \\
& = \quad [P(-\Delta)^{1/4}[Q] + (-\Delta)^{1/4}[P]Q - (-\Delta)^{1/4}[PQ]]v \\
& + \quad T(PQ, v) - T(P, Qv).
\end{aligned}
$$

Finally the estimates (2.2.17), (11.5.4) follow from Theorems 3.2 and 3.3 in [10].

An analogous property holds for the operator $\mathcal{R}S$. We just state the Theorem and we refer for proof to Theorem 3.10 in [10].

**Theorem 2.19. [Multiplication of $\mathcal{R}S$ by a rotation $P \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$]** *Let $P, Q \in H^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v \in L^2(\mathbb{R})$. Then*

$$
P\mathcal{R}[S(Q, v)] \quad = \quad \mathcal{A}_S(P, Q)v + J_S(P, Q, v) \tag{2.2.19}
$$

*where $\mathcal{A}_S(P, Q) \in L^{2,1}$, $J_S(P, Q, v) \in \mathcal{H}^1(\mathbb{R})$ with*

$$
\|\mathcal{A}_S(P, Q)\|_{L^{2,1}} \leqslant C\|(-\Delta)^{1/4}[P]\|_{L^2}\|(-\Delta)^{1/4}[Q]\|_{L^2},
$$

*and*

$$
\|J_S(P, Q, v)\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C(\|P\|_{L^\infty} + \|Q\|_{L^\infty}) \left(\|(-\Delta)^{1/4}[P]\|_{L^2} + \|(-\Delta)^{1/4}[Q]\|_{L^2}\right) \|v\|_{L^2}.
$$

We just mention that the operators $\mathcal{A}_S(P, Q), J_S(P, Q, v)$ and $\mathcal{A}_T(P, Q), J_T(P, Q, v)$ can be expressed in turn as a combinations of the operators $F, T, S$.

**Remark 2.1.** We remark without going into detail that in 2-D the Jacobian $J(a, b) = \nabla(a)\,\nabla^\perp(b)$ satisfies a stability property enjoyed by the operators (2.2.2), (2.2.3), (2.2.10) with respect to the multiplication by $P \in W^{1,2}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ as well. More precisely we may define the following two zero-order pseudo-differential operators: $\mathrm{Grad}(X) := \nabla \mathrm{div}(-\Delta)^{-1}(X)$, $\mathrm{Rot}(Y) = \nabla^\perp \mathrm{curl}(-\Delta)^{-1}(Y)$. If $a, b \in W^{1,2}(\mathbb{R}^2)$ and $P \in W^{1,2}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ then

$$
\begin{aligned}
J(a, b) \quad & = \quad \nabla(a)\,\nabla^\perp(b) \\
& = \quad \mathrm{Grad}(\nabla(a))\,\mathrm{Rot}(\nabla^\perp(b)) - \mathrm{Rot}(\nabla(a))\,\mathrm{Grad}(\nabla^\perp(b));
\end{aligned} \tag{2.2.20}
$$

and

$$
\begin{aligned}
P\,J(a, b) \quad & = \quad P\,\nabla(a)\,\nabla^\perp(b) \\
& = \quad \underbrace{[P\mathrm{Grad}(\nabla(a)) - \mathrm{Grad}(P\nabla(a))]}_{\in L^{2,1}(\mathbb{R}^2)}\,\mathrm{Rot}(\nabla^\perp(b)) \\
& + \quad \underbrace{\mathrm{Grad}(P\nabla(a))\,\mathrm{Rot}(\nabla^\perp(b)) - \mathrm{Rot}(P\nabla(a))\,\mathrm{Grad}(\nabla^\perp(b))}_{\in \mathcal{H}^1(\mathbb{R}^2)}.
\end{aligned} \tag{2.2.21}
$$

## 2.3 Regularity of Horizontal 1/2-harmonic Maps and Applications

In this section we describe the regularity results we have obtained respectively in [8, 9, 10].

### 2.3.1 Case of 1/2-harmonic maps with values into a sphere

In [8] we started the investigation of weak 1/2-harmonic maps $u \in H^{1/2}(\mathbb{R}, S^{m-1})$ with values into the sphere $S^{m-1}$ which are critical points of the Lagrangian

$$\mathcal{L}^{1/2}(u) = \int_{\mathbb{R}} |(-\Delta)^{1/4} u(x)|^2 \, dx. \tag{2.3.1}$$

The main novelty in [8] is the rewriting of the Euler-Lagrange equation. To this purpose we recall the following equivalent relations.

**Theorem 2.20.** *All weak* $1/2$*-harmonic maps* $u \in H^{1/2}(\mathbb{R}, S^{m-1})$ *satisfy in a weak sense*

*i) the equation*

$$\int_{\mathbb{R}} (-\Delta)^{1/2} u \cdot v \, dx = 0, \tag{2.3.2}$$

*for every* $v \in H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ *and* $v \in T_{u(x)} S^{m-1}$ *almost everywhere, or in a equivalent way*

*ii) the equation*

$$(-\Delta)^{1/2} u \wedge u = 0 \ \text{ in } \mathcal{D}', \tag{2.3.3}$$

*or*

*iii) the equation*

$$(-\Delta)^{1/4}(u \wedge (-\Delta)^{1/4} u) = T(Q, u) \text{ in } \mathcal{D}', \tag{2.3.4}$$

