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Representations of integers as sums of an even number of squares

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1. Introduction

For positive integers s and n, let $r_s(n)$ be the number of representations of n as a sum of s integral squares. An old and classical problem in number theory is to find closed and exact formulas for $r_s(n)$. For small s this was accomplished by works of Fermat, Gauss, Lagrange, Jacobi, Hardy, Mordell, Rankin and many others. For a good survey see e.g., [5].

Let

$$\theta(z) = \sum_{n\in\mathbf{Z}} q^{n^2} \qquad (q = e^{2\pi i z},\ z \in \mathcal{H} = \text{upper half-plane})$$

be the standard theta function. Then θ^s which is the generating function for $r_s(n)$ $(n \in \mathbf{N})$, is a modular form of weight $\frac{s}{2}$ and level 4 and hence for $s \geq 4$ can be expressed as the sum of a modular form in the space generated by Eisenstein series of weight $\frac{s}{2}$ and level 4 and a corresponding cusp form. Recall that the Fourier coefficients of Eisenstein series of integral weight ≥ 2 on congruence subgroups are given explicitly in terms of elementary divisor functions. Thus if the space of cusp forms happens to be zero, for even $s \geq 4$ this leads to exact formulas for $r_s(n)$ in terms of those functions. In general, however, one gets only asymptotic results for $r_s(n)$ when n is large, since the coefficients of cusp forms are mysterious and in general no simple arithmetic expressions are known for them.

Recently, Milne [7–9] took up the subject again from a completely different point of view and using tools from the theory of elliptic functions, Lie theory, the theory of hypergeometric functions and other devices obtained exact formulas for $r_s(n)$ whenever $s = 4j^2$ or $s = 4j^2 + 4j$ with $j \in \mathbf{N}$, in terms of explicit finite sums of products of j modified elementary divisor functions. In [8], [9], using

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similar methods, he also proved a conjecture of Kac and Wakimoto on the number of representations of n as a sum of triangular numbers. This conjecture was also proved independently in a different way, using the theory of modular forms, by Zagier [14]. In this connection see also the work of Ono [10] where the arguments used are slightly different.

On the other hand, in another recent paper Chan and Chua [3] made a remarkable conjecture which says that θ^s for $8|s$, $s \geq 16$ is a unique rational linear combination of $\frac{s}{8} - 1$ products of *two* specific Eisenstein series of level 4 (nothing, however, is said how to obtain the coefficients in these linear combinations). There are also corresponding conjectures for $s \equiv 2, 4, 6 \pmod 8$. The paper of Chan and Chua was motivated by the new 2×2 determinant formula for θ^{24} in Theorem 1.6 of [9] and the $n = 2$ case of Theorem 2.3 in [8]. In [3] the case $s = 32$ is treated in detail. We shall briefly recall the conjecture in Sect. 2 below.

The purpose of this paper is to give a result in the flavor of the above conjecture in the case $8|s$ (the other cases can be treated in a similar way). More precisely, we shall show that for each $s \equiv 0 \pmod 8$ with $s \geq 16$, θ^s up to the addition of an Eisenstein series can be expressed as a rational linear combination of $\frac{s}{4} - 3$ modular forms of weight $\frac{s}{2}$ and level 4, each of which is a product of two specific Eisenstein series of level 4, of type slightly different from that suggested in [3](Sect. 2, Theorem 2). Although the coefficients in these linear combinations for $s > 16$ are not uniquely determined, for each given s they all can be universally computed by solving a certain system of linear equations. We would like to point out that we make use of the fortunate circumstance that θ^s has highest order of vanishing at one cusp. This implies that we can obtain finite explicit formulas for $r_s(n)$ for all n, *without* having any pre-knowledge of $r_s(n)$ for some small n. As numerical examples, we give the cases $s = 16, 24, 32, 40, 48$ explicitly.

What in fact we shall prove is that for each even integer $k \geq 8$ the products $E^0_{k-2\ell} E^{i\infty}_{2\ell}$ $(\ell = 2, 3, \dots, \frac{k}{2} - 2)$ form a set of generators of the space of cusp forms of weight k and level 2 (Sect. 2, Theorem 1). Here $E^0_{2\ell}$ resp., $E^{i\infty}_{2\ell}$ are the normalized Eisenstein series of weight 2ℓ and level 2 for the cusp zero resp., infinity. The proof is based both on the Rankin-Selberg method and on the Eichler-Shimura theory identifying spaces of cusp forms with spaces of periods.

Let

$$T(z) := q^{1/8} \sum_{n \geq 0} q^{n(n+1)/2} \qquad (z \in \mathcal{H}) \tag{1}$$

be the generating function (up to the factor $q^{1/8}$) for the triangular numbers. Then it is well known that T^s $(8|s)$ is a modular form of weight $\frac{s}{2}$ and level 2 and is transformed up to a constant factor into θ^s by acting with the matrix $\begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$. Our assertion about θ^s can be deduced from the above assertions together with the fact that $\operatorname{ord}_\infty T^s = \frac{s}{8}$.

We would like to make one final comment. One easily shows that $T^8 = \frac{1}{64}E_4^0$ –so $T^s = (\frac{1}{64}E_4^0)^{s/8}$ for $8|s$– and in this way for $8|s$ obtains explicit formulas for $r_s(n)$ in terms of products of $\frac{s}{8}$ elementary divisor functions. Two important points of Milne's papers (loc. cit.) are that for $r_s(n)$ with $s = 4j^2$ and resp., $s = 4j^2 + 4j$ he needs only j such functions, and that in his corresponding Eisenstein series expansions of θ to these powers, the maximum weight of any Eisenstein series that appears is $4j - 2$ and $4j$, respectively. The sums of squares and triangular numbers formulas, as well as the corresponding powers of theta series expansions in [4], [10], [12], [14] also all have this same maximum weight of their Eisenstein series. Even though the maximum weight of the Eisenstein series in our expansions is half the power of the theta functions involved, our results seem to be rather optimal from another standpoint. In fact, for arbitary s with $8|s,\ s \geq 16$ to represent $r_s(n)$ we essentially need only two functions of the above type.

The paper is organized as follows. In sect. 2 we state our results in detail. In sect. 3 we compute scalar products of a product of two Eisenstein series of level 2 against a cuspidal Hecke eigenform using Rankin's method as given in [13], while in sect. 4 we apply the Eichler–Shimura theory on periods of cusp forms in the special case of level 2. Finally, sect. 5 gives the proofs of the statements of sect. 2.

Notations. The letter k always denotes an *even integer at least 4*.

If $f : \mathcal{H} \to \mathbf{C}$ is a function and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$, we put as usual

$$(f|_k\gamma)(z) := (ad - bc)^{k/2}(cz + d)^{-k} f(\frac{az + b}{cz + d}) \qquad (z \in \mathcal{H}).$$

We let $\Gamma(1) = SL_2(\mathbf{Z})$ and $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod{N} \right\}$ $(N \in \mathbf{N})$. In this paper, only the cases $N = 1, 2, 4$ matter.

We denote by $M_k(N)$ resp., $S_k(N)$ the space of modular forms resp., cusp forms of weight k with respect to $\Gamma_0(N)$. If $N = 1$ we simply write M_k resp., S_k.

We put $W_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ and denote by $f \mapsto f|_k W_N$ the Fricke involution on $M_k(N)$.

If $f \in S_k(N)$, we let $L(f, s)$ $(s \in \mathbf{C})$ be the Hecke L-function of f defined by analytic continuation of the Dirichlet series $\sum_{n\geq 1} a(n)n^{-s}$ $(\Re(s) >> 0;\ a(n) =$ nth Fourier coefficient of f). Recall that

$$L^*(f, s) := (2\pi)^{-s}N^{s/2}\Gamma(s)L(f, s)$$

satisfies the functional equation

$$L^*(f, k - s) = (-1)^{k/2}L^*(f|_k W_N, s). \qquad (2)$$

If $n \mapsto a(n)\,(n \in \mathbf{N})$ is a number-theoretic function, we put $a(x) := 0$ if $x \notin \mathbf{N}$.

For $\ell \in 2\mathbf{N},\ \ell \geq 4$ we let B_ℓ be the ℓth Bernoulli number. For $\nu \in \mathbf{N}$ we put $\sigma_\nu(n) := \sum_{d|n} d^\nu\ (n \in \mathbf{N})$.

If A is a matrix with complex entries, we denote by A' the transpose of A.

2. Statement of results

We denote by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n \quad (z \in \mathcal{H})$$

the normalized Eisenstein series in M_k.

We let

$$E_k^{i\infty}(z) := \frac{1}{2^k - 1}(2^k E_k(2z) - E_k(z)) \qquad (z \in \mathcal{H})$$

and

$$E_k^0(z) := E_k^{i\infty}|_k W_2(z) = \frac{2^{k/2}}{2^k - 1}(E_k(z) - E_k(2z)) \qquad (z \in \mathcal{H})$$

be the normalized Eisenstein series of weight k on $\Gamma_0(2)$ for the cusp infinity and the cusp zero, respectively.

Recall that $\dim_{\mathbf{C}} M_k(2) = [\frac{k}{4}] + 1$ and $\dim_{\mathbf{C}} S_k(2) = [\frac{k}{4}] - 1$, cf. e.g., [11, Thm., 7.1.4., p. 222].

The functions $E_{k-2\ell}^0 E_{2\ell}^{i\infty}$ $(\ell = 2, 3, \ldots, \frac{k}{2} - 2)$ clearly are in $S_k(2)$.

Theorem 1. *The products* $E_{k-2\ell}^0 E_{2\ell}^{i\infty}$ $(\ell = 2, 3, \ldots, \frac{k}{2} - 2)$ *generate* $S_k(2)$.

Now suppose that $k \geq 8$ and denote by $\epsilon_k(n; \ell)$ $(n \in \mathbf{N};\ \ell = 2, 3, \ldots, \frac{k}{2} - 2)$ the Fourier coefficients of $E_{k-2\ell}^0 E_{2\ell}^{i\infty}$. Define a rational $([\frac{k}{4}] - 1, \frac{k}{2} - 3)$-matrix by

$$A_k := \big(\epsilon_k(n; \ell)\big)_{n=1,\ldots,[\frac{k}{4}]-1;\ell=2,\ldots,\frac{k}{2}-2}.$$

Proposition 1. *The matrix A_k has maximal rank.*

We denote by $e_k^0(n)$ $(n \in \mathbf{N})$ the Fourier coefficients of E_k^0.