*with* $Q = u \wedge$ *.*

**Proof of Theorem 2.20**

**i)** The proof of (2.3.2) is analogous of Lemma 1.4.10 in [17].

Let $v \in H^{1/2}(\mathbb{R}, \mathbb{R}^m) \cap L^\infty(\mathbb{R}, \mathbb{R}^m)$ and $v \in T_{u(x)} S^{m-1}$. We have

$$\Pi_{S^{m-1}}(u + tv) = u + t w_t \, ,$$

where $\Pi_{S^{m-1}}$ is the orthogonal projection onto $S^{m-1}$ and

$$w_t = \int_0^1 \frac{\partial \Pi_{S^{m-1}}}{\partial y_j}(u + tsv) v^j \, ds \, .$$

Hence

$$\mathcal{L}^{1/2}(\Pi_{S^{m-1}}(u+tv)) = \int_{\mathbb{R}} |(-\Delta)^{1/4}u|^2 dx + 2t\int_{\mathbb{R}} (-\Delta)^{1/2}u \cdot w_t dx + o(t)\,,$$

as $t \to 0$.

Thus to be a critical point of (11.2.1) is equivalent to

$$\lim_{t\to 0}\int_{\mathbb{R}} (-\Delta)^{1/2}u \cdot w_t dx = 0\,.$$

Since $\Pi_{S^{m-1}}$ is smooth it follows that $w_t \to w_0 = d\Pi_{S^{m-1}}(u)(v)$ in $H^{1/2}(\mathbb{R},\mathbb{R}^m) \cap L^\infty(\mathbb{R},\mathbb{R}^m)$ and therefore

$$\int_{\mathbb{R}} (-\Delta)^{1/4}u\, d\Pi_{S^{m-1}}(u)(v) dx = 0\,.$$

Since $v \in T_{u(x)}S^{m-1}$ a.e., we have $d\Pi_{S^{m-1}}(u)(v) = v$ a.e. and thus equation (2.3.2) follows immediately.

**ii)** We prove (2.3.3). We take $\varphi \in C_0^\infty(\mathbb{R}, \bigwedge_{m-2}(\mathbb{R}^m))$. The following holds

$$\int_{\mathbb{R}} \varphi \wedge u \wedge (-\Delta)^{1/2}u\, dx = \left(\int_{\mathbb{R}} *(\varphi \wedge u)\cdot(-\Delta)^{1/2}u\, dx\right) e_1 \wedge \ldots \wedge e_m\,. \tag{2.3.5}$$

**Claim :** $v = *(\varphi \wedge u) \in \dot{H}^{1/2}(\mathbb{R},\mathbb{R}^m)$ [2.4] and $v(x) \in T_{u(x)}S^{m-1}$ a.e.

**Proof of the claim.**

The fact that $v \in H^{1/2}(\mathbb{R},\mathbb{R}^m) \cap L^\infty(\mathbb{R},\mathbb{R}^m)$ follows form the fact that its components are the product of two functions which are in $\dot{H}^{1/2}(\mathbb{R},\mathbb{R}^m) \cap L^\infty(\mathbb{R},\mathbb{R}^m)$, which is an algebra .

We have

$$v \cdot u = *(u \wedge \varphi)\cdot u = *(u \wedge \varphi \wedge u) = 0\,. \tag{2.3.6}$$

It follows from (2.3.2) and (2.3.5) that

$$\int_{\mathbb{R}} \varphi \wedge u \wedge (-\Delta)^{1/2}u\, dx = 0\,.$$

This shows that $(-\Delta)^{1/2}u \wedge u = 0$ in $\mathcal{D}'$ , and we can conclude .

**iii)** As far as equation (2.3.4) is concerned it is enough to observe that $(-\Delta)^{1/2}u \wedge u = 0$ and $(-\Delta)^{1/4}u \wedge (-\Delta)^{1/4}u = 0$ . □

The Euler Lagrange equation (2.3.4) will often be completed by the following "structure equation" which is a consequence of the fact that $u \in S^{m-1}$ almost everywhere:

**2.4** the symbol $*$ we denote the Hodge-star operator, $*\colon \bigwedge_p(\mathbb{R}^m) \to \bigwedge_{m-p}(\mathbb{R}^m)$, defined by $*\beta = (e_1 \wedge \ldots \wedge e_n) \bullet \beta$, the symbol $\bullet$ is the first order contraction between multivectors, for every $p = 1,\ldots,m$, $\bigwedge_p(\mathbb{R}^m)$ is the vector space of $p$-vectors.

**Proposition 2.21.** *All maps in $\dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ satisfy the following identity*

$$(-\Delta)^{1/4}(u \cdot (-\Delta)^{1/4}u) = S(u\cdot, u) - \overline{\mathcal{R}}((-\Delta)^{1/4}u \cdot \mathcal{R}(-\Delta)^{1/4}u), \tag{2.3.7}$$

*where, in general for an arbitrary integer n, for every $Q \in \dot{H}^{1/2}(\mathbb{R}^n, \mathcal{M}_{\ell\times m}(\mathbb{R}))$, $\ell \geqslant 1$ and $u \in \dot{H}^{1/2}(\mathbb{R}^n, \mathbb{R}^m)$, $S$ is the operator defined by (2.2.3).*