Theorem 2. *Suppose that* $s \in \mathbf{N}$ *with* $8|s,\ s \geq 16$. *Put*

$$e^0 := (e_{s/2}^0(1), \ldots, e_{s/2}^0(s/8 - 1)) \in \mathbf{Q}^{s/8-1}$$

and let

$$\lambda = (\lambda_2, \ldots, \lambda_{s/4-2}) \in \mathbf{Q}^{s/4-3}$$

be any rational vector such that

$$A_{s/2}\lambda' = -e^{0'}.$$

Let $T(z)$ *be defined by (1). Then*

$$T^s(z) = 2^{-3s/4}\Big(E^0_{s/2}(z) + \sum_{\ell=2}^{s/4-2} \lambda_\ell E^0_{s/2-2\ell}(z) E^{i\infty}_{2\ell}(z)\Big).$$

Corollary. *Suppose that* $s \in \mathbf{N}$ *with* $8|s,\ s \geq 16$. *Then in the above notation*

$$\theta^s(z) = E^{i\infty}_{s/2}(z + \frac{1}{2}) + \sum_{\ell=2}^{s/4-2} \lambda_\ell E^{i\infty}_{s/2-2\ell}(z + \frac{1}{2}) E^0_{2\ell}(z + \frac{1}{2}).$$

Examples. Using *Mathematica*, we have checked that the following identities hold:

i)

$$\theta^{16}(z) = E^{i\infty}_8(z + \frac{1}{2}) - \frac{2^3}{17} E^{i\infty}_4(z + \frac{1}{2}) E^0_4(z + \frac{1}{2}),$$

ii)

$$\begin{aligned}\theta^{24}(z) = E^{i\infty}_{12}(z + \frac{1}{2}) &- \frac{2 \cdot 17 \cdot 251}{3^2 \cdot 691} E^{i\infty}_8(z + \frac{1}{2}) E^0_4(z + \frac{1}{2}) \\ &- \frac{2^5 \cdot 11^2}{3^2 \cdot 691} E^{i\infty}_6(z + \frac{1}{2}) E^0_6(z + \frac{1}{2}),\end{aligned}$$

iii)

$$\begin{aligned}\theta^{32}(z) = E^{i\infty}_{16}(z + \frac{1}{2}) & \\ &- \frac{2^5 \cdot 691 \cdot 761543}{3^3 \cdot 5^2 \cdot 7 \cdot 257 \cdot 3617} E^{i\infty}_{12}(z + \frac{1}{2}) E^0_4(z + \frac{1}{2}) \\ &- \frac{2^9 \cdot 23 \cdot 31 \cdot 37 \cdot 41}{3^3 \cdot 7 \cdot 257 \cdot 3617} E^{i\infty}_{10}(z + \frac{1}{2}) E^0_6(z + \frac{1}{2}) \\ &- \frac{2^5 \cdot 11 \cdot 17 \cdot 43 \cdot 97}{3^3 \cdot 5^2 \cdot 257 \cdot 3617} E^{i\infty}_8(z + \frac{1}{2}) E^0_8(z + \frac{1}{2}),\end{aligned}$$

iv)

$$\theta^{40}(z) = E_{20}^{i\infty}(z+\frac{1}{2})$$
$$-\frac{2\cdot 73\cdot 257\cdot 3617\cdot 4801\cdot 11281469}{3^6\cdot 5^4\cdot 7^2\cdot 11^2\cdot 31\cdot 41\cdot 283\cdot 617}E_{16}^{i\infty}(z+\frac{1}{2})$$
$$\times E_4^0(z+\frac{1}{2})$$
$$-\frac{2^6\cdot 13\cdot 17\cdot 43\cdot 127\cdot 3833\cdot 782209}{3^6\cdot 5^3\cdot 7\cdot 11^2\cdot 31\cdot 41\cdot 283\cdot 617}E_{14}^{i\infty}(z+\frac{1}{2})$$
$$\times E_6^0(z+\frac{1}{2})$$
$$-\frac{2^4\cdot 17\cdot 691\cdot 228915913339}{3^6\cdot 5^4\cdot 7\cdot 11\cdot 31\cdot 41\cdot 283\cdot 617}E_{12}^{i\infty}(z+\frac{1}{2})E_8^0(z+\frac{1}{2})$$
$$-\frac{2^8\cdot 31\cdot 127\cdot 1213\cdot 68147}{3^5\cdot 5^3\cdot 7^2\cdot 11\cdot 41\cdot 283\cdot 617}E_{10}^{i\infty}(z+\frac{1}{2})E_{10}^0(z+\frac{1}{2}),$$

v)

$$\theta^{48}(z) = E_{24}^{i\infty}(z+\frac{1}{2})$$
$$-\frac{2^3\cdot 19\cdot 31\cdot 41\cdot 283\cdot 617\cdot 4003\cdot 130223\cdot 1679351}{3^7\cdot 5^3\cdot 7^5\cdot 11\cdot 13\cdot 17\cdot 103\cdot 241\cdot 2294797}$$
$$\times E_{20}^{i\infty}(z+\frac{1}{2})E_4^0(z+\frac{1}{2})$$
$$-\frac{2^7\cdot 73\cdot 797\cdot 1361\cdot 5323\cdot 43867\cdot 104297}{3^5\cdot 5^3\cdot 7^5\cdot 11\cdot 13\cdot 103\cdot 241\cdot 2294797}$$
$$\times E_{18}^{i\infty}(z+\frac{1}{2})E_6^0(z+\frac{1}{2})$$
$$-\frac{2^3\cdot 17\cdot 149\cdot 257\cdot 3617\cdot 327876882136907}{3^7\cdot 5^4\cdot 7^5\cdot 11\cdot 13\cdot 103\cdot 241\cdot 2294797}$$
$$\times E_{16}^{i\infty}(z+\frac{1}{2})E_8^0(z+\frac{1}{2})$$
$$-\frac{2^9\cdot 31\cdot 43\cdot 127\cdot 1019\cdot 2003\cdot 139279297}{3^7\cdot 5^3\cdot 7^5\cdot 11\cdot 103\cdot 241\cdot 2294797}$$
$$\times E_{14}^{i\infty}(z+\frac{1}{2})E_{10}^0(z+\frac{1}{2})$$
$$-\frac{2^7\cdot 691^2\cdot 65686986763}{3^7\cdot 5^4\cdot 7^5\cdot 103\cdot 2294797}E_{12}^{i\infty}(z+\frac{1}{2})E_{12}^0(z+\frac{1}{2}).$$

Remarks. The conjecture in [3] referred to in sect. 1 can be stated by saying that for $8|s,\ s\geq 16$, $\theta^s(z)$ can be uniquely expressed as a rational linear combination of the functions $E_{s/2-2\ell}^{i\infty}(z+\frac{1}{2})E_{2\ell}^{i\infty}(z+\frac{1}{2})$ $(\ell = 2, 3, \ldots, \frac{s}{8})$. Note that $\frac{s}{8}-1$ is the dimension of the subspace $\{f \in M_k(2) \mid \mathrm{ord}_\infty f \geq 2\}$. We have not been able so far to prove this conjecture.

In the case of the examples given above, i.e. for $s = 16, 24, \ldots, 48$ it turned out that the functions $E^0_{s/2-2\ell} E^{i\infty}_{2\ell}$ ($\ell = 2, \ldots, \frac{s}{8}$) already give a basis for $S_{s/2}(2)$. In fact using *Mathematica* we have checked that the latter is true for all $s \leq 176$. Given a system of generators for an m dimensional vector space it is not surprising that m elements of this system picked out at random form a basis. What may be more surprising is that for $16 \leq s \leq 176$ all the coefficients of λ_ℓ of $\theta^s(z) - E^{i\infty}_{s/2}(z + \frac{1}{2})$ turned out to be negative.

3. Computation of scalar products

If $f, g \in S_k(2)$, we denote by

$$\langle f, g\rangle = \int_{\Gamma_0(2)\backslash\mathcal{H}} f(z)\overline{g(z)} y^{k-2} dx dy \qquad (x = \Re(z),\ y = \Im(z))$$

the Petersson scalar product of f and g.

For $k \geq 8$ and $\ell \in \{2, 3, \ldots, [\frac{k}{4}]\}$ put

$$c_{k,\ell} := \frac{(k-2)!}{(4\pi)^{k-1}} \cdot \frac{4\ell}{B_{2\ell}} \cdot \frac{2^\ell}{1-2^{2\ell}} \cdot \frac{1}{(1-2^{2\ell-k})\zeta(k-2\ell)}. \tag{3}$$

Proposition 2. *Let $k \geq 8$, $\ell \in \{2, 3, \ldots, [\frac{k}{4}]\}$ and define $c_{k,\ell}$ by (3).*

i) Let f be a newform in $S_k(2)$ which is a normalized Hecke eigenform and let $\epsilon_f \in \{\pm 1\}$ be the eigenvalue of f under W_2. Then

$$\langle f, E^0_{2\ell} E^{i\infty}_{k-2\ell}\rangle = c_{k,\ell} \cdot (\epsilon_f + 2^{-k/2}) L(f, k-1) L(f, k-2\ell).$$

ii) Let f be a normalized Hecke eigenform in S_k and denote by $\lambda_{2,f}$ the eigenvalue of f under the Hecke operator $T_k(2)$ of index 2 on S_k (normalized in the usual way). Then

$$\langle f, E^0_{2\ell} E^{i\infty}_{k-2\ell}\rangle = c_{k,\ell} \cdot \big(1 + 2^{-k+1}(1-\lambda_{2,f})\big) L(f, k-1) L(f, k-2\ell)$$

and

$$\langle f|_k W_2, E^0_{2\ell} E^{i\infty}_{k-2\ell}\rangle = c_{k,\ell} \cdot \big(1 + 2^{-k+1}(1-\lambda_{2,f})\big) L(f, k-1) L(f|_k W_2, k-2\ell).$$

Proof. Denote by $e^0_{2\ell}(n)$ ($n \in \mathbf{N}$) the Fourier coefficients of $E^0_{2\ell}$. For any $f(z) = \sum_{n\geq 1} a(n) q^n \in S_k(2)$ define

$$\mathcal{L}_{f,\ell}(s) := \sum_{n\geq 1} e^0_{2\ell}(n)\overline{a(n)} n^{-s} \qquad \left(\Re(s) > 2\ell + \frac{k-1}{2}\right). \tag{4}$$