**Proof of Proposition 2.21.** We observe that if $u \in H^{1/2}(\mathbb{R}, \mathbb{R}^{m-1})$ then the Leibniz's rule holds. Thus

$$\nabla|u|^2 = 2u \cdot \nabla u \ \text{ in } \ \mathcal{D}'. \tag{2.3.8}$$

Indeed the equality (2.3.8) trivially holds if $u \in C_0^\infty(\mathbb{R}, \mathbb{R}^{m-1})$. Let $u \in H^{1/2}(\mathbb{R}, \mathbb{R}^{m-1})$ and $u_j \in C_0^\infty(\mathbb{R}, \mathbb{R}^m)$ be such that $u_j \to u$ as $j \to +\infty$ in $H^{1/2}(\mathbb{R}, \mathbb{R}^m)$ . Then $\nabla u_j \to \nabla u$ as $j \to +\infty$ in $H^{-1/2}(\mathbb{R}, \mathbb{R}^{m-1})$. Thus $u_j \cdot \nabla u_j \to u \cdot \nabla u$ in $\mathcal{D}'$ and (2.3.8) follows.
If $u \in H^{1/2}(\mathbb{R}, S^{m-1})$, then $\nabla|u|^2 = 0$ and thus $u \cdot \nabla u = 0$ in $\mathcal{D}'$ as well. Thus $u$ satisfies equation (2.3.7) and this conclude the proof. □

We remark that in the sphere case the term $\overline{\mathcal{R}}((-\Delta)^{1/4}u \cdot \mathcal{R}(-\Delta)^{1/4}u)$ is in the Hardy-Space $\mathcal{H}^1(\mathbb{R})$ as well (see Corollary 3.1 in [8]). The estimates (2.2.4) and (2.2.5) imply in particular that if $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{S}^{m-1})$ is a $1/2$-harmonic map then

$$\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} \leqslant C\|(-\Delta)^{1/4}u\|^2_{L^2(\mathbb{R})} . \tag{2.3.9}$$

where the constant $C$ is independent of $u$.

From the inequality (2.3.9) it follows that if $\varepsilon_0 := \|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}$ is small enough so that

$$C\varepsilon_0 < 1 \tag{2.3.10}$$

then the solution is constant. This the so-called *bootstrap test* and it is the key observation to prove Morrey-type estimates and to deduce Hölder regularity of $1/2$-harmonic maps.

Indeed by combining Theorem 2.20, Proposition 2.21 and suitable localization estimates obtained in Section 4 in [8] we get the local Hölder regularity of weak $1/2$-harmonic maps.

**Theorem 2.22. [Theorem 5.2, [8]]** *Let $u \in \dot{H}^{1/2}(\mathbb{R}, S^{m-1})$ be a weak $1/2$-harmonic map. Then $u \in C^{0,\alpha}_{loc}(\mathbb{R}, S^{m-1})$, for all $\alpha \in (0, 1)$.*

**Sketch of Proof of 2.22.** The strategy of proof is to show some decrease energy estimates. From Proposition 4.1 and 4.2 in [8] by using the fact that $u \wedge (-\Delta)^{1/4}u$ and $u \cdot (-\Delta)^{1/4}u$ satisfy respectively (2.3.4) and (2.3.7) one deduces that there exist $C > 0$ depending on $\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})}$, $\bar{k} \in \mathbb{Z}$ depending on $\varepsilon_0$ in (2.3.10), such that that for every $x_0 \in \mathbb{R}$, for all $k < \bar{k}$ the following estimate holds

$$||(-\Delta)^{1/4}u||^2_{L^2(B_{2^k})} \leqslant C \sum_{h=k}^{\infty} (2^{\frac{k-h}{2}})||(-\Delta)^{1/4}u||^2_{L^2(A_h)} \tag{2.3.11}$$

where $B_{2^k} = B(x_0, 2^k)$, $A_h = B_{2^{h+1}} \backslash B_{2^{h-1}}$. On the other hand one has

$$2^{-1} \sum_{h=-\infty}^{k-1} ||(-\Delta)^{1/4} u||^2_{L^2(A_h)} \leqslant ||(-\Delta)^{1/4} u||^2_{L^2(B_{2^k})} \leqslant \sum_{h=-\infty}^{k-1} ||(-\Delta)^{1/4} u||^2_{L^2(A_h)} \tag{2.3.12}$$

By combining (2.3.11) and (2.3.12) we get

$$\sum_{h=-\infty}^{k-1} ||(-\Delta)^{1/4} u||^2_{L^2(A_h)} \leqslant C \sum_{h=k}^{\infty} (2^{\frac{k-h}{2}}) ||(-\Delta)^{1/4} u||^2_{L^2(A_h)}.$$

This implies by an iteration argument (see Proposition A.1 in [8], or Lemma A.1 in [24])

$$\sup_{\substack{x \in B(x_0, \rho) \\ 0 < r < \rho/8}} r^{-\beta} \int_{B(x,r)} |(-\Delta)^{1/4} u|^2 dx \leqslant C\,, \tag{2.3.13}$$

for $\rho$ small enough, for some $0 < \beta < 1$ independent on $x_0$ and $C > 0$ depending only on the dimension and on $\|(-\Delta)^{1/4} u\|^2_{L^2(\mathbb{R})}$.
Condition (2.3.13) yields that $u \in C^{0,\beta/2}_{loc}(\mathbb{R})$ , (see for instance [1] or [11] for the details). By bootstrapping into the equations (2.3.4) and (2.3.7) we can deduce that $u \in C^{0,\alpha}_{loc}(\mathbb{R})$ for all $\alpha \in (0, 1)$. □

We mention that Schikorra in [24] and the author and Schikorra in [12] extended the local the Hölder continuity of respectively $k/2$-harmonic maps ($k > 1$ odd) and $k/p$-harmonic maps ($p \in (1, \infty)$, $k/p \in (0, k)$) from subsets of $\mathbb{R}^k$ into a sphere.