Then $\mathcal{L}_{f,\ell}(s)$ has meromorphic continuation to the entire plane, its value at $s = k-1$ is finite (if $k \equiv 2 \pmod 4$, the series (4) of course is already finite at $s = k-1$) and by Rankin's method as applied in [13, p. 144–146] one has

$$\langle f, E^0_{2\ell} E^{i\infty}_{k-2\ell} \rangle = \frac{(k-2)!}{(4\pi)^{k-1}} \mathcal{L}_{f,\ell}(k-1). \tag{5}$$

From the definition of $E^0_{2\ell}$ we find that

$$\mathcal{L}_{f,\ell}(s) = \frac{4\ell}{B_{2\ell}} \cdot \frac{2^\ell}{1-2^{2\ell}} \Big(\sum_{n\geq 1} \sigma_{2\ell-1}(n)\overline{a(n)} n^{-s} - \sum_{n \geq 1} \sigma_{2\ell-1}(\frac{n}{2}) \overline{a(n)} n^{-s} \Big). \tag{6}$$

Now assume that f is a newform in $S_k(2)$ which is a normalized Hecke eigenform. Then in a similar way as in [13, p. 146, eq. (72)] we infer that

$$\sum_{n\geq 1} \sigma_{2\ell-1}(n) a(n) n^{-s} = \frac{L(f,s)L(f,s-2\ell+1)}{\zeta^{(2)}(2s-2\ell-k+2)} \tag{7}$$

where $\zeta^{(2)}(s) := (1-2^{-s})\zeta(s)$.

Indeed, if p is a prime > 2, then the local Euler factor at p on the left-hand side of (7) is computed in exactly the same way as in the case of a Hecke eigenform on $\Gamma(1)$. If $p = 2$, this local Euler factor equals

$$\begin{aligned} \sum_{\nu \geq 0} \sigma_{2\ell-1}(2^\nu) a(2^\nu) 2^{-\nu s} &= \sum_{\nu\geq 0} \sigma_{2\ell-1}(2^\nu)(a(2)2^{-s})^\nu \\ &= (1-a(2)2^{-s})^{-1}(1-a(2)2^{-(s-2\ell+1)})^{-1}. \end{aligned}$$

This proves our claim.

As is well known, $a(2) = -\epsilon_f 2^{k/2-1}$. Therefore we find that

$$\begin{aligned} \sum_{n\geq 1} \sigma_{2\ell-1}(\frac{n}{2}) a(n) n^{-s} &= 2^{-s} \sum_{n\geq 1} \sigma_{2\ell-1}(n) a(2n) n^{-s} \\ &= -\epsilon_f \cdot 2^{-s+k/2-1} \sum_{n \geq 1} \sigma_{2\ell-1}(n) a(n) n^{-s} \\ &= -\epsilon_f \cdot 2^{-s+k/2-1} \cdot \frac{L(f,s)L(f,s-2\ell+1)}{\zeta^{(2)}(2s-2\ell-k+2)}. \end{aligned} \tag{8}$$

Combining (7) and (8) and specializing to $s = k-1$, the assertion of i) now follows from (5).

Let us now prove ii) and assume that $f \in S_k$ is a normalized Hecke eigenform. We shall write $\lambda_2 = \lambda_{2,f}$.

We have

$$\sum_{n\geq 1} \sigma_{2\ell-1}(n) a(n) n^{-s} = \frac{L(f,s)L(f,s-2\ell+1)}{\zeta(2s-2\ell-k+2)} \tag{9}$$

[13, p. 146, eq. (72)].

To compute the second term on the right-hand side of (6), we observe that

$$a(2n) = \lambda_2 a(n) - 2^{k-1} a(\frac{n}{2}) \qquad (n \in \mathbf{N}),$$

hence

$$\sum_{n\geq 1} \sigma_{2\ell-1}(\frac{n}{2}) a(n) n^{-s} = 2^{-s}\Big(\lambda_2 \sum_{n\geq 1} \sigma_{2\ell-1}(n) a(n) n^{-s} - 2^{-s+k-1} \sum_{n\geq 1} \sigma_{2\ell-1}(2n) a(n) n^{-s}\Big). \tag{10}$$

Also

$$\sigma_{2\ell-1}(2n) = (1 + 2^{2\ell-1})\sigma_{2\ell-1}(n) - 2^{2\ell-1}\sigma_{2\ell-1}(\frac{n}{2}) \qquad (n \in \mathbf{N}). \tag{11}$$

Inserting (11) into (10) we deduce that

$$\sum_{n\geq 1} \sigma_{2\ell-1}(\frac{n}{2}) a(n) n^{-s} = \frac{2^{-s}\big(\lambda_2 - (1 + 2^{2\ell-1}) 2^{-s+k-1}\big)}{1 - 2^{-2s+k+2\ell-2}} \sum_{n\geq 1} \sigma_{2\ell-1}(n) a(n) n^{-s}. \tag{12}$$

If we observe (9) and (12) and then specialize (6) to $s = k - 1$, we find after an easy calculation taking into account (5) that the first assertion of ii) holds.