$k/p$-harmonic maps with values into a sphere are defined as critical points of the following nonlocal Lagrangian

$$\int_{\mathbb{R}^k} |(-\Delta)^{\frac{k}{2p}} u|^p \, dx^k\,,$$

where $u(x) \in S^{m-1}$, a.e. and $\int_{\mathbb{R}^k} |(-\Delta)^{\frac{k}{2p}} u|^p \, dx^k < +\infty$.

## 2.3.2 Case of 1/2-harmonic maps into a closed manifold

We consider the case of $1/2$-harmonic maps with values into a closed $C^2$ $n$-dimensional manifold $\mathcal{N} \subset \mathbb{R}^m$. Let $\Pi_{\mathcal{N}}$ be the orthogonal projection on $\mathcal{N}$. We denote by $P_T$ and $P_N$ respectively the tangent and the normal projection to the manifold $\mathcal{N}$.

They verify the following properties: $(P_T)^t = P_T$, $(P_N)^t = P_N$ (namely they are symmetric operators), $(P_T)^2 = P_T$, $(P_N)^2 = P_N$, $P_T + P_N = Id$, $P_N P_T = P_T P_N = 0$.

In this case the Euler-Lagrange equation associated to the energy (11.2.1) and the structural equation can be expressed as follows:

$$\begin{cases} P_T(u)(-\Delta)^{1/2} u = 0 & \text{in } \mathcal{D}'(\mathbb{R}) \\ P_N \nabla u = 0 & \text{in } \mathcal{D}'(\mathbb{R}). \end{cases} \tag{2.3.14}$$

The second step is to reformulate the two equations in (2.3.14) by using the commutators introduced in the previous section. The Euler equation (2.3.4) and structural equation (2.3.7) become in this case respectively

$$(-\Delta)^{1/4}(P^T(-\Delta)^{1/4}u) = T(P^T, u) - \underbrace{((-\Delta)^{1/4}P^T)(-\Delta)^{1/4}u}_{(1)} . \tag{2.3.15}$$

and

$$(-\Delta)^{1/4}(\mathcal{R}(P^N(-\Delta)^{1/4}u)) = \mathcal{R}(S(P^N, u)) - \underbrace{((-\Delta)^{1/4}P^N)(\mathcal{R}(-\Delta)^{1/4}u)}_{(2)} . \tag{2.3.16}$$

Unlike the sphere case the term $(1)$ in (2.3.15) is not zero and term $(2)$ in (2.3.16) is not in the Hardy Space.

The main idea in Proposition 1.1 in [9] is the re-writing of the terms $(1)$ and $(2)$ and to show that $v = (P_T(-\Delta)^{1/4}u, \mathcal{R}P_N(-\Delta)^{1/4}u)^t$ satisfies a nonlocal Schrödinger type system with a antisymmetric potential. Precisely, we obtained the following result.

**Proposition 2.23. [Proposition 1.1, [9]]** *Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ be a weak $1/2$-harmonic map. Then the following equation holds*

$$\begin{aligned}(-\Delta)^{1/4}v = (-\Delta)^{1/4}\begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix} &= \tilde{\Omega} + \Omega_1 \begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix} \\ &+ \Omega \begin{pmatrix} P_T(-\Delta)^{1/4}u \\ \mathcal{R}P_N(-\Delta)^{1/4}u \end{pmatrix},\end{aligned} \tag{2.3.17}$$

*where $\Omega = \Omega \in L^2(\mathbb{R}, so(2m))$, $\Omega_1 = \Omega_1 \in L^{2,1}(\mathbb{R}, \mathcal{M}_{m\times m})$ with*

$$\|\Omega\|_{L^2}, \|\Omega_1\|_{L^{2,1}} \leqslant C(\|P_T\|_{\dot{H}^{1/2}} + \|P_T\|^2_{\dot{H}^{1/2}}),$$

$\tilde{\Omega} =$

$$\begin{pmatrix} -2F(\omega_1, (P_N\Delta^{1/4}u)) + T(P_T, (-\Delta)^{1/4}u) \\ -2F(\mathcal{R}((-\Delta)^{1/4}P_N), \mathcal{R}((-\Delta)^{1/4}u)) - 2F(\omega_2, P_N((-\Delta)^{1/4}u) + \mathcal{R}(S(P_N, (-\Delta)^{1/4}u)) \end{pmatrix}$$

*$\omega_1, \omega_2 \in L^2(\mathbb{R}, \mathcal{M}_{m\times m})$ and*

$$\|\omega_1\|_{L^2},\ \|\omega_2\|_{L^2} \leqslant C(\|P_T\|_{\dot{H}^{1/2}} + \|P_T\|^2_{\dot{H}^{1/2}}).$$ [2.5]

We would like to make some comments on Proposition 2.23.

**2.5** The matrices $\Omega$, $\Omega_1$, $\omega_1$ and $\omega_2$ are constructed out of the projection $P_T$.

In [20] and [21] the author proved the sub-criticality of local *a-priori* critical Schödinger systems of the form

$$\forall i = 1 \cdots m \qquad - \Delta u^i = \sum_{j=1}^{m} \Omega^i_j \cdot \nabla u^j \,, \tag{2.3.18}$$

where $u = (u^1, \cdots, u^m) \in W^{1,2}(D, \mathbb{R}^m)$ and $\Omega \in L^2(D, \mathbb{R}^2 \otimes so(m))$, or of the form

$$\forall i = 1 \cdots m \qquad - \Delta v^i = \sum_{j=1}^{m} \Omega^i_j \, v^j \,, \tag{2.3.19}$$

where $v \in L^{n/(n-2)}(B^n, \mathbb{R}^m)$ and $\Omega \in L^{n/2}(B^n, so(m))$. In each of these two situations the antisymmetry of $\Omega$ was responsible for the regularity of the solutions or for the stability of the system under weak convergence.

One of the main result in the paper [9] was to establish the sub-criticality of non-local Schrödinger systems of the form

$$(-\Delta)^{1/4} v = \Omega v + \Omega_1 v + \mathcal{Z}(Q, v) + g(x) \tag{2.3.20}$$

where $v \in L^2(\mathbb{R})$, $Q \in \dot{H}^{1/2}(\mathbb{R})$, $\mathcal{Z}\colon \dot{H}^{1/2}(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathcal{H}^1(\mathbb{R})$ is a linear combination of the operators (2.2.10), (2.2.2) and (2.2.3) introduced in the previous section, $\Omega \in L^2(\mathbb{R}, so(m))$, $\Omega_1 \in L^{2,1}(\mathbb{R})$. Precisely we prove the following theorem which extends to a non-local setting the phenomena observed in [20] and [21] for the above local systems.

**Theorem 2.24. [Theorem 1.1, [9]]** *Let $v \in L^2(\mathbb{R})$ be a weak solution of (2.3.20). Then $v \in L^p_{loc}(\mathbb{R})$ for every $1 \leqslant p < +\infty$.*

From Theorem 2.24 it follows that $(-\Delta)^{1/4} u \in L^p_{loc}(\mathbb{R})$, for all $p \geqslant 1$ as well, ($u$ as in Proposition 2.23). This implies that $u \in C^{0,\alpha}_{loc}$ for all $0 < \alpha < 1$, since $W^{1/2,p}_{loc}(\mathbb{R}) \hookrightarrow C^{0,\alpha}_{loc}(\mathbb{R})$ if $p > 2$ (see for instance [1]).

The main technique to prove Theorem 2.24 is to perform a *change of gauge* by rewriting the system after having multiplied $v$ by a well chosen rotation valued map $P \in H^{1/2}(\mathbb{R}, SO(m))$ . [2.6] In [20] the choice of $P$ for systems of the form (2.3.18) was given by the geometrically relevant *Coulomb Gauge* satisfying

$$div \left[ P^{-1} \nabla P + P^{-1} \Omega P \right] = 0 \,. \tag{2.3.21}$$

In this context there is not hope to solve an equation of the form (2.3.21) with the operator $\nabla$ replaced by $(-\Delta)^{1/4}$, since for $P \in SO(m)$ the matrix $P^{-1}(-\Delta)^{1/4} P$ is not in general antisymmetric. The novelty in [9] was to choose the gauge $P$ satisfying the

**2.6** $SO(m)$ is the space of $m \times m$ matrices $R$ satisying $R^t R = R R^t = Id$ and $det(R) = +1$

following (maybe less geometrically relevant) equation which involves the antisymmetric part of $P^{-1}(-\Delta)^{1/4}P$[2.7]:

$$Asymm\left(P^{-1}(-\Delta)^{1/4}P\right) := 2^{-1}\left[P^{-1}(-\Delta)^{1/4}P - (-\Delta)^{1/4}P^{-1}P\right] = \Omega\,. \tag{2.3.22}$$

The local existence of such $P$ is given by the following theorem.

**Theorem 2.25.** *There exists $\varepsilon > 0$ and $C > 0$ such that for every $\Omega \in L^2(\mathbb{R}; so(m))$ satisfying $\int_{\mathbb{R}} |\Omega|^2\,dx \leqslant \varepsilon$, there exists $P \in \dot{H}^{1/2}(\mathbb{R}, SO(m))$ such that*

$$\begin{cases} (i) \quad P^{-1}(-\Delta)^{1/4}P - (-\Delta)^{1/4}P^{-1}P = 2\,\Omega\,; \\ (ii) \quad \displaystyle\int_{\mathbb{R}} |(-\Delta)^{1/4}P|^2\,dx \leqslant C\int_{\mathbb{R}} |\Omega|^2\,dx\,. \end{cases} \tag{2.3.23}$$

□

The proof of this theorem is established by following an approach introduced by K.Uhlenbeck in [29] to construct *Coulomb Gauges* for $L^2$ curvatures in 4 dimension. The construction does not provide the continuity of the map which to $\Omega \in L^2$ assigns $P \in \dot{H}^{1/2}$. This illustrates the difficulty of the proof of Theorem 10.4.5 which is not a direct consequence of an application of the local inversion theorem but requires more elaborated arguments.

Thus if the $L^2$ norm of $\Omega$ is small, Theorem 10.4.5 gives a $P$ for which $w := Pv$ satisfies

$$\begin{aligned} (-\Delta)^{1/4}w &= -\left[P\Omega P^{-1} - (-\Delta)^{1/4}P\,P^{-1}\right] w + T(P, P^{-1}w) + P\Omega_1 P^{-1}w \\ &+ PZ(Q, P^{-1}w) = -Symm\left(((-\Delta)^{1/4}P)\,P^{-1}\right) w + T(P, P^{-1}w) \\ &+ P\Omega_1 P^{-1}w + PZ(Q, P^{-1}w)\,. \end{aligned} \tag{2.3.24}$$

The matrix $Symm\left(((-\Delta)^{1/4}P)\,P^{-1}\right)$ belongs to $L^{2,1}(\mathbb{R})$ and this fact comes from the combination of the following lemma according to which

$$(-\Delta)^{1/4}(Symm\left(((-\Delta)^{1/4}P)\,P^{-1}\right)) \in \mathcal{H}^1(\mathbb{R})$$

and the sharp Sobolev embedding [2.8] which says that $f \in \mathcal{H}^1(\mathbb{R})$ implies that $(-\Delta)^{-1/4}f \in L^{2,1}$. Precisely we have

**2.7** Given a $m \times m$ matrix $M$, we denote by $Asymm(M)$ and by $Symm(M)$ respectively the antisymmetric and the symmetric part of $M$, namely $Asymm(M) := \frac{M-M^t}{2}$ and $Symm(M) := \frac{M+M^t}{2}$, $M^t$ is the transpose of $M$.