To prove the second assertion of ii), we note that

$$(f|_k W_2)(z) = 2^{k/2} f(2z) \tag{13}$$

and hence find from (6) with f replaced by $f|_k W_2$ that

$$\mathcal{L}_{f|_k W_2,\ell}(s) = 2^{k/2} \cdot \frac{4\ell}{B_{2\ell}} \cdot \frac{2^\ell}{1 - 2^{2\ell}} \Big(\sum_{n\geq 1} \sigma_{2\ell-1}(n) a(\frac{n}{2}) n^{-s} - \sum_{n\geq 1} \sigma_{2\ell-1}(\frac{n}{2}) a(\frac{n}{2}) n^{-s}\Big),$$

i.e.

$$\mathcal{L}_{f|_k W_2,\ell}(s) = 2^{k/2} \cdot \frac{4\ell}{B_{2\ell}} \cdot \frac{2^\ell}{1 - 2^{2\ell}} \cdot 2^{-s} \Big(\sum_{n\geq 1} \sigma_{2\ell-1}(2n) a(n) n^{-s} - \sum_{n\geq 1} \sigma_{2\ell-1}(n) a(n) n^{-s}\Big). \tag{14}$$

By (11), we obtain for the first sum on the right-hand side of (14) the expression

$$(1+2^{2\ell-1})\sum_{n\geq 1}\sigma_{2\ell-1}(n)a(n)n^{-s} - 2^{2\ell-1}\sum_{n\geq 1}\sigma_{2\ell-1}(\frac{n}{2})a(n)n^{-s}.$$

Taking into account (12) and (9) we therefore deduce that

$$\begin{aligned}\mathcal{L}_{f|_kW_2,\ell}(s) = 2^{k/2}\cdot\frac{4\ell}{B_{2\ell}}\cdot\frac{2^\ell}{1-2^{2\ell}} \\ \cdot 2^{-s+2\ell-1}\left(1-\frac{2^{-s}\big(\lambda_2-(1+2^{2\ell-1})2^{-s+k-1}\big)}{1-2^{-2s+k+2\ell-2}}\right) \\ \cdot\frac{L(f,s)L(f,s-2\ell+1)}{\zeta(2s-2\ell-k+2)}.\end{aligned} \tag{15}$$

By (13) clearly

$$L(f|_kW_2,s) = 2^{-s+k/2}L(f,s)$$

and so

$$L(f|_kW_2,k-2\ell) = 2^{-k/2+2\ell}L(f,k-2\ell). \tag{16}$$

Specializing (15) to $s=k-1$ and taking into account (5) with f replaced by $f|_kW_2$ and (16), an easy computation now implies the second assertion of ii). □

4. Periods of cusp forms of level 2

We put $w := k-2$. Recall that if $f\in S_k(N)$ and $n\in\mathbf{Z}$ with $0\leq n\leq w$, then one defines the nth period of f by

$$r_n(f) := \int_0^{i\infty} f(z)z^n dz.$$

Clearly one has

$$r_n(f) = i^{n+1}\cdot n!(2\pi)^{-n-1}L(f,n+1). \tag{17}$$

More generally, if $\gamma\in\Gamma_0(N)\backslash\Gamma(1)$ and $\epsilon := \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, one puts

$$\begin{aligned} r_{n,\gamma}(f) &:= \int_0^{i\infty}(f|_k\gamma)(z)z^n dz,\\ r^+_{n,\gamma}(f) &:= \frac{1}{2i}\big(r_{n,\gamma}(f)+(-1)^n r_{n,\epsilon\gamma\epsilon}(f)\big)\end{aligned}$$

and

$$r^-_{n,\gamma}(f) := \frac{1}{2}\big(r_{n,\gamma}(f)-(-1)^n r_{n,\epsilon\gamma\epsilon}(f)\big).$$

Then an easy consequence of the theory of Eichler–Shimura–Manin is that either the conditions

$$r^{+}_{n,\gamma}(f) = 0 \qquad (\forall\ 0 \le n \le w,\ \forall\ \gamma \in \Gamma_0(N)\backslash\Gamma(1))$$

or the conditions

$$r^{-}_{n,\gamma}(f) = 0 \qquad (\forall\ 0 \le n \le w,\ \forall\ \gamma \in \Gamma_0(N)\backslash\Gamma(1))$$

imply that $f = 0$ ([6], for a good survey see also [1]).

Here in the special case $N = 2$ we shall prove

Proposition 3. *Let $f \in S_k(2)$ and suppose that $r_3(f) = r_5(f) = \cdots = r_{w-3}(f) = 0$. Then $f = 0$.*

Proof. We shall first prove that the conditions stated imply that $r_n(f) = 0$ for all odd n with $1 \le n \le w-1$. We may assume that $w > 2$.

For all $n \in \mathbf{Z},\ 0 \le n \le w$ we have

$$\begin{aligned}\int_0^{-1} f(z) z^n dz &= \int_0^1 f\Big(\frac{-z}{2z-1}\Big)\Big(\frac{-z}{2z-1}\Big)^n d\Big(\frac{-z}{2z-1}\Big)\\ &= (-1)^n \int_0^1 f(z)(2z-1)^{w-n} z^n dz \end{aligned} \tag{18}$$

where in the last line we have used the modularity of f. (Alternatively, to obtain (18) we could have also used that the element $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ has order 2 in $\Gamma_0(2)/\{\pm 1\}$.)