**2.8** The fact that $v \in \mathcal{H}^1$ implies $(-\Delta)^{-1/4}v \in L^{2,1}$ is deduced by duality from the fact that $(-\Delta)^{1/4}v \in L^{2,\infty}$ implies that $v \in BMO(\mathbb{R})$. This last embedding has been proved by Adams in [1]

**Lemma 2.26.** *Let $P \in H^{1/2}(\mathbb{R}, SO(m))$ then $(-\Delta)^{1/4}(Symm\left((-\Delta)^{1/4}P\,P^{-1}\right))$ is in the Hardy space $\mathcal{H}^1(\mathbb{R})$ and the following estimates hold*

$$\|(-\Delta)^{1/4}[(-\Delta)^{1/4}P\,P^{-1} + P\,(-\Delta)^{1/4}P^{-1}]\|_{\mathcal{H}^1} \leqslant C\|P\|^2_{\dot{H}^{1/2}}\,,$$

*where $C > 0$ is a constant independent of $P$. This implies in particular that*

$$\|Symm\left(((-\Delta)^{1/4}P)\,P^{-1}\right)\|_{L^{2,1}} \leqslant C\|P\|^2_{\dot{H}^{1/2}}\,. \tag{2.3.25}$$

The proof of Lemma 2.26 is a consequence of the Theorem 1.5 in [9].

By combining the different properties of the commutators (2.2.2), (2.2.3), (2.2.10) mentioned in section 2.2, in [10] we proved that the system (2.3.20) is “equivalent" to a conservation law.

**Theorem 2.27.** *Let $v \in L^2(\mathbb{R}, \mathbb{R}^m)$ be a solution of (2.3.20), where $\Omega \in L^2(\mathbb{R}, so(m))$, $\Omega_1 \in L^{2,1}(\mathbb{R})$, $\mathcal{Z}$ is a linear combination of the operators (2.2.10), (2.2.2) and (2.2.3), $\mathcal{Z}(Q, v) \in \mathcal{H}^1$ for every $Q \in \dot{H}^{1/2}$, $v \in L^2$ with*

$$\|\mathcal{Z}(Q, v))\|_{\mathcal{H}^1} \quad \leqslant \quad C\|Q\|_{\dot{H}^{1/2}}\|v\|_{L^2}.$$

*There exists $\varepsilon_0 > 0$ such that if*

$$(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{\dot{H}^{1/2}}) < \varepsilon_0,$$

*then there exist $A \in \dot{H}^{1/2}(\mathbb{R}, GL_m(\mathbb{R})))$ and an operator $B \in \dot{H}^{1/2}(\mathbb{R})$ (both constructed out of $(\Omega, \Omega_1, Q)$) such that*

$$\begin{aligned} \|A\|_{\dot{H}^{1/2}} + \|B\|_{\dot{H}^{1/2}} &\leqslant C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{\dot{H}^{1/2}}) \qquad (2.3.26)\\ dist(\{A, A^{-1}\}, SO(m)) &\leqslant C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{\dot{H}^{1/2}}) \qquad (2.3.27) \end{aligned}$$

*and*

$$(-\Delta)^{1/4}[Av] = \mathcal{J}(B, v) + Ag, \tag{2.3.28}$$

*where $\mathcal{J}$ is a linear operator in $B$, $v$, $\mathcal{J}(B, v) \in \mathcal{H}^1(\mathbb{R})$ and*

$$\|\mathcal{J}(B, v)\|_{\mathcal{H}^1(\mathbb{R})} \leqslant C\|B\|_{\dot{H}^{1/2}}\|v\|_{L^2}\,. \tag{2.3.29}$$

We mention that the case of $k/2$-harmonic maps ($k \geqslant 3$ odd) with values into a closed manifold has been considered in [4].

### 2.3.3 Case of horizontal 1/2-harmonic maps

We release the assumption that the field of orthogonal projection $P_T$ is *integrable* and associated to a sub-manifold $\mathcal{N}$ and to consider the equation (2.3.14) for a general field of orthogonal projections $P_T$ defined on the whole of $\mathbb{R}^m$ and for *horizontal maps u* satisfying $P_T(u)\nabla u = \nabla u$.

Precisely we consider $P_T \in C^1(\mathbb{R}^m, \mathcal{M}_m(\mathbb{R}))$ and $P_N \in C^1(\mathbb{R}^m, \mathcal{M}_m(\mathbb{R}))$ such that

$$\begin{cases} P_T \circ P_T = P_T \quad P_N \circ P_N = P_N \\ P_T + P_N = I_m \\ \forall\, z \in \mathbb{R}^m \quad \forall\, U, V \in T_z(\mathbb{R}^m) \qquad < P_T(z)U, P_N(z)V >= 0 \\ \|\partial_z P_T\|_{L^\infty(\mathbb{R}^m)} < +\infty \end{cases} \tag{2.3.30}$$

For such a distribution of projections $P_T$ we denote by

$$n := \mathrm{rank}(P_T).$$

Such a distribution identifies naturally with the distribution of $n-$planes given by the images of $P_T$ (or the Kernel of $P_T$) and conversely, any $C^1$ distribution of $n-$dimensional planes defines uniquely $P_T$ satisfying (2.3.30).