Writing

$$\int_0^{\pm 1} = \int_0^{i\infty} - \int_{\pm 1}^{i\infty}$$

and then substituting $z \mapsto z \pm 1$ in the second integral, we find that (18) is equivalent to

$$\begin{aligned}(-1)^n \sum_{\nu=0}^{n-1} (-1)^{n-\nu} \binom{n}{\nu} r_\nu(f) &= \int_0^{i\infty} f(z)(2z+1)^{w-n}(z+1)^n dz\\ &\quad - \int_0^{i\infty} f(z)(2z-1)^{w-n} z^n dz. \end{aligned} \tag{19}$$

For $n = w$, Eq. (19) says that

$$\sum_{\nu=0}^{w-1} (-1)^\nu \binom{w}{\nu} r_\nu(f) = \sum_{\nu=0}^{w-1} \binom{w}{\nu} r_\nu(f).$$

We deduce that

$$\sum_{1\le\nu\le w-1,\,\nu\equiv 1\pmod 2}\binom{w}{\nu}r_\nu(f)=0,$$

or in other words that

$$r_1(f)+r_{w-1}(f)=-\frac{1}{w}\Big(\sum_{3\le\nu\le w-3,\,\nu\equiv 1\pmod 2}\binom{w}{\nu}r_\nu(f)\Big). \tag{20}$$

Next in (19) we take $n=0$ and obtain

$$\int_0^{i\infty}f(z)\big((2z+1)^w-(2z-1)^w\big)dz=0. \tag{21}$$

In (21) the coefficient at $r_1(f)$ is equal to $4w$ and that at $r_{w-1}(f)$ is equal to $2^w w$. From this we find that we can express $4wr_1(f)+2^w wr_{w-1}(f)$ as a linear combination of $r_3(f),\dots,r_{w-3}(f)$.

Since the matrix

$$\begin{pmatrix}1 & 1\\ 4w & 2^w w\end{pmatrix}$$

is invertible (observe that $w>2$ by assumption), this proves our claim.

We now have to show that the assumption $r_n(f)=0$ for all $1\le n\le w-1$ with n odd implies that $f=0$. For this we use the Eichler-Shimura isomorphism identifying $S_k(2)$ with $H^1_P(\Gamma_0(N),V_{\mathbf{C}})^-$. Here $H^1_P(\Gamma_0(N),V_{\mathbf{C}})$ is the first parabolic cohomology group of $\Gamma_0(N)$ with coefficients in the space $V_{\mathbf{C}}$ consisting of polynomials over $\mathbf{C}$ of degree $\le w$ with the usual left action of $\Gamma_0(N)$ given by

$$\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\circ P\right)(X)=(-cX+a)^w P\left(\frac{dX-b}{-cX+a}\right)$$

and $H^1_P(\Gamma_0(N),V_{\mathbf{C}})^-$ is the minus eigenspace of the involution induced by ϵ. The isomorphism is explicitly given by sending f to the class of $\frac{1}{2}(\pi_f-\tilde{\pi}_f)$ where π_f resp., $\tilde{\pi}_f$ are the cocycles given by

$$\gamma\mapsto\int_0^{\gamma\circ 0}f(z)(X-z)^w dz$$

resp.,

$$\gamma\mapsto\int_0^{-\gamma\circ 0}f(z)(X+z)^w dz.$$

It suffices to show that $\frac{1}{2}(\pi_f-\tilde{\pi}_f)=0$ under our given assumptions. A cocycle is determined by its values on the generators of the group. Since $\Gamma_0(2)$ is generated by $\gamma_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\gamma_2 := \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ (cf., e.g., [2, Thm., 4.3., p. 78]) and $\gamma_1 \circ 0 = 1,\ \gamma_2 \circ 0 = -1$, we have to show that

$$\int_0^1 f(z)(X - z)^w dz - \int_0^{-1} f(z)(X + z)^w = 0. \tag{22}$$

This, however, is obvious. Indeed, writing

$$\int_0^{\pm 1} = \int_0^{i\infty} - \int_{\pm 1}^{i\infty}$$

and then substituting $z \mapsto z \pm 1$ in the second integral, we see that the expression on the left-hand side of (22) is a polynomial in X where the coefficients involve only the odd periods. This proves the assertion of Proposition 3. □

5. Proof of main results

Let us *prove Theorem 1*. One has

$$S_k(2) = S_k^+(2) \oplus S_k^-(2)$$

where $S_k^{\pm}(2)$ denotes the (± 1)-eigenspace of W_2. Since W_2 interchanges the cusps $i\infty$ and zero, it is sufficient to show that the products $E_{2\ell}^0 E_{k-2\ell}^{i\infty}|_k(1 \pm W_2)$ ($\ell = 2, 3, \ldots, [\frac{k}{4}]$) generate $S_k^{\pm}(2)$.

Let $\{f_1, \ldots, f_r, g_1, \ldots, g_s\}$ be a basis of the subspace of newforms of $S_k(2)$ consisting of normalized Hecke eigenforms and suppose that $f_1, \ldots, f_r$ has eigenvalue $+1$ and $g_1, \ldots, g_s$ has eigenvalue -1 under W_2. Let $\{h_1, \ldots, h_t\}$ be a basis of S_k consisting of normalized Hecke eigenforms. Then $\{f_1, \ldots, f_r, h_1|_k(1+W_2), \ldots, h_t|_k(1+W_2)\}$ is a basis of S_k^+ and $\{g_1, \ldots, g_s, h_1|_k(1-W_2), \ldots, h_t|_k(1-W_2)\}$ is a basis of $S_k^-(2)$.