We will present here the proof of the $C^\alpha_{loc}$ of horizontal $1/2$-harmonic maps which directly uses the conservation law (2.3.28) and which is a refinement of the arguments used in Theorem 2.24 (Theorem 1.1 in [9]). We premise the following result.

**Theorem 2.28.** *Let $m \in \boldsymbol{N}^*$, then there exists $\delta > 0$ such that for any $P_T, P_N \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_m)$ satisfying*

$$\begin{cases} P_T \circ P_T = P_T, \quad P_N = I_m - P_T \\ \forall\, X, Y \in \mathbb{R}^m,\ \text{for a.e } x \in \mathbb{R} \qquad < P_T(x)X, P_N(x)Y >= 0 \end{cases} \tag{2.3.31}$$

*and*

$$\int_{\mathbb{R}} |(-\Delta)^{1/4} P_T|^2 \, d\vartheta \leqslant \delta \tag{2.3.32}$$

*then for any $f \in H^{-1/2}(\mathbb{R})$*

$$(P_T + P_N \mathcal{R})\, f = 0 \quad \Longrightarrow \quad f = 0. \tag{2.3.33}$$

**Proof of Theorem 2.28.**
We first set $f := (-\Delta)^{1/2} u$. From (2.3.33) it follows that

$$\begin{cases} P_T(-\Delta)^{1/2} u = 0 \\ P_N \mathcal{R}(-\Delta)^{1/2} u = 0 \end{cases} \tag{2.3.34}$$

Then set $v = (P_T(-\Delta)^{1/4} u, \mathcal{R}(P_N(-\Delta)^{1/4} u))^t$. Therefore $v$ satisfies a system of the form (2.3.20) with $\Omega \in L^2(\mathbb{R}, so(\mathbb{R}^m))$ $\Omega_1 \in L^{2,1}$, ($\Omega$ and $\Omega_1$ depend on $P_T$), $\mathcal{Z}(P_T, v)$ is a linear operator in $P_T, v$, $\mathcal{Z}(P_T, v) \in \mathcal{H}^1$ with

$$\|\Omega\|_{L^2} \quad = \quad \|\Omega\|_{L^2} \leqslant C\|P_T\|_{\dot{H}^{1/2}}$$

$$\begin{array}{rcl} \|\Omega_1\|_{L^{2,1}} & = & \|\Omega_1\|_{L^{2,1}} \leqslant C\|P_T\|_{\dot{H}^{1/2}} \\ \|\mathcal{Z}(P_T, v))\|_{\mathcal{H}^1} & \leqslant & C\|P_T\|_{\dot{H}^{1/2}}\|v\|_{L^2} \end{array}$$

From Theorem 2.27 it follows that if $\delta$ is small enough then there exist $A \in L^\infty \cap \dot{H}^{1/2}(\mathbb{R}, GL_m(\mathbb{R}))$ and $B \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{M}_{m\times m}(\mathbb{R}))$ such that

$$(-\Delta)^{1/4}[Av] = \mathcal{J}(B, v) \tag{2.3.35}$$

and

$$\begin{array}{rcl} \|A\|_{\dot{H}^{1/2}} + \|B\|_{\dot{H}^{1/2}} & \leqslant & C\|P_T\|_{\dot{H}^{1/2}} \\ dist(\{A, A^{-1}\}, SO(m)) & \leqslant & \leqslant C\|P_T\|_{\dot{H}^{1/2}} \\ \|\mathcal{J}(B, v)\|_{\mathcal{H}^1(\mathbb{R})} & \leqslant & C\|B\|_{\dot{H}^{1/2}}\|v\|_{L^2}. \end{array} \tag{2.3.36}$$

From (2.3.35) and (2.3.36) it follows that

$$\begin{array}{rcl} \|v\|_{L^2} & = & \|A^{-1}Av\|_{L^2} \leqslant C\|A^{-1}\|_{L^\infty}\|Av\|_{L^2} \\ & \leqslant & C\|(-\Delta)^{-1/4}\mathcal{J}(B, v)\|_{L^{2,1}} \leqslant C\|B\|_{\dot{H}^{1/2}}\|v\|_{L^2} \\ & \leqslant & C\|P_T\|_{\dot{H}^{1/2}}\|v\|_{L^2} \leqslant C\delta\|v\|_{L^2}. \end{array} \tag{2.3.37}$$

Again if $\delta$ is small enough then (2.3.37) yields $v \equiv 0$ a.e. and therefore $f = 0$ a.e. as well. □

**Proof of Theorem 2.9.** The proof of Theorem 2.9 follows by combining Theorem 2.28 and localization arguments used in [9]. □

## 2.3.4 Applications

In this section we mention two geometric applications related to $1/2$-harmonic maps. We start by proving Theorem 2.7 .

**Proof of Theorem 2.7 .** 1) (see [5, 14, 18] ). If $\mathcal{N} = S^1$, then its harmonic extension $\tilde{u}$, which is conformal thanks to Theorem 2.6, maps the unit disk $B^2(0, 1)$ into itsself because of the maximum principle. On the other hand it turns out that every conformal transformation with finite energy from $B^2(0, 1)$ into $B^2(0, 1)$ and sending $S^1$ into $S^1$ has to be a finite Blaschke product, namely there exist $d > 0, \vartheta_0 \in \mathbb{R}, a_1, \dots, a_d \in B^2(0, 1)$ such that

$$\tilde{u}(z) = \prod_{i=1}^{d} e^{i\vartheta_0}\frac{z - a_i}{1 - z\overline{a}_i}.$$

Since $deg(u) = 1$ then $d = 1$ and $\tilde{u}$ coincides with a Möbius transformation of the disk.