Now let $f \in S_k^+(2)$ say and suppose that

$$\langle f, E_{2\ell}^0 E_{k-2\ell}^{i\infty}|_k(1 + W_2)\rangle = 0 \qquad \left(\forall\ \ell = 2, 3, \ldots, [\frac{k}{4}]\right).$$

Write

$$f = \sum_{\nu=1}^{r} \alpha_\nu f_\nu + \sum_{\mu=1}^{t} \beta_\mu h_\mu|_k(1 + W_2)$$

with $\alpha_\nu, \beta_\mu \in \mathbf{C}$. From Proposition 2 and since W_2 is a hermitian involution, we then infer that

$$\sum_{\nu=1}^{r} \alpha_\nu (1 + 2^{-k/2}) L(f_\nu, k-1) L(f_\nu, k-2\ell)$$
$$+ \sum_{\mu=1}^{t} \beta_\mu \big(1 + 2^{-k+1}(1 - \lambda_{2,h_\mu})\big) L(h_\mu, k-1) L(h_\mu|_k(1 + W_2), k - 2\ell) = 0$$

for all $\ell = 2, 3, \ldots, [\frac{k}{4}]$. In other words, if we put

$$F := \sum_{\nu=1}^{r} \alpha_\nu (1 + 2^{-k/2}) L(f_\nu, k-1) f_\nu + \sum_{\mu=1}^{t} \beta_\mu \big(1 + 2^{-k+1}(1 - \lambda_{2,h_\mu})\big) L(h_\mu, k-1) h_\mu|_k (1 + W_2), \quad (23)$$

then

$$L(F, k - 2\ell) = 0 \qquad (\forall\ \ell = 2, 3, \ldots, [\frac{k}{4}]).$$

Since $F \in S_k^+(2)$, by the functional equation (2) we deduce that

$$L(F, 2\ell) = 0 \qquad (\forall\ \ell = 2, 3, \ldots, \frac{k}{2} - 2).$$

Observing (17) and Proposition 3 we therefore find that $F = 0$. Note that the numbers $(1+2^{-k/2})L(f_\nu, k-1)$ and $\big(1+2^{-k+1}(1-\lambda_{2,h_\mu})\big)L(h_\mu, k-1)$ appearing on the right-hand side of (23) are non-zero. Indeed, since $k > 2$ the Euler products for $L(f_\nu, k-1)$ and $L(h_\mu, k-1)$ are absolutely convergent and by Deligne's theorem one has $|\lambda_{2,h_\mu}| \le 2 \cdot 2^{(k-1)/2}$. We therefore obtain that $\alpha_\nu = 0$ for all ν and $\beta_\mu = 0$ for all μ, i.e. $f = 0$.

If $f \in S_k^-(2)$ we proceed in the same way to conclude that $f = 0$. This finishes the proof of Theorem 1.

Next we shall give the *proof of Proposition 1*. We claim that the functions

$$(E_4^0)^{t-1} E_4^{i\infty}, (E_4^0)^{t-2} (E_4^{i\infty})^2, \ldots, E_4^0 (E_4^{i\infty})^{t-1}$$

in the case $k \equiv 0 \pmod 4$ and

$$(E_4^0)^{t-1} E_6^{i\infty}, (E_4^0)^{t-2} E_4^{i\infty} E_6^{i\infty}, \ldots, E_4^0 (E_4^{i\infty})^{t-2} E_6^{i\infty}$$

in the case $k \equiv 2 \pmod 4$ where $t := [\frac{k}{4}]$ form a basis of $S_k(2)$. Indeed, their number is $[\frac{k}{4}] - 1 = \dim_{\mathbf{C}} S_k(2)$ and the square matrix of size $[\frac{k}{4}] - 1$ whose columns consists of their first (the constant terms being ignored) $[\frac{k}{4}] - 1$ Fourier coefficients is a matrix whose transpose is upper triangular with non-zero entries on the diagonal, hence invertible.

Since every generating system contains a basis, one basis is transformed into another one by acting with an invertible matrix and finally this action is transported over to the matrix of Fourier coefficients, it follows that A_k has maximal rank.

Let us finally give the *proof of Theorem 2 and the Corollary*. As is well known, for $8|s$ one has $T^s \in M_{s/2}(2)$ and

$$(T^s|_{s/2} W_2)(z) = 2^{-3s/4} \theta^s (z + \frac{1}{2}) \quad (24)$$

(cf., e.g., [8]). It also follows from definition (1) that

$$\mathrm{ord}_\infty T^s = \frac{s}{8}. \tag{25}$$

In particular $T^s - 2^{-3s/4} E^0_{s/2} \in S_{s/2}(2)$ and hence by Theorem 1 we can write

$$T^s - 2^{-3s/4} E^0_{s/2} = \sum_{\ell=2}^{s/4-2} \lambda_\ell E^0_{s/2-2\ell} E^{i\infty}_{2\ell} \tag{26}$$

for some complex numbers λ_ℓ ($\ell = 2, 3, \ldots, \frac{s}{4} - 2$). Comparing Fourier coefficients on both sides of (26) and using (25) and Proposition 1, Theorem 2 follows.

The Corollary immediately follows from (24).

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