2) We are going to use the following result by Nitsche [19]: if $\Sigma$ is a regular minimal immersion in $B^3(0, 1) \subset \mathbb{R}^3$ that meets $B^3(0, 1)$ orthogonally then $\partial\Sigma$ is a great circle.

Let $\tilde{u}\colon B^2(0,1) \to B^3(0,1)$ be the harmonic extension of $u$. In [11] it has been shown that $u \in C^{1,\alpha}(S^1)$, therefore $\tilde{u} \in C^{1,\alpha}(\overline{B}^2)$. Moreover $\tilde{u}$ is conformal in $\overline{B}^2(0,1)$ (see Proposition 2.29 below)[2.9] and by Maximum Principle $\tilde{u}$ takes values in $B^3(0,1)$. We set $h = |\tilde{u}|^2$. We have $-\Delta h \leqslant 0$, and $h = 1$ on $S^2$. By Hopf Boundary Lemma we have $\frac{\partial h}{\partial r} \neq 0$ on $S^1$. Since $\tilde{u}$ is conformal up to the boundary, this implies in particular $\nabla \tilde{u} \neq 0$ on $S^1$ and therefore $\tilde{u}$ is a minimal immersion up to the boundary. Since it meets $B^3(0,1)$ orthogonally then by Nitsche's result [19] $\tilde{u}(S^1) = u(S^1)$ is an equatorial circle. Let $T\colon S^2 \to S^2$ be an isometry,[2.10] $\sigma := \{az + by + cx = 0, a, b, c \in \mathbb{R}\}$ be a plane in $\mathbb{R}^3$ such that $u(S^1) = \sigma \cap S^2$. Define $\tau = T|_{\sigma \cap S^2}\colon \sigma \cap S^2 \to S^1$. Let $v := \tau \circ u\colon S^1 \to S^1$ and we show that it is $1/2$-harmonic in $S^1$.

$$\begin{cases} \Delta(\widetilde{\tau \circ u}) = 0 & \text{in } B^2 \\ \widetilde{\tau \circ u} = \tau \circ u & \text{in } \partial B^2 \end{cases} \tag{2.3.38}$$

Since $\tau$ can be identified with a rotation in $\mathbb{R}^3$, we have

$$\frac{\partial \widetilde{\tau \circ u}}{\partial \nu} = \tau \frac{\partial \tilde{u}}{\partial \nu}.$$

It follows that

$$\begin{aligned} (-\Delta)^{1/2}(\tau \circ u) &= \frac{\partial \widetilde{\tau \circ u}}{\partial \nu} = \tau \frac{\partial \tilde{u}}{\partial \nu} \\ &= \tau (-\Delta)^{1/2} u \parallel \tau \circ u\,. \end{aligned}$$

We can conclude the proof. □

**Proposition 2.29. [Proposition 1.1, [10]]** *An element in $\mathfrak{H}^{1/2}$ satisfying*

$$P_T(u)\,(-\Delta)^{1/2} u = 0 \quad \text{in } \mathcal{D}'(S^1) \tag{2.3.39}$$

*has a harmonic extension $\tilde{u}$ in $B^2(0,1)$ which is conformal in $\overline{B}^2(0,1)$ and hence it is the boundary of a minimal disk whose exterior normal derivative $\partial_r \tilde{u}$ is orthogonal to the plane distribution given by $P_T$.*

**Proof of Proposition 2.29.** We prove the result by assuming that $P_T \in C^2(\mathbb{R}^m)$. In that case we have that $u \in C^{1,\alpha}(S^1)$, (see [11]). Denote $\tilde{u}$ the harmonic extension of $u$. It is well known that the *Hopf differential* of $\tilde{u}$

$$|\partial_{x_1} \tilde{u}|^2 - |\partial_{x_2} \tilde{u}|^2 - 2\,i\,\langle \partial_{x_1} \tilde{u}, \partial_{x_2} \tilde{u}\rangle = f(z)$$

is holomorphic. Considering on $S^1 = \partial B^2$

$$2\,\langle \partial_r \tilde{u}, \partial_\vartheta \tilde{u}\rangle = -\sin 2\vartheta \left(|\partial_{x_1} \tilde{u}|^2 - |\partial_{x_2} \tilde{u}|^2\right) - \cos 2\vartheta\,(-2\,\langle \partial_{x_1} \tilde{u}, \partial_{x_2} \tilde{u}\rangle) = -\operatorname{Im}\left(z^2 f(z)\right).$$

**2.9** We refer to the book [22] for an overview of the the regularity of minimal disks up to the boundary (solution of the Plateau problem)

**2.10** The isometry group of the sphere $S^2$ is isomorphic to the group $SO(3)$ of orthogonal matrices.

Since $0 = P_T(u)\,(-\Delta)^{1/2}u = P_T(u)\,\partial_r\tilde{u}$ and $0 = P_N(u)\,\partial_\vartheta u = P_N(u)\,\partial_\vartheta\tilde{u}$ on $S^1$ we have that

$$\operatorname{Im}\left(z^2 f(z)\right) = 0 \quad \text{on } S^1.$$

Hence the holomorphic function $z^2 f(z)$ is equal to a real constant. Since $f(z)$ cannot have a pole at the origin we have that $z^2 f(z)$ is identically equal to zero and thus $\tilde{u}$ is conformal. □

